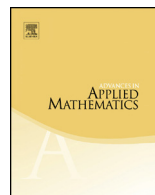




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journal homepage: www.elsevier.com/locate/yaamaOn the ordering of the Markov numbers[☆]Kyungyong Lee^{a,b}, Li Li^{c,*}, Michelle Rabideau^d, Ralf Schiffler^e^a Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487, USA^b School of Mathematics, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea^c Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309-4479, USA^d Department of Mathematics, University of Hartford, West Hartford, CT 06117-1599, USA^e Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, USA

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ABSTRACT

The Markov numbers are the positive integers that appear in the solutions of the equation $x^2 + y^2 + z^2 = 3xyz$. These numbers are a classical subject in number theory and have important ramifications in hyperbolic geometry, algebraic geometry and combinatorics.

It is known that the Markov numbers can be labeled by the lattice points (q, p) in the first quadrant and below the diagonal whose coordinates are coprime. In this paper, we consider the following question. Given two lattice points, can we say which of the associated Markov numbers is larger? A complete answer to this question would solve the uniqueness conjecture formulated by Frobenius in 1913. We give a partial answer in terms of the slope of the line segment that connects the two lattice points. We prove that the Markov number with the greater x -coordinate is larger than the other if the slope is at least $-\frac{8}{7}$ and that it is smaller than the other if the slope is at most $-\frac{5}{4}$.

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As a special case, namely when the slope is equal to 0 or 1, we obtain a proof of two conjectures from Aigner's book "Markov's theorem and 100 years of the uniqueness conjecture".

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1. Introduction

In 1879, Andrey Markov studied the equation

$$x^2 + y^2 + z^2 = 3xyz, \quad (1.1)$$

which is now known as the *Markov equation*. A positive integer solution (m_1, m_2, m_3) of (1.1) is called a *Markov triple* and the integers that appear in the Markov triples are called *Markov numbers*. For example $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 5)$, $(1, 5, 13)$, $(2, 5, 29)$ are Markov triples and 1, 2, 5, 13, 29 are Markov numbers.

The Markov numbers are related to approximation theory. Given a real number α , its Lagrange number $L(\alpha)$ is defined as the supremum of all real numbers L for which there exist infinitely many rational numbers $\frac{p}{q}$ such that $|\alpha - \frac{p}{q}| < \frac{1}{Lq^2}$. Thus the Lagrange number measures how well the real number α can be approximated by rational numbers.

The *Lagrange spectrum* is defined as the set of all Lagrange numbers $L(\alpha)$, where α ranges over all irrational real numbers. Considering it as a subset of the real line, it is known that the Lagrange spectrum is discrete below 3, it is fractal between 3 and a number $F \approx 4.5278$ called the *Freiman number*, and it is continuous above F .

Markov proved in 1879 [13] that the Lagrange spectrum below 3 is precisely the set of all $\sqrt{9m^2 - 4}/m$, where m ranges over all Markov numbers.

In 1913, Frobenius conjectured that, for every Markov number m , there exists a unique Markov triple in which m is the largest number, see [9]. This uniqueness conjecture is still open today. It has inspired a considerable amount of research and the Markov numbers have important ramifications in number theory, hyperbolic geometry, combinatorics and algebraic geometry. For an overview, we refer to the recent textbooks [1, 21]. This uniqueness conjecture also is equivalent to a conjecture saying that exceptional bundles on $\mathbb{P}^2$ are uniquely determined by their ranks up to shift and dual (see [22] and [11, Section 3]; we thank Michael Shapiro for pointing out to us the equivalence of the two conjectures).

1.1. Aigner's conjectures

The Markov numbers can be represented in a binary tree called the *Markov tree*. This Markov tree is combinatorially equivalent to the Farey or Stern-Brocot tree of rational

numbers. Thus there is a correspondence between $\mathbb{Q}_{[0,1]}$ and the Markov numbers by considering corresponding positions in these trees. We henceforth will refer to a Markov number as $m_{\frac{p}{q}}$ where $p < q$ for relatively prime positive integers p and q . For example, $m_{\frac{1}{1}} = 2, m_{\frac{1}{2}} = 5, m_{\frac{1}{3}} = 13, m_{\frac{2}{3}} = 29$.

In his textbook [1], Martin Aigner formulates the following three conjectures on the ordering of the Markov numbers.

Conjecture 1.1.

- (a) $m_{\frac{p}{q}} < m_{\frac{p}{q+i}}$ *constant numerator conjecture*,
- (b) $m_{\frac{p}{q}} < m_{\frac{p+i}{q}}$ *constant denominator conjecture*,
- (c) $m_{\frac{p}{q}} < m_{\frac{p-i}{q+i}}$ *constant sum conjecture*,

for all $i > 0$ such that all the fraction in the index are reduced, positive and less than 1.

The third and fourth authors proved the constant numerator conjecture recently using continued fractions [20]. The denominator conjecture and constant sum conjecture follows from the main result below (see Corollary 1.3).

1.2. Main results

In this article, we prove Conjecture 1.1. Indeed, we prove a much stronger result that we explain next.

Let $F = \{(q, p) \in \mathbb{Z}^2 \mid 1 \leq p < q, \gcd(p, q) = 1\}$ denote the domain of the Markov number function $(q, p) \mapsto m_{\frac{p}{q}}$. For any slope $a \in \mathbb{Q}$ and any y -intercept $b \in \mathbb{Q}$, define the function

$$M_{a,b}: \{(x, y) \in F \mid y = ax + b\} \longrightarrow \mathbb{Z}, (x, y) \mapsto m_{\frac{y}{x}}.$$

We say that the Markov numbers *increase with* x along the line $y = ax + b$ if

$$M_{a,b}(x, y) < M_{a,b}(x', y')$$

whenever $x < x'$. Similarly, we say the Markov numbers *decrease with* x along the line if

$$M_{a,b}(x, y) > M_{a,b}(x', y')$$

whenever $x < x'$. We say that the Markov numbers are *monotonic* along the line $y = ax + b$ if they increase with x or decrease with x along the line.

With this terminology, the three conjectures above can be restated by saying that the Markov numbers increase with x along any line of slope $a = 0$, $a = \infty$ and $a = -1$, respectively.

We are now ready to state our main result.

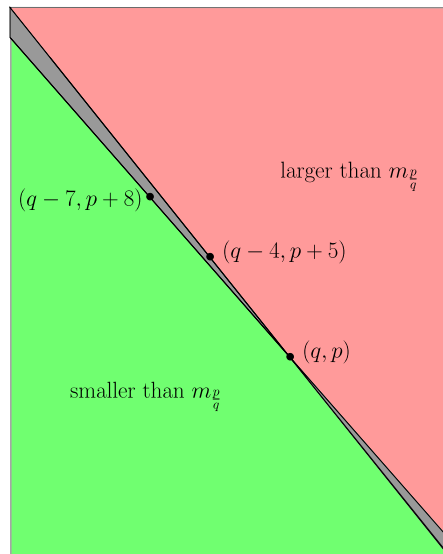


Fig. 1. Relations between Markov numbers. Markov numbers at lattice points in the green region are strictly smaller than $m_{\frac{p}{q}}$ and those at lattice points in the red region are strictly larger than $m_{\frac{p}{q}}$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Theorem 1.2.

- (a) *The Markov numbers increase with x along any line $y = ax + b$ with slope $a \geq -\frac{8}{7} = -1.142857 \dots$.*
- (b) *The Markov numbers decrease with x along any line $y = ax + b$ with slope $a \leq -\frac{5}{4} = -1.25$.*
- (c) *There exist lines of slope $-\frac{6}{5}$ and $-\frac{7}{6}$ along which the Markov numbers are not monotonic.*

Corollary 1.3. *The Conjectures 1.1 hold.*

We also obtain the following result that strengthens both the constant numerator and the constant denominator conjecture.

Theorem 1.4. *We have*

$$m_{\frac{p+i}{q+i}} \geq m_{\frac{p}{q+i}} + m_{\frac{p+i}{q}},$$

whenever the indices are reduced fractions smaller than one.

The order relations between the Markov numbers are illustrated in Fig. 1. The figure shows a neighborhood of the point (q, p) in the plane together with the lines of slope $-\frac{8}{7}$ and $-\frac{5}{4}$, respectively. The lattice points in the green area correspond to Markov numbers that are strictly smaller than $m_{\frac{p}{q}}$ and the lattice points in the red area correspond

to Markov numbers that are strictly larger than $m_{\frac{p}{q}}$. In particular, if the uniqueness conjecture fails, then the second lattice point whose Markov number is equal to $m_{\frac{p}{q}}$ must lie in the grey area. So we have the following corollary of Theorem 1.2.

Corollary 1.5. (a) If $m_{\frac{p}{q}} = m_{\frac{p'}{q'}}$ and $(p, q) \neq (p', q')$, then

$$-\frac{5}{4} < \frac{p-p'}{q-q'} < -\frac{8}{7}$$

(b) Given any Markov number $m_{\frac{p}{q}}$, there are at most $\lfloor \frac{5q+4p+44}{54} \rfloor$ pairs (p', q') satisfying

$$m_{\frac{p'}{q'}} = m_{\frac{p}{q}}, \quad (0 < p' < q', \gcd(p', q') = 1)$$

Proof. (a) follows immediately from Theorem 1.2.

For (b), denote (p_i, q_i) ($1 \leq i \leq \ell$, $1 \leq p_1 \leq p_2 \leq \dots \leq p_\ell$) the set of all pairs (p', q') satisfying the condition in (b). For any $1 \leq i < \ell$, denote $a = p_{i+1} - p_i$, $b = q_i - q_{i+1}$. Then (a) implies $(a, b) \in \mathbb{Z}_{>0}^2$ and $8/7 < a/b < 5/4$. The smallest possible value of a is 6 (in which case $b = 5$), so $p_{i+1} - p_i \geq 6$, thus $p_\ell \geq 1 + 6(\ell - 1)$. On the other hand, p_ℓ is strictly less than the y -coordinate of the intersection point of the two lines $y = x$ and $y - p = -\frac{5}{4}(x - q)$ (which has slope $-5/4$ and passes through the point (q, p)). Since the intersection is $(\frac{5q+4p}{9}, \frac{5q+4p}{9})$, we have $\frac{5q+4p}{9} > p_\ell$. Then $\frac{5q+4p-1}{9} \geq p_\ell \geq 1 + 6(\ell - 1)$, which implies $\ell \leq \lfloor \frac{1}{6}(\frac{5q+4p-1}{9} - 1) + 1 \rfloor = \lfloor \frac{5q+4p+44}{54} \rfloor$. $\square$

Note that in the above proof we have not used the fact that $\gcd(p', q') = 1$. So the corollary also holds if we replace $m_{\frac{p}{q}}$ and $m_{\frac{p'}{q'}}$ by $m_{q,p}$ and $m_{q',p'}$ respectively, and without assuming $\gcd(p, q) = \gcd(p', q') = 1$.

1.3. Ingredients for the proof

The proof uses a connection to cluster algebras. It was observed in [19,2] that the Markov numbers can be obtained from the cluster variables in the cluster algebra of the once-punctured torus by specializing the initial cluster variables to 1. Moreover, the clusters in the cluster algebra then specialize to the Markov triples. On the other hand, the cluster variables can be computed by a combinatorial formula given as a summation over the perfect matchings of a so-called snake graph [16].

Inspired by these results, we introduce a semi-metric, which we call the *Markov distance*, that associates to a pair (A, B) of lattice points in the plane an integer $|AB|$ defined as the number of perfect matchings of an associated snake graph. If the point A is the origin and the point B has coordinates (q, p) with $\gcd(p, q) = 1$ then the Markov distance $|AB|$ is equal to the Markov number $m_{\frac{p}{q}}$. This interpretation allows us to define a number $m_{q,p}$ for pairs of integers that are not relatively prime. We prove all our inequalities in the more general setting of the numbers $m_{q,p}$.

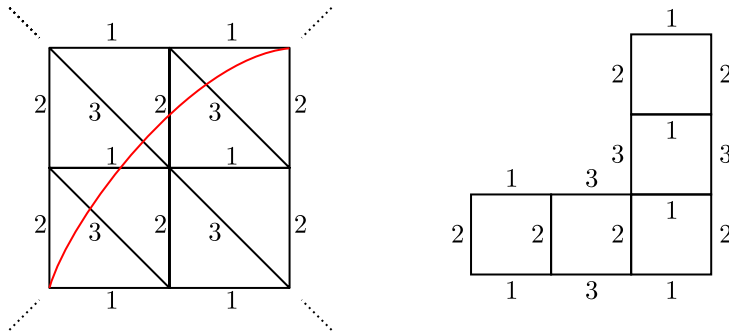


Fig. 2. On the left, the triangulation of the plane $(\mathbb{R}^2, \mathbb{Z}^2)$ in black and an arc in red; on the right, the snake graph of the red arc.

We then use the skein relations, a result from cluster algebras proved in [18], to show that the Markov distance satisfies some fundamental relations. Once these are established, our main result follows from elementary planar geometry arguments.

The paper is organized as follows. Section 2 reviews the connection to cluster algebras, the definition of snake graphs and the skein relations. In Section 3, we introduce the Markov distance and deduce several key properties. We then give an elementary proof of Conjecture 1.1 and Theorem 1.4 in Section 4. Section 5 is devoted to the proof of our main result Theorem 1.2. We end the paper with a comment on the Markov distance in Section 7.

Some of the results of this paper have also been proved recently in several papers using methods quite different from ours. In an independent and simultaneous work [12], C. Lagisquet, E. Pelantová, S. Tavenas, and L. Vuillon proved Conjecture 1.1 using transformations of lattice paths. Shortly thereafter in [14], G. McShane gave a proof of Conjecture 1.1 using methods from hyperbolic geometry, namely a relationship between Markov numbers and the lengths of closed simple geodesics on the punctured torus. More recently in [10], J. Gaster has proved the Conjectures 6.8, 6.11, and 6.12 for Markov numbers (where the full conjectures are for Markov distances); note that Gaster discovered those bounds independently since we did not put these conjectures in the first draft of the paper on arXiv.

2. Relation to cluster algebras

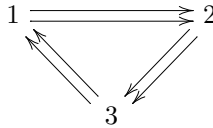
2.1. Cluster algebras

In [8] the authors associate a cluster algebra to an arbitrary surface with marked points. In this paper we only need the cluster algebra of the torus with one puncture.

Let (S, p) be the torus with one puncture p and let $\pi: (\mathbb{R}^2, \mathbb{Z}^2) \rightarrow (S, p)$ be the universal cover. Define a triangulation T of $(\mathbb{R}^2, \mathbb{Z}^2)$ as follows, see Fig. 2. The vertices are the integer lattice points $\mathbb{Z}^2$, and the labeled edges come in the following three families;

- (1) horizontal edges are of the form $(i, j) \rightarrow (i + 1, j)$ and are labeled 1;
- (2) vertical edges are of the form $(i, j) \rightarrow (i, j + 1)$ and are labeled 2;
- (3) diagonal edges are of the form $(i, j) \rightarrow (i + 1, j - 1)$ and are labeled 3.

Via the covering map π , the triangulation T induces a triangulation on the torus (T, p) . The quiver Q of this triangulation is the following, see [8].



Let $\mathcal{A} = \mathcal{A}(Q)$ be the cluster algebra with trivial coefficients associated to this quiver.

2.1.1. Arcs

Arcs can be defined on an arbitrary surface with marked points. We shall only need them here in the plane $(\mathbb{R}^2, \mathbb{Z}^2)$ with marked points the lattice points $\mathbb{Z}^2$, and in the torus with one puncture (S, p) with marked point p .

An *arc* in a $(\mathbb{R}^2, \mathbb{Z}^2)$ is a curve γ from a lattice point A to a lattice point B which does not pass through a third lattice point and does not cross itself. An *generalized arc* in $(\mathbb{R}^2, \mathbb{Z}^2)$ is a curve γ from a lattice point A to a lattice point B which does not pass through a third lattice point and has at most finitely many self-crossings.

An *arc* in (S, p) is a curve γ from p to p that does not visit p except for its endpoints and that does not cross itself. It is well-known that the arcs on (S, p) are precisely the images of line segments OA in $(\mathbb{R}^2, \mathbb{Z}^2)$ from the origin O to a point $A = (q, p)$ such that q, p are coprime integers.

Arcs and generalized arcs are considered up to isotopy, where isotopies do not move curves over marked points. In other words, two curves in $(\mathbb{R}^2, \mathbb{Z}^2)$ represent the same arc if and only if they both start at the same lattice point, they both end at the same lattice point and the region bounded by the two curves does not contain any other lattice points. We shall always assume that our (generalized) arcs are represented by curves that have a minimal number of crossing points with the triangulation T as well as a minimal number of self-crossings. In particular, we assume that our generalized arcs do not contain any kinks (subcurves that are contractible loops).

A *closed loop* is a closed curve that is disjoint from all lattice points and has a finite number of self-crossings. A *multicurve* is a finite multiset of generalized arcs and closed loops such that there are only finitely many crossings among the curves.

It was shown in [8] that the cluster variables in $\mathcal{A}$ are in bijection with the arcs in (S, p) and that the clusters are in bijection with triangulations. In [16] a combinatorial formula was given for the cluster variables and in [17] this formula was used to associate cluster algebra elements to generalized arcs and closed loops as well as to multicurves. If C is a multicurve, we denote by x_C the associated cluster algebra element.

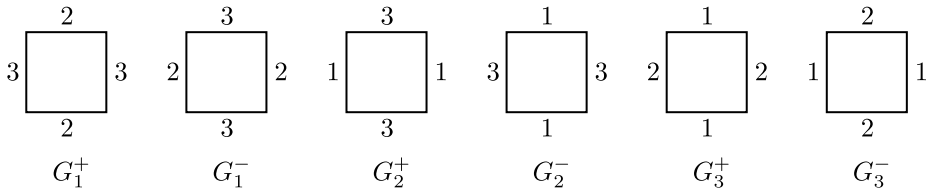


Fig. 3. The six tiles that are the building blocks for the snake graphs.

2.1.2. Skein relations

Let C be a multicurve and let x be a crossing point in C . Thus x could be a crossing point between two curves that are generalized arcs or closed loops, or x could be a self-crossing point of a single generalized arc or a closed loop. The *smoothing* of C at the point x is defined as the pair of multicurves C_+ and C_- , where C_+, C_- are the same as C except for the local change that replaces the crossing $\times$ at the point x with the pair of segments $\succ$ or $\supset$ respectively. Examples are shown in Figs. 7-9.

The following theorem is a special case of a result proved in [18] using hyperbolic geometry and in [3-5] in a purely combinatorial way using snake graphs.

Theorem 2.1 (Skein relations). *Let C be a multicurve with a crossing point x and let C_-, C_+ be the multicurves obtained by smoothing C at x . Then we have the following identity in the cluster algebra associated to the surface.*

$$x_C = x_{C_+} + x_{C_-}$$

2.1.3. Snake graphs

Let γ be a generalized arc in $(\mathbb{R}^2, \mathbb{Z}^2)$ from a point A to a point B . Denote by $p_1, p_2, \dots, p_n$ the crossing points of γ with T in order along γ . Let $i_j \in \{1, 2, 3\}$ be the label of the edge in T that contains p_j . The sequence $(i_1, i_2, \dots, i_n)$ is called the *crossing sequence* of γ . It determines γ up to translation and rotation by 180° . Note that $i_{j+1} \neq i_j$, for all $j = 1, 2, \dots, n-1$, because γ has a minimal number of crossings with T . Moreover, since there are exactly three labels for the edges in T , we can define b_j to be the unique label such that $\{i_j, i_{j+1}, b_j\} = \{1, 2, 3\}$.

To every (generalized) arc γ one can associate a planar graph $\mathcal{G}(\gamma)$ called the *snake graph* of γ as follows. This construction was introduced in [15,16] for arcs on arbitrary surfaces. Here we give a simpler description adapted to our situation. For each $i = 1, 2, 3$, we define two *tiles* G_i^ϵ , with $\epsilon \in \{+, -\}$, as the labeled graphs shown in Fig. 3. We remark that adding the diagonal from the northwest corner to the southeast corner in the tile G_i^+ and labeling this diagonal by i will result in a graph that is a subgraph of the triangulation T . On the other hand, adding the same labeled diagonal to the tile G_i^- will produce the mirror image of this subgraph.

The snake graph $\mathcal{G}(\gamma)$ is defined recursively as follows. Let $i_1, i_2, \dots, i_n$ be the crossing sequence of γ . We start by laying down the tile $G_{i_1}^+$ exactly as it is shown in Fig. 3. Next,

we join the tile $G_{i_2}^-$ by identifying the unique edge labeled b_1 on the north or east of $G_{i_1}^+$ to the unique edge labeled b_1 on the south or west of $G_{i_2}^-$, where we use the notation b_1 defined in the previous subsection. Note that this gluing edge is the north edge in $G_{i_1}^+$ if and only if it is the south edge in $G_{i_2}^-$, and it is the east edge in $G_{i_1}^+$ if and only if it is the west edge in $G_{i_2}^-$. Recursively, we join the tile $G_{i_{j+1}}^\epsilon$, with $\epsilon \in \{+, -\}$, to the graph consisting of $G_{i_1}^+, G_{i_2}^-, G_{i_3}^+, \dots, G_{i_j}^\epsilon$ by identifying the unique edge labeled b_j on the north or east of $G_{i_j}^\epsilon$ to the unique edge labeled b_j on the south or west of $G_{i_{j+1}}^\epsilon$.

For example, if γ is the arc shown in Fig. 2 then the crossing sequence is 3, 1, 3, 2, 3 and the snake graph is built from the tiles $G_3^+, G_1^-, G_3^+, G_2^-, G_3^+$. This graph is shown in the right picture in Fig. 2.

It is known that the unlabeled snake graphs are in bijection with continued fractions [6]. This relation was crucial in the proof of part (a) of Conjecture 1.1 in [20].

2.1.4. Length function

In order to introduce our length function we need the following concept from graph theory. A *perfect matching* of a graph is a subset of the set of edges such that every vertex of the graph belongs to exactly one edge of the perfect matching.

Definition 2.2. Let γ be a generalized arc in $(\mathbb{R}^2, \mathbb{Z}^2)$. We define its *length* $|\gamma|$ to be the number of perfect matchings of its snake graph $\mathcal{G}(\gamma)$.

Note that $|\gamma|$ is a positive integer and $|\gamma| = 1$ if and only if $\gamma \in T$.

2.1.5. Relation between the cluster algebra and the Markov numbers

As we have seen above, the cluster variables in the cluster algebra $\mathcal{A}$ of the once-punctured torus correspond to the line segments $\ell_{\frac{p}{q}}$ from $(0, 0)$ to (q, p) in $(\mathbb{R}^2, \mathbb{Z}^2)$ with $\gcd(p, q) = 1$. We denote the cluster variable of $\ell_{\frac{p}{q}}$ by $x_{\frac{p}{q}}$. Each cluster variable is a Laurent polynomial in three variables x_1, x_2, x_3 , which can be computed by a combinatorial formula as a sum over all perfect matchings of the snake graph $\mathcal{G}(\ell_{\frac{p}{q}})$ associated to $\ell_{\frac{p}{q}}$ [16].

It is shown in [2, 19] that the specialization of the cluster variable $x_{\frac{p}{q}}$ at $x_1 = x_2 = x_3 = 1$ is equal to the Markov number $m_{\frac{p}{q}}$ of slope $\frac{p}{q}$, where $0 < p < q$ and $\gcd(p, q) = 1$. Therefore, the Markov number is equal to the number of perfect matchings of the snake graph $\mathcal{G}(\ell_{\frac{p}{q}})$. In terms of our length function, this can be restated as follows.

$$m_{\frac{p}{q}} = |\ell_{\frac{p}{q}}|. \quad (2.1)$$

Thus looking at the notation $m_{\frac{p}{q}}$ from another perspective, $\frac{p}{q}$ is not only the corresponding rational from the Farey tree, but it also refers to the slope of the line from the origin to (q, p) from which a Markov snake graph can be constructed and the associated continued fraction yields the Markov number $m_{\frac{p}{q}}$ itself. In Section 3, we shall generalize this correspondence to pairs of integers (q, p) that are not necessarily coprime.

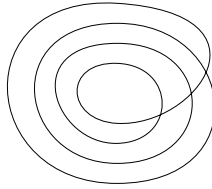


Fig. 4. A 4-bracelet.

It follows from the above discussion that every skein relation $x_C = x_{C_-} + x_{C_+}$ induces an integer equation $|x_C| = |x_{C_-}| + |x_{C_+}|$. In particular, we have the following special case.

Corollary 2.3 (*Generalized Ptolemy equality*). *Let $\alpha, \beta, \delta, \epsilon$ be arcs in $(\mathbb{R}^2, \mathbb{Z}^2)$ that form a quadrilateral Q that does not contain any lattice points besides the four endpoints of the arcs. Suppose that α, δ are opposite sides, and β, ϵ are opposite sides of Q . Let γ_1 and γ_2 be the two diagonals of Q . Then the length function satisfies*

$$|\gamma_1| |\gamma_2| = |\alpha| |\delta| + |\beta| |\epsilon|.$$

2.1.6. Bracelets

Let ζ be a closed simple curve. Following [17], we let $\text{Brac}_k \zeta$ be the k -fold concatenation of ζ with itself, see Fig. 4. $\text{Brac}_k \zeta$ is called the k -bracelet of ζ .

Lemma 2.4. *Let ζ be a closed simple curve in $(\mathbb{R}^2, \mathbb{Z}^2)$ that is contractible to a lattice point. Then $|\text{Brac}_k \zeta| = 2$, for all $k > 0$.*

Proof. This follows easily from the skein relations and induction. $\square$

3. The Markov distance

In this section we introduce a semi-metric called Markov distance between to lattice points as the number of perfect matchings of an associated snake graph.

Let A and B be two lattice points and ℓ_{AB} the line segment from A to B . Denote by (s, r) the coordinates of $B - A$. Thus ℓ_{AB} has slope $\frac{r}{s}$. If $\gcd(r, s) = 1$, then there are no lattice points on the line segment ℓ_{AB} besides A and B , and in this case, the line segment ℓ_{AB} is an arc. Otherwise, we let p, q be relatively prime integers such that $\frac{r}{s} = \frac{p}{q}$ and denote by $P_0 = A, P_1, P_2, \dots, P_t = B$ the sequence of lattice points along ℓ_{AB} . Thus $t = \gcd(r, s)$ and $P_i = A + i(q, p)$, for $i = 1, 2, \dots, t$.

Definition 3.1. With the notation above, we define the *left deformation* γ_{AB}^L of the line segment ℓ_{AB} to be the arc from A to B that is an infinitesimal deformation of ℓ_{AB} passing on the left of the points $P_1, P_2, \dots, P_t$ with respect to the orientation from A to B , see Fig. 5.

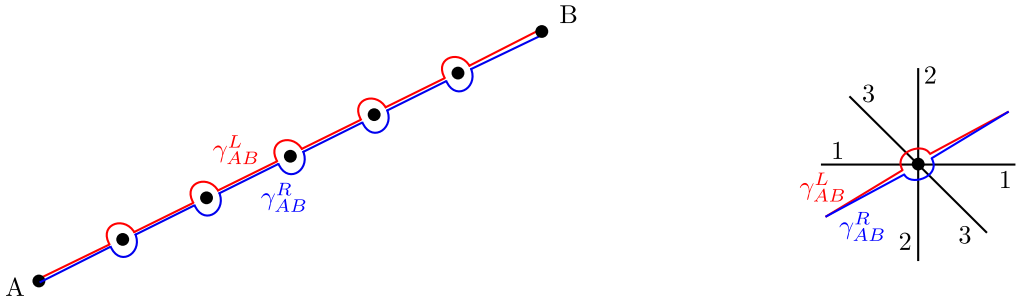


Fig. 5. The arcs γ_{AB}^L and γ_{AB}^R (left) and their local crossing pattern with the triangulation in the vicinity of a lattice point (right).

Similarly, the *right deformation* γ_{AB}^R of ℓ_{AB} is the arc from A to B that is an infinitesimal deformation of the line segment ℓ_{AB} passing on the right of the points $P_1, P_2, \dots, P_t$.

In the case where ℓ_{AB} is already an arc, we have $\gamma_{AB}^L = \gamma_{AB}^R = \ell_{AB}$.

Lemma 3.2. *Let A and B be two lattice points in the plane and let γ_{AB}^L and γ_{AB}^R be the left and right deformations of the line segment ℓ_{AB} . Then both deformations have the same length, thus*

$$|\gamma_{AB}^L| = |\gamma_{AB}^R|.$$

Proof. The line segment ℓ_{AB} can be decomposed as the concatenation of line segments $\ell_{AB} = \ell_{AP_1} \ell_{P_1P_2} \ell_{P_2P_3} \cdots \ell_{P_{t-1}B}$. Each of the line segments $\ell_{P_iP_{i+1}}$ is an arc with the same crossing sequence $(i_1, i_2, \dots, i_n)$. Moreover, since the crossing sequences of line segments are palindromic, we have $(i_1, i_2, \dots, i_n) = (i_n, i_{n-1}, \dots, i_1)$.

Every lattice point P_i is incident to six edges of the triangulation, see the right picture in Fig. 5. The arc γ_{AB}^L crosses the three edges labeled 1, 3, 2 in order on the left of P_i and the arc γ_{AB}^R crosses the edges labeled 2, 3, 1 in order on the right of P_i . Thus the crossing sequence of γ_{AB}^L is $(i_1, i_2, \dots, i_n, 1, 3, 2, i_1, i_2, \dots, i_n, 1, 3, 2, \dots, i_1, i_2, \dots, i_n)$ and the crossing sequence of γ_{AB}^R is $(i_1, i_2, \dots, i_n, 2, 3, 1, i_1, i_2, \dots, i_n, 2, 3, 1, \dots, i_1, i_2, \dots, i_n)$. Note that the second sequence is the reverse of the first, since $(i_1, i_2, \dots, i_n) = (i_n, i_{n-1}, \dots, i_1)$. This implies that the snake graph of γ_{AB}^L is obtained from the snake graph of γ_{AB}^R by a rotation of 180° [7, Proposition 3.1]. In particular, the two snake graphs have the same number of perfect matchings, thus $|\gamma_{AB}^L| = |\gamma_{AB}^R|$. $\square$

The lemma motivates the following definition.

Definition 3.3. The *Markov distance* $|AB|$ between two lattice points A, B in the plane is the semi-metric defined by the integer

$$|AB| = |\gamma_{AB}^L|.$$

We also set $|AA| = 0$.

The Markov distance between the origin $O = (0, 0)$ and a point $A = (q, p)$ will be denoted by

$$m_{q,p} = |OA|.$$

Remark 3.4. Thus if q, p are coprime then $m_{q,p} = m_{\frac{p}{q}}$ is the Markov number of slope $\frac{p}{q}$.

We are now ready for the main result of this section. It can be thought of as the analogue of the statement that the straight line is the shortest curve between two points in the plane, when replacing the Euclidean distance by the Markov distance. This result is the key to the proof of the conjectures. Recall that a generalized arc is allowed to cross itself finitely many times [18].

Theorem 3.5. Let A and B be two lattice points in the plane and γ_{AB}^L the left deformation of the line segment ℓ_{AB} . Let γ be a generalized arc from A to B . Then

$$|AB| \leq |\gamma|,$$

and equality holds if and only if γ is homotopic to γ_{AB}^L or γ_{AB}^R . (In the special case when $A = B$, the equality holds if and only if γ is homotopic to a contractible curve starting and ending in A .)

Proof. Let γ be a generalized arc from A to B and such that $|\gamma|$ is minimal among all arcs from A to B . We want to show that $\gamma = \gamma_{AB}^L$ or $\gamma = \gamma_{AB}^R$.

We start by showing that γ has no self-crossing. Suppose to the contrary that γ does have a self-crossing at a point x . Then we can write γ as a concatenation of subcurves $\gamma = \gamma_1\gamma_2\gamma_3$, where γ_1 runs from A to the crossing point x , γ_2 is a non-contractible closed curve starting and ending at x and γ_3 runs from x to B . The skein relation (Theorem 2.1) applied to the smoothing of the crossing at x implies

$$|\gamma| = |\gamma_1\gamma_3| + |\epsilon|$$

where ϵ is the generalized arc obtained in the smoothing. Note that, since γ has no kink, ϵ also does not contain a kink and thus $|\epsilon| > 0$. Moreover $|\gamma_2| > 0$, since γ_2 is non-contractible. Thus

$$|\gamma| > |\gamma_1\gamma_3|$$

with $\gamma_1\gamma_3$ a generalized arc from A to B . This contradicts the minimality of $|\gamma|$.

Thus γ is a simple curve in the plane from A to B that does not meet any other lattice points besides A and B . We may think of the plane as a board with a peg in each lattice point and γ as a string lying on this board. If we now pull this string taut it will touch a

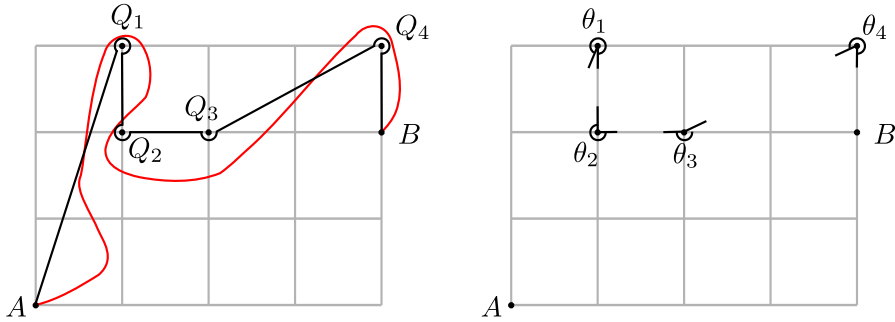


Fig. 6. Proof of Theorem 3.5. The construction of the path and the angles. The left picture shows an arc γ in red and the associated path $l_1a_1 \dots, l_m$ in black. The right picture shows the angles θ_i . Here $\theta_1, \theta_4 \in [-2\pi, -\pi]$ and $\theta_2, \theta_3 \in [\pi, 2\pi]$.

sequence of pegs in the board. This image shows that there exists a unique sequence of lattice points $A = Q_0, Q_1, \dots, Q_m = B$ each equipped with a circle C_i of infinitesimally small radius and centered at $Q_i, i = 1, 2, \dots, m - 1$ such that γ is homotopic to the path

$$l_1a_1l_2a_2 \dots l_{m-1}a_{m-1}l_m$$

defined as follows, see Fig. 6.

- l_1 is the straight line segment that starts at $Q_0 = A$ and ends at a point R_1 on the circle C_1 , and a_1 is an arc along the circle C_1 from R_1 to a point S_1 ;
- for $i = 2, \dots, m - 1$, l_i is the straight line segment that starts at S_{i-1} on the circle C_{i-1} and ends at a point R_i on the circle C_i , and a_i is an arc along the circle C_i from R_i to a point S_i ;
- l_m is the straight line segment that starts at S_{m-1} on the circle C_{i-1} and ends at $Q_m = B$.

We define a sequence of angles $\theta_1, \theta_2, \dots, \theta_{m-1}$ by setting θ_i equal to the angle between the line segments $Q_{i-1}Q_i$ and Q_iQ_{i+1} following the curve γ . It follows that $\theta_i \in [-2\pi, -\pi] \cup [\pi, 2\pi]$, indeed, if $-\pi < \theta < \pi$ then the point Q_i would not be a point in the above sequence.

With this notation, the arc γ is equal to the arc γ_{AB}^L , or γ_{AB}^R respectively, if and only if all angles θ_i are equal to π , or $-\pi$ respectively. Assume γ is not equal to γ_{AB}^L or γ_{AB}^R . Without loss of generality, we may assume $\theta_1 > 0$. Let θ_{s+1} be the first angle that is not equal to π . We consider three cases.

(1) Suppose $\theta_{s+1} \in [-2\pi, -\pi]$. Since $\theta_1 = \theta_2 = \dots = \theta_s = \pi$, there is a pair (p, q) of relatively prime integers such that each vector $\overrightarrow{Q_iQ_{i+1}} = \begin{pmatrix} p \\ q \end{pmatrix}$, for $i = 1, 2, \dots, s$. Let $D = Q_s + (p, q)$, see Fig. 7, and consider the arc γ_{AD}^L . Note that $D = Q_{s+1}$, because the angle $\theta_s = \pi$. Applying the skein relations (Theorem 2.1) to the product of γ and γ_{AD}^L by smoothing their crossing near D , we obtain

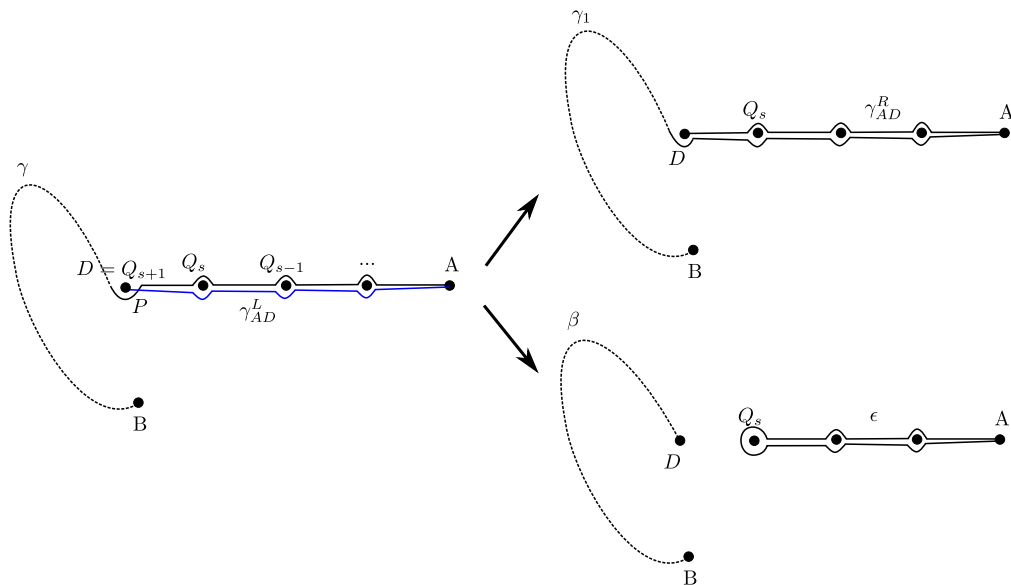


Fig. 7. Proof of Theorem 3.5. Step 3, Case 1.

$$|\gamma_{AD}^L| |\gamma| = |\gamma_1| |\gamma_{AD}^R| + |\epsilon| |\beta|,$$

where $\gamma_1, \beta, \epsilon$ are the arcs obtained from the smoothing as shown in Fig. 7. Lemma 3.2 implies $|\gamma_{AD}^L| = |\gamma_{AD}^R|$ and thus $|\gamma| > |\gamma_1|$, a contradiction to the minimality of $|\gamma|$.

(2) Suppose $\theta_{s+1} \in [\pi, 2\pi)$. Again there is a pair (p, q) of relatively prime integers such that each vector $\overrightarrow{Q_i Q_{i+1}} = \begin{pmatrix} p \\ q \end{pmatrix}$, for $i = 1, 2, \dots, s$, and we let $D = Q_{s+1} = Q_s + (p, q)$, see Fig. 8. Let $E = D + (p, q)$. Note that E is not one of the points Q_i . Applying the skein relation to the product of γ and γ_{AE}^L by smoothing their crossing near D , we obtain

$$|\gamma_{AE}^L| |\gamma| = |\gamma_1| |\gamma_{AE}^R| + |\epsilon| |\beta|,$$

where $\gamma_1, \beta, \epsilon$ are the arcs obtained from the smoothing as shown in Fig. 8. Again Lemma 3.2 allows us to conclude $|\gamma| > |\gamma_1|$, which is a contradiction to the minimality of $|\gamma|$.

(3) Suppose $\theta_{s+1} = 2\pi$. In this case $Q_s = Q_{s+2}$. Let $D = Q_{s+1}$, see Fig. 9. If all subsequent angles are equal to π then B lies on the line through A and D and γ is obtained from the concatenation of $\gamma_{AD}^R \gamma_{DB}^R$ by an infinitesimal deformation avoiding the point D on the right. Clearly $|\gamma| > \gamma_{AB}^L$ in this case, a contradiction.

Suppose therefore that at least one of the angles after θ_{s+1} is different from π . Let $t > 0$ be the least integer such that $\theta_{s+1+t} \neq \pi$.

If $t < s$ then we must have $\theta_{s+1+t} \in [-2\pi, -\pi]$ since γ has no self-crossing. This case is illustrated in Fig. 9. Similarly to the case (1), applying the skein relation to the product of γ and γ_{AD}^L by smoothing their crossing near D , we obtain

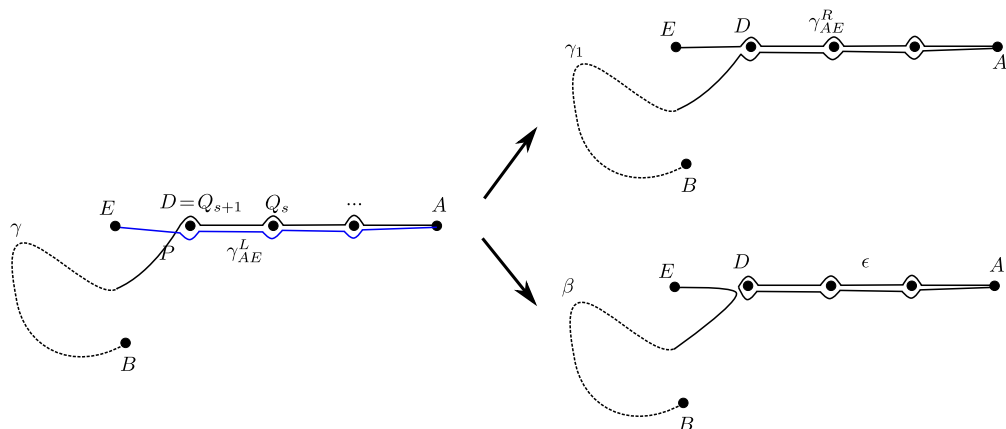


Fig. 8. Proof of Theorem 3.5 Case 2.

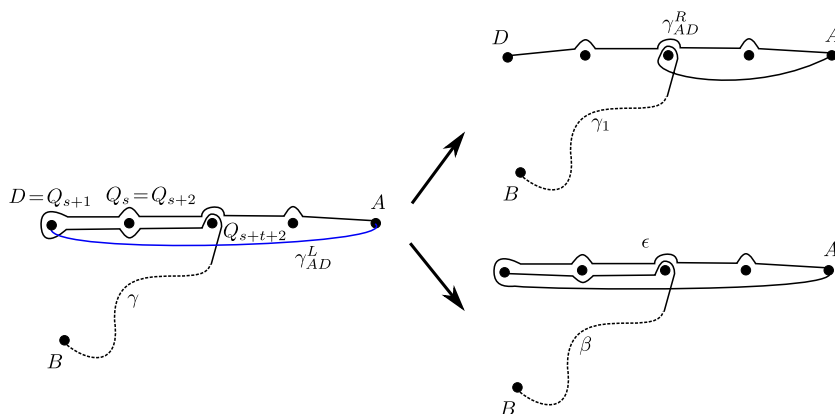


Fig. 9. Proof of Theorem 3.5. Case 3. with $t < s$.

$$|\gamma_{AD}^L| |\gamma| = |\gamma_1| |\gamma_{AD}^R| + |\epsilon| |\beta|,$$

where $\gamma_1, \epsilon, \beta$ are the arcs obtained in the smoothing, see Fig. 9. Again we conclude $|\gamma| > |\gamma_1|$, which is a contradiction to the minimality of $|\gamma|$.

We do not illustrate the case $t > s$ here, but it also leads to a contradiction using an argument similar to the one in case (2) above. Thus each of the cases (1)-(3) leads to a contradiction, and therefore we must have that all angles θ_i are equal to π . Thus $\gamma = \gamma_{AB}^L$. $\square$

As an immediate consequence, we obtain the following Ptolemy inequality.

Corollary 3.6 (Ptolemy inequality). *Given four points A, B, C, D in the plane such that the straight line segments $\ell_{AB}, \ell_{BC}, \ell_{CD}, \ell_{DA}$ form a convex quadrilateral with diagonals ℓ_{AC} and ℓ_{BD} , we have*

$$|AC||BD| \geq |AB||CD| + |AD||BC|.$$

Proof. Consider the arcs γ_{AC}^L and γ_{BD}^L given as the left deformations of the diagonals. Let α_{AB} be the arc from A to B defined by following γ_{AC}^L up to its crossing point with γ_{BD}^L and then following γ_{BD}^L up to B . Similarly, we can define arcs α_{BC}, α_{CD} and α_{DA} to obtain a curved quadrilateral with corner vertices A, B, C, D that does not contain any other lattice points. Then Corollary 2.3 implies

$$|\gamma_{AC}^L||\gamma_{BD}^L| = |\alpha_{AB}||\alpha_{CD}| + |\alpha_{BC}||\alpha_{DA}|.$$

The left hand side of this equation is equal to $|AC||BD|$, by definition of the Markov distance, and thus the result follows from Theorem 3.5. $\square$

Remark 3.7. The Markov distance is a semi-metric but not a metric. It does not satisfy the triangle inequality because the deformations may create too many crossing points. For example, take the three colinear points $O = (0, 0), A = (1, 0), B = (2, 0)$. Then $|OA| = |AB| = 1$ but $|OB| = 3$, because the deformation γ_{OB}^L crosses two arcs of the triangulation. For a non-colinear example, take $D = (3, 0), C = (1, 1)$. Then each of the line segments ℓ_{OC}, ℓ_{CD} crosses exactly one arc of the triangulation, hence $|OC| = |CD| = 2$. However $|OD| = 8$.

However, there is a variation of the triangle inequality which we shall prove in Section 7.

4. Proof of Aigner's conjectures

We shall now prove Conjecture 1.1 from Aigner's book in the following theorem. We actually prove more general statements in Theorem 4.1 about the numbers $m_{q,p}$ introduced in Definition 3.3 for arbitrary lattice point $0 \leq p \leq q$. Recall that $m_{q,p} = m_{\frac{p}{q}}$ is a Markov number if $0 < p < q$ and p, q are relatively prime.

Note that the theorem is proved independently by C. Lagisquet, E. Pelantová, S. Tavenas, L. Vuillon in [12] using a different method.

Theorem 4.1. *For all integers $0 \leq p \leq q$, we have the following inequalities.*

$$m_{q,p} < m_{q,p+1}, \tag{4.1}$$

$$m_{q,p} < m_{q+1,p}, \tag{4.2}$$

$$m_{q,p} < m_{q+1,p-1}, \quad (\text{if } p \geq 1). \tag{4.3}$$

Proof. Let $A = (0, 0), B = (q, p), C = (q+1, p)$ and $D = (q+1, p-1)$, see Fig. 10. The Ptolemy inequality (Corollary 3.6) implies

$$|AC||BD| \geq |AB||CD| + |AD||BC|.$$

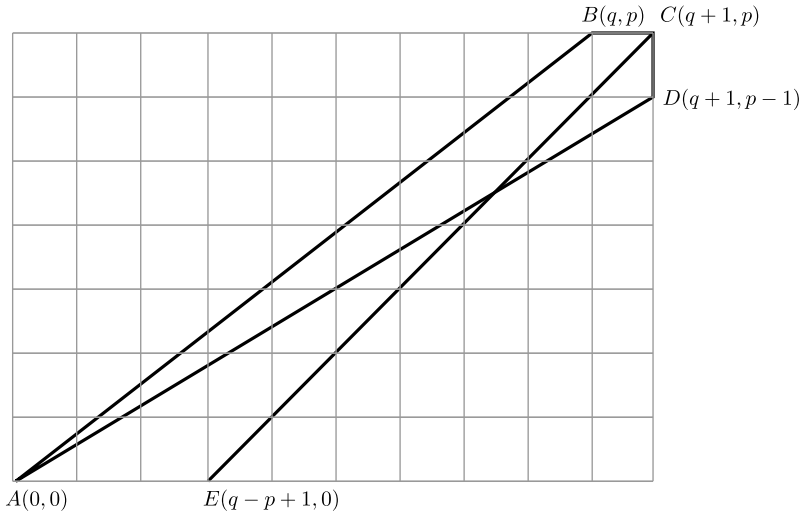


Fig. 10. Proof of Theorem 4.1.

But $|BD| = |CD| = |BC| = 1$, since the corresponding line segments are arcs of the triangulation T . So

$$|AC| \geq |AB| + |AD|. \quad (4.4)$$

In particular, we have two strict inequalities $|AC| > |AB|$ and $|AC| > |AD|$, and the former proves (4.2) while the latter proves (4.1).

In order to show (4.3), let $E = (q-p+1, 0)$, see Fig. 10. Then the coordinates of $B-E$ and $D-E$ are $(p-1, p)$ and $(p, p-1)$, respectively, and therefore we have $|BE| = |DE|$.

Consider the quadrilateral defined by the points A, B, D, E . The Ptolemy inequality implies

$$|AD| |BE| \geq |AB| |DE| + |BD| |AE|, \quad (4.5)$$

hence

$$|AD| |BE| > |AB| |DE|,$$

and thus the inequality (4.3) follows since $|BE| = |DE|$. $\square$

Corollary 4.2. *The Conjectures 1.1 hold.* $\square$

From our proof, we also obtain the following stronger result.

Theorem 4.3. *For all integers $0 \leq p \leq q$, we have*

$$m_{q+1,p} \geq m_{q,p} + m_{q+1,p-1}.$$

Proof. This follows directly from equation (4.4). $\square$

Example 4.4. $433 = m_{5,3} > m_{4,3} + m_{5,2} = 169 + 194 = 363$.

If we use this result repeatedly

$$\begin{aligned} m_{q+i,p+i} &\geq m_{q+i-1,p+i} + m_{q+i,p+i-1} \\ &\geq m_{q+i-2,p+i} + 2m_{q+i-1,p+i-1} + m_{q+i,p+i-2} \\ &\geq m_{q+i-3,p+i} + 3m_{q+i-2,p+i-1} + 3m_{q+i-1,p+i-2} + m_{q+i,p+i-3} \\ &\vdots \end{aligned}$$

we have proved the following corollary.

Corollary 4.5. For all integers $0 \leq p \leq q$ and $i > 0$ we have

$$m_{q+i,p+i} \geq \sum_{j=0}^i \binom{i}{j} m_{q+j,p+i-j}.$$

In particular, Theorem 1.4 holds.

5. The main result

In this section we study the monotonicity of the Markov numbers on lines of a given slope. To be more precise, let $F = \{(q, p) \in \mathbb{Z}^2 \mid 1 \leq p < q, \gcd(p, q) = 1\}$ denote the domain of the Markov number function $(q, p) \mapsto m_{\frac{p}{q}}$. For any slope $a \in \mathbb{Q}$ and any y -intercept $b \in \mathbb{Q}$, define the function

$$M_{a,b}: \{(x, y) \in F \mid y = ax + b\} \longrightarrow \mathbb{Z}, \quad (x, y) \mapsto m_{\frac{y}{x}}.$$

We say that the Markov numbers *increase with x* , respectively *decrease with x* , along the line $y = ax + b$ if $M_{a,b}(x, y) < M_{a,b}(x', y')$ whenever $x < x'$, respectively if $M_{a,b}(x, y) > M_{a,b}(x', y')$ whenever $x < x'$. We say that the Markov numbers are monotonic along the line $y = ax + b$ if they increase with x or decrease with x along the line.

It seems that there are critical slopes $a_1 < a_2 < -1$ such that

- (i) the Markov numbers increase with x along any line of slope $a \geq a_2$;
- (ii) the Markov numbers decrease with x along any line of slope $a \leq a_1$;
- (iii) for every a with $a_1 < a < a_2$, there exists a line of slope a along which the Markov numbers are not monotonic.

At this time, we do not know what the numbers a_1 and a_2 should be, but the following theorem settles the question for slopes of the form $a = -\frac{n+1}{n}$.

Theorem 5.1 (Theorem 1.2).

- (a) The Markov numbers increase with x along any line $y = ax + b$ with slope $a \geq -\frac{8}{7}$.
- (b) The Markov numbers decrease with x along any line $y = ax + b$ with slope $a \leq -\frac{5}{4}$.
- (c) There exist lines of slope $-\frac{6}{5}$ and $-\frac{7}{6}$ along which the Markov numbers are not monotonic.

Remark 5.2. It is not hard to see that the three conjectures proved in Theorem 4.1 imply that the Markov numbers increase with x on all lines of slope $a \geq -1$. Thus part (a) of Theorem 1.2 sharpens the results of Theorem 4.1.

Remark 5.3. Part (c) of the theorem is proved by the following examples.

$$\begin{aligned} m_{\frac{13}{14}} &> m_{\frac{7}{19}} < m_{\frac{1}{24}} & \text{on } y = -\frac{6}{5}x + \frac{149}{5} \\ m_{\frac{16}{17}} &> m_{\frac{9}{23}} < m_{\frac{2}{29}} & \text{on } y = -\frac{7}{6}x + \frac{215}{6} \end{aligned}$$

Indeed,

$$m_{\frac{13}{14}} = 7\,645\,370\,045 > m_{\frac{7}{19}} = 6\,684\,339\,842 < m_{\frac{1}{24}} = 7\,778\,742\,049$$

and

$$m_{\frac{16}{17}} = 1\,513\,744\,654\,945 > m_{\frac{9}{23}} = 1\,490\,542\,435\,045 < m_{\frac{2}{29}} = 2\,076\,871\,684\,802$$

6. Proof of Theorem 1.2

Before we start, we recall some facts about Fibonacci numbers and Pell numbers. Let $\mathcal{F}_0 = 0, \mathcal{F}_1 = \mathcal{F}_2 = 1, \mathcal{F}_3 = 2, \mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$ denote the Fibonacci sequence, and $\mathcal{P}_0 = 0, \mathcal{P}_1 = 1, \mathcal{P}_2 = 2, \mathcal{P}_3 = 5, \mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2}$ the Pell sequence. It is well known that the odd indexed Fibonacci and Pell numbers are Markov numbers. In fact $m_{\frac{1}{q}} = \mathcal{F}_{2q+1}$ and $m_{\frac{n}{n+1}} = \mathcal{P}_{2n+1}$.

This rest of the section is devoted to the proof of the main result. We prove part (b) first. The general strategy is as follows. We want to study the difference $m_{q,p} - m_{q+s,p-t}$ where $E(q,p)$ and $F = (q+s, p-t)$ are two neighboring lattice points on a line of slope $a = -\frac{s}{t}$. Let us define a new pair of points (E', F') by translating the pair (E, F) , we write $E' = (q', t+1)$, $F' = (q' + s, 1)$. Then E', F' lie on a different line of same slope, and the point F' corresponds to a Fibonacci number. We compare the difference between the numbers associated to the points E and F with the difference between the numbers associated to the points E' and F' and then let q' go to infinity. The fact that F' corresponds to a Fibonacci number gives us control over the limit.

For part (a), the proof is similar except that we translate that pair (E, F) to a pair (E', F') where now $E' = (q', q' - 1)$ corresponds to a Pell number.

We start with the following observation.

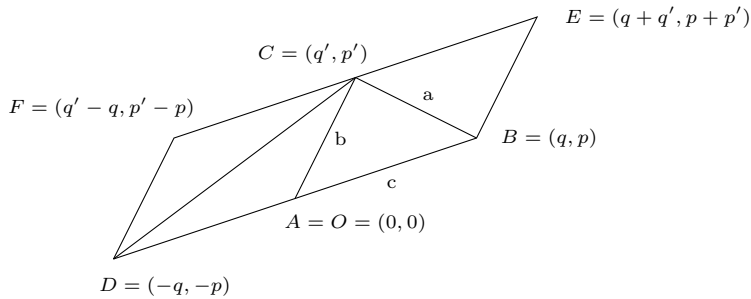


Fig. 11. Picture for Lemma 6.2.

Lemma 6.1. Assume that A, B, C are non-colinear lattice points such that the triangle ABC does not contain other lattice points besides A, B, C . Then

$$|AB|^2 + |AC|^2 + |BC|^2 = 3|AB| |AC| |BC|$$

Proof. The points A, B, C define a triangulation of the torus and the corresponding cluster variables form a cluster in the cluster algebra. Hence the Markov numbers $|AB|, |AC|, |BC|$ form a Markov triple, see [19,2]. $\square$

Lemma 6.2. Let p, q be coprime positive integers and let $f_n = m_{nq, np}$. Thus $f_0 = 0$ and $f_1 = m_{q, p}$ is the Markov number. Then, for $n \geq 2$,

$$f_n = 3f_1 f_{n-1} - f_{n-2}$$

As a consequence, $f_n = \frac{c}{\sqrt{9c^2 - 4}}(\alpha^n - \alpha^{-n})$ where $c = m_{q, p}$ and $\alpha = (3c + \sqrt{9c^2 - 4})/2$ is the larger root of $x^2 - 3cx + 1 = 0$.

Proof. The statement is trivially true for $n = 0, 1$. Assume $n = 2$. Let $A = (0, 0)$, $B = (q, p)$ and let $C = (q', p')$ be a lattice point such that the triangle ABC does not contain any lattice points besides A, B and C .

Let E be such that $ABEC$ is a parallelogram, thus $E = (q + q', p + p')$, and let D, F be such that $ACFD$ is the parallelogram obtained by translation of $ABEC$ by $(-q, -p)$, see Fig. 11. Note that $f_1 = |AB|$ and $f_2 = |BD|$.

The skein relations imply the following identities

$$|CD| = (|AC| |DF| + |AD| |CF|) / |AF| = (|AC|^2 + |AB|^2) / |BC| \quad (6.1)$$

and

$$|BD| = (|AD| |BC| + |AB| |CD|) / |AC| = |AB| (|BC| + |CD|) / |AC| \quad (6.2)$$

and using equation (6.1) this implies

$$|BD| = |AB| \frac{|BC|^2 + |AC|^2 + |AB|^2}{|BC||AC|}$$

Now Lemma 6.1 implies

$$|BD| = |AB| \frac{3|BC||AC||AB|}{|BC||AC|} = 3|AB|^2$$

This proves the case $n = 2$.

For $n > 2$, let $A = (0, 0)$, $B = (q, p)$, $C = (2q, 2p)$ and $D = (nq, np)$. The skein relations imply

$$|\gamma_{AC}^L||\gamma_{BD}^L| = |\gamma_{AD}^L||\gamma_{BC}^L| + |\gamma_{AB}^L||\gamma_{CD}^L|$$

thus

$$|AC||BD| = |AD||BC| + |AB||CD|$$

and hence

$$f_2 f_{n-1} = f_n f_1 + f_1 f_{n-2}$$

and the result follows since $f_2 = 3f_1^2$. $\square$

Remark 6.3. An alternative proof of Lemma 6.2 can be given using bracelets in the torus as follows. If ζ is a closed curve in the torus then its k -bracelet $\text{Brac}_k \zeta$ is defined as the k -fold concatenation of ζ with itself. It is known, see [17], that the bracelets satisfy the following Chebyshev recursion

$$\text{Brac}_k \zeta = \zeta \text{Brac}_{k-1} \zeta - \text{Brac}_{k-2} \zeta. \quad (6.3)$$

Now let ζ be the closed curve obtained from the arc $\ell_{\frac{p}{q}}$ by moving the arc infinitesimally away from the puncture. Then is shown in [7] that $|\zeta| = 3|\ell_{\frac{p}{q}}| = 3f_1$. Similarly, the bracelet $\text{Brac}_n \zeta$ is obtained from the arc $\gamma_{nq, np}^L$ by moving the arc infinitesimally away from the puncture. Again we have $|\text{Brac}_n \zeta| = 3f_n$. Now equation (6.3) implies $3f_n = 3f_1 3f_{n-1} - 3f_{n-2}$ and we are done.

Lemma 6.4. Assume O, E, F, E', F' be lattice points such that O, E, F are not colinear, F' is such that $\overrightarrow{FF'} = s\overrightarrow{EO} + t\overrightarrow{OF}$ with $s, t \in \mathbb{R}_{>0}$, E' is such that $\overrightarrow{OE'} = \overrightarrow{OE} + \overrightarrow{FF'}$. Thus $EFF'E'$ is a parallelogram. See Fig. 12. Then

$$|OE||OF'| \geq |OE'||OF|.$$

Proof. Let O' be the lattice point such that $\overrightarrow{OO'} = \overrightarrow{FF'}$. Then the Ptolemy inequality implies $|O'E'||OF'| \geq |OE'||O'F'|$, and the result follows from $|O'E'| = |OE|$ and $|O'F'| = |OF|$. $\square$

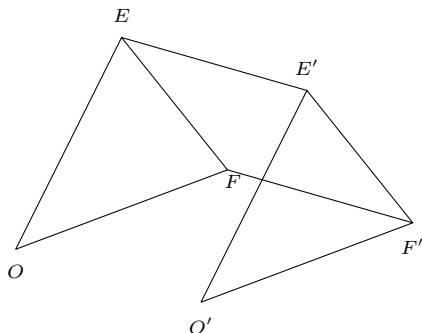


Fig. 12. Figure for Lemma 6.4 and Lemma 6.5.

Lemma 6.5. Let q, p, s, t, q' be positive integers with $\gcd(s, t) = 1$ and $q' > q + t$ and define the lattice points $E = (q, p + s)$, $F = (q + t, p)$, $E' = (q', s + 1)$ and $F' = (q' + t, 1)$. Thus $EFF'E'$ is a parallelogram as shown in Fig. 12 and the slope of the line segment ℓ_{EF} is $-\frac{s}{t}$. Assume that $\lim_{q' \rightarrow \infty} \frac{|OE'|}{|OF'|} > 1$. Then $|OE| > |OF|$.

Proof. For every $q' > q + t$, the points E, F, F', E' satisfy the conditions in Lemma 6.4. Thus $|OE||OF'| \geq |OE'||OF|$, and hence

$$\frac{|OE|}{|OF|} \geq \frac{|OE'|}{|OF'|}.$$

The result now follows since, by our assumption on the limit, the right hand side is greater than 1 when q' is large. $\square$

Lemma 6.6. We have

$$\lim_{q \rightarrow \infty} \frac{m_{q,6}}{m_{q+4,1}} > 1.$$

Proof. We claim that, for any positive integer a ,

$$\lim_{q \rightarrow \infty} \frac{m_{q,a}}{m_{q+a-2,1}} = \lim_{n \rightarrow \infty} \frac{m_{an,a}}{m_{an+a-2,1}} = \left(\frac{3}{\sqrt{5}\phi}\right)^{a-1} \phi^2, \text{ where } \phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots \quad (6.4)$$

The first equality follows from Lemma 6.4, which asserts that the sequence of positive numbers $\{\frac{m_{q,a}}{m_{q+a-2,1}}\}$ is weakly decreasing as q increases.

Next, we prove the second equality of (6.4). The sequence $\{m_{q,1}\} = (1, 2, 5, 13, 34, 89, \dots)$ is odd-indexed Fibonacci sequence $\{\mathcal{F}_{2q-1}\}$. So $m_{q,1} = (\phi^{2q+1} + \phi^{-2q-1})/\sqrt{5}$. For any two functions $f(n)$ and $g(n)$ with variable n , we write $f \sim g$ if $\lim f(n)/g(n) = 1$ as $n \rightarrow \infty$. Then $c := m_{n,1} \sim \phi^{2n+1}/\sqrt{5}$, $\alpha = (3c + \sqrt{9c^2 - 4})/2 \sim 3c$, thus

$$m_{an,a} = \frac{c}{\sqrt{9c^2 - 4}}(\alpha^a - \alpha^{-a}) \quad (\text{by Lemma 6.2})$$

$$\sim \frac{c}{3c} \alpha^a \sim \frac{1}{3}(3c)^a \sim 3^{a-1} \phi^{(2n+1)a} / (\sqrt{5})^a$$

On the other hand,

$$m_{an+a-2,1} \sim \phi^{2(an+a-2)+1} / \sqrt{5}$$

So

$$\lim_{n \rightarrow \infty} \frac{m_{an,a}}{m_{an+a-2,1}} = \frac{3^{a-1} \phi^{(2n+1)a} / (\sqrt{5})^a}{\phi^{2(an+a-2)+1} / \sqrt{5}} = \left(\frac{3}{\sqrt{5}}\right)^{a-1} \phi^{-a+3} = \left(\frac{3}{\sqrt{5}\phi}\right)^{a-1} \phi^2$$

This proves (6.4).

Finally, for $a = 6$, the right hand side of (6.4) is $1.026 \dots > 1$. $\square$

Remark 6.7. Let a' be the solution of the equation $\lim_{q \rightarrow \infty} \frac{m_{q,a'}}{m_{q+a'-2,1}} = 1$. Equivalently, by (6.4), a' satisfies the equation $(\frac{3}{\sqrt{5}\phi})^{a'-1} \phi^2 = 1$, thus $a' = \frac{-2 \ln \phi}{\ln(\frac{3}{\sqrt{5}\phi})} + 1$. Let b denote the corresponding slope, thus

$$b = -\frac{a' - 1}{a' - 2} = -\frac{\frac{-2 \ln \phi}{\ln(\frac{3}{\sqrt{5}\phi})}}{\frac{-2 \ln \phi}{\ln(\frac{3}{\sqrt{5}\phi})} - 1} = \frac{-2}{2 + \frac{\ln(\frac{3}{\sqrt{5}\phi})}{\ln \phi}} = \frac{-2}{2 + \frac{\ln(3/\sqrt{5}) - \ln \phi}{\ln \phi}} = \frac{-2}{1 + \frac{\ln(3/\sqrt{5})}{\ln \phi}}$$

Or we can express b as

$$b = \frac{-2 \ln \phi}{\ln \phi + \ln(3/\sqrt{5})} = -\frac{\ln((\frac{1+\sqrt{5}}{2})^2)}{\ln(\frac{1+\sqrt{5}}{2} \cdot \frac{3}{\sqrt{5}})} = -\frac{\ln \frac{3+\sqrt{5}}{2}}{\ln \frac{3(1+\sqrt{5})}{2\sqrt{5}}} = -1.241668489 \dots$$

We conjecture that b is best replacement of $-5/4$ in Theorem 1.2 (b):

Conjecture 6.8. The Markov distances decrease with x along any line with slope $\leq -\frac{\ln \frac{3+\sqrt{5}}{2}}{\ln \frac{3(1+\sqrt{5})}{2\sqrt{5}}}$.

6.1. Proof of Theorem 1.2 (b)

By Lemma 6.6(c), the Markov numbers are decreasing on any line with slope $-5/4$. So we are left to show the decreasing property on lines of slope $< -5/4$.

Let $(q_1, p_1), (q_2, p_2)$ be two lattice points satisfying $q_i > p_i > 0$ (for $i = 1, 2$), $q_2 > q_1$, $(p_2 - p_1)/(q_2 - p_1) < -\frac{5}{4}$. We claim that $m_{q_1, p_1} > m_{q_2, p_2}$.

Consider the line L through the point (q_2, p_2) with slope $-\frac{5}{4}$. If there exists an integer point on L , denoted (q_3, p_3) , such that $q_3 \leq q_1$ and $p_3 \leq p_1$. In other

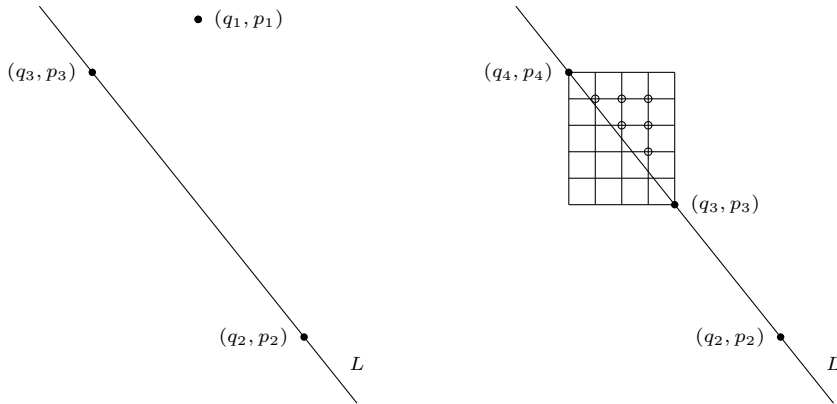


Fig. 13. Proof of Theorem 1.2 (b). Slope of L is $-5/4$. Left: the case $q_3 \leq q_1$ and $p_3 \leq p_1$; Right: the other case, where (q_1, p_1) is one of the 6 circled points.

words, the point (q_3, p_3) lies (weakly) southwest of the point (q_1, p_1) . Then we have $m_{q_1, p_1} > m_{q_3, p_1} > m_{q_3, p_3} > m_{q_2, p_2}$ where the first inequality follows from the constant numerator Theorem 4.2, the second from the constant denominator Theorem 4.1, while the third inequality uses the decreasing property when the slope is equal to $-5/4$, see the left picture in Fig. 13. This completes the proof in this case.

Now suppose that there is no integer point on the line L that lies southwest of (q_1, p_1) . Then there are two consecutive integer points (q_3, p_3) and $(q_4, p_4) = (q_3 - 4, p_3 + 5)$ such that $q_4 < q_1 < q_3$ and $p_4 > p_1 > p_3$, see the right picture in Fig. 13. Then there are only 6 possibilities for (q_1, p_1)

$$(q_3 + 1, p_3 - 1), (q_3 + 2, p_3 - 1), (q_3 + 3, p_3 - 1), (q_3 + 2, p_3 - 2), (q_3 + 3, p_3 - 2), (q_3 + 3, p_3 - 3).$$

It follows from Theorem 4.1 that the smallest $m_{q, p}$ for the above 6 pairs (q, p) is obtained when $(q, p) = (q_3 + 1, p_3 - 1)$. Thus we have $m_{q_1, p_1} \geq m_{q_3 + 1, p_3 - 1} > m_{q_4, p_4} \geq m_{q_2, p_2}$ where the second inequality follows from Theorem 4.3 and the last inequality from Lemma 6.6(c) since the slope of the line L is $-5/4$. This completes the proof of part (b) of Theorem 1.2.

6.2. Proof of Theorem 1.2 (a)

The proof of part (a) of the theorem is similar to the proof of part (b) except that instead of using Fibonacci numbers $m_{q, 1}$ we will use Pell numbers $m_{q, q-1}$.

We first show an analogue of Lemma 6.6:

Lemma 6.9. *We have*

$$\lim_{q \rightarrow \infty} \frac{m_{q+7, q-9}}{m_{q, q-1}} > 1.$$

Proof. Similar to the proof of Lemma 6.6, we claim that, for any positive integer a ,

$$\lim_{q \rightarrow \infty} \frac{m_{q+a-1, q-a-1}}{m_{q, q-1}} = \lim_{n \rightarrow \infty} \frac{m_{2an, 2an-2a}}{m_{2an-a+1, 2an-a}} = \frac{3^{2a-1}}{2^{3a-1}}(2 - \sqrt{2}). \quad (6.5)$$

The first equality follows from Lemma 6.4, which asserts that the sequence of positive numbers $\{\frac{m_{q+a-1, q-a-1}}{m_{q, q-1}}\}$ is weakly decreasing as q increases.

Next, we prove the second equality of (6.5). The Markov number $m_{q, q-1}$ is the Pell number $\mathcal{P}_{2q-1}$. The sequence $\{m_{q, q-1}\} = (1, 5, 29, 169, \dots)$ (for $q \geq 1$) is defined by the recursive relation $a_q = 6a_{q-1} - a_{q-2}$ with initial conditions $a_1 = 1$, $a_2 = 5$. Using the well-known formula $\mathcal{P}_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$ we get

$$m_{q, q-1} = \mathcal{P}_{2q-1} = \frac{(1 + \sqrt{2})^{2q-1} - (1 - \sqrt{2})^{2q-1}}{2\sqrt{2}} = \frac{2 - \sqrt{2}}{4} \psi^q + \frac{2 + \sqrt{2}}{4} \psi^{-q},$$

where $\psi = 3 + \sqrt{8}$. By Lemma 6.2 (with $c = m_{n, n-1}$, $\alpha = (3c + \sqrt{9c^2 - 4})/2 \sim 3c$):

$$\begin{aligned} m_{2an, 2a(n-1)} &= \frac{c}{\sqrt{9c^2 - 4}} (\alpha^{2a} - \alpha^{-2a}) \sim \frac{1}{3} (3c)^{2a} \sim 3^{2a-1} (m_{n, n-1})^{2a} \\ &\sim 3^{2a-1} \left(\frac{2-\sqrt{2}}{4} \right)^{2a} \psi^{2an}. \end{aligned}$$

On the other hand

$$m_{2an-a+1, 2an-a} \sim \left(\frac{2-\sqrt{2}}{4} \right) \psi^{2an-a+1}.$$

So

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{m_{2an, 2an-2a}}{m_{2an-a+1, 2an-a}} &= \frac{3^{2a-1} \left(\frac{2-\sqrt{2}}{4} \right)^{2a} \psi^{2an}}{\left(\frac{2-\sqrt{2}}{4} \right) \psi^{2an-a+1}} = \left(\frac{3(2 - \sqrt{2})}{4} \right)^{2a-1} \psi^{a-1} \\ &= \frac{3^{2a-1}}{2^{3a-1}} (2 - \sqrt{2}) \end{aligned}$$

This proves (6.5).

Finally, for $a = 8$, the right hand side of (6.5) is $1.00200118 \dots > 1$. $\square$

Remark 6.10. Let a'' be the solution of the equation $\lim_{q \rightarrow \infty} \frac{m_{q+a''-1, q-a''-1}}{m_{q, q-1}} = 1$. Equivalently, by (6.5), a'' satisfies the equation $\frac{3^{2a''-1}}{2^{3a''-1}}(2 - \sqrt{2}) = 1$, thus $(\frac{9}{8})^{a''} = \frac{3}{2(2-\sqrt{2})}$, $a'' = \frac{\ln \frac{3}{2(2-\sqrt{2})}}{\ln(9/8)}$. Let b be the corresponding slope, that is,

$$b = -\frac{a''}{a'' - 1} = -\frac{\frac{\ln \frac{3}{2(2-\sqrt{2})}}{\ln(9/8)}}{\frac{\ln \frac{3}{2(2-\sqrt{2})}}{\ln(9/8)} - 1} = -\frac{\ln \frac{3}{2(2-\sqrt{2})}}{\ln \frac{3}{2(2-\sqrt{2})} - \ln \frac{9}{8}} = -\frac{\ln \frac{3(2+\sqrt{2})}{4}}{\ln \frac{2(2+\sqrt{2})}{3}} = -1.14320438 \dots$$

We conjecture that b is best replacement of $-8/7$ in Theorem 1.2 (a):

Conjecture 6.11. *The Markov distances increase with x along any line with slope $\geq -\frac{\ln \frac{3(2+\sqrt{2})}{4}}{\ln \frac{2(2+\sqrt{2})}{3}}$.*

For a line whose slope is not within the range given in Conjectures 6.8 and 6.11, we conjecture that the Markov numbers form a strictly anti-unimodal sequence provided there are enough lattice points on the line. More precisely, we propose the following.

Conjecture 6.12. *Given a line $y = ax + b$ whose slope a is rational and satisfies $-\frac{\ln \frac{3+\sqrt{5}}{2}}{\ln \frac{3(1+\sqrt{5})}{2\sqrt{5}}} < a < -\frac{\ln \frac{3(2+\sqrt{2})}{4}}{\ln \frac{2(2+\sqrt{2})}{3}}$, we let $(q_1, p_1), \dots, (q_n, p_n)$ be the lattice points on the line that satisfy $1 \leq p_i \leq q_i$, and we arrange them such that $q_1 < q_2 < \dots < q_n$. Then there exists $b_0 > 0$ such that for all lines $y = ax + b$ with $b \geq b_0$ and contain at least a lattice points, the sequence of Markov distances m_{q_i, p_i} is strictly anti-unimodal, that is,*

$$m_{q_1, p_1} > m_{q_2, p_2} > \dots > m_{q_j, p_j} < m_{q_{j+1}, p_{j+1}} < \dots < m_{q_n, p_n}, \text{ for some } 2 \leq j \leq n-1.$$

Remark 6.13. In this conjecture we do not require that the coordinates (q_i, p_i) are relatively prime.

We now return to the proof of our main result. The next lemma proves Theorem 1.2 (a) in the case where the slope is equal to $-\frac{7}{8}$.

Lemma 6.14. *Let $p < q$ be positive integers. Then $m_{q,p} < m_{q+7, p-8}$. In particular, the Markov numbers increase with x on any line of slope $a = -\frac{7}{8}$.*

Proof. Consider the points $E = (q+7, p-8)$, $F = (q, p)$, $E' = (q'+7, q'-9)$, $F' = (q', q'-1)$. We verify that, when q' is sufficiently large, the condition $\overrightarrow{FF'} = s\overrightarrow{EO} + t\overrightarrow{OF}$ with $s, t \in \mathbb{R}_{>0}$ in Lemma 6.4 holds. Indeed,

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} -q-7 & q \\ -p+8 & p \end{bmatrix}^{-1} \begin{bmatrix} q'-q \\ q'-p-1 \end{bmatrix} = \frac{1}{7p+8q} \begin{bmatrix} (q-p)q'-q \\ (q-p+15)q'-7p-8q-7 \end{bmatrix}$$

which is in $\mathbb{R}_{>0}^2$ as $q' \rightarrow \infty$ because $q > p$. Thus we can apply Lemma 6.4 to get an inequality

$$|OE| |OF'| \geq |OE'| |OF| \quad \text{and thus} \quad \frac{m_{q+7, p-8}}{m_{q,p}} \geq \frac{m_{q'+7, q'-9}}{m_{q', q'-1}}.$$

The limit as $q' \rightarrow \infty$ of the right hand side in the last inequality is greater than 1, by Lemma 6.9. Thus the left hand side is also greater than 1. $\square$

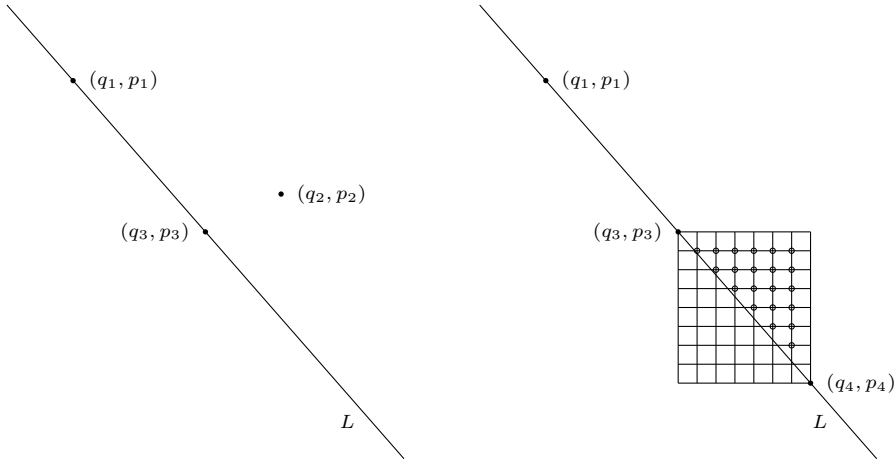


Fig. 14. Proof of Theorem 1.2 (a). Slope of L is $-8/7$. Left: the case $q_3 \leq q_2$ and $p_3 \leq p_2$; Right: the other case, where (q_2, p_2) is one of the 21 circled points.

Proof of Theorem 1.2(a). Let $(q_1, p_1), (q_2, p_2)$ be two lattice points satisfying $0 < p_1 < q_1$, $0 < p_2 < q_2$ such that $q_1 < q_2$ and the slope of the line segment from (q_1, p_1) to (q_2, p_2) is greater or equal to $-\frac{8}{7}$, that is $(p_2 - p_1)/(q_2 - p_1) \geq -\frac{8}{7}$. We need to show that $m_{q_1, p_1} < m_{q_2, p_2}$. Since we already know that the conclusion is true if $p_2 \geq p_1$, we assume $p_2 < p_1$ in the rest of the proof. We also may assume that the slope is strictly smaller than $-\frac{8}{7}$ because of Lemma 6.14.

Consider the line L through the point (q_1, p_1) with slope $-\frac{8}{7}$. If there exists a lattice point (q_3, p_3) on L that lies (weakly) southwest of (q_2, p_2) , then Theorem 4.1 implies $m_{q_1, p_1} < m_{q_3, p_3} < m_{q_2, p_2}$ and we are done. See the left picture in Fig. 14.

The only unproved situation is when there are two consecutive lattice points (q_3, p_3) and $(q_4, p_4) = (q_3 + 7, p_3 - 8)$ such that $q_3 < q_2 < q_4$ and $p_3 > p_2 > p_4$. Then there are precisely 21 possibilities for (q_2, p_2) , namely the lattice points in the interior of the right triangle with vertices $(q_3, p_3), (q_4, p_3), (q_4, p_4)$; see the right picture in Fig. 14. Observe that any line that passes through (q_3, p_3) and one of these 21 points has slope ≥ -1 . Thus $m_{q_2, p_2} > m_{q_3, p_3}$. On the other hand, we have $m_{q_1, p_1} < m_{q_3, p_3}$ from Lemma 6.14 and thus we conclude that $m_{q_2, p_2} > m_{q_1, p_1}$. $\square$

7. A comment on the Markov distance

In this section we show that a modified version of the Markov distance satisfies the triangle inequality.

Corollary 7.1. *Let A, B, C be distinct lattice points. Then*

$$3|AB| \cdot |BC| \geq |AC|$$

and the equality holds if and only if $\overrightarrow{AB} = \overrightarrow{BC} = \begin{pmatrix} p \\ q \end{pmatrix}$ where p, q are relatively prime.

As a consequence, if we define $d : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}_{\geq 0}$ as

$$d(A, B) = \begin{cases} \ln(3|AB|), & \text{if } A \neq B; \\ 0, & \text{if } A = B, \end{cases}$$

then $d(A, B) + d(B, C) \geq d(A, C)$ for any $A, B, C \in \mathbb{Z}^2$. Moreover, the equality holds if $A = B$, or $B = C$, or $\overrightarrow{AB} = \overrightarrow{BC} = (p, q)$ where p, q are relatively prime.

Proof. First we prove the inequality for the special case when B is the midpoint of the segment ℓ_{AC} (that is, $\overrightarrow{AB} = \overrightarrow{BC}$). For this, we shall use Lemma 6.2 and the notations therein. Let $\frac{p}{q}$ be the slope of the line segment ℓ_{AB} . Then ℓ_{AB} runs through $n+1$ lattice points $A, A + (q, p), A + 2(q, p), \dots, A + n(q, p) = B$. Then we want to show

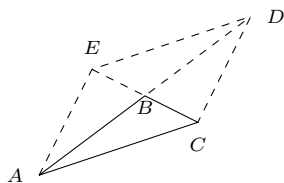
$$3|AB|^2 \geq |AC|^2$$

or equivalently, $3f_n^2 \geq f_{2n}$ where $f_n = \frac{c}{\sqrt{9c^2-4}}(\alpha^n - \alpha^{-n})$ where $\alpha = (3c + \sqrt{9c^2-4})/2 > 1$ is the larger root of $x^2 - 3cx + 1 = 0$ (where $c \in \mathbb{Z}_{>0}$). We have

$$\begin{aligned} 3f_n^2 - f_{2n} &= 3 \left(\frac{c}{\sqrt{9c^2-4}}(\alpha^n - \alpha^{-n}) \right)^2 - \frac{c}{\sqrt{9c^2-4}}(\alpha^{2n} - \alpha^{-2n}) \\ &= \frac{c}{\sqrt{9c^2-4}}(\alpha^n - \alpha^{-n}) \left(3 \frac{c}{\sqrt{9c^2-4}}(\alpha^n - \alpha^{-n}) - (\alpha^{2n} - \alpha^{-2n}) \right) \\ &= \frac{c}{\sqrt{9c^2-4}}(\alpha^n - \alpha^{-n}) \left((3 \frac{c}{\sqrt{9c^2-4}} - 1)(\alpha^n - \alpha^{-n}) \right) \end{aligned}$$

and each term in this product is positive, since $\frac{c}{\sqrt{9c^2-4}} > 1, \alpha > 1$ and $n > 1$. Thus $3f_n^2 \geq f_{2n}$.

Now for A, B, C in general, consider the parallelogram $ACDE$ with center B .



Then by the first part of the proof, $3|AB|^2 \geq |AD|^2$ and $3|BC|^2 \geq |BE|^2$. By Corollary 3.6, $|AD| \cdot |CE| \geq |AC|^2 + |AE|^2 \geq |AC|^2$. So we have $(3|AB|^2)(3|BC|^2) \geq |AC|^2$, taking square roots on both sides gives $3|AB| \cdot |BC| \geq |AC|$.

The consequence follows immediately. $\square$

At the end of the paper, we propose a conjecture that implies the uniqueness conjecture of Markov numbers.

We call a real number r a Markov distance if there exist two lattice points A, B such that r is the Markov distance $|AB|$ between A and B . Note that a Markov distance is always a nonnegative integer.

Conjecture 7.2 (*Uniqueness conjecture of Markov distances*). *Every Markov distance is equal to $m_{q,p}$ for a unique the pair of integers (p, q) satisfying $0 \leq p \leq q$.*

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