

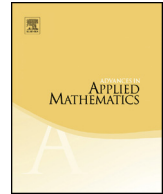


ELSEVIER

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama



Cambrian combinatorics on quiver representations (type A)



Emily Barnard^a, Emily Gunawan^{b,*}, Emily Meehan^c,
Ralf Schiffler^{d,1}

^a Department of Mathematical Sciences, DePaul University, Chicago, IL 60614-3210, USA

^b Department of Mathematics, University of Oklahoma, Norman, OK 73019-3103, USA

^c School of Science, Technology, Accessibility, Mathematics, and Public Health, Gallaudet University, Washington, DC 20002-3600, USA

^d Department of Mathematics, University of Connecticut, Storrs, CT 06269-1009, USA

ARTICLE INFO

Article history:

Received 2 May 2021

Received in revised form 30 July 2022

Accepted 22 August 2022

Available online xxxx

MSC:

05E10

13F60

16G20

16G70

ABSTRACT

This paper presents a geometric model of the Auslander–Reiten quiver of a type A quiver together with a stability function for which all indecomposable modules are stable. We also introduce a new Catalan object which we call a maximal almost rigid representation. We show that its endomorphism algebra is a tilted algebra of type A. We define a partial order on the set of maximal almost rigid representations and use our new geometric model to show that this partial order is a Cambrian lattice.

© 2022 Elsevier Inc. All rights reserved.

* Corresponding author.

E-mail addresses: e.barnard@depaul.edu (E. Barnard), egunawan@ou.edu (E. Gunawan), emily.meehan@gallaudet.edu (E. Meehan), schiffler@math.uconn.edu (R. Schiffler).

¹ The fourth author was supported by the NSF grant DMS-1800860 and by the University of Connecticut.

Contents

1.	Introduction	2
2.	Construction of the polygon $P(Q)$ and the map η_Q	5
3.	Representations of quivers of type $\mathbb{A}$	6
3.1.	Indecomposable representations	7
3.2.	Irreducible morphisms in terms of hooks and cohooks	8
3.3.	Maximal increasing and decreasing paths	10
4.	A geometric model for $\text{rep } Q$	12
4.1.	The category of line segments $\mathcal{C}_{P(Q)}$	14
4.2.	The functor $F: \mathcal{C}_{P(Q)} \rightarrow \text{ind } Q$	15
5.	Stability function	17
5.1.	A geometric model for the derived category	20
6.	Maximal almost rigid representations	22
7.	Endomorphism algebras of maximal almost rigid representations	25
8.	Representation theoretic version of the Cambrian lattice and the η map	31
8.1.	Poset structure on $\text{mar}(Q)$	31
8.2.	The poset on $\text{mar}(Q)$ is a Cambrian lattice	32
8.3.	The η map	33
	Acknowledgments	35
	References	36

1. Introduction

Let Q be a Dynkin quiver of type $\mathbb{A}_{n+2}$. Thus Q is an oriented graph whose underlying unoriented graph is the Dynkin diagram of type $\mathbb{A}_{n+2}$. In [25], Reading introduced a family of lattices called Cambrian lattices which depend on a choice of Dynkin diagram together with an orientation of that diagram. In this paper we focus on the type $\mathbb{A}$ case, and use the orientation given by Q . In this context, Reading defined an $(n+3)$ -gon $P(Q)$ and a surjective map

$$\eta_Q: S_{n+1} \rightarrow \{\text{triangulations of } P(Q)\}$$

from permutations to triangulations, following and modifying an explication [27, Sec. 4.3] of an iterated fiber-polytope construction given in [11]. In the special case where Q is linearly oriented, this map had been well-studied as a map to various Catalan objects — see [36], [12, Sec. 9], [35, Sec. 1.5].

One of the objectives of this paper is to give a representation theoretic interpretation of the map η_Q . In order to achieve this goal, we develop several other concepts and results along the way.

Inspired by the map η_Q , we define a category $\mathcal{C}_{P(Q)}$ whose objects are line segments in the polygon $P(Q)$ and whose morphisms are generated by pivots of line segments modulo mesh relations. We show that the category $\mathcal{C}_{P(Q)}$ is equivalent to the abelian category $\text{ind } Q$ of indecomposable representations of the quiver Q .

Theorem A (Theorem 4.6). *There is an equivalence of categories $F: \mathcal{C}_{P(Q)} \rightarrow \text{ind } Q$.*

The reader familiar with the construction of the cluster category of type A_n as the category of diagonals in an $(n+3)$ -gon in [13] will notice a strong similarity between the two constructions. Note however that our construction here is different in several ways. For one, we include the boundary edges, which was not the case in [13], and secondly, our category is hereditary and contains no cycles.

On the other hand, the result of Theorem A is very similar to the construction of the derived category of Q by Oppen, Plamondon and Schroll in [21]. In that paper, the authors give a geometric model for the derived category of an arbitrary gentle algebra. Since the path algebra of a quiver Q of type A is a special kind of gentle algebra, the category of representations of Q can be recovered as a heart of the derived category in [21]. However, the two constructions have some fundamental differences. While our polygon $P(Q)$ is homeomorphic to the surface of [21], our construction is rigid in the sense that it fixes the relative position of each vertex of the polygon in the plane. This rigidity is essential here, since we are using the geometric data of the slope of the line segments in $P(Q)$ to describe the morphisms in the category.

The slope is also the key to our next result proving that the functor F naturally induces a stability condition on the category of representations of Q . First, a central charge is obtained by considering each oriented line segment in $P(Q)$ as a complex number. Then the stability function of an indecomposable module is given by the angle of the oriented line segment with the positive real axis. This stability function has the property that every indecomposable representation is stable. Reineke conjectured in [28] that every Dynkin quiver admits a stability function with that property and Apruzzese and Igusa proved the conjecture in type A in [4].

Theorem B (Theorem 5.3). *Let Q be a Dynkin quiver of type A . Then the function that associates to every indecomposable representation M the angle of the corresponding oriented line segment $F^{-1}(M)$ is a stability function for which every indecomposable representation of Q is stable.*

Using the stability function, we extend our model to a model of the derived category of $\text{rep } Q$ recovering the (special case of) the result in [21].

We then introduce a class of representations which we call *maximal almost rigid* representations. We say that the direct sum of two indecomposable representations M and N is *almost rigid* if they do not have any nonsplit extensions, or if all extensions between M and N are indecomposable. We say that a representation T is *maximal almost rigid* if, for every nonzero representation M , the representation $T \oplus M$ is not almost rigid. Let $\text{mar}(Q)$ denote the set of all maximal almost rigid representations of Q .

Our third main result is the following.

Theorem C (Theorem 6.8). *The functor F induces a bijection*

$$F: \{\text{triangulations of } P(Q)\} \rightarrow \text{mar}(Q).$$

By definition the notion of an almost rigid representation is a generalization of that of a *rigid* representation. For quivers without oriented cycles, the *maximal rigid representations* are precisely the *tilting modules* over the path algebra of the quiver, and their endomorphism algebras are the *tilted algebras* introduced in [16]. On the other hand, the endomorphism algebras of triangulations in cluster categories from marked surfaces are cluster-tilted algebras, or more generally, 2-Calabi-Yau tilted algebras, [13,8,33,1,20]. It is shown in [3] that if C is a tilted algebra, then its trivial extension $C \ltimes \text{Ext}_C^2(DC, C)$ is cluster-tilted, and every cluster-tilted algebra arises this way.

Both tilted and cluster-tilted algebras are extensively studied and occupy a central place in the representation theory of finite dimensional algebras. It is therefore natural to consider the endomorphism algebras of maximal almost rigid representations. Our fourth main result is the following.

Theorem D (Theorem 7.3). *Let Q be a Dynkin quiver of type A_{n+2} and T a maximal almost rigid representation of Q . Then the endomorphism algebra $C = \text{End}_{\text{rep } Q} T$ is a tilted algebra of type A_{2n+3} . Moreover, the quiver of its associated cluster-tilted algebra $C \ltimes \text{Ext}_C^2(DC, C)$ is the adjacency quiver of the triangulation $F^{-1}(T)$.*

We then come back to our original goal to give a representation theoretic interpretation of Cambrian lattices and the map η_Q . In general, a Cambrian lattice is a certain quotient of the weak order on a finite Coxeter group. The Tamari lattice is an example in type A . The Cambrian lattice for type A is a partial order on triangulations of a polygon whose cover relations are given by diagonal flips. The Hasse diagram for each Cambrian lattice is isomorphic to the 1-skeleton of the generalized associahedron for the corresponding Coxeter group, see [30]. Furthermore, the elements of each Cambrian lattice of a finite Coxeter group are in bijection with several families of objects related to cluster algebras. For example, the clusters of the corresponding cluster algebra of finite type, the (isoclasses of) cluster-tilting objects in the corresponding cluster category, the finitely generated torsion classes over the path algebra of the Dynkin quiver, as well as the non-crossing partitions associated with the quiver, see [17].

We add a new item to this list by proving the following.

Theorem E (Theorem 8.4). *If Q is a Dynkin quiver of type A , the set of maximal almost rigid representations of Q with the covering relation given in Definition 8.1 is isomorphic to the Cambrian lattice coming from Q .*

We also introduce the representation theoretic map η_Q^{rep} as the composition $F \circ \eta_Q$ of the map η_Q with our functor F . We show that η_Q^{rep} is described via degenerations and extensions of representations of Q of dimension vector $(1, 1, \dots, 1)$.

The paper is organized as follows. We start by recalling the construction of the polygon $P(Q)$ and the map η_Q in section 2. In section 3, we review several facts on representations of quivers of type A . Section 4 is devoted to the construction of the category $\mathcal{C}_{P(Q)}$ of line

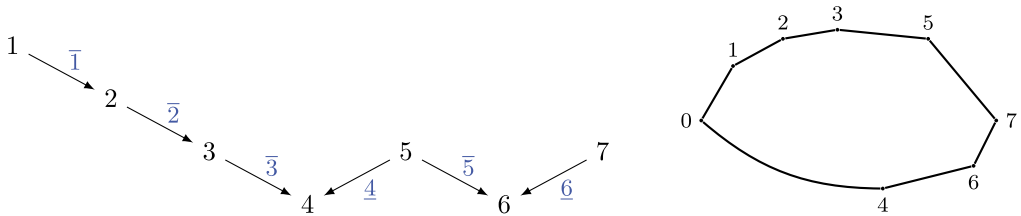


Fig. 1. Quiver Q with $\underline{[6]} = \{4, 6\}$ and $\overline{[6]} = \{1, 2, 3, 5\}$ and the polygon $P(Q)$.

segments in $P(Q)$ and the proof of Theorem A. We introduce the stability function in section 5 and prove Theorem B. In the same section, we extend our model to the derived category. In section 6, we define maximal almost rigid representations and prove Theorem C, and in section 7, we study their endomorphism algebras and prove Theorem D. In the last section, we give our representation theoretic interpretations of the Cambrian lattice and the η_Q map and prove Theorem E. We also produce the interpretation of the η_Q map via degenerations and extensions in that section.

2. Construction of the polygon $P(Q)$ and the map η_Q

We recall the construction of the $(n+3)$ -gon $P(Q)$ from [25], where Q is a Dynkin quiver of type A_{n+2} . Label the vertices $1, 2, \dots, n+2$ in linear order as in Fig. 1(left). Let $[n+1]$ denote the set $\{1, 2, \dots, n+1\}$ and let S_{n+1} be the symmetric group.

First, use Q to partition the set $[n+1]$ into two sets, the *upper-barred* integers $\overline{[n+1]}$ and the *lower-barred* integers $\underline{[n+1]}$, as follows. Let $\overline{[n+1]}$ be the set of all vertices i such that $i \rightarrow (i+1)$ is in Q and let $\underline{[n+1]}$ be the set of all vertices i such that $i \leftarrow (i+1)$ is in Q .

Next, associate to Q an $(n+3)$ -gon $P(Q)$ with vertex labels $0, 1, 2, \dots, n+2$. Draw the vertices $0, 1, \dots, n+2$ in order from left to right so that: (1) the vertices 0 and $n+2$ are placed on the same horizontal line L ; (2) the upper-barred vertices are placed above L ; (3) the lower-barred vertices are placed below L . See Fig. 1.

Given a permutation $\pi \in S_{n+1}$, we write π in one-line notation as $\pi_1 \pi_2 \dots \pi_{n+1}$, where $\pi_i = \pi(i)$ for $i \in [n+1]$. For each $i \in \{0, \dots, n+1\}$, define $\lambda_i(\pi)$ to be a path from the left-most vertex 0 to the right-most vertex $n+2$ as follows. Let $\lambda_0(\pi)$ be the path from the vertex 0 to the vertex $n+2$ passing through all lower-barred vertices $\underline{i} \in \underline{[n+1]}$ in numerical order. Thus $\lambda_0(\pi)$ is the path along the lower boundary edges of $P(Q)$. Define $\lambda_1(\pi)$ as the piecewise linear path from 0 to $n+2$ passing through the vertices

$$\begin{cases} \underline{[n+1]} \cup \{\pi_1\}, & \text{if } \pi_1 \in \overline{[n+1]}; \\ \underline{[n+1]} \setminus \{\pi_1\}, & \text{if } \pi_1 \in \underline{[n+1]}, \end{cases}$$

maintaining the numerical order of the vertices visited. Repeating this process recursively, the final path $\lambda_{n+1}(\pi)$ passes from 0 to $n+2$ through all upper-barred vertices $\overline{i} \in \overline{[n+1]}$. Thus $\lambda_{n+1}(\pi)$ is the path along the upper boundary edges of $P(Q)$.

Definition 2.1. [25] Define a map $\eta_Q: S_{n+1} \rightarrow \{\text{triangulations of } P(Q)\}$, $\pi \mapsto \eta_Q(\pi)$, where $\eta_Q(\pi)$ is the triangulation (including the boundary edges) of $P(Q)$ that arises as the union of the paths $\lambda_0(\pi), \dots, \lambda_{n+1}(\pi)$.

Remark 2.2. It is shown in [25] that η_Q is surjective and that its fibers correspond to the congruence classes of a certain lattice congruence on the weak order on the symmetric group. The induced poset structure is a lattice called a *Cambrian lattice* of type A. More precisely, two triangulations are ordered $\mathcal{T} \leq \mathcal{T}'$ if there exist permutations $\pi \leq \pi'$ in the weak order such that $\eta_Q(\pi) = \mathcal{T}$ and $\eta_Q(\pi') = \mathcal{T}'$.

Example 2.3. Let Q be the quiver in Fig. 1, where $n = 5$. Then $\overline{[6]} = \{\overline{1}, \overline{2}, \overline{3}, \overline{5}\}$ and $\underline{[6]} = \{\underline{4}, \underline{6}\}$. Let $\pi = 453126 \in S_6$, written in one-line notation.

Then the paths $\lambda_i(\pi)$ described above are as follows.

$$\begin{array}{ll} \lambda_0(\pi) = 0, \underline{4}, \underline{6}, 7 & \\ \lambda_1(\pi) = 0, \underline{6}, 7 & \text{delete } \pi(1) = \underline{4} \text{ from } \lambda_0(\pi) \\ \lambda_2(\pi) = 0, \overline{5}, \underline{6}, 7 & \text{add } \pi(2) = \overline{5} \text{ to } \lambda_1(\pi) \\ \lambda_3(\pi) = 0, \overline{3}, \overline{5}, \underline{6}, 7 & \text{add } \pi(3) = \overline{3} \text{ to } \lambda_2(\pi) \\ \lambda_4(\pi) = 0, \overline{1}, \overline{3}, \overline{5}, \underline{6}, 7 & \text{add } \pi(4) = \overline{1} \text{ to } \lambda_3(\pi) \\ \lambda_5(\pi) = 0, \overline{1}, \overline{2}, \overline{3}, \overline{5}, \underline{6}, 7 & \text{add } \pi(5) = \overline{2} \text{ to } \lambda_4(\pi) \\ \lambda_6(\pi) = 0, \overline{1}, \overline{2}, \overline{3}, \overline{5}, 7 & \text{delete } \pi(6) = \underline{6} \text{ from } \lambda_5(\pi) \end{array}$$

The triangulation $\mathcal{T} = \eta_Q(\pi)$ is given in Fig. 2. Note that the fiber of $\mathcal{T}$ is

$$\eta_Q^{-1}(\mathcal{T}) = \{453126, 453162, 453612, 456312\}.$$

3. Representations of quivers of type A

Let $\mathbb{k}$ be an algebraically closed field, for example, $\mathbb{k} = \mathbb{C}$. Given a quiver Q , we denote by Q_0 the set of its vertices and by Q_1 its set of arrows. For $\alpha \in Q_1$, let $s(\alpha)$ be the source of α and $t(\alpha)$ be its target. A *path* from i to j in Q is a sequence of arrows $\alpha_1\alpha_2\dots\alpha_\ell$ such that $s(\alpha_1) = i$, $t(\alpha_\ell) = j$, and $t(\alpha_h) = s(\alpha_{h+1})$, for all $1 \leq h \leq \ell - 1$. The integer ℓ is called the *length* of the path. Paths of length zero are called *constant paths* and are denoted by e_i , $i \in Q_0$.

A *representation* $M = (M_i, \varphi_\alpha)$ of Q consists of a $\mathbb{k}$ -vector space M_i , for each vertex $i \in Q_0$, and a $\mathbb{k}$ -linear map $\varphi_\alpha: M_{s(\alpha)} \rightarrow M_{t(\alpha)}$, for each arrow $\alpha \in Q_1$. If each vector space M_i is finite dimensional, we say that M is *finite dimensional*, and the *dimension vector* $\underline{\dim} M$ of M is the vector $(\dim M_i)_{i \in Q_0}$ of the dimensions of the vector spaces. For example, the representation in Fig. 3 is a representation with dimension vector $(0, 1, 1, 1, 1, 0, 0)$ of the type A_7 quiver in Fig. 1. Let $\text{rep } Q$ denote the category of

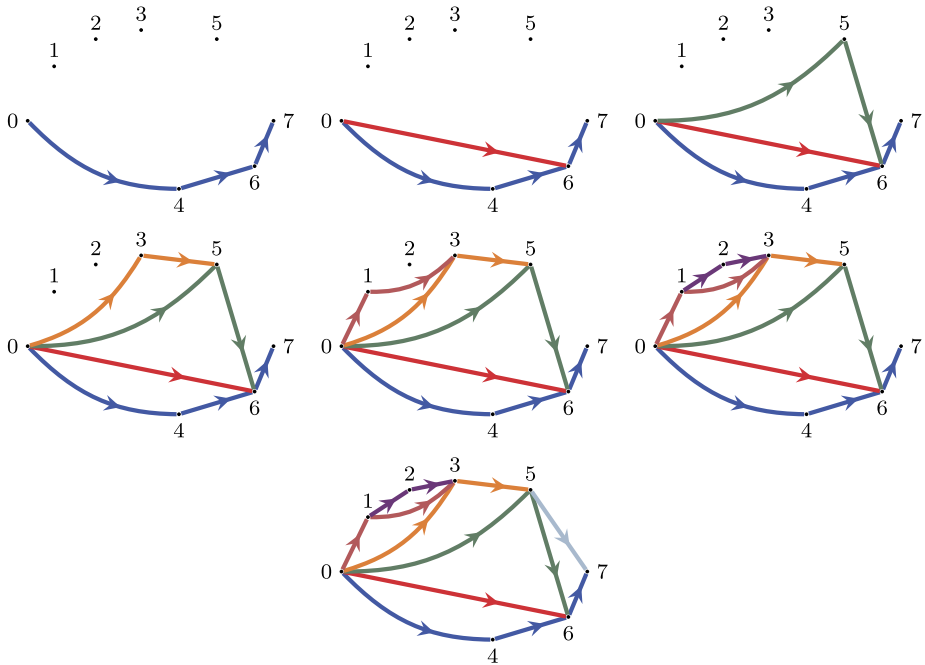


Fig. 2. The paths $\lambda_i(453126)$ and triangulation for $\eta_Q(453126)$ from Example 2.3.

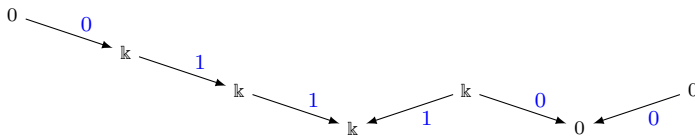


Fig. 3. A representation of the quiver in Fig. 1.

finite dimensional representations of Q and let $\text{ind } Q$ denote a full subcategory whose objects are one representative of the isoclass of each indecomposable representation. The category $\text{rep } Q$ is equivalent to the category of finitely generated modules over the path algebra $\mathbb{k}Q$ of Q . The Auslander–Reiten quiver $\Gamma_{\text{rep } Q}$ of $\text{rep } Q$ has the isoclasses of indecomposable representations as vertices and irreducible morphisms as arrows. For more information about representations of quivers we refer to the textbooks [5,34].

From now on, let Q be a Dynkin quiver of type A_{n+2} , labeled as before (see Fig. 1).

In the remainder of this section, we recall the classification of indecomposable representations and irreducible morphisms in $\text{rep } Q$, and give an interpretation of hooks and cohooks as boundary edges of the polygon $P(Q)$.

3.1. Indecomposable representations

For each $1 \leq i \leq j \leq n+2$, let $M(i, j)$ denote the indecomposable representation supported on the vertices between i and j . Thus $M(i, j) = (M_\ell, \varphi_\alpha)$ with

$$M_\ell = \begin{cases} \mathbb{k} & \text{if } i \leq \ell \leq j; \\ 0 & \text{otherwise;} \end{cases}$$

and $\varphi_\alpha = 1$, whenever $M_{s(\alpha)}$ and $M_{t(\alpha)}$ are nonzero, and $\varphi_\alpha = 0$, otherwise. The representations $M(i, i)$ are *simple* representations (containing no proper nonzero subrepresentations) and are denoted by $S(i)$, $i \in Q_0$.

We have the following proposition, which is a special case of Gabriel's theorem [14].

Proposition 3.1. *Let Q be a Dynkin quiver of type $\mathbb{A}_{n+2}$. Up to isomorphism, the indecomposable representations of Q are precisely the representations $M(i, j)$ with $1 \leq i \leq j \leq n+2$.*

3.2. Irreducible morphisms in terms of hooks and cohooks

A morphism from a representation (M_ℓ, ϕ_α) to (N_ℓ, ψ_α) is a sequence of linear maps (f_ℓ) such that for each arrow $\alpha \in Q_1$ the following diagram commutes.

$$\begin{array}{ccc} M_{s(\alpha)} & \xrightarrow{\phi_\alpha} & M_{t(\alpha)} \\ f_{s(\alpha)} \downarrow & & \downarrow f_{t(\alpha)} \\ N_{s(\alpha)} & \xrightarrow{\psi_\alpha} & N_{t(\alpha)} \end{array}$$

In this subsection, we describe irreducible morphisms in type $\mathbb{A}$.

Roughly speaking, an irreducible morphism between indecomposable representations is a morphism that does not factor non-trivially. For a precise definition, see for example the textbooks [5,34]. The work of Butler and Ringel from [9] shows how each irreducible morphism in a string algebra is determined by adding a hook or removing a cohook from an indecomposable representation to produce a new representation. Recall that each irreducible morphism in $\text{rep}(Q)$ is an arrow in the Auslander–Reiten quiver. In Fig. 5 we show an example. Each (blue) h and (red) c indicates an arrow that corresponds to adding a hook and removing a cohook, respectively. In the following, we recall this construction in the special case of type $\mathbb{A}$.

Definition 3.2 ([9]). Let Q be a quiver of type $\mathbb{A}$ and let α be an arrow in Q . The *hook* of α , denoted by $\text{hook}(\alpha)$ is given by the maximal path of Q starting at $x = s(\alpha)$ that does not use α . Note that this maximal path may consist of only one vertex $x = s(\alpha)$ and no arrows.

The *cohook* of α , denoted by $\text{cohook}(\alpha)$ is given by the maximal path of Q ending at $y = t(\alpha)$ that does not use α . Note that this maximal path may consist of only one vertex $y = t(\alpha)$ and no arrows.

That is, $\text{hook}(\alpha)$ and $\text{cohook}(\alpha)$ are of the form shown in Fig. 4 (without the arrow α).

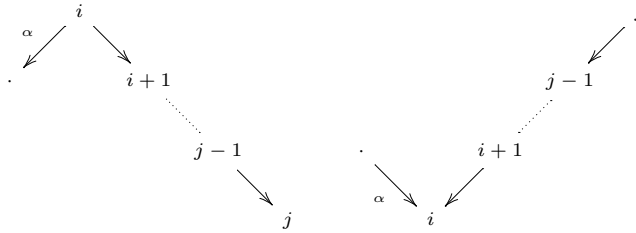


Fig. 4. The hook of the arrow α , starting from $i = s(\alpha)$ (left); the cohook of the arrow α , ending at $i = t(\alpha)$ (right).

Denote by $H(\alpha)$, respectively $C(\alpha)$, the indecomposable representation of Q supported on the hook, respectively cohook, of the arrow α , more precisely

$$H(\alpha)_x = \begin{cases} \mathbb{k} & \text{if } x \text{ is a vertex in } \text{hook}(\alpha); \\ 0 & \text{otherwise;} \end{cases}$$

$$C(\alpha)_x = \begin{cases} \mathbb{k} & \text{if } x \text{ is a vertex in } \text{cohook}(\alpha); \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.3. It may happen that $H(\alpha) = C(\beta)$. For the quiver in Fig. 1, we have $H(2 \rightarrow 3) = C(1 \rightarrow 2) = S(2)$ is the simple representation at vertex 2, and $H(4 \leftarrow 5) = C(6 \leftarrow 7) = M(5, 6)$ is the indecomposable representation supported at vertices 5 and 6.

Let α be an arrow in Q_1 whose target or source is either i or j , and assume that α does not lie in the support of $M = M(i, j)$. Adding a hook to M is an operation which produces a new indecomposable representation N whose support is $\text{supp}(M) \sqcup \text{supp}(H(\alpha))$. Similarly removing a cohook from a representation $N = M(i, \ell)$ or $M(h, j)$ is an operation which produces a new indecomposable representation M whose support is $\text{supp}(N) \setminus \text{supp}(C(\alpha))$. In the following proposition, we fix the representation M , and we either add a hook to M to produce a “bigger” representation, or we remove a cohook from a “bigger” representation to obtain M .

Proposition 3.4. Let Q be a quiver of type $\mathbb{A}$, let $M = M(i, j)$, and let N be an indecomposable representation satisfying $\text{supp}(M) \subset \text{supp}(N)$.

- (1) If $f : M \rightarrow N$ is an irreducible morphism then N is obtained by adding a hook to M . This can be done in at most two ways.
 - (a) If $\alpha : j \leftarrow (j+1)$ and $H(\alpha) = M(j+1, \ell)$ then $N = M(i, \ell)$.
 - (b) If $\alpha : (i-1) \rightarrow i$ and $H(\alpha) = M(h, i-1)$ then $N = M(h, j)$.

- (2) If $f : N \rightarrow M$ is an irreducible morphism then M is obtained by removing a cohook from N . This can be done in at most two ways.
- (a) If $\alpha : j \rightarrow (j+1)$ and $C(\alpha) = M(j+1, \ell)$ then $N = M(i, \ell)$.
 - (b) If $\alpha : (i-1) \leftarrow i$ and $C(\alpha) = M(h, i-1)$ then $N = M(h, j)$.

Moreover all irreducible morphisms between indecomposable representations are of this form.

Example 3.5. In the example of Fig. 5, we can illustrate the four different types of irreducible morphisms as follows. In each example, M is the representation with the smaller support.

$$\begin{array}{ll}
 (1)(a) \quad \frac{2}{3} \rightarrow \frac{2}{3} {}_4 5_6, \alpha : 5 \rightarrow 4, H(\alpha) = \frac{5}{6} & (2)(a) \quad \frac{5}{6} {}_7 \rightarrow 5, \alpha : 5 \rightarrow 6, C(\alpha) = \frac{7}{6} \\
 (1)(b) \quad \frac{2}{3} \rightarrow \frac{1}{3}, \alpha : 1 \rightarrow 2, H(\alpha) = 1 & (2)(b) \quad \frac{1}{3} {}_5 \rightarrow 5, \alpha : 5 \rightarrow 4, C(\alpha) = \frac{1}{3}
 \end{array}$$

3.3. Maximal increasing and decreasing paths

A path $i \rightarrow \dots \rightarrow j$ in Q is said to be *increasing* if $i \leq j$, and *decreasing* if $i \geq j$. Note that the constant paths e_i are both increasing and decreasing. A decreasing path is called a *maximal decreasing path* if it is not a proper subpath of a decreasing path. Similarly, an increasing path is called a *maximal increasing path* if it is not a proper subpath of an increasing path. Note that if c is a maximal increasing path or a maximal decreasing path in a Dynkin quiver of type $\mathbb{A}$, then c is a maximal path of Q or a constant path. See Example 3.8.

The following lemma gives an interpretation of maximal increasing paths and maximal decreasing paths as boundary edges in the labeled polygon $P(Q)$. In the polygon $P(Q)$, let $\gamma(a, b)$ denote the line segment between vertices a and b for $0 \leq a < b \leq n+2$.

Lemma 3.6. Let Q be a Dynkin quiver of type $\mathbb{A}_{n+2}$.

- (1) The map $(i \rightarrow \dots \rightarrow j) \mapsto \gamma(i-1, j)$ is a bijection between the maximal increasing paths in Q and the lower boundary edges of $P(Q)$.
- (2) The map $(i \rightarrow \dots \rightarrow j) \mapsto \gamma(j-1, i)$ is a bijection between the maximal decreasing paths in Q and the upper boundary edges of $P(Q)$.

Proof. Recall that $\underline{[n+1]}$ is the set of vertices on the lower part of the boundary of $P(Q)$ and $\overline{[n+1]}$ is the set of vertices on the upper part of the boundary.

(1) Let c be a maximal increasing path in Q . Since Q is of type $\mathbb{A}$, either $c = e_i$ is constant or c is a maximal path in Q .

First, suppose $c = e_i$ is a maximal increasing path which is constant. Then either $(i+1) \rightarrow i \rightarrow (i-1)$ is a decreasing path in Q , or $i = 1$ and is a sink, or $i = n+2$ and is a source. Therefore $i \in [n+1]$ (if $i \neq n+2$) and $i-1 \in [n+1]$ (if $i \neq 1$). Then the line segment $\gamma(i-1, i)$ is a lower boundary edge.

Next, let c be a non-constant maximal increasing path $i \rightarrow \cdots \rightarrow j$ in Q . Then c is a maximal path in Q , so the vertex i is a source and the vertex j is a sink of Q . We want to show that the line segment $\gamma(i-1, j)$ is a bottom boundary edge. Note that the vertices $i, \dots, j-1$ are in $[n+1]$, by construction. Since i is a source, either $i = 1$ or $i-1 \in [n+1]$. Since j is a sink, either $j \in [n+1]$ or $j = n+2$. Then the vertices $i, \dots, (j-1)$ form a maximal consecutive sequence of integers which label $j-i$ consecutive vertices of the upper boundary of the polygon $P(Q)$, so $\gamma(i-1, j)$ is a lower boundary edge. It follows from the construction that this map is a bijection.

The proof of (2) is analogous to part (1). $\square$

Corollary 3.7. *A Dynkin quiver of type $\mathbb{A}_{n+2}$ has exactly $n+3$ maximal increasing and maximal decreasing paths.*

Proof. This follows from Lemma 3.6, since $P(Q)$ has exactly $n+3$ boundary edges. $\square$

Example 3.8. Let Q be the quiver $1 \rightarrow 2$, so $n = 0$. Then there is exactly one maximal increasing path $1 \rightarrow 2$, and two maximal decreasing paths, namely the two constant paths e_1 and e_2 .

If Q is the quiver from Fig. 1, then $n = 5$ and there are $n+3 = 8$ maximal increasing and maximal decreasing paths.

Recall that for each $\alpha \in Q_1$, the $\text{hook}(\alpha)$ and $\text{cohook}(\alpha)$ are defined to be maximal paths. Therefore, in type $\mathbb{A}$ each hook and cohook is a maximal increasing or a maximal decreasing path. We make this precise in the next lemma.

Lemma 3.9. *Let Q be a quiver of type $\mathbb{A}_{n+2}$. Then the set*

$$\bigcup_{\alpha \in Q_1} \{H(\alpha), C(\alpha)\}$$

of all indecomposable representations supported on the hooks and cohooks is in bijection with

- (a) *the set $\{S(1), \dots, S(n+2)\}$ of all simple representations of Q , if Q is the linear orientation of $\mathbb{A}_{n+2}$ (all arrows pointing in the same direction);*
- (b) *the set of maximal increasing paths and maximal decreasing paths, for all other choices of Q .*

Proof. We have noted earlier that every hook and every cohook is either a maximal increasing path or a maximal decreasing path. To prove (a) it suffices to notice that for a linearly oriented quiver, say, $1 \rightarrow 2 \rightarrow \dots \rightarrow n+2$, the maximal decreasing paths are the $n+2$ constant paths, and the unique maximal increasing path from 1 to $n+2$ is neither a hook nor a cohook.

To prove (b) it remains to show that if Q is not the linear orientation then every maximal increasing path and every maximal decreasing path is a hook or cohook. Suppose c is a maximal increasing path in Q . Since Q is of type $\mathbb{A}$, either c is constant or c is a (non-constant) maximal path in Q .

First, suppose $c = e_i$ is a maximal increasing path which is constant. Then either $(i-1) \leftarrow i \leftarrow (i+1)$ is a decreasing path in Q , or $i = 1$ and is a sink, or $i = n+2$ and is a source. In the first situation, c is the hook of the arrow $(i-1) \leftarrow i$ in Q (and also the cohook of the arrow $i \leftarrow (i+1)$ in Q). If $i = 1$ is a sink, then c is the cohook of the arrow $1 \leftarrow 2$ in Q . If $i = n+2$ is a source, then c is the hook of the arrow $(n+1) \leftarrow (n+2)$ in Q .

Next, suppose $c = i \rightarrow \dots \rightarrow j$ is a non-constant maximal increasing path in Q . Then, as noted earlier, c must be a maximal path in Q , so the vertex i is a source and the vertex j is a sink of Q . By assumption, the quiver Q is not the linear orientation $1 \rightarrow 2 \rightarrow \dots \rightarrow n+2$. Thus if $i > 1$ we have an arrow $\alpha: (i-1) \leftarrow i$ in Q , since i is a source. In this case, c is the hook of the arrow α , as illustrated in the left picture of Fig. 4. On the other hand, if $i = 1$ then $j < n+2$ and we have an arrow $\beta: j \leftarrow (j+1)$ in Q , since j is a sink. In this case, c is the cohook of the arrow β (not illustrated).

The proof that a maximal decreasing path corresponds to a hook or cohook is similar. $\square$

Corollary 3.10. *Let Q be a quiver of type $\mathbb{A}_{n+2}$. Then the set of all representations supported on the hooks and cohooks is in bijection with*

- (a) *the set of all boundary edges of the polygon $P(Q)$ except for the long boundary edge $\gamma(0, n+2)$, if Q is the linear orientation;*
- (b) *the set of all boundary edges of the polygon $P(Q)$, for all other choices of Q .*

Proof. This follows from Lemmas 3.9 and 3.6. $\square$

4. A geometric model for $\text{rep } Q$

In this section, we construct a category of line segments in the polygon $P(Q)$ and prove that it is equivalent to the category $\text{ind } Q$ of indecomposable representations of Q . The construction is illustrated in an example in Figs. 5 and 6.

4.1. The category of line segments $\mathcal{C}_{P(Q)}$

In this subsection, we construct a category $\mathcal{C}_{P(Q)}$ whose objects are line segments in the polygon $P(Q)$ and whose morphisms are generated by pivots of line segments modulo mesh relations.

4.1.1. Translation quivers and mesh categories

We start by reviewing the notions of translation quiver and mesh category, from [29,15]. Recall that for any quiver Γ , we denote the set of vertices by Γ_0 and the set of arrows by Γ_1 . A *loop* is an arrow that starts and ends at the same vertex.

A *translation quiver* (Γ, τ) is a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ without loops together with an injective map $\tau: \Gamma'_0 \rightarrow \Gamma_0$ (the *translation*) from a subset Γ'_0 of Γ_0 to Γ_0 such that, for all vertices $x \in \Gamma'_0$, $y \in \Gamma_0$, the number of arrows from $y \rightarrow x$ is equal to the number of arrows from $\tau x \rightarrow y$. Given a translation quiver (Γ, τ) , a *polarization* of Γ is an injective map $\sigma: \Gamma'_1 \rightarrow \Gamma_1$, where Γ'_1 is the set of all arrows $\alpha: y \rightarrow x$ with $x \in \Gamma'_0$, such that $\sigma(\alpha): \tau x \rightarrow y$ for every arrow $\alpha: y \rightarrow x \in \Gamma_1$.

From now on we assume that Γ has no multiple arrows. In that case, there is a unique polarization of Γ .

The *path category* of a translation quiver (Γ, τ) is the category whose objects are the vertices Γ_0 of Γ , and, given $x, y \in \Gamma_0$, the $\mathbb{k}$ -vector space of morphisms from x to y is given by the $\mathbb{k}$ -vector space with basis the set of all paths from x to y . The composition of morphisms is induced from the usual composition of paths. The *mesh ideal* in the path category of Γ is the ideal generated by the *mesh relations*

$$m_x = \sum_{\alpha: y \rightarrow x} \sigma(\alpha)\alpha.$$

for all $x \in \Gamma'_0$

For example, in the Auslander–Reiten quiver in Fig. 5, we have the commutativity relation

$$m_{\begin{smallmatrix} 3 & 5 \\ 4 & 6 \end{smallmatrix}} = (4 \rightarrow \begin{smallmatrix} 5 \\ 4 & 6 \end{smallmatrix} \rightarrow \begin{smallmatrix} 3 & 5 \\ 4 & 6 \end{smallmatrix}) + (4 \rightarrow \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \rightarrow \begin{smallmatrix} 3 & 5 \\ 4 & 6 \end{smallmatrix})$$

and the zero relation $m_2 = (3 \rightarrow \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \rightarrow 2)$.

The *mesh category* $\mathcal{M}(\Gamma, \tau)$ of (Γ, τ) is the quotient of the path category of (Γ, τ) by the mesh ideal.

Example 4.1. If Q is a quiver of Dynkin type then its Auslander–Reiten quiver $\Gamma_{\text{rep } Q}$ together with the Auslander–Reiten translation τ is a translation quiver, where Γ'_0 is the set of all non-projective indecomposable representations. In this case, the mesh category $\mathcal{M}(\Gamma_{\text{rep } Q}, \tau)$ is equivalent to the category $\text{ind } Q$, and the additive closure of $\mathcal{M}(\Gamma_{\text{rep } Q}, \tau)$ is equivalent to $\text{rep } Q$.

4.1.2. Definition of $\mathcal{C}_{P(Q)}$

We define the set $\mathcal{E}$ of all line segments $\gamma(i, j)$ of $P(Q)$ as follows

$$\mathcal{E} = \{\gamma(i, j) \mid 0 \leq i < j \leq n + 2\}, \quad (4.1)$$

where $\gamma(i, j)$ denotes the line segment between the vertices i and j . Notice that $\mathcal{E}$ contains all boundary edges and all diagonals of $P(Q)$. Also notice that our line segments are oriented by our choice of labeling, and that $\gamma(i, j) \in \mathcal{E}$ if and only if $\gamma(j, i) \notin \mathcal{E}$.

For any vertex ℓ of $P(Q)$, denote by $R(\ell)$ and by $R^{-1}(\ell)$, respectively, the clockwise and counterclockwise neighbor of ℓ on the boundary of $P(Q)$.

Definition 4.2. Let $\gamma(i, j) \in \mathcal{E}$. The line segment $\gamma(i, R^{-1}(j))$ is called a *pivot* of $\gamma(i, j)$ if it lies in the set $\mathcal{E}$, that is, if $i < R^{-1}(j)$. Similarly, the line segment $\gamma(R^{-1}(i), j)$ is called a *pivot* of $\gamma(i, j)$ if it lies in the set $\mathcal{E}$, that is, if $R^{-1}(i) < j$.

Thus a pivot of a line segment γ is given by fixing one of the endpoints of γ while moving the other endpoint to its counterclockwise neighbor.

Remark 4.3. Notice that a line segment $\gamma(i, j)$ may have two, one or zero pivots. In the example in Fig. 6, the line segment $\gamma(3, 4)$ has the two pivots $\gamma(2, 4)$ and $\gamma(3, 6)$. On the other hand, the line segment $\gamma(0, 4)$ has only one pivot $\gamma(0, 6)$, and the line segment $\gamma(4, 5)$ has no pivots at all.

We now define a translation quiver $(\Gamma_{P(Q)}, R)$. The vertices of the quiver are the line segments in $\mathcal{E}$. There is an arrow $\gamma(i, j) \rightarrow \gamma(i', j')$ if and only if $\gamma(i', j')$ is a pivot of $\gamma(i, j)$, thus if and only if $i' < j'$ and $(i', j') = (R^{-1}(i), j)$ or $(i', j') = (i, R^{-1}(j))$. Finally, the translation R is defined by

$$R(\gamma(i, j)) = \begin{cases} \gamma(R(i), R(j)) & \text{if } R(i) < R(j); \\ 0 & \text{otherwise.} \end{cases} \quad (4.2)$$

Thus R acts on $\mathcal{E}$ by rotation.

The following is the main definition of this section.

Definition 4.4. Let $\mathcal{C}_{P(Q)}$ be the mesh category of the translation quiver $(\Gamma_{P(Q)}, R)$. We call $\mathcal{C}_{P(Q)}$ the *category of line segments* of $P(Q)$.

4.2. The functor $F: \mathcal{C}_{P(Q)} \rightarrow \text{ind } Q$

In this subsection, we construct an equivalence of categories between $\mathcal{C}_{P(Q)}$ and $\text{ind } Q$.

Definition 4.5. Let $F: \mathcal{C}_{P(Q)} \rightarrow \text{ind } Q$ be the functor defined as follows. On *objects*, let

$$F(\gamma(i, j)) = M(i + 1, j).$$

To define F on *morphisms*, it suffices to define it on the pivots introduced in Definition 4.2. Define $F(\gamma(i, j) \rightarrow \gamma(R^{-1}(i), j))$ to be the irreducible morphism $M(i+1, j) \rightarrow M(R^{-1}(i)+1, j)$ given by

- (a) adding the hook corresponding to the boundary edge $\gamma(R^{-1}(i), i)$, if $R^{-1}(i) < i$;
- (b) removing the cohook corresponding to the boundary edge $\gamma(i, R^{-1}(i))$, if $i < R^{-1}(i)$.

Similarly, let $F(\gamma(i, j) \rightarrow \gamma(i, R^{-1}(j)))$ be the irreducible morphism $M(i+1, j) \rightarrow M(i+1, R^{-1}(j))$ given by

- (a) adding the hook corresponding to the boundary edge $\gamma(j, R^{-1}(j))$, if $j < R^{-1}(j)$;
- (b) removing the cohook corresponding to the boundary edge $\gamma(R^{-1}(j), j)$, if $R^{-1}(j) < j$.

We are now ready for our first main result.

Theorem 4.6. *The functor F is an equivalence of categories*

$$F: \mathcal{C}_{P(Q)} \rightarrow \text{ind } Q.$$

In particular,

- (1) F induces an isomorphism of translation quivers $(\Gamma_{P(Q)}, R) \rightarrow (\Gamma_{\text{rep } Q}, \tau)$;
- (2) F induces bijections

$$\begin{aligned} \{\text{line segments in } P(Q)\} &\rightarrow \text{ind } Q; \\ \{\text{pivots in } P(Q)\} &\rightarrow \{\text{irreducible morphisms in } Q\}; \end{aligned}$$

- (3) *the rotation R corresponds to the Auslander–Reiten translation τ in the following sense*

$$F \circ R = \tau \circ F;$$

- (4) F is an exact functor with respect to the induced abelian structure on $\mathcal{C}_{P(Q)}$.

Proof. Proposition 3.1 implies that the indecomposable representations are of the form $M(i, j)$ with $1 \leq i \leq j \leq n+2$. On the other hand, the line segments $\gamma(i, j) \in \mathcal{E}$ are parametrized by $0 \leq i < j \leq n+2$ by equation (4.1). Thus F is a bijection between the objects of the categories $\mathcal{C}_{P(Q)}$ and $\text{ind } Q$ and thus a bijection between the vertices of the translation quivers.

Proposition 3.4 combined with Corollary 3.10 shows that F is a bijection between pivots and irreducible morphisms, hence a bijection between the arrows of the translation quivers. Moreover, $f: \gamma(i, j) \rightarrow \gamma(i', j')$ is an arrow in $\Gamma_{P(Q)}$ if and only if

$F(f): M(i+1, j) \rightarrow M(i'+1, j')$ is an arrow in $\Gamma_{\text{rep } Q}$. Thus F is an isomorphism of quivers $\Gamma_{P(Q)} \rightarrow \Gamma_{\text{rep } Q}$.

Next we check that the translations of the two quivers correspond to each other. Let $\gamma(i, j) \in \mathcal{E}$ and assume that $R^{-1}(\gamma(i, j)) \in \mathcal{E}$ as well. Note that

$$F(R^{-1}(\gamma(i, j))) = F(\gamma(R^{-1}(i), R^{-1}(j))) = M(R^{-1}(i) + 1, R^{-1}(j)). \quad (4.3)$$

Since $R^{-1}(\gamma(i, j)) \in \mathcal{E}$ then $\gamma(i, j)$ has at least one pivot, see Definition 4.2. Without loss of generality, we may assume that $\gamma(i, j) \rightarrow \gamma(R^{-1}(i), j)$ is a pivot. Then $\gamma(R^{-1}(i), j) \rightarrow \gamma(R^{-1}(i), R^{-1}(j))$ is a pivot too. Under F , the composition of these two pivots is mapped to the composition of two irreducible morphisms

$$M(i+1, j) \rightarrow M(R^{-1}(i) + 1, j) \rightarrow M(R^{-1}(i) + 1, R^{-1}(j)).$$

Since $i \neq R^{-1}(i)$ and $j \neq R^{-1}(j)$, it follows from the structure of the Auslander–Reiten quivers of type $\mathbb{A}$ that the last representation in this sequence is the inverse Auslander–Reiten translate of the first, thus

$$M(R^{-1}(i) + 1, R^{-1}(j)) = \tau^{-1}M(i+1, j) = \tau^{-1}F(\gamma(i, j)).$$

Now Equation (4.3) yields $F(R^{-1}(\gamma(i, j))) = \tau^{-1}F(\gamma(i, j))$. This proves (1), (2) and (3).

Since both categories $\mathcal{C}_{P(Q)}$ and $\text{ind } Q$ are the mesh categories of their translation quivers $\Gamma_{P(Q)}$ and $\Gamma_{\text{rep } Q}$, statement (1) implies that F is an equivalence of categories. In particular, this equivalence induces the structure of an abelian category on $\mathcal{C}_{P(Q)}$. With respect to this structure F is exact, since every equivalence between abelian categories is exact. $\square$

Remark 4.7. In [13], the authors defined a category of unoriented diagonals (no boundary edges) in an $(n+3)$ -gon, whose morphisms are also generated by pivots modulo mesh relations. That construction yields the cluster category of type $\mathbb{A}_n$. Our construction here is different in the following sense. First, we also include boundary edges, and our quiver is of rank $n+2$. Second, our line segments are oriented, which implies that certain pivots that are allowed in the cluster category are not allowed in the category $\mathcal{C}_{P(Q)}$. This is the reason why $\mathcal{C}_{P(Q)}$ is a hereditary abelian category, while the cluster category is triangulated and not abelian.

We investigate the relation between the two constructions further in section 7.

5. Stability function

Stability conditions were introduced in [32,18,31,10] and have important applications in algebraic geometry and representation theory. We recall the definition in the setting of the category $\text{rep } Q$ of finite-dimensional representations of a quiver Q .

Let K_0 denote the Grothendieck group of $\text{rep } Q$. For $M \in \text{rep } Q$, we denote its class in K_0 by $[M]$. Thus for $M, N \in \text{rep } Q$, we have $[M] = [N]$ if and only if M and N have the same dimension vector. K_0 has a basis given by the classes of the simple representations $S(x)$, $x \in Q_0$.

Definition 5.1. (1) A *central charge* is a group homomorphism $Z: K_0 \rightarrow \mathbb{C}$ such that for all nonzero representations M the complex number $Z([M])$ lies in the strict right half plane²

$$\mathbb{H} = \{r e^{i\pi\phi} \mid r > 0 \text{ and } -\frac{1}{2} < \phi < \frac{1}{2}\}.$$

(2) Given a central charge Z , we obtain an associated *stability function*

$$\phi(M) = \frac{1}{\pi} \arg(Z([M])).$$

(3) Given a stability function ϕ , a nonzero representation M is called ϕ -*stable* if every nonzero proper subrepresentation $L \subsetneq M$ satisfies $\phi(L) < \phi(M)$.

Remark 5.2. (1) Since Z is a group homomorphism it is determined by its values on the classes of simple representations.

(2) If M is ϕ -stable then M is indecomposable. Indeed, since direct summands are subrepresentations, this follows directly from the additivity of the central charge.

It is natural to ask if there exist stability functions for which all indecomposable representations are stable. Reineke conjectured in [28, Conjecture 7.1] that such a stability function exists for every Dynkin quiver. In the same paper, he gave a stability function for the linearly oriented type A quiver. Apruzzese and Igusa proved the conjecture for all type A quivers in [4], and, very recently, Kinser described all such stability functions for type A in [19], see also [22–24].

We are going to show that our geometric model also provides such a stability function for every quiver of Dynkin type A . The stability function is given via the functor F of Theorem 4.6 simply by the slope of the oriented line segment of an indecomposable representation.

From now on let Q be a quiver of type A_{n+2} and let $P(Q)$ be the $(n+3)$ -gon constructed in Section 2. Let $\mathcal{E}$ be the set of oriented line segments

$$\mathcal{E} = \{\gamma(j, k) \mid 0 \leq j < k \leq n+2\}$$

as in section 4. Recall that $\gamma(j, k)$ denotes the oriented line segment from vertex j to vertex k of the polygon $P(Q)$. Since the vertices of $P(Q)$ are points in the plane, we can define a map $\text{vec}: \mathcal{E} \rightarrow \mathbb{C}$, $\gamma(j, k) \mapsto \vec{\gamma}(j, k)$, where $\vec{\gamma}(j, k)$ is the complex number $re^{i\theta}$

² [10] uses the strict *upper* half plane.

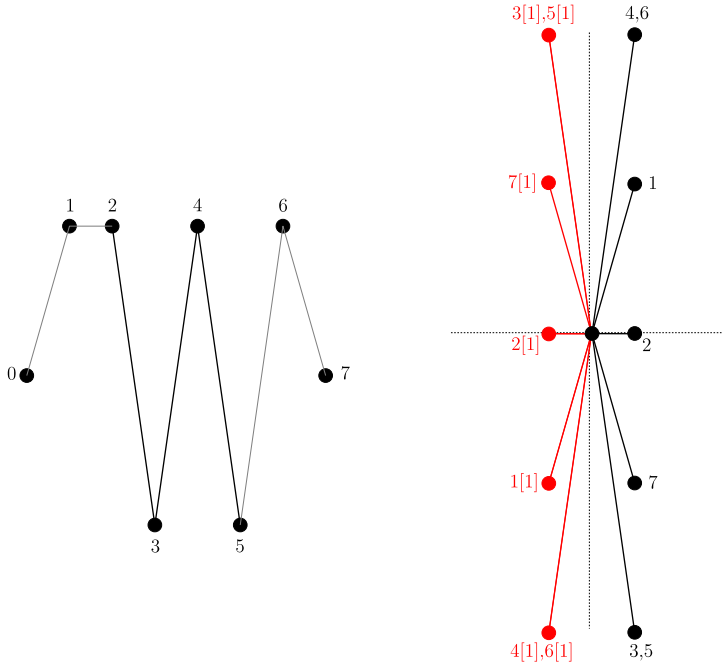


Fig. 7. Line segments of the form $\gamma(x, x + 1)$ in the polygon $P(Q)$ (left); the corresponding central charges of the simple representations in black as well as their first shifts in the derived category in red (right).

given by the vector with the same direction and magnitude as the oriented line segment from point j to point k .

Let $F: \mathcal{E} \rightarrow \text{ind } Q$ be the bijection of Theorem 4.6 and denote its inverse by G . Furthermore, let $\vec{G} = \text{vec} \circ G$ denote its composition with the map defined above. Thus, for a representation $M = M(j, k)$ we have

$$\vec{G}(M) = \vec{\gamma}(j - 1, k) = r(M) e^{i\theta(M)},$$

where $r(M)$ is the length of the vector $\vec{G}(M) = \vec{\gamma}(j - 1, k)$ and $\theta(M)$ is the angle from the positive real axis to the vector $\vec{G}(M)$ in the complex plane. An example is given in Fig. 7.

Then $\vec{G}$ induces a group homomorphism $Z: K_0 \rightarrow \mathbb{C}$ defined on the basis of simples by $Z([S(x)]) = \vec{\gamma}(x - 1, x)$ and extended additively to all of K_0 . By definition, we have

$$Z([M(j, k)]) = Z\left(\sum_{x=j}^k [S(x)]\right) = \sum_{x=j}^k \vec{\gamma}(x - 1, x) = \vec{\gamma}(j - 1, k),$$

where the last identity is the addition in $\mathbb{C}$. Thus for every indecomposable representation M , we have

$$Z([M]) = \vec{G}(M) = r(M) e^{i\theta(M)}. \quad (5.1)$$

Note that Z maps every representation of Q to the strict right half plane, and thus Z is a central charge. The corresponding stability function ϕ is given by

$$\phi(M) = \frac{1}{\pi} \theta(M) \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Now, for every nonzero proper subrepresentation $L \subsetneq M$, the inclusion is a nonzero morphism in $\text{Hom}(L, M)$. Thus the slope of $G(L)$ is strictly smaller than the slope of $G(M)$, by the same argument as in the proof of Lemma 8.5. Therefore $\phi(L) < \phi(M)$ and it follows that M is ϕ -stable. We have shown the following.

Theorem 5.3. *Let Q be a Dynkin quiver of type A . Then the function that associates to every indecomposable representation M the angle of the corresponding oriented line segment $G(M)$ is a stability function for which every indecomposable representation of Q is stable.*

Remark 5.4. In [19], Kinser characterizes the stability conditions in question by two sets of inequalities that are formulated using a decomposition of the quiver into horizontal and vertical segments. The horizontal segments in Kinser's quiver correspond to the lower boundary segments in our polygon in order from left to right, and his vertical segments correspond to our upper boundary segments also in order from left to right. Kinser's inequalities in [19, Theorem 1.13] translate in our setting to the condition that the central charge (=slope) of the upper boundary segments is decreasing when moving along the upper boundary from left to right, and the central charge of the lower boundary segments is increasing.

5.1. A geometric model for the derived category

The stability function naturally allows for a geometric model for the derived category which we describe in this subsection. Geometric models for derived categories for arbitrary gentle algebras were given in [21], and our result here is equivalent to (a special case) of their construction. Note however, that the stability function is new.

Let $\mathcal{D} = \mathcal{D}^b(\text{rep } Q)$ be the derived category of bounded complexes in $\text{rep } Q$. Since $\text{rep } Q$ is a hereditary category, the indecomposable objects of $\mathcal{D}$ are of the form $M[i]$, with M an indecomposable representation of Q and i an integer, called the shift. Moreover if M, N are indecomposable representations then

$$\text{Hom}_{\mathcal{D}}(M[i], N[j]) = \begin{cases} \text{Hom}_{\text{rep } Q}(M, N) & \text{if } j = i; \\ \text{Ext}_{\text{rep } Q}^1(M, N) & \text{if } j = i + 1; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the Auslander-Reiten quiver of $\mathcal{D}$ is the translation quiver $\mathbb{Z}Q$, and in this quiver, each arrow corresponds either to (a shift of) an irreducible morphism in $\text{rep } Q$ or

to (a shift of) a short exact sequence of the form $0 \rightarrow P(x) \rightarrow E \rightarrow I(y) \rightarrow 0$ with $P(x)$ an indecomposable projective, $I(y)$ an indecomposable injective, E an indecomposable representation and there exists an arrow $y \rightarrow x$ in Q .

We know from Theorem 4.6 that the irreducible morphisms in $\text{rep } Q$ correspond to the pivots $\gamma(i, j) \rightarrow \gamma(R^{-1}(i), j)$ or $\gamma(i, j) \rightarrow \gamma(i, R^{-1}(j))$ of oriented line segments in $\mathcal{E}$, and these are defined only if $R^{-1}(i) < j$ or $i < R^{-1}(j)$, respectively. This condition can be reformulated by saying that the stability function ϕ evaluated on the line segment after the pivot is still between $-\frac{1}{2}$ and $\frac{1}{2}$.

Now let us consider a pivot $\gamma(i, j) \rightarrow \gamma(R^{-1}(i), j)$ that violates this condition, thus $R^{-1}(i) > j$. This happens precisely when i lies on the lower boundary (or $i = 0$), all vertices $i + 1, i + 2, \dots, j$ lie on the upper boundary, so that $R^{-1}(i) = k \geq j + 1$ is the first vertex after i on the lower boundary (possibly $k = n + 2$). In particular, the quiver contains the following subquiver

$$(i + 1) \rightarrow \cdots \rightarrow j \rightarrow (j + 1) \rightarrow \cdots \rightarrow k,$$

where the vertex $(i + 1)$ is a source and the vertex k is a sink. In particular, the representation $F(\gamma(i, j)) = M(i + 1, j) = I(j)$ is the injective at vertex j , and the reverse $\gamma(j, R^{-1}(i)) \in \mathcal{E}$ of the pivot $\gamma(R^{-1}(i), j)$ corresponds to the representation $F(\gamma(j, R^{-1}(i))) = M(j + 1, k) = P(j + 1)$, which is the projective at vertex $j + 1$. In particular, there is an arrow $j \rightarrow j + 1$ and a short exact sequence

$$0 \longrightarrow P(j + 1) \xrightarrow{f} M(i + 1, k) \xrightarrow{g} I(j) \longrightarrow 0.$$

Moreover, the line segment corresponding to the middle term of this sequence is $\gamma(i, k)$ is a boundary segment on the lower boundary.

We define a generalized pivot to be a map $\gamma(i, j) \rightarrow \gamma(R^{-1}(i), j)$ or $\gamma(i, j) \rightarrow \gamma(i, R^{-1}(j))$ with $0 \leq i, j \leq n + 1$ but without any other condition on i and j . Thus in the derived category a generalized pivot $\gamma(i, j) \rightarrow \gamma(R^{-1}(i), j)$ with $i < j$ and $R^{-1}(i) > j$ corresponds to a morphism $I(j) \rightarrow P(j + 1)[1]$. Note that $P(j + 1)$ and its shift $P(j + 1)[1]$ are represented by the same line segment, but with opposite orientations. Similarly, a generalized pivot $\gamma(i, j) \rightarrow \gamma(i, R^{-1}(j))$ with $i < j$ and $i > R^{-1}(j)$ corresponds to a morphism $I(j) \rightarrow P(j - 1)[1]$.

More generally, consider the following translation quiver $(\Gamma_{P(Q)}^{\mathbb{Z}}, R^{\mathbb{Z}})$. The quiver $\Gamma_{P(Q)}^{\mathbb{Z}}$ has as vertices the pairs $(\gamma, j) \in \mathcal{E} \times \mathbb{Z}$, and there is an arrow $(\gamma, j) \rightarrow (\gamma', j)$ if there is a pivot $\gamma \rightarrow \gamma'$, and there is an arrow $(\gamma, j) \rightarrow (\overline{\gamma'}, j + 1)$ if there is a generalized pivot $\gamma \rightarrow \gamma'$ but $\gamma' \notin \mathcal{E}$, where $\overline{\gamma'}$ is the same line segment as γ' but in the opposite direction. The translation $R^{\mathbb{Z}}$ is defined by

$$R^{\mathbb{Z}}(\gamma, j) = \begin{cases} (R(\gamma), j) & \text{if } R(\gamma) \in \mathcal{E}; \\ (\overline{R(\gamma)}, j - 1) & \text{if } R(\gamma) \notin \mathcal{E}, \end{cases}$$

where R is the rotation of a line segment defined in equation (4.2).

Let $\mathcal{C}_{P(Q)}^{\mathbb{Z}}$ be the mesh category of $\Gamma_{P(Q)}^{\mathbb{Z}}$. Then the functor F of Theorem 4.6 induces an equivalence of categories

$$F^{\mathbb{Z}}: \mathcal{C}_{P(Q)}^{\mathbb{Z}} \rightarrow \text{ind } \mathcal{D}$$

that maps the pair (γ, j) to the object $F(\gamma)[j]$.

Our stability function ϕ on $\text{rep } Q$ induces a stability function on the derived category which we also denote by ϕ , see [10, Proposition 5.3]. This stability function is again given by the angle of the corresponding line segment. More precisely, for every indecomposable representation M we define the angle $\theta(M[j])$ and the stability function $\phi(M[j])$ of the j -th shift of M in $\mathcal{D}$ as

$$\theta(M[j]) = \theta(M) + j\pi \quad \text{and} \quad \phi(M[j]) = \phi(M) + j.$$

Note that $\phi(M[j]) = \frac{1}{\pi}\theta(M[j])$.

6. Maximal almost rigid representations

In this section, we give a characterization of the representations that correspond to the triangulations of $P(Q)$ under the equivalence $F: \mathcal{C}_{P(Q)} \rightarrow \text{ind } Q$. It seems that this class of representations has not been studied so far.

Recall that for $A, B \in \text{rep } Q$, the vector space $\text{Ext}(B, A)$ can be identified with the space of all short exact sequences of the form $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$. Given such a short exact sequence, we call A, B the end terms and E the middle term of the sequence. Recall further that a representation is called *basic* if it has no repeated direct summands.

Definition 6.1. We say that a quiver representation T is *almost rigid* if it is basic and satisfies the following: For each pair A, B of indecomposable summands of T , either $\text{Ext}(A, B) = 0$ or $\text{Ext}(A, B) = \mathbb{k}$ and it is generated by a short exact sequence of the form $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$ whose middle term E is indecomposable.

We say that a representation T is *maximal almost rigid* if, for every nonzero representation M , the representation $T \oplus M$ is not almost rigid.

Let $\text{mar}(Q)$ be the set of all maximal almost rigid representations of Q .

Example 6.2. Let Q be the quiver $1 \rightarrow 2$ of type $\mathbb{A}_2$. Up to isomorphism, there are precisely three indecomposable representations of Q , namely $1, 2$ and $\frac{1}{2}$. The representation $T' = 1 \oplus 2$ is almost rigid since the extension $\frac{1}{2}$ is indecomposable. The representation $T = 1 \oplus \frac{1}{2} \oplus 2$ is the only maximal almost rigid representation (up to isomorphism).

Example 6.3. Let Q be the quiver $1 \rightarrow 2 \rightarrow 3$. Then the representation $\frac{1}{2} \oplus \frac{2}{3}$ is not almost rigid, because there is an extension

$$0 \rightarrow \frac{2}{3} \rightarrow \frac{1}{2} \oplus 2 \rightarrow \frac{1}{2} \rightarrow 0$$

whose middle term is not indecomposable.

Remark 6.4. At first sight, the definition of maximal almost rigid representations does not seem very natural from a representation theoretical perspective. However, at least for the path algebras of a Dynkin type $\mathbb{A}$ quiver, our Theorem 6.8 shows that these representations are important from a combinatorial perspective, since they do correspond to the triangulations of the polygon.

It would be interesting to see if this correlation extends to other types of algebras beyond the path algebras of type $\mathbb{A}$. As we shall see in Example 7.7, this does not seem to be the case for path algebras of Dynkin type $\mathbb{D}$. The reason for this difference is that an indecomposable representation in type $\mathbb{D}$ may contain a vertex with an incoming arrow and two outgoing arrows such that the composition of the incoming arrow with either one of the outgoing arrows is nonzero.

Therefore, in order to find a more general class of algebras for which the maximal almost rigid representations may play a similar combinatorial role as in type $\mathbb{A}$, we need to look for algebras whose indecomposable representations are locally of the same form as those of type $\mathbb{A}$. These algebras are known as gentle algebras. They are given by quivers with relations (see Example 7.8). A combinatorial model for the module category of a gentle algebra has been found in [6] by Baur and Coelho Simões, and a combinatorial model for the derived category of a gentle algebra in [21] by Oppermann, Plamondon and Schroll. It would be interesting to see if the maximal almost rigid representations (or complexes) of a gentle algebra realize combinatorially notable configurations in this model.

In order to prove Theorem 6.8, we first need to recall the structure of extensions in type $\mathbb{A}$. The following result is well-known. For a proof see for example [34, Chapter 3].

Proposition 6.5. *Let Q be a quiver of type $\mathbb{A}_{n+2}$, and let A, B be indecomposable representations of Q .*

- (a) *There exists a non-split short exact sequence with end terms A, B and indecomposable middle term E if and only if the relative position of A, B, E in the Auslander–Reiten quiver defines a rectangle with one point missing as on the left of Fig. 8.*
- (b) *There exists a non-split short exact sequence with endterms A, B and decomposable middle term E if and only if E has two indecomposable summands E_1, E_2 , and the relative position of A, B, E_1, E_2 in the Auslander–Reiten quiver defines a rectangle as on the right of Fig. 8.*

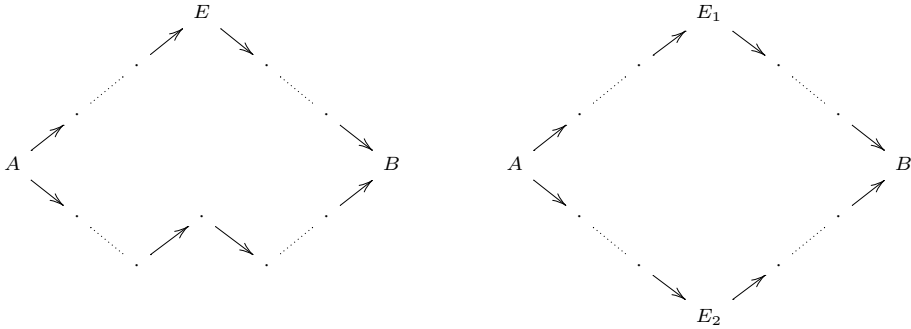


Fig. 8. Short exact sequences $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ seen in the Auslander–Reiten quiver. On the left, the middle term E is indecomposable, on the right $E = E_1 \oplus E_2$.

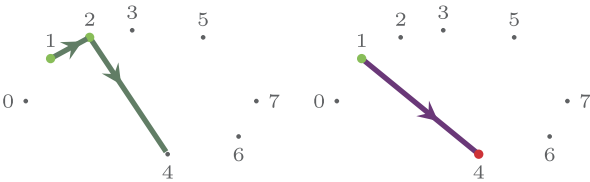


Fig. 9. Line segments $\gamma(1, 2)$ and $\gamma(2, 4)$ share an endpoint at the vertex 2, and their extension is the line segment $\gamma(1, 4)$. Applying F yields the short exact sequence $0 \rightarrow M(3, 4) \rightarrow M(2, 4) \rightarrow S(2) \rightarrow 0$.

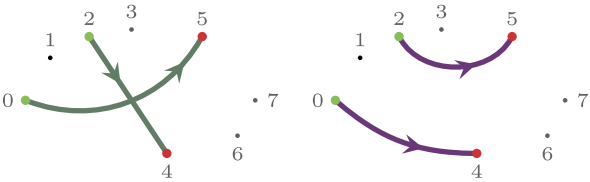


Fig. 10. Diagonals $\gamma(2, 4)$ and $\gamma(0, 5)$ cross each other, and their extension is the direct sum of $\gamma(0, 4)$ and $\gamma(2, 5)$. Applying F yields the short exact sequence $0 \rightarrow M(3, 4) \rightarrow M(1, 4) \oplus M(3, 5) \rightarrow M(1, 5) \rightarrow 0$.

Example 6.6. Consider P_2, I_4, I_6 and I_7 in Fig. 5. The rectangle defined by P_2 and I_4 lies completely within the AR-quiver and therefore there is an extension $0 \rightarrow P_2 \rightarrow M(1, 4) \oplus M(2, 5) \rightarrow I_4 \rightarrow 0$.

The rectangle defined by P_2 and I_6 is missing precisely one point, thus there is an extension $0 \rightarrow P_2 \rightarrow M(2, 7) \rightarrow I_6 \rightarrow 0$ with indecomposable middle term.

On the other hand, the rectangle defined by P_2 and I_7 is missing three points, and hence $\text{Ext}(I_7, P_2) = 0$.

Example 6.7. Figs. 9 and 10 illustrate short exact sequences in terms of line segments. It may happen that two line segments share an endpoint but there is no extension between them. See Fig. 11.

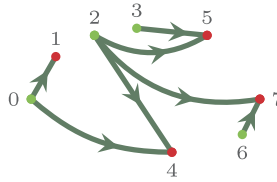


Fig. 11. $\text{Ext}^1(A, B) = 0$ for all A, B in $\{S(1), M(1, 4), M(3, 4), M(3, 7), M(3, 5), M(4, 5), S(7)\}$ which corresponds to the set of illustrated line segments.

We have the following realization of maximal almost rigid representations.

Theorem 6.8. F induces a bijection also denoted by F

$$F: \{\text{triangulations of } P(Q)\} \rightarrow \text{mar}(Q).$$

Proof. Let $M(i, j), M(i', j')$ be two non-isomorphic indecomposable representations of Q . Using Theorem 4.6 and Proposition 6.5 (b), we see that there is a short exact sequence of the form

$$0 \rightarrow M(i, j) \rightarrow E_1 \oplus E_2 \rightarrow M(i', j') \rightarrow 0$$

if and only if there exist positive integers s, t such that $E_1 = M(i, R^{-t}(j))$, $E_2 = M(R^{-s}(i), j)$ and $M(i', j') = M(R^{-s}(i), R^{-t}(j))$. In particular, $i \leq R^{-t}(j)$ and $R^{-s}(i) \leq j$ and thus the line segments $F^{-1}(M(i, j)) = \gamma(i-1, j)$ and $F^{-1}(M(i', j')) = \gamma(R^{-s}(i)-1, R^{-t}(j))$ are crossing diagonals in $P(Q)$.

This shows that the sum of two indecomposable representations is almost rigid if and only if the corresponding line segments in $P(Q)$ do not cross. Consequently, maximal almost rigid representations correspond to maximal sets of noncrossing line segments, hence triangulations. $\square$

Corollary 6.9. If Q is of type $\mathbb{A}_{n+2}$, every maximal almost rigid representation of Q has exactly $2n+3$ summands. The maximal almost rigid representations of Q are counted by the Catalan number $\binom{2n+2}{n+1} \frac{1}{n+2}$.

Proof. The first statement follows from the theorem, because each triangulation of the $(n+3)$ -gon $P(Q)$ has n diagonals and $n+3$ boundary edges. The second statement holds because the number of triangulations of $P(Q)$ is given by the Catalan number. $\square$

7. Endomorphism algebras of maximal almost rigid representations

Recall that a module T is *rigid* provided that $\text{Ext}(T, T) = 0$. We say that T is a *maximal rigid module* if $T \oplus M$ is not rigid for any nonzero module M . A maximal rigid module over the path algebra $\mathbb{k}Q$ of a quiver Q is called a *tilting module*, and the endomorphism algebras of tilting modules over path algebras are called *tilted algebras*,

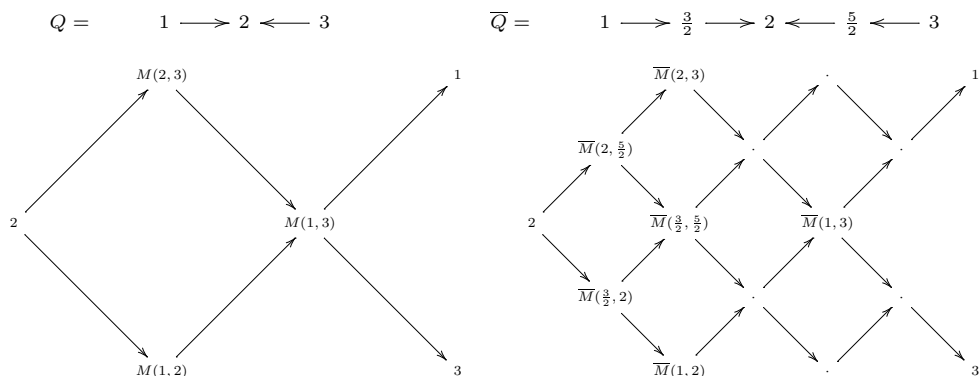


Fig. 12. A quiver Q and its Auslander–Reiten quiver on the left; the corresponding quiver $\overline{Q}$ and its Auslander–Reiten quiver on the right. The image of the functor G are the representation $\overline{M}(i, j)$ with i and j integers.

see [16]. On the other hand, the endomorphism algebras of triangulations of polygons are *cluster-tilted algebras* [13]. Both tilted algebras and cluster-tilted algebras have been studied extensively.

It is therefore natural to study the endomorphism algebras of the maximal *almost* rigid representations. The purpose of this section is to show that in type A_{n+2} these algebras are tilted algebras of type A_{2n+3} . For more details on endomorphism algebras and tilted algebras, see textbooks [5, Chapter 8] and [34, Chapter 6].

To every quiver Q of type A_{n+2} we associate a quiver $\overline{Q}$ of type A_{2n+3} by replacing each arrow $i \rightarrow (i+1)$ by a path of length two $i \rightarrow (\frac{2i+1}{2}) \rightarrow (i+1)$ and each arrow $i \leftarrow (i+1)$ by a path of length two $i \leftarrow (\frac{2i+1}{2}) \leftarrow (i+1)$, see Fig. 12 for an example. Since Q has $n+2$ vertices and $n+1$ arrows, $\overline{Q}$ has $2n+3$ vertices which are labeled by the half-integers $i \in \frac{1}{2}\mathbb{Z}$ with $1 \leq i \leq n+2$. Note that Proposition 3.1 implies that the indecomposable representations of $\overline{Q}$ are of the form $\overline{M}(i, j)$ with $i \leq j$ and $i, j \in \frac{1}{2}\mathbb{Z}$.

Define a functor $G: \text{rep } Q \rightarrow \text{rep } \overline{Q}$ as follows. On indecomposable objects, we let $G(M(i, j)) = \overline{M}(i, j)$ and we extend it additively to all objects. If $f: M(i, j) \rightarrow M(i', j')$ is a morphism between indecomposables in $\text{rep } Q$, we let $G(f) = \overline{f}: \overline{M}(i, j) \rightarrow \overline{M}(i', j')$ defined by $\overline{f}_i = f_i$ if $i \in \mathbb{Z}$, and for the new vertices we let

$$\overline{f}_{\frac{2i+1}{2}} = \begin{cases} 1 & \text{if } f_i = 1 \text{ and } f_{i+1} = 1; \\ 0 & \text{otherwise.} \end{cases}$$

For example, the irreducible morphism $2 \rightarrow M(2, 3)$ in $\text{rep } Q$ in the example of Fig. 12 is mapped under G to the composition of two irreducible morphisms $2 \rightarrow \overline{M}(2, \frac{5}{2}) \rightarrow \overline{M}(2, 3)$.

Lemma 7.1. *The functor G is full and faithful.*

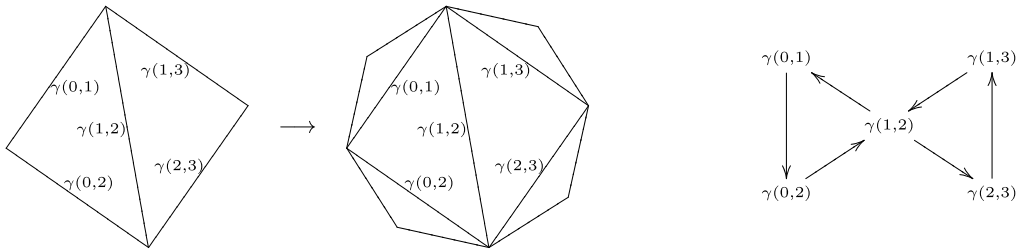


Fig. 13. Left: The embedding of a polygon $P(Q)$ with triangulation $\mathcal{T}$ into the polygon $\bar{P}$. Right: The quiver $Q(\mathcal{T})$.

Proof. Faithfulness is clear from the definition. To show that G is full, take a nonzero morphism $\bar{f} \in \text{Hom}_{\bar{Q}}(G(M(i, j)), G(M(i', j')))$, thus $\bar{f}: \bar{M}(i, j) \rightarrow \bar{M}(i', j')$. We want to show that $\bar{f} = G(f)$ for some $f \in \text{Hom}_Q(M(i, j), M(i', j'))$. Let $\bar{I} \subset \bar{Q}_0$ be the set of all vertices on which both $\bar{M}(i, j)$ and $\bar{M}(i', j')$ are nonzero. Then $\bar{f}_x = 1$, if $x \in \bar{I}$, and $\bar{f}_x = 0$, otherwise. Let $I \subset \bar{I}$ be the subset of vertices from Q and define $f: M(i, j) \rightarrow M(i', j')$ by $f_x = 1$, if $x \in I$, and $f_x = 0$, otherwise. Then f is a morphism in $\text{rep } Q$ and $G(f) = \bar{f}$. $\square$

Corollary 7.2. For all $T \in \text{rep } Q$, we have $\text{End}_{\text{rep } Q} T \cong \text{End}_{\text{rep } \bar{Q}} G(T)$.

Proof. This follows since G is full and faithful. $\square$

To state our next result, we recall that the *trivial extension* $C \ltimes E$ of an algebra C by a C -bimodule E is the algebra whose underlying vector space is the direct sum $C \oplus E$, and whose multiplication is defined as $(c, e)(c', e') = (cc', ce' + ec')$. For more details, see for example the textbook [34, Chapter 6.2]

We also need the notion of the *adjacency quiver* $Q(\mathcal{T})$ of a triangulation $\mathcal{T}$ of a polygon. The vertices of $Q(\mathcal{T})$ are given by the line segments in $\mathcal{T}$. And there is an arrow $\gamma \rightarrow \gamma'$ in $Q(\mathcal{T})$ if the line segments γ, γ' bound the same triangle in $\mathcal{T}$ such that γ' follows γ when going along the boundary of the triangle in counterclockwise direction. For an example see Fig. 13.

We are ready for the main theorem of this section.

Theorem 7.3. Let Q be a quiver of type $\mathbb{A}_{n+2}$, and let T be a maximal almost rigid representation of Q with endomorphism algebra $C = \text{End}_{\text{rep } Q} T$. Then

- (1) C is a tilted algebra of type $\bar{Q}$.
- (2) The quiver of the cluster-tilted algebra $B = C \ltimes \text{Ext}_C^2(DC, C)$ is the adjacency quiver $Q(\mathcal{T})$ of the triangulation $\mathcal{T} = F^{-1}(T)$. Moreover, every arrow of $Q(\mathcal{T})$ lies in a (unique) 3-cycle.
- (3) The quiver $Q(C)$ of C is obtained from $\mathcal{T}$ by deleting the arrows of $Q(\mathcal{T})$ of the form $\gamma(i, j) \rightarrow \gamma(j, k)$ and $\gamma(j, k) \rightarrow \gamma(i, j)$, $i < j < k$. Moreover, every arrow of $Q(C)$ lies in a (unique) relation.

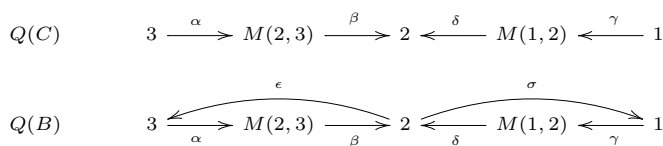
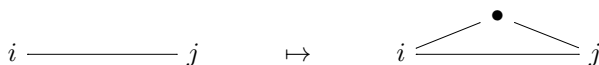


Fig. 14. The quivers of C and B in Example 7.4. To obtain $Q(C)$ from $Q(B)$ remove the arrows ϵ and σ as described in part (3) of Theorem 7.3.

Example 7.4. Let Q be the quiver of Fig. 12 and let T be the maximal almost rigid representation $T = 2 \oplus M(1, 2) \oplus M(2, 3) \oplus 1 \oplus 3$. The corresponding triangulation $\mathcal{T}$ is shown left in Fig. 13. Then $C = \text{End}_{\text{rep } Q} T$ is given by the quiver in Fig. 14(top) bound by the relations $\alpha\beta = 0$ and $\gamma\delta = 0$. Its cluster-tilted algebra $B = C \ltimes \text{Ext}_C^2(DC, C)$ is given by the quiver in Fig. 14(bottom) bound by the relations $\alpha\beta = \beta\epsilon = \epsilon\alpha = 0$ and $\gamma\delta = \delta\sigma = \sigma\gamma = 0$.

Proof of Theorem 7.3. Let $\overline{Q}$ be the quiver of type A_{2n+3} defined in the beginning of this section. The cluster category $\overline{\mathcal{C}}$ of $\overline{Q}$ can be realized in two ways. For one, $\overline{\mathcal{C}} = \mathcal{D}^b(\text{rep } \overline{Q})/\tau^{-1}[1]$ is an orbit category of the bounded derived category of representations of $\overline{Q}$, see [7]. On the other hand, $\overline{\mathcal{C}}$ is the category of diagonals in a $(2n+6)$ -gon $\overline{P}$, see [13]. Using the realization via the derived category, we get a functor $\tilde{G}: \text{rep } Q \rightarrow \overline{\mathcal{C}}$, which maps a representation to its orbit in the cluster category. This functor is not full, since, in the cluster category, there exist morphisms between the orbits of two representations that are not given by morphisms between the representations themselves.

On the other hand, using the realization via diagonals of $\overline{P}$, we see that the composition $\tilde{G} \circ F: \mathcal{C}_{P(Q)} \rightarrow \overline{\mathcal{C}}$ is a functor from the category of line segments (=diagonals and boundary edges) in the $(n+3)$ -gon $P(Q)$ to the category of diagonals (not including boundary edges) in the $(2n+6)$ -gon $\overline{P}$. We now explain this construction on the level of the polygons. We embed $P(Q)$ into $\overline{P}$ by adding one vertex for each boundary edge; thus each boundary edge of $P(Q)$ becomes an interior edge of $\overline{P}$ which bounds a triangle whose other two sides are boundary edges, as follows, see Fig. 13 for an example.



The functor $\tilde{G} \circ F$ induces a map $\mathcal{T} \rightarrow \overline{\mathcal{T}}$ from triangulations of $P(Q)$ that include all boundary edges to triangulations of $\overline{P}$ that do not include any boundary edges.

Now let T be a maximal almost rigid representation of Q . By Theorem 6.8, there exists a triangulation $\mathcal{T}$ of $P(Q)$ such that $T = F(\mathcal{T})$. Let $C = \text{End}_{\text{rep } Q} T$ be its endomorphism algebra. Denote by $\overline{\mathcal{T}}$ the triangulation (without boundary edges) of $\overline{P}$ given as the image of $\mathcal{T}$ under $\tilde{G} \circ F$. We have the following commutative diagram.

$$\begin{array}{ccccc}
 \mathcal{C}_{P(Q)} & & & & \\
 \downarrow F & \searrow \tilde{G} \circ F & & & \\
 \text{rep } Q & \xrightarrow{G} & \text{rep } \overline{Q} & \xrightarrow{\iota} & \overline{\mathcal{C}} \\
 & \searrow \tilde{G} & & &
 \end{array}$$

Thus the object $\tilde{G}(T) = \iota \circ G(T)$ in the cluster category is given by the triangulation $\overline{\mathcal{T}}$. Therefore $\tilde{G}(T)$ is a cluster-tilting object in $\overline{\mathcal{C}}$, by [13]. Let $B = \text{End}_{\overline{\mathcal{C}}} \tilde{G}(T)$ denote its cluster-tilted algebra.

Furthermore, we know that $\tilde{G}(T)$ is induced from the representation $G(T) \in \text{rep } \overline{Q}$, and therefore $G(T)$ is a tilting module over the path algebra $\mathbb{k}\overline{Q}$ of $\overline{Q}$, by [7,2]. Its endomorphism algebra $\overline{C} = \text{End}_{\text{rep } \overline{Q}} G(T)$ is a tilted algebra of type $\overline{Q}$ and its trivial extension $\overline{C} \ltimes \text{Ext}_{\overline{C}}^2(D\overline{C}, \overline{C})$ is the cluster-tilted algebra B , by [3]. Corollary 7.2 implies that $C \cong \overline{C}$, so C is tilted of type $\overline{Q}$ and $B \cong C \ltimes \text{Ext}_C^2(DC, C)$. The quiver of B is the quiver of the triangulation $\overline{\mathcal{T}}$, by [13], and thus it is also the quiver of the triangulation $\mathcal{T}$. This shows (1) and (2).

To show (3) we need to consider the morphisms in the category $\mathcal{C}_{P(Q)}$. By definition, they are given by composition of pivots of the form $\gamma(i, j) \rightarrow \gamma(i, R^{-1}(j))$ or $\gamma(i, j) \rightarrow \gamma(R^{-1}(i), j)$. Given a triangle in $\mathcal{T}$ with vertices $i < j < k$ in the labeling of $P(Q)$, and such that traveling from i to j to k to i is going counterclockwise around the triangle, there exists a morphism $\gamma(i, j) \rightarrow \gamma(i, k)$ given by a sequence of pivots that fix the endpoint i , and a morphism $\gamma(i, k) \rightarrow \gamma(j, k)$ given by a sequence of pivots that fix the endpoint k . In the cluster category $\overline{\mathcal{C}}$ there also is a nonzero morphism $\gamma(j, k) \rightarrow \gamma(j, i)$ given by a sequence of pivots fixing the endpoint j , however, this morphism is zero in the category $\mathcal{C}_{P(Q)}$ because the diagonal $\gamma(j, i)$ is not in $\mathcal{E}$, since $j > i$.

Similarly, when the triangle $\mathcal{T}$ has vertices $i < j < k$ in the labeling of $P(Q)$ such that traveling from i to j to k is going clockwise around the triangle, there is a morphism $\gamma(i, k) \rightarrow \gamma(i, j)$ given by a sequence of pivots that fix i , and there is a morphism $\gamma(k, j) \rightarrow \gamma(i, k)$ given by a sequence of pivots that fix k . $\square$

Remark 7.5. Not every tilted algebra of type $\mathbb{A}_{2n+3}$ is realizable as the endomorphism algebra of a maximal almost rigid representation of type $\mathbb{A}_{n+2}$. For example the tilted algebra given by the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \longleftarrow 4 \longleftarrow 5$ with relation $\alpha\beta = 0$ is not, because not every arrow lies in a relation.

Remark 7.6. It would be interesting to see how the maximal almost rigid representations behave for other quivers or more generally for bound quiver algebras. The first example below shows that in Dynkin type $\mathbb{D}_4$ the number of summands in a maximal almost rigid representation is not always the same.

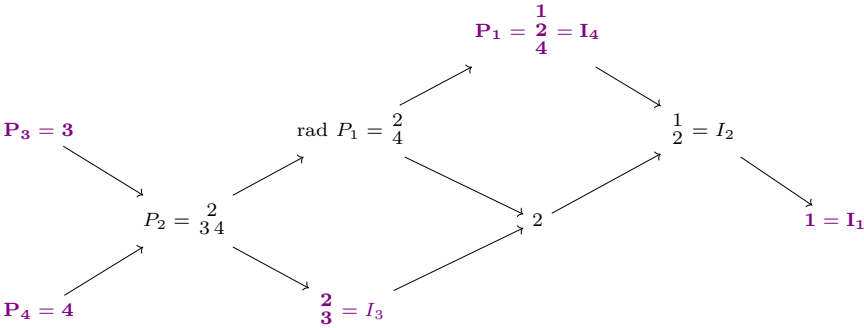
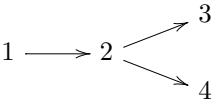


Fig. 15. Auslander–Reiten quiver for the bound quiver in Example 7.8.

Example 7.7. The $\mathbb{D}_4$ quiver Q

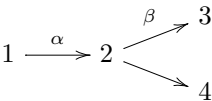


admits the following maximal almost rigid representations, one of which has 7 direct summands and the other has 9.

$$T = P(1) \oplus P(2) \oplus P(3) \oplus P(4) \oplus \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 1 \quad \text{and}$$
$$T' = P(1) \oplus P(3) \oplus P(4) \oplus \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus 2 \oplus 1$$

The endomorphism algebra of T is tilted of affine type $\widetilde{\mathbb{E}}_6 = \overline{Q}$, which is encouraging for a possible generalization of the type A results. However, the endomorphism algebra of T' does not seem to be a tilted algebra.

Example 7.8. Consider the $\mathbb{D}_4$ quiver



bound by the relation $\alpha\beta = 0$. The Auslander–Reiten quiver is illustrated in Fig. 15.

If an indecomposable representation M has at most one incoming arrow and one outgoing arrow in the Auslander–Reiten quiver, then M is a direct summand of every maximal almost rigid representation. Therefore, the five indecomposable representations written in bold in Fig. 15 are direct summands of every maximal almost rigid representation.

The maximal almost rigid representations are as follows. Each of them has 7 direct summands.

$$3 \oplus 4 \oplus \frac{2}{3} \oplus \frac{1}{2} \oplus 1 \oplus \frac{2}{4} \oplus \frac{2}{34}$$

$$3 \oplus 4 \oplus \frac{2}{3} \oplus \frac{1}{2} \oplus 1 \oplus \frac{2}{4} \oplus 2$$

$$3 \oplus 4 \oplus \frac{2}{3} \oplus \frac{1}{2} \oplus 1 \oplus 2 \oplus \frac{1}{2}$$

8. Representation theoretic version of the Cambrian lattice and the η map

In this section, we come back to our initial motivation and give a new realization of the η map in terms of maximal almost rigid representations.

8.1. Poset structure on $\text{mar}(Q)$

A *flip* is a transformation of a triangulation $\mathcal{T}$ that removes a diagonal γ and replaces it with a (unique) different diagonal γ' that, together with the remaining diagonals, forms a new triangulation $\mathcal{T}'$. Note that the two diagonals γ and γ' involved in such a flip must cross.

In [25], Reading defined the following poset structure on the set of triangulations of $P(Q)$. A triangulation $\mathcal{T}_1$ is said to be *covered by* a triangulation $\mathcal{T}_2$ if there exist two diagonals $\gamma_1 \neq \gamma_2$ such that $\mathcal{T}_1 \setminus \{\gamma_1\} = \mathcal{T}_2 \setminus \{\gamma_2\}$ and the slope of γ_1 is smaller than the slope of γ_2 . This covering relation induces a partial order on the set of triangulations, and the resulting poset is called the *Cambrian lattice of type A*. See also the survey paper [26].

Definition 8.1. We define a partial order on $\text{mar}(Q)$ as follows. For $T_1, T_2 \in \text{mar}(Q)$, we say that T_1 is *covered by* T_2 if there exist indecomposable summands M_i of T_i such that $T_1/M_1 \cong T_2/M_2$ and there is a non-split short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{f} E \oplus E' \xrightarrow{g} M_2 \longrightarrow 0$$

with E, E' indecomposable summands of T_1/M_1 .

Remark 8.2. In the short exact sequence in Definition 8.1, the morphism f is a minimal $\text{add}(T_1/M_1)$ approximation of M_1 . For a definition, see for example [7].

Remark 8.3. It follows from Theorem 6.8 that every maximal almost rigid representation contains all hooks and all cohooks of Q , since these correspond to boundary edges of the triangulation of $P(Q)$ by Corollary 3.10. Therefore, there is a unique minimal element in the poset $\text{mar}(Q)$ given by the basic representation whose direct summands are the indecomposable projective representations together with the hooks and cohooks.

The covering relation given by the short exact sequence in Definition 8.1 then gives an algorithm to construct all maximal almost rigid representations using approximations. The unique maximal element in the poset on $\text{mar}(Q)$ is given by the basic representation whose direct summands are the indecomposable injective representations together with the hooks and cohooks.

8.2. The poset on $\text{mar}(Q)$ is a Cambrian lattice

We have the following result.

Theorem 8.4. *Let Q be a Dynkin quiver of type A_{n+2} . Then the poset on $\text{mar}(Q)$ is isomorphic to a Cambrian lattice.*

Proof. By Theorem 6.8, the functor F is a bijection from triangulations of $P(Q)$ to $\text{mar}(Q)$. It only remains to prove that F preserves the covering relations. This is done in the next lemma. $\square$

Lemma 8.5. *Let Q be a quiver of type A_{n+2} . Let M_1, M_2 be two indecomposable representations of Q and γ_1, γ_2 be the line segments in $P(Q)$ such that $F(\gamma_i) = M_i$, $i = 1, 2$. Suppose there is a non-split short exact sequence $0 \rightarrow M_1 \rightarrow E \rightarrow M_2 \rightarrow 0$. Then the following conditions are equivalent.*

- (1) $\text{Hom}(M_1, M_2) \neq 0$.
- (2) E is decomposable.
- (3) γ_1 and γ_2 cross and the slope of γ_2 is larger than the slope of γ_1 .

Proof. (1) $\Leftrightarrow$ (2). Applying the functor $\text{Hom}(M_1, -)$ to the short exact sequence yields an exact sequence

$$0 \rightarrow \mathbb{k} = \text{Hom}(M_1, M_1) \rightarrow \text{Hom}(M_1, E) \rightarrow \text{Hom}(M_1, M_2) \rightarrow \text{Ext}(M_1, M_1) = 0, \quad (8.1)$$

where the first term is one-dimensional and the last term is zero, because M_1 is an indecomposable representation of a Dynkin quiver. Since M_1 and M_2 are indecomposable and the short exact sequence is non-split, there is a nonzero morphism from M_1 to every indecomposable summand of E . Thus the dimension of $\text{Hom}(M_1, E)$ is at least equal to the number of indecomposable summands of E . However, Q being of type A implies that the dimension of Hom between indecomposables is at most one, and thus the dimension of $\text{Hom}(M_1, E)$ is *exactly* equal to the number of indecomposable summands of E . Thus the exactness of (8.1) implies that E is decomposable if and only if $\text{Hom}(M_1, M_2) \neq 0$.

(1)&(2) $\Rightarrow$ (3). Suppose there is a nonzero morphism from M_1 to M_2 . By Theorem 4.6, this means that we can get from γ_1 to γ_2 by a sequence of pivots, each of which is moving one of the endpoints of a diagonal to its counterclockwise neighbor (in a way

that preserves the ordering of the endpoints). Each such pivot increases the slope of a diagonal, thus the slope of γ_2 is larger than the slope of γ_1 . Furthermore γ_1 and γ_2 must cross, because otherwise the extension E would be indecomposable.

(3) $\Rightarrow$ (1). Suppose γ_1 and γ_2 cross and that γ_2 has larger slope than γ_1 . Then it is possible to get from γ_1 to γ_2 by a sequence of counterclockwise pivots which preserve the ordering of the endpoints. By Theorem 4.6, there is a nonzero morphism from M_1 to M_2 . $\square$

8.3. The η map

Recall the map η_Q from the symmetric group S_{n+1} to the set of triangulations of $P(Q)$ from Definition 2.1. We define a new realization of this map via the composition of η_Q with the equivalence F of Theorem 4.6.

Definition 8.6. Let $\eta_Q^{\text{rep}} = F \circ \eta_Q: S_{n+1} \rightarrow \text{mar}(Q)$.

We shall give an alternate, equivalent description of η_Q^{rep} using extensions and degenerations. A representation M is an *extension* of a representation N if there exists a non-split short exact sequence $0 \rightarrow N_1 \rightarrow M \rightarrow N_2 \rightarrow 0$ such that $N = N_1 \oplus N_2$. A representation M is a *degeneration* of a representation N if M is contained in the Zariski closure of the isomorphism class of N . In Dynkin type $\mathbb{A}$, a representation M is a degeneration of a representation N if there exists a non-split short exact sequence $0 \rightarrow M_1 \rightarrow N \rightarrow M_2 \rightarrow 0$ with $M = M_1 \oplus M_2$. Note that in both cases, the dimension vectors of M and N are equal.

Recall that the map η_Q is defined by taking the union of a list of paths. This list is created by a recursive process of removing and adding a vertex of $P(Q)$ to a path. Instead of a list of paths, we will now create a list of representations of Q each of which has dimension vector $(1, 1, \dots, 1)$. Instead of removing (respectively adding) a vertex, we will now apply an extension (respectively a degeneration) to a representation. Because every representation on this list has dimension vector $(1, 1, \dots, 1)$, an extension is the same as changing the assignment of the linear map on an arrow in Q from the zero map to the identity map. Conversely, a degeneration is simply changing the assignment of the linear map on an arrow in Q from the identity map to the zero map. Finally, instead of taking the union of the list of paths, we take the direct sum of the union of all indecomposable summands of our list of representations.

Let α_ℓ denote the arrow of Q between vertices ℓ and $(\ell + 1)$ for all ℓ . Given a permutation $\pi = \pi_1 \pi_2 \pi_3 \dots \pi_{n+1}$ written in one-line notation, we define a representation $\lambda_i^{\text{rep}}(\pi)$ of Q for each $i \in \{0, \dots, n+1\}$. Let $\lambda_0^{\text{rep}}(\pi)$ be the representation with dimension $(1, 1, \dots, 1)$ with the 0 map on each arrow α_i for all $i \in [n+1]$, and the identity map everywhere else. That is, it is the direct sum of the indecomposable representations which correspond to the lower boundary edges of $P(Q)$ (or maximal increasing paths in Q , due to Lemma 3.6). These indecomposable representations are the ones in the τ -orbit of the

projective at 1. In the Auslander–Reiten quiver example of Fig. 5, they are drawn at the bottom row.

Define $\lambda_1^{\text{rep}}(\pi)$ to be

- (a) the degeneration of $\lambda_0^{\text{rep}}(\pi)$ obtained by replacing the identity map on α_{π_1} with the zero map, if $\pi_1 \in \overline{[n+1]}$, or
- (b) the extension of $\lambda_0^{\text{rep}}(\pi)$ obtained by replacing the zero map on α_{π_1} with the identity map, if $\pi_1 \in \underline{[n+1]}$.

Repeat this process recursively, until we get to $\lambda_{n+1}^{\text{rep}}(\pi)$ which is the representation with dimension vector $(1, 1, \dots, 1)$ with the 0 map on all arrows α_i with $i \in \overline{[n+1]}$, and the identity map everywhere else. That is, $\lambda_{n+1}^{\text{rep}}(\pi)$ is the direct sum of the indecomposable representations which correspond to the upper boundary edges of $P(Q)$ (or maximal decreasing paths in Q , due to Lemma 3.6). These indecomposable representations are the ones in the τ -orbit of the projective at $n+2$. In the Auslander–Reiten quiver example of Fig. 5, they are drawn at the top row.

Proposition 8.7. $\eta_Q^{\text{rep}}(\pi)$ is the maximal almost rigid representation of Q whose indecomposable direct summands are exactly those appearing in the list $\lambda_0^{\text{rep}}(\pi), \dots, \lambda_{n+1}^{\text{rep}}(\pi)$.

Proof. We need to check that the recursive definition of $\lambda_\ell^{\text{rep}}$ corresponds to the recursive definition of λ_ℓ under the functor F . For $\ell = 0$, this is clear. Suppose $\ell > 0$ and that $F(\lambda_\ell) = \lambda_\ell^{\text{rep}}$. We want to show $F(\lambda_{\ell+1}) = \lambda_{\ell+1}^{\text{rep}}$. Suppose first that $\pi_{\ell+1} \in \overline{[n+1]}$. Then the difference between the paths λ_ℓ and $\lambda_{\ell+1}$ is that $\lambda_{\ell+1}$ passes through the vertex $\pi_{\ell+1}$ of $P(Q)$. This means that λ_ℓ contains a line segment $\gamma(i, j)$ which is replaced in $\lambda_{\ell+1}$ by two line segments $\gamma(i, \pi_{\ell+1}), \gamma(\pi_{\ell+1}, j)$, where $i < \pi_{\ell+1} < j$. Applying the functor F , the indecomposable direct summand $M(i+1, j)$ is replaced by two indecomposable direct summands $M(i+1, \pi_{\ell+1}), M(\pi_{\ell+1}+1, j)$. This change coincides with the difference between $\lambda_\ell^{\text{rep}}$ and $\lambda_{\ell+1}^{\text{rep}}$ which is given by a degeneration that replaces the identity map on the arrow $\alpha_{\pi_{\ell+1}}$ by the zero map.

Now suppose that $\pi_{\ell+1} \in \underline{[n+1]}$. Then the difference between the paths λ_ℓ and $\lambda_{\ell+1}$ is that λ_ℓ passes through the vertex $\pi_{\ell+1}$ of $P(Q)$. This means that λ_ℓ contains two line segments $\gamma(i, \pi_{\ell+1}), \gamma(\pi_{\ell+1}, j)$, with $i < \pi_{\ell+1} < j$, which are replaced in $\lambda_{\ell+1}$ by a single line segment $\gamma(i, j)$. Applying the functor F , two indecomposable direct summands $M(i+1, \pi_{\ell+1}), M(\pi_{\ell+1}+1, j)$ are replaced by a single indecomposable direct summand $M(i+1, j)$. This change coincides with the difference between $\lambda_\ell^{\text{rep}}$ and $\lambda_{\ell+1}^{\text{rep}}$ which is given by an extension that replaces the zero map on the arrow $\alpha_{\pi_{\ell+1}}$ by the identity map. $\square$

Example 8.8. Let Q be the quiver in Fig. 1. Then $\overline{[6]} = \{1, 2, 3, 5\}$ and $\underline{[6]} = \{4, 6\}$. Let $\pi = 453126 \in S_6$. The paths $\lambda_i(\pi)$ are listed in Example 2.3. The list of the representations $\lambda_i^{\text{rep}}(\pi)$ is as follows; see also Fig. 16.

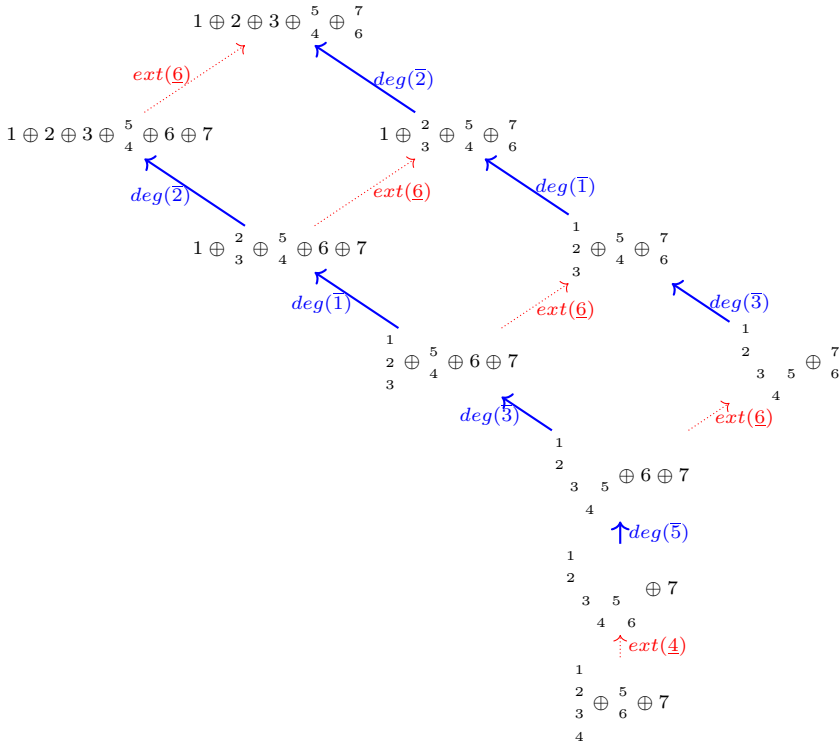


Fig. 16. The four maximal chains correspond to the four permutations in the fiber of $\eta_Q^{\text{rep}}(\pi)$ given in Example 8.8. The left-most maximal chain corresponds to the permutation 453126.

$$\begin{aligned}\lambda_0^{\text{rep}} &= M(1, 4) \oplus M(5, 6) \oplus S(7) \\ \lambda_1^{\text{rep}} &= M(1, 6) \oplus S(7) \\ \lambda_2^{\text{rep}} &= M(1, 5) \oplus S(6) \oplus S(7) \\ \lambda_3^{\text{rep}} &= M(1, 3) \oplus M(4, 5) \oplus S(6) \oplus S(7) \\ \lambda_4^{\text{rep}} &= S(1) \oplus M(2, 3) \oplus M(4, 5) \oplus S(6) \oplus S(7) \\ \lambda_5^{\text{rep}} &= S(1) \oplus S(2) \oplus S(3) \oplus M(4, 5) \oplus S(6) \oplus S(7) \\ \lambda_6^{\text{rep}} &= S(1) \oplus S(2) \oplus S(3) \oplus M(4, 5) \oplus M(6, 7)\end{aligned}$$

As we have seen earlier the fiber of $\eta_Q(453126)$ is the set $\{453126, 453162, 453612, 456312\}$. These four permutations correspond to the four maximal chains in Fig. 16.

Acknowledgments

We thank Bernhard Keller for suggesting an outline for proving Theorem A. We also thank Nathan Reading and Hugh Thomas for answering our questions about Cambrian lattices, and Ana Garcia Elsener, Kaveh Mousavand, Pierre-Guy Plamondon, and Gorana Todorov for helpful suggestions. Special thanks go to Alastair King for stimulating

discussions on stability. Finally, we thank the anonymous reviewer whose suggestions helped improve and clarify this paper.

References

- [1] I. Assem, T. Brüstle, G. Charbonneau Jodoin, P.-G. Plamondon, Gentle algebras arising from surface triangulations, *Algebra Number Theory* 4 (2) (2010) 201–229.
- [2] I. Assem, T. Brüstle, R. Schiffler, Cluster-tilted algebras and slices, *J. Algebra* 319 (8) (2008) 3464–3479.
- [3] I. Assem, T. Brüstle, R. Schiffler, Cluster-tilted algebras as trivial extensions, *Bull. Lond. Math. Soc.* 40 (1) (2008) 151–162.
- [4] P. Apruzzese, K. Igusa, Stability conditions for affine type A, *Algebr. Represent. Theory* (2019) 1–33.
- [5] I. Assem, D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras. Vol. 1. Techniques of Representation Theory*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006.
- [6] K. Baur, R. Coelho Simões, A geometric model for the module category of a gentle algebra, *Int. Math. Res. Not.* 2021 (15) (2021) 11357–11392.
- [7] A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, *Adv. Math.* 204 (2) (2006) 572–618.
- [8] A. Buan, R. Marsh, I. Reiten, Cluster-tilted algebras, *Trans. Am. Math. Soc.* 359 (1) (2007) 323–332.
- [9] M.C.R. Butler, C.M. Ringel, Auslander-Reiten sequences with few middle terms and applications to string algebras, *Commun. Algebra* 15 (1–2) (1987) 145–179.
- [10] T. Bridgeland, Stability conditions on triangulated categories, *Ann. Math.* (2) 166 (2) (2007) 317–345.
- [11] L. Billera, B. Sturmfels, Iterated fiber polytopes, *Mathematika* 41 (2) (1994) 348–363.
- [12] A. Björner, M. Wachs, Shellable nonpure complexes and posets. II, *Trans. Am. Math. Soc.* 349 (10) (1997) 3945–3975.
- [13] P. Caldero, F. Chapoton, R. Schiffler, Quivers with relations arising from clusters (A_n case), *Trans. Am. Math. Soc.* 358 (3) (2006) 1347–1364.
- [14] P. Gabriel, Unzerlegbare Darstellungen. I, *Manuscr. Math.* 6 (1972) 71–103, correction, *Manuscr. Math.* 6 (1972) 309.
- [15] D. Happel, *Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.
- [16] D. Happel, C.M. Ringel, Tilted algebras, *Trans. Am. Math. Soc.* 274 (2) (1982) 399–443.
- [17] C. Ingalls, H. Thomas, Noncrossing partitions and representations of quivers, *Compos. Math.* 145 (6) (2009) 1533–1562.
- [18] A.D. King, Moduli of representations of finite-dimensional algebras, *Q. J. Math. Oxf. Ser.* (2) 45 (180) (1994) 515–530.
- [19] R. Kinser, Total stability functions for type A quivers, *Algebr. Represent. Theory* 25 (4) (2022) 835–845.
- [20] D. Labardini-Fragoso, Quivers with potentials associated to triangulated surfaces, *Proc. Lond. Math. Soc.* (3) 98 (3) (2009) 797–839.
- [21] S. Oppen, P.-G. Plamondon, S. Schroll, A geometric model for the derived category of gentle algebras, Preprint, arXiv:1801.09659, 2018.
- [22] Y. Qiu, Stability conditions and quantum dilogarithm identities for Dynkin quivers, *Adv. Math.* 269 (2015) 220–264.
- [23] Y. Qiu, Global dimension function on stability conditions and Gepner equations, arXiv:1807.00010, 2018.
- [24] Y. Qiu, X. Zhang, Geometric classification of total stability spaces, Preprint, arXiv:2202.00092, 2022.
- [25] N. Reading, Cambrian lattices, *Adv. Math.* 205 (2) (2006) 313–353.
- [26] N. Reading, From the Tamari lattice to Cambrian lattices and beyond, in: *Associahedra, Tamari Lattices and Related Structures*, in: *Prog. Math. Phys.*, vol. 299, Birkhäuser/Springer, Basel, 2012, pp. 293–322.
- [27] V. Reiner, Equivariant fiber polytopes, *Doc. Math.* 7 (2002) 113–132.

- [28] M. Reineke, The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli, *Invent. Math.* 152 (2) (2003) 349–368.
- [29] C. Riedtmann, Algebren, Darstellungsköcher, Überlagerungen und zurück, *Comment. Math. Helv.* 55 (2) (1980) 199–224.
- [30] N. Reading, D. Speyer, Cambrian fans, *J. Eur. Math. Soc.* 11 (2) (2009) 407–447.
- [31] A. Rudakov, Stability for an abelian category, *J. Algebra* 197 (1) (1997) 231–245.
- [32] A. Schofield, Semi-invariants of quivers, *J. Lond. Math. Soc.* (2) 43 (3) (1991) 385–395.
- [33] R. Schiffler, A geometric model for cluster categories of type D_n , *J. Algebraic Comb.* 27 (1) (2008) 1–21.
- [34] R. Schiffler, *Quiver Representations*, CMS Books in Mathematics (Ouvrages de Mathématiques de la SMC.), Springer, Cham, 2014.
- [35] R. Stanley, *Enumerative Combinatorics. Volume 1*, second edition, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.
- [36] A. Tonks, Relating the associahedron and the permutohedron, in: *Operads: Proceedings of Renaissance Conferences*, Hartford, CT/Luminy, 1995, in: *Contemp. Math.*, vol. 202, Amer. Math. Soc., Providence, RI, 1997, pp. 33–36.