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Rational Exponents Near Two

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Abstract: A longstanding conjecture of Erdős and Simonovits states that for every rational r between 1 and 2 there is a graph H such that the largest number of edges in an H-free graph on n vertices is $\Theta(n^r)$. Answering a question raised by Jiang, Jiang and Ma, we show that the conjecture holds for all rationals of the form $2-a/b$ with b sufficiently large in terms of a.

Key words and phrases: extremal numbers, rational exponents

1 Introduction

Given a positive integer n and a graph H, the *extremal number* $\mathrm{ex}(n,H)$ is the largest number of edges in an H-free graph on n vertices. In this short paper, we will be concerned with one of the standard conjectures about extremal numbers, the rational exponents conjecture of Erdős and Simonovits (see, for example, [4]), which states that every rational number r between 1 and 2 is *realisable* in the sense that there exists a graph H such that $\mathrm{ex}(n,H)=\Theta(n^r)$.

Conjecture 1.1 (Rational exponents conjecture)**.** *For every rational number $r\in[1,2]$, there exists a graph H with $\mathrm{ex}(n,H)=\Theta(n^r)$.*

The main result towards this conjecture is arguably the result of Bukh and Conlon [1] saying that for any rational number $r\in[1,2]$ there exists a finite family $\mathcal{H}$ of graphs such that $\mathrm{ex}(n,\mathcal{H})=\Theta(n^r)$, where $\mathrm{ex}(n,\mathcal{H})$ denotes the largest number of edges in an n-vertex graph which does not contain any $H\in\mathcal{H}$ as a subgraph. However, the conjecture remains open in its original form, which asks for a single graph rather than a family.

Nevertheless, following the breakthrough in [1], progress on the single graph case has been swift, with substantial contributions, each extending the range of exponents for which the conjecture is known,

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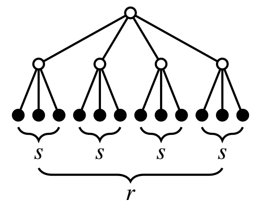

Figure 1: The rooted graph $F_{r,s}$, with black vertices representing roots.

made by Jiang, Ma and Yepremyan [9], Kang, Kim and Liu [12], Conlon, Janzer and Lee [2], Janzer [7], Jiang and Qiu [10, 11] and, most recently, Jiang, Jiang and Ma [8]. For now, we highlight only one of these results, due to Jiang and Qiu [11] saying that any rational of the form $1 + p/q$ with $q > p^2$ is realisable. Proving a conjecture of Jiang, Jiang and Ma [8, Conjecture 11] in a strong form, we show that a similar phenomenon holds near two.

Theorem 1.2. *All rationals of the form $r = 2 - a/b$ with $b \geq \max(a, (a-1)^2)$ are realisable.*

To say more, we must first explain the context in which the recent progress has been made. We will be interested in *rooted graphs* (F,R) consisting of a graph F together with a proper subset R of the vertex set $V(F)$ that we refer to as the *roots*. We will usually just write F if the roots are clear from context. For each $S \subseteq V(F) \setminus R$, let $\rho_F(S) := \frac{e_S}{|S|}$, where e_S is the number of edges in F incident with a vertex of S. The *density* of F is then $\rho(F) := \rho_F(V(F) \setminus R)$ and we say that (F,R) is *balanced* if $\rho_F(S) \geq \rho(F)$ for all $S \subseteq V(F) \setminus R$. Finally, given a rooted graph (F,R) and a positive integer t, the *t-blowup* F^t is the graph obtained by taking t vertex-disjoint copies of F and identifying the different copies of v for each $v \in R$. The following result of Bukh and Conlon [1] now yields a lower bound for the extremal number of F^t provided F is balanced and t is sufficiently large in terms of F.

Lemma 1.3 (Bukh–Conlon)**.** *For every balanced rooted graph F with density ρ, there exists a positive integer t_0 such that $\mathrm{ex}(n, F^t) = \Omega(n^{2-\frac{1}{\rho}})$ for all integers $t \geq t_0$.*

Paired to this result is the following conjecture, saying that Lemma 1.3 is tight up to the constant for balanced rooted *trees*. If true, this conjecture would easily imply Conjecture 1.1.

Conjecture 1.4 (Bukh–Conlon)**.** *For every balanced rooted tree F with density ρ and all positive integers t, $\mathrm{ex}(n, F^t) = O(n^{2-\frac{1}{\rho}})$.*

The recent progress then has centred on proving Conjecture 1.4 for particular choices of the rooted tree F, with many novel and interesting ideas going into each new case. Here, we consider a family of rooted trees first studied in this setting by Jiang, Jiang and Ma [8]. More precisely, for every pair of integers (r,s) with $r,s \geq 1$, we write $F_{r,s}$ for the rooted graph with vertices y, z_i for $1 \leq i \leq r$ and $w_{i,j}$ for $1 \leq i \leq r$, $1 \leq j \leq s$, with the $w_{i,j}$ roots, and edges yz_i for all $1 \leq i \leq r$ and $z_i w_{i,j}$ for all $1 \leq i \leq r, 1 \leq j \leq s$. For a picture with $r = 4$ and $s = 3$, we refer the reader to Figure 1, where the roots are drawn in black. It is easy to verify that $F_{r,s}$ is balanced provided $s \leq r$. Therefore, since $\rho(F_{r,s}) = (rs+r)/(r+1)$, Lemma 1.3 implies that

$$\mathrm{ex}(n, F_{r,s}^t) = \Omega(n^{2-\frac{r+1}{rs+r}})$$

for $s \le r$ and t sufficiently large. Our main technical result is the corresponding upper bound for a certain range of parameters.

Theorem 1.5. *For any integers $r \ge s+2 \ge 3$ and $t \ge 1$, $\mathrm{ex}(n, F^t_{r,s}) = O(n^{2-\frac{r+1}{rs+r}})$.*

This improves on a result of Jiang, Jiang and Ma [8], who proved a result similar to Theorem 1.5, but under the more restrictive assumption that $r \ge s^3 - 1$. While our argument, which we outline in the next subsection, shares some ideas with theirs, it is considerably simpler.

To see that Theorem 1.5 implies Theorem 1.2, we require one more ingredient, a key observation of Kang, Kim and Liu [12] saying that if the exponent $2 - \frac{a}{ap_0+q}$ is realisable by a power of a balanced rooted graph, then so is $2 - \frac{a}{ap+q}$ for all $p \ge p_0$. But

$$2 - \frac{r+1}{rs+r} = 2 - \frac{r+1}{(r+1)s + (r-s)},$$

so the observation of Kang, Kim and Liu implies that the exponent $2 - \frac{r+1}{(r+1)p+(r-s)}$ is realisable for all $p \ge s$. Since $r-s$ ranges from 2 to $r-1$, this means that we get all exponents of the form $2 - \frac{r+1}{d}$ with $r \ge 3$, $d \ge r^2$ and $d \not\equiv -1, 0, 1 \pmod{r+1}$. Therefore, setting $a = r+1$, we see that $2 - \frac{a}{b}$ is realisable provided $a \ge 4$, $b \ge (a-1)^2$ and $b \not\equiv -1, 0, 1 \pmod{a}$. The remaining cases, where $a \in \{1,2,3\}$ or $b \equiv -1, 0, 1 \pmod{a}$, have all previously appeared in the literature (see, for instance, [12]). It therefore remains to prove Theorem 1.5.

1.1 An outline of the proof

Let G be an n-vertex graph with $Cn^{2-\frac{r+1}{rs+r}}$ edges, where C is taken sufficiently large in terms of r, s and t. We want to show that G contains $F^t_{r,s}$ as a subgraph. As is usual when estimating extremal numbers, we may assume that G is K-almost-regular for some constant K depending only on r and s, by which we mean that every vertex in G has degree at most K times the minimum degree $\delta(G)$.

Suppose that G does not contain $F^t_{r,s}$ as a subgraph. First, we show that, among all stars in G with $s+1$ leaves, the proportion of those in which the leaves have codegree at least $|V(F^t_{r,s})|$ is only $o(1)$. Indeed, otherwise we could find a vertex $u \in V(G)$ such that a positive proportion of the $(s+1)$-sets in $N(u)$ have codegree at least $|V(F^t_{r,s})|$. However, since $F^t_{r,s}$ is the subdivision of an $(s+1)$-partite $(s+1)$-uniform hypergraph, this would imply that $F^t_{r,s}$ can be embedded into G with one part of the bipartition mapped to a subset of $N(u)$.

We call a copy of $F_{r,s}$ in G *nice* if, for each $1 \le i \le r$, the codegree of the images of $y, w_{i,1}, \dots, w_{i,s}$ is at most $|V(F^t_{r,s})|$. By the previous paragraph and since G is almost regular, almost all copies of $F_{r,s}$ in G are nice.

Suppose now that we have a large collection of nice copies of $F_{r,s}$ in G all of which have the same leaf set, i.e., they all map the $w_{i,j}$ to the same vertices $x_{i,j}$. Since G is $F^t_{r,s}$-free, there cannot be t of these copies of $F_{r,s}$ which are pairwise vertex-disjoint apart from the $x_{i,j}$. Hence, a positive proportion of them must map one of $y, z_1, \dots, z_r$ to the same vertex in G. However, there cannot exist many nice copies of $F_{r,s}$ which map y and all the $w_{i,j}$ to the same set of vertices. Hence, we find that a positive proportion of the nice $F_{r,s}$ rooted at the $x_{i,j}$ must map some z_k to the same vertex $v \in V(G)$. For the sake of notational simplicity, we will assume that a positive proportion of the copies rooted at the $x_{i,j}$ map z_r to v. The crucial observation is that this means that v sends many edges to a relatively small set that depends only on the vertices $x_{i,j}$ for $1 \le i \le r-1, 1 \le j \le s$. More precisely, v is clearly a neighbour of the image of y in every copy of $F_{r,s}$ that maps z_r to v. However, the locus of the possible images of y is rather restricted: if u is such an image, then, for each $1 \le i \le r-1$, the vertices $u, x_{i,1}, \dots, x_{i,s}$ have a common neighbour.

Fix a "typical" collection of vertices $x_{i,j}$, $1 \le i \le r-1, 1 \le j \le s$, and let X be the locus of the possible images of y in embeddings of $F_{r,s}$ that map $w_{i,j}$ to $x_{i,j}$ for all $1 \le i \le r-1, 1 \le j \le s$. For any $u \in X$, there are around $\delta(G)^{s+1}$ embeddings of $F_{r,s}$ that map y to u and $w_{i,j}$ to $x_{i,j}$ for each $1 \le i \le r-1, 1 \le j \le s$, since we can "freely" choose how $z_r, w_{r,1}, \dots, w_{r,s}$ are embedded. If we assume that $|X|$ is about as large as it would be in a random graph with the same edge density, then, on average, for each embedding of $F_{r,s}$ which maps $w_{i,j}$ to $x_{i,j}$ for each $1 \le i \le r-1, 1 \le j \le s$, there are a large constant number of copies of $F_{r,s}$ with the same leaves. Assuming that these copies are nice, the previous paragraph shows that there are many embeddings of $F_{r,s}$ which map $w_{i,j}$ to $x_{i,j}$ for all $1 \le i \le r-1, 1 \le j \le s$ with the property that the image of z_r has a large constant number of neighbours in X. This then allows us to conclude that there are many edges $uv \in E(G)$ with $u \in X$ such that v has a large constant number of neighbours in X, which in turn yields a very unbalanced bipartite subgraph of G with parts X and Y where every $v \in Y$ has many neighbours in X. This subgraph contains many stars with $s+1$ leaves centred in Y and, for most of them, the leaves have large codegree, contradicting the observation made in the second paragraph.

2 Proof of Theorem 1.5

Fix $r \ge s+2 \ge 3$, $t \ge 1$ and let $H = F^t_{r,s}$. We begin our proof by defining what it means for a star with $s+1$ leaves to be heavy and then showing that there cannot be too many such stars. Originating in work of Conlon and Lee [3] and Janzer [6] on extremal numbers of subdivisions, similar definitions and results appear often in the recent literature on the rational exponents conjecture.

Definition 2.1. We call a star with $s+1$ leaves *heavy* if the leaves have codegree at least $|V(H)|$ and *light* otherwise.

Lemma 2.2. *For any $\varepsilon > 0$, there is a constant $C = C(\varepsilon, H)$ such that the following holds. Let G be an H-free bipartite graph with parts X and Y and minimum degree at least C on side Y. Then the proportion of heavy $(s+1)$-stars among all $(s+1)$-stars centred in Y is at most ε.*

Proof. It suffices to prove that for each $u \in Y$, the proportion of heavy stars among all stars centred at u is at most ε. Define an $(s+1)$-uniform hypergraph $\mathcal{G}$ on vertex set $N(u)$ by setting $S \subset N(u)$ with $|S| = s+1$ to be an edge of $\mathcal{G}$ if and only if the common neighbourhood (in G) of the vertices in S has order at least $|V(H)|$. We also define an $(s+1)$-uniform hypergraph $\mathcal{H}$ with vertices y_k for $1 \le k \le t$ and $w_{i,j}$ for $1 \le i \le r, 1 \le j \le s$ whose edges are $\{y_k w_{i,j} : 1 \le j \le s\}$ for every $1 \le k \le t, 1 \le i \le r$. It is easy to see that if $\mathcal{G}$ contains a copy of $\mathcal{H}$, then there exists a copy of H in G. Moreover, $\mathcal{H}$ is $(s+1)$-partite (the parts being $\{y_1, \dots, y_t\}$ and $\{w_{i,j} : 1 \le i \le r\}$ for each $1 \le j \le s$), so $\mathrm{ex}(n, \mathcal{H}) = o(n^{s+1})$. It follows that if $|N(u)|$ is large enough in terms of ε and $\mathcal{H}$, then there are at most $\varepsilon\binom{|N(u)|}{s+1}$ heavy $(s+1)$-stars in G with centre u. Since $\mathcal{H}$ depends only on H, the proof is complete. □

We now make a few definitions which capture some of the main ideas in our proof.

Definition 2.3. Let F be a labelled copy of $F_{r,s}$ with vertices $y, z_i, w_{i,j}$ as before. We call F *nice* if, for each $1 \le i \le r$, the $(s+1)$-star with centre z_i and leaves $y, w_{i,1}, \dots, w_{i,s}$ is light.

Definition 2.4. For distinct vertices $x_{i,j}$ with $1 \le i \le r-1$, $1 \le j \le s$ in a graph G, let $S(x_{1,1}, \dots, x_{1,s}, x_{2,1}, \dots, x_{2,s}, \dots, x_{r-1,1}, \dots, x_{r-1,s})$ be the set of vertices $u \in V(G)$ for which there are vertices $v_1, \dots, v_{r-1}$ such that u, the v_i and the $x_{i,j}$ are all distinct, $uv_i \in E(G)$ for all i and $v_i x_{i,j} \in E(G)$ for all i, j.

Definition 2.5. Let F be a nice labelled copy of $F_{r,s}$ with vertices $y, z_i, w_{i,j}$ and let q be the number of nice labelled copies of $F_{r,s}$ with the same labelled leaf set as F. For $c > 0$ and $1 \le k \le r$, we call F *(c,k)-rich* if z_k has at least cq neighbours in $S(w_{1,1}, \dots, w_{1,s}, \dots, w_{k-1,1}, \dots, w_{k-1,s}, w_{k+1,1}, \dots, w_{k+1,s}, \dots, w_{r,1}, \dots, w_{r,s})$.

The next lemma shows that if an H-free graph G has many nice copies of $F_{r,s}$ sharing the same leaves, then many of those copies of $F_{r,s}$ are rich.

Lemma 2.6. *There exist positive constants $c = c(H)$ and $C = C(H)$ such that the following holds. Let G be an H-free graph and let $x_{i,j}$, for $1 \le i \le r$, $1 \le j \le s$, be distinct vertices in G. Assume that there are $q \ge C$ nice labelled copies of $F_{r,s}$ in G with $w_{i,j}$ mapped to $x_{i,j}$ for all i, j. Then there is some $1 \le k \le r$ such that the number of (c,k)-rich labelled copies of $F_{r,s}$ with $w_{i,j}$ mapped to $x_{i,j}$ for all i, j is at least cq.*

Proof. Let $C = (t-1)(r+1)^2|V(H)|^r + 1$ and $c = 1/((t-1)(r+1)^2|V(H)|^r)$. Since G is H-free, there cannot be more than $t-1$ copies of $F_{r,s}$ which all have the same leaves $x_{i,j}$ but are otherwise pairwise vertex-disjoint. This means that any maximal collection of copies of $F_{r,s}$ with leaves $x_{i,j}$ which are otherwise pairwise disjoint cover a set R of at most $(t-1)(r+1)$ vertices in addition to $\{x_{i,j} : 1 \le i \le r, 1 \le j \le s\}$. Because of the maximality, any labelled copy of $F_{r,s}$ with leaves $x_{i,j}$ must map one of $y, z_1, \dots, z_r$ to an element of R. By the pigeonhole principle, there are therefore at least $q/(|R|(r+1)) \ge q/((t-1)(r+1)^2)$ nice copies of $F_{r,s}$ with leaves $x_{i,j}$ in which one of the vertices $y, z_1, \dots, z_r$ is mapped to the same vertex v in G. By the condition that these copies are nice, y cannot be mapped to the same vertex in more than $|V(H)|^r$ copies. Hence, since $q \ge C > (t-1)(r+1)^2|V(H)|^r$, there is some $1 \le k \le r$ such that z_k is mapped to the same vertex v in at least $q/((t-1)(r+1)^2)$ copies. Again using the fact that y is mapped to the same vertex at most $|V(H)|^r$ many times, it follows that there are at least $q/((t-1)(r+1)^2|V(H)|^r) = cq$ different images of y in these copies. All of these vertices are in $S(x_{1,1}, \dots, x_{1,s}, \dots, x_{k-1,1}, \dots, x_{k-1,s}, x_{k+1,1}, \dots, x_{k+1,s}, \dots, x_{r,1}, \dots, x_{r,s})$ and all of them are neighbours of v. Thus, all nice copies of $F_{r,s}$ mapping $w_{i,j}$ to $x_{i,j}$ for every i, j and z_k to v are (c,k)-rich. □

The upshot of what we have done so far is the following lemma, which says that, under a mild technical condition on the degrees (that we will in any case be able to assume), any H-free graph must have many rich copies of $F_{r,s}$.

Lemma 2.7. *For any positive real number K, there are positive constants $c = c(H)$ and $C = C(K,H)$ such that the following holds. Let G be an H-free n-vertex bipartite graph with minimum degree $\delta \ge Cn^{1-\frac{r+1}{rs+r}}$ and maximum degree at most $K\delta$. Then G has at least $cn\delta^{rs+r}$ (c,r)-rich labelled copies of $F_{r,s}$.*

Proof. The number of labelled copies of $F_{r,s}$ in G is at least $\frac{1}{2}n\delta^{rs+r}$. Let $\varepsilon = \frac{1}{4rK^{rs+r}}$. By Lemma 2.2, if C is sufficiently large compared to K and H, then the proportion of heavy $(s+1)$-stars in G is at most ε. Then, by the maximum degree condition, there are at most $\varepsilon n(K\delta)^{s+1}$ labelled heavy $(s+1)$-stars. Thus, again using the maximum degree assumption, there are at most $r \cdot \varepsilon n(K\delta)^{s+1} \cdot (K\delta)^{rs+r-(s+1)} = \frac{1}{4}n\delta^{rs+r}$ labelled copies of $F_{r,s}$ in G which contain a heavy $(s+1)$-star. It follows that there are at least $\frac{1}{4}n\delta^{rs+r} \geq \frac{C^{rs+r}}{4}n^{rs}$ nice labelled copies of $F_{r,s}$ in G. Let C' be the constant $C(H)$ from Lemma 2.6. Clearly, there are at most $C'n^{rs}$ nice labelled copies of $F_{r,s}$ whose leaves $w_{i,j}$ are mapped to some $x_{i,j}$ for all $1 \leq i \leq r$, $1 \leq j \leq s$ with the property that there are fewer than C' nice labelled copies of $F_{r,s}$ with $w_{i,j}$ mapped to $x_{i,j}$. Hence, if C is sufficiently large, then these nice labelled copies of $F_{r,s}$ amount to at most half of all nice labelled copies of $F_{r,s}$. The statement then follows from Lemma 2.6 by noting that the number of (c,k)-rich labelled copies of $F_{r,s}$ in G is the same for every k. □

The following lemma is the last ingredient needed for the proof of Theorem 1.5.

Lemma 2.8. *There is a constant $C_0 = C_0(H)$ such that the following holds. Let G be a bipartite graph with parts X and Y such that there are at least $|X|p$ edges xy for which $x \in X$, $y \in Y$ and y has degree at least q in G. If $q \geq C_0$ and $pq^s \geq C_0|X|^s$, then G contains H as a subgraph.*

We will prove Lemma 2.8 using Lemma 2.2, but we remark that it can also be proved directly using dependent random choice.

Proof. We may assume, by shrinking Y if necessary, that each $y \in Y$ has degree at least q. Then any edge in G can be extended in at least $\binom{q-1}{s}$ ways to an $(s+1)$-star centred in Y. Hence, the conditions of the lemma guarantee that G has at least $|X|p\binom{q-1}{s}/(s+1)$ stars with $s+1$ leaves centred in Y. Suppose that G is H-free. If C_0 is sufficiently large, then Lemma 2.2 implies that at least half of the $(s+1)$-stars centred in Y are light. If again C_0 is sufficiently large, then, since $pq^s \geq C_0|X|^s$, there are more than $|V(H)||X|^{s+1}$ light $(s+1)$-stars centred in Y. However, since there are at most $|X|^{s+1}$ choices for the set of $s+1$ leaves and, given such a choice, there are at most $|V(H)|$ possibilities for the centre, this is a contradiction. □

We are now ready to complete the proof of Theorem 1.5. By a reduction going back to work of Erdős and Simonovits [5], we may assume that our graph is *K-almost-regular* for some constant K depending only on r and s, by which we mean that $\max_{v\in V(G)} \deg(v) \leq K \min_{v\in V(G)} \deg(v)$. As noted in [3], we may also assume that the graph is bipartite, reducing our task to proving the following result.

Theorem 2.9. *For any positive real number K, there is a constant $C = C(K,H)$ such that if G is an n-vertex bipartite graph with minimum degree $\delta \geq Cn^{1-\frac{r+1}{rs+r}}$ and maximum degree at most $K\delta$, then G contains H as a subgraph.*

Proof. Let C be sufficiently large and suppose, for the sake of contradiction, that G is H-free. By Lemma 2.7, there is a positive constant $c = c(H)$ such that G has at least $cn\delta^{rs+r}$ (c,r)-rich labelled copies of $F_{r,s}$.

Claim. There are distinct vertices $x_{i,j} \in V(G)$ for $1 \leq i \leq r-1$, $1 \leq j \leq s$ such that the number of (c,r)-rich labelled copies of $F_{r,s}$ mapping $w_{i,j}$ to $x_{i,j}$ for $1 \leq i \leq r-1, 1 \leq j \leq s$ is

1. at least $\frac{1}{2}cn\delta^{rs+r}n^{-(r-1)s}$ and

2. at least $c/(2K^{rs+r})$ times the number of all labelled copies of $F_{r,s}$ mapping $w_{i,j}$ to $x_{i,j}$ for $1 \le i \le r-1, 1 \le j \le s$.

Proof of Claim. Clearly, the number of (c,r)-rich labelled copies of $F_{r,s}$ which agree with fewer than $\frac{1}{2}cn\delta^{rs+r}n^{-(r-1)s}$ (c,r)-rich labelled copies of $F_{r,s}$ on the images of $w_{i,j}$ $(1 \le i \le r-1, 1 \le j \le s)$ is less than $\frac{1}{2}cn\delta^{rs+r}$. Hence, there are at least $\frac{1}{2}cn\delta^{rs+r}$ (c,r)-rich labelled copies of $F_{r,s}$ such that each of them agrees with at least $\frac{1}{2}cn\delta^{rs+r}n^{-(r-1)s}$ other (c,r)-rich labelled copies of $F_{r,s}$ on the images $w_{i,j}$ $(1 \le i \le r-1, 1 \le j \le s)$. Moreover, the total number of labelled copies of $F_{r,s}$ in G is at most $n(K\delta)^{rs+r}$. Since $\frac{\frac{1}{2}cn\delta^{rs+r}}{n(K\delta)^{rs+r}} = c/(2K^{rs+r})$, there are vertices $x_{i,j}$ satisfying the two conditions in the claim. □

Fix some vertices $x_{i,j}$ $(1 \le i \le r-1, 1 \le j \le s)$ satisfying the conclusion of the claim and let $X = S(x_{1,1},\dots,x_{1,s},x_{2,1},\dots,x_{2,s},\dots,x_{r-1,1},\dots,x_{r-1,s})$. Moreover, let $\mathcal{A}$ be the set of (c,r)-rich labelled copies of $F_{r,s}$ mapping $w_{i,j}$ to $x_{i,j}$ for all $1 \le i \le r-1, 1 \le j \le s$. Observe that

$$|\mathcal{A}| \le |X|(K\delta)^{s+1}|V(H)|^{r-1}. \tag{2.1}$$

Indeed, there are at most $|X|$ ways to embed $y \in V(F_{r,s})$, by the maximum degree condition there are at most $(K\delta)^{s+1}$ ways to embed $z_r, w_{r,1}, w_{r,2}, \dots, w_{r,s}$ and, finally, since the copy needs to be nice, there are at most $|V(H)|$ ways to embed each of $z_1, z_2, \dots, z_{r-1}$. On the other hand, property 1 of the claim asserts that $|\mathcal{A}| \ge \frac{1}{2}cn\delta^{rs+r}n^{-(r-1)s}$, so, by comparing this with (2.1), we get

$$|X|(K\delta)^{s+1}|V(H)|^{r-1} \ge \frac{1}{2}cn\delta^{rs+r}n^{-(r-1)s}. \tag{2.2}$$

Note also that the total number of labelled copies of $F_{r,s}$ mapping $w_{i,j}$ to $x_{i,j}$ for all $1 \le i \le r-1, 1 \le j \le s$ is at least $|X|\delta^{s+1}/2$, since, after embedding y to any vertex in X, there are at least $\delta^{s+1}/2$ ways to complete the embedding. It follows from property 2 of the claim that

$$|\mathcal{A}| \ge \frac{c}{4K^{rs+r}}|X|\delta^{s+1}.$$

The number of those elements of $\mathcal{A}$ which agree with fewer than $\frac{c}{8K^{rs+r}}|X|\delta^{s+1}n^{-s}$ elements of $\mathcal{A}$ on the images of $w_{r,1},\dots,w_{r,s}$ is at most $\frac{c}{8K^{rs+r}}|X|\delta^{s+1}$. Hence, there are at least $\frac{c}{8K^{rs+r}}|X|\delta^{s+1}$ elements of $\mathcal{A}$ such that each of them agrees with at least $\frac{c}{8K^{rs+r}}|X|\delta^{s+1}n^{-s}$ elements of $\mathcal{A}$ on the images of $w_{r,1},\dots,w_{r,s}$. By the definition of (c,r)-richness, for all these copies, the image of z_r has at least $c \cdot \frac{c}{8K^{rs+r}}|X|\delta^{s+1}n^{-s}$ neighbours in X. By the maximum degree condition in G and since any (c,r)-rich copy of $F_{r,s}$ is nice, we see that for any $u,v \in V(G)$, there are at most $|V(H)|^{r-1}(K\delta)^s$ elements of $\mathcal{A}$ which map y to u and z_r to v. Hence, G has at least $\frac{\frac{c}{8K^{rs+r}}|X|\delta^{s+1}}{|V(H)|^{r-1}(K\delta)^s} = \frac{c}{8K^{rs+r+s}|V(H)|^{r-1}}|X|\delta$ edges uv with $u \in X$ and $v \in V(G)$ such that v has at least $c \cdot \frac{c}{8K^{rs+r}}|X|\delta^{s+1}n^{-s}$ neighbours in X. Set $Y = V(G) \setminus X$. Since G is bipartite, any neighbour of a vertex in X is in Y.

We now want to apply Lemma 2.8 to the bipartite graph $G[X,Y]$. By the previous paragraph, we can take

$$p = \frac{c}{8K^{rs+r+s}|V(H)|^{r-1}}\delta$$

and

$$q = \frac{c^2}{8K^{rs+r}}|X|\delta^{s+1}n^{-s}$$

and we just need to verify that $q \geq C_0$ and $pq^s \geq C_0|X|^s$, where $C_0 = C_0(H)$ is the constant provided by Lemma 2.8.

But, by equation (2.2),

$$q \geq \frac{c^3}{16K^{rs+r+s+1}|V(H)|^{r-1}}\delta^{rs+r}n^{1-rs} \geq \frac{c^3}{16K^{rs+r+s+1}|V(H)|^{r-1}}C^{rs+r}.$$

When C is sufficiently large, this is indeed at least C_0. Moreover,

$$\begin{aligned} pq^s &= \frac{c^{2s+1}}{8^{s+1}K^{rs+r+s+s(rs+r)}|V(H)|^{r-1}}\delta^{s^2+s+1}n^{-s^2}|X|^s \\ &\geq \frac{c^{2s+1}}{8^{s+1}K^{rs+r+s+s(rs+r)}|V(H)|^{r-1}}C^{s^2+s+1}n^{(s^2+s+1)(1-\frac{r+1}{rs+r})-s^2}|X|^s. \end{aligned}$$

Since $r \geq s+2$, we have $(s^2+s+1)(1-\frac{r+1}{rs+r})-s^2 \geq 0$, so we get that

$$pq^s \geq \frac{c^{2s+1}}{8^{s+1}K^{rs+r+s+s(rs+r)}|V(H)|^{r-1}}C^{s^2+s+1}|X|^s \geq C_0|X|^s,$$

provided that C is sufficiently large. Hence, we can indeed apply Lemma 2.8 to find a copy of H in G, which is a contradiction. □

3 Concluding remarks

Let $T_{r,s,s'}$ be the rooted tree obtained from $F_{r,s}$ by attaching s' leaves to the vertex y, all of which are taken to be roots. It is easy to verify that $T_{r,s,s'}$ is balanced if and only if $s'-1 \leq s \leq r+s'$. In their paper, Jiang, Jiang and Ma [8] actually studied this family of graphs, which clearly includes $F_{r,s} = T_{r,s,0}$, showing that if $T_{r,s,s'}$ is balanced and $r \geq s^3-1$, then $\mathrm{ex}(n, T^t_{r,s,s'}) = O(n^{2-1/\rho})$ holds, where $\rho = \rho(T_{r,s,s'}) = \frac{rs+r+s'}{r+1}$. We can prove the same upper bound under the relaxed condition $r \geq s-s'+1$ (except in the case $s'=0$, where we need $r \geq s+2$), almost matching the inequality $r \geq s-s'$ required for balancedness.

Theorem 3.1. *For any integers $s' \geq 1$, $s \geq s'-1$, $r \geq s-s'+1$ and $t \geq 1$, $\mathrm{ex}(n, T^t_{r,s,s'}) = O(n^{2-\frac{r+1}{rs+r+s'}})$.*

Proof sketch. Since the proof is very similar to that of Theorem 1.5, we only mention the necessary adjustments. Taking $H = T^t_{r,s,s'}$, Lemma 2.2 still holds, although in the proof we need to consider the common neighbourhood of s' vertices rather than that of a single vertex. The auxiliary hypergraphs $\mathcal{G}$ and $\mathcal{H}$ can then be defined identically (except that the vertex set of $\mathcal{G}$ is the common neighbourhood of s' vertices). By making use of the extra s' vertices whose common neighbourhood we considered, the existence of a subgraph $\mathcal{H}$ inside $\mathcal{G}$ still provides a copy of H.

The next substantial change is in Definition 2.4, where an additional s' vertices are taken as inputs, corresponding to the images of the s' new leaves, and the vertices in S are required to be common neighbours of these s' vertices (on top of the previous requirements). Similarly, for the claim in (the analogue of) Theorem 2.9, we choose and fix the s' new leaves as well as the $(r-1)s$ leaves that were fixed before.

Finally, although Lemma 2.8 does not directly provide a copy of $H = T^t_{r,s,s'}$, we can still use Lemma 2.8 in the proof of Theorem 2.9 to find a copy of $F^t_{r,s}$ in $G[X,Y]$ with the t copies of y embedded into X. But X is the common neighbourhood of s' fixed vertices, so using those vertices we can extend $F^t_{r,s}$ to H.

The remaining changes are numerical, so we do not detail them here. □

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References

[1] B. Bukh and D. Conlon, Rational exponents in extremal graph theory, *J. Eur. Math. Soc.* **20** (2018), 1747–1757. 1, 2

[2] D. Conlon, O. Janzer and J. Lee, More on the extremal number of subdivisions, *Combinatorica* **41** (2021), 465–494. 2

[3] D. Conlon and J. Lee, On the extremal number of subdivisions, *Int. Math. Res. Not.* (2021), 9122–9145. 4, 6

[4] P. Erdős, On the combinatorial problems which I would most like to see solved, *Combinatorica* **1** (1981), 25–42. 1

[5] P. Erdős and M. Simonovits, Some extremal problems in graph theory, in Combinatorial theory and its applications, I (Proc. Colloq., Balatonfüred, 1969), 377–390, North-Holland, Amsterdam, 1970. 6

[6] O. Janzer, Improved bounds for the extremal number of subdivisions, *Electron. J. Combin.* **26** (2019), Paper No. 3.3, 6 pp. 4

[7] O. Janzer, The extremal number of the subdivisions of the complete bipartite graph, *SIAM J. Discrete Math.* **34** (2020), 241–250. 2

[8] T. Jiang, Z. Jiang and J. Ma, Negligible obstructions and Turán exponents, *Ann. Appl. Math.* **38** (2022), 356–384. 2, 3, 8

[9] T. Jiang, J. Ma and L. Yepremyan, On Turán exponents of bipartite graphs, *Combin. Probab. Comput.* **31** (2022), 333–344. 2

[10] T. Jiang and Y. Qiu, Turán numbers of bipartite subdivisions, *SIAM J. Discrete Math.* **34** (2020), 556–570. 2

[11] T. Jiang and Y. Qiu, Many Turán exponents via subdivisions, *Combin. Probab. Comput.*, to appear. 2

[12] D. Y. Kang, J. Kim and H. Liu, On the rational Turán exponents conjecture, *J. Combin. Theory Ser. B* **148** (2021), 149–172. 2, 3

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