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RESEARCH ARTICLE

Distributions on partitions arising from Hilbert schemes and hook lengths

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Abstract

Recent works at the interface of algebraic combinatorics, algebraic geometry, number theory and topology have provided new integer-valued invariants on integer partitions. It is natural to consider the distribution of partitions when sorted by these invariants in congruence classes. We consider the prominent situations that arise from extensions of the Nekrasov–Okounkov hook product formula and from Betti numbers of various Hilbert schemes of n points on $\mathbb{C}^2$. For the Hilbert schemes, we prove that homology is equidistributed as $n \to \infty$. For t-hooks, we prove distributions that are often not equidistributed. The cases where $t \in \{2, 3\}$ stand out, as there are congruence classes where such counts are zero. To obtain these distributions, we obtain analytic results of independent interest. We determine the asymptotics, near roots of unity, of the ubiquitous infinite products

$$F_1(\xi; q) := \prod_{n=1}^{\infty}(1 - \xi q^n), \quad F_2(\xi; q) := \prod_{n=1}^{\infty}(1 - (\xi q)^n) \quad \text{and} \quad F_3(\xi; q) := \prod_{n=1}^{\infty}\left(1 - \xi^{-1}(\xi q)^n\right).$$

Contents

1. Introduction and statement of results

A *partition* of a nonnegative integer n, denoted $\lambda \vdash n$, is any nonincreasing sequence of positive integers, say $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, that satisfies $|\lambda| := \lambda_1 + \cdots + \lambda_m = n$. As usual, we let $p(n)$ denote the number of such partitions. One hundred years ago, Hardy and Ramanujan [16] proved their striking asymptotic formula

$$p(n) \sim \frac{1}{4\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}}, \tag{1.1}$$

as $n \to \infty$. This work marked the birth of the so-called Circle Method.

Partitions appear in many areas of mathematics. We consider recently discovered structures that arise at the interface of algebraic combinatorics, algebraic geometry, number theory and topology, where the size n partitions play a prominent role in defining various integer-valued invariants. These invariants can be sorted by congruence conditions, resulting in identities of the form

$$p(n) = C(0, b; n) + C(1, b; n) + \cdots + C(b-1, b; n), \tag{1.2}$$

where $C(a, b; n)$ counts those partitions whose invariant is in the congruence class $a \pmod b$. In the spirit of Dirichlet's Theorem on primes, where primes are equidistributed over admissible congruence classes, one may ask how the partitions are distributed, as $n \to \infty$, over the arithmetic progressions modulo b. We answer these questions for t-hooks, which arise in work of Han [14] that refined the Nekrasov–Okounkov hook product formula and for Betti numbers of various Hilbert schemes on n points in $\mathbb{C}^2$, as established by Göttsche [10, 11] and Buryak, Feigin and Nakajima [4, 10, 11].

We first consider the distribution of t-hooks. Each partition has a *Ferrers–Young diagram*

$$\begin{array}{lll} \bullet\ \bullet\ \bullet\ \dots\ \bullet & \leftarrow & \lambda_1 \text{ many nodes} \\ \bullet\ \bullet\ \dots\ \bullet & \leftarrow & \lambda_2 \text{ many nodes} \\ \vdots\ \ \vdots\ \ \vdots & & \\ \bullet\ \dots\ \bullet & \leftarrow & \lambda_m \text{ many nodes,} \end{array}$$

and each node has a *hook length*. The node in row k and column j has hook length $h(k, j) := (\lambda_k - k) + (\lambda'_j - j) + 1$, where λ'_j is the number of nodes in column j. These numbers play many significant roles in combinatorics, number theory and representation theory (for example, see [17, 26]).

We investigate those hook lengths that are multiples of a fixed positive integer t, the so-called *t-hooks*. We let $\mathcal{H}_t(\lambda)$ denote the multiset of t-hooks of a partition λ. In recent work, the second author and Pun [7] analysed the t-hook partition functions

$$p_t^e(n) := \#\{\lambda \vdash n \ : \ \#\mathcal{H}_t(\lambda) \text{ is even}\}, \quad p_t^o(n) := \#\{\lambda \vdash n \ : \ \#\mathcal{H}_t(\lambda) \text{ is odd}\},$$

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which divide the partitions of n into two subsets: those with an even (respectively, odd) number of t-hooks. For even t, they proved that partitions are equidistributed between these subsets as $n \to \infty$. Namely, they showed that

$$\lim_{n\to\infty} \frac{p_t^e(n)}{p(n)} = \lim_{n\to\infty} \frac{p_t^o(n)}{p(n)} = \frac{1}{2}.$$

However, for odd t, they found that the partitions are not equidistributed. More precisely, if $a \in \{0, 1\}$, then they proved that[1]

$$\lim_{n\to\infty} \frac{p_t^e(2n+a)}{p(2n+a)} = \frac{1}{2} + \frac{(-1)^a}{2^{\frac{1}{2}(t+1)}}.$$

In view of this unexpected result, it is natural to consider the more general t-hook partition functions

$$p_t(a, b; n) := \#\{\lambda \vdash n \ : \ \#\mathcal{H}_t(\lambda) \equiv a \pmod b\}.$$

The $p_t(a, b; n)$ are clear generalisations of $p_t^e(n)$ and $p_t^o(n)$. In this setting, (1.2) is

$$p(n) = p_t(0, b; n) + p_t(1, b; n) + p_t(2, b; n) + \cdots + p_t(b-1, b; n).$$

For odd primes b, we determine the distribution of these decompositions as $n \to \infty$, and in many situations they turn out to be nonuniform. To this end, we first obtain asymptotic formulas for $p_t(a, b; n)$. For this, we define a modified indicator function $\mathbb{I}$ by

$$\mathbb{I}(a, b, t, n) := \begin{cases} b-1 & \text{if } \frac{1}{24}(1-t^2)(1-b^2) + at - n \equiv 0 \pmod b, \\ -1 & \text{otherwise,} \end{cases} \tag{1.3}$$

and a distribution function

$$c_t(a, b; n) := \frac{1}{b} + \begin{cases} 0 & \text{if } b|t, \\ (-1)^{\frac{(1-t)(b-1)}{4}} \mathbb{I}(a, b, t, n) b^{-\frac{t+1}{2}} \left(\frac{t}{b}\right) & \text{if } b \nmid t \text{ and } t \text{ is odd}, \\ i^{\frac{(1-t)(b-1)}{2}} \varepsilon_b b^{-\frac{t}{2}} \left(\frac{\frac{1}{24}(1-t^2)(1-b^2)+at-n}{b}\right) & \text{if } b \nmid t \text{ and } t \text{ is even}, \end{cases} \tag{1.4}$$

where $(\frac{\bullet}{b})$ is the Legendre symbol and $\varepsilon_d := 1$ if $d \equiv 1 \pmod 4$ and $\varepsilon_d := i$ if $d \equiv 3 \pmod 4$. This function exactly characterises the distribution properties of the $p_t(a, b; n)$. In particular, the second summand in equation (1.4) represents the obstruction to equidistribution.

We prove the following asymptotic formulae for $p_t(a, b; n)$.

Theorem 1.1. *If $t > 1$, b is an odd prime and $0 \le a < b$, then as $n \to \infty$, we have*

$$p_t(a, b; n) \sim \frac{c_t(a, b; n)}{4\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}}.$$

Remark. Thanks to equation (3.7) in the proof of Theorem 1.1, we actually obtain an exact formula for $p_t(a, b; n)$ as a complicated convergent infinite sum.

As a corollary, we obtain the following limiting distributions.

[1]This claim is trivial if $t = 1$, as $p_1^e(n) = p(n)$ (respectively, $p_1^o(n) = p(n)$) if n is even (respectively, odd).

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Corollary 1.2. *Assuming the hypotheses in Theorem 1.1, if* $0 \le a_1 < b$ *and* $0 \le a_2 < b$, *then*

$$\lim_{n\to\infty} \frac{p_t(a_1, b; bn + a_2)}{p(bn + a_2)} = c_t(a_1, b; a_2).$$

Example. For 4-hooks with $b = 3$, the collection of values $c_4(a_1, 3; a_2)$ in Corollary 1.2 implies that

$$\lim_{n\to\infty} \frac{p_4(a, 3; 3n)}{p(3n)} = \begin{cases} \frac{4}{9} & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a = 1, \\ \frac{2}{9} & \text{if } a = 2. \end{cases}$$

Further examples are offered in Section 5.

The cases where $t \in \{2, 3\}$ are particularly striking. In addition to many instances of nonuniform distribution, there are situations where certain counts are identically zero.

Theorem 1.3. *The following are true.*

(1) *If* ℓ *is an odd prime and* $0 \le a_1, a_2 < \ell$ *satisfy* $(\frac{-16a_1+8a_2+1}{\ell}) = -1$, *then for every nonnegative integer n, we have*

$$p_2(a_1, \ell; \ell n + a_2) = 0.$$

(2) *If* $\ell \equiv 2 \pmod 3$ *is prime and* $0 \le a_1, a_2 < \ell^2$ *have the property that* $\operatorname{ord}_\ell(-9a_1 + 3a_2 + 1) = 1$, *then for every nonnegative integer n, we have*

$$p_3\left(a_1, \ell^2; \ell^2 n + a_2\right) = 0.$$

Example. For $\ell = 3$, Theorem 1.3 (1) implies that

$$p_2(0, 3; 3n + 2) = p_2(1, 3; 3n + 1) = p_2(2, 3; 3n) = 0.$$

More generally, for every odd prime ℓ and each $0 \le a_1 < \ell$, there are $\frac{1}{2}(\ell - 1)$ choices of $0 \le a_2 < \ell$ satisfying the given hypotheses. In particular, there are $\frac{1}{2}(\ell^2 - \ell)$ many pairs of a_1 and a_2 giving rise to vanishing arithmetic progressions for 2-hooks.

Example. For $\ell = 2$, Theorem 1.3 (2) gives

$$p_3(0, 4; 4n + 3) = p_3(1, 4; 4n + 2) = p_3(2, 4; 4n + 1) = p_3(3, 4; 4n) = 0.$$

Moreover, for each ℓ and each $0 \le a_1 < \ell^2$, there are $\ell - 1$ choices for a_2.

Remark. Theorem 1.3 depends on the paucity of 2-core and 3-core partitions. Recall that a partition λ is a *t-core* if $\mathcal{H}_t(\lambda) = \emptyset$. There are no such vanishing results for $t \ge 4$. This follows from the proof of the t-core conjecture by Granville and the fourth author [12]. McSpirit and Scheckelhoff [18] have found a beautiful combinatorial proof of Theorem 1.3 that makes use of the theory of abaci, t-cores and t-quotients.

We now turn to applications of partitions in algebraic geometry and topology. The fundamental goal of topology is to determine whether two spaces have the same topological, differentiable or complex analytic structure. One seeks invariants that distinguish dissimilar spaces. For complex manifolds, the Hodge numbers are one class of invariants. For any n-dimensional complex manifold M and any $0 \le s, t, \le n$, the Hodge number $h^{s,t}(M)$ gives the dimension of a certain vector space of differential forms on M. For the manifolds we consider, the Betti numbers arise as linear combinations of the Hodge numbers (for example, see [29]). We shall determine the asymptotics and modular distribution properties of certain Betti numbers.

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We consider examples occurring in the algebraic geometry of Hilbert schemes (for example, see [20]). The nth *Hilbert scheme* of a projective variety S is a projective variety $\mathrm{Hilb}^n(S)$ that is a 'smoothed' version of the nth symmetric product of S (for example, see [11, 20]). The nth symmetric product of a manifold M admits a simple combinatorial interpretation: outside of a negligible subset, the symmetric product is the collection of subsets of M of size n assembled as a manifold on its own. Rather nicely, the Hodge numbers of a complex projective surface S determine the Hodge numbers of $\mathrm{Hilb}^n(S)$ in a beautiful combinatorial way. This is captured by the pleasing formula of Göttsche [10, 11]

$$\sum_{n,s,t}(-1)^{s+t}h^{s,t}(\mathrm{Hilb}^n(S))x^{s-n}y^{t-n}q^n=\prod_{n=1}^{\infty}\frac{\prod_{s+t\ \mathrm{odd}}\left(1-x^{s-1}y^{t-1}q^n\right)^{h^{s,t}(S)}}{\prod_{s+t\ \mathrm{even}}\left(1-x^{s-1}y^{t-1}q^n\right)^{h^{s,t}(S)}}.$$

These q-infinite products often essentially specialise to modular forms, which then leads to asymptotics and distribution results via a standard application of the Circle Method. Indeed, the fourth author and his collaborators carried this out in [9]. Here we consider a prominent situation involving partitions, where modular forms do not arise, a fact that complicates the computation of asymptotics and distributions. Namely, we investigate the Hilbert schemes that arise from n points on $\mathbb{C}^2$ that have been considered recently by Göttsche [10, 11] and Buryak, Feigin and Nakajima [3, 4].

We denote the Hilbert scheme of n points of $\mathbb{C}^2$ by $(\mathbb{C}^2)^{[n]}$. For $0\le a<b$, we consider the modular sums of Betti numbers

$$B\left(a,b;\left(\mathbb{C}^2\right)^{[n]}\right):=\sum_{j\equiv a\pmod b}b_j\left(\left(\mathbb{C}^2\right)^{[n]}\right)=\sum_{j\equiv a\pmod b}\dim\left(H_j\left(\left(\mathbb{C}^2\right)^{[n]},\mathbb{Q}\right)\right).$$

We also consider their quasihomogeneous versions. To define them, we use the torus $(\mathbb{C}^\times)^2$-action on $\mathbb{C}^2$ defined by scalar multiplication (i.e., $(t_1,t_2)\cdot(x,y):=(t_1x,t_2y)$). This action lifts to $(\mathbb{C}^2)^{[n]}$. For relatively prime $\alpha,\beta\in\mathbb{N}$, we let $T_{\alpha,\beta}:=\{(t^\alpha,t^\beta)\ :\ t\in\mathbb{C}^\times\}$, a one-dimensional subtorus. The quasihomogeneous Hilbert scheme $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ is the fixed point set of $(\mathbb{C}^2)^{[n]}$. We consider their modular sums of Betti numbers

$$\begin{aligned}B\left(a,b;\left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{\alpha,\beta}}\right)&:=\sum_{j\equiv a\pmod b}b_j\left(\left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{\alpha,\beta}}\right)\\&=\sum_{j\equiv a\pmod b}\dim\left(H_j\left(\left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{\alpha,\beta}},\mathbb{Q}\right)\right).\end{aligned}$$

Remark. The odd index Betti numbers for these Hilbert schemes are always zero. In fact, for a odd and b even, simple calculations using Corollary 4.3 reveal that both $B(a,b;(\mathbb{C}^2)^{[n]})$ and $B(a,b;((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}})$ identically vanish. Moreover, in accord with (1.2), we have the homology decompositions for $p(n)$

$$p(n)=\sum_{a=0}^{b-1}B\left(a,b;\left(\mathbb{C}^2\right)^{[n]}\right)=\sum_{a=0}^{b-1}B\left(a,b;\left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{\alpha,\beta}}\right).\tag{1.5}$$

These results require the rational numbers

$$d(a,b):=\begin{cases}\frac{1}{b} & \text{if } b \text{ is odd,}\\ \frac{2}{b} & \text{if } a \text{ and } b \text{ are even,}\\ 0 & \text{if } a \text{ is odd and } b \text{ is even.}\end{cases}\tag{1.6}$$

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Theorem 1.4. *Assuming the notation above, the following are true.*

(1) *As $n \to \infty$, we have*

$$B\left(a, b; \left(\mathbb{C}^2\right)^{[n]}\right) \sim \frac{d(a,b)}{4\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}}.$$

(2) *If $\alpha, \beta \in \mathbb{N}$ are relatively prime, then as $n \to \infty$, we have*

$$B\left(a, b; \left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{\alpha,\beta}}\right) \sim \frac{d(a,b)}{4\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}}.$$

As a consequence of Theorem 1.4, we obtain distributions (i.e., see (1.5)) for the proportions

$$\delta(a, b; n) := \frac{B\left(a, b; \left(\mathbb{C}^2\right)^{[n]}\right)}{p(n)} \quad \text{and} \quad \delta_{\alpha,\beta}(a, b; n) := \frac{B\left(a, b; \left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{\alpha,\beta}}\right)}{p(n)}.$$

Corollary 1.5. *If $0 \le a < b$, then the following are true.*

(1) *We have that*

$$\lim_{n\to\infty} \delta(a, b; n) = d(a, b).$$

(2) *If $\alpha, \beta \in \mathbb{N}$ are relatively prime, then we have*

$$\lim_{n\to\infty} \delta_{\alpha,\beta}(a, b; n) = d(a, b).$$

This paper is organised as follows. In Section 2, we state and prove a general theorem (see Theorem 2.1) on the asymptotic properties (near roots of unity) of the three infinite products given in the abstract, a result that is of independent interest. The proof is obtained by suitably adapting the method of Euler–Maclaurin summation in two cases and via modularity in the other. In Section 3, we recall recent work of Han extending the Nekrasov–Okounkov partition formula, and we prove Theorems 1.1 and 1.3. To show Theorem 1.1 and Corollary 1.2, we employ Theorem 2.1 (2) and results of Zuckerman pertaining to exact formulas for Fourier coefficients of modular forms. In Section 4, we recall the work of Göttsche and Buryak, Feigin and Nakajima on homogeneous and quasihomogeneous Hilbert schemes for n points, which we then employ to prove Theorem 1.4 and Corollary 1.5 using Theorem 2.1 (1) and (3) and results of Ngo–Rhoades using Wright's Circle Method. Finally, in Section 5, we offer numerical examples of these results.

2. Asymptotics for special q-infinite products

The Hardy–Ramanujan asymptotic formula given in equation (1.1) marked the birth of the Circle Method. Its proof relied on the modular transformation properties of *Dedekind's eta-function* $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$, where $q := e^{2\pi i\tau}$ (for example, see Chapter 1 of [23]). Their work has been thoroughly developed in the theory of modular forms and harmonic Maass forms (for example, see Chapter 15 of [1]) and has been generalised beyond this setting in papers by Grosswald, Meinardus, Richmond, Roth and Szekeres [13, 19, 27, 28], to name a few.

2.1. Statement of the results

Generalising the infinite product that defines η, we consider the ubiquitous q-infinite products

$$F_1(\xi; q) := \prod_{n=1}^{\infty}(1 - \xi q^n), \quad F_2(\xi; q) := \prod_{n=1}^{\infty}(1 - (\xi q)^n), \quad \text{and} \quad F_3(\xi; q) := \prod_{n=1}^{\infty}\left(1 - \xi^{-1}(\xi q)^n\right).$$

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These infinite products are common as factors of generating functions in combinatorics, number theory and representation theory. We obtain the asymptotic properties for $F_1(\xi;q), F_2(\xi;q)$ and $F_3(\xi;q)$, where ξ is a root of unity, which are generally required for implementing the Circle Method to such generating functions. This result is of independent interest.

To make this precise, we recall *Lerch's transcendent*

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}.$$

Moreover, for coprime $h, k \in \mathbb{N}$, we define

$$\omega_{h,k} := \exp(\pi i \cdot s(h,k)), \tag{2.1}$$

using the *Dedekind sum*

$$s(h,k) := \sum_{\mu \pmod k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

Here we use the standard notation

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

For arbitrary positive integers m and n, we define $\omega_{m,n} := \omega_{\frac{m}{\gcd(m,n)}, \frac{n}{\gcd(m,n)}}$. Note that $s(h,k)$ only depends on $h \pmod k$ and that $s(0,1) = 0$. Moreover, we let

$$\lambda_{t,a,b,h,k} := \gcd(k,t) \begin{cases} 1 & \text{if } k = 1 \text{ or } \left(k > 1 \text{ and } b \nmid \frac{k}{\gcd(k,t)}\right), \\ b & \text{if } b \mid \frac{k}{\gcd(k,t)} \text{ and } \frac{ht}{\gcd(k,t)} + a\frac{k}{b\gcd(k,t)} \not\equiv 0 \pmod b, \\ b^2 & \text{if } b \mid \frac{k}{\gcd(k,t)} \text{ and } \frac{ht}{\gcd(k,t)} + a\frac{k}{b\gcd(k,t)} \equiv 0 \pmod b. \end{cases} \tag{2.2}$$

For $0 \le \theta < \frac{\pi}{2}$, we define the domain

$$D_\theta := \left\{ z = re^{i\alpha} : r \ge 0 \text{ and } |\alpha| \le \theta \right\}. \tag{2.3}$$

Theorem 2.1. *Assume the notation above. For $b > 0$, let ξ be a primitive bth root of unity; then the following are true.*

(1) *As $z \to 0$ in D_θ, we have*

$$F_1(\xi; e^{-z}) = \frac{1}{\sqrt{1-\xi}}\, e^{-\frac{\xi \Phi(\xi,2,1)}{z}} (1 + O(|z|)).$$

(2) *Suppose that b is an odd prime, and let $\xi = e^{\frac{2\pi i a}{b}}$, $t \in \mathbb{N}$, $q = e^{\frac{2\pi i}{k}(h+iz)}$ for $0 \le h < k$ with $\gcd(h,k) = 1$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. Then as $z \to 0$, we have*

$$F_2(\xi; q^t) \sim \omega^{-1}_{\frac{hbt+ak}{\lambda_{t,a,b,h,k}}, \frac{kb}{\lambda_{t,a,b,h,k}}} \left(\frac{\lambda_{t,a,b,h,k}}{tbz}\right)^{\frac{1}{2}} e^{-\frac{\pi \lambda^2_{t,a,b,h,k}}{12b^2ktz}}.$$

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(3) *As $z \to 0$ in D_θ, we have*

$$F_3(\xi; e^{-z}) = \frac{\sqrt{2\pi}\left(b^2 z\right)^{\frac{1}{2}-\frac{1}{b}}}{\Gamma\left(\frac{1}{b}\right)} \prod_{j=1}^{b-1} \frac{1}{(1-\xi^j)^{\frac{j}{b}}} e^{-\frac{\pi^2}{6b^2 z}} (1 + O(|z|)).$$

Remark. If $\xi = 1$ and $q = e^{2\pi i \tau}$, then we have

$$F_1(1; q) = F_2(1; q) = F_3(1; q) = q^{-\frac{1}{24}} \eta(\tau).$$

Asymptotic properties in this case are well-known consequences of the modularity of $\eta(\tau)$.

2.2. The Euler–Maclaurin summation formula

We require the following generalisation of the Euler–Maclaurin summation formula. To state it, we need some notation. For $s, z \in \mathbb{C}$ with $\mathrm{Re}(s) > 1, \mathrm{Re}(z) > 0$, we recall the *Hurwitz zeta function* $\zeta(s, z) := \sum_{n=0}^{\infty} \frac{1}{(n+z)^s}$, the *digamma function* $\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$ and the Euler–Mascheroni constant γ. Furthermore, we let $B_n(x)$ denote the *nth Bernoulli polynomial* defined via its generating function $\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$. The consequence of the Euler–Maclaurin summation formula required is described by the following lemma. A function f on a domain in $\mathbb{C}$ is of *sufficient decay* if there exists $\varepsilon > 0$ such that $f(w) \ll w^{-1-\varepsilon}$ as $|w| \to \infty$ in the domain. Throughout, we say that

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^n$$

if for any $N \in \mathbb{N}_0$, $f(z) = \sum_{n=0}^{N} a_n z^n + O(|z|^{N+1})$.

Lemma 2.2. *Let $0 < a \leq 1$ and $A \in \mathbb{R}^+$, and let D_θ be defined by equation (2.3). Assume that $f(z) \sim \sum_{n=n_0}^{\infty} c_n z^n$ ($n_0 \in \mathbb{Z}$) as $z \to 0$ in D_θ. Furthermore, assume that f and all of its derivatives are of sufficient decay in D_θ in the above sense. Then we have that*

$$\sum_{n=0}^{\infty} f((n+a)z) \sim \sum_{n=n_0}^{-2} c_n \zeta(-n, a) z^n + \frac{I^*_{f,A}}{z} - \frac{c_{-1}}{z}(\mathrm{Log}(Az) + \psi(a) + \gamma) - \sum_{n=0}^{\infty} c_n \frac{B_{n+1}(a)}{n+1} z^n,$$

as $z \to 0$ uniformly in D_θ, where

$$I^*_{f,A} := \int_0^\infty \left(f(u) - \sum_{n=n_0}^{-2} c_n u^n - \frac{c_{-1} e^{-Au}}{u} \right) du.$$

Remark. Note that for $a = 1$, we have that $\psi(a) + \gamma = 0$.

Proof of Lemma 2.2. A generalisation of an observation of Zagier [32, Proposition 3] is that of [2, Theorem 1.2], which states the following. Let h be a holomorphic function on a domain containing D_θ so that in particular h is holomorphic at the origin, such that h and all of its derivatives have sufficient decay and $h(z) \sim \sum_{n=0}^{\infty} c_n z^n$ as $z \to 0$ in D_θ. Furthermore, let $I_h := \int_0^\infty h(x) dx$. Then we have for $a \in \mathbb{R}$

$$\sum_{n=0}^{\infty} h((n+a)z) \sim \frac{I_h}{z} - \sum_{n=0}^{\infty} c_n \frac{B_{n+1}(a)}{n+1} z^n, \tag{2.4}$$

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as $z \to 0$ in D_θ. For the given A, write

$$f(z) = g(z) + \frac{c_{-1}e^{-Az}}{z} + \sum_{n=n_0}^{-2} c_n z^n, \tag{2.5}$$

which means that

$$g(z) = f(z) - \frac{c_{-1}e^{-Az}}{z} - \sum_{n=n_0}^{-2} c_n z^n.$$

The final term in equation (2.5) yields the first term in the right-hand side of the lemma. Since g has no pole, equation (2.4) gives that

$$\sum_{n=0}^{\infty} g((n+a)z) \sim \frac{I_g}{z} - \sum_{n=0}^{\infty} c_n(g)\frac{B_{n+1}(a)}{n+1} z^n,$$

where $c_n(g)$ are the coefficients of g. Note that $I_g = I^*_{f,A}$. We compute that

$$-\sum_{n=0}^{\infty} c_n(g)\frac{B_{n+1}(a)}{n+1} z^n = -\sum_{n=0}^{\infty}\left(c_n - \frac{(-A)^{n+1}c_{-1}}{(n+1)!}\right)\frac{B_{n+1}(a)}{n+1} z^n.$$

Combining the contribution from the second term with the contribution from the second term from equation (2.5), we obtain

$$\frac{c_{-1}}{z}\left(\sum_{n=0}^{\infty}\frac{e^{-A(n+a)z}}{n+a} + \sum_{n=1}^{\infty}\frac{B_n(a)}{n\cdot n!}(-Az)^n\right).$$

Using [2, equation (5.10)], the term in the parentheses is equal to $-(\mathrm{Log}(Az) + \psi(a) + \gamma)$. Combining the contributions yields the statement of the lemma. □

2.3. An integral evaluation

We require the following integral evaluation.

Lemma 2.3. *We have for* $N \in \mathbb{R}^+$

$$\int_0^\infty \left(\frac{e^{-x}}{x(1-e^{Nx})} - \frac{1}{Nx^2} + \left(\frac{1}{N} - \frac{1}{2}\right)\frac{e^{-x}}{x}\right)dx$$
$$= \log\left(\Gamma\left(\frac{1}{N}\right)\right) + \left(\frac{1}{2} - \frac{1}{N}\right)\log\left(\frac{1}{N}\right) - \frac{1}{2}\log(2\pi).$$

Proof. Making the change of variables $x \mapsto \frac{x}{N}$, the left-hand side equals

$$\int_0^\infty \left(\frac{e^{-\frac{x}{N}}}{x(1-e^{-x})} - \frac{1}{x^2} + \left(\frac{1}{N} - \frac{1}{2}\right)\frac{e^{-\frac{1}{N}}}{x}\right)dx.$$

Now write

$$\frac{1}{x(1-e^{-x})} = \frac{1}{x} + \frac{1}{x(e^x - 1)}.$$

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Thus the integral becomes

$$\int_0^\infty \left(\frac{1}{e^x-1}+\frac{1}{2}-\frac{1}{x}\right)\frac{e^{-\frac{x}{N}}}{x}dx+\int_0^\infty \left(\frac{e^{-\frac{x}{N}}}{x}-\frac{1}{x^2}+\left(\frac{1}{N}-\frac{1}{2}\right)\frac{e^{-\frac{x}{N}}}{x}-\frac{e^{-\frac{x}{N}}}{2x}+\frac{e^{-\frac{x}{N}}}{x^2}\right)dx.$$

We evaluate the second integral as $-\frac{1}{N}$. The claim now follows, using Binet's first integral formula (see 12.31 of [31]). □

2.4. Proof of Theorem 2.1

We employ the generalised Euler–Maclaurin summation formula to prove Theorem 2.1 (1) and (3); for part (2), we use modularity.

2.4.1. Proof of Theorem 2.1 (1)

Let $|z|<1$. Taking logarithms, we have

$$G_\xi(e^{-z}) := \operatorname{Log}(F_1(\xi;e^{-z})) = -z\sum_{j=1}^{b}\xi^j\sum_{m=0}^{\infty} f\left(\left(m+\frac{j}{b}\right)bz\right),$$

where

$$f(z) := \frac{e^{-z}}{z(1-e^{-z})} = \frac{1}{z^2}-\frac{1}{2z}+\sum_{n=0}^{\infty}\frac{B_{n+2}}{(n+2)!}z^n.$$

By Lemma 2.2, it follows that

$$\sum_{m=0}^{\infty} f\left(\left(m+\frac{j}{b}\right)bz\right) = \frac{\zeta\left(2,\frac{j}{b}\right)}{b^2z^2}+\frac{I^*_{f,1}}{bz}+\frac{1}{2bz}\left(\operatorname{Log}(bz)+\psi\left(\frac{j}{b}\right)+\gamma\right)+O(1).$$

Therefore, we find that

$$G_\xi(e^{-z}) = -\frac{1}{b^2z}\sum_{j=1}^{b}\xi^j\zeta\left(2,\frac{j}{b}\right)-\frac{I^*_{f,1}}{b}\sum_{j=1}^{b}\xi^j-\frac{1}{2b}\sum_{j=1}^{b}\xi^j\left(\operatorname{Log}(bz)+\psi\left(\frac{j}{b}\right)+\gamma\right)+O(|z|).$$

Now note that $\sum_{j=1}^{b}\xi^j=0$. Moreover, we require the identity [5, p. 39] (correcting a minus sign and erroneous k on the right-hand side)

$$\sum_{j=1}^{b}\psi\left(\frac{j}{b}\right)\xi^j = b\operatorname{Log}(1-\xi). \tag{2.6}$$

Combining these observations, we obtain

$$G_\xi(e^{-z}) = -\frac{1}{b^2z}\sum_{j=1}^{b}\xi^j\zeta\left(2,\frac{j}{b}\right)-\frac{1}{2}\operatorname{Log}(1-\xi)+O(|z|).$$

After noting that

$$\sum_{j=1}^{b}\xi^j\zeta\left(2,\frac{j}{b}\right) = b^2\xi\Phi(\xi,2,1),$$

the claim follows by exponentiation. □

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2.4.2. Proof of Theorem 2.1 (2)
Note that

$$F_2(\xi; q^t) = (\xi q^t; \xi q^t)_\infty,$$

where $(q;q)_\infty := \prod_{j=1}^{\infty}(1-q^j)$. The classical modular transformation law for the Dedekind η-function (see 5.8.1 of [6]) along with the identity $\eta(\tau) = q^{\frac{1}{24}}(q;q)_\infty$ implies that

$$(q;q)_\infty = \omega_{h,k}^{-1} z^{-\frac{1}{2}} e^{\frac{\pi}{12k}\left(z-\frac{1}{z}\right)} (q_1;q_1)_\infty, \tag{2.7}$$

where $q_1 := e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}$, where $0 \le h' < k$ is defined by $hh' \equiv -1 \pmod{k}$ and $\omega_{h,k}$ is defined as in equation (2.1). In particular, this implies that

$$(q;q)_\infty \sim \omega_{h,k}^{-1} z^{-\frac{1}{2}} e^{-\frac{\pi}{12kz}} \tag{2.8}$$

as $z \to 0$ with $\mathrm{Re}(z) > 0$. Now, by using the definitions of ξ, q given in the statement of Theorem 2.1 (2), we have

$$\xi q^t = e^{\frac{2\pi i}{kb}(hbt+ak+itbz)}.$$

We claim that $\lambda_{t,a,b,h,k}$ as defined in equation (2.2) satisfies $\lambda_{t,a,b,h,k} = \gcd(kb, hbt+ak)$. If $k = 1$, then the claim is clear, so we assume that $k > 1$. Write $k = \gcd(k,t)k_1$ and $t = \gcd(k,t)t_1$. Then we have

$$\gcd(kb, hbt+ak) = \gcd(k,t)\gcd(k_1 b, hbt_1 + ak_1).$$

Noting that $\gcd(k_1, b)$ divides each of $k_1 b, hbt_1$ and ak_1, it follows that

$$\gcd(kb, hbt+ak) = \gcd(k,t)\gcd(k_1,b)\gcd\left(\frac{k_1 b}{\gcd(k_1,b)}, \frac{hbt_1}{\gcd(k_1,b)} + a\frac{k_1}{\gcd(k_1,b)}\right).$$

Note that since b is prime, $\gcd(k_1,b) \in \{1,b\}$. If $\gcd(k_1,b) = 1$, then

$$\gcd(k_1 b, hbt_1 + ak_1) = \gcd(k_1, hbt_1)\gcd(b, ak_1) = 1.$$

If, on the other hand $\gcd(k_1,b) = b$, then write $k_1 = b^\kappa k_2$ with $\gcd(k_2,b) = 1$. Then

$$\begin{aligned}\gcd\left(k_1, ht_1 + a\frac{k_1}{b}\right) &= \gcd\left(b^\kappa k_2, ht_1 + ak_2 b^{\kappa-1}\right) = \gcd\left(b^\kappa, ht_1 + ak_2 b^{\kappa-1}\right)\gcd(k_2, ht_1)\\ &= \gcd\left(b^\kappa, ht_1 + ak_2 b^{\kappa-1}\right).\end{aligned}$$

If $\kappa > 1$, then $\gcd(b^\kappa, ht_1 + ak_2 b^{\kappa-1}) = 1$ since $\gcd(b, ht_1) = 1$. If $\kappa = 1$, then we are left with $\gcd(b, ht_1 + ak_2)$. Therefore, we obtain

$$\gcd(kb, hbt+ak) = \gcd(k,t)\begin{cases} 1 & \text{if } b \nmid \frac{k}{\gcd(k,t)},\\ b & \text{if } b \mid \frac{k}{\gcd(k,t)} \text{ and } \frac{ht}{\gcd(k,t)} + a\frac{k}{b\gcd(k,t)} \not\equiv 0 \pmod{b},\\ b^2 & \text{if } b \mid \frac{k}{\gcd(k,t)} \text{ and } \frac{ht}{\gcd(k,t)} + a\frac{k}{b\gcd(k,t)} \equiv 0 \pmod{b},\end{cases}$$

which is equal to $\lambda_{t,a,b,h,k}$.

It follows that $\gcd\left(\frac{kb}{\lambda_{t,a,b,h,k}}, \frac{hbt+ak}{\lambda_{t,a,b,h,k}}\right) = 1$. Therefore, by making the replacements $h \mapsto \frac{hbt+ak}{\lambda_{t,a,b,h,k}}$, $k \mapsto \frac{kb}{\lambda_{t,a,b,h,k}}$ and $z \mapsto \frac{tbz}{\lambda_{t,a,b,h,k}}$ in equation (2.8), the result follows. □

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2.4.3. Proof of Theorem 2.1 (3)
Again assume that $|z| < 1$. Writing

$$F_3(\xi;q) = \prod_{j=1}^{b}\prod_{n=0}^{\infty}\left(1-\xi^{j-1}q^{bn+j}\right),$$

we compute

$$\operatorname{Log}(F_3(\xi;e^{-z})) = -z\sum_{1\le j,r\le b}\xi^{(j-1)r}\sum_{m=0}^{\infty} f_j\left(\left(m+\frac{r}{b}\right)bz\right),$$

where $f_j(z) := \frac{e^{-jz}}{z(1-e^{-bz})}$. By Lemma 2.2, we obtain

$$\sum_{m=0}^{\infty} f_j\left(\left(m+\frac{r}{b}\right)bz\right) \sim \frac{\zeta\left(2,\frac{r}{b}\right)}{b^3z^2} + \frac{I^*_{f_j,1}}{bz} + \frac{B_1\left(\frac{j}{b}\right)}{bz}\left(\operatorname{Log}(bz)+\psi\left(\frac{r}{b}\right)+\gamma\right)+O(1).$$

The first term contributes $-\frac{\pi^2}{6b^2z}$. By Lemma 2.3, the second term contributes

$$-\frac{1}{b}\sum_{j=1}^{b} I^*_{f_j,1}\sum_{r=1}^{b}\xi^{(j-1)r} = -I^*_{f_1,1} = -\log\left(\Gamma\left(\frac{1}{b}\right)\right) - \left(\frac{1}{2}-\frac{1}{b}\right)\log\left(\frac{1}{b}\right)+\frac{1}{2}\log(2\pi)$$
$$= \log\left(\frac{b^{\frac{1}{2}-\frac{1}{b}}(2\pi)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{b}\right)}\right).$$

Next we evaluate

$$-\frac{1}{b}(\operatorname{Log}(bz)+\gamma)\sum_{1\le j\le b}B_1\left(\frac{j}{b}\right)\sum_{1\le r\le b}\xi^{(j-1)r} = -B_1\left(\frac{1}{b}\right)(\operatorname{Log}(bz)+\gamma).$$

Finally we are left to compute

$$-\frac{1}{b}\sum_{1\le j,r\le b}\xi^{(j-1)r}\left(\frac{j}{b}-\frac{1}{2}\right)\psi\left(\frac{r}{b}\right) = -\frac{1}{b}\sum_{\substack{0\le j\le b-1\\ 1\le r\le b}}\xi^{jr}\left(\frac{j}{b}+\frac{1}{b}-\frac{1}{2}\right)\psi\left(\frac{r}{b}\right).$$

The $(\frac{1}{b}-\frac{1}{2})$-term yields $\gamma(\frac{1}{b}-\frac{1}{2})$. Thanks to equation (2.6), the $\frac{j}{b}$ term contributes

$$-\frac{1}{b^2}\sum_{0\le j\le b-1} j\sum_{1\le r\le b}\psi\left(\frac{r}{b}\right)\xi^{jr} = -\frac{1}{b}\sum_{1\le j\le b-1} j\operatorname{Log}\left(1-\xi^j\right).$$

Combining these observations yields that

$$\operatorname{Log}(F_3(\xi;e^{-z})) = \log\left(\frac{b^{\frac{1}{2}-\frac{1}{b}}(2\pi)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{b}\right)}\right) - \frac{\pi^2}{6b^2z} - B_1\left(\frac{1}{b}\right)\operatorname{Log}(bz)$$
$$-\sum_{1\le j\le b-1}\frac{j}{b}\operatorname{Log}\left(1-\xi^j\right)+O(|z|).$$

Exponentiating gives the desired claim. □

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3. Proof of Theorem 1.1, Corollary 1.2 and Theorem 1.3

Here we recall a beautiful q-series identity of Han, who offered the generating functions we require for Theorems 1.1 and 1.3 and Corollary 1.2. Apart from factors that naturally correspond to quotients of Dedekind's eta-function, these generating functions have factors of the form $F_2(\xi;q^t)^{-t}$. The proof of Theorem 1.3 follows directly from this fact along with known identities for the 2-core and 3-core generating functions. To prove Theorem 1.1, we apply Zuckerman's exact formulas to these functions, making strong use of Theorem 2.1 (2).

3.1. Work of Han

Here we derive the generating functions for the modular t-hook functions $p_t(a,b;n)$. To this end, we recall the following beautiful formula of Han that he derived in his work on extensions of the celebrated Nekrasov–Okounkov formula[2] (see (6.12) of [22]) with $w \in \mathbb{C}$:

$$\sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{w}{h^2}\right) = \prod_{n=1}^{\infty}(1-q^n)^{w-1}.$$

Here $\mathcal{P}$ denotes the set of all integer partitions, including the empty partition, and $\mathcal{H}(\lambda)$ denotes the multiset of hook lengths for λ. Han [14] proved the following beautiful identity for the generating function for t-hooks in partitions

$$H_t(\xi;q) := \sum_{\lambda\in\mathcal{P}} \xi^{\#\mathcal{H}_t(\lambda)} q^{|\lambda|}.$$

Theorem 3.1 (Corollary 5.1 of [14]). *As formal power series, we have*

$$H_t(\xi;q) = \frac{1}{F_2(\xi;q^t)^t} \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n}.$$

As a corollary, we obtain the following generating function for $p_t(a,b;n)$.

Corollary 3.2. *If $t>1$ and $0\le a<b$, then as formal power series we have*

$$H_t(a,b;q) := \sum_{n=0}^{\infty} p_t(a,b;n)q^n = \frac{1}{b}\sum_{r=0}^{b-1} \zeta_b^{-ar} H_t\left(\zeta_b^r;q\right),$$

where $\zeta_b := e^{\frac{2\pi i}{b}}$.

Proof. We have that

$$\frac{1}{b}\sum_{r=0}^{b-1} \zeta_b^{-ar} H_t(\zeta_b^r;q) = \frac{1}{b}\sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \sum_{r=0}^{b-1} \zeta_b^{(\#\mathcal{H}_t(\lambda)-a)r} = H_t(a,b;q). \qquad \square$$

The dependence of $H_t(\xi;q)$ on $F_2(\xi;q^t)$ enables us to compute asymptotic behavior of $H_t(\xi;q)$ using Theorem 2.1 (2) and, by Corollary 3.2, the asymptotic behavior of $H_t(a,b;q)$.

3.2. Proof of Theorem 1.3

Here we prove Theorem 1.3. We first consider the case (1), where ℓ is an odd prime. We consider the generating function, using Corollary 3.2

$$H_2(a_1,\ell;q) = \sum_{n=0}^{\infty} p_2(a_1,\ell;n)q^n = \frac{1}{\ell}\sum_{r_1=0}^{\ell-1} \zeta_\ell^{-a_1r_1} H_2\left(\zeta_\ell^{r_1};q\right).$$

[2]This formula was also obtained by Westbury (see Proposition 6.1 and 6.2 of [30]).

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Applying again orthogonality of roots of unity, keeping only those terms $a_2 \pmod{\ell}$, where $a_2 \in \{0, 1, \dots, \ell - 1\}$, we find that

$$\sum_{n=0}^{\infty} p_2(a_1, \ell; \ell n + a_2) q^{\ell n + a_2} = \frac{1}{\ell^2} \sum_{r_1, r_2 \pmod{\ell}} \zeta_\ell^{-a_1 r_1 - a_2 r_2} H_2\left(\zeta_\ell^{r_1}; \zeta_\ell^{r_2} q\right).$$

Making use of the definition of $H_t(\xi; q)$, if we define $\mathcal{B}_2(q)$ and $\mathcal{C}_2(q)$ by

$$\mathcal{B}_2(q) = \sum_{n=0}^{\infty} b_2(n) q^n := \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} \quad \text{and} \quad \mathcal{C}_2(q) := \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{1-q^n}, \tag{3.1}$$

then we have

$$\sum_{\substack{n \geq 0 \\ n \equiv a_2 \pmod{\ell}}} p_2(a_1, \ell; n) q^n = \frac{1}{\ell^2} \sum_{r_1, r_2 \pmod{\ell}} \zeta_\ell^{-a_1 r_1 - a_2 r_2} \mathcal{B}_2\left(\zeta_\ell^{r_1 + 2r_2} q^2\right) \mathcal{C}_2\left(\zeta_\ell^{r_2} q\right).$$

Thanks to the classical identity of Jacobi

$$\mathcal{C}_2(q) = \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}},$$

for $N \equiv a_2 \pmod{\ell}$, we find that

$$\begin{aligned} p_2(a_1, \ell; N) &= \frac{1}{\ell^2} \sum_{r_1, r_2 \pmod{\ell}} \zeta_\ell^{-a_1 r_1 - a_2 r_2} \sum_{\substack{k, m \geq 0 \\ 2m + \frac{k(k+1)}{2} = N}} b_2(m) \zeta_\ell^{(r_1 + 2r_2) m + r_2 \frac{k(k+1)}{2}} \\ &= \sum_{\substack{m \equiv a_1 \pmod{\ell} \\ 2m + \frac{k(k+1)}{2} = N}} b_2(m), \end{aligned} \tag{3.2}$$

by making the linear change of variables $r_1 \mapsto r_1 - 2r_2$ and again using orthogonality of roots of unity. This then requires the solvability of the congruence $a_2 - 2a_1 \equiv \frac{k(k+1)}{2} \pmod{\ell}$. Completing the square produces the quadratic residue condition that prohibits this solvability and hence completes the proof of (1).

The proof of (2) follows similarly, with ℓ replaced by ℓ^2 for primes $\ell \equiv 2 \pmod{3}$. The functions in (3.1) are replaced with

$$\mathcal{B}_3(q) = \sum_{n=0}^{\infty} b_3(n) q^n := \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^3} \quad \text{and} \quad \mathcal{C}_3(q) := \prod_{n=1}^{\infty} \frac{(1-q^{3n})^3}{1-q^n}.$$

It is well-known that (for example, see Section 3 of [12] or [15, Lemma 2.5]),

$$\mathcal{C}_3(q) =: \sum_{n=0}^{\infty} c_3(n) q^n = \sum_{n=0}^{\infty} \sum_{d \mid (3n+1)} \left(\frac{d}{3}\right) q^n.$$

For primes $\ell \equiv 2 \pmod{3}$, this implies that $c_3(\ell^2 n + a) = 0$ for every positive integer n, whenever $\mathrm{ord}_\ell(3a + 1) = 1$. For example, this means that $c_3(4n + 3) = 0$ if $\ell = 2$.

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Let $0 \le a_1, a_2 < \ell^2$. In direct analogue with (3.2), a calculation reveals that nonvanishing for $N \equiv a_2 \pmod{\ell^2}$ relies on sums of the form

$$\sum_{\substack{m \equiv a_1 \pmod{\ell^2} \\ 3m+k=N}} b_3(m)c_3(k).$$

If $\mathrm{ord}_\ell(3a+1) = 1$ and $a_2 - 3a_1 \equiv a \pmod{\ell^2}$, then $p_3(a_1, \ell^2; \ell^2 + a) = 0$. This is claim (2).

3.3. Evaluating certain Kloosterman sums

The proof of Theorem 1.1 relies on the arithmetic of the Kloosterman sums

$$K(a, b, t; n) := \sum_{h=1}^{b-1} \frac{\omega_{h,b}}{\omega_{th,b}^t} \zeta_b^{(at-n)h},$$

where b is an odd prime and $s \ge 1$, $t > 1$ are integers. We evaluate this sum if t is coprime to b. We start by computing $\omega_{h,b}\omega_{th,b}^{-t}$.

Proposition 3.3. *Let b be an odd prime, h, t integers coprime to b, and let $\omega_{h,k}$ be defined by equation (2.1). Then we have*

$$\frac{\omega_{h,b}}{\omega_{th,b}^t} = \left(\frac{h}{b}\right)\left(\frac{th}{b}\right)^t e^{\pi i \frac{(1-t)(b-1)}{4}} e^{\frac{2\pi i}{b}\frac{1}{24}(1-t^2)(1-b^2)h}.$$

Proof. The proof of this proposition uses the η-multiplier, which we label ψ. Theorem 5.8.1 of [6] yields that for $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma > 0$ odd, we have

$$\psi\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \left(\frac{\delta}{\gamma}\right) e^{\frac{\pi i}{12}((\alpha+\delta)\gamma - \beta\delta(\gamma^2-1)-3\gamma)}.$$

We also have from formula (57b) of [25] that for $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$\psi\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = e^{\pi i\left(\frac{\alpha+\delta}{12\gamma} - \frac{1}{4}\right)} \omega_{\delta,\gamma}^{-1}.$$

By letting $\delta = h$, $\gamma = b$, we obtain

$$\omega_{h,b} = \left(\frac{h}{b}\right) e^{\pi i\left(\frac{1}{12b}(\alpha+h-\beta hb)(1-b^2)+\frac{b-1}{4}\right)},$$

where α, β satisfy $\alpha h - \beta b = 1$. We therefore may conclude that

$$\frac{\omega_{h,b}}{\omega_{th,b}^t} = \left(\frac{h}{b}\right)\left(\frac{th}{b}\right)^t e^{\pi i \frac{(1-t)(b-1)}{4}} e^{\frac{\pi i}{12b}((\alpha - tA)(1-b^2)+h(1-\beta b - t^2(1-Bb))(1-b^2))},$$

where $\alpha h - \beta b = Ath - Bb = 1$. A straightforward calculation then gives the claim. □

We now turn to evaluating the Kloosterman sum $K(a, b, t; n)$.

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Proposition 3.4. *Suppose that b is an odd prime, a, n are integers, and $t > 1$ is an integer coprime to b. Then we have*

$$K(a,b,t;n) = \begin{cases} \mathbb{I}(a,b,t,n)(-1)^{\frac{(1-t)(b-1)}{4}}\left(\frac{t}{b}\right) & \text{if } t \text{ is odd,} \\ (-1)^{\frac{(1-t)(b-1)}{4}}\varepsilon_b\left(\frac{\frac{1}{24}(1-t^2)(1-b^2)+at-n}{b}\right)\sqrt{b} & \text{if } t \text{ is even,} \end{cases}$$

where $\mathbb{I}(a,b,t,n)$ is defined by equation (1.3).

Proof. By Proposition 3.3, we have

$$K(a,b,t;n) = e^{\frac{\pi i}{4}(1-t)(b-1)}\sum_{h=1}^{b-1}\left(\frac{h}{b}\right)\left(\frac{th}{b}\right)^t \zeta_b^{(at-n)h+\frac{1}{24}(1-t^2)(1-b^2)h}.$$

The multiplicativity of the Legendre symbol implies

$$\left(\frac{h}{b}\right)\left(\frac{th}{b}\right)^t = \left(\frac{h}{b}\right)^{t+1}\left(\frac{t}{b}\right)^t = \begin{cases} \left(\frac{t}{b}\right) & \text{if } t \text{ is odd,} \\ \left(\frac{h}{b}\right) & \text{if } t \text{ is even.} \end{cases}$$

We proceed to distinguish the parity of t. Suppose first that t is odd. Then since b is odd, $\frac{1}{4}(1-t)(b-1)$ is an integer, and the claim directly follows.

Suppose next that t is even. Then we have

$$K(a,b,t;n) = e^{\pi i\frac{(1-t)(b-1)}{4}}\sum_{h=1}^{b-1}\left(\frac{h}{b}\right)\zeta_b^{h\left(\frac{1}{24}(1-t^2)(1-b^2)+at-n\right)}.$$

Using the classical evaluation of the Gauss sum (see for example pages 12-13 of [8]), we obtain

$$\sum_{h=1}^{b-1}\left(\frac{h}{b}\right)\zeta_b^{\left(\frac{1}{24}(1-t^2)(1-b^2)+at-n\right)h} = \left(\frac{\frac{1}{24}(1-t^2)(1-b^2)+at-n}{b}\right)\varepsilon_b\sqrt{b}. \qquad \square$$

3.4. An exact formula of Zuckerman

Here we recall a result of Zuckerman [33], building on work of Rademacher [24]. Using the Circle Method, Zuckerman computed exact formulae for Fourier coefficients for weakly holomorphic modular forms of arbitrary nonpositive weight on finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ in terms of the cusps of the underlying subgroup and the principal parts of the form at each cusp. Let F be a weakly holomorphic modular form of weight $\kappa \leq 0$ with transformation law

$$F(\gamma\tau) = \chi(\gamma)(c\tau+d)^\kappa F(\tau),$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in some finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The transformation law can be viewed alternatively in terms of the cusp $\frac{h}{k} \in \mathbb{Q}$. Let h' be defined through the congruence $hh' \equiv -1 \pmod{k}$. Taking $\tau = \frac{h'}{k} + \frac{i}{kz}$ and choosing $\gamma = \gamma_{h,k} := \left(\begin{smallmatrix} h & \beta \\ k & -h' \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we obtain the equivalent transformation law

$$F\left(\frac{h}{k}+\frac{iz}{k}\right) = \chi(\gamma_{h,k})(-iz)^{-\kappa}F\left(\frac{h'}{k}+\frac{i}{kz}\right).$$

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Let F have the Fourier expansion at $i\infty$ given by

$$F(\tau) = \sum_{n \gg -\infty} a(n) q^{n+\alpha}$$

and Fourier expansions at each rational number $0 \leq \frac{h}{k} < 1$ given by

$$F|_\kappa \gamma_{h,k}(\tau) = \sum_{n \gg -\infty} a_{h,k}(n) q^{\frac{n+\alpha_{h,k}}{c_k}}.$$

Furthermore, let I_α denote the usual I-Bessel function. In this framework, the relevant theorem of Zuckerman [33, Theorem 1] may be stated as follows.

Theorem 3.5. *Assume the notation and hypotheses above. If $n + \alpha > 0$, then we have*

$$a(n) = 2\pi (n+\alpha)^{\frac{\kappa-1}{2}} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \chi(\gamma_{h,k}) e^{-\frac{2\pi i (n+\alpha)h}{k}}$$

$$\times \sum_{m+\alpha_{h,k} \leq 0} a_{h,k}(m) e^{\frac{2\pi i}{kc_k}(m+\alpha_{h,k})h'} \left(\frac{|m+\alpha_{h,k}|}{c_k} \right)^{\frac{1-\kappa}{2}} I_{-\kappa+1}\left(\frac{4\pi}{k} \sqrt{\frac{(n+\alpha)|m+\alpha_{h,k}|}{c_k}} \right).$$

3.5. Proof of Theorem 1.1 and Corollary 1.2

We next provide proofs of both Theorem 1.1 and Corollary 1.2. Our main tool is the powerful theorem of Zuckerman described in Section 3.4.

Proof of Theorem 1.1. Using Corollary 3.2, we have

$$H_t(a,b;q) = \frac{1}{b(q;q)_\infty} + \sum_{r=1}^{b-1} \zeta_b^{-ar} H_t\left(\zeta_b^r; q\right). \tag{3.3}$$

From Theorem 3.1, we conclude

$$H_t\left(\zeta_b^r; q\right) = \frac{\left(q^t; q^t\right)_\infty^t}{\left(\zeta_b^r q^t; \zeta_b^r q^t\right)_\infty^t (q;q)_\infty}.$$

To obtain the transformation formula for $H_t(\zeta_b^r; q)$ at the cusp $\frac{h}{k}$, we write

$$q^t = e^{\frac{2\pi i t}{k}(h+iz)} = e^{\frac{\frac{2\pi i}{k}}{\gcd(k,t)}\left(h\frac{t}{\gcd(k,t)} + i\frac{t}{\gcd(k,t)}z\right)},$$

where we note that $\gcd(h\frac{t}{\gcd(k,t)}, \frac{k}{\gcd(k,t)}) = 1$. Thus we may use equation (2.7) with $k \mapsto \frac{k}{\gcd(k,t)}$, $h \mapsto h\frac{t}{\gcd(k,t)}$, $z \mapsto \frac{t}{\gcd(k,t)}z$ to obtain

$$\left(q^t; q^t\right)_\infty = \omega^{-1}_{h\frac{t}{\gcd(k,t)}, \frac{k}{\gcd(k,t)}} \left(\frac{t}{\gcd(k,t)} z \right)^{-\frac{1}{2}} e^{\frac{\pi \gcd(k,t)}{12k}\left(\frac{t}{\gcd(k,t)}z - \frac{\gcd(k,t)}{tz}\right)}$$

$$\times \left(e^{\frac{2\pi i \gcd(k,t)}{k}\left(h_{k,t} + i\frac{\gcd(k,t)}{tz}\right)}; e^{\frac{2\pi i \gcd(k,t)}{k}\left(h_{k,t} + i\frac{\gcd(k,t)}{tz}\right)} \right)_\infty, \tag{3.4}$$

where $0 \leq h_{k,t} < \frac{k}{\gcd(k,t)}$ is defined by $h\frac{t}{\gcd(k,t)} h_{k,t} \equiv -1 \pmod{\frac{k}{\gcd(k,t)}}$.

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Similarly, for $\left(\zeta_b^r q^t;\zeta_b^r q^t\right)_\infty$ the proof of Theorem 2.1 (2) implies that we may use equation (2.7) with $h \mapsto \frac{hbt+rk}{\lambda_{t,r,b,h,k}}$, $k \mapsto \frac{kb}{\lambda_{t,r,b,h,k}}$, $z \mapsto \frac{tbz}{\lambda_{t,r,b,h,k}}$ and obtain

$$\left(\zeta_b^r q^t;\zeta_b^r q^t\right)_\infty = \omega^{-1}_{\frac{hbt+rk}{\lambda_{t,r,b,h,k}},\frac{kb}{\lambda_{t,r,b,h,k}}}\left(\frac{tbz}{\lambda_{t,r,b,h,k}}\right)^{-\frac{1}{2}} e^{\frac{\pi\lambda_{t,r,b,h,k}}{12kb}\left(\frac{tbz}{\lambda_{t,r,b,h,k}}-\frac{\lambda_{t,r,b,h,k}}{tbz}\right)} \times \left(e^{\frac{2\pi i\lambda_{t,r,b,h,k}}{kb}\left(h_{k,t,b,r}+i\frac{\lambda_{t,r,b,h,k}}{tbz}\right)};e^{\frac{2\pi i\lambda_{t,r,b,h,k}}{kb}\left(h_{k,t,b,r}+i\frac{\lambda_{t,r,b,h,k}}{tbz}\right)}\right)_\infty, \tag{3.5}$$

where $0 \le h_{k,t,b,r} < \frac{kb}{\lambda_{t,r,b,h,k}}$ is defined by $\frac{hbt+rk}{\lambda_{t,r,b,h,k}} h_{k,t,b,r} \equiv -1 \pmod{\frac{kb}{\lambda_{t,r,b,h,k}}}$.

Combining equation (2.7), equation (3.4), and equation (3.5) yields

$$H_t\left(\zeta_b^r;q\right) = \Omega_{b,t}(r;h,k)\left(\frac{\gcd(k,t)b}{\lambda_{t,r,b,h,k}}\right)^{\frac{t}{2}} z^{\frac{1}{2}} e^{\frac{\pi}{12k}\left(-z+\left(1-\gcd(k,t)^2+\frac{\lambda_{t,r,b,h,k}^2}{b^2}\right)\frac{1}{z}\right)} \times \frac{\left(e^{\frac{2\pi i\gcd(k,t)}{k}\left(h_{k,t}+i\frac{\gcd(k,t)}{tz}\right)};e^{\frac{2\pi i\gcd(k,t)}{k}\left(h_{k,t}+i\frac{\gcd(k,t)}{tz}\right)}\right)_\infty^t}{\left(e^{\frac{2\pi i\lambda_{t,r,b,h,k}}{kb}\left(h_{k,t,b,r}+i\frac{\lambda_{t,r,b,h,k}}{tbz}\right)};e^{\frac{2\pi i\lambda_{t,r,b,h,k}}{kb}\left(h_{k,t,b,r}+i\frac{\lambda_{t,r,b,h,k}}{tbz}\right)}\right)_\infty^t \left(e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)};e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}\right)_\infty}, \tag{3.6}$$

where

$$\Omega_{b,t}(r;h,k) := \frac{\omega^t_{\frac{hbt+rk}{\lambda_{t,r,b,h,k}},\frac{kb}{\lambda_{t,r,b,h,k}}}\omega_{h,k}}{\omega^t_{h\frac{t}{\gcd(k,t)},\frac{k}{\gcd(k,t)}}}.$$

As usual, we define $P_t(q) := (q;q)_\infty^t =: \sum_{n=0}^\infty q_t(n)q^n$ and $P(q)^t =: \sum_{n=0}^\infty p_t(n)q^n$. Then we see that the principal part of equation (3.6) is governed by the sum

$$\sum_{\substack{n_1,n_2,n_3\ge 0\\ r_{k,h,t,b}(n_1,n_2,n_3)\ge 0}} q_t(n_1)p_t(n_2)p(n_3)\zeta_{kb}^{\gcd(k,t)bh_{k,t}n_1+\lambda_{t,r,b,h,k}h_{k,t,b,r}n_2+bh'n_3} e^{\frac{\pi}{12kz}r_{k,h,t,b}(n_1,n_2,n_3)},$$

where

$$r_{k,h,t,b}(n_1,n_2,n_3) := 1-\gcd(k,t)^2+\frac{\lambda_{t,r,b,h,k}^2}{b^2}-24\left(\frac{\gcd(k,t)^2}{t}n_1+\frac{\lambda_{t,r,b,h,k}^2}{tb^2}n_2+n_3\right).$$

We denote the Fourier coefficients of $H_t(\zeta_b^r;q)$ by $c_{t,b,r}(n)$. Using Theorem 3.5, we conclude that

$$c_{t,b,r}(n) = \frac{2\pi}{n^{\frac{3}{4}}}b^{\frac{t}{2}}\sum_{k=1}^\infty\frac{\gcd(k,t)^{\frac{t}{2}}}{k}\sum_{\substack{0\le h<k\\ \gcd(h,k)=1}}\Omega_{b,t}(r;h,k)e^{-\frac{2\pi inh}{k}}\lambda_{t,r,b,h,k}^{-\frac{t}{2}}\sum_{\substack{n_1,n_2,n_3\ge 0\\ r_{k,h,t,b}(n_1,n_2,n_3)\ge 0}} q_t(n_1)p_t(n_2)p(n_3)$$
$$\times\zeta_{kb}^{\gcd(k,t)bh_{k,t}n_1+\lambda_{t,r,b,h,k}h_{k,t,b,r}n_2+bh'n_3}\left(\frac{r_{k,h,t,b}(n_1,n_2,n_3)}{24}\right)^{\frac{3}{4}} I_{\frac{3}{2}}\left(\frac{\pi}{k}\sqrt{\frac{2nr_{k,h,t,b}(n_1,n_2,n_3)}{3}}\right). \tag{3.7}$$

Since $x^\alpha I_\alpha(x)$ is monotonically increasing as $x\to\infty$ for any fixed α, the terms that dominate asymptotically are those which have the largest possible value of $\frac{1}{k}\sqrt{r_{k,h,t,b}(n_1,n_2,n_3)}$. In particular

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for this we require $n_1 = n_2 = n_3 = 0$. Note that we have $q_t(0) = p_t(0) = p(0) = 1$. Since the expression in question is positive, we can maximise its square: that is, we maximise

$$\frac{r_{k,h,t,b}(0,0,0)}{k^2} = \frac{1}{k^2}\left(1 - \gcd(k,t)^2 + \frac{\lambda_{t,r,b,h,k}^2}{b^2}\right).$$

We consider the three possible values of $\lambda_{t,r,b,h,k}$. If $\lambda_{t,r,b,h,k} = \gcd(k,t)$; then

$$\frac{r_{k,h,t,b}(0,0,0)}{k^2} = \frac{1}{k^2}\left(1 + \left(\frac{1}{b^2} - 1\right)\gcd(k,t)^2\right) \leq \left(1 + \left(\frac{1}{9} - 1\right)\right) < 1.$$

If $\lambda_{t,r,b,h,k} = b\gcd(k,t)$, then (noting that in this case $k > 1$)

$$\frac{r_{k,h,t,b}(0,0,0)}{k^2} = \frac{1}{k^2} < 1.$$

Finally, if $\lambda_{t,r,b,h,k} = b^2\gcd(k,t)$, then we have

$$\frac{r_{k,h,t,b}(0,0,0)}{k^2} = \frac{1}{k^2}\left(1 + \left(b^2 - 1\right)\gcd(k,t)^2\right).$$

Since $b \mid\mid \dfrac{k}{\gcd(k,t)}$ in this case, we may write $\gcd(k,t) = b^{\varrho}d$, where $\gcd(b,d) = 1$, $b^{\varrho} \mid\mid t$ and $k = b^{\varrho+1}dk_0$ for $\gcd(k_0, \frac{t}{\gcd(k,t)}) = \gcd(k_0, b) = 1$. Therefore,

$$\frac{r_{k,h,t,b}(0,0,0)}{k^2} = \frac{1 + \left(b^2 - 1\right)b^{2\varrho}d^2}{b^{2\varrho+2}d^2k_0^2},$$

which is maximised if $k_0 = 1$. In this case, we have $k = b\gcd(k,t)$, and therefore we may write

$$\frac{r_{k,h,t,b}(0,0,0)}{k^2} = \frac{1 + \left(b^2 - 1\right)\gcd(k,t)^2}{b^2\gcd(k,t)^2} = \frac{b^2 - 1}{b^2} + \frac{1}{b^2\gcd(k,t)^2}.$$

To maximise this, we need to minimise $\gcd(k,t)$, which is $\gcd(k,t) = 1$. Note that in this case

$$\frac{r_{k,h,t,b}(0,0,0)}{k^2} = 1.$$

Since $ht + r \equiv 0 \pmod{b}$, we have

$$\Omega_{b,t}(r;h,b) = \frac{\omega_{\frac{ht+r}{b},1}^t\omega_{h,b}}{\omega_{ht,b}^t} = \frac{\omega_{-r\bar{t},b}}{\omega_{-r,b}^t},$$

where $\bar{t}$ denotes the inverse of $t \pmod{b}$. Then by equation (3.7), we have

$$c_{t,b,r}(n) \sim \frac{2\pi b^{\frac{t}{2}}\omega_{-r\bar{t},b}e^{\frac{2\pi inr\bar{t}}{b}}}{(24n)^{\frac{3}{4}}\omega_{-r,b}^t b^{t+1}} I_{\frac{3}{2}}\left(\pi\sqrt{\frac{2n}{3}}\right) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}nb^{\frac{t}{2}+1}}\frac{\omega_{-r\bar{t},b}}{\omega_{-r,b}^t}e^{\frac{2\pi inr\bar{t}}{b}},$$

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as $n \to \infty$, where we use that $I_\alpha(x) \sim \frac{e^x}{\sqrt{2\pi x}}$ as $x \to \infty$. Using equation (1.1), we obtain

$$\frac{c_{t,b,r}(n)}{p(n)} \sim \begin{cases} \dfrac{1}{b^{\frac{t}{2}+1}} \dfrac{\omega_{-r\bar{t},b}}{\omega_{-r,b}^t} e^{\frac{2\pi i n r\bar{t}}{b}} & \text{if } b \nmid t, \\ 0 & \text{otherwise.} \end{cases}$$

By equation (3.3), we have

$$p_t(a,b;n) = \frac{1}{b}p(n) + \frac{1}{b}\sum_{r=1}^{b-1} \zeta_b^{-ar} c_{t,b,r}(n),$$

and so dividing through by $p(n)$ yields

$$\frac{p_t(a,b;n)}{p(n)} = \frac{1}{b} + \frac{1}{b}\sum_{r=1}^{b-1} \zeta_b^{-ar} \frac{c_{t,b,r}(n)}{p(n)} \sim \begin{cases} \dfrac{1}{b} + \dfrac{1}{b^{\frac{t}{2}+2}} \displaystyle\sum_{r=1}^{b-1} \frac{\omega_{-r\bar{t},b}}{\omega_{-r,b}^t} \zeta_b^{(n\bar{t}-a)r} & \text{if } b \nmid t, \\ \dfrac{1}{b} & \text{otherwise} \end{cases}$$

as $n \to \infty$. This completes the proof in the case where $b|t$. Otherwise, setting $h = -r\bar{t}$ shows

$$\frac{p_t(a,b;n)}{p(n)} \sim \frac{1}{b} + \frac{1}{b^{\frac{t}{2}+2}} \sum_{h=1}^{b-1} \frac{\omega_{h,b}}{\omega_{th,b}^t} \zeta_b^{(at-n)h} = \frac{1}{b}\left(1 + \frac{K(a,b,t;n)}{b^{\frac{t}{2}+1}}\right)$$

as $n \to \infty$. The evaluation of $K(a,b,t;n)$ in Proposition 3.4 then completes the proof. □

Proof of Corollary 1.2. To derive Corollary 1.2, it is enough to consider the leading constants in Theorem 1.1. Namely, it suffices to show that for a, b fixed, $c_t(a,b;n)$ depends only on $n \pmod b$, which is clear from the definition of equation (1.4). □

4. Proof of Theorem 1.4 and Corollary 1.5

Here we recall the relevant generating functions for the Poincaré polynomials of the Hilbert schemes that pertain to Theorem 1.4.

4.1. Work of Göttsche and Buryak, Feigin and Nakajima

For convenience, we let $P(X;T)$ be the usual *Poincaré polynomial*

$$P(X;T) := \sum_j b_j(X) T^j = \sum_j \dim\big(H_j(X,\mathbb{Q})\big) T^j,$$

which is the generating function for the Betti numbers of X. For the various Hilbert schemes on n points we consider, the work of Göttsche and Buryak, Feigin and Nakajima [3, 4, 10, 11] offers the generating function of these Poincaré polynomials as a formal power series in q. Namely, we have the following.

Theorem 4.1 (Göttsche). *We have that*

$$G(T;q) := \sum_{n=0}^{\infty} P\left(\left(\mathbb{C}^2\right)^{[n]}; T\right) q^n = \prod_{m=1}^{\infty} \frac{1}{1 - T^{2m-2}q^m} = \frac{1}{F_3(T^2;q)}.$$

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Theorem 4.2 (Buryak and Feigin). *If $\alpha, \beta \in \mathbb{N}$ are relatively prime, then we have that*

$$G_{\alpha,\beta}(T;q) := \sum_{n=0}^{\infty} P\left(\left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{\alpha,\beta}}; T\right) q^n = \frac{1}{F_1(T^2; q^{\alpha+\beta})} \prod_{m=1}^{\infty} \frac{1-q^{(\alpha+\beta)m}}{1-q^m}.$$

Remark. The Poincaré polynomials in these cases only have even degree terms. The odd index Betti numbers are always zero. Moreover, letting $T = 1$ in these generating functions give Euler's generating function for $p(n)$. Therefore, we directly see that

$$p(n) = P\left(\left(\mathbb{C}^2\right)^{[n]}; 1\right) = P\left(\left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{\alpha,\beta}}; 1\right),$$

confirming (1.5). Of course, the proofs of these theorems begin with partitions of size n.

Arguing as in the proof of Corollary 3.2, we obtain the following generating functions for the modular sums of Betti numbers.

Corollary 4.3. *For $0 \leq a < b$, the following are true.*

(1) We have that

$$\sum_{n=0}^{\infty} B\left(a, b; \left(\mathbb{C}^2\right)^{[n]}\right) q^n = \frac{1}{b} \sum_{r=0}^{b-1} \zeta_b^{-ar} G(\zeta_b^r; q).$$

(2) If $\alpha, \beta \in \mathbb{N}$ are relatively prime, then we have

$$\sum_{n=0}^{\infty} B\left(a, b; \left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{\alpha,\beta}}\right) q^n = \frac{1}{b} \sum_{r=0}^{b-1} \zeta_b^{-ar} G_{\alpha,\beta}(\zeta_b^r; q).$$

4.2. Wright's variant of the Circle Method

The classical Circle Method, as utilised by Hardy–Ramanujan and many others, derives asymptotic or exact formulas for the Fourier coefficients of q-series by leveraging modular properties of the generating functions. More recently, a variation of the Circle Method due to Wright has grown increasingly important in number theory. For the proof of Theorem 1.4 and Corollary 1.5, we use Wright's variation, which obtains asymptotic formulas for generating functions carrying suitable analytic properties.

Remark. Ngo and Rhoades [21] proved a more restricted version[3] of the following proposition where the generating function F split as two functions. Our purposes do not require such a splitting, so we state the proposition in terms of a single function F.

Proposition 4.4. *Suppose that $F(q)$ is analytic for $q = e^{-z}$, where $z = x + iy \in \mathbb{C}$ satisfies $x > 0$ and $|y| < \pi$, and suppose that $F(q)$ has an expansion $F(q) = \sum_{n=0}^{\infty} c(n) q^n$ near 1. Let $c, N, M > 0$ be fixed constants. Consider the following hypotheses:*

(1) *As $z \to 0$ in the bounded cone $|y| \leq Mx$ (major arc), we have*

$$F(e^{-z}) = z^B e^{\frac{A}{z}} \left(\sum_{j=0}^{N-1} \alpha_j z^j + O_\delta\left(|z|^N\right) \right),$$

where $\alpha_s \in \mathbb{C}$, $A \in \mathbb{R}^+$ and $B \in \mathbb{R}$.

[3] We note that hypothesis 4 in Proposition 1.8 of [21] is stated differently than our hypothesis 2 in Proposition 4.4 below.

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(2) *As $z \to 0$ in the bounded cone $Mx \le |y| < \pi$ (minor arc), we have*

$$|F(e^{-z})| \ll_\delta e^{\frac{1}{\mathrm{Re}(z)}(A-\kappa)}.$$

for some $\kappa \in \mathbb{R}^+$.

If (1) and (2) hold, then as $n \to \infty$, we have for any $N \in \mathbb{R}^+$

$$c(n) = n^{\frac{1}{4}(-2B-3)} e^{2\sqrt{An}} \left(\sum_{r=0}^{N-1} p_r n^{-\frac{r}{2}} + O\left(n^{-\frac{N}{2}}\right) \right),$$

where $p_r := \sum_{j=0}^{r} \alpha_j c_{j,r-j}$ *and* $c_{j,r} := \dfrac{(-\frac{1}{4\sqrt{A}})^r \sqrt{A}^{j+B+\frac{1}{2}}}{2\sqrt{\pi}} \dfrac{\Gamma(j+B+\frac{3}{2}+r)}{r!\Gamma(j+B+\frac{3}{2}-r)}$.

Proof. By Cauchy's Theorem, we have

$$c(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{F(q)}{q^{n+1}} dq,$$

where $\mathcal{C}$ is a circle centred at the origin inside the unit circle surrounding zero exactly once counter-clockwise. We choose $|q| = e^{-\lambda}$ with $\lambda := \sqrt{\frac{A}{n}}$. Set

$$A_j(n) := \frac{1}{2\pi i} \int_{\mathcal{C}_1} \frac{z^{B+j} e^{\frac{A}{z}}}{q^{n+1}} dq,$$

where $\mathcal{C}_1$ is the major arc. We claim that

$$c(n) = \sum_{j=0}^{N-1} \alpha_j A_j(n) + O\left(n^{\frac{1}{2}(-B-N-1)} e^{2\sqrt{An}}\right). \tag{4.1}$$

For this, write

$$c(n) - \sum_{j=0}^{N-1} \alpha_j A_j(n) = \mathcal{E}_1(n) + \mathcal{E}_2(n),$$

where

$$\mathcal{E}_1(n) := \frac{1}{2\pi i} \int_{\mathcal{C}_2} \frac{F(q)}{q^{n+1}} dq, \quad \mathcal{E}_2(n) := \frac{1}{2\pi i} \int_{\mathcal{C}_1} \left(F(q) z^{-B} e^{-\frac{A}{z}} - \sum_{j=0}^{N-1} \alpha_j z^j \right) z^B e^{\frac{A}{z}} q^{-n-1} dq,$$

where $\mathcal{C}_2$ is the minor arc.

We next bound $\mathcal{E}_1(n)$ and $\mathcal{E}_2(n)$. For $\mathcal{E}_2(n)$, we have, by condition (1)

$$\left| F(e^{-z}) z^{-B} e^{-\frac{A}{z}} - \sum_{j=0}^{N-1} \alpha_j z^j \right| \ll_\delta |z|^N.$$

Note that on $\mathcal{C}$, $x = \lambda$ and that

$$\left| \exp\left(\frac{A}{z} + nz\right) \right| \le \exp\left(2\sqrt{An}\right).$$

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Since the length of $\mathcal{C}_1$ is $\approx \lambda$, we obtain

$$\mathcal{E}_2(n) \ll \lambda |z|^{N+B} \exp\Big(2\sqrt{An}\Big).$$

On $\mathcal{C}_1$, we have $y \ll \lambda$, implying $|z| \sim \frac{1}{\sqrt{n}}$. This gives $\mathcal{E}_1(n)$ satisfies the bound required in equation (4.1).

On $\mathcal{C}_2$, we estimate

$$|F(q)| \ll e^{\frac{1}{\lambda}(A-\kappa)}.$$

Therefore,

$$\mathcal{E}_1(n) \ll |F(q)||q|^{-n} \ll e^{\frac{1}{\lambda}(A-\kappa)+n\lambda} \ll e^{(2-\kappa)\sqrt{An}}.$$

The required bound equation (4.1) follows. Using Lemma 3.7 of [21] to estimate the integrals $A_j(n)$ now gives the claim. □

4.3. Proof of Theorem 1.4 and Corollary 1.5

We now apply the Circle Method to the generating functions in Theorems 4.1 and 4.2.

Proof of Theorem 1.4. Using first Corollary 4.3 (1) and then Theorem 4.1, we obtain

$$H_{a,b}(q) := \sum_{n=0}^{\infty} B\Big(a, b; \Big(\mathbb{C}^2\Big)^{[n]}\Big) q^n = \frac{1}{b}\big(1+\delta_{2|b}\big)\frac{1}{(q;q)_\infty} + \frac{1}{b}\sum_{\substack{1\le r\le b-1\\ r\neq \frac{b}{2}}} \zeta_b^{-ar}\frac{1}{F_3\Big(\zeta_b^{2r};q\Big)}.$$

We want to apply Proposition 4.4. For this, we first show ($M > 0$ arbitrary) that we have as $z \to 0$ on the major arc $|y| \le Mx$

$$H_{a,b}(e^{-z}) = \frac{1}{b}\big(1+\delta_{2|b}\big)\sqrt{\frac{z}{2\pi}}e^{\frac{\pi^2}{6z}}(1+O(|z|)). \tag{4.2}$$

Recall that we have $P(q) := \sum_{n=0}^{\infty} p(n)q^n = (q;q)_\infty^{-1}$. First we note the well-known bound (for $|y| \le Mx$, as $z \to 0$)

$$P(e^{-z}) = \sqrt{\frac{z}{2\pi}}e^{\frac{\pi^2}{6z}}(1+O(|z|)).$$

Next we consider $\frac{1}{F_3(\zeta_b^{2r};q)}$ for $\zeta_b^{2r} \neq 1$ on the major arc. By Theorem 2.1 (3)

$$\frac{1}{F_3\Big(\zeta_b^{2r};e^{-z}\Big)} = \frac{(b^2z)^{\frac{1}{b}-\frac{1}{2}}\Gamma\Big(\frac{1}{b}\Big)}{\sqrt{2\pi}}\prod_{j=1}^{b-1}\Big(1-\zeta_b^{2rj}\Big)^{\frac{j}{b}}e^{\frac{\pi^2}{6b^2z}}(1+O(|z|)) \ll |z|^{-N}e^{\frac{\pi^2}{6z}}$$

for any $N \in \mathbb{N}$. This gives equation (4.2).

Next we show that we have as $z \to 0$ on the minor arc $|y| \ge Mx$

$$H_{a,b}(e^{-z}) \ll e^{\left(\frac{\pi^2}{6}-\kappa\right)\frac{1}{x}}. \tag{4.3}$$

It is well-known (and follows by logarithmic differentiation) that for some $\mathcal{C} > 0$

$$|P(e^{-z})| \le x^{\frac{1}{2}}e^{\frac{\pi}{6x}-\frac{\mathcal{C}}{x}}.$$

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We are left to bound $\frac{1}{F_3(\zeta_b^{2r};q)}$ on the minor arc. For this, we write

$$\operatorname{Log}\left(\frac{1}{F_3\left(\zeta_b^{2r};q\right)}\right)=\sum_{m=1}^{\infty}\frac{q^m}{m\left(1-\zeta_b^{2rm}q^m\right)}.$$

Noting that $|1-\zeta_b^{2rm}q^m|\geq 1-|q|^m$, we obtain

$$\left|\operatorname{Log}\left(\frac{1}{F_3\left(\zeta_b^{2r};q\right)}\right)\right|\leq\left|\frac{q}{1-\zeta_b^{2r}q}\right|-\frac{|q|}{1-|q|}+\log(P|q|)$$

so we are done once we show that

$$\left|\frac{q}{1-\zeta_b^{2r}q}\right|-\frac{|q|}{1-|q|}<-\frac{\mathcal{C}}{x}$$

for some $\mathcal{C}>0$. Note that

$$\frac{1}{1-\zeta_b^{2r}q}=O_{b,r}(1)$$

and thus

$$\left|\frac{q}{1-\zeta_b^{2r}q}\right|-\frac{|q|}{1-|q|}=-\frac{1}{x}+O_{b,r}(1),$$

giving equation (4.3). The claim of (1) now follows by Proposition 4.4.

(2) By Corollary 4.3 (2) and Theorem 4.2, we have

$$\begin{aligned}\mathcal{H}_{a,b,\alpha,\beta}(q)&:=\sum_{n=0}^{\infty}B\left(a,b;\left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{\alpha,\beta}}\right)q^n\\&=\frac{1}{b}(1+\delta_{2|b})P(q)+\frac{1}{b}\sum_{\substack{1\leq r\leq b-1\\ r\neq\frac{b}{2}}}\zeta_b^{-ar}\frac{\left(q^{\alpha+\beta};q^{\alpha+\beta}\right)_\infty}{F_1\left(\zeta_b^{2r};q^{\alpha+\beta}\right)(q;q)_\infty}.\end{aligned}$$

We show the same bounds as in (1) with the only additional condition that

$$M<\frac{2\pi^2}{b^2}\min_{1\leq r<\frac{b}{2}}\frac{r(b-2r)}{\left|\sum_{n=1}^{\infty}\frac{\sin\left(\frac{4\pi r}{b}\right)}{n^2}\right|}. \tag{4.4}$$

We only need to prove the bounds for

$$\mathcal{H}_{\alpha,\beta}(q):=\frac{\left(q^{\alpha+\beta};q^{\alpha+\beta}\right)_\infty}{F_1\left(\zeta_b^{2r};q^{\alpha+\beta}\right)(q;q)_\infty}$$

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for $\zeta_b^{2r} \neq 1$. We may assume without loss of generality that $1 \leq 2r < b$. We start by showing the major arc bound. By Theorem 2.1 (1) and equation (2.8), we have, for z on the major arc

$$\mathcal{H}_{\alpha,\beta}(q) \ll \left| e^{\frac{\pi^2}{6z} - \frac{\pi^2}{6(\alpha+\beta)z} + \frac{\zeta_b^{2r}\phi\left(\zeta_b^{2r},2,1\right)}{(\alpha+\beta)z}} \right|.$$

So to prove the major arc bound, we need to show that for some $\varepsilon > 0$

$$\left(\frac{\pi^2}{6} - \varepsilon\right)\mathrm{Re}\left(\frac{1}{z}\right) - \mathrm{Re}\left(\frac{\zeta_b^{2r}\phi\left(\zeta_b^{2r},2,1\right)}{z}\right) > 0.$$

We first rewrite

$$\zeta_b^{2r}\phi\left(\zeta_b^{2r},2,1\right) = \sum_{n=1}^{\infty} \frac{\cos\left(\frac{4\pi rn}{b}\right) + i\sin\left(\frac{4\pi rn}{b}\right)}{n^2}.$$

Now note the evaluation for $0 \leq \theta \leq 2\pi$ (see, for example, [32])

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2} = \frac{\pi^2}{6} - \frac{\theta(2\pi-\theta)}{4}.$$

Thus we are left to show

$$\frac{2\pi^2 r}{b^2}(b-2r)x \geq \left| \sum_{n=1}^{\infty} \frac{\sin\left(\frac{4\pi rn}{b}\right)}{n^2} \right| y.$$

This follows by the definition of M given in equation (4.4). □

Proof of Corollary 1.5. This follows immediately from Theorem 1.4 and the definition of $d(a,b)$ in equation (1.6). □

5. Examples

This section includes examples of the main results in this paper.

5.1. *Examples of Theorem 1.1 and Corollary 1.2*

This subsection pertains to Han's t-hook generating functions. For convenience, we define the proportion functions

$$\Psi_t(a,b;n) := \frac{p_t(a,b;n)}{p(n)}.$$

Example. In the case of $t = 3$, we find that

$$H_3(\xi;q) = 1 + q + 2q^2 + 3\xi q^3 + (2+3\xi)q^4 + (1+6\xi)q^5 + \left(2+9\xi^2\right)q^6 + \left(6\xi+9\xi^2\right)q^7 + \left(1+3\xi+18\xi^2\right)q^8 + \ldots.$$

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and the three generating functions $H_3(a, 3; q)$ begin with the terms

$$H_3(0, 3; q) = 1 + q + 2q^2 + 2q^4 + q^5 + 2q^6 + q^8 + \dots,$$
$$H_3(1, 3; q) = 3q^3 + 3q^4 + 6q^5 + 6q^7 + 3q^8 + \dots,$$
$$H_3(2, 3; q) = 9q^6 + 9q^7 + 18q^8 + \dots.$$

Theorem 1.1 implies (independently of a) that

$$p_3(a, 3; n) \sim \frac{1}{12\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}} \sim \frac{1}{3} \cdot p(n).$$

The next table illustrates the conclusion of Corollary 1.2, that the proportions $\Psi_3(a, b; n) \to \frac{1}{3}$.

n	$\Psi_3(0, 3; n)$	$\Psi_3(1, 3; n)$	$\Psi_3(2, 3; n)$
100	≈ 0.4356	≈ 0.1639	≈ 0.4003
⋮	⋮	⋮	⋮
500	≈ 0.3234	≈ 0.3670	≈ 0.3096
600	≈ 0.3318	≈ 0.3114	≈ 0.3567
⋮	⋮	⋮	⋮
2100	≈ 0.3320	≈ 0.3348	≈ 0.3332
2300	≈ 0.3330	≈ 0.3345	≈ 0.3325
2500	≈ 0.3324	≈ 0.3337	≈ 0.3339

Example. We consider a typical case where the modular sums of t-hook functions are not equidistributed. We consider $t = 2$, where we have

$$H_2(\xi; q) = 1 + q + 2\xi q^2 + (1 + 2\xi)q^3 + 5\xi^2 q^4 + \left(2\xi + 5\xi^2\right)q^5 + \left(1 + 10\xi^3\right)q^6 + \left(5\xi^2 + 10\xi^3\right)q^7 + \left(2\xi + 20\xi^4\right)q^8 + \dots.$$

The three generating functions $H_2(a, 3; q)$ begin with the terms

$$H_2(0, 3; q) = 1 + q + q^3 + 11q^6 + 10q^7 + \dots,$$
$$H_2(1, 3; q) = 2q^2 + 2q^3 + 2q^5 + 22q^8 + \dots,$$
$$H_2(2, 3; q) = 5q^4 + 5q^5 + 5q^7 + \dots.$$

Theorem 1.1 implies that

$$p_2(a, 3; n) \sim \frac{A(a, n)}{12\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}} \sim \frac{A(a, n)}{3} \cdot p(n),$$

where $A(a, n) \in \{0, 1, 2\}$ satisfies the congruence $A(a, n) \equiv 2 - a - n \pmod 3$. This explains the uneven distribution established by Corollary 1.2 in this case. In particular, we have that

$$\lim_{n\to\infty} \frac{p_t(a, 3; 3n + 2 - a)}{p(n)} = 0.$$

Of course, this zero distribution is weaker than the vanishing obtained in Theorem 1.3.

The next table illustrates the uneven asymptotics for $n \equiv 0 \pmod 3$.

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n	$\Psi_2(0,3;n)$	$\Psi_2(1,3;n)$	$\Psi_2(2,3;n)$
300	≈ 0.7347	≈ 0.2653	0
⋮	⋮	⋮	⋮
600	≈ 0.6977	≈ 0.3022	0
900	≈ 0.6837	≈ 0.3163	0
⋮	⋮	⋮	⋮
4500	≈ 0.6669	≈ 0.3330	0
4800	≈ 0.6669	≈ 0.3330	0
5100	≈ 0.6668	≈ 0.3331	0

Example. We consider another typical case where the modular sums of t-hook functions are not equidistributed. We consider $t = 4$, where we have

$$H_4(\xi;q) = 1 + q + 2q^2 + 3q^3 + (1+4\xi)q^4 + (3+4\xi)q^5 + (3+8\xi)q^6 + (3+12\xi)q^7 + \left(4 + 4\xi + 14\xi^2\right)q^8 + \dots.$$

The three generating functions $H_4(a,3;q)$ begin with the terms

$$\begin{aligned} H_4(0,3;q) &= 1 + q + 2q^2 + 3q^3 + q^4 + 3q^5 + 3q^6 + 3q^7 + 4q^8 + \dots, \\ H_4(1,3;q) &= 4q^4 + 4q^5 + 8q^6 + 12q^7 + 4q^8 + \dots, \\ H_4(2,3;q) &= 14q^8 + \dots. \end{aligned}$$

Theorem 1.1, restricted to partitions of integers that are multiples of 12, gives

$$p_4(a,3;12n) \sim \begin{cases} \frac{4}{9} \cdot p(12n) & \text{if } a = 0, \\ \frac{1}{3} \cdot p(12n) & \text{if } a = 1, \\ \frac{2}{9} \cdot p(12n) & \text{if } a = 2. \end{cases}$$

The next table illustrates these asymptotics.

n	$\Psi_4(0,3;12n)$	$\Psi_4(1,3;12n)$	$\Psi_4(2,3;12n)$
10	≈ 0.4804	≈ 0.3373	≈ 0.1823
⋮	⋮	⋮	⋮
50	≈ 0.4500	≈ 0.3381	≈ 0.2119
60	≈ 0.4485	≈ 0.3373	≈ 0.2142
⋮	⋮	⋮	⋮
180	≈ 0.4447	≈ 0.3340	≈ 0.2212
190	≈ 0.4447	≈ 0.3339	≈ 0.2214
200	≈ 0.4446	≈ 0.3338	≈ 0.2215

5.2. Examples of Theorem 1.4 and Corollary 1.5

Finally, we consider examples of the asymptotics and distributions in the setting of Hilbert schemes on n points.

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Example. By Göttsche's Theorem (i.e., Theorem 4.1), we have

$$G(T;q) := \sum_{n=0}^{\infty} P\left(\left(\mathbb{C}^2\right)^{[n]};T\right)q^n = \prod_{m=1}^{\infty}\frac{1}{1-T^{2m-2}q^m} = \frac{1}{F_3(T^{-2};T^2q)}$$
$$= 1+q+\left(1+T^2\right)q^2+\left(1+T^2+T^4\right)q^3+\left(1+T^2+2T^4+T^6\right)q^4+\dots.$$

Theorem 1.4 (1) implies that

$$B\left(a,3;\left(\mathbb{C}^2\right)^{[n]}\right) \sim \frac{1}{12\sqrt{3}n}\cdot e^{\pi\sqrt{\frac{2n}{3}}},$$

and so Corollary 1.5 implies that $\delta(a,3;n) \to \frac{1}{3}$. The next table illustrates this phenomenon.

n	$\delta(0,3;n)$	$\delta(1,3;n)$	$\delta(2,3;n)$
1	1	0	0
2	0.5000	0	0.500
⋮	⋮	⋮	⋮
18	≈ 0.3377	≈ 0.3325	≈ 0.3299
19	≈ 0.3367	≈ 0.3306	≈ 0.3327
20	≈ 0.3333	≈ 0.3317	≈ 0.3349

Example. By Theorem 4.2, for $\alpha = 2$ and $\beta = 3$, we have

$$G_{2,3}(T;q) := \sum_{n=0}^{\infty} P\left(\left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{2,3}};T\right)q^n = \frac{1}{F_1(T^2;q^5)}\prod_{m=1}^{\infty}\frac{\left(1-q^{5m}\right)}{1-q^m}$$
$$= 1+q+2q^2+\dots+\left(6+T^2\right)q^5+\left(10+T^2\right)q^6+\left(13+2T^2\right)q^7+\dots.$$

Theorem 1.4 (2) implies that

$$B\left(a,3;\left(\left(\mathbb{C}^2\right)^{[n]}\right)^{T_{\alpha,\beta}}\right) \sim \frac{1}{12\sqrt{3}n}\cdot e^{\pi\sqrt{\frac{2n}{3}}},$$

so Corollary 1.5 yields that $\delta_{2,3}(a,3;n) \to \frac{1}{3}$. The next table illustrates this phenomenon.

n	$\delta_{2,3}(0,3;n)$	$\delta_{2,3}(1,3;n)$	$\delta_{2,3}(2,3;n)$
1	1	0	0
2	1	0	0
⋮	⋮	⋮	⋮
100	≈ 0.3693	≈ 0.2658	≈ 0.3649
200	≈ 0.3343	≈ 0.3176	≈ 0.3481
300	≈ 0.3313	≈ 0.3293	≈ 0.3393
400	≈ 0.3318	≈ 0.3324	≈ 0.3358
500	≈ 0.3324	≈ 0.3332	≈ 0.3343

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