LIMITING BETTI DISTRIBUTIONS OF HILBERT SCHEMES ON n POINTS

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Abstract. Hausel and Rodriguez-Villegas [7] recently observed that work of Göttsche, combined with a classical result of Erdős and Lehner on integer partitions, implies that the limiting Betti distribution for the Hilbert schemes $(\mathbb{C}^2)^{[n]}$ on n points, as $n \to +\infty$, is a *Gumbel distribution.* In view of this example, they ask for further such Betti distributions. We answer this question for the quasihomogeneous Hilbert schemes $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ that are cut out by torus actions. We prove that their limiting distributions are also of Gumbel type. To obtain this result, we combine work of Buryak, Feigin, and Nakajima on these Hilbert schemes with our generalization of the result of Erdős and Lehner, which gives the distribution of the number of parts in partitions that are multiples of a fixed integer $A \geq 2$. Furthermore, if $p_k(A;n)$ denotes the number of partitions of n with exactly k parts that are multiples of A, then we obtain the asymptotic

$$p_k(A,n) \sim \frac{24^{\frac{k}{2}-\frac{1}{4}}(n-Ak)^{\frac{k}{2}-\frac{3}{4}}}{\sqrt{2}\left(1-\frac{1}{A}\right)^{\frac{k}{2}-\frac{1}{4}} k! A^{k+\frac{1}{2}}(2\pi)^k} e^{2\pi\sqrt{\frac{1}{6}\left(1-\frac{1}{A}\right)(n-Ak)}},$$

a result which is of independent interest.

1. Introduction and statement of results

We consider the Hilbert schemes of n points on $\mathbb{C}^2$, denoted $X^{[n]} = (\mathbb{C}^2)^{[n]}$, that have been studied by Göttsche [9, 10], and Buryak, Feigin, and Nakajima [2, 3]. Each $X^{[n]}$ is a nonsingular, irreducible, quasiprojective dimension $2n$ algebraic variety. Moreover, they enjoy the convenient description

$$X^{[n]} = \{I \subset \mathbb{C}[x,y] \ : \ I \text{ is an ideal with } \dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = n\}, \tag{1.1}$$

which reduces the calculation of its Betti numbers to problems on integer partitions. To investigate these Betti numbers, it is natural to combine them to form the generating function

$$P\left(X^{[n]};T\right) := \sum_{j=0}^{2n-2} b_j(n)T^j = \sum_{j=0}^{2n-2} \dim\left(H_j\left(X^{[n]},\mathbb{Q}\right)\right)T^j, \tag{1.2}$$

known as its *Poincaré polynomial.* Due to the connection with integer partitions, it turns out that these polynomial generating functions equivalently keep track of the number of parts among the size n partitions.

In their work on the statistical properties of certain varieties, Hausel and Rodriguez-Villegas [7] observed that a classical result of Erdős and Lehner on partitions [4] gives (see Section 4.3 of [7]) the limiting distribution for the Betti numbers of $X^{[n]}$ as $n \to +\infty$. Using Göttsche's generating function [9, 10] for the $P(X^{[n]};T)$, it is straightforward to compute examples that offer glimpses of this result. For example, we find that

$$P\left(X^{[50]};T\right) = 1 + T^2 + 2T^4 + \cdots + 5427T^{88} + 2611T^{90} + 920T^{92} + 208T^{94} + 25T^{96} + T^{98}.$$

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The renormalized even degree[1] coefficients are plotted in Figure 1. As $P\left(X^{[50]};1\right)=p(50)$, the number of partitions of 50, the plot consists of the points $\left\{\left(\frac{2m}{98},\frac{b_{2m}(50)}{p(50)}\right)\ :\ 0\leq m\leq 49\right\}$.

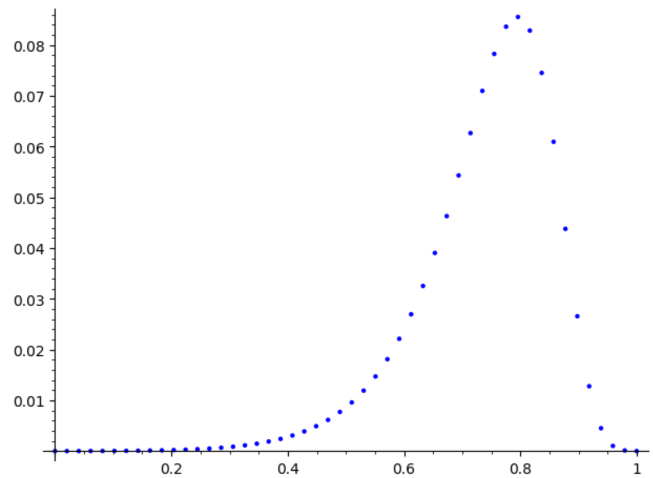

FIGURE 1. Betti distribution for $X^{[50]}$

These distributions, when properly renormalized, converge to a *Gumbel distribution* as $n\to+\infty$.

Hausel and Rodriguez-Villegas asked for further such n-aspect Betti distributions. We answer this question for the quasihomogeneous n point Hilbert schemes that are cut out by torus actions. To define them, we use the torus $(\mathbb{C}^{\times})^2$-action on $\mathbb{C}^2$ defined by scalar multiplication

$$(t_1,t_2)\cdot(x,y):=(t_1x,t_2y),$$

which lifts to $X^{[n]}=(\mathbb{C}^2)^{[n]}$. For relatively prime $\alpha,\beta\in\mathbb{N}$, we have the one-dimensional subtorus

$$T_{\alpha,\beta}:=\{(t^{\alpha},t^{\beta})\ :\ t\in\mathbb{C}^{\times}\}.$$

The quasihomogeneous Hilbert scheme $X^{[n]}_{\alpha,\beta}:=((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ is the fixed point set of $X^{[n]}$.

To define Betti distributions, we make use of the Poincaré polynomials

$$P\left(X^{[n]}_{\alpha,\beta};T\right):=\sum_{j=0}^{2\lfloor\frac{n}{\alpha+\beta}\rfloor}b_j(\alpha,\beta;n)T^j=\sum_{j=0}^{2\lfloor\frac{n}{\alpha+\beta}\rfloor}\dim\left(H_j\left(X^{[n]}_{\alpha,\beta},\mathbb{Q}\right)\right)T^j. \tag{1.3}$$

As $P\left(X^{[n]}_{\alpha,\beta};1\right)=p(n)$, we have that the discrete measure $d\mu^{[n]}_{\alpha,\beta}$ for $X^{[n]}_{\alpha,\beta}$ is

$$\Phi_n(\alpha,\beta;x):=\frac{1}{p(n)}\cdot\int_{-\infty}^{x}d\mu^{[n]}_{\alpha,\beta}=\frac{1}{p(n)}\cdot\sum_{j\leq x}b_j(\alpha,\beta;n). \tag{1.4}$$

The following theorem gives the limiting Betti distributions (as functions in x) we seek.

Theorem 1.1. *If α and β are relatively prime positive integers, then*

$$\lim_{n\to+\infty}\Phi_n(\alpha,\beta;2\sqrt{n}x+\delta_n(\alpha,\beta))=\exp\left(-\frac{\sqrt{6}}{\pi(\alpha+\beta)}\cdot\exp\left(-\frac{\pi(\alpha+\beta)}{\sqrt{6}}x\right)\right),$$

where $\delta_n(\alpha,\beta):=\frac{\sqrt{6}}{\pi(\alpha+\beta)}\sqrt{n}\log(n)$.

[1]The coefficients $b_{2j+1}(n)$ for odd degree terms identically vanish.

Two Remarks.
(1) The limiting cumulative distribution in Theorem 1.1 is of Gumbel type [5, 6]. Such distributions are often used to study the maximum (resp. minimum) of a number of samples of a random variable. Letting $A := \alpha + \beta$, we have mean $\frac{\sqrt{6}}{A\pi}\left(\log\left(\frac{\sqrt{6}}{A\pi}\right)+\gamma\right)$, where γ is the Euler-Mascheroni constant, and variance $1/A^2$.

(2) Gillman, Gonzalez, Schoenbauer and two of the authors studied a different kind of distribution for Hilbert schemes of surfaces in [8]. In that work equidistribution results were obtained for the Hodge numbers organized by congruence conditions.

Example. For example, let $\alpha = 1$ and $\beta = 2$. For $n = 20$, we have

$$P\left(X_{1,2}^{[20]};T\right) = 202 + 212T^2 + 126T^4 + 56T^6 + 22T^8 + 7T^{10} + 2T^{12}.$$

This small degree polynomial is not very suggestive. However, for $n = 1000$ the renormalized even degree[2] coefficients displayed in Figure 2 is quite illuminating. As $P\left(X_{1,2}^{[1000]};1\right) = p(1000)$, the plot consists of the 334 points $\left\{\left(\frac{2m}{666}, \frac{b_{2m}(1000)}{p(1000)}\right) \ : \ 0 \leq m \leq 333\right\}$.

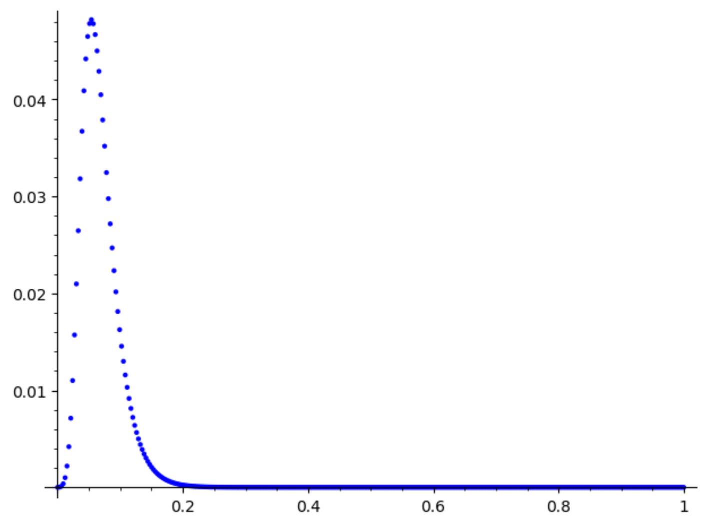

FIGURE 2. Betti distribution for $X_{1,2}^{[1000]}$

Theorem 1.1 gives the cumulative distribution corresponding to such plots as $n \to +\infty$. In this case, the theorem asserts that

$$\lim_{n\to+\infty} \Phi_n\left(1,2;2\sqrt{n}x + \frac{\sqrt{6n}}{3\pi}\cdot\log(n)\right) = \exp\left(-\frac{\sqrt{6}}{3\pi}\cdot\exp\left(-\frac{3\pi x}{\sqrt{6}}\right)\right).$$

Theorem 1.1 follows from a result which is of independent interest that generalizes a theorem of Erdős and Lehner on the distribution of the number of parts in partitions of fixed size. Using the celebrated Hardy-Ramanujan asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}}\cdot\exp(C\sqrt{n}),$$

[2] The odd degree coefficients terms identically vanish.

where $C := \pi\sqrt{2/3}$, Erdős and Lehner determined the distribution of the number of parts in partitions of size n. More precisely, if $k_n = k_n(x) := C^{-1}\sqrt{n}\log(n) + \sqrt{n}x$, they proved (see Theorem 1.1 of [4]) that

$$\lim_{n\to+\infty} \frac{p_{\leq k_n}(n)}{p(n)} = \exp\left(-\frac{2}{C}e^{-\frac{1}{2}Cx}\right), \tag{1.5}$$

where $p_{\leq k}(n)$ denotes[3] the number of partitions of n with at most k parts. In particular, the normal order for the number of parts of a partition of size n is $C^{-1}\sqrt{n}\log(n)$.

To prove Theorem 1.1, the generalization of the observation of Hausel and Rodriguez-Villegas, we require the distribution of the number of parts in partitions that are multiples of a fixed integer $A \geq 2$. The next theorem describes these distributions.

Theorem 1.2. *If $A \geq 2$ and $p_{\leq k}(A;n)$ denotes the number of partitions of n with at most k parts that are multiples of A, then for $k_{A,n} = k_{A,n}(x) := \frac{1}{AC}\sqrt{n}\log(n) + x\sqrt{n}$, we have*

$$\lim_{n\to+\infty} \frac{p_{\leq k_{A,n}}(A;n)}{p(n)} = \exp\left(-\frac{2}{AC}\exp\left(-\frac{1}{2}xAC\right)\right).$$

Remark. The distribution functions in Theorem 1.2 are of Gumbel type with mean $\frac{2}{AC}\left(\log\left(\frac{2}{AC}\right)+\gamma\right)$ and variance $1/A^2$.

Example. Here we illustrate Theorem 1.2 with $A = 2$ and $n = 600$. In this case we have

$$k_{2,600}(x) := \frac{\sqrt{600}\log(600)}{2C} + \sqrt{600}x.$$

For real numbers x, we let

$$\delta_{k_{2,600}}(x) := \frac{\#\{\lambda \vdash 600 \text{ with } \leq k_{2,n}(x) \text{ many even parts}\}}{p(n)}.$$

The theorem indicates that these proportions are approximated by the Gumbel values

$$G_{2,600}(x) := \exp\left(-\frac{1}{C}\cdot e^{-Cx}\right).$$

The table below illustrates the strength of these approximations for various values of x.

x	$\lfloor k_{2,600}(x)\rfloor$	$\delta_{k_{2,600}}(x)$	$G_{2,600}(x)$
-0.1	28	$0.597\ldots$	$0.604\ldots$
0.0	30	$0.663\ldots$	$0.677\ldots$
0.1	32	$0.721\ldots$	$0.739\ldots$
0.2	35	$0.791\ldots$	$0.792\ldots$
0.3	37	$0.830\ldots$	$0.835\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1.5	67	$0.994\ldots$	$0.992\ldots$
2.0	79	$0.998\ldots$	$0.998\ldots$

We note that Theorem 1.2 does not offer the asymptotics for $p_k(A;n)$, the number of partitions of n with exactly k parts that are multiples of A. For completeness, we offer such asymptotics, a result which is of independent interest. To make this precise, we recall the q-Pochhammer symbol

$$(a;q)_k := \prod_{n=0}^{k-1}(1-aq^n).$$

[3]We note that $p_{\leq k}(n)$ is denoted $p_k(n)$ in [4].

Theorem 1.3. *If $A \geq 2$ is an integer, then the following are true.*
(1) We have that $p_k(A;n)$ is the coefficient of $T^k q^n$ in the infinite product

$$\frac{(q^A;q^A)_\infty}{(q;q)_\infty (Tq^A;q^A)_\infty}.$$

(2) For every non-negative integer n, we have $p_k(A;n) = p_{\leq k}(A;n-Ak)$. Moreover, we have

$$\frac{(q^A;q^A)_\infty}{(q;q)_\infty (q^A;q^A)_k} = \sum_{n\geq 0} p_{\leq k}(A;n) q^n.$$

(3) For fixed k, as $n \to +\infty$, we have the asymptotic formulas

$$p_{\leq k}(A;n) \sim \frac{24^{\frac{k}{2}-\frac{1}{4}} n^{\frac{k}{2}-\frac{3}{4}}}{\sqrt{2}\left(1-\frac{1}{A}\right)^{\frac{k}{2}-\frac{1}{4}} k! A^{k+\frac{1}{2}} (2\pi)^k} e^{2\pi\sqrt{\frac{1}{6}\left(1-\frac{1}{A}\right)n}},$$

$$p_k(A;n) \sim \frac{24^{\frac{k}{2}-\frac{1}{4}} (n-Ak)^{\frac{k}{2}-\frac{3}{4}}}{\sqrt{2}\left(1-\frac{1}{A}\right)^{\frac{k}{2}-\frac{1}{4}} k! A^{k+\frac{1}{2}} (2\pi)^k} e^{2\pi\sqrt{\frac{1}{6}\left(1-\frac{1}{A}\right)(n-Ak)}}.$$

Example. Here we illustrate the convergence of the asymptotic for $p_1(3;n)$. Theorem 1.3 (3) gives

$$p_1(3;n) \sim \frac{1}{6\pi(n-3)^{\frac{1}{4}}} e^{\frac{2\pi\sqrt{n-3}}{3}}.$$

For convenience, we let $p_1^*(3;n)$ denote the right hand side of this asymptotic. The table below illustrates the convergence of the asymptotic.

n	$p_1(3;n)$	$p_1^*(3;n)$	$p_1(3;n)/p_1^*(3;n)$
200	93125823847	≈ 82738081118	≈ 1.126
400	$\approx 1.718 \times 10^{16}$	$\approx 1.579 \times 10^{16}$	≈ 1.088
600	$\approx 1.928 \times 10^{20}$	$\approx 1.799 \times 10^{20}$	≈ 1.071
800	$\approx 5.058 \times 10^{23}$	$\approx 4.764 \times 10^{23}$	≈ 1.062
1000	$\approx 5.232 \times 10^{26}$	$\approx 4.959 \times 10^{26}$	≈ 1.055

Table 1. Asymptotics for $p_1(3;n)$

This paper is organized as follows. In Section 2 we prove Theorem 1.2, the generalization of the classical limiting distribution (1.5) of Erdős and Lehner. In Section 3, we recall the work of Buryak, Feigin, and Nakajima [2, 3], which gives the infinite product generating functions for the Poincaré polynomials $P\left(X_{\alpha,\beta}^{[n]};T\right)$. These generating functions relate the Betti numbers to the partition functions $p_{\leq k}(\cdot)$. We use these facts, combined with Theorem 1.2, to obtain Theorem 1.1. Finally, in Section 4 we obtain Theorem 1.3, the asymptotic formulas for the $p_{\leq k}(A;n)$ and $p_k(A;n)$ partition functions. These asymptotics follow from an application of Ingham's Tauberian theorem.

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2. Generalization of a theorem of Erdős and Lehner

Here we prove Theorem 1.2. To prove the theorem we combine some elementary observations about integer partitions with delicate asymptotic analysis.

2.1. **Elementary considerations.** First we begin with an elementary convolution involving the partition functions $p_{\leq k}(A;\cdot)$, $p_{\leq k}(\cdot)$, and $p_{\mathrm{reg}}(A;n)$, the number of A-regular partitions of size n. Recall that a partition is A-regular if all of its parts are not multiples of A.

Proposition 2.1. *If $A \geq 2$ is a positive integer, then for every positive integer n we have*

$$p_{\leq k}(A;n) = \sum_{j=0}^{\lfloor \frac{n}{A} \rfloor} p_{\leq k}(j) \cdot p_{\mathrm{reg}}(A; n - Aj).$$

Proof. Every partition of n with at most k parts that are multiples of A can be represented as the direct product of an A-regular partition and a partition into at most k parts that are all multiples of A. If the sum of these multiples of A is Aj, then the A-regular partition has size $n - Aj$. Moreover, by dividing by A, the multiples of A are represented by a partition of j into at most k parts. This proves the claimed convolution. □

We also require an elegant inclusion-exclusion formula due to Erdős and Lehner [4] for $p_{\leq k}(n)$.

Proposition 2.2. *If k and j are positive integers, then*

$$p_{\leq k}(j) = \sum_{m=0}^{\infty} (-1)^m S_k(m;j),$$

where[4]

$$S_k(m;j) := \sum_{\substack{1 \leq r_1 < r_2 < \cdots < r_m \\ T_m \leq r_1 + r_2 + \cdots + r_m \leq j - mk}} p\left(j - \sum_{i=1}^{m} (k + r_i)\right) \tag{2.1}$$

and $T_m := m(m+1)/2$.

Proof. By definition, $p_{\leq k}(j)$ is the number of partitions of j with at most k parts. By considering conjugates of partitions, one can equivalently define $p_{\leq k}(j)$ as the number of partitions of j with no parts $\geq k+1$. Since the number of partitions of size j that contain a part of size $k + r$, where $r \geq 1$, equals $p(j - (k + r))$, we find that $S_k(1, j)$ is generally an overcount for the number of partitions of j with at least one part $\geq k + 1$. Due to this overcounting, we find that

$$p(j) - S_k(1;j) \leq p_{\leq k}(j) \leq p(j) - S_k(1;j) + S_k(2;j),$$

which is obtained by taking into account those partitions which have at least two parts of distinct size $\geq k + 1$. The claim follows in this way by inclusion-exclusion. □

[4]The $S_k(m;j)/p(j)$ are denoted S_m in [4].

2.2. **Proof of Theorem 1.2.** To prove Theorem 1.2, we require Propositions 2.1 and 2.2, and the asymptotics for $p_{\mathrm{reg}}(A;n)$. Thanks to the identity

$$\prod_{n=1}^{\infty}(1+q^n+q^{2n}+\cdots+q^{(A-1)n})=\prod_{n=1}^{\infty}\frac{(1-q^{An})}{(1-q^n)}=\sum_{n=0}^{\infty}p_{\mathrm{reg}}(A;n)q^n,$$

we find that $p_{\mathrm{reg}}(A;n)$ equals the number of partitions of n where no part occurs more than $A-1$ times. Hagis [11] obtained asymptotics for the number of partitions where no part is repeated more than t times, and letting $t=A-1$ in Corollary 4.2 of [11] gives the following theorem.

Theorem 2.3. *If $A\geq 2$, then we have*

$$p_{\mathrm{reg}}(A;n)=C_A(24n-1+A)^{-\frac{3}{4}}\exp\left(C\sqrt{\frac{A-1}{A}\left(n+\frac{A-1}{24}\right)}\right)\left(1+O(n^{-\frac{1}{2}})\right),$$

where $C:=\pi\sqrt{2/3}$ and $C_A:=\sqrt{12}A^{-\frac{3}{4}}(A-1)^{\frac{1}{4}}$, and the implied constant is independent of A.

Proof of Theorem 1.2. Thanks to Propositions 2.1 and 2.2, we have that

$$\frac{p_{\leq k}(A;n)}{p(n)}=\sum_{j=0}^{\lfloor\frac{n}{A}\rfloor}\frac{\left(\sum_{m=0}^{\infty}(-1)^m S_k(m;j)\right)p_{\mathrm{reg}}(A;n-Aj)}{p(n)}. \tag{2.2}$$

The proof follows directly from this expression by a sequence of observations involving the asymptotics for $p(\cdot)$ and $p_{\mathrm{reg}}(A;\cdot)$, combined with the earlier work of Erdős and Lehner on the sums $S_k(m;j)$. Thanks to the special choice of $k_n=k_n(x)$, this expression yields the Taylor expansion of the claimed cumulative Gumbel distribution in x, as $n\to+\infty$. In other words, these asymptotics conspire so that the dependence on n vanishes in the limit.

For convenience, we let $S^*_k(m;j):=S_k(m;j)/p(j)$. In terms of $S^*_k(m,j)$, (2.2) becomes

$$\frac{p_{\leq k}(A;n)}{p(n)}=\sum_{j=0}^{\lfloor\frac{n}{A}\rfloor}\frac{\left(\sum_{m=0}^{\infty}(-1)^m S^*_k(m;j)\right)p(j)p_{\mathrm{reg}}(A;n-Aj)}{p(n)}. \tag{2.3}$$

To make use of this formula, we begin by employing the method of Erdős and Lehner *mutatis mutandis*, which we briefly recapitulate here. For $k\to+\infty$, with j and m fixed, Erdős and Lehner proved (see (2.5) of [4]) that

$$S^*_k(m;j)=\frac{1}{m!}\left(\frac{2}{C}\sqrt{j}\exp\left(-\frac{C}{2\sqrt{j}}k\right)\right)^m+o_{j,m}(1). \tag{2.4}$$

For every positive integer m, this effectively gives

$$S^*_k(m;j)=\frac{1}{m!}\cdot S^*_k(1;j)^m+o_{j,m}(1)\sim\frac{1}{m!}\cdot S^*_k(1;j)^m,$$

which Erdős and Lehner show produces, as functions in x, the asymptotic

$$\sum_{m=0}^{\infty}(-1)^m S^*_{k_n}(m;j)=\exp(-S^*_{k_n}(1;j))\left(1+o_n(1)\right). \tag{2.5}$$

We recall the choice of $k=k_{A,n}=k_{A,n}(x)=\frac{1}{AC}\sqrt{n}\log(n)+x\sqrt{n}$. This is the exponential which arises in the exponential of the claimed cumulative distribution.

To make use of (2.5), it is convenient to recenter the sum on j in (2.3) by setting $j=\frac{n}{A^2}+y$. As (2.5) only involves $S^*_{k_n}(1;j)$, it suffices to note that when $m=1$, (2.4) becomes

$$S^*_{k_{A,n}}(1;j)=\frac{2}{AC}\sqrt{n+A^2y}\cdot\exp\left(-\frac{\log(n)}{2\sqrt{1+yA^2/n}}-\frac{xAC}{2\sqrt{1+yA^2/n}}\right)+o_n(1). \tag{2.6}$$

As the proof relies on (2.3), we must also estimate the quotients

$$\frac{p(j)p_{\mathrm{reg}}(A;n-Aj)}{p(n)}.$$

Thanks to the Hardy-Ramanujan asymptotic for $p(n)$ and Theorem 2.3, we have

$$\begin{aligned}
&\frac{p(j)p_{\mathrm{reg}}(A;n-Aj)}{p(n)}\\
&=\frac{C_A}{(24n-24Aj-1+A)^{\frac{3}{4}}}\frac{n}{j}\exp\left(C\left(\sqrt{j}-\sqrt{n}+\sqrt{\frac{A-1}{A}\left(n-Aj+\frac{A-1}{24}\right)}\right)\right)\cdot\left(1+O_j(n^{-\frac{1}{2}})\right)\\
&=\frac{C_A}{(24n-24n/A-24Ay-1+A)^{\frac{3}{4}}}\frac{A^2n}{n+A^2y}\\
&\qquad\times\exp\left(C\left(\sqrt{n/A^2+y}-\sqrt{n}+\sqrt{\frac{A-1}{A}\left(n-n/A-Ay+\frac{A-1}{24}\right)}\right)\right)\cdot\left(1+O_y(n^{-\frac{1}{2}})\right).
\end{aligned}\tag{2.7}$$

The last manipulation uses the change of variable for j.

We will make use of (2.5), (2.6) and (2.7) to complete the proof. To this end, we let $j=\lfloor n/A^2\rfloor+y$ essentially as above, but now modified[5] so that the y are integers. We then rewrite (2.3) as

$$\frac{p_{\le k_{A,n}}(A;n)}{p(n)}=\Sigma_1+\Sigma_2+\Sigma_3,$$

where Σ_1 is the sum over $-n/A^2\le y<-n^{3/4}\log(n)$, Σ_2 is the sum over $-n^{3/4}\log(n)\le y\le n^{3/4}\log(n)$, and Σ_3 is the sum over $n^{3/4}\log(n)\le y\le n(1/A-1/A^2)$. We shall show that the main contribution will come from Σ_2, and that Σ_1 and Σ_3 vanish as $n\to+\infty$.

To establish the vanishing of $\Sigma_1+\Sigma_3$, we consider the case that $|y|>n^{3/4}\log(n)$. For such y we have

$$\sqrt{n/A^2+y}-\sqrt{n}+\sqrt{\frac{A-1}{A}\left(n-n/A-Ay+\frac{A-1}{24}\right)}=O_y(\sqrt{n}),$$

where the implied constant is negative. Moreover, (2.6) implies that $S^*_{k_{A,n}}(1;n/A^2+y)=O(\sqrt{n})$, where the implied constant is positive. Thus, for y in these ranges, both $\frac{p(j)}{p(n)}p_{\mathrm{reg}}(A;n-Aj)$ and $\sum_{m=0}^{\infty}(-1)^mS^*_{k_{A,n}}(m;j)$ decay sub-exponentially, and so

$$\lim_{n\to\infty}\Sigma_1+\Sigma_3=0.$$

We now consider Σ_2, where $|y|\le n^{3/4}\log(n)$. In this range, (2.6) becomes

$$S^*_{k_{A,n}}(1;j)=\frac{2}{AC}\sqrt{n+A^2y}\cdot\exp\left(-\frac{\log(n)}{2\sqrt{1+yA^2/n}}-\frac{xAC}{2\sqrt{1+yA^2/n}}\right)+o_n(1)\tag{2.8}$$

$$=\frac{2}{AC}\exp\left(-\frac{1}{2}xAC\right)+o_n(1).\tag{2.9}$$

Using (2.5), we obtain

$$\sum_{m=0}^{\infty}(-1)^mS^*_{k_{A,n}}(m;j)=\exp\left(-\frac{2}{AC}\exp\left(-\frac{1}{2}xAC\right)\right)(1+o_n(1)).\tag{2.10}$$

[5]We can ignore the difference between $\lfloor n/A^2\rfloor$ with n/A^2 as it makes no difference for our limit calculations.

We now estimate (2.7) for these $|y| \leq n^{3/4}\log(n)$. Since we have

$$\sqrt{n/A^2+y}-\sqrt{n}+\sqrt{\frac{A-1}{A}\left(n-n/A-Ay+\frac{A-1}{24}\right)}=-\frac{A^4}{8(A-1)}y^2n^{-3/2}+O_A(y^3n^{-5/2}),$$

the hypothesis on y allows us to turn (2.7) into

$$\frac{p(j)p_{\mathrm{reg}}(A;n-Aj)}{p(n)}=\frac{A^2C_A}{(24n\frac{A-1}{A})^{3/4}}\times\exp\left(-C\frac{A^4}{8(A-1)}\frac{y^2}{n^{3/2}}\right)\cdot\left(1+O_A(n^{-\frac{1}{4}+\varepsilon})\right).$$

Combined with (2.10), and using $C_A=\sqrt{12}A^{-\frac{3}{4}}(A-1)^{\frac{1}{4}}$, we obtain

$$\lim_{n\to\infty}\Sigma_2$$
$$=\lim_{n\to\infty}\sum_{|y|<n^{3/4}\log(n)}\frac{A^2}{96^{1/4}\sqrt{A-1}}\cdot\frac{1}{n^{3/4}}\cdot\exp\left(-\frac{CA^4}{8(A-1)}\frac{y^2}{n^{3/2}}-\frac{2}{AC}\exp\left(-\frac{1}{2}xAC\right)\right)\cdot(1+o_A(1)).$$

Approximating the right hand side as a Riemann sum, we obtain

$$\lim_{n\to+\infty}\Sigma_2=\lim_{n\to+\infty}\frac{A^2}{96^{1/4}\sqrt{A-1}}\int_{-\log(n)}^{\log(n)}\exp\left(-\frac{CA^4}{8(A-1)}t^2-\frac{2}{AC}\exp\left(-\frac{1}{2}xAC\right)\right)dt, \tag{2.11}$$

where n only appears in the limits of integration. To obtain this, we have used the substitutions $t=yn^{-3/4}$ and $dt=n^{-3/4}dy$, and employ the fact that the widths of the subintervals defining the Riemann sums tend to 0. Expanding as an integral over $\mathbb{R}$, this expression simplifies to

$$\exp\left(-\frac{2}{AC}\exp\left(-\frac{1}{2}xAC\right)\right).$$

Therefore, as a function in x, we have

$$\lim_{n\to+\infty}\frac{p_{\leq k_{A,n}}(A;n)}{p(n)}=\exp\left(-\frac{2}{AC}\exp\left(-\frac{1}{2}xAC\right)\right).$$

This completes the proof of the theorem. □

3. Application to the Hilbert schemes $X_{\alpha,\beta}^{[n]}$

Here we recall the relevant generating functions for the Poincaré polynomials of the Hilbert schemes that pertain to Theorem 1.1. For the various Hilbert schemes on n points, Göttsche, Buryak, Feigin, and Nakajima [2, 3, 9, 10] proved infinite product generating functions for these Poincaré polynomials. For Theorem 1.1, we require the following theorem.

Theorem 3.1. (Buryak and Feigin) *If $\alpha,\beta\in\mathbb{N}$ are relatively prime, then we have that*

$$G_{\alpha,\beta}(T;q):=\sum_{n=0}^{\infty}P\left(X_{\alpha,\beta}^{[n]};T\right)q^n=\frac{(q^{\alpha+\beta};q^{\alpha+\beta})_\infty}{(q;q)_\infty(T^2q^{\alpha+\beta};q^{\alpha+\beta})_\infty}.$$

Remark. The Poincaré polynomials in these cases only have even degree terms (i.e. odd index Betti numbers are zero). Moreover, letting $T=1$ in these generating functions give Euler's generating function for $p(n)$. Therefore, we directly see that

$$p(n)=P\left(X_{\alpha,\beta}^{[n]};1\right).$$

Of course, the proof of Theorem 3.1 begins with partitions of size n.

Corollary 3.2. *Assuming the notation and hypotheses above, if $d\mu_{\alpha,\beta}^{[n]}$ is the discrete measure for $X_{\alpha,\beta}^{[n]}$, then*

$$\Phi_n(\alpha,\beta;x) = \frac{1}{p(n)} \cdot \int_{-\infty}^{x} d\mu_{\alpha,\beta}^{[n]} = \frac{p_{\leq \frac{x}{2}}(\alpha+\beta;n)}{p(n)}.$$

Proof. By Theorem 3.1, the Poincaré polynomial $P\left(X_{\alpha,\beta}^{[n]};T\right)$ is the coefficient of q^n of

$$\frac{(q^{\alpha+\beta};q^{\alpha+\beta})_\infty}{(q;q)_\infty (T^2 q^{\alpha+\beta};q^{\alpha+\beta})_\infty}.$$

Part (1) of Theorem 1.3 applied to $A = \alpha+\beta$ gives that the coefficient of T^{2k} in this expression is $p_k(\alpha+\beta;n)$ (the odd powers of T do not appear in this product as it is a function of T^2). Therefore, (1.3) becomes

$$P\left(X_{\alpha,\beta}^{[n]};T\right) = \sum_{j=0}^{\lfloor \frac{n}{\alpha+\beta} \rfloor} p_j(\alpha+\beta;n)T^{2j} = \sum_{j=0}^{2\lfloor \frac{n}{\alpha+\beta} \rfloor} \dim\left(H_j\left(X_{\alpha,\beta}^{[n]},\mathbb{Q}\right)\right)T^j.$$

Thus, the sum of coefficients up to x, divided by $p(n)$, is

$$\frac{1}{p(n)} \cdot \sum_{j\leq x} b_j(\alpha,\beta;n) = \frac{1}{p(n)} \cdot \sum_{j \leq x/2} p_j(\alpha+\beta;n) = \frac{p_{\leq \frac{x}{2}}(\alpha+\beta;n)}{p(n)}.$$

This completes the proof. □

Proof of Theorem 1.1. To prove Theorem 1.1, we remind the reader that Theorem 1.2 gives the cumulative asymptotic distribution function for $p_{\leq k}(A;n)$ when $A \geq 2$. Corollary 3.2, with $A = \alpha+\beta$, identifies this partition distribution with the Betti distribution for the n point Hilbert schemes cut out by the α,β torus action. The theorem follows by combining these two results. □

4. Asymptotic formulae for the $p_k(A;n)$ partition functions

Here we prove Theorem 1.3. To this end, we make use of Ingham's Tauberian theorem [12]. We note that this theorem is misstated in a number of places in the literature. Condition (3) in the statement below is often omitted. The reader is referred to the discussion in [1]. Here we use a special case[6] of Theorem 1.1 of [1].

Theorem 4.1 (Ingham)**.** *Let $f(q) = \sum_{n\geq 0} a(n)q^n$ be a holomorphic function in the unit disk $|q| < 1$ satisfying the following conditions:*
(1) The sequence $\{a(n)\}_{n\geq 0}$ is positive and weakly monotonically increasing.
(2) There exist $c \in \mathbb{C}$, $d \in \mathbb{R}$, and $N > 0$, such that as $t \to 0^+$ we have

$$f(e^{-t}) \sim \lambda \cdot t^d \cdot e^{\frac{N}{t}}.$$

(3) For any $\Delta > 0$, in the cone $|y| \leq \Delta x$ with $x > 0$ and $z = x+iy$, we have, as $z \to 0$

$$f(e^{-z}) \ll |z|^d \cdot e^{\frac{N}{|z|}}.$$

Then as $n \to +\infty$ we have

$$a(n) \sim \frac{\lambda \cdot N^{\frac{d}{2}+\frac{1}{4}}}{2\sqrt{\pi} \cdot n^{\frac{d}{2}+\frac{3}{4}}} e^{2\sqrt{Nn}}.$$

[6]In the notation of [1], we let $d = \beta$, $N = \gamma$, and we let $\alpha = 0$ in the case of weak monotonicity of Theorem 1.1.

Proof of Theorem 1.3. We prove the claims one-by-one.
(1) We begin by recalling the q-Pochhammer symbol

$$(a;q)_k := \prod_{n=0}^{k-1}(1-aq^n).$$

Clearly, we have

$$\frac{(q^A;q^A)_\infty}{(q;q)_\infty} = \prod_{j=1}^{A-1}\frac{1}{(q^j;q^A)_\infty},$$

which in turn gives

$$\frac{(q^A;q^A)_\infty}{(q;q)_\infty(Tq^A;q^A)_\infty} = \prod_{n\not\equiv 0 \pmod{A}}\frac{1}{1-q^n} \quad\times \prod_{n\equiv 0 \pmod{A}}\frac{1}{1-Tq^n}.$$

Expanding each term as a geometric series, we find that the coefficient of T^k collects those partitions which have k parts which are 0 (mod A).
(2) We make use of the q-binomial theorem, which asserts that

$$\sum_{n\geq 0}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

Hence, if we let $[T^k]$ denote the coefficient of T^k, this theorem allows us to conclude that

$$\frac{(q^A;q^A)_\infty}{(q;q)_\infty}[T^k]\left(\frac{1}{(Tq^A;q^A)_\infty}\right) = \frac{(q^A;q^A)_\infty}{(q;q)_\infty}[T^k]\left(\sum_{n\geq 0}\frac{(Tq^A)^n}{(q^A;q^A)_n}\right) = \frac{q^{Ak}(q^A;q^A)_\infty}{(q;q)_\infty(q^A;q^A)_k}.$$

Arguing as in the proof of (1), we find the claimed generating function identity

$$\frac{(q^A;q^A)_\infty}{(q;q)_\infty(q^A;q^A)_k} = \sum_{n\geq 0}p_{\leq k}(A;n)q^n. \tag{4.1}$$

These two q-series identities, combined with (1), imply that $p_k(A;n) = p_{\leq k}(A;n-Ak)$.
(3) To establish the desired asymptotics, we apply Theorem 4.1 to (4.1), which is facilitated by the modularity of Dedekind's eta-function

$$\eta(\tau) := q^{\frac{1}{24}}(q;q)_\infty.$$

This function is well-known to satisfy

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau).$$

As a consequence of this transformation and the q-expansion $\eta(\tau) = q^{\frac{1}{24}} + O(q^{\frac{25}{24}})$ near $\tau = i\infty$ (for example, see p. 53 of [14]), for $q = e^{-t}$, $t \to 0^+$, we find that

$$\log\left(\frac{1}{(q;q)_\infty}\right) = \frac{\pi^2}{6t} - \frac{1}{2}\log\left(\frac{2\pi}{t}\right) + O(t). \tag{4.2}$$

Thus, letting $t \mapsto At$ and taking a difference yields

$$\log\left(\frac{(q^A;q^A)_\infty}{(q;q)_\infty}\right) = \frac{\pi^2}{6t}\left(1-\frac{1}{A}\right) - \frac{\log(A)}{2} + O(t). \tag{4.3}$$

This calculation gives the behavior in the radial limit as $t \to 0^+$ of the infinite Pochhammer symbols in (4.1).

To satisfy condition (3) of Theorem 4.1, we also need to estimate the quotient on the left hand side of (4.3) for the regions $|y| \leq \Delta x$. This is given directly in Section 3.1 of [1]. Namely, they show that in these regions, one has

$$\frac{1}{(e^{-z};e^{-z})_\infty} = \sqrt{\frac{z}{2\pi}} \cdot \; e^{\frac{\pi^2}{6z}} \left(1 + O_\Delta\left(\left|e^{-\frac{4\pi^2}{z}}\right|\right)\right)$$

and

$$e^{-\frac{1}{z}} \leq e^{-\frac{1}{(1+\Delta^2)|z|}}.$$

Thus, we have

$$\frac{1}{(e^{-z};e^{-z})_\infty} = \sqrt{\frac{z}{2\pi}} \cdot \; e^{\frac{\pi^2}{6z}} \left(1 + O_\Delta\left(e^{-\frac{4\pi^2}{(1+\Delta^2)|z|}}\right)\right). \tag{4.4}$$

Changing variables to let $z \mapsto Az$, we then find

$$\frac{(e^{-Az};e^{-Az})_\infty}{(e^{-z};e^{-z})_\infty} = \sqrt{A} \cdot e^{-\frac{\pi^2}{6z}\left(1-\frac{1}{A}\right)} \cdot \frac{\left(1 + O_\Delta\left(e^{-\frac{4\pi^2}{A(1+\Delta^2)|z|}}\right)\right)}{\left(1 + O_\Delta\left(e^{-\frac{4\pi^2}{(1+\Delta^2)|z|}}\right)\right)} = \sqrt{A} \cdot e^{-\frac{\pi^2}{6z}\left(1-\frac{1}{A}\right)} \left(1 + O_\Delta\left(e^{-\frac{4\pi^2}{A(1+\Delta^2)|z|}}\right)\right). \tag{4.5}$$

Now we turn to estimating the remaining factor in (4.1), namely, $1/(q^A;q^A)_k$. On the line $t \to 0^+$, an important result of Zhang (see Theorem 2 of [15]) gives that for $0 < t \to 0$ and $w \in \mathbb{C}$,

$$(e^{-wt};e^{-t})_\infty \sim \frac{\sqrt{2\pi}}{\Gamma(w)} e^{-\frac{\pi^2}{6t} - \left(w - \frac{1}{2}\right)\log(t)}.$$

Letting $w = k+1$ and combining with (4.2), we conclude that

$$\frac{1}{(q;q)_k} = \frac{(q^{k+1};q)_\infty}{(q;q)_\infty} \sim \frac{\sqrt{2}\pi}{k!} e^{-\frac{\pi^2}{6\varepsilon} - (k+1/2)\log(t) + \frac{\pi^2}{6t} - \frac{1}{2}\log(2\pi/t)} = \frac{t^{-k}}{k!}.$$

Letting $t \mapsto At$, we have

$$\frac{1}{(q^A;q^A)_k} \sim \frac{1}{k!A^k} t^{-k}. \tag{4.6}$$

Turning to estimate $1/(q^A;q^A)_k$ in the regions $|y| \leq \Delta x$, we use the same argument in the proof of Theorem 2 of [15]. One merely modifies the proof by replacing x with $|z|$ in Zhang's setting to obtain

$$(e^{-A(k+1)z};e^{-Az})_\infty \ll \frac{\sqrt{2\pi}}{k!} e^{-\frac{\pi^2}{6|z|} - \left(k+1-\frac{1}{2}\right)\log|z|},$$

as $z \to 0$. Moreover, by combining with (4.4), we have

$$\frac{1}{(e^{-Az};e^{-Az})_k} = \frac{(e^{-A(k+1)z};e^{-Az})_\infty}{(e^{-z};e^{-z})_\infty} \ll \frac{|z|^{-k}}{k!}. \tag{4.7}$$

Then multiplying (4.5) and (4.7), we find that

$$\frac{(e^{-Az};e^{-Az})_\infty}{(e^{-z};e^{-z})_\infty (e^{-Az};e^{-Az})_k} \ll \frac{\sqrt{A}}{k!} |z|^{-k} e^{\frac{\pi^2}{6|z|}\left(1-\frac{1}{A}\right)}, \tag{4.8}$$

which shows that condition (3) of Theorem 4.1 is satisfied.

Multiplying (4.3) with (4.6), where $q := e^{-t}$, we obtain

$$\frac{(q^A;q^A)_\infty}{(q;q)_\infty (q^A;q^A)_k} \sim \frac{1}{k!A^{k+\frac{1}{2}}} t^{-k} e^{\frac{\pi^2}{6t}\left(1-\frac{1}{A}\right)}.$$

Moreover, the coefficients $\frac{(q^A;q^A)_\infty}{(q;q)_\infty (q^A;q^A)_k}$ are clearly positive as they count partitions. They are weakly increasing as there is an easy injection from the set of partitions of n with at most k parts which are multiples of A into the set of partitions of $n+1$ which have at most k parts which are multiples of A; simply add 1 to the partition, which doesn't affect the number of multiples of A among the parts.

We are thus in the situation of Theorem 4.1, where we interprete (4.8) with

$$\lambda = \frac{1}{k! A^{k+\frac{1}{2}}}, \quad d = -k, \quad N = \frac{\pi^2}{6}\left(1 - \frac{1}{A}\right).$$

Plugging these into the Theorem 4.1 gives the desired asymptotic for $p_{\leq k}(A;n)$. The asymptotics for $p_k(A;n)$ follows from the identity $p_k(A;n) = p_{\leq k}(A; n - Ak)$ obtained in (2).

$\square$

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