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Coherence as entropy increment for Tsallis and Rényi entropies

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Abstract
Relative entropy of coherence can be written as an entropy difference of the original state and the incoherent state closest to it when measured by relative entropy. The natural question is, if we generalize this situation to Tsallis or Rényi entropies, would it define good coherence measures? In other words, we define a difference between Tsallis entropies of the original state and the incoherent state closest to it when measured by Tsallis relative entropy. Taking Rényi entropy instead of the Tsallis entropy, leads to the well-known distance-based Rényi coherence, which means this expression defined a good coherence measure. Interestingly, we show that Tsallis entropy does not generate even a genuine coherence monotone, unless it is under a very restrictive class of operations. Additionally, we provide continuity estimate for Rényi coherence. Furthermore, we present two coherence measures based on the closest incoherent state when measures by Tsallis or Rényi relative entropy.

Keywords Coherence · Rényi entropy · Tsallis entropy · Genuine coherence

1 Introduction

Quantum coherence describes the existence of quantum interference, and it is often used in thermodynamics [1, 6, 15], transport theory [23, 34], and quantum optics [10, 25], among few applications. Recently, problems involving coherence included quantification of coherence [2, 18, 21, 22, 26, 37], distribution [20], entanglement [5, 29], operational resource theory [3, 5, 9, 33], correlations [13, 16, 30], with only a few references mentioned in each. See [28] for a more detailed review.

The golden standard for any "good" coherence measure is for it to satisfy four criteria presented in [2]: vanishing on incoherent states; monotonicity under

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incoherent operations; strong monotonicity under incoherent operations; and convexity. Alternatively, the last two properties can be substituted by an additivity for subspace-independent states, which was shown in [37]. See Preliminaries for more details.

A number of ways have been proposed as a coherence measure, but only a few satisfy all necessary criteria [2, 38, 39]. A broad class of coherence measures are defined as the minimal distance D to the set of incoherent states $\mathcal{I}$, as

$$\mathrm{CD}(\rho) = \min_{\delta \in \mathcal{I}} D(\rho, \delta).$$

Here, "distance" is understood in a rather broad term, more of a distinguishability measure. We discuss the properties it should satisfy in chapters below. It was shown in [2] that for a relative entropy there is a closed expression of a distance-based coherence:

$$\min_{\delta \in \mathcal{I}} S(\rho \| \delta) = S(\rho \| \Delta(\rho)) = S(\Delta(\rho)) - S(\rho) \tag{1.1}$$

where $\Delta(\rho)$ is the dephased state in a pre-fixed basis, see Notation 2.2.

Different sets of incoherent operations generate other physically relevant coherence measures. The largest set one considers is the set of incoherent operations (IO) [2], which have Kraus operators that each preserve the set of incoherent states (see Definition 2.3). A smaller set is called genuine incoherent operations (GIO) [8], which act trivially on incoherent states, see Definition 2.4. See [4] for a larger list of incoherent operations and their comparison. For these types of incoherent operations, one may look at similar properties as the ones presented in [2]. Restricted to GIO, one would obtain a measure of genuine coherence when it is non-negative and monotone, or a coherence monotone when it is also strongly monotone under GIO.

Motivated by the last expression in (1.1), similar expressions were considered in [7] for Tsallis and Rényi entropies:

$$S_\alpha^R(\Delta(\rho)) - S_\alpha^R(\rho)$$
$$S_\alpha^T(\Delta(\rho)) - S_\alpha^T(\rho).$$

It was found that these expressions define genuine coherence monotones (definition will come later). They have advantage over distance-based measures by being the explicit expressions, easy to calculate. Moreover, they can be regarded as measurement-induced entropy increment related to the quantum thermodynamics [14].

In [31], the following generalized genuine coherence monotone was proposed:

$$\mathcal{C}_f(\rho) = S_f(\Delta(\rho)) - S_f(\rho)$$

where $S_f(\rho)$ is a quasi-entropy, which could be defined in two ways, one of which is $S_f(\rho) = -S_f(\rho \| I)$.

Here, we show the operational meaning of this f-coherence, by showing that it is not possible to distill a higher coherence states from a lower coherence state via

GIO (Theorem 3.4). To prove this result, we first show the continuity of f-coherence (Theorem 3.2).

If one looks at (1.1) again, the last expression is the difference in entropies of the state ρ and its closest incoherent state $\Delta(\rho)$, when measured by the relative entropy. So we ask a question, if we change the entropy and relative entropy in this expression to the Tsallis ones, would that generate a good coherence monotone/measure? Note that this change will change the closest incoherent state as well. In other words, we investigate the properties of the following Tsallis coherence

$$\mathrm{CT}_\alpha(\rho) := S_\alpha^T(\Delta_\alpha(\rho)) - S_\alpha^T(\rho)$$

where $\Delta_\alpha(\rho)$ is the closest incoherent state to ρ when measured by Tsallis relative entropy, i.e.,

$$S_\alpha^T(\rho\|\Delta_\alpha(\rho)) := \min_{\delta\in\mathcal{I}} S_\alpha^T(\rho\|\delta).$$

The explicit form of Δ_α is given in [22], and it is the same for Rényi and Tsallis relative entropies.

Surprisingly, taking Rényi entropies above leads to the well-known distance-based Rényi coherence:

$$\begin{aligned}\mathrm{CR}_\alpha(\rho) &= \min_{\delta\in\mathcal{I}} S_\alpha^R(\rho\|\delta)\\ &= S_\alpha^R(\rho\|\Delta_\alpha(\rho))\\ &= S_\alpha^R(\Delta_\alpha(\rho)) - S_\alpha^R(\rho).\end{aligned}$$

We provide a continuity estimate for this Rényi coherence 4.1.

This means that the entropy increment for von Neumann entropy (with relative entropy) and Rényi entropy is good coherence measures; however, we show that a similar Tsallis entropy does not lead even to a good genuine coherence monotone. It is a coherence monotone under a very restrictive class of operations.

At the end, we propose two new coherence measures, inspired by the expression for the closest incoherent state when measured by the Tsallis or Rényi relative entropy.

2 Preliminaries

2.1 Coherence

Let $\mathcal{H}$ be a d-dimensional Hilbert space. Let us fix an orthonormal basis $\mathcal{E} = \{|j\rangle\}_{j=1}^d$ of vectors in $\mathcal{H}$.

Definition 2.1 A state δ is called *incoherent* if it can be represented as follows: $\delta = \sum_j \delta_j |j\rangle\langle j|$.

Notation 2.2 *Denote the set of incoherent states for a fixed basis* $\mathcal{E} = \{|j\rangle\}_j$ *as* $\mathcal{I} = \{\rho = \sum_j p_j |j\rangle \langle j|\}$. *A dephasing operation in* $\mathcal{E}$ *basis is the following map:*

$$\Delta(\rho) = \sum_j \langle j| \rho |j\rangle |j\rangle \langle j| .$$

Definition 2.3 A CPTP map Φ with the following Kraus operators

$$\Phi(\rho) = \sum_n K_n \rho K_n^*$$

is called *the incoherent operation (IO)* or incoherent CPTP (ICPTP), when the Kraus operators satisfy

$$K_n \mathcal{I} K_n^* \subset \mathcal{I}, \text{ for all } n$$

besides the regular completeness relation $\sum_n K_n^* K_n = \mathbb{1}$.

Considering each K_n, in [36] it was shown that condition $K_n \mathcal{I} K_n^* \subset \mathcal{I}$ implies that there exists at most one nonzero entry in every column of K_n.

Any reasonable measure of coherence $\mathcal{C}(\rho)$ should satisfy the following conditions:

- (C1) $\mathcal{C}(\rho) \geq 0$, and $\mathcal{C}(\rho) = 0$ if and only if $\rho \in \mathcal{I}$;
- (C2) Non-selective monotonicity under IO (monotonicity): for all IO Φ and all states ρ,

$$\mathcal{C}(\rho) \geq \mathcal{C}(\Phi(\rho))$$

- (C3) Selective monotonicity under IO (strong monotonicity): for all IO Φ with Kraus operators K_n, and all states ρ,

$$\mathcal{C}(\rho) \geq \sum_n p_n \mathcal{C}(\rho_n)$$

where p_n and ρ_n are the outcomes and post-measurement states

$$\rho_n = \frac{K_n \rho K_n^*}{p_n}, \quad p_n = \mathrm{Tr} K_n \rho K_n^*.$$

- (C4) Convexity,

$$\sum_n p_n \mathcal{C}(\rho_n) \geq \mathcal{C}\left(\sum_n p_n \rho_n\right)$$

for any sets of states $\{\rho_n\}$ and any probability distribution $\{p_n\}$.

Conditions (C3) and (C4) together imply (C2) [2].

Alternatively, instead of the last two conditions, one can impose the following one:

- (C5) Additivity for subspace-independent states: For $p_1 + p_2 = 1$, $p_1, p_2 \geq 0$, and any two states ρ_1 and ρ_2,

$$\mathcal{C}(p_1\rho_1 \oplus p_2\rho_2) = p_1\mathcal{C}(\rho_1) + p_2\mathcal{C}(\rho_2).$$

In [37], it was shown that (C3) and (C4) are equivalent to (C5) condition.

These properties are parallel with the entanglement measure theory, where the average entanglement is not increased under the local operations and classical communication (LOCC). Notice that coherence measures that satisfy conditions (C3) and (C4) also satisfy condition (C2).

In [8], a class of incoherent operations was defined, called genuinely incoherent operations (GIO) as quantum operations that preserve all incoherent states.

Definition 2.4 An IO map Λ is called a *genuinely incoherent operation (GIO)* which is for any incoherent state $\delta \in \mathcal{I}$,

$$\Lambda(\delta) = \delta.$$

Additionally, it was shown that an operation Λ is GIO if and only if all Kraus representations of Λ have all Kraus operators diagonal in a pre-fixed basis [8].

Conditions (C2), (C3), and (C4) can be restricted to GIO and obtain different classes of coherence measures.

Definition 2.5 In this case, a *genuine coherence monotone* satisfies at least (C1) and (C2). And if a coherence measure fulfills conditions (C1), (C2), and (C3), it is called *measure of genuine coherence*.

A larger class than GIO, called SIO, was defined in [33, 35].

Definition 2.6 An IO Λ is called *strictly incoherent operation (SIO)* if its Kraus representation operators commute with dephasing, i.e., for $\Lambda(\rho) = \sum_j K_j \rho K_j^*$, we have for any j,

$$K_j \Delta(\rho) K_j^* = \Delta(K_j \rho K_j^*).$$

Since Kraus operators of GIO are diagonal in $\mathcal{E}$ basis, any GIO map is SIO as well, i.e., GIO $\subset$ SIO, [8].

A class of operators generalizing SIO, called DIO, was introduced in [3].

Definition 2.7 An IO Λ is called *dephasing-incoherent operation (DIO)* if it itself commute with dephasing operator, i.e.,

$$\Lambda(\Delta(\rho)) = \Delta(\Lambda(\rho)).$$

Thus, we have GIO $\subset$ SIO $\subset$ DIO.

One may consider an additional property, closely related to the entanglement theory:

- (C6) Uniqueness for pure states: for any pure state $|\psi\rangle$ coherence takes the form:

$$\mathcal{C}(\psi) = S(\Delta(\psi))$$

where S is the von Neumann entropy and Δ is the dephasing operation defined as

$$\Delta(\rho) = \sum_j \langle j|\,\rho\,|j\rangle\,|j\rangle\,\langle j|\,.$$

However, for other coherence measures the von Neumann entropy in (C6) may change to another one, and the dephased state may also change to another free state.

2.2 Rényi and Tsallis coherences

As mentioned before, relative entropy of coherence can be defined using three expressions:

$$C(\rho) = \min_{\delta\in\mathcal{I}} S(\rho\|\delta) = S(\rho\|\Delta(\rho)) = S(\Delta(\rho)) - S(\rho). \tag{2.1}$$

Let us point out that $\Delta(\rho)$ is the closest incoherent state to ρ when measured by relative entropy, which was shown in [2].

Recall, that Tsallis entropy is defined as for $\alpha \in (0, 2]$

$$S_\alpha^T(\rho) = \frac{1}{1-\alpha}\left[\mathrm{Tr}\rho^\alpha - 1\right]$$

Tsallis relative entropy is defined as

$$S_\alpha^T(\rho\|\delta) = \frac{1}{\alpha-1}\left[\mathrm{Tr}\left(\rho^\alpha\delta^{1-\alpha}\right) - 1\right].$$

Rényi entropy is defined as for $\alpha \in (0, \infty)$

$$S_\alpha^R(\rho) = \frac{1}{1-\alpha}\log\mathrm{Tr}\rho^\alpha$$

and Rényi relative entropy is defined as

$$S_\alpha^R(\rho\|\delta) = \frac{1}{\alpha-1}\log\mathrm{Tr}\left(\rho^\alpha\delta^{1-\alpha}\right).$$

Motivated by different forms involved in the definition of relative entropy of coherence (2.1), Rényi coherence has been defined as

$$\mathrm{CR}_\alpha^1(\rho) = \min_{\delta\in\mathcal{I}} S_\alpha^R(\rho\|\delta), \tag{2.2}$$

$$\mathrm{CR}_\alpha^2(\rho) = S_\alpha^R(\Delta(\rho)) - S_\alpha^R(\rho), \tag{2.3}$$

$$\mathrm{CR}_\alpha^3(\rho) = S_\alpha^R(\rho\|\Delta(\rho)). \tag{2.4}$$

The first definition CR_α^1 is a particular case of any distance-based coherence [2] and was separately discussed in [27]. The second definition CR_α^2 was introduced in [7]. The third definition CR_α^3 was introduced in [4].

Similarly, Tsallis coherence has been defined as

$$\mathrm{CT}_\alpha^1(\rho) = \min_{\delta\in\mathcal{I}} S_\alpha^T(\rho\|\delta), \tag{2.5}$$

$$\mathrm{CT}_\alpha^2(\rho) = S_\alpha^T(\Delta(\rho)) - S_\alpha^T(\rho). \tag{2.6}$$

The first definition CT_α^1 is a particular case of any distance-based coherence [2]. The second definition CT_α^2 was introduced in [7].

These definitions are all different, in particular, due to the fact that the closest incoherent state to a state ρ, when measured by either Rényi or Tsallis relative entropy, is not a state $\Delta(\rho)$. From [4, 22], the closest incoherent state to a state ρ for either Rényi or Tsallis relative entropies is

$$\Delta_\alpha(\rho) = \frac{1}{N(\rho)} \sum_j \langle j|\,\rho^\alpha\,|j\rangle^{1/\alpha}\,|j\rangle\,\langle j| \;\in \mathcal{I} \tag{2.7}$$

where $N(\rho) = \sum_j \langle j|\,\rho^\alpha\,|j\rangle^{1/\alpha}$. The corresponding relative entropy becomes

$$\mathrm{CT}_\alpha^1(\rho) = S_\alpha^T(\rho\|\Delta_\alpha(\rho)) = \frac{1}{\alpha-1}\left[N(\rho)^\alpha - 1\right] \tag{2.8}$$

and

$$\mathrm{CR}_\alpha^1(\rho) = S_\alpha^R(\rho\|\Delta_\alpha(\rho)) = \frac{\alpha}{\alpha-1}\log N(\rho). \tag{2.9}$$

Interestingly, enough difference-based Tsallis coherence when $\alpha = 2$ is related to the distance-based coherence induced by the Hilbert–Schmidt distance [8]

$$C_2^{HS}(\rho) := \min_{\delta\in\mathcal{I}} \|\rho-\delta\|_2^2 = S_2^T(\Delta(\rho)) - S_2^T(\rho)$$

where $\|\rho-\delta\|_2^2 = \mathrm{Tr}(\rho-\delta)^2$.

2.3 Generalized coherences

Any proper distance $D(\rho,\sigma)$ between two quantum states can induce a potential candidate for coherence. The distance-based coherence measure is defined as follows [2]:

Definition 2.8

$$\mathrm{CD}(\rho) := \min_{\delta\in\mathcal{I}} D(\rho,\delta)$$

i.s. the minimal distance between the state ρ and the set of incoherent states $\mathcal{I}$ measured by the distance D.

- (C1) is satisfied whenever $D(\rho,\delta)=0$ iff $\rho=\delta$.
- (C2) is satisfied whenever D is contracting under CPTP maps, i.e., $D(\rho,\sigma)\geq D(\Phi(\rho),\Phi(\sigma))$.
- (C4) is satisfied whenever D is jointly convex.

Since the relative entropy and Rényi and Tsallis relative entropies satisfy all three above conditions for $\alpha\in[0,1)$, (C1), (C2), and (C4) are satisfied for $C(\rho)$, CR^1_α, CT^1_α.

Another generalization was considered in [31], which is based on quasi-relative entropy.

Definition 2.9 For strictly positive bounded operators A and B acting on a finite-dimensional Hilbert space $\mathcal{H}$, and for any continuous function $f:(0,\infty)\to\mathbb{R}$, *the quasi-relative entropy* (or sometimes referred to as *the f-divergence*) is defined as

$$S_f(A||B)=\mathrm{Tr}(f(L_BR_A^{-1})A)$$

where left and right multiplication operators are defined as $L_B(X)=BX$ and $R_A(X)=XA$.

Having the spectral decomposition of operators, one can calculate the quasi-relative entropy explicitly [12, 32]. Let A and B have the following spectral decomposition

$$A=\sum_j\lambda_j\left|\phi_j\right\rangle\!\left\langle\phi_j\right|,\quad B=\sum_k\mu_k\left|\psi_k\right\rangle\left\langle\psi_k\right|. \tag{2.10}$$

Here, the sets $\{|\phi_k\rangle\left\langle\psi_j\right|\}_{j,k}$, $\{|\psi_k\rangle\left\langle\psi_j\right|\}_{j,k}$ form orthonormal bases of $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators. By [32], the quasi-relative entropy is calculated as follows:

$$S_f(A||B)=\sum_{j,k}\lambda_j f\left(\frac{\mu_k}{\lambda_j}\right)|\left\langle\psi_k\right|\left|\phi_j\right\rangle|^2. \tag{2.11}$$

Assumption 2.10 To define f-coherence, we assume that the function f is operator convex and operator monotone decreasing and $f(1)=0$.

f-entropy was defined in two ways in [31]

$$S^1_f(\rho):=-S_f(\rho\|I)=-\sum_j\lambda_j f\left(\frac{1}{\lambda_j}\right) \tag{2.12}$$

$$S_f^2(\rho) := f(1/d) - S_f(\rho\|I/d) = f(1/d) - \sum_j \lambda_j f\left(\frac{1}{d\lambda_j}\right), \tag{2.13}$$

where $\{\lambda_j\}_j$ are the eigenvalues of ρ.

Definition 2.11 For either f-entropy, the f-coherence is then defined as

$$C_f(\rho) := S_f(\Delta(\rho)) - S_f(\rho). \tag{2.14}$$

If $\{\lambda_j\}$ are the eigenvalues of ρ, and the diagonal elements of ρ in $\mathcal{E}$ basis are $\chi_j = \langle j|\rho|j\rangle$, then from (2.12), we have

$$C_f^1(\rho) = \sum_j \lambda_j f\left(\frac{1}{\lambda_j}\right) - \sum_j \chi_j f\left(\frac{1}{\chi_j}\right)$$
$$C_f^2(\rho) = \sum_j \lambda_j f\left(\frac{1}{d\lambda_j}\right) - \sum_j \chi_j f\left(\frac{1}{d\chi_j}\right),$$

Since $f(x) = -\log(x)$ is operator convex, coherence measure defined above coincides with the relative entropy of coherence (2.1) [2]:

$$C_{\log}(\rho) = S_{\log}(\Delta(\rho)) - S_{\log}(\rho) = S(\Delta(\rho)) - S(\rho) = C(\rho).$$

The function $f(x) = \frac{1}{1-\alpha}(1 - x^{1-\alpha})$ is operator convex for $\alpha \in (0, 2)$. The coherence monotone then becomes the Tsallis relative entropy of coherence

$$C_\alpha^1(\rho) = \frac{1}{1-\alpha}\left[\sum_j \chi_j^\alpha - \sum_j \lambda_j^\alpha\right] = CT_\alpha^2(\rho).$$

2.4 Properties

Here, we list which properties (C1–C5) are satisfied by which coherences and under which conditions. For Rényi and Tsallis entropies, we do not consider a case when $\alpha = 1$ and the entropies reduce to the relative entropy of coherence.

	(C1)	(C2) Under	(C3) Under	(C4)	(C5)
CD	✓	IO [2]	X	✓	
CR_α^1 $\alpha \in [0, 1)$	✓	IO	X [27]	✓	
CR_α^2 $\alpha \in (0, 2]$	✓	GIO	see (a)		X
CR_α^3	✓	DIO [4]			
CT_α^1 $\alpha \in [0, 1)$	✓	IO	X	✓	
CT_α^2 $\alpha \in (0, 2]$	✓	GIO	see (a)		X
C_f	✓	GIO	see (a)		X

The fact that CT^2_α and CR^2_α are monotone under GIO can be derived from GIO monotonicity of C_f [31], or it was shown separately in [7]. There are examples when the monotonicity of both is violated under a larger class of operators when $\alpha > 1$, [7].

CT^2_α satisfies a modified version of additivity (C5), which CR^2_α also violates [7],

$$\mathrm{CT}^2_\alpha(p_1\rho_1 \oplus p_2\rho_2) = p_1^\alpha \mathrm{CT}^2_\alpha(\rho_1) + p_2^\alpha \mathrm{CT}^2_\alpha(\rho_2).$$

(a) In [31], it was shown that C_f, and in particular, CR^2_α and CT^2_α reach equality in the strong monotonicity under a convex mixture of diagonal unitaries in any dimension, which implies these coherences reach equality in strong monotonicity under GIO in two and three dimensions. Moreover, these coherences are strongly monotone under GIO on pure states in any dimension.

$\mathrm{CR}^1_\alpha(\rho)$, $\mathrm{CT}^1_\alpha(\rho)$ violate strong monotonicity [22, 27]. In [22], it was shown that $\mathrm{CT}^1_\alpha(\rho)$ satisfies a modified version of the strong monotonicity: for $\alpha \in (0, 2]$

$$\sum_n p_n^\alpha q_n^{1-\alpha} \mathrm{CT}^1_\alpha(\rho_n) \leq \mathrm{CT}^1_\alpha(\rho)$$

where $p_n = \mathrm{Tr}(K_n\rho K_n^*)$, $q_n = \mathrm{Tr}(K_n\Delta_\alpha(\rho)K_n^*)$ and ρ_n is a post-measurement state.

Clearly, (C6) is not satisfied for any Rényi or Tsallis coherences in its original form; therefore, it was not included in the list. However, the values of coherences on pure states can be easily calculated in some cases.

3 *f*-Coherence distillation

3.1 Continuity of *f*-entropy and *f*-coherence

In addition to the above list of properties of the f-coherence, one can add its continuity in the following form (this is a direct application of result in [19]).

Lemma 3.1 *Let ρ and σ be two states such that $\epsilon := \frac{1}{2}\|\rho - \sigma\|_1$. Then,*

$$|S^1_f(\rho) - S^1_f(\sigma)| \leq - (1-\epsilon) f\left(\frac{1}{1-\epsilon}\right) - \epsilon f\left(\frac{d-1}{\epsilon}\right)$$

$$|S^2_f(\rho) - S^2_f(\sigma)| \leq f\left(\frac{1}{d}\right) - (1-\epsilon) f\left(\frac{1}{d(1-\epsilon)}\right) - \epsilon f\left(\frac{d-1}{d\epsilon}\right).$$

Denote either of the right-hand sides as $H(\epsilon)$, and note that H is continuous in ϵ and goes to zero when $\epsilon \to 0$.

Proof Recall that for any convex function f, the transpose of it $\tilde{f}(x) = xf(1/x)$ is also convex. We adapt a convention $0 \cdot \infty = 0$, so for a convex function f such that $f(1) = 0$, we have $\tilde{f}(0) = \tilde{f}(1) = 0$. Then, f-entropy (2.12) can be written using a transpose function as

$$S^1_f(\rho) = -S_f(\rho\|I) = -\mathrm{Tr}(\rho f(\rho^{-1})) = -\mathrm{Tr}(\tilde{f}(\rho))$$

and

$$\begin{aligned} S^2_f(\rho) &= -S_f(\rho\|I/d) = f(1/d) - \mathrm{Tr}(\rho f(\{d\rho\}^{-1})) \\ &= f(1/d) - \frac{1}{d}\mathrm{Tr}(\tilde{f}(d\rho)). \end{aligned}$$

In [19] Theorem 1, it was proven that for $S_f(\rho) = -\mathrm{Tr}g(\rho)$ and any convex function g, the following holds

$$|S_g(\rho) - S_g(\sigma)| \leq g(1) - g(1-\epsilon) - (d-1)\left(g\left(\frac{\epsilon}{d-1}\right) - g(0)\right)$$

when $\epsilon = \frac{1}{2}\|\rho - \sigma\|_1$. And in Corollary 3, the result was generalized for non-unit trace density matrices: Let ρ and σ be two states of the same trace t, and let $\epsilon = \frac{1}{2}\|\rho - \sigma\|_1 \in [0, t]$, then

$$|S_g(\rho) - S_g(\sigma)| \leq g(t) - g(t-\epsilon) - (d-1)\left(g\left(\frac{\epsilon}{d-1}\right) - g(0)\right).$$

Adapting this result to our situation, it holds that

$$|S^1_f(\rho) - S^1_f(\sigma)| \leq -(1-\epsilon)f\left(\frac{1}{1-\epsilon}\right) - \epsilon f\left(\frac{d-1}{\epsilon}\right).$$

And similarly, for $\tilde{\epsilon} := d\epsilon = \frac{1}{2}\|\mathrm{d}\rho - \mathrm{d}\sigma\|_1 \in [0, d]$

$$\begin{aligned} &|S^2_f(\rho) - S^2_f(\sigma)| \\ &= \frac{1}{d}\left|\mathrm{Tr}(\tilde{f}(\mathrm{d}\rho)) - \mathrm{Tr}(\tilde{f}(\mathrm{d}\sigma))\right| \\ &\leq \frac{1}{d}\left[\tilde{f}(d) - \tilde{f}(d-\tilde{\epsilon}) - (d-1)\left(\tilde{f}(\tilde{\epsilon}/(d-1)) - \tilde{f}(0)\right)\right] \\ &= f\left(\frac{1}{d}\right) - (1-\epsilon)f\left(\frac{1}{d(1-\epsilon)}\right) - \epsilon f\left(\frac{d-1}{d\epsilon}\right). \end{aligned}$$

□

From this continuity result, one can obtain continuity of the f-coherence.

Theorem 3.2 *Let ρ and σ be two states such that $\epsilon := \frac{1}{2}\|\rho - \sigma\|_1$. Let $H(\epsilon)$ be as in the previous theorem for the corresponding f-entropy. Then, for f-coherences we obtain*

$$|C_f(\rho) - C_f(\sigma)| \leq 2H(\epsilon).$$

Proof Let ρ and σ be two states with $\epsilon = \frac{1}{2}\|\rho - \sigma\|_1$. Since trace-norm is monotone under CPTP maps, in particular, under dephasing operation, it follows that

$$\|\Delta(\rho) - \Delta(\sigma)\|_1 \leq \|\rho - \sigma\|_1 \leq 2\epsilon.$$

Therefore, from continuity results above Theorem 3.1, for either f-coherence and the corresponding f-entropy, we obtain

$$\begin{aligned} &|C_f(\rho) - C_f(\sigma)| \\ &\leq |S_f(\Delta(\rho)) - S_f(\Delta(\sigma))| + |S_f(\rho) - S_f(\sigma)| \\ &\leq 2H(\epsilon). \end{aligned}$$

□

3.2 Coherence distillation

In [8], it was shown that it is not possible to distill a higher coherence state σ from a lower coherence state ρ via GI operations when coherence is measured by a relative entropy of coherence (which equal to the distillable coherence). The same result holds for f-coherences as well, which relies on the continuity property of coherence above, and the GIO monotonicity of f-coherence [31]. For completeness sake, we present the adapted proof from [8] below.

Definition 3.3 A state σ can be distilled from the state ρ at rate $0 < R \leq 1$ if there exists an operation $\rho^{\otimes n} \to \tau$ such that $\|\mathrm{Tr}_{ref}\tau - \sigma^{\otimes nR}\|_1 \leq \epsilon$ and $\epsilon \to 0$ as $n \to \infty$. The optimal rate at which distillation is possible is the supremum of R over all protocols fulfilling the aforementioned conditions.

Theorem 3.4 *Given two states ρ and σ such that*

$$C_f(\rho) < C_f(\sigma)$$

it is not possible to distill σ from ρ at any rate $R > 0$ via GIO operations.

Proof Supposing the contradiction holds, assume that there are two states ρ and σ such that $C_f(\rho) < C_f(\sigma)$ and that the distillation is possible. In particular, for large enough n, it is possible to approximate one copy of σ. In other words, for any $\epsilon > 0$, there is a GIO Λ such that

$$\|\mathrm{Tr}_{n-1}\Lambda(\rho^{\otimes n}) - \sigma\|_1 \leq \epsilon.$$

By Lemma 12 in [8], there exists a GIO $\tilde{\Lambda}$ acting only on one copy of ρ, such that

$$\mathrm{Tr}_{n-1}\Lambda(\rho^{\otimes n}) = \tilde{\Lambda}(\rho).$$

Thus, for any $\epsilon > 0$, there is a GIO $\tilde{\Lambda}$ such that

$$\|\tilde{\Lambda}(\rho) - \sigma\|_1 \leq \epsilon.$$

Using the asymptotic continuity of f-coherence, Theorem 3.2, for these two ϵ-close states, we obtain

$$\left|C_f(\tilde{\Lambda}(\rho)) - C_f(\sigma)\right| \leq 2H(\epsilon/2).$$

Recall that $H(\epsilon)$ for either f-coherence is continuous in $\epsilon \in (0, 1)$ and it goes to zero when $\epsilon \to 0$. Therefore, summarizing from the beginning, for any $\delta > 0$, there is a GIO $\tilde{\Lambda}$ such that

$$\left|C_f(\tilde{\Lambda}(\rho)) - C_f(\sigma)\right| < \delta. \tag{3.1}$$

Take $\delta := \frac{1}{2}(C_f(\sigma) - C_f(\rho)) > 0$. Since C_f is GIO monotone, for any GIO Λ, we have

$$C_f(\tilde{\Lambda}(\rho)) \leq C_f(\rho).$$

Therefore,

$$\delta \leq \frac{1}{2}(C_f(\sigma) - C_f(\tilde{\Lambda}(\rho)) < C_f(\sigma) - C_f(\tilde{\Lambda}(\rho)).$$

This is a contradiction to (3.1). □

4 New Rényi and Tsallis coherences

Playing off the last expression in the definition of the relative entropy of coherence 2.1, we define coherence measure as follows:

$$\mathrm{CT}_\alpha(\rho) := S^T_\alpha(\Delta_\alpha(\rho)) - S^T_\alpha(\rho)$$

for Tsallis entropy, and

$$\mathrm{CR}_\alpha(\rho) := S^R_\alpha(\Delta_\alpha(\rho)) - S^R_\alpha(\rho)$$

for Rényi entropy. Recall that here $\Delta_\alpha(\rho)$ is the closest incoherent state to ρ when measured by the Rényi or Tsallis relative entropy, i.e.,

$$S_\alpha(\rho\|\Delta_\alpha(\rho)) := \min_{\delta\in\mathcal{I}} S_\alpha(\rho\|\delta).$$

Recall from (2.7) that

$$\Delta_\alpha(\rho) = \frac{1}{N(\rho)} \sum_j \langle j|\,\rho^\alpha\,|j\rangle^{1/\alpha}\,|j\rangle\,\langle j|$$

where $N(\rho) = \sum_j \langle j| \rho^\alpha |j\rangle^{1/\alpha}$. Having this explicit form of $\Delta_\alpha(\rho)$, both coherences can be explicitly calculated

$$\begin{aligned}
\mathrm{CT}_\alpha(\rho) &= \frac{1}{1-\alpha}\left[\mathrm{Tr}\left(\Delta_\alpha(\rho)^\alpha\right) - \mathrm{Tr}\rho^\alpha\right] \\
&= \frac{1}{1-\alpha}\left[\frac{1}{N(\rho)^\alpha} - 1\right]\mathrm{Tr}\rho^\alpha \\
&= \frac{N(\rho)^\alpha - 1}{\alpha - 1}\frac{\mathrm{Tr}\rho^\alpha}{N(\rho)^\alpha} \\
&= S_\alpha^T(\rho\|\Delta_\alpha(\rho))\frac{\mathrm{Tr}\rho^\alpha}{N(\rho)^\alpha} \\
&= \mathrm{CT}_\alpha^1(\rho)\frac{\mathrm{Tr}\rho^\alpha}{N(\rho)^\alpha} \\
&\geq 0.
\end{aligned}$$

The last two equalities come from (2.8). Similarly, from (2.9) for the Rényi coherence

$$\begin{aligned}
\mathrm{CR}_\alpha(\rho) &= \frac{1}{1-\alpha}\left[\log\mathrm{Tr}\left(\Delta_\alpha(\rho)^\alpha\right) - \log\mathrm{Tr}\rho^\alpha\right] \\
&= \frac{1}{1-\alpha}\left[\log\left(\frac{1}{N(\rho)^\alpha}\mathrm{Tr}\rho^\alpha\right) - \log\mathrm{Tr}\rho^\alpha\right] \\
&= \frac{\alpha}{\alpha-1}\log N(\rho) \\
&= S_\alpha^R(\rho\|\Delta_\alpha(\rho)) \\
&= \mathrm{CR}_\alpha^1(\rho).
\end{aligned}$$

This means that for Rényi entropy of coherence we have a similar expressions to the relative entropy of coherence (2.1)

$$\mathrm{CR}_\alpha^1(\rho) = \min_{\delta\in\mathcal{I}} S_\alpha^R(\rho\|\delta) = S_\alpha^R(\rho\|\Delta_\alpha(\rho)) = S_\alpha^R(\Delta_\alpha(\rho)) - S_\alpha^R(\rho).$$

Therefore, the distance-based Rényi coherence $\mathrm{CR}_\alpha^1(\rho)$ coincides with the new definition $\mathrm{CR}_\alpha(\rho)$. Before moving on to investigation of the new Tsallis coherence, let us show one result on Rényi coherence.

Theorem 4.1 *Let $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$ be pure states on $\mathbb{C}^d$ such that $\frac{1}{2}\|\rho - \sigma\|_1 = \epsilon$. Then, we obtain*

$$\left|\mathrm{CR}_\alpha^1(\rho) - \mathrm{CR}_\alpha^1(\sigma)\right| \leq \frac{\alpha}{1-\alpha}\log\left(d^{1-\frac{1}{\alpha}} + H(\epsilon)\right) + \log d \;\; for 0 < \alpha < 1$$

and

$$\left|\mathrm{CR}_\alpha^1(\rho) - \mathrm{CR}_\alpha^1(\sigma)\right| \leq \frac{\alpha}{\alpha-1}\log(1 - H(\epsilon)) \;\; for 1 < \alpha < 2$$

where $H(\epsilon) = 1 - (1-\epsilon)^{1/\alpha} - \epsilon^{1/\alpha}(d-1)^{1-\frac{1}{\alpha}}$. Both right-hand sides converge to zero when ϵ goes to zero.

Proof Denote $\chi_j = |\langle\psi|j\rangle|^2$ and $\xi_j = |\langle\phi|j\rangle|^2$. Then,

$$|\mathrm{CR}^1_\alpha(\rho) - \mathrm{CR}^1_\alpha(\sigma)| = \frac{\alpha}{|1-\alpha|}\left|\log\left(\sum_j \chi_j^{1/\alpha}\right) - \log\left(\sum_j \xi_j^{1/\alpha}\right)\right|$$
$$= \frac{1}{|1-\alpha|}\left|\log \mathrm{Tr} f(\Delta(\rho)) - \log \mathrm{Tr} f(\Delta(\sigma))\right|,$$

where $f(x) = x^{1/\alpha}$ is convex function for $0 < \alpha < 1$ and $-f$ is convex for $\alpha > 1$, and recall that $\Delta(\rho) = \sum_j \chi_j |j\rangle\langle j|$ and $\Delta(\sigma) = \sum_j \xi_j |j\rangle\langle j|$.

Since trace-norm is monotone under CPTP maps, and Δ is a CPTP map, we obtain

$$\frac{1}{2}\|\Delta(\rho) - \Delta(\sigma)\|_1 \le \frac{1}{2}\|\rho - \sigma\|_1 = \epsilon.$$

By continuity of f-entropy [19], the difference for $0 < \alpha < 1$ is bounded by

$$|\mathrm{Tr} f(\Delta(\rho)) - \mathrm{Tr} f(\Delta(\sigma))| \le H(\epsilon)$$

where $H(\epsilon)$ is calculated for $f(x) = x^{1/\alpha}$, and therefore has expression as in the theorem statement. For $\alpha > 1$, $-f$ is convex, and therefore,

$$|\mathrm{Tr} f(\Delta(\rho)) - \mathrm{Tr} f(\Delta(\sigma))| \le -H(\epsilon)$$

where the right-hand side is positive for $\alpha > 1$.

For $0 < \alpha < 1$, notice that the constant sequence is majorized by both $(\frac{1}{d})_j \prec (\chi)_j$ and $(\frac{1}{d})_j \prec (\xi)_j$; therefore, since $f(x) = x^{1/\alpha}$ is a convex function, by results on Schur concavity [11, 17, 24], we have $\sum_j \chi_j^{1/\alpha}, \sum_j \xi_j^{1/\alpha} > d^{1-\frac{1}{\alpha}}$. For $\alpha > 1$, since $x < x^{1/\alpha}$, then $\sum_j \chi_j^{1/\alpha}, \sum_j \xi_j^{1/\alpha} > 1$.

For the function $g(x) = \log x$ and any $0 < s < c < 1$, we have $|g'(s)| > |g'(c)|$, and therefore, by the mean value theorem, there exist $s, c \in (0, 1]$, such that $d^{1-\frac{1}{\alpha}} \le s \le c$, and

$$\left|[\mathrm{Tr} f(\Delta(\rho))]^{-\alpha} - [\mathrm{Tr} f(\Delta(\sigma))]^{-\alpha}\right| = |\mathrm{Tr} f(\Delta(\rho)) - \mathrm{Tr} f(\Delta(\sigma))|\,|g'(c)|$$
$$\le H(\epsilon)|g'(s)|$$
$$= \left|\log d^{1-\frac{1}{\alpha}} - \log\left(d^{1-\frac{1}{\alpha}} + H(\epsilon)\right)\right|.$$

Therefore, by the mean value theorem, there exists $s, c \ge 1$, such that

$$\begin{aligned} \left|[\mathrm{Tr} f(\Delta(\rho))]^{-\alpha} - [\mathrm{Tr} f(\Delta(\sigma))]^{-\alpha}\right| &= |\mathrm{Tr} f(\Delta(\rho)) - \mathrm{Tr} f(\Delta(\sigma))|\,|g'(c)| \\ &\leq -H(\epsilon)|g'(s)| \\ &= \log(1 - H(\epsilon)). \end{aligned}$$

Thus, we obtain the statement of the theorem. □

5 Tsallis coherence

5.1 Positivity

As we noted above, the Tsallis coherence is non-negative. Note that this is a non-trivial statement that cannot be directly observed by the monotonicity of entropy under linear CPTP maps, as it was done for CT_α^2, CR_α^2, C_f, since the map $\rho \to \Delta_\alpha(\rho)$ is nonlinear.

5.2 Vanishing only on incoherent states

Proposition 5.1 $\mathrm{CT}_\alpha(\rho) = 0$ *if and only if* $\rho \in \mathcal{I}$ *is incoherent.*

Proof First, suppose that the state $\rho \in \mathcal{I}$ is incoherent, then $\Delta_\alpha(\rho) = \rho$. Therefore, $\mathrm{CT}_\alpha(\rho) = S_\alpha^T(\Delta_\alpha(\rho)) - S_\alpha^T(\rho) = 0$.

Now, suppose that $\mathrm{CT}_\alpha(\rho) = 0$. From calculations above, since $\mathrm{Tr}\rho^\alpha > 0$ for a nonzero state, this means that $S_\alpha^T(\rho\|\Delta_\alpha(\rho)) = 0$, which happens only when $\rho = \Delta_\alpha(\rho) \in \mathcal{I}$. Therefore, $\rho \in \mathcal{I}$ is incoherent. □

5.3 Value on pure states

Let $\rho = |\psi\rangle\langle\psi|$ be a pure state. Since $\rho^\alpha = \rho$,

$$\mathrm{CT}_\alpha(\rho) = \frac{1}{1-\alpha}\left[\mathrm{Tr}\left(\Delta_\alpha(\rho)^\alpha\right) - \mathrm{Tr}\rho^\alpha\right] = S_\alpha^T(\Delta_\alpha(\rho)).$$

To calculate this Tsallis entropy explicitly, we note that $\mathrm{Tr}(\Delta_\alpha(|\psi\rangle\langle\psi|)^\alpha) = N(|\psi\rangle\langle\psi|)^{-\alpha}$, where $N(|\psi\rangle\langle\psi|) = \sum_j |\langle\psi|j\rangle|^{2/\alpha}$. Thus,

$$\mathrm{CT}_\alpha(\rho) = \frac{1}{1-\alpha}\left[\left(\sum_j |\langle\psi|j\rangle|^{2/\alpha}\right)^{-\alpha} - 1\right].$$

5.4 Comparison with CT_α^1

Recall that from our previous calculations,

$$\mathrm{CT}_\alpha(\rho) = CT_\alpha^1(\rho)\,\frac{\mathrm{Tr}\rho^\alpha}{N(\rho)^\alpha}.$$

Let us denote as $\lambda_j := \langle j| \rho^\alpha |j\rangle$. Then,

$$\mathrm{Tr}(\rho^\alpha) = \sum_j \langle j| \rho^\alpha |j\rangle = \|\lambda\|_1.$$

And

$$N(\rho)^\alpha = \left(\sum_j \langle j| \rho^\alpha |j\rangle^{1/\alpha} \right)^\alpha = \|\lambda\|_{1/\alpha}.$$

Here, $\|\cdot\|_p$ denotes the Schatten p-norm. Since Schatten p-norms are monotone decreasing in p, we have that

$$\mathrm{CT}_\alpha(\rho) \geq \mathrm{CT}^1_\alpha(\rho) \text{ for } 0 < \alpha < 1$$

and

$$\mathrm{CT}_\alpha(\rho) \leq \mathrm{CT}^1_\alpha(\rho) \text{ for } 1 < \alpha < 2.$$

5.5 Monotonicity

Theorem 5.2 $\mathrm{CT}_\alpha(\rho)$ *is invariant under diagonal unitaries.*

Proof Let $U = \sum_n e^{i\phi_n} |n\rangle \langle n|$ be a unitary diagonal in $\mathcal{E}$ basis. Then,

$$\begin{aligned}\Delta_\alpha(U\rho U^*) &= \frac{1}{\sum \langle j| U\rho^\alpha U^* |j\rangle^{1/\alpha}} \sum \langle j| U\rho^\alpha U^* |j\rangle^{1/\alpha} |j\rangle \langle j| \\ &= \frac{1}{\sum \langle j| e^{i\phi_j} \rho^\alpha e^{-i\phi_j} |j\rangle^{1/\alpha}} \sum \langle j| e^{i\phi_j} \rho^\alpha e^{-i\phi_j} |j\rangle^{1/\alpha} |j\rangle \langle j| \\ &= \Delta_\alpha(\rho).\end{aligned}$$

Since the Tsallis entropy is invariant under unitaries itself, we have

$$\mathrm{CT}_\alpha(U\rho U^*) = \mathrm{CT}_\alpha(\rho).$$

□

Theorem 5.3 *Tsallis coherence is not monotone under GIO.*

Proof Let us fix the basis $\mathcal{E} = \{|0\rangle, |1\rangle\}$ Let $\rho = |\psi\rangle \langle\psi|$ be a pure state with $|\langle\psi|0\rangle|^2 = \chi = 3/4$ and $|\langle\psi|1\rangle|^2 = 1 - \chi = 1/4$.

For a pure state ρ, the entropy is zero, and therefore,

$$\mathrm{CT}_\alpha(\rho) = S^T_\alpha(\Delta_\alpha(\rho)) - S^T_\alpha(\rho)$$

$$\begin{aligned}
&= S_\alpha^T(\Delta_\alpha(\rho)) \\
&= \frac{1}{1-\alpha}\left[\mathrm{Tr}\left\{\Delta_\alpha(\rho))^\alpha\right\} - 1\right] \\
&= \frac{1}{1-\alpha}\left[\frac{1}{\left(\sum_j \chi_j^{1/\alpha}\right)^\alpha} - 1\right] \\
&= \frac{1}{1-\alpha}\left[\frac{4}{\left(3^{1/\alpha}+1\right)^\alpha} - 1\right].
\end{aligned}$$

Let Λ be GIO, with Kraus operators $\Lambda(\rho) = K_1\rho K_1^* + K_2\rho K_2^*$ where Kraus operators are diagonal in $\mathcal{E}$ basis

$$K_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \qquad K_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Clearly $\sum_n K_n^* K_n = I$. Then,

$$\Lambda(\rho) = \begin{pmatrix} \frac{3}{4} & a \\ a & \frac{1}{4} \end{pmatrix}$$

where $a = \frac{3+\sqrt{3}}{8\sqrt{2}}$. The eigenvalues of this matrix are $\beta_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{\frac{1}{4} + 4a^2}\right)$. And the normalized eigenvector corresponding to $\beta_{1,2}$ is

$$\left|\psi_{1,2}\right\rangle = \frac{1}{\sqrt{a^2 + (\beta_{1,2} - \frac{3}{4})^2}} \begin{pmatrix} a \\ \beta_{1,2} - \frac{3}{4} \end{pmatrix}.$$

Therefore, $\mathrm{Tr}(\Lambda(\rho)^\alpha) = \beta_1^\alpha + \beta_2^\alpha$, and

$$N(\Lambda(\rho)) = \sum_j \beta_1 |\langle j|\psi_1\rangle|^{2/\alpha} + \beta_2 |\langle j|\psi_2\rangle|^{2/\alpha}.$$

And the Tsallis coherence is then

$$\mathrm{CT}_\alpha(\Lambda(\rho)) = \frac{1}{1-\alpha}\left[\frac{1}{N(\Lambda(\rho))^\alpha} - 1\right]\mathrm{Tr}(\Lambda(\rho)^\alpha).$$

From Fig. 1, we see that, for example, for $\alpha = 0.2$, monotonicity has failed

$$\mathrm{CT}_\alpha(\rho) < 0.5 < \mathrm{CT}_\alpha(\Lambda(\rho)).$$

□

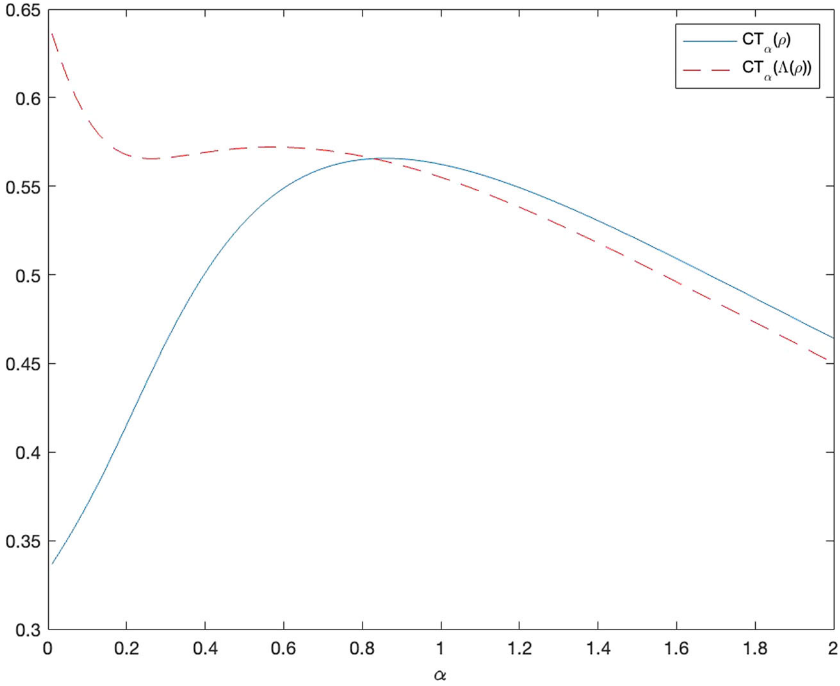

Fig. 1 Failure of monotonicity under GIO for small α

Definition 5.4 A GIO map Λ that commutes with Δ_α is called $\alpha - GIO$.

A unitary diagonal under a fixed basis $\mathcal{E}$ is an α-GIO for any α. For $\alpha = 1$, $\Delta_\alpha(\rho) = \Delta(\rho)$, which commutes with any GIO.

Theorem 5.5 *Tsallis coherence is monotone under α-GIO.*

Proof By definition

$$\mathrm{CT}_\alpha(\rho) - \mathrm{CT}_\alpha(\Lambda(\rho)) = S_\alpha^T(\Lambda(\rho)) - S_\alpha^T(\rho) + S_\alpha^T(\Delta_\alpha(\rho)) - S_\alpha^T(\Delta_\alpha(\Lambda(\rho))).$$

Since Tsallis entropy is monotone under CPTP maps, $S_\alpha^T(\Lambda(\rho)) - S_\alpha^T(\rho) \geq 0$. Λ commutes with Δ_α, and Λ is GIO, so it leaves the incoherent states, such as $\Delta_\alpha(\rho)$, invariant, therefore

$$S_\alpha^T(\Delta_\alpha(\rho)) - S_\alpha^T(\Delta_\alpha(\Lambda(\rho))) = S_\alpha^T(\Delta_\alpha(\rho)) - S_\alpha^T(\Lambda(\Delta_\alpha(\rho))) = 0.$$

□

5.6 Strong monotonicity

Theorem 5.6 *Tsallis coherence $CT_\alpha(\rho)$ reaches equality in strong monotonicity for convex mixtures of diagonal unitaries. Therefore, $CT_\alpha(\rho)$ reaches equality in strong* *monotonicity under GIO in two and three dimensions, when Kraus operators are proportional to diagonal unitaries.*

Proof Consider a GIO Λ that is a probabilistic mixture of diagonal unitaries, i.e., let

$$\Lambda(\rho) = \sum_k \alpha_k U_k \rho U_k^*$$

where $\alpha_j \in [0, 1]$ with $\sum \alpha_k = 1$, and the unitaries U_k are diagonal in $\mathcal{E}$. Then, from Theorem 5.2, since CT_α is invariant under diagonal unitaries, we have

$$\sum_k \alpha_k CT_\alpha(U_k \rho U_k^*) = \left(\sum_k \alpha_k\right) CT_\alpha(\rho) = CT_\alpha(\rho).$$

□

In general, CT_α fails strong monotonicity for IO maps.

Theorem 5.7 *Tsallis coherence $CT_\alpha(\rho)$ fails strong monotonicity under IO maps.*

Proof We use example from [27], which was used to show that CR_α^1 fails strong monotonicity under IO maps. Consider a three-dimensional space spanned by standard orthonormal basis $\mathcal{E} = \{|0\rangle, |1\rangle, |2\rangle\}$. Let the density matrix be

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let the Kraus operators of the IO map be

$$K_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \qquad K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, $|a|^2 + |b|^2 = 1$ to satisfy the condition $K_1^* K_1 + K_2^* K_2 = I$. It is straightforward to check that these Kraus operators leave the space of incoherent states $\mathcal{I}$ invariant. The output states are

$$\rho_1 = \frac{1}{p_1} K_1 \rho K_1^* = \frac{1}{2 + |a|^2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |a|^2 \end{pmatrix} \qquad \rho_2 = \frac{1}{p_2} K_2 \rho K_2^* = \frac{1}{1 + |b|^2}\begin{pmatrix} 1 & b^* & 0 \\ b & |b|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $p_1 = \frac{2+|a|^2}{4}$ and $p_2 = \frac{1+|b|^2}{4}$. Notice that $\rho_1 \in \mathcal{I}$ is diagonal and therefore incoherent, and $\rho_2 = |\psi\rangle\langle\psi|$ is the pure state with $|\psi\rangle = \frac{1}{\sqrt{1+|b|^2}}(|0\rangle + b\,|1\rangle)$.

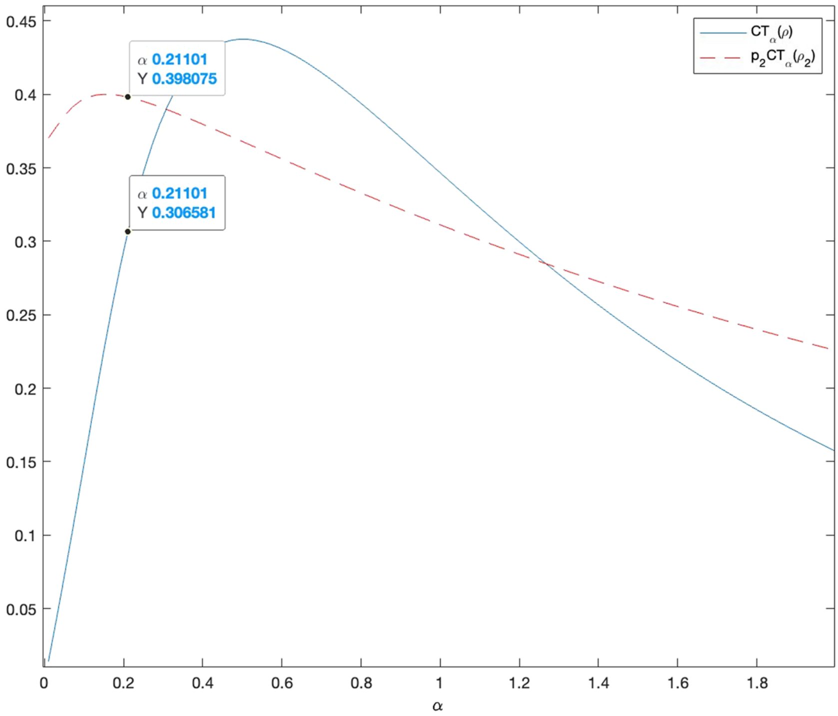

Fig. 2 Failure of strong monotonicity under IO

The α power of ρ is the state

$$\rho^{\alpha} = \frac{1}{2^{1+\alpha}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

And therefore, the Tsallis coherence is

$$\mathrm{CT}_{\alpha}(\rho) = S_{\alpha}^{T}(\Delta_{\alpha}(\rho)) - S_{\alpha}^{T}(\rho) = \frac{4}{1-\alpha}\left[(2+2^{1/\alpha})^{-\alpha} - 2^{-(1+\alpha)}\right].$$

Since $\rho \in \mathcal{I}$ is incoherent, $\mathrm{CT}_{\alpha}(\rho_1) = 0$. And since ρ_2 is a pure state, the Tsallis coherence is

$$p_2\mathrm{CT}_{\alpha}(\rho_2) = p_2 S_{\alpha}^{T}(\Delta_{\alpha}(\rho_2)) = \frac{1}{1-\alpha}\frac{1+|b|^2}{4}\left[(1+|b|^2)(1+|b|^{2/\alpha})^{-\alpha} - 1\right].$$

From Fig. 2, for example, for $b = 0.9$ and $\alpha = 0.21101$, we have

$$\mathrm{CT}_{\alpha}(\rho) < 0.35 < p_2\mathrm{CT}_{\alpha}(\rho_2) = \sum_j p_j \mathrm{CT}_{\alpha}(\rho_j).$$

□

For strong monotonicity property, it is important how the quantum channel is written in terms of its Kraus operators. We showed that in two or three dimensions, if GIO is written as a convex mixture of diagonal unitaries, then Tsallis coherence reaches equality. However, if GIO is written in some other way, we show that Tsallis coherence may fail strong monotonicity.

Theorem 5.8 *Tsallis coherence fails strong monotonicity under GIO, even on pure states, if Kraus operators are not proportional to unitaries.*

Proof We are going to use the same example as in Theorem 5.3. Let us fix the basis $\mathcal{E} = \{|0\rangle, |1\rangle\}$ Let $\rho = |\psi\rangle\langle\psi|$ be a pure state with $|\langle\psi|0\rangle|^2 = \chi = 3/4$ and $|\langle\psi|1\rangle|^2 = 1 - \chi = 1/4$.

For a pure state ρ, the entropy is zero, and therefore,

$$\begin{aligned}
\mathrm{CT}_\alpha(\rho) &= S_\alpha^T(\Delta_\alpha(\rho)) \\
&= \frac{1}{1-\alpha}\left[\mathrm{Tr}\left\{\Delta_\alpha(\rho))^\alpha\right\} - 1\right] \\
&= \frac{1}{1-\alpha}\left[\frac{1}{\left(\sum_j \chi_j^{1/\alpha}\right)^\alpha} - 1\right] \\
&= \frac{1}{1-\alpha}\left[\frac{4}{(3^{1/\alpha}+1)^\alpha} - 1\right].
\end{aligned}$$

Let Λ be GIO, with Kraus operators $\Lambda(\rho) = K_1\rho K_1^* + K_2\rho K_2^*$ where Kraus operators are diagonal in $\mathcal{E}$ basis

$$K_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \qquad K_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Clearly $\sum_n K_n^* K_n = I$. Then, the post-measurement states $\rho_n = \frac{1}{p_n} K_n \rho K_n^* = |\psi_n\rangle\langle\psi_n|$ are also pure, where $|\psi_n\rangle = \frac{1}{\sqrt{p_n}} K_n |\psi\rangle$ and $p_n = \langle\psi| K_n^* K_n |\psi\rangle$. Let us denote $|\langle\psi_n|j\rangle|^2 = \xi_{nj} = \frac{1}{p_n}|\langle j| K_n |\psi\rangle|^2 = \frac{1}{p_n}|k_{nj}|^2\chi_j$, and $p_n = \sum_j |k_{nj}|^2\chi_j$. Then, $p_1 = \frac{9}{16}$ and $p_2 = \frac{7}{16}$, and

$$\xi_{11} = \frac{2}{3}\ \xi_{12} = \frac{1}{3} \qquad \xi_{21} = \frac{6}{7}\ \xi_{22} = \frac{1}{7}.$$

Therefore,

$$\begin{aligned}
\mathrm{CT}_\alpha(\rho_1) &= S_\alpha^T(\Delta_\alpha(\rho_1)) \\
&= \frac{1}{1-\alpha}\left[\mathrm{Tr}\left\{\Delta_\alpha(\rho_1))^\alpha\right\} - 1\right]
\end{aligned}$$

$$= \frac{1}{1-\alpha}\left[\frac{1}{\left(\sum_j \xi_{1j}^{1/\alpha}\right)^{\alpha}} - 1\right]$$
$$= \frac{1}{1-\alpha}\left[\frac{3}{\left(2^{1/\alpha}+1\right)^{\alpha}} - 1\right].$$

Similarly,

$$\begin{aligned} \mathrm{CT}_\alpha(\rho_2) &= S_\alpha^T(\Delta_\alpha(\rho_2)) \\ &= \frac{1}{1-\alpha}\left[\mathrm{Tr}\left\{\Delta_\alpha(\rho_2))^\alpha\right\} - 1\right] \\ &= \frac{1}{1-\alpha}\left[\frac{1}{\left(\sum_j \xi_{2j}^{1/\alpha}\right)^{\alpha}} - 1\right] \\ &= \frac{1}{1-\alpha}\left[\frac{7}{\left(6^{1/\alpha}+1\right)^{\alpha}} - 1\right]. \end{aligned}$$

From Fig. 3, for example, for $\alpha = 0.20303$, strong monotonicity fails since

$$\mathrm{CT}_\alpha(\rho) < 0.42 < p_1\mathrm{CT}_\alpha(\rho_1) + p_2\mathrm{CT}_\alpha(\rho_2).$$

□

6 Improved α-coherence measure

Note that even though $\Delta_1 = \Delta$, these two operators scale differently, in the following sense: $\Delta(p\rho) = p\Delta(\rho)$, and $\Delta_\alpha(p\rho) = \Delta(\rho)$. For this reason, define the "unnormalized" Δ_α,

$$\tilde{\Delta}_\alpha(\rho) = \sum_j \langle j|\,\rho^\alpha\,|j\rangle^{1/\alpha}\,|j\rangle\,\langle j|\,. \tag{6.1}$$

Note that $\tilde{\Delta}_\alpha(\rho) = \Delta(\rho^\alpha)^{1/\alpha}$.

In [7], a coherence measure was proposed

$$\mathrm{Tr}\left|\Delta(\rho)^\alpha - \rho^\alpha\right|^{1/\alpha} \tag{6.2}$$

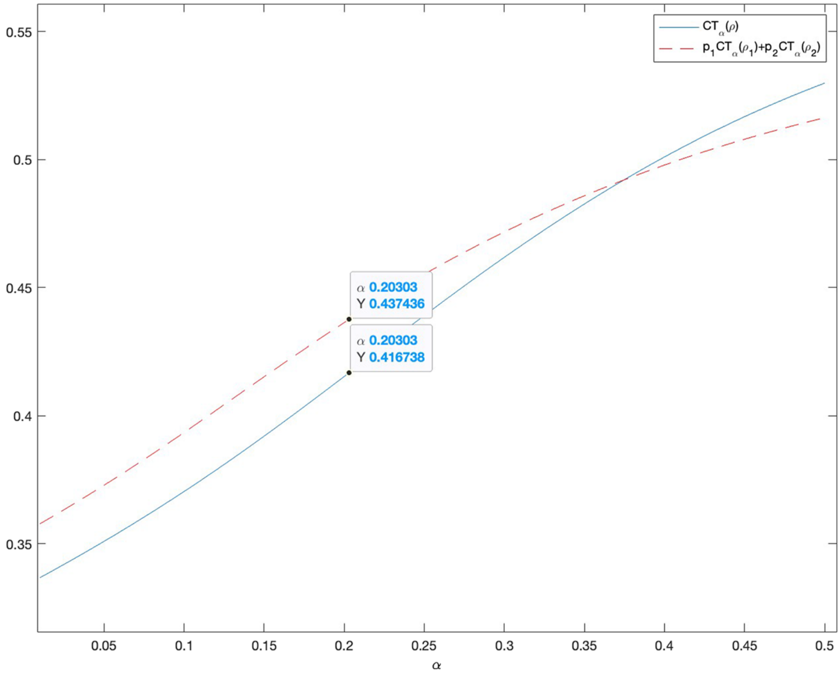

Fig. 3 Failure of strong monotonicity under GIO for small α

which was shown to satisfy (C5). Since (C5) is equivalent to (C3) and (C4), and the later two imply (C2), satisfying (C5) implies that the expression is a coherence measure.

Similar to this, we propose the following coherence measures

$$C_\alpha^1(\rho) = \mathrm{Tr}\left|\tilde{\Delta}_\alpha(\rho) - \rho\right| = \mathrm{Tr}\left|\Delta(\rho^\alpha)^{1/\alpha} - \rho\right| \tag{6.3}$$

and

$$C_\alpha^2(\rho) = \mathrm{Tr}\left|\tilde{\Delta}_\alpha(\rho)^\alpha - \rho^\alpha\right|^{\frac{1}{\alpha}} = \mathrm{Tr}\left|\Delta(\rho^\alpha) - \rho^\alpha\right|^{\frac{1}{\alpha}}. \tag{6.4}$$

Both C_α^1 and C_α^2 can be easily shown to satisfy (C5): for $p_1 + p_2 = 1$, $p_1, p_2 \geq 0$ and any two states ρ_1 and ρ_2,

$$\mathcal{C}(p_1\rho_1 \oplus p_2\rho_2) = p_1\mathcal{C}(\rho_1) + p_2\mathcal{C}(\rho_2).$$

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Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The author has no competing interests or conflict of interest to declare that are relevant to the content of this article.

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