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Series Bwww.elsevier.com/locate/jctbCounting r -graphs without forbidden configurationsJózsef Balogh^{a,1}, Felix Christian Clemen^{a,2}, Letícia Mattos^{b,3}^a Department of Mathematics, University of Illinois at Urbana-Champaign,
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ABSTRACT

One of the major problems in combinatorics is to determine the number of r -uniform hypergraphs (r -graphs) on n vertices which are free of certain forbidden structures. This problem dates back to the work of Erdős, Kleitman and Rothschild, who showed that the number of K_r -free graphs on n vertices is $2^{\text{ex}(n, K_r) + o(n^2)}$. Their work was later extended to forbidding graphs as induced subgraphs by Prömel and Steger.

Here, we consider one of the most basic counting problems for 3-graphs. Let E_1 be the 3-graph with 4 vertices and 1 edge. What is the number of induced $\{K_4^3, E_1\}$ -free 3-graphs on n vertices? We show that the number of such 3-graphs is of order $n^{\Theta(n^2)}$. More generally, we determine asymptotically the number of induced $\mathcal{F}$ -free 3-graphs on n vertices for all families $\mathcal{F}$ of 3-graphs on 4 vertices. We also provide upper bounds on the number of r -graphs on n vertices which do not induce $i \in L$ edges on any set of k vertices, where $L \subseteq \{0, 1, \dots, \binom{k}{r}\}$ is a list which does not contain 3 consecutive integers in its complement. Our bounds are best possible up to a constant multiplicative factor in the exponent when $k =$

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$r+1$. The main tool behind our proof is counting the solutions of a constraint satisfaction problem.

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1. Introduction

1.1. History

For an r -uniform hypergraph (r -graph) F , let $\text{ex}(n, F)$ denote the maximum number of edges in an F -free r -graph on n vertices. One of the central questions in extremal combinatorics is to determine the extremal number $\text{ex}(n, F)$. For $r = 2$, the extremal number is well-understood for all non-bipartite graphs, see [15] and [38]. However, determining the extremal number for general r -graphs is a well-known and hard problem. The simplest and still not answered question posed by Turán asks to determine the extremal number of K_4^3 , the complete 3-graph on 4 vertices. It is widely believed that

$$\text{ex}(n, K_4^3) = \left(\frac{5}{9} + o(1) \right) \binom{n}{3}.$$

In a series of papers, different K_4^3 -free 3-graphs on n vertices and $\frac{5}{9}\binom{n}{3} + o(n^3)$ edges were constructed by Brown [11], Kostochka [23] and Fon-der-Flaass [17] and Razborov [34]. In 2008, Frohmader [19] showed that there are $\Omega(6^{n/3})$ non-isomorphic r -graphs which are conjectured to be extremal. This is believed to be one of the reasons of the difficulty of this problem. For other related papers, see [3,30,34].

The problem of determining the extremal number can also be extended to families of induced r -graphs. For a family of r -graphs $\mathcal{F}$, let $\text{ex}_I(n, \mathcal{F})$ denote the maximum number of edges in an induced $\mathcal{F}$ -free r -graph on n vertices. In 2010, Razborov [34] used the method of flag algebras to determine $\text{ex}_I(n, \{K_4^3, E_1\})$, where E_1 denotes the 3-graph with 4 vertices and 1 edge. In his paper, he showed that

$$\text{ex}_I(n, \{K_4^3, E_1\}) = \left(\frac{5}{9} + o(1) \right) \binom{n}{3}.$$

Later, this result was extended by Pikhurko [30], who obtained the corresponding stability result and proved that there is only one extremal induced $\{K_4^3, E_1\}$ -free 3-graph on n vertices, up to isomorphism. Sometimes referred to as Turán's construction and here denoted by C_n , the extremal induced $\{K_4^3, E_1\}$ -free 3-graph on $[n]$ is obtained as follows. Let $V_1 \cup V_2 \cup V_3$ be a partition of $[n]$ with $||V_i| - |V_j|| \leq 1$ for all $i, j \in [3]$. An edge is placed in C_n if it intersects each of the classes V_1, V_2 and V_3 , or if for some $i \in [3]$ it contains two elements of V_i and one of V_{i+1} , where the indices are understood modulo 3. See Fig. 1 for an illustration of C_n .

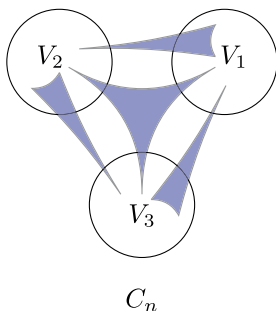


Fig. 1. Illustration of C_n .

In this paper, we first consider the problem of counting induced $\{K_4^3, E_1\}$ -free 3-graphs on n vertices, which is the counting problem related to the results of Razborov [34] and Pikhurko [30]. Recently, Balogh and Mubayi [7] observed that a standard application of the hypergraph container method [4,35] shows that the number of induced $\{K_4^3, E_1\}$ -free graphs on n vertices is $2^{O(n^{8/3})}$. From the other side, we can construct a family $Q(n)$ with $2^{\Omega(n^2 \log n)}$ subgraphs of C_n which are induced $\{K_4^3, E_1\}$ -free. A 3-graph is in $Q(n)$ if it is obtained from a complete tripartite 3-graph with classes V_1, V_2, V_3 by removing a linear⁴ 3-graph with the additional property that every edge contains one element from each of the classes V_1, V_2 and V_3 . It is not hard to show that every 3-graph in $Q(n)$ is in fact induced $\{K_4^3, E_1\}$ -free and that $|Q(n)| = 2^{\Omega(n^2 \log n)}$ (see the proof of Theorem 1.2 in Section 4). Balogh and Mubayi [7] conjectured that almost all induced $\{K_4^3, E_1\}$ -free 3-graphs are in this family, up to isomorphism.

Conjecture 1.1 (Balogh and Mubayi [7]). *Almost all induced $\{K_4^3, E_1\}$ -free 3-graphs on $[n]$ are in $Q(n)$, up to isomorphism.*

The motivation behind this conjecture comes from similar results. In particular, Person and Schacht [29] proved that almost all Fano-plane free 3-graphs are bipartite, and Balogh and Mubayi [6] proved that almost all F_5 -free triple systems are tripartite, where F_5 is the 5-vertex 3-graph with edge set $\{123, 124, 345\}$. See also [5] for results along the same line.

The problem of counting r -graphs which are free of forbidden structures dates back to the work of Erdős, Kleitman and Rothschild [14] in the context of graphs. They showed that the number of K_s -free graphs on n vertices is $2^{(1+o(1))\text{ex}(n, K_s)}$. Their work was later extended to all non-bipartite graphs by Erdős, Frankl and Rödl [13] using the Szemerédi regularity lemma. For other related results, see [8,9,13,16,21,28,36]. In a sequence of papers [31–33], Prömel and Steger studied the corresponding problem for induced graphs. Their results were stated in terms of a different notion of extremal number, which was latter generalized by Dotson and Nagle [12] as follows. Given a family

⁴ A 3-graph H is *linear* if every pair of distinct edges $e_1, e_2 \in E(H)$ satisfies $|e_1 \cap e_2| \leq 1$.

of r -graphs $\mathcal{F}$, let M and N be r -sets⁵ in $[n] := \{1, \dots, n\}$ with the following properties: (i) $M \cap N = \emptyset$; and (ii) for $G \subseteq \binom{[n]}{r} \setminus (M \cup N)$, the r -graph $G \cup M$ is induced $\mathcal{F}$ -free. The notation $\binom{[n]}{r}$ stands for $\{S \subseteq [n] : |S| = r\}$. The $*$ -extremal number $\text{ex}^*(n, \mathcal{F})$ is defined as

$$\text{ex}^*(n, \mathcal{F}) := \binom{n}{r} - \min_{M, N} (|M| + |N|),$$

where the minimum is over all r -sets $M, N \subseteq [n]$ satisfying (i) and (ii). In 1992, Prömel and Steger [32] showed that the number of induced $\mathcal{F}$ -free graphs on n vertices is $2^{\text{ex}^*(n, \mathcal{F}) + o(n^2)}$. This result was later extended by Alekseev [1] and Bollobás and Thomason [10] for families of graphs, and by Kohayakawa, Nagle and Rödl [22] for 3-graphs. In 2009, Dotson and Nagle [12] generalized these results, showing that for all families of r -graphs $\mathcal{F}$ the number of induced $\mathcal{F}$ -free r -graphs is $2^{\text{ex}^*(n, \mathcal{F}) + o(n^r)}$.

For a family $\mathcal{F}$ of r -graphs such that $\text{ex}^*(n, \mathcal{F}) = o(n^r)$, the counting results mentioned above are not precise. In the case of graphs, Alon, Balogh, Bollobás and Morris [2] obtained a more refined result. They showed that the number of induced $\mathcal{F}$ -free graphs on n vertices is $2^{\text{ex}^*(n, \mathcal{F}) + O(n^{2-\varepsilon})}$, where $\varepsilon > 0$ depends only on the family $\mathcal{F}$. Terry [37] generalized this result to finite relational languages which in particular covers r -graphs. For a family of r -graphs $\mathcal{F}$, her result says that the number of induced $\mathcal{F}$ -free 3-graphs is either $2^{\Theta(n^r)}$ or there exists $\varepsilon > 0$ such that for all large enough n , the number of induced $\mathcal{F}$ -free 3-graphs is at most $2^{n^{r-\varepsilon}}$.

1.2. Our results

Our first theorem determines the number of induced $\{K_4^3, E_1\}$ -free graphs up to a constant factor on the exponent, making progress towards Conjecture 1.1.

Theorem 1.2. *The number of induced $\{K_4^3, E_1\}$ -free 3-graphs on n vertices is $2^{\Theta(n^2 \log n)}$.*

More generally, we also determine the number of induced $\mathcal{F}$ -free 3-graphs on n vertices for all families $\mathcal{F}$ of 3-graphs on 4 vertices. Since every 3-graph on 4 vertices is determined by its number of edges, our result is stated in terms of forbidden number of edges. For a set $L \subseteq \{0, 1, 2, 3, 4\}$, let $f(n, 3, 4, L)$ be the number of 3-graphs on n vertices which do not induce $i \in L$ edges on any set of 4 vertices. Our result can be stated as follows, where we do not attempt to optimize the constants in the exponent.

Theorem 1.3. *Let $L \subseteq \{0, 1, 2, 3, 4\}$ be a set. Then, the following holds for $n \geq 13$.*

- (a) *If $\{0, 4\} \subseteq L$ or $\{1, 2, 3\} \subseteq L$, then $f(n, 3, 4, L) \in \{0, 1, 2\}$;*
- (b) *If $L = \{0, 2, 3\}$ or $L = \{1, 2, 4\}$, then $f(n, 3, 4, L) = n + 1$;*

⁵ An r -set or r -subset is a set with exactly r elements.

- (c) If $L = \{0, 1, 3\}$ or $L = \{1, 3, 4\}$, then $f(n, 3, 4, L) = 2^{\Theta(n \log n)}$;
- (d) If $L = \{1, 3\}$, then $f(n, 3, 4, L) = 2^{\binom{n-1}{2}}$;
- (e) If $L \in \{\emptyset, \{0\}, \{1\}, \{3\}, \{4\}, \{0, 1\}, \{3, 4\}\}$, then $f(n, 3, 4, L) = 2^{\Theta(n^3)}$;
- (f) For all the remaining cases, we have $f(n, 3, 4, L) = 2^{\Theta(n^2 \log n)}$.

Note that some of the statements in Theorem 1.3 are trivial and others are known. We included those for the sake of completeness. For a list $L \subseteq \{0, 1, 2, 3, 4\}$, define $L^c = \{4 - i : i \in L\}$. It is not hard to check that if a list L belongs to item (f), then some $T \in \{L, L^c\}$ satisfies $T = \{1, 4\}$ or $\{2\} \subseteq T \subseteq \{0, 1, 2\}$. See Table 1 for a detailed list with the bounds on $f(n, 3, 4, L)$ and the references of statements which proves each of them.

Before we state our next theorem, we need some notation. Let $k, r \in \mathbb{N}$ and $L \subseteq \{0, 1, \dots, \binom{k}{r}\}$ be a set, which we refer to as a *list*. We say that an r -graph G is (L, k) -free if for all $i \in L$ there is no set of k vertices in G inducing exactly i edges. By generalizing our previous notation, we denote by $f(n, r, k, L)$ the number of (L, k) -free r -graphs on n vertices. Our next theorem extends Theorem 1.2 to r -graphs and 3-good lists. We say that a list L is 3-good if $\{i, i+1, i+2\} \cap L \neq \emptyset$ for all $i \in \{0, 1, \dots, \binom{k}{r} - 2\}$. That is, the complement of L does not contain three consecutive integers. Throughout this paper, all logarithms are in base 2.

Theorem 1.4. *Let $n \geq k > r \geq 2$ be integers and $L \subseteq \{0, 1, \dots, \binom{k}{r}\}$ be a list. If L is 3-good, then*

$$f(n, r, k, L) \leq 2^{2kn^{r-1} + n^{r-1} \log n}.$$

The main tool behind the proof of Theorem 1.4 is a lemma which counts the solutions of a certain constraint satisfaction problem, see Lemma 3.1. For $L = \{2, 3, \dots, r+1\}$, we observe that $f(n, r, r+1, L)$ is equal to the number of r -graphs such that, for every pair of edges, the size of their intersection is not $r-1$. This is related to the problem of counting designs, a heavily studied object in combinatorics, see [20,24,25].

The rest of this paper is organized as follows. In Section 2 we discuss the sharpness of Theorem 1.4; in Section 3 we present the proof of Theorem 1.4; in Section 4 we prove Theorems 1.2 and 1.3.

2. Sharpness discussion of Theorem 1.4

In this section, we provide three examples which show that Theorem 1.4 is sharp for $k = r+1$. Our first lemma shows that there is a 3-good list that achieves the upper bound given by Theorem 1.4.

Lemma 2.1. *For $r \geq 2$ we have*

$$f(n, r, r+1, \{2, 3, \dots, r+1\}) = 2^{\Theta(n^{r-1} \log n)}.$$

Proof. The list $L := \{2, 3, \dots, r+1\}$ is 3-good and therefore Theorem 1.4 can be applied, which gives the upper bound. Now, let $M(n, r)$ be the set of r -graphs on $[n]$ such that every $(r-1)$ -subset of $[n]$ is contained in at most one edge. Note that the number of $(L, r+1)$ -free r -graphs on $[n]$ is equal to $|M(n, r)|$. Thus, to lower bound $f(n, r, r+1, L)$ it suffices to show that $|M(n, r)| = 2^{\Omega(n^{r-1} \log n)}$.

One way to lower bound $M(n, r)$ is to use a result of Keevash (Theorem 6.1 in [20]) on *designs*. An r -graph G on $[n]$ is an (n, r, t, λ) -*design* if every t -subset of $[n]$ is contained in exactly λ elements of $E(G)$. If certain divisibility conditions involving n and r are satisfied, namely $(r-i) \mid \binom{n-i}{r-1-i}$ for each $0 \leq i \leq r-2$, then Keevash's result implies that the number of $(n, r, r-1, 1)$ -designs is $2^{\Theta(n^{r-1} \log n)}$. In fact, let n' be the largest integer smaller than n such that $n' \equiv r-2 \pmod{(r!)^2}$. For each $0 \leq i \leq r-2$, there exists an integer A_i such that $\binom{n'-i}{r-1-i} = A_i \cdot \frac{n'-r+2}{r!}$. By the choice of n' , it follows that $r! \mid \binom{n'-i}{r-1-i}$ and hence all divisibility conditions are satisfied. In particular, we have $|M(n, r)| \geq |M(n', r)| \geq 2^{\Omega(n^{r-1} \log n)}$. For the sake of self-completeness, we present here another simple way to derive this inequality. To do so, we build a subfamily of r -graphs in $M(n, r)$ via the following greedy procedure. Let e_1 be an r -subset in $[n]$. For $i > 1$, let e_i be an r -subset in $[n]$ such that $|e_i \cap e_j| \neq r-1$ for all $j \in [i-1]$. The procedure stops when an edge e_i with this property cannot be found. As there are at most rni sets of size r which intersect some element of $\{e_1, \dots, e_{i-1}\}$ in exactly $r-1$ vertices, we have at least $\binom{n}{r} - rni$ choices for e_i . It follows that the procedure lasts for at least $\frac{n^{r-1}}{2r^{r+1}}$ steps. As we have $\binom{n}{r} - rni \geq \frac{1}{2} \binom{n}{r}$ for all $i \leq \frac{n^{r-1}}{2r^{r+1}}$, it follows that the number of r -graphs in $M(n, r)$ is at least

$$|M(n, r)| \geq \frac{\left(\frac{1}{2} \binom{n}{r}\right)^{\frac{n^{r-1}}{2r^{r+1}}}}{\left(\frac{n^{r-1}}{2r^{r+1}}\right)!} \geq \left(\frac{r^{r+1} \binom{n}{r}}{n^{r-1}}\right)^{\frac{n^{r-1}}{2r^{r+1}}} \geq n^{\frac{n^{r-1}}{2r^{r+1}}} = 2^{\Theta(n^{r-1} \log n)}.$$

The factorial term above takes the double counting into consideration. Combining this bound with the upper bound from Theorem 1.4, we obtain $f(n, r, r+1, L) = 2^{\Theta(n^{r-1} \log n)}$. $\square$

Our next example shows that there is a list L which is not 3-good such that the bound presented in Theorem 1.4 does not hold.

Lemma 2.2. *For $r \geq 2$ we have*

$$f(n, r, r+1, \{3, 4, \dots, r+1\}) = 2^{\Theta(n^r)}.$$

Proof. Fix an r -partition $(V_i)_{i \in [r]}$ of the vertex set $[n]$ such that $||V_i| - |V_j|| \leq 1$ for all $i, j \in [r]$. Let G be the r -graph whose set of edges is given by the r -sets with one vertex in each class V_i . This r -graph has $\Theta(n^r)$ edges and every subgraph of it is $(L, r+1)$ -free, where $L := \{3, 4, \dots, r+1\}$. We conclude that $f(n, r, r+1, L) = 2^{\Theta(n^r)}$. $\square$

When $k = n$, there is a 3-good list $L \subseteq \{0, 1, 2, \dots, \binom{n}{r}\}$ such that the bound on $f(n, r, n, L)$ given by Theorem 1.4 is sharp.

Claim 2.3. *Let $r \geq 2$ and L be the set of all odd integers in $\{0, 1, 2, \dots, \binom{n}{r}\}$. Then, we have*

$$f(n, r, n, L) = 2^{\Theta(n^r)}.$$

Proof. The number of (L, n) -free r -graphs on $[n]$ is equal to the number of r -graphs on $[n]$ with an even number of edges. Clearly, there are $2^{\Theta(n^r)}$ of those. $\square$

3. Proof of Theorem 1.4

We will start by proving a combinatorial lemma. To state it we use the language of *constraint satisfaction problems* (CSP). Let $\mathcal{P}$ be the family of all subsets of $\{0, 1\} \times \{0, 1\}$. We refer to the elements of $\mathcal{P}$ as *constraints*. A CSP on $[m]$ is a pair $([m], f)$, where $f : \binom{[m]}{2} \rightarrow \mathcal{P}$ is a function assigning a constraint for each pair of vertices. An *assignment* on $[m]$ is a function $g : [m] \rightarrow \{0, 1\}$ which assigns for every vertex $v \in [m]$ an integer (or *color*) from $\{0, 1\}$. We say that an assignment $g : [m] \rightarrow \{0, 1\}$ is *satisfying* for $([m], f)$ if we have $(g(a), g(b)) \notin f(\{a, b\})$ for any pair $a, b \in [m]$ such that $a < b$.

For a CSP $G = ([m], f)$, let $A(G)$ be the set of satisfying assignments for G . Now, define

$$\mathcal{C} := \left\{ \{(1, 0), (0, 1)\}, \{(0, 0)\}, \{(1, 1)\} \right\}.$$

Observe that $\mathcal{C}$ is a subset of $\mathcal{P}$. Our next lemma shows that for $f : \binom{[m]}{2} \rightarrow \mathcal{C}$ the CSP $G = ([m], f)$ satisfies $|A(G)| \leq m + 1$. This bound is best possible, as we can see by the following example. Let $f : \binom{[m]}{2} \rightarrow \mathcal{C}$ be the function given by $f(\{i, j\}) = \{(1, 1)\}$ for all $i < j$ and let $G = ([m], f)$ be the corresponding CSP. A function $g : [m] \rightarrow \{0, 1\}$ is a satisfying assignment for G if and only if $|g^{-1}(1)| \leq 1$. As there are exactly $m + 1$ choices of $g^{-1}(1)$ for which $|g^{-1}(1)| \leq 1$, we have $|A(G)| = m + 1$.

Lemma 3.1. *Let $m \in \mathbb{N}$, $f : \binom{[m]}{2} \rightarrow \mathcal{C}$ and $G = ([m], f)$ be a CSP. The size of the set $A(G)$ of satisfying assignments for G is bounded by*

$$|A(G)| \leq m + 1.$$

Proof. We proceed by induction on m . The base case is $m = 1$. For a CSP G on one vertex we have $|A(G)| \leq 2$, as there is only one vertex to be assigned with a color from $\{0, 1\}$. Now, let $m \geq 2$ and assume that for every $i < m$ and every CSP G on i vertices we have $|A(G)| \leq i + 1$.

Fix a function $f: \binom{[m]}{2} \rightarrow \mathcal{C}$ and the CSP given by $G = ([m], f)$. We partition the set of satisfying assignments into $A(G) = A_0 \cup A_1$, where

$$A_0 := \{g \in A(G) : g(m) = 0\} \quad \text{and} \quad A_1 := \{g \in A(G) : g(m) = 1\}.$$

We gain extra information about the satisfying assignments in each of these sets based on the constraints given by f . Note that if $f(\{j, m\}) = \{(1, 0), (0, 1)\}$, then we automatically have $g(j) = i$ for all $g \in A_i$ and $i \in \{0, 1\}$. If $f(\{j, m\}) = \{(i, i)\}$, then we must have $g(j) = 1 - i$ for all $g \in A_i$ and $i \in \{0, 1\}$. This motivates us to consider the following subsets of $[m]$:

$$F_0 := \{j < m : f(\{j, m\}) = \{(1, 1)\}\} \quad \text{and} \quad F_1 := \{j < m : f(\{j, m\}) = \{(0, 0)\}\}.$$

For each $i \in \{0, 1\}$, the set F_i corresponds to the values of $j < m$ which are *free*, that is, the values of j for which $g(j)$ might not be the same for all functions $g \in F_i$. More precisely, our analysis shows that for $g_1, g_2 \in A_i$ and $j \notin F_i$ we have $g_1(j) = g_2(j)$, for $i \in \{0, 1\}$.

Let $G[F_0] := (F_0, f|_{\binom{F_0}{2}})$ and $G[F_1] := (F_1, f|_{\binom{F_1}{2}})$ be the *induced CSP* of G on F_0 and F_1 , respectively. It follows that

$$|A(G)| \leq |A(G[F_0])| + |A(G[F_1])|. \quad (1)$$

As F_0 and F_1 are subsets of $[m - 1]$, by the induction hypothesis we have

$$|A(G[F_0])| \leq |F_0| + 1 \quad \text{and} \quad |A(G[F_1])| \leq |F_1| + 1. \quad (2)$$

As F_0 and F_1 are disjoint, we have $|F_0| + |F_1| \leq m - 1$. Combining (1) and (2), we obtain

$$|A(G)| \leq |F_0| + 1 + |F_1| + 1 \leq m + 1.$$

This completes our proof of Lemma 3.1. $\square$

Lemma 3.1 will be used in the proof of Lemma 3.3 below. For the rest of this section, we fix natural numbers $k > r$ and a 3-good list $L \subseteq \{0, 1, \dots, \binom{k}{r}\}$. For simplicity, denote $f(n) := f(n, r, k, L)$ and let $\mathcal{F}(n)$ be the family of (L, k) -free r -graphs on $[n]$. In order to bound $f(n)$, we introduce other related quantities. For a set $A \subseteq [n]$ and an r -graph $H \in \mathcal{F}(n)$, define

$$D(A, H, n) := \left\{ G \in \mathcal{F}(n) : A \subseteq e \quad \forall e \in E(G) \Delta E(H) \right\},$$

where Δ denotes the symmetric difference. In words, $D(A, H, n)$ is the set of (L, k) -free graphs on $[n]$ for which the edges and non-edges agree with H on the r -sets not

containing A . When $|A| = 1$, $D(A, H, n)$ is the set of r -graphs which extend $H|_{[n] \setminus A}$ to an (L, k) -free r -graph on $[n]$. It is also convenient to define

$$d(a, n) := \max \{|D(A, H, n)| : H \in \mathcal{F}(n)\},$$

where A is an arbitrary set of size a . As the maximum is taken over all $H \in \mathcal{F}(n)$ and the set $\mathcal{F}(n)$ is closed for permuting the vertices of the graphs, it does not depend on the set A . When $a = 1$, we can think of $d(1, n)$ as the maximum number of extensions that an r -graph may have. We can easily see that

$$f(n) \leq d(1, n) \cdot f(n-1) \quad (3)$$

for all $n \in \mathbb{N}$. Therefore, to bound $f(n)$ we will bound $d(1, n)$ for all $n \in \mathbb{N}$.

In order to upper bound $d(1, n)$, we first provide an upper bound on $d(a, n)$ which depends on $d(a+1, v)$, for $v \in \{r, \dots, n\}$.

Lemma 3.2. *Let n, r, k and a be natural numbers such that $k > r$ and $n > r - 1 \geq a$. Then, we have*

$$d(a, n) \leq \prod_{v=r}^n d(a+1, v).$$

Proof. Let H be an (L, k) -free r -graph on $[n]$ and let $A \subseteq [n-1]$ be a set of size a . For an r -graph G on $[n]$, denote by G_{n-1} the r -graph on $[n-1]$ induced by G . Observe that if $G \in D(A, H, n)$, then $G_{n-1} \in D(A, H_{n-1}, n-1)$. We then partition the set $D(A, H, n)$ according to the r -graph induced on the vertex set $[n-1]$. For each r -graph $J \in D(A, H_{n-1}, n-1)$, let $T_H(A, J, n)$ be the set of r -graphs G in $D(A, H, n)$ for which $G_{n-1} = J$. Then,

$$D(A, H, n) = \bigcup_J T_H(A, J, n), \quad (4)$$

where the union is over $J \in D(A, H_{n-1}, n-1)$.

We claim that $T_H(A, J, n) \subseteq D(\{n\} \cup A, G, n)$ for all $G \in T_H(A, J, n)$. To prove this, let G and G' be r -graphs in $T_H(A, J, n)$. As $G_{n-1} = J = G'_{n-1}$, we have $n \in e$ for all $e \in E(G') \Delta E(G)$. Moreover, as $G, G' \in D(A, H, n)$, we have $A \subseteq e$ for all $e \in E(G) \Delta E(G')$. This implies that $\{n\} \cup A \subseteq e$ for all $e \in E(G) \Delta E(G')$, which proves our claim. As $|A \cup \{n\}| = a+1$, we obtain

$$|T_H(A, J, n)| \leq d(a+1, n) \quad (5)$$

for all H and $J \in D(A, H_{n-1}, n-1)$. Combining (4) and (5), we get

$$|D(A, H, n)| \leq |D(A, H_{n-1}, n-1)| \cdot d(a+1, n) \leq d(a, n-1) \cdot d(a+1, n). \quad (6)$$

As (6) holds for every (L, k) -free r -graph H on n vertices, (6) implies that

$$\frac{d(a, n)}{d(a, n-1)} \leq d(a+1, n).$$

Let $m \geq r$ be an integer. Performing the telescopic product with n going from r to m , we have

$$\frac{d(a, m)}{d(a, r-1)} \leq \prod_{v=r}^m d(a+1, v).$$

As $d(a, r-1) = 1$, this completes our proof. $\square$

Recall that we need to bound $d(1, n)$ in order to bound $f(n)$. To do so, we use ‘backward’ induction. If we bound $d(a+1, n)$ for every $n \geq k$, then Lemma 3.2 yields a simple bound on $d(a, n)$ in terms of $d(a+1, n)$. As $d(r, n) \leq 2$, if we apply the recursion in Lemma 3.2 using this bound then we obtain $d(1, n) = 2^{O(n^{r-1})}$. However, this only implies that $f(n) = 2^{O(n^r)}$. We overcome this problem in our next lemma, where we obtain a polynomial bound for $d(r-1, n)$ instead.

Lemma 3.3. *Let n, r and k be natural numbers such that $n \geq r \geq 2$ and $k > r$. If $L \subseteq \{0, 1, \dots, \binom{k}{r}\}$ is a 3-good list, then*

$$d(r-1, n) \leq 2^k n.$$

Proof. Let H be an (L, k) -free r -graph. Recall that $D([r-1], H, n)$ is the set of (L, k) -free r -graphs G on $[n]$ for which the following holds. If $e \in E(H) \triangle E(G)$, then $[r-1] \subseteq e$. We associate the problem of counting the r -graphs in $D([r-1], H, n)$ to the problem of counting the 2-colorings of the vertices in $[n] \setminus [r-1]$ under certain restrictions. The first step is to define for each $G \in D([r-1], H, n)$ a coloring $C_G : [n] \setminus [r-1] \rightarrow \{0, 1\}$ as follows:

$$\begin{cases} C_G(i) = 1, & \text{if } [r-1] \cup \{i\} \in E(G), \\ C_G(i) = 0, & \text{otherwise.} \end{cases}$$

Observe that the number of r -graphs in $D([r-1], H, n)$ is equal to the number of colorings $c : [n] \setminus [r-1] \rightarrow \{0, 1\}$ for which

$$(H \cup \{i \cup [r-1] : i \in c^{-1}(1)\}) \setminus \{i \cup [r-1] : i \in c^{-1}(0)\}$$

forms an (L, k) -free r -graph. As $|D([r-1], H, n)| \leq 2^k$ when $n \leq k$, from now on we assume that $n > k$. We partition the set $D([r-1], H, n)$ according to a partial coloring

on the set $S := [k-2] \setminus [r-1]$. For each coloring $c : S \rightarrow \{0, 1\}$, let $D(c)$ be the set of r -graphs $G \in D([r-1], H, n)$ for which $C_G|_S = c$. Now,

$$D([r-1], H, n) = \bigcup_{c: S \rightarrow \{0,1\}} D(c).$$

We may think that all the edges and non-edges corresponding to r -sets of the form $[r-1] \cup \{i\}$, for $i \in S$, are fixed inside $D(c)$. Our objective now is to bound the size of $D(c)$ via a constraint satisfaction problem on $[n] \setminus [k-2]$. To do so, we need to introduce some notation. For each pair of distinct vertices $\{i, j\} \subseteq [n] \setminus [k-2]$, let $R_{i,j}$ be the collection of all r -sets in $[k-2] \cup \{i, j\}$ which are different from $[r-1] \cup \{i\}$ and $[r-1] \cup \{j\}$. Observe that all r -graphs in $D(c)$ coincide on the r -sets in $R_{i,j}$. More precisely, for $G_1, G_2 \in D(c)$ and $e \in R_{i,j}$, we have $e \in E(G_1)$ if and only if $e \in E(G_2)$. Let $\|R_{i,j}\|_c$ be the number of edges in $R_{i,j}$ which are in common for every r -graph in $D(c)$. Note that we cannot have r -graphs G_1, G_2 and G_3 in $D(c)$ which induce $\|R_{i,j}\|_c$, $\|R_{i,j}\|_c + 1$ and $\|R_{i,j}\|_c + 2$ edges in $[k-2] \cup \{i, j\}$, respectively, as this would contradict the assumption that $L \cap \{a-1, a, a+1\} \neq \emptyset$ for all $a \in \left[\binom{k}{r} - 1\right]$.

For each $\{i, j\} \subseteq [n] \setminus [k-2]$, let $t_{i,j}^c \in \{0, 1, 2\}$ be such that $\|R_{i,j}\|_c + t_{i,j}^c \in L$. Then, for every $G \in D(c)$ we must have $C_G(i) + C_G(j) \neq t_{i,j}^c$, otherwise G would have a forbidden structure on $[k-2] \cup \{i, j\}$. If $t_{i,j}^c = 0$ or 2 , this is equivalent to the constraint $(C_G(i), C_G(j)) \neq (t_{i,j}^c/2, t_{i,j}^c/2)$; and if $t_{i,j}^c = 1$, it is equivalent to the constraint $(C_G(i), C_G(j)) \notin \{(0, 1), (1, 0)\}$. We can then define the following constraint function $f_c : \binom{[n] \setminus [k-2]}{2} \rightarrow \mathcal{C}$:

$$f(\{i, j\}) := \begin{cases} \{(0, 0)\}, & \text{if } t_{i,j}^c = 0, \\ \{(1, 0), (0, 1)\}, & \text{if } t_{i,j}^c = 1, \\ \{(1, 1)\}, & \text{if } t_{i,j}^c = 2. \end{cases}$$

Let $G_c = ([n] \setminus [k-2], f_c)$ be a CSP. It follows that $|D(c)| \leq |A(G_c)|$, where $A(G_c)$ is the set of satisfying assignments of the CSP G_c . By Lemma 3.1, we have $|A(G_c)| \leq n - k + 3$ and hence

$$|D([r-1], H, n)| \leq \sum_{c: S \rightarrow \{0,1\}} |D(c)| \leq n \cdot 2^{|S|}.$$

As $|S| \leq k$, this proves our lemma. $\square$

Combining Lemmas 3.2 and 3.3, we obtain the following corollary.

Corollary 3.4. *Let n, k and r be natural numbers such that $n, k \geq r \geq 2$ and let $L \subseteq \{0, 1, \dots, \binom{k}{r}\}$ be a 3-good list. Then, for all $i \in [r-1]$ we have*

$$d(r-i, n) \leq 2^{kn^{i-1} + n^{i-1} \log n}.$$

Proof. We proceed by induction on i . By Lemma 3.3, the statement holds for $i = 1$. Now, assume that the lemma holds for some $i \in [r-2]$. By Lemma 3.2, we obtain

$$d(r-(i+1), n) \leq \prod_{v=r}^n d(r-i, v) \leq \prod_{v=r}^n 2^{kv^{i-1} + v^{i-1} \log v} \leq 2^{kn^i + n^i \log n},$$

as required. $\square$

We are now ready to complete the proof of Theorem 1.4. Let $n \geq k > r \geq 2$ be integers and $L \subseteq \{0, 1, \dots, \binom{k}{r}\}$ be a 3-good list. By (3), we have $f(n) \leq d(1, n) \cdot f(n-1)$ and hence

$$f(n) \leq f(k) \cdot \prod_{v=k+1}^n d(1, v).$$

As $d(v) = d(1, v)$, by Corollary 3.4 we have $d(v) \leq 2^{kv^{r-2} + v^{r-2} \log v}$. From this together with the trivial bound $f(k) \leq 2^{k^r}$, we obtain

$$f(n) \leq 2^{k^r} \prod_{v=k+1}^n 2^{kv^{r-2} + v^{r-2} \log v} \leq 2^{k^r + kn^{r-1} + n^{r-1} \log n},$$

as required.

4. Proof of Theorems 1.2 and 1.3

In this section we prove Theorem 1.3, that is, we determine $f(n, 3, 4, L)$ asymptotically for all possible L . In particular, we prove Theorem 1.2. For simplicity, we denote $f(n, L) = f(n, 3, 4, L)$ and assume that $n \geq 4$ throughout this section.

For a list $L \subseteq \{0, 1, 2, 3, 4\}$, recall that $L^c = \{4-i : i \in L\}$. Observe that $f(n, L) = f(n, L^c)$, as a 3-graph G does not induce i edges on 4 vertices if and only if its complement G^c does not induce $4-i$ edges on 4 vertices. In light of this, to prove Theorem 1.3 it is sufficient to bound $f(n, L)$ for only one set L in each row of Table 1 below. When $\{0, 4\} \subseteq L$, our proof trivially follows from Ramsey's theorem, see Claim 4.1. When $\{2\} \subseteq L \subseteq \{0, 1, 2\}$, L is 3-good and hence the upper bound on $f(n, L)$ follows from Theorem 1.4. The lower bound is obtained via the same greedy procedure used in the proof of Lemma 2.1, see Claim 4.2 for more details. These and the trivial cases already fill a good proportion of the table. The remaining cases are more delicate and we need to deal with each of them separately. We start with the proof of Theorem 1.2.

Proof of Theorem 1.2. The upper bound follows immediately from Theorem 1.4. For the lower bound, we consider the family of 3-graphs $Q(n)$ presented in the introduction. Let

Table 1This table shows the values for $f(n, 3, 4, L)$ for all possible L .

L	$f(n, 3, 4, L)$	Proof
$\emptyset$	$2^{\binom{n}{3}}$	trivial
$\{0\}, \{4\}$	$2^{\Theta(n^3)}$	Claim 4.6
$\{1\}, \{3\}$	$2^{\Theta(n^3)}$	follows from $\{3, 4\}$
$\{0, 1\}, \{3, 4\}$	$2^{\Theta(n^3)}$	Claim 4.6
$\{2\}$	$2^{\Theta(n^2 \log n)}$	Claim 4.2
$\{0, 2\}, \{2, 4\}$	$2^{\Theta(n^2 \log n)}$	Claim 4.2
$\{0, 3\}, \{1, 4\}$	$2^{\Theta(n^2 \log n)}$	Theorem 1.2
$\{1, 2\}, \{2, 3\}$	$2^{\Theta(n^2 \log n)}$	Claim 4.2
$\{0, 1, 2\}, \{2, 3, 4\}$	$2^{\Theta(n^2 \log n)}$	Claim 4.2
$\{1, 3\}$	$2^{\binom{n-1}{2}}$	Lemma 4.5
$\{0, 1, 3\}, \{1, 3, 4\}$	$2^{\Theta(n \log n)}$	Lemma 4.4
$\{0, 2, 3\}, \{1, 2, 4\}$	$n + 1$ for $n \geq 5$	Claim 4.8
$\{1, 2, 3\}$	2	Claim 4.7
$\{1, 2, 3, 4\}, \{0, 1, 2, 3\}$	1	trivial
$\{0, 4\}$	0 for $n \geq 13$	Claim 4.1
$\{0, 1, 4\}, \{0, 3, 4\}$	0 for $n \geq 13$	Claim 4.1
$\{0, 2, 4\}$	0 for $n \geq 13$	Claim 4.1
$\{0, 2, 3, 4\}, \{0, 1, 2, 4\}$	0 for $n \geq 13$	Claim 4.1
$\{0, 1, 3, 4\}$	0 for $n \geq 13$	Claim 4.1
$\{0, 1, 2, 3, 4\}$	0	trivial

$[n] = V_1 \cup V_2 \cup V_3$ be a partition of $[n]$ with $||V_i| - |V_j|| \leq 1$ for all $i, j \in [3]$, which is the same partition as the one used in the definition of C_n . Recall that a 3-graph is in $Q(n)$ if it is obtained from C_n by removing a linear 3-graph with the additional property that every edge contains one vertex from each of the classes V_1, V_2 and V_3 . Note that every 4-set in C_n has either 0, 2 or 3 edges. Moreover, 4-sets containing at least one vertex of each of the classes have 3 edges. This implies that for a 3-graph $H \in Q(n)$ every 4-set has either 0, 2 or 3 edges. As $Q(n)$ is a family of $(\{1, 4\}, 4)$ -free 3-graphs on n vertices, to lower bound $f(n, \{1, 4\})$ it suffices to lower bound $|Q(n)|$.

Let $L(n)$ be the family of linear 3-graphs on $[n]$ with the additional property that every edge intersects each of the classes V_1, V_2 and V_3 . Clearly, we have $|Q(n)| = |L(n)|$. Now, we lower bound $|L(n)|$ via the following greedy procedure, which is similar to the one in the proof of Lemma 2.3. Let e_1 be a 3-set in $[n]$ such that $|e_1 \cap V_j| = 1$ for all $j \in [3]$. For $i > 1$, let e_i be a 3-set in $[n]$ such that $|e_i \cap V_j| = 1$ for all $j \in [3]$ and such that $|e_i \cap e_k| \leq 1$ for all $k \in [i-1]$. When an edge e_i with this property cannot be found, the procedure stops and outputs $\{e_1, \dots, e_{i-1}\}$. Observe that every 3-graph obtained from this procedure belongs to $L(n)$. As there are at most ni sets of size 3 which intersect some element of $\{e_1, \dots, e_{i-1}\}$ in 2 vertices or more, we have at least $|V_1||V_2||V_3| - ni$ choices for e_i . This implies that the procedure lasts for at least $\frac{n^2}{27} - \frac{n^2}{\log n}$ steps. Moreover, as we have $|V_1||V_2||V_3| - ni \geq \frac{n^3}{2 \log n}$ for all $i \leq \frac{n^2}{27} - \frac{n^2}{\log n}$, the number of 3-graphs in $L(n)$ and hence in $Q(n)$ is at least

$$|Q(n)| \geq \frac{\left(\frac{n^3}{2 \log n}\right)^{\frac{n^2}{27} - \frac{n^2}{\log n}}}{\left(\left\lfloor \frac{n^2}{27} - \frac{n^2}{\log n} \right\rfloor\right)!} \geq \left(\frac{n}{\log n}\right)^{\frac{n^2}{27} - \frac{n^2}{\log n}} = 2^{(\frac{1}{27} + o(1))n^2 \log n}. \quad (7)$$

The factorial term above takes the double counting into consideration. Combining this bound with the upper bound from Theorem 1.4, we obtain $f(n, \{1, 4\}) = 2^{\Theta(n^2 \log n)}$. $\square$

Claim 4.1. *If $n \geq 13$ and $\{0, 4\} \subseteq L$, then $f(n, L) = 0$.*

Proof. Let $R_3(4, 4)$ be the smallest integer n such that every red and blue edge-coloring of the complete 3-graph K_n^3 contains a red copy of K_4^3 or a blue copy of K_4^3 . The hypergraph Ramsey number $R_3(4, 4)$ was determined by McKay and Radziszowski in [27], where they showed that $R_3(4, 4) = 13$. We conclude that for all sets L such that $\{0, 4\} \subseteq L$ there is no $(L, 4)$ -free 3-graph on at least 13 vertices. $\square$

Claim 4.2. *Let L be a list such that $\{2\} \subseteq L \subseteq \{0, 1, 2\}$. Then, $f(n, L) = 2^{\Theta(n^2 \log n)}$.*

Proof. Let L be a list such that $\{2\} \subseteq L \subseteq \{0, 1, 2\}$. Then, for every $i \in [3]$ we have $L \cap \{i-1, i, i+1\} \neq \emptyset$ and hence L is 3-good. By Theorem 1.4, it follows that $f(n, L) \leq 2^{\Theta(n^2 \log n)}$.

To show a lower bound of the same order, we define the set $M(n, 3)$ of 3-graphs on $[n]$ such that every pair of vertices is contained in at most one edge. That is, $M(n, 3)$ is the set of all linear 3-graphs on $[n]$. For a graph $H \in M(n, 3)$, we have that the graph $K_n^3 - H$ is $(L, 4)$ -free and hence $f(n, L) \geq |M(n, 3)|$. A lower bound on $|M(n, 3)|$ was already obtained in the proof of Lemma 2.1, where we showed that $|M(n, 3)| \geq 2^{\Theta(n^2 \log n)}$. $\square$

To bound $f(n, L)$ for $L = \{0, 1, 3\}$ and $L = \{1, 3\}$, it is convenient to define the link graph of a vertex. For a 3-graph H and a vertex v of H , we define $S_H(v)$ to be the graph with vertex set $V(H) \setminus \{v\}$ and edge set

$$E(S_H(v)) := \{e \setminus \{v\} : v \in e, e \in E(H)\}.$$

We refer to $S_H(v)$ as the *link graph* of v in H . For a list L , let $\mathcal{F}(n, L)$ be the family of $(L, 4)$ -free 3-graphs on $[n]$ and $\mathcal{S}_L := \{S_G(n) : G \in \mathcal{F}(n, L)\}$ be a family of link graphs of n associated to $\mathcal{F}(n, L)$. The following claim establishes a bijection between $\mathcal{F}(n, L)$ and $\mathcal{S}_L$ when $L = \{0, 1, 3\}$ or $L = \{1, 3\}$.

Claim 4.3. *If $L = \{0, 1, 3\}$ or $L = \{1, 3\}$, then for every $A \in \mathcal{S}_L$ there is a unique $G \in \mathcal{F}(n, L)$ such that $S_G(n) = A$. Moreover, for every 3-set $\{a_1, a_2, a_3\}$ in $[n-1]$, the following holds:*

- (1) *If A has 1 or 3 edges in $\{a_1, a_2, a_3\}$, then $a_1 a_2 a_3 \in E(G)$;*
- (2) *If A has 2 edges in $\{a_1, a_2, a_3\}$, then $a_1 a_2 a_3 \notin E(G)$;*
- (3) *If $L = \{0, 1, 3\}$, then A has at least one edge in $\{a_1, a_2, a_3\}$;*
- (4) *If $L = \{1, 3\}$ and A has no edge in $\{a_1, a_2, a_3\}$, then $a_1 a_2 a_3 \notin E(G)$.*



Fig. 2. All non-isomorphic graphs on 4 vertices satisfying (iii).

Proof. If some item of this claim is not satisfied, then we create a forbidden structure. Moreover, from items (1)–(4) it follows that for $L = \{0, 1, 3\}$ or $L = \{1, 3\}$ and for each $A \in \mathcal{S}$ there is unique $G \in \mathcal{F}(n, L)$ such that $S_G(n) = A$. $\square$

Let $L = \{0, 1, 3\}$ or $L = \{1, 3\}$ and $A \in \mathcal{S}_L$. From now on, we denote by G_A an unique graph such that $S_{G_A}(n) = A$, which was given by Claim 4.3.

Lemma 4.4. *If $L = \{0, 1, 3\}$, then $f(n, L) = 2^{\Theta(n \log n)}$.*

Proof. By Claim 4.3, there is a bijection between $\mathcal{F}(n, L)$ and $\mathcal{S}_L$, and hence to bound $|\mathcal{F}(n, L)|$ it suffices to determine all graphs which belong to $\mathcal{S}_L$. Fix some $A \in \mathcal{S}_L$ and let A^c be its complement, that is, the graph with vertex set $[n - 1]$ and edge set $\binom{[n-1]}{2} \setminus E(A)$. By Claim 4.3(3), we already know that A^c must be triangle-free. To see which other conditions A must satisfy, we analyze the graph induced by A in each 4-set in $[n - 1]$. We first observe that graphs on 4 vertices can be divided into 3 categories: (i) the complement contains a triangle; (ii) the graph induces a C_4 ; (iii) the assumptions in items (i) and (ii) do not hold.

Let $\{a, b, c, d\}$ be a set of size 4 in $[n - 1]$. As A^c is triangle-free, we already know that $\{a, b, c, d\}$ does not satisfy item (i) in A . Now, we claim that A does not induce a C_4 in $\{a, b, c, d\}$, hence item (ii) does not hold. Indeed, if A induces a C_4 in $\{a, b, c, d\}$, then it follows from Claim 4.3(2) that G_A has no edge in $\{a, b, c, d\}$, which is a contradiction. We conclude that every set of 4 vertices in A satisfies item (iii). This is equivalent to saying that A^c is free of triangles and free of induced matchings of size 2.

For $m \in \mathbb{N}$, let $\mathcal{F}_{\Delta, M}(m)$ be the family of graphs on $[m]$ which are free of triangles and induced matchings of size 2. We have seen that if $A \in \mathcal{S}$, then $A^c \in \mathcal{F}_{\Delta, M}(n - 1)$. Now, we claim that the converse also holds. Let H be a graph such that $H^c \in \mathcal{F}_{\Delta, M}(n - 1)$ and denote by G^H the 3-graph on $[n]$ which satisfies items (1)–(3), with A replaced by H and G replaced by G^H . By the definition of G^H , all 4-sets in $[n]$ containing n do not induce a forbidden structure in G^H . Now, let $\{a, b, c, d\}$ be an arbitrary 4-set in $[n - 1]$. As $H^c \in \mathcal{F}_{\Delta, M}(n - 1)$, the graph induced by H in $\{a, b, c, d\}$ satisfies item (iii). We represent in Fig. 2 all non-isomorphic graphs that H can induce on $\{a, b, c, d\}$. Using Claim 4.3(1) and (2), a careful analysis on the number of edges in $\{a, b, c, d\}$ shows that G^H does not induce any forbidden structure. Therefore, we have $G^H \in \mathcal{F}(n, L)$, which implies that $H \in \mathcal{S}$. In particular, $G^H = G_H$.

Now it remains to bound the size of $\mathcal{F}_{\Delta, M}(m)$. To do so, we first claim that a graph $G \in \mathcal{F}_{\Delta, M}(m)$ has chromatic number at most 3. Indeed, fix $G \in \mathcal{F}_{\Delta, M}(m)$ and let uv be any edge of G . Let $N(u)$ and $N(v)$ be the neighborhoods of u and v , respectively. These neighborhoods cannot intersect, otherwise we create a triangle. Moreover, the set

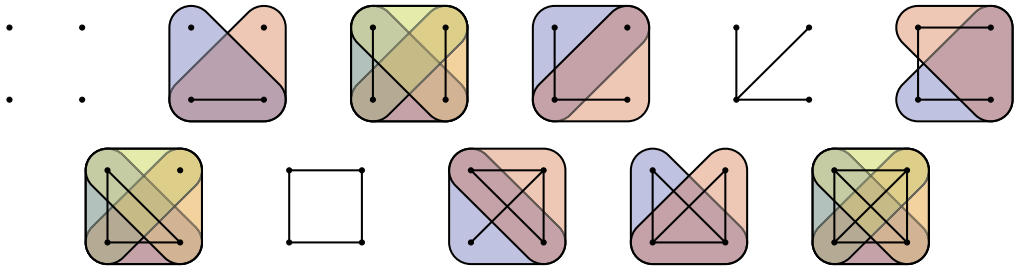


Fig. 3. All possible non-isomorphic link graphs and 3-graphs induced by G^H on $\{a, b, c, d\}$.

$[m] \setminus (\{u, v\} \cup N(u) \cup N(v))$ cannot have an edge, otherwise we create an induced matching of size 2 with uv . It follows that we can divide the graph into three disjoint independent sets: $A_1 = \{u\} \cup N(v)$, $A_2 = \{v\} \cup N(u)$ and $A_3 := [m] \setminus (\{u, v\} \cup N(u) \cup N(v))$. This proves our claim.

Let $b(m)$ be the number of bipartite graphs with m vertices in each class and with no induced matching of size 2. As every graph in $\mathcal{F}_{\Delta, M}(m)$ has chromatic number at most 3 and does not induce a matching of size 2, we have

$$b(\lfloor m/2 \rfloor) \leq |\mathcal{F}_{\Delta, M}(m)| \leq 3^m \cdot (b(m))^3. \quad (8)$$

The factor of 3^m in the upper bound accounts for the number of ways to partition the set $[m]$ into 3 parts. To bound $b(m)$, we use an argument which appears in [26]. In [26], the authors observed that a bipartite graph with parts A and B has no induced matching if and only if for every $a_1, a_2 \in A$ we have $N(a_1) \subseteq N(a_2)$ or $N(a_2) \subseteq N(a_1)$. That is, the set $\{N(a) : a \in A\}$ forms a chain. Observe that the number of chains of length m is equal to the number of ways to distribute the elements of $[m]$ into m disjoint labeled sets $S_1, \dots, S_m$. On one hand, this number is at least $m! = m^{\Theta(m)}$, which is the number of ways to place exactly one element in each S_i . On the other hand, we have the trivial upper bound m^m , hence the number of chains of length m is of order $m^{\Theta(m)}$. By (8), we obtain $|\mathcal{F}_{\Delta, M}(m)| = m^{\Theta(m)}$ and therefore

$$|\mathcal{F}(n, 3, 4, L)| = |\mathcal{F}_{\Delta, M}(n-1)| = n^{\Theta(n)}. \quad \square$$

Lemma 4.5. If $L = \{1, 3\}$, then $f(n, L) = 2^{\binom{n-1}{2}}$.

Proof. By Claim 4.3, there is a bijection between $\mathcal{F}(n, L)$ and $\mathcal{S}_L$, and hence to bound $|\mathcal{F}(n, L)|$ it suffices to determine all graphs which belong to $\mathcal{S}_L$. We claim that $\mathcal{S}_L$ contains all graphs on $[n-1]$. Let H be a graph on $[n-1]$ and denote by G^H the 3-graph on $[n]$ which satisfies items (1), (2) and (4) in Claim 4.3, with A replaced by H and G replaced by G^H . By the definition of G^H , every 4-set in $[n]$ containing n does not induce a forbidden structure in G^H . Now, let $\{a, b, c, d\}$ be an arbitrary 4-set in $[n-1]$. In Fig. 3, we show all possible non-isomorphic graphs on $\{a, b, c, d\}$ and their associated 3-graphs

satisfying items (1), (2) and (4) in Claim 4.3. We can see that for any graph induced by H on $\{a, b, c, d\}$, the 3-graph G^H does not induce any forbidden structure on $\{a, b, c, d\}$. As the number of graphs on $[n-1]$ is $2^{\binom{n-1}{2}}$, we have $|\mathcal{F}(n, L)| = 2^{\binom{n-1}{2}}$. $\square$

Claim 4.6. *If $L = \{4\}$ or $L = \{3, 4\}$, then $f(n, L) = 2^{\Theta(n^3)}$.*

Proof. Denote K_4^{3-} the 3-graph on 4 vertices with 3 edges. For $L = \{3, 4\}$, $f(n, L)$ counts the K_4^{3-} -free n -vertex 3-graphs and $f(n, \{4\})$ counts the K_4^3 -free n -vertex 3-graphs. Observe that

$$\frac{2}{7} \binom{n}{3} (1 + o(1)) \leq \text{ex}(n, K_4^{3-}) \leq \text{ex}(n, K_4^3) \leq \binom{n}{3},$$

where the lower bound comes from a construction by Frankl and Füredi [18]. It follows that

$$2^{\frac{2}{7} \binom{n}{3} (1+o(1))} \leq f(n, \{3, 4\}) \leq f(n, \{4\}) \leq 2^{\binom{n}{3}}. \quad \square$$

Claim 4.7. *Let $n \geq 4$ and $L = \{1, 2, 3\}$. Then, $f(n, L) = 2$.*

Proof. Let G be an $(L, 4)$ -free 3-graph on $[n]$ with at least one edge. Let C be a maximal clique in G and suppose for contradiction that there is a vertex $v \notin C$. Note that because G contains an edge, $|C| \geq 3$. For any 3 distinct vertices i, j and k in C , we have that $\{i, j, k, v\}$ induces a complete 3-graph. Thus, $C \cup \{v\}$ is a clique as well, which contradicts the maximality of C . It follows that all the vertices in $[n]$ are in C , and hence G is complete. We conclude that the only $(L, 4)$ -free 3-graphs on $[n]$ are the complete 3-graph and the 3-graph with no edges. $\square$

Claim 4.8. *Let $n \geq 5$ and $L = \{0, 2, 3\}$. Then, $f(n, L) = n + 1$.*

Proof. Let G be an $(L, 4)$ -free 3-graph on $[n]$. As $0 \in L$, the 3-graph G has at least one edge, hence we can fix a maximal clique C in G . We claim that there is no edge containing exactly 2 vertices of C . Suppose for contradiction that there is $v \notin C$ and $\{i, j\} \subseteq C$ such that $vij \in E(G)$. Then, for every $k \in C \setminus \{i, j\}$ the set $\{v, i, j, k\}$ spans at least 2 edges. As $2, 3 \in L$, we have no other choice but to have a complete 3-graph on $\{v, i, j, k\}$ for all $k \in C \setminus \{i, j\}$. As vik is an edge for all $k \in C \setminus \{i, j\}$, we can repeat the same argument and show that $\{v, i, k, \ell\}$ induces a complete graph for all $k, \ell \in C \setminus \{i\}$. We conclude that $vk\ell \in E(G)$ for all $k, \ell \in C$, and hence $C \cup \{v\}$ must be a clique. This contradicts the maximality of C .

We now claim that $|V(G) \setminus C| \leq 1$. Suppose for contradiction that there exists distinct vertices $i, j \notin C$. Let c_1 and c_2 be distinct vertices in C . As c_1c_2i and c_1c_2j are not edges of G and $0, 2 \in L$, we have that either ijc_1 or ijc_2 is an edge of G , otherwise we create a forbidden structure in $\{i, j, c_1, c_2\}$. Without loss of generality, suppose that $ijc_1 \in E(G)$.

We can now take another vertex $c_3 \in C \setminus \{c_1, c_2\}$ and repeat the same argument. By analyzing the 3-graph induced by G on $\{i, j, c_2, c_3\}$, we conclude that either ijc_2 or ijc_3 is an edge of G . As we assumed that $ijc_2 \notin E(G)$, we have $ijc_3 \in E(G)$ and hence the set $\{i, j, c_1, c_3\}$ induces exactly 2 edges in G . Since $2 \in L$, this is a contradiction.

If $|C| = n$, then G is the complete 3-graph. If $|C| = n - 1$, then G is the union of a clique of size $n - 1$ and an isolated vertex. There are n such 3-graphs depending on which vertex the isolated vertex is. We conclude that the number of $(L, 4)$ -free 3-graphs on $[n]$ is $n + 1$, for all $n \geq 5$. $\square$

5. Concluding remarks

When analyzing the proof of Theorem 1.4 in the case where $L = \{1, 4\}$, $k = 4$ and $r = 3$, it can be observed that it actually gives

$$f(n, 3, 4, \{1, 4\}) \leq \prod_{m=1}^{n-1} m! = G(n+1) = 2^{\frac{n^2}{2} \log n(1+o(1))},$$

where G is the Barnes G function. On the other hand, we have seen in the proof of Theorem 1.2 that

$$f(n, 3, 4, \{1, 4\}) \geq |Q(n)| = 2^{\frac{n^2}{27} \log n(1+o(1))}.$$

Towards solving Conjecture 1.1, it would be interesting to first determine the constant in front of the main term of the exponent.

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