Discrete Applied Mathematics 321 (2022) 72–81

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Chain method for panchromatic colorings of hypergraphs

Margarita Akhmejanova [a,*], József Balogh [b], Dmitrii Shabanov [a,c]

[a] *Moscow Institute of Physics and Technology, Laboratory of Combinatorial and Geometric Structures, Laboratory of Advanced Combinatorics and Network Applications, 141700, Institutskiy per. 9, Dolgoprudny, Moscow Region, Russia*
[b] *Department of Mathematical Sciences, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA*
[c] *National Research University Higher School of Economics (HSE), Faculty of Computer Science, 101000, Myasnitskaya Str. 20, Moscow, Russia*

A R T I C L E I N F O

Article history:
Received 20 February 2021
Received in revised form 28 May 2022
Accepted 3 June 2022
Available online 2 July 2022

Keywords:
Panchromatic coloring
Property B
Proper coloring
Uniform hypergraph

A B S T R A C T

We deal with an extremal problem concerning panchromatic colorings of hypergraphs. A vertex r-coloring of a hypergraph H is *panchromatic* if every edge meets every color. We prove that for every $2 \leq r < \sqrt[3]{\frac{n}{100 \ln n}}$, every n-uniform hypergraph H with $|E(H)| \leq \frac{c}{r^2}\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}}\left(\frac{r}{r-1}\right)^{n-1}$ has a panchromatic coloring with r colors, where $c > 0$ is an absolute constant.

1. Introduction and related work

We study colorings of uniform hypergraphs. Let us recall some definitions.

A vertex r-coloring of a hypergraph $H = (V, E)$ is a mapping from the vertex set V to a set of r colors. An r-coloring of H is *panchromatic* if each edge has at least one vertex of each color.

The first sufficient condition on the existence of a panchromatic coloring of a hypergraph was obtained in 1975 by Erdős and Lovász [6]. They proved that if every edge of an n-uniform hypergraph intersects at most

$$\frac{r^{n-1}}{4(r-1)^n} \tag{1}$$

other edges then the hypergraph has a panchromatic coloring with r colors.

The next generalization of the problem was formulated in 2002 by Kostochka [9], who posed the following question: *What is the minimum possible number of edges in an n-uniform hypergraph that does not admit a panchromatic coloring with r colors?* He denoted this number by $p(n, r)$.

Following closely behind this problem is a related one: a hypergraph $H = (V, E)$ has property B if there is a coloring of V by 2 colors so that no edge $f \in E$ is monochromatic. Erdős and Hajnal [5] (1961) proposed to find the value $m(n)$ equal to the minimum possible number of edges in a n-uniform hypergraph without property B. Erdős [4] (1963–1964) found bounds $\Omega\left(2^n\right) \leq m(n) = O\left(2^n n^2\right)$ and Radhakrishnan and Srinivasan [11] (2000) proved $m(n) \geq \Omega\left(2^n (n/\ln n)^{1/2}\right)$. Clearly, $m(n) = p(n, 2)$.

We return to the panchromatic coloring. Kostochka [9] has found connections between $p(n, r)$ and minimum possible number of vertices in a k-partite graph with list chromatic number greater than r. Using results of Erdős, Rubin and

* Corresponding author.
E-mail addresses: mechmathrita@gmail.com (M. Akhmejanova), jobal@illinois.edu (J. Balogh), dmitry.shabanov@phystech.edu (D. Shabanov).

https://doi.org/10.1016/j.dam.2022.06.005
0166-218X/

Taylor [7] and also Alon's result [2] Kostochka [9] proved the existence of constants c_1 and c_2 that for every large n and fixed r:

$$\frac{e^{c_1 \frac{n}{r}}}{r} \le p(n,r) \le r e^{c_2 \frac{n}{r}}. \tag{2}$$

In 2010, bounds (2) were considerably improved in the paper of Shabanov [14]:

$$p(n,r) \geqslant \frac{\sqrt{21}-3}{4r}\left(\frac{n}{(r-1)^2 \ln n}\right)^{1/3}\left(\frac{r}{r-1}\right)^n, \quad \text{for all } \ r<n,$$

$$p(n,r) \leqslant \frac{1}{r}\left(\frac{r}{r-1}\right)^n e(\ln r)\frac{n^2}{2(r-1)}\varphi_1, \quad \text{when } \ r=o(\sqrt{n}),$$

$$p(n,r) \leqslant \frac{1}{r}\left(\frac{r}{r-1}\right)^n e(\ln r) n^{3/2}\varphi_2, \quad \text{when } \ n=o\left(r^2\right),$$

where φ_1, φ_2 are some functions of n and $r(n)$, tending to one at $n \to \infty$.

In 2012, Rozovskaya and Shabanov [13] improved Shabanov's lower bound by proving that for $r < n/(2\ln n)$

$$\frac{1}{2r^2}\left(\frac{n}{\ln n}\right)^{1/2}\left(\frac{r}{r-1}\right)^n \leqslant p(n,r) \leqslant c_2 n^2 \left(\frac{r}{r-1}\right)^n \ln r. \tag{3}$$

Note that bounds (3) are the same as the best known bounds for $r = 2$. Further research was conducted by Cherkashin [3] in 2018. In his work, Cherkashin introduced the auxiliary value $p'(n,r)$, which is numerically equal to the minimum number of edges in the class of n-uniform hypergraphs $H = (V, E)$, in which any subset of vertices $V' \subset V$ with $\left|V'\right| \ge \left[\frac{r-1}{r}|V|\right]$ must contain an edge. Analyzing the value $p'(n,r)$ and using Sidorenko's [15] estimates on the Turan numbers, Cherkashin proved that for $n \ge 2, r \ge 2$

$$p(n,r) \le c\frac{n^2 \ln r}{r}\left(\frac{r}{r-1}\right)^n.$$

Cherkashin also proved that for $r \le c \cdot \frac{n}{\ln n}$

$$p(n,r) \ge c \max\left(\frac{n^{1/4}}{r\sqrt{r}}, \frac{1}{\sqrt{n}}\right)\left(\frac{r}{r-1}\right)^n. \tag{4}$$

And repeating the ideas of Gebauer [8] Cherkashin constructed an example of a hypergraph that has few edges and does not admit a panchromatic coloring in r colors. The reader is referred to the survey [12] for the detailed history of panchromatic colorings.

It is thus natural to consider the local case. Formally, *the degree of an edge A* is the number of hyperedges intersecting A. Let $d(n,r)$ be the minimum possible value of the maximum edge degree in an n-uniform hypergraph that does not admit panchromatic coloring with r colors. Then, the Erdős and Lovász result (1) can be easily translated into following form:

$$d(n,r) \ge \frac{r^{n-1}}{4(r-1)^n}. \tag{5}$$

However, the bound (5) appeared not to be sharp. The restriction on $d(n,r)$ have been improved by Rozovskaya and Shabanov [13]. In their work they achieved that

$$d(n,r) > \frac{\sqrt{11}-3}{4r(r-1)}\left(\frac{n}{\ln n}\right)^{1/2}\left(\frac{r}{r-1}\right)^n, \qquad \text{when } r \leqslant n/(2\ln n). \tag{6}$$

2. Our results

The main result of our paper improves the estimate (3) as follows.

Theorem 1. *Suppose $2 \le r \le \sqrt[3]{\frac{n}{100\ln n}}$. Then we have*

$$p(n,r) \ge \frac{1}{20e^3 r^2}\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}}\left(\frac{r}{r-1}\right)^n. \tag{7}$$

Corollary 1. *There is an absolute constant C so that for every $n > 2$ and $\ln n < r < \sqrt[3]{\frac{n}{100\ln n}}$*

$$p(n,r) \ge \frac{Cn}{r^2 \ln n} \cdot e^{\frac{n}{r}+\frac{n}{2r^2}}.$$

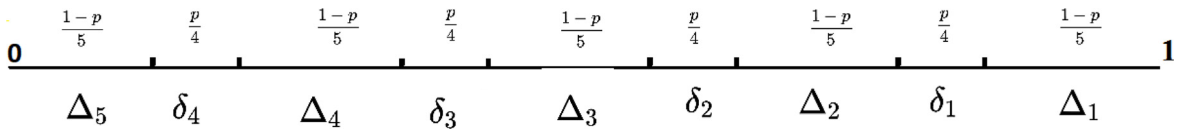

Fig. 1. Partition of $[0, 1]$ into $\Delta_5, \delta_4, \Delta_4, \delta_3, \dots, \Delta_1$ when $r = 5$.

We refine the bound (6) as follows.

Theorem 2. *For every* $2 < r < \sqrt[3]{\frac{n}{100 \ln n}}$

$$d(n, r) \geq \frac{1}{20e^3r^3} \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} \left(\frac{r}{r-1}\right)^n. \tag{8}$$

2.1. Methods

In the work, we propose a new idea based on the Pluhar's ordered chain method [10]. In the case of panchromatic coloring, the resulting structure is no longer a real ordered chain, but rather an intricate "snake ball". Nevertheless, with the help of probabilistic analysis, we managed to obtain a strong lower bound.

The rest of the paper is organized as follows. Section 3 describes a coloring algorithm. Section 4 is devoted to an analysis of the algorithm. In Section 5 we collect some inequalities that will be subsequently useful. Sections 6 and 7 contain proofs of Theorems 1 and 2.

3. A coloring algorithm

We may and will assume that $r \geq 3$, because case $r = 2$ corresponds to the case $m(n)$. Let $H = (V, E)$ be an n-uniform hypergraph with less than $\frac{1}{20e^3r^2} \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} \left(\frac{r}{r-1}\right)^n$ edges and let $r < \sqrt[3]{\frac{n}{100 \ln n}}$. We will show that H has a panchromatic coloring with r colors.

We define a special random order on the set V of vertices of hypergraph H using a mapping $\sigma : V \to [0, 1]$, where $\sigma(v), v \in V$ — i.i.d. with the uniform distribution on $[0, 1]$. The value $\sigma(v)$ we call the *weight* of vertex v. Reorder the vertices so that $\sigma(v_1) < \cdots < \sigma(v_{|V|})$. Put

$$p = \left(\frac{r-1}{r}\right) \frac{(r-1)^2 \ln(\frac{n}{\ln n})}{n}. \tag{9}$$

We divide the unit intercept $[0, 1]$ into subintervals $\Delta_r, \delta_{r-1}, \Delta_{r-1}, \delta_{r-2}, \dots, \Delta_1$ as in Fig. 1, i.e.

$$\Delta_{r-i+1} = \left[(i-1)\left(\frac{1-p}{r} + \frac{p}{r-1}\right), i \cdot \frac{1-p}{r} + (i-1) \cdot \frac{p}{r-1}\right), \ i = 1, \dots, r;$$

$$\delta_{r-i} = \left[i \cdot \frac{1-p}{r} + (i-1) \cdot \frac{p}{r-1}, i\left(\frac{1-p}{r} + \frac{p}{r-1}\right)\right), \ i = 1, \dots, r-1.$$

Since $p < \frac{1}{100r}$ under the given assumptions on r, we can see that the intervals $\Delta_1, \dots, \Delta_r$ are each wider than the intervals $\delta_1 \dots, \delta_{r-1}$. The length of each large subinterval Δ_i is equal to $\frac{1-p}{r}$ and every small subinterval δ_i has length equal to $\frac{p}{r-1}$. For convenience, we denote belonging $\sigma(v)$ to a certain segment $[c, d]$ by a shorthand $v \in [c, d]$ and say that v belongs to $[c, d]$. We also note that the similar division of the segment $[0, 1]$ has already been used by the first author for proving some bounds on proper colorings [1].

We color the vertices of hypergraph H according to the following algorithm, which consists of two steps.

1. First, each $v \in \Delta_i$ is colored with color i for every $i \in \{r, \dots, 1\}$.
2. Then, using the vertex ordering determined by σ', we color a vertex $v \in \delta_i$ with color i if there is no edge e, $v \in e$ that does not meet color $i + 1$. Otherwise we color v with $i + 1$.

4. Analysis of the algorithm

4.1. Short edge

We say that an edge A is *short* if $A \cap (\Delta_i \cup \delta_i) = \emptyset$ or $A \cap (\Delta_{i+1} \cup \delta_i) = \emptyset$ for some $i \in \{1, \dots, r-1\}$. The probability of this event for fixed edge A and fixed i is at most $2\left(1 - \left(\frac{1-p}{r} + \frac{p}{r-1}\right)\right)^n = 2\left(\frac{r-1}{r} - \frac{p}{r(r-1)}\right)^n$. Summing up this upper

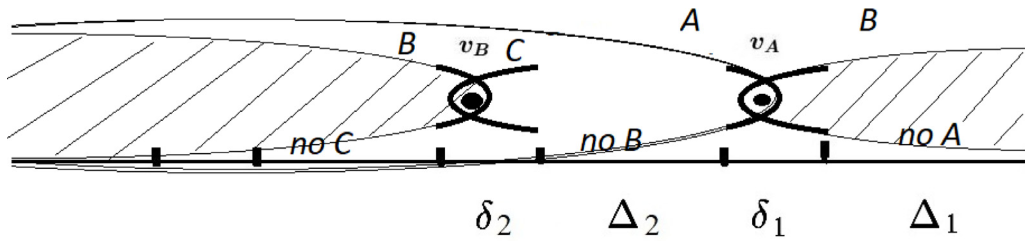

Fig. 2. Edges A and B in a snake ball.

bound over all edges and $i \in \{1, \ldots, r-1\}$ we get

$$2(r-1)|E|\left(1-\left(\frac{1-p}{r}+\frac{p}{r-1}\right)\right)^n \leq \frac{2(r-1)}{20e^3r^2}\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}}\left(\frac{r}{r-1}\right)^n\left(\frac{r-1}{r}-\frac{p}{r(r-1)}\right)^n$$
$$\leq \frac{1}{e}\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}}\left(1-\frac{p}{(r-1)^2}\right)^n \leq \frac{1}{e}\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} e^{-pn(r-1)^2} = \frac{1}{e}\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} e^{-\ln\left(\frac{n}{\ln n}\right)\frac{r-1}{r}} \leq \frac{1}{e}.$$

Hence, we conclude that the expected number of short edges is less than $1/e$. Applying Markov's inequality, we get that with probability at least $1-1/e$ there is no short edge.

4.2. Snake ball

Suppose our algorithm fails to produce a panchromatic r-coloring and there are no short edges. Let A be an edge, which does not contain some color i. Now we have two possibilities:

- $i \in \{2, \ldots, r\}$, in this situation edge A is disjoint from the interval $\Delta_i \cup \delta_{i-1}$, which means that A is short, a contradiction.
- $i = 1$.

Edge A is not short, so $A \cap (\delta_1 \cup \Delta_1) \neq \emptyset$. Since A does not contain color 1 we have $A \cap \Delta_1 = \emptyset$. Denote v_A the last vertex of $A \cap \delta_1$. We note that v_A could receive color 2 only if at the moment of coloring v_A there was an edge B without color 2 and v_A was the first vertex of $B \cap \delta_1$ (see Fig. 2). Again, edge B is not short and did not contain color 2 at the moment of coloring v_A, so $B \cap (\delta_2 \cup \Delta_2) \neq \emptyset$ and $B \cap \Delta_2 = \emptyset$. For v_B, the last vertex of $B \cap \delta_2$, there exists an edge C, which at the moment of coloring v_B was without color 2 and v_B was the first vertex of $C \cap \delta_2$.

Repeating the above arguments, we obtain an ordering r-tuple of edges $(C_1 = A, C_2 = B, \ldots, C_r)$ that witnesses the failure of the Algorithm 1. Note that this r-tuple has three properties:

1. (local placing): $C_i \cap \Delta_i = \emptyset$ and C_i is not short edge for all $i \in \{1, \ldots, r\}$.
2. (local order): $\sigma(u) \leq \sigma(w)$ for all $u \in C_i \cap \delta_i$, $w \in C_{i+1} \cap \delta_i$ and $i \in \{1, \ldots, r-1\}$.
3. (local linearity): $|C_i \cap C_{i+1} \cap \delta_i| = 1$ for all $i \in \{1, \ldots, r-1\}$.

Now we call any ordered edge sequence $H' = (C_1 = A, C_2 = B, \ldots, C_r)$ with above three properties *snake ball* and conclude the following claim:

Claim 1. *If for injective $\sigma : V \to [0, 1]$ there are neither snake balls nor short edges then Algorithm 1 produces a panchromatic r-coloring.*

Lemma 1. *Let $H' = (C_1, \ldots, C_r)$ be an ordered r-tuple of edges in H. Then*

$$\mathbb{P}\left(H' \text{ forms a snake ball}\right) \leq \left(\frac{r-1}{r}\right)^{(n-r)r}\left(\frac{p}{r-1}\right)^{r-1} \cdot \prod_{i=1}^{r-1}|C_i \cap C_{i+1}| \cdot \max H',$$

where $\max H' = \max_{\tilde{H}' \subset H' : |\tilde{H}'| = |H'| - (r-1)} \prod_{v \in \tilde{H}' : s(v) \geq 2} \frac{\left(1 - s(v)\frac{1-p}{r}\right)}{\left(1-\left(\frac{1-p}{r}+\frac{2p}{r-1}\right)\right)^{s(v)}}$ *and $s(v)$ is the number of edges of H' that contain vertex v.*

The scheme of the proof is following:

- fix vertex $v_j \in C_j \cap C_{j+1}$ and its weight $\sigma(v_j)$ for all $j \in \{1, \ldots, r-1\}$. Calculate conditional probability given weights of $v_1, \ldots, v_{r-1}$.

- sum up (integrate) the previous probability over all possible values of weights, using that $\sigma(v_j) \in \delta_j, j \in \{1, \dots, r-1\}$, as this is needed for H' to be a snake ball.
- Finally, sum over all choices of $v_1, \dots, v_{r-1}$.

Now formally. We assume that there exists only one vertex in each intersection $C_j \cap C_{j+1} \cap \delta_j, j \in \{1, \dots, r-1\}$ as this is needed for H' to be a snake ball (property 3). We denote by v_j such vertex in each $C_j \cap C_{j+1} \cap \delta_j, j \in \{1, \dots, r-1\}$ and assume for a moment that its weights equal some real numbers $y_i \in \delta_j$. Just for uniform notation of intercepts, we introduce two extra numbers: $y_0 = 1$ and $y_r = 1$. For convenience, we also denote by $\tilde{H}'$ and $\tilde{C}_j, j \in \{1, \dots, r-1\}$, the set of the remaining vertices in H' and in C_j, i.e. $\tilde{H}' = H' \setminus \{v_1, \dots, v_{r-1}\}$ and $C_j \setminus \{v_1, \dots, v_{r-1}\}$. Using the above notations we are ready to upper bound the conditional probability of the event "H' forms a snake ball" given weights of $v_1, \dots, v_{r-1}$ are equal to $y_1 \dots, y_{r-1}$, respectively.

$$\mathbb{P}(H' \text{ is a snake ball } | \sigma(v_1) = y_1, \dots, \sigma(v_{r-1}) = y_{r-1}, y_0 = 1, y_r = 1) \le \\ \Big[\prod_{i=1}^{r} \prod_{v \in \tilde{C}_i : s(v)=1} \mathbb{P}(\sigma(v) \notin [y_i, y_{i-1}])\Big] \times \prod_{v \in \tilde{H}' : s(v) \ge 2} \mathbb{P}(\sigma(v) \notin \cup_{j: v \in C_j} \Delta_j) \tag{10}$$

Since for every v its weight $\sigma(v)$ is uniformly sampled, for $v \in \tilde{C}_i$, $\mathbb{P}(\sigma(v) \notin [y_i, y_{i-1}]) \ge \mathbb{P}(\sigma(v) \notin [\Delta_i \cup \delta_i \cup \delta_{i+1}]) = \big(1 - (\frac{1-p}{r} + \frac{2p}{r-1})\big)$. So,

$$\frac{\prod_{i: v \in \tilde{C}_i} \mathbb{P}(\sigma(v) \notin [y_i, y_{i-1}])}{\big(1 - (\frac{1-p}{r} + \frac{2p}{r-1})\big)^{s(v)}} \ge 1 \quad \text{and}$$

$$1 \le \prod_{v \in \tilde{H}' : s(v) \ge 2} \frac{\prod_{i: v \in \tilde{C}_i} \mathbb{P}(\sigma(v) \notin [y_i, y_{i-1}])}{\big(1 - (\frac{1-p}{r} + \frac{2p}{r-1})\big)^{s(v)}} = \frac{\prod_{i=1}^{r} \prod_{v \in \tilde{C}_i : s(v) \ge 2} \mathbb{P}(\sigma(v) \notin [y_i, y_{i-1}])}{\prod_{v \in \tilde{H}' : s(v) \ge 2} \big(1 - (\frac{1-p}{r} + \frac{2p}{r-1})\big)^{s(v)}}. \tag{11}$$

Multiplying bound (10) on right side of Eq. (11), we get a convenient bound on the conditional probability, which will be a bit easier to analyze:

$$\mathbb{P}(H' \text{ is a snake ball } | \sigma(v_1) = y_1, \dots, \sigma(v_{r-1}) = y_{r-1}, y_0 = 1, y_r = 1) \le \\ \left[\prod_{i=1}^{r} \prod_{v \in \tilde{C}_i} \mathbb{P}(\sigma(v) \notin [y_i, y_{i-1}])\right] \times \prod_{v \in \tilde{H}' : s(v) \ge 2} \frac{\mathbb{P}(\sigma(v) \notin \cup_{j: v \in \tilde{C}_j} \Delta_j)}{\big(1 - (\frac{1-p}{r} + \frac{2p}{r-1})\big)^{s(v)}} \tag{12}$$

To simplify computations, we put $[\alpha_j, \beta_j) = \delta_j$ and $x_j = \beta_j - \sigma(v_j) = \beta_j - y_j$ for all $j \in \{1, \dots, r-1\}$. Then the length of intercept $[y_i, y_{i-1}]$ equals exactly $\big(\frac{1-p}{r} + x_1\big)$ for $i = 1$, $\frac{1-p}{r} + x_i + \frac{p}{r-1} - x_{i-1}$ for $i \in \{2, \dots, r-2\}$ and $\frac{1-p}{r} + \frac{p}{r-1} - x_{r-1}$ for $i = r-1$. Therefore, using that for every v its weight $\sigma(v)$ is uniformly sampled, (12) does not exceed

$$\Big(1 - \Big(\frac{1-p}{r} + x_1\Big)\Big)^{|\tilde{C}_1|} \cdot \Big(1 - \Big(\frac{1-p}{r} + x_2 + \frac{p}{r-1} - x_1\Big)\Big)^{|\tilde{C}_2|} \cdot \dots \\ \Big(1 - \Big(\frac{1-p}{r} + x_{r-1} + \frac{p}{r-1} - x_{r-2}\Big)\Big)^{|\tilde{C}_{r-2}|} \cdot \Big(1 - \Big(\frac{1-p}{r} + \frac{p}{r-1} - x_{r-1}\Big)\Big)^{|\tilde{C}_r|}. \\ \prod_{v \in \tilde{H}' : s(v) \ge 2} \frac{\big(1 - s(v)\frac{1-p}{r}\big)}{\big(1 - (\frac{1-p}{r} + \frac{2p}{r-1})\big)^{s(v)}}.$$

Now we lower bound the size of each $\tilde{C}_i = C_i \setminus \{v_1, \dots, v_{r-1}\}, i \in \{1, \dots, r\}$ by $n-r$ and take out factor $((r-1)/r)^{(n-r)r}$:

$$\Big(\frac{r-1}{r}\Big)^{(n-r)r} \Big(1 + \frac{p}{r-1} - \frac{x_1 r}{r-1}\Big)^{n-r} \left(1 + \frac{p}{r-1} - \frac{pr}{(r-1)^2} - \frac{(x_2 - x_1)r}{r-1}\right)^{n-r} \cdot \dots \cdot \\ \Big(1 + \frac{p}{r-1} - \frac{pr}{(r-1)^2} - \frac{x_{r-1} r}{r-1}\Big)^{n-r} \prod_{v \in \tilde{H}' : s(v) \ge 2} \frac{\Big(1 - s(v)\frac{1-p}{r}\Big)}{\Big(1 - \big(\frac{1-p}{r} + \frac{2p}{r-1}\big)\Big)^{s(v)}}$$

Use estimate $(1+y)^s \le \exp\{ys\}$:

$$\Big(\frac{r-1}{r}\Big)^{(n-r)r} \exp\Big\{(n-r)\Big(\sum_{i=1}^{r} \frac{p}{r-1} - \sum_{i=1}^{r-1} \frac{pr}{(r-1)^2}\Big)\Big\} \prod_{v \in \tilde{H}' : s(v) \ge 2} \frac{\big(1 - s(v)\frac{1-p}{r}\big)}{\big(1 - (\frac{1-p}{r} + \frac{2p}{r-1})\big)^{s(v)}} \\ = \Big(\frac{r-1}{r}\Big)^{r(n-r)} \prod_{v \in \tilde{H}' : s(v) \ge 2} \frac{\big(1 - s(v)\frac{1-p}{r}\big)}{\big(1 - (\frac{1-p}{r} + \frac{2p}{r-1})\big)^{s(v)}}.$$

To obtain the final estimate, we have to integrate over the weights $y_1, y_2, \dots, y_{r-1}$ (factor $(p/(r-1))^{r-1}$), sum up over all possible choices for the $v_1, \dots, v_{r-1}$ (factor $\prod_{i=1}^{r-1} |C_i \cap C_{i+1}|$) and take the maximum over all $\tilde{H}' \subset H' : |\tilde{H}'| = |H'| - (r-1)$.

5. Auxiliary calculations

Under the assumptions of Theorem 1 we will formulate and prove three auxiliary lemmas needed to prove Theorem 1. In particular, in Lemma 2, we replace product of pairwise intersections by their sum $\sum_{i<j} |C_i \cap C_j|$ and in Lemma 4, we will use double-counting for estimating the sum $\sum_{i<j} |C_i \cap C_j|$, which can be large with n, by special bounded terms.

Lemma 2. *Let $H' = (C_1, \dots, C_r)$ be an ordered r-tuple of edges in the hypergraph H. Then*

$$\sum_{\pi \in S_r} |C_{i_1} \cap C_{i_2} \| C_{i_2} \cap C_{i_3}| \cdot \dots \cdot |C_{i_{r-1}} \cap C_{i_r}| \le \left(\frac{2\sum_{i<j} |C_i \cap C_j| + r}{r} \right)^r, \tag{13}$$

where S_r denotes all permutations $\pi = (i_1, \dots, i_r)$ of $(1, 2, \dots, r)$.

Proof. Denote the cardinality of the edge intersection $|C_i \cap C_j|$ by $x_{i,j}$. Then, we have to prove that

$$\sum_{\pi \in S_r} x_{i_1,i_2} x_{i_2,i_3} \cdot \dots \cdot x_{i_{r-1},i_r} \le \left(\frac{2\sum_{i<j} x_{i,j} + r}{r} \right)^r.$$

First, we will show that

$$\sum_{\pi \in S_r} x_{i1,i2} x_{i_2,i_3} \cdot \dots \cdot x_{i_{r-1},i_r} \le (x_{1,2} + \dots + x_{1,r} + 1) \cdot \dots \cdot (x_{r,1} + \dots + x_{r,r-1} + 1). \tag{14}$$

Let us call $(x_{i,1} + \dots + x_{i,r} + 1)$ from (14) the *bracket number i*. We define a mapping f between elements from the left-hand side of (14) and ordered sets that are obtained after performing the multiplication in (14).

Let $f : x_{i_1,i_2} x_{i_2,i_3} \dots x_{i_{r-1},i_r} \mapsto x_{1,t_1} x_{2,t_2} \dots x_{r,t_r}$, where $x_{1,t_1} x_{2,t_2} \dots x_{r,t_r}$ is the product of the following r elements: x_{i_{r-1},i_r} from the bracket number i_{r-1}, $x_{i_{r-2},i_{r-1}}$ from the bracket number i_{r-2} and so forth, finally we take the factor 1 from the unused bracket. For example, $x_{5,6} x_{6,1} x_{1,4} x_{4,3} x_{3,2}$ is mapped to $x_{1,4} \cdot 1 \cdot x_{3,2} \cdot x_{4,3} \cdot x_{5,6} \cdot x_{6,1}$.

We note that f is an injection. Indeed, for each $x_{1,t_1} x_{2,t_2} \dots x_{r,t_r}$ there exists at most one sequence $x_{i_1,i_2} x_{i_2,i_3} \dots x_{i_{r-1},i_r}$ with $i1, \dots, ir$ distinct, such as $f(x_{i_1,i_2} x_{i_2,i_3} \dots x_{i_{r-1},i_r}) = x_{1,t_1} \dots x_{r,t_r}$.

So, since f does not change the product and f is an injection we get that the right-hand side of (14) is not less than the left-hand side.

Finally, by the inequality on the arithmetic–geometric means and by $x_{i,j} = x_{j,i}$

$$(x_{1,2} + \dots + x_{1,r} + 1) \cdot \dots \cdot (x_{r,1} + \dots + x_{r,r-1} + 1) \le \left(\frac{2\sum_{i<j} x_{i,j} + r}{r} \right)^r. \quad \square$$

Lemma 3. *For all $s \in \{2, \dots, r\}$*

$$\frac{\left(1 - s\frac{1-p}{r}\right)}{\left(1 - \left(\frac{1-p}{r} + \frac{2p}{r-1}\right)\right)^s} \le e^{-\frac{s^2}{20r^2}}. \tag{15}$$

Proof. First prove the case $s \ge 3$.

$$\frac{\left(1 - \frac{s(1-p)}{r}\right)}{\left(1 - \left(\frac{1-p}{r} + \frac{2p}{r-1}\right)\right)^s} = \frac{\left(1 - \frac{s(1-p)}{r}\right)}{\left(1 - \left(\frac{1-p}{r}\right)\right)^s \left(1 - \frac{2pr}{(r-1)(r-1+p)}\right)^s} \le \frac{\left(1 - \frac{s(1-p)}{r}\right)\left(1 + \frac{1-p}{r-1+p}\right)^s}{\left(1 - \frac{2pr}{(r-1)^2}\right)^s}. \tag{16}$$

Now we deal with factors in (16) separately:

$$\left(1 + \frac{1-p}{r-1+p}\right)^s \le \left(1 + \frac{1-p}{r-1}\right)^s = |\text{Apply Taylor's formula with Lagrange Remainder}| =$$

$$1 + \frac{s(1-p)}{r-1} + \frac{s(s-1)(1-p)^2}{2(r-1)^2} + \frac{s(s-1)(s-2)(1-p)^3(1+\theta \cdot \frac{1-p}{r-1})^{s-3}}{6(r-1)^3} \quad \text{for some } \theta \in (0, 1).$$

Bound $(s-1)/(r-1)$ by s/r, $(s-1)(s-2)/(r-1)^2$ by s^2/r^2, $(1 + \theta/(r-1))^{s-3}$ with $\theta \in (0, 1)$ by e and $(1-p)$ factor by 1 in the second and the third terms:

$$\le 1 + \frac{s(1-p)}{r-1} + \frac{s^2(1-p)}{2r(r-1)} + \frac{s^3(1-p)^2 e}{6r^2(r-1)}.$$

Hence, the numerator of (16) does not exceed

$$\left(1-\frac{s(1-p)}{r}\right)\left(1+\frac{s(1-p)}{r-1}+\frac{s^2(1-p)}{2r(r-1)}+\frac{s^3(1-p)^2}{2r^2(r-1)}\right)<1-\frac{s^2(1-p)}{r(r-1)}(1-p-1/2)+\frac{s(1-p)}{r(r-1)}=1-\frac{s^2(1-p)}{r(r-1)}(1/2-1/s-p)<1-\frac{s^2(1/6-p)(1-p)}{r^2}<1-\frac{s^2}{7r^2}\le\exp\left\{-\frac{s^2}{7r^2}\right\}.$$

Using bounds $1/(1-x)<1+2x$ for $x<1/2$ and estimating $pr<1/100$, which follows from restrictions on r, we finally get

$$\frac{\left(1-s\frac{1-p}{r}\right)}{\left(1-\left(\frac{1-p}{r}+\frac{2pr}{r-1}\right)\right)^s}\le\exp\left\{-\frac{s^2}{7r^2}\right\}\left(1-\frac{2pr}{(r-1)^2}\right)^{-s}<\exp\left\{-\frac{s^2}{7r^2}\right\}\left(1+\frac{4pr}{(r-1)^2}\right)^{s}\le$$
$$\exp\left\{\frac{4prs}{(r-1)^2}-\frac{s^2}{7r^2}\right\}\le\exp\left\{\frac{s}{25(r-1)^2}-\frac{s^2}{7r^2}\right\}<\exp\left\{\frac{4}{25s}\cdot\frac{s^2}{r^2}-\frac{s^2}{7r^2}\right\}<\exp\left\{-\frac{s^2}{20r^2}\right\}.$$

Consider the case $s=2$.

$$\frac{1-2(1-p)/r}{\left(1-\left(\frac{1-p}{r}+\frac{2p}{r-1}\right)\right)^2}\le\frac{1-2(1-p)/r}{1-\frac{2(1-p)}{r}-\frac{4p}{(r-1)}+\frac{1}{2r^2}}\le\frac{1-2(1-p)/r}{1-\frac{2(1-p)}{r}-\frac{1}{4r^2}+\frac{1}{2r^2}}=1-\frac{1/4r^2}{1-2\frac{1-p}{r}+\frac{1}{4r^2}}$$
$$\le 1-1/4r^2\le\exp\{-1/4r^2\}<\exp\{-1/5r^2\},$$

where we used that $4p/(r-1)<8p/r=8pr/r^2<8/100r^2<1/4r^2$. □

Lemma 4.

$$\max H'\cdot\sum_{\sigma\in S_r}|C_{i_1}\cap C_{i_2}|\cdot\ldots\cdot|C_{i_{r-1}}\cap C_{i_r}|\le\left(20r^2\right)^r e^{1/20}, \tag{17}$$

where $\max H'=\max_{\tilde{H}'\subset H':|\tilde{H}'|=|H'|-(r-1)}\prod_{v\in\tilde{H}':s(v)\ge2}\frac{\left(1-s(v)\frac{1-p}{r}\right)}{\left(1-\left(\frac{1-p}{r}+\frac{2p}{r-1}\right)\right)^{s(v)}}$ *and* $s(v)$ *is the number of edges of* H' *that contain vertex* v.

Proof. By Lemmas 2 and 3 the left hand side of (17) does not exceed

$$\left(\frac{2\sum_{i<j}|C_i\cap C_j|+r}{r}\right)^r\max_{\tilde{H}'\subset H':|\tilde{H}'|=|H'|-(r-1)}\exp\Big\{\sum_{v\in\tilde{H}':s(v)\ge2}-s^2(v)/20r^2\Big\}. \tag{18}$$

Using that $|\tilde{H}'|=|H'|-(r-1)$ and $s(v)\le r$ for all $v\in H$, we can upper bound the second term from (18). Namely,

$$\sum_{v\in\tilde{H}':s(v)\ge2}-\frac{s^2(v)}{20r^2}\le\sum_{v\in H':s(v)\ge2}-\frac{s^2(v)}{20r^2}+r,$$

$$\max_{\tilde{H}'\subset H':|\tilde{H}'|=|H'|-(r-1)}\exp\Big\{\sum_{v\in\tilde{H}':s(v)\ge2}-s^2(v)/20r^2\Big\}\le\exp\Big\{r+\sum_{v\in H':s(v)\ge2}-s^2(v)/20r^2\Big\}.$$

Now we will use the following double-counting: $\sum_{i<j}|C_i\cap C_j|$ is equal to $\sum_{v\in H':s(v)\ge2}\binom{s(v)}{2}<1/2\sum_{v\in H':s(v)\ge2}s^2(v)$. Hence, the left hand side of (17) does not exceed

$$(er)^r\exp\Big\{\sum_{v\in H':s(v)\ge2}-s^2(v)/20r^2\Big\}\left(\frac{\sum_{v\in H':s(v)\ge2}s^2(v)+r}{r^2}\right)^r\le(er)^re^{-\frac{t}{20}}(t+1)^r\le(20r^2)^re^{\frac{1}{20}},$$

where we used $t=\sum_{v\in H':s(v)\ge2}s^2(v)/r^2$ and observed that the expression $(t+1)^re^{-t/20}$ is maximized when $t=20r-1$. □

6. Proof of Theorem 1

It remains to show that the if H is a hypergraph with less than $\frac{1}{20e^3r^2}\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}}\left(\frac{r}{r-1}\right)^n$ edges then with positive probability, H does not contain neither a short edge nor a snake ball.

Denote $\sum^{unordered}$ the sum over all r-sets $J\subseteq(1,2,\ldots,|E|)$, $\sum^{ordered}$ the sum over all ordered r-tuples $(j_1,\ldots,j_r)$, with $\{j_1,\ldots,j_r\}$ forming such a J and $\sum_{\pi\in S_r}$ denote the sum over all permutations $\pi=(i_1,\ldots,i_r)$ of $(1,2,\ldots,r)$.

Then,

$$\sum_{\pi\in S_r}\mathbb{P}\left((C_{i_1},\dots,C_{i_r})\ \text{is a snake ball}\right)\le$$
$$\sum_{\pi\in S_r}\left(\frac{r-1}{r}\right)^{(n-r)r}\left(\frac{p}{r-1}\right)^{r-1}\cdot|C_{i_1}\cap C_{i_2}|\cdot\dots\cdot|C_{i_{r-1}}\cap C_{i_r}|\cdot\max H'=$$
$$\left(\frac{r-1}{r}\right)^{(n-r)r}\left(\frac{p}{r-1}\right)^{r-1}\max H'\cdot\sum_{\pi\in S_r}|C_{i_1}\cap C_{i_2}|\dots|C_{i_{r-1}}\cap C_{i_r}|\le$$
$$\left(\frac{r-1}{r}\right)^{(n-r)r}\left(\frac{p}{r-1}\right)^{r-1}(20r^2)^r e^{1/20}, \tag{19}$$

where for the first inequality we used Lemma 1 and for the second Lemma 4.

Finally, the expected number of snake balls can be upper bounded as follows:

$$\sum^{\text{ordered}}\mathbb{P}\left((C_{j_1},\dots,C_{j_r})\ \text{is a snake ball}\right)=\sum^{\text{unordered}}\sum_{\pi\in S_r}\mathbb{P}\left((C_{i_1},\dots,C_{i_r})\ \text{is a snake ball}\right)$$
$$\le\binom{|E|}{r}\left(\frac{r-1}{r}\right)^{(n-r)r}\left(\frac{p}{r-1}\right)^{r-1}(20r^2)^r e^{1/20}\le$$
$$\frac{\left(\frac{1}{20e^3r^2}\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}}\left(\frac{r}{r-1}\right)^n\right)^r}{r!}\cdot\left(\frac{r-1}{r}\right)^{(n-r)r}\cdot\left(\frac{(r-1)^2\ln(\frac{n}{\ln n})}{rn}\right)^{r-1}(20r^2)^r e^{\frac{1}{20}}\le$$
$$\frac{1}{e^{3r}r!}\left(\frac{r}{r-1}\right)^{r^2}\left(\frac{(r-1)^2}{r}\right)^{r-1}e^{\frac{1}{20}}\le\frac{1}{e^{2r}}\left(\frac{r}{r-1}\right)^{r^2}e^{\frac{1}{20}}\le e^{-2r+\frac{r^2}{r-1}+\frac{1}{20}}<e^{-r+2}\le\frac{1}{e}.$$

Applying Markov's inequality, we get that with probability at least $1-1/e$ there are no snake balls. Combining this with the result from Section 4.1, we get that with positive probability the algorithm creates a panchromatic coloring with r colors, which proves Theorem 1.

Corollary 1. *There is an absolute constant C so that for every $n>2$ and $\ln n<r<\sqrt[3]{\frac{n}{100\ln n}}$*

$$p(n,r)\ge\frac{Cn}{r^2(\ln n)}\cdot e^{\frac{n}{r}+\frac{n}{2r^2}}.$$

Proof. By applying Taylor's formula with Peano remainder, we obtain

$$\left(1+\frac{1}{r-1}\right)e^{-\frac{1}{r}-\frac{1}{2r^2}}=1+\frac{1}{3r^3}+O\left(\frac{1}{r^4}\right).$$

Thus, $\left(1+\frac{1}{r-1}\right)>e^{\frac{1}{r}+\frac{1}{2r^2}}$. Finally, we use $\left(\frac{n}{\ln n}\right)^{-\frac{1}{r}}>\frac{1}{e}$ when $r>\ln n$ and Theorem 1. $\square$

7. Local variant: proof of Theorem 2

A useful parameter of H is its *maximal edge degree*

$$D:=D(H)=\max_{e\in E(H)}\left|\left\{e'\in E(H):e\cap e'\ne 0\quad\text{and}\quad e\ne e'\right\}\right|.$$

We show that for $3<r<\sqrt[3]{\frac{n}{100\ln n}}$ every n-uniform hypergraph with $D\le\frac{1}{20e^3r^3}\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}}\left(\frac{r}{r-1}\right)^n$ has a panchromatic coloring with r colors, which implies Theorem 2.

We recall the asymmetric version of the Lovász Local Lemma, published by Spencer [16], in which it is allowed that different events can have different probability bounds. Spencer considered arbitrary events $A_1,\dots,A_n$, and expressed their dependencies by a dependency graph $G=(V,E)$. In this graph, $V=\{1,\dots,n\}$, and each event A_i is assumed mutually independent of the set of events $\left\{A_j\mid(i,j)\notin E\right\}$.

Lemma 5 (*Asymmetric LLL, [16]*)**.** *Let $A_1,\dots,A_n$ be a system of events with dependency graph $G=(V,E)$. Suppose there are real numbers $x_1,\dots,x_n\in[0,1)$, such that*

$$\mathbb{P}(A_i)\le x_i\prod_{(i,j)\in E}(1-x_j)\quad(\forall i)$$

Then,

$$\mathbb{P}\left(\bar{A}_1 \ldots \bar{A}_n\right) \geq \prod_{i=1}^{n} (1 - x_i).$$

In particular, $\mathbb{P}\left(\bar{A}_1 \ldots \bar{A}_n\right) > 0$ *holds.*

To prove Theorem 2 we will use the following special case of Lemma 5.

Lemma 6. *If all events have probability* $\mathbb{P}(A_i) < \frac{1}{2}$ *and for all* $i \in [n]$

$$\sum_{(i,j)\in E} \mathbb{P}\left(A_j\right) \leq \frac{1}{4}, \tag{20}$$

then there is a positive probability that no A_i *holds.*

For the sake of completeness, we give the proof of Lemma 6 here.

Proof. Put $x_i = 2\mathbb{P}(A_i)$. Then, for all $i \in [n]$

$$x_i \prod_{(i,j)\in E} \left(1 - x_j\right) = 2\mathbb{P}(A_i) \prod_{(i,j)\in E} \left(1 - 2\mathbb{P}(A_j)\right) = 2\mathbb{P}(A_i)\, e^{\sum_{(i,j)\in E} \ln\left(1-\mathbb{P}(A_j)\right)} \geq 2\mathbb{P}(A_i)\, e^{-\frac{1}{4}} \geq \mathbb{P}(A_i),$$

where we used that $\ln(1-x) \geq -x$ for all $x > 0$. □

In our case the set of bad events has two types: short edges and snake balls. Let $\mathcal{Q}(C)$ be the event "edge C is short" and $\mathcal{W}(C_1, \ldots, C_r)$ be the event "$(C_1, \ldots, C_r)$ forms a snake ball".

Note that $\mathcal{Q}(C)$ depends at most on D events $\mathcal{Q}(C'), C' \in E(H)$ and at most on rD^r events $\mathcal{W}(C_1, \ldots, C_r), (C_1, \ldots, C_r) \in [E(H)]^r$. The second bound follows from the fact that when we fix index $i \in \{1, \ldots, r\}$ for which $C \cap C_i \neq \emptyset$, we have at most D choices for the each of label $C_i, \ldots, C_1$ and $C_{i+1}, \ldots, C_r$. We also recall that, by definition, snake ball do not contain short edges, so $C \neq C_i$ for all $i \in \{1, \ldots, r\}$.

Similarly, $\mathcal{W}(C_1, \ldots, C_r)$ depends at most on rD events $\mathcal{Q}(C'), C' \in E(H)$ and at most on $r^2(D+1)D^{r-1}$ events $\mathcal{W}(C'_1, \ldots, C'_r), (C'_1, \ldots, C'_r) \in [E(H)]^r$. In the second bound we again fixed indexes $i, j \in \{1, \ldots, r\}$ such that $C_i \cap C'_j \neq \emptyset$ and then pick r edges. But in this case C_i can coincide with C'_j so we have $D+1$ choices for the edge C'_j and D^{r-1} choices for the rest.

Finally, using bound (19) and bound $2(r-1)\left(1 - \left(\frac{1-p}{r} + \frac{p}{r-1}\right)\right)^n$ from Section 4.1, we have

if $A_i = \mathcal{W}(C_1, \ldots, C_r)$:

$$\sum_{(i,j)\in E} \mathbb{P}(A_j) \leq rD \cdot 2(r-1)\left(1 - \left(\frac{1-p}{r} + \frac{p}{r-1}\right)\right)^n$$

$$+ r^2(D+1)D^{r-1} \cdot \left(\frac{r-1}{r}\right)^{(n-r)r} \cdot \left(\frac{p}{r-1}\right)^{r-1} 20^r r^{2r} e^{1/20} \leq$$

$$r\left(\frac{1}{20e^3r^3}\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}}\left(\frac{r}{r-1}\right)^n\right) \cdot 2(r-1)\left(\frac{r-1}{r}\right)^n\left(1 - \frac{p}{(r-1)^2}\right)^n$$

$$+ 2r^2\left(\frac{1}{20e^3r^3}\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}}\left(\frac{r}{r-1}\right)^n\right)^r\left(\frac{r-1}{r}\right)^{(n-r)r}\left(\frac{(r-1)^2\ln(\frac{n}{\ln n})}{rn}\right)^{r-1}(20r^2)^r e^{\frac{1}{20}}$$

$$\leq \frac{2r(r-1)}{20e^3r^3} + \frac{2r^2}{e^{3r}}\left(\frac{r}{r-1}\right)^{r^2} e^{\frac{1}{20}} \leq \frac{1}{10} + \frac{2r^2}{e^{3r}} e^{\frac{r^2}{r-1}+\frac{1}{20}} \leq \frac{1}{10} + \frac{2r^2e^2}{e^{2r}} < \frac{1}{4}.$$

if $A_i = \mathcal{Q}(C)$:

$$\sum_{(i,j)\in E} \mathbb{P}(A_j) \leq D \cdot 2(r-1)\left(1 - \left(\frac{1-p}{r} + \frac{p}{r-1}\right)\right)^n$$

$$+ rD^r \cdot \left(\frac{r-1}{r}\right)^{(n-r)r}\left(\frac{p}{r-1}\right)^{r-1} 20^r r^{2r} e^{1/20} < \frac{1}{4}.$$

In both cases inequality (20) holds, completing the proof of Theorem 2.

Acknowledgments

The work of the first and the third authors was funded by Russian Foundation for Basic Research No. 20-31-70039 and No. 20-31-90029 and the Council for the Support of Leading Scientific Schools of the President of the Russian Federation

No. N.Sh.-775.2022.1.1. The first author is a Young Russian Mathematics award winner and would like to thank its sponsors and jury.

References

[1] M. Akhmejanova, D.A. Shabanov, Equitable colorings of hypergraphs with few edges, Discrete Appl. Math. 276 (2020) 2–12.
[2] N. Alon, Choice number of graphs: a probabilistic approach, Combin. Probab. Comput. 1 (2) (1992) 107–114.
[3] D. Cherkashin, A note on panchromatic colorings, Discrete Math. 341 (3) (2018) 652–657.
[4] P. Erdős, On a combinatorial problem II, in: In J. Spencer (Ed.), Paul Erdős The Art of Counting, MIT Press, 1973, pp. 445–447.
[5] P. Erdős, A. Hajnal, On a property of families of sets, Acta Math. Hungar. 12 (1) (1961) 87–123.
[6] P. Erdős, L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, in: Infinite and Finite Sets, Vol. 10, Colloquia Mathematica Societatis János Bolyai, 1975, pp. 609–627.
[7] P. Erdős, A. Rubin, H. Taylor, Choosability in graphs, proceedings of the West Coast conference on combinatorics, in: Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979), Congr. Numer., Vol. 26, Utilitas Math. Publ., Winnipeg, Man, 1980, pp. 125–157.
[8] H. Gebauer, On the construction of 3-chromatic hypergraphs with few edges, J. Combin. Theory Ser. A 120 (7) (2013) 1483–1490.
[9] A. Kostochka, On a theorem of erdős, Rubin, and taylor on choosability of complete bipartite graphs, Electron. J. Combin. 9 (1) (2002) 1–4.
[10] A. Pluhár, Greedy colorings for uniform hypergraphs, Random Struct. Algorithms 35 (2) (2009) 216–221.
[11] J. Radhakrishnan, A. Srinivasan, Improved bounds and algorithms for hypergraph two-coloring, Random Struct. Algorithms 16 (1) (2000) 4–32.
[12] A. Raigorodskii, D. Cherkashin, Extremal problems in hypergraph colourings, Russian Math. Surveys 75 (1) (2020) 89–146.
[13] A. Rozovskaya, D. Shabanov, Extremal problems for panchromatic colourings of uniform hypergraphs, Discrete Math. Appl. 22 (2) (2012) 185–206.
[14] D. Shabanov, On a generalization of Rubin's theorem, J. Graph Theory 67 (3) (2011) 226–234.
[15] A. Sidorenko, What we know and what we do not know about Turan numbers, Graphs Combin. 11 (2) (1995) 179–199.
[16] J. Spencer, Asymptotic lower bounds for Ramsey functions, Discrete Math. 20 (1997) 69–76.