Powers of Rational Matrices

Every power A^k of the rational matrix

$$A = \begin{bmatrix} 2 & -\frac{89}{97} \\ \frac{97}{89} & 0 \end{bmatrix}$$

has a nonintegral entry for $k = 1, 2, \ldots, 8632$, but

$$A^{8633} = \begin{bmatrix} 8634 & -7921 \\ 9409 & -8632 \end{bmatrix}.$$

We provide a recipe for constructing such examples. Let $\mathbb{Z}$ be the set of integers and let $\mathsf{M}_n(\mathbb{Z})$ be the set of $n \times n$ matrices with entries in $\mathbb{Z}$.

Theorem. *Let $B, C \in \mathsf{M}_n(\mathbb{Z})$ with $\det B = \pm 1$ and $m = \det C \neq 0$. Let $A = CBC^{-1}$. Then there is a positive $r \le m^{2n^2}$ such that $A^r \in \mathsf{M}_n(\mathbb{Z})$.*

Proof. Since $B \in \mathsf{M}_n(\mathbb{Z})$ and $\det B = \pm 1$, it follows that B is invertible and $B^{-1} = (\operatorname{adj} B)/(\det B) = \pm \operatorname{adj} B \in \mathsf{M}_n(\mathbb{Z})$, in which $\operatorname{adj} B$ is the adjugate of B. Thus, $B^k \in \mathsf{M}_n(\mathbb{Z})$, and hence $mA^k = CB^k(\operatorname{adj} C) \in \mathsf{M}_n(\mathbb{Z})$, for all $k \in \mathbb{Z}$. There are $(m^2)^{n^2} = m^{2n^2}$ elements in $\mathsf{M}_n(\mathbb{Z}/m^2\mathbb{Z})$, so the pigeonhole principle yields positive integers $j > k$ such that $mA^j \equiv mA^k \pmod{m^2}$. Thus, there is an $X \in \mathsf{M}_n(\mathbb{Z})$ such that $mA^j = mA^k + m^2X$. Therefore, $A^{j-k} = I + (mA^{-k})X \in \mathsf{M}_n(\mathbb{Z})$, in which $1 \le j - k \le m^{2n^2}$. Let $r = j - k$ to complete the proof. ■

In the opening example, $B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 89 & 0 \\ 0 & 97 \end{bmatrix}$.

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