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Rectifiability of singular sets of noncollapsed limit spaces with Ricci curvature bounded below

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Abstract

This paper is concerned with the structure of Gromov-Hausdorff limit spaces $(M^n_i, g_i, p_i) \xrightarrow{d_{GH}} (X^n, d, p)$ of Riemannian manifolds satisfying a uniform lower Ricci curvature bound $\mathrm{Ric}_{M^n_i} \geq -(n-1)$ as well as the noncollapsing assumption $\mathrm{Vol}(B_1(p_i)) > \mathrm{v} > 0$. In such cases, there is a filtration of the singular set, $S^0 \subset S^1 \cdots S^{n-1} := S$, where $S^k := \{x \in X :$ no tangent cone at x is $(k+1)$-symmetric$\}$. Equivalently, S^k is the set of points such that no tangent cone splits off a Euclidean factor $\mathbb{R}^{k+1}$. It is classical from Cheeger-Colding that the Hausdorff dimension of S^k satisfies $\dim S^k \leq k$ and $S = S^{n-2}$, i.e., $S^{n-1} \setminus S^{n-2} = \emptyset$. However, little else has been understood about the structure of the singular set S.

Our first result for such limit spaces X^n states that S^k is k-rectifiable for all k. In fact, we will show for $\mathcal{H}^k$-a.e. $x \in S^k$ that *every* tangent cone X_x at x is k-symmetric, i.e., that $X_x = \mathbb{R}^k \times C(Y)$ where $C(Y)$ might depend on the particular X_x. Here $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. As an application we show for all $0 < \epsilon < \epsilon(n, \mathrm{v})$ there exists an $(n-2)$-rectifiable closed set S^{n-2}_ϵ with $\mathcal{H}^{n-2}(S^{n-2}_\epsilon) < C(n, \mathrm{v}, \epsilon)$, such that $X^n \setminus S^{n-2}_\epsilon$ is ϵ-bi-Hölder equivalent to a smooth Riemannian manifold. Moreover, $S = \bigcup_\epsilon S^{n-2}_\epsilon$. As another application, we show that tangent cones are unique $\mathcal{H}^{n-2}$-a.e.

In the case of limit spaces X^n satisfying a 2-sided Ricci curvature bound $|\mathrm{Ric}_{M^n_i}| \leq n-1$, we can use these structural results to give a new proof of a conjecture from Cheeger-Colding stating that S is $(n-4)$-rectifiable with uniformly bounded measure. We can also conclude from this structure that tangent cones are unique $\mathcal{H}^{n-4}$-a.e.

Our analysis builds on the notion of quantitative stratification introduced by Cheeger-Naber, and the neck region analysis developed by Jiang-Naber-Valtorta. Several new ideas and new estimates are required, including a sharp cone-splitting theorem and a geometric transformation theorem, which will allow us to control the degeneration of harmonic functions on these neck regions.

Keywords: Ricci, curvature, stratification, rectifiability
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Contents

1. **Introduction and statement of results**

This paper is concerned with the structure of noncollapsed limit spaces with a lower bound on Ricci curvature:

$$\mathrm{Ric}_{M_i^n} \geq -(n-1), \tag{1.1}$$

$$\mathrm{Vol}(B_1(p_i)) > \mathrm{v} > 0. \tag{1.2}$$

Our results represent both a qualitative and quantitative improvement over what was previously known about noncollapsed Gromov Hausdorff limit spaces with Ricci curvature bounded below. For 2-sided Ricci curvature bounds,

$$|\mathrm{Ric}_{M_i^n}| \leq n-1, \tag{1.3}$$

we are able to combine our techniques with the Codimension 4 Conjecture, proved in [CN15], in order to give a new proof that the singular set is rectifiable with a definite bound on its $(n-4)$-dimensional Hausdorff measure, a result originally proved by the second and third named authors of this article in [JN21].

1.1. *The classical stratification.* Let $C(Y)$ denote the metric cone on the metric space Y. We begin by recalling the following definition.

Definition 1.1. The metric space X is called *k-symmetric* if X is isometric to $\mathbb{R}^k \times C(Z)$ for some Z.

Remark 1.2. We say X is k-symmetric at $x \in X$ if there is an isometry of X with $\mathbb{R}^k \times C(Z)$ which carries x to a vertex of the cone $\mathbb{R}^k \times C(Z)$.

In [CC97] a *filtration* on the singular set S was defined. Namely,

$$\emptyset \subset S^0 \subseteq \cdots \subseteq S^{n-1} := S \subseteq X^n, \tag{1.4}$$

where

$$S^k := \{x \in X : \text{no tangent cone at } x \text{ is } (k+1)\text{-symmetric}\}. \tag{1.5}$$

The set $S^k \setminus S^{k-1}$ is called the kth *stratum* of the singular set. A key result of [CC97] is the Hausdorff dimension bound

$$\dim S^k \leq k \quad \text{ for all } k. \tag{1.6}$$

In [CC97], [CN15], by showing that $S^{n-1} \setminus S^{n-2} = \emptyset$, respectively $S^{n-1} \setminus S^{n-4} = \emptyset$, the following sharper estimates were proved:

$$\dim S \leq n-2 \ \text{ if } \ \mathrm{Ric}_{M_i^n} \geq -(n-1), \tag{1.7}$$

$$\dim S \leq n-4 \ \text{ if } \ |\mathrm{Ric}_{M_i^n}| \leq (n-1). \tag{1.8}$$

Note that for noncollapsed limit spaces satisfying the lower Ricci bound (1.1), *the singular set can be dense and one can have* $\mathcal{H}^{n-2}(S \cap B_1(p)) = \infty$; see Example 3.4. For general strata, essentially nothing else beyond the dimension estimate in (1.6) was previously known about the structure of the sets S^k. In the present paper, we will show that S^k is k-rectifiable *for all* k and in addition, that for $\mathcal{H}^k$-a.e. $x \in S^k$, *every* tangent cone at x is k-symmetric; see Theorems 1.9 and 1.12.[1]

[1]At the above-mentioned points, uniqueness of tangent cones can actually fail to hold for $k < n-2$; see Example 3.3. Namely, the non-Euclidean factor need not be unique. However, as a consequence of Theorem 1.12, it will follow that the tangent cones are unique $\mathcal{H}^{n-2}$-a.e..

For the case in which the lower Ricci bound (1.1) is strengthened to the 2-sided Ricci bound (1.3), the singular set is closed. In this case, we will give new proofs of conjectures stated in [CC97]. Specifically, the singular set $S = S^{n-4}$ is $(n-4)$-rectifiable and has an a priori bound on its $(n-4)$-dimensional Hausdorff measure:

$$\mathcal{H}^{n-4}(S \cap B_1(p)) \le C(n, \mathrm{v}).$$

The first proofs of those conjectures were given by W. Jiang and A. Naber in [JN21], who even proved a priori L^2 curvature estimates on M^n; for earlier results in which integral bounds on curvature were *assumed*, see [Che03], [CCT02]. The proofs in the present paper are based on new estimates, which assume only a lower bound on Ricci. In that case, the stronger estimates proved [JN21], which require assuming a 2-sided bound, can fail to hold.

1.2. *The quantitative stratification.* The quantitative stratification involves sets $S^k_{\epsilon,r}$, whose definition will be recalled below. The quantitative stratification was introduced in [CN13a] in the context of Ricci curvature, in order to state and prove new *effective* estimates on noncollapsed manifolds with Ricci curvature bounded below and, in particular, Einstein manifolds. These quantitative stratification ideas have been since used in a variety contexts (see [CN13b], [CHN13], [CHN15], [CNV15], [NV17b], [BL15], [Chu16], [Wan16], [NV19], [EE19]) to prove similar results in other areas including minimal submanifolds, harmonic maps, mean curvature flow, harmonic map flow, critical sets of linear elliptic PDE's, bi-harmonic maps, stationary Yang-Mills and free boundary problems.

Next, we recall some relevant definitions; compare (1.4). Let X denote a metric space.

Definition 1.3. Given $\epsilon > 0$ we say a ball $B_r(x) \subset X$ is (k, ϵ)-*symmetric* if there exists a k-symmetric metric cone $X' = \mathbb{R}^k \times C(Z)$, with x' a vertex of $\mathbb{R}^k \times C(Z)$, such that $d_{GH}(B_r(x), B_r(x')) < \epsilon r$.

Remark 1.4. If $\iota : B_r(x') \to B_r(x)$ is the ϵr-GH map and $\mathcal{L}_{x,r} := \iota(\mathbb{R}^k \times \{x'\} \cap B_r(x'))$, then we say $B_r(x)$ is (k, ϵ)-symmetric with respect to $\mathcal{L}_{x,r}$.

Definition 1.5.

(1) For $\epsilon, r > 0$, we define the k^{th} (ϵ, r)-stratum to be $S^k_{\epsilon,r} \setminus S^{k-1}_{\epsilon,r}$, where $S^{-1} := \emptyset$ and for $k \ge 0$,

$$S^k_{\epsilon,r} := \{x \in B_1(p) : \text{ for no } r \le s < 1 \text{ is } B_s(x) \text{ a } (k+1, \epsilon)\text{-symmetric ball}\}. \tag{1.9}$$

This should be seen as a first step toward a conjecture of [CN13], [Nab14], stating that tangent cones are unique away from a set of codimension three. Theorems 1.9 and 1.12 give the precise results in this context.

(2) For $\epsilon > 0$, we define the k^{th} ϵ-stratum to be $S^k_\epsilon \setminus S^{k-1}_\epsilon$, where $S^{-1} := \emptyset$ and for $k \geq 0$,

(1.10)
$$\begin{aligned} S^k_\epsilon &:= \bigcap_{r>0} S^k_{\epsilon,r}(X) \\ &:= \{x \in B_1(p) :\ \text{for no } 0 < r < 1 \text{ is } B_r(x) \text{ a } (k+1,\epsilon)\text{-symmetric ball}\}. \end{aligned}$$

Remark 1.6. The standard and quantitative stratification are related as follows:

$$S^k = \bigcup_{\epsilon>0} S^k_\epsilon. \tag{1.11}$$

One can see this through a simple, instructive (though not a priori obvious) contradiction argument.

To summarize,

- *The sets S^k are defined by grouping together all points $x \in X$, all of whose tangent cones fail to have $k+1$ independent translational symmetries.*
- *The sets S^k_ϵ are defined by grouping together all points $x \in X$ such that all balls fail by a definite amount to have at most $k+1$ independent translational symmetries.*
- *The sets $S^k_{\epsilon,r}$ are defined by grouping together points of $x \in X$ such that all balls $B_s(x)$ of radius at least r fail by a definite amoun to have at most $k+1$ translational symmetries.*

1.3. *Significance of the quantitative stratification.* According to (1.10), (1.11), the quantitative stratification carries more information than the standard stratification. Thus, estimates proved for the quantitative stratification have immediate consequences for the standard stratification. The latter, however, are significantly weaker. In order to illustrate this, we introduce the following notation.

Notation. Let $B_r(A) = \bigcup_{a\in A} B_r(a)$ denote tubular neighborhood of $A \subset X$ with radius r.

In [CN13a], the Hausdorff dimension estimates (1.6) on S^k were improved to the Minkowski type estimate,

$$\mathrm{Vol}(B_r(S^k_{\epsilon,r} \cap B_1(p))) \leq c(n, \mathrm{v}, \epsilon, \eta) \cdot r^{n-k-\eta} \qquad (\text{for all } \eta > 0). \tag{1.12}$$

This is further sharpened in the present paper; see Theorem 1.7, where the η in (1.12) is removed.

A complementary point to (1.12), which is *crucial for various applications*, accounts for much of the significance of the quantitative stratification. Namely, for solutions of various geometric equations, we have on the complement of the tubular neighborhood (1.12) that the solution has a definite amount of regularity, as measured by the so called *regularity scale*; see also Theorem 1.7 for the improved version. Essentially, this means that if x lies in the complement of $B_r(S^k_{\epsilon,r} \cap B_1(p))$, then on $B_{r/2}(x)$ the solution satisfies *uniform* scale invariant estimates on its derivatives. A key element of this is the existence of an ϵ-regularity theorem, stated in scale invariant form. For balls of radius 2, the ϵ-regularity theorem typically states the following: There exists k (whose value depends on the particular equation being considered) such that

If $B_2(x)$ is (k,ϵ)-symmetric, then $B_1(x)$ has bounded regularity.

In the context of the present paper, see Theorem 4.35 for the appropriate ϵ-regularity theorem for spaces with 2-sided Ricci curvature bounds. Such results allow us to turn estimates on the quantitative stratification into classical regularity estimates on the solution itself. See Theorem 1.16, as well as the L^p estimates proved in [CN13a], [CN15].

1.4. *Main results on the quantitative stratification.* In this subsection, we give our main results on the quantitative stratification for limit spaces satisfying the lower Ricci bound (1.1) and the noncollapsing condition (1.2). Our first result gives us k-dimensional Minkowski estimates on the quantitative stratification. That is, we can remove the constant $\eta > 0$ in (1.12).

THEOREM 1.7 (Measure bound for $S^k_{\epsilon,r}$). *For each $\epsilon > 0$, there exists $C_\epsilon = C_\epsilon(n, \mathrm{v}, \epsilon)$ such that the following holds. Let $(M^n_i, g_i, p_i) \xrightarrow{d_{GH}} (X, d, p)$ satisfy $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$ and $\mathrm{Ric}_{M^n_i} \geq -(n-1)$. Then*

$$\mathrm{Vol}\Big(B_r\big(S^k_{\epsilon,r}\big) \cap B_1(p)\Big) \leq c(n, \mathrm{v}, \epsilon) \cdot r^{n-k}. \tag{1.13}$$

Showing that one can replace $(n-k-\eta)$ in (1.12) by $n-k$ in (1.13) requires techniques which are fundamentally different from those used to establish (1.12) and arguments which are significantly harder. This is because such estimates are tied in with the underlying structure of the singular set itself. On the other hand, the new techniques enable us to prove much more. Our next result states that the set S^k_ϵ is rectifiable. Let us recall the definition of rectifiablity for our context.

Definition 1.8. A metric space Z is *k-rectifiable* if there exists a countable collection of $\mathcal{H}^k$-measurable subsets $Z_i \subset Z$, and bi-Lipschitz maps $\phi_i : Z_i \to \mathbb{R}^k$ such that $\mathcal{H}^k(Z \setminus \bigcup_i Z_i) = 0$.

For further details on rectifiability, especially for subsets of Euclidean space, see [Fed69]. Our main theorem on the structure of the quantitative stratification S^k_ϵ is now the following:

THEOREM 1.9 (ϵ-Stratification). *There exists $C_\epsilon = C_\epsilon(n, \mathrm{v}, \epsilon)$ for each $\epsilon > 0$ such that the following holds. Let $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X, d, p)$ satisfy $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$ and $\mathrm{Ric}_{M_i^n} \geq -(n-1)$. Then*

$$\mathrm{Vol}\Big(B_r\big(S_\epsilon^k(X)\big) \cap B_1(p)\Big) \leq C_\epsilon \cdot r^{n-k}. \tag{1.14}$$

In particular,

$$\mathcal{H}^k(S_\epsilon^k \cap B_1(p)) \leq C_\epsilon. \tag{1.15}$$

Moreover, the set S_ϵ^k is k-rectifiable, and for $\mathcal{H}^k$-a.e. $x \in S_\epsilon^k$, every tangent cone at x is k-symmetric.

Remark 1.10. The techniques used in proving the above results provide an even stronger estimate than the Minkowski estimate of (1.14). Namely, they lead to a uniform *k-dimensional packing content estimate*: Let $\{B_{r_i}(x_i)\}$ denote *any* collection of *disjoint* balls such that $x_i \in S_\epsilon^k$. Then

$$\sum r_i^k \leq C_\epsilon. \tag{1.16}$$

Remark 1.11. The structural results above are actually sharp. In Example 3.2 we will explain a construction from [LN20] of a noncollapsed limit space X^n such that

$$S = S^k = S_\epsilon^k,$$
$$0 < \mathcal{H}^k(S) < \infty,$$

for which S_ϵ^k is both k-rectifiable and bi-Lipschitz to a k-dimensional (fat) Cantor set. In particular, the singular set has no manifold points. However, it is still an open question to show that in the presence of a 2-sided bound on Ricci curvature, the singular set must contain manifold points.[2]

1.5. *Results for the classical stratification.* We now state our main results for the classical stratification S^k. They follow as special cases of the preceding results on the quantitative stratification.

Since $S^k = \bigcup_\epsilon S_\epsilon^k$, the following theorem is essentially an immediate consequence of Theorem 1.9.

THEOREM 1.12 (Stratification). *Let $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X^n, d, p)$ satisfy $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$ and $\mathrm{Ric}_{M_i^n} \geq -(n-1)$. Then S^k is k-rectifiable and for $\mathcal{H}^k$-a.e. $x \in S^k$, every tangent cone at x is k-symmetric.*

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[2]If M^n is Kähler with a polarization, then it has been shown in [DS14], [Tia13] that the singular set is *topologically* a variety. However, the smoothness or even the bi-Lipschitz structure of the singular set is still unknown even in this case.

Remark 1.13. Note that unlike in the Hausdorff measure bound on $S^k_\epsilon \subset S^k$ given in (1.15), we are not asserting a finite measure bound on all of S^k. Example 3.4 shows that such a bound need not hold. However, as will become clear in the proof of Theorem 1.12, to prove results which concern the structure of the sets S^k, it is crucial to be able to break the stratification into the well-behaved finite measure subsets S^k_ϵ.

We end this subsection with two results which are essentially direct applications of Theorems 1.9 and 1.12.

THEOREM 1.14 (Manifold structure). *Let* $(M^n_i, g_i, p_i) \xrightarrow{d_{GH}} (X^n, d, p)$ *satisfy* $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$ *and* $\mathrm{Ric}_{M^n_i} \geq -(n-1)$. *Then there exists a subset* $S_\epsilon \subseteq X^n$ *which is* $(n-2)$*-rectifiable with* $\mathcal{H}^{n-2}\Big(S_\epsilon \cap B_1(p)\Big) \leq C(n, \mathrm{v}, \epsilon)$ *and such that* $X^n \backslash S_\epsilon$ *is bi-Hölder homeomorphic to a smooth Riemannian manifold.*

THEOREM 1.15 (Tangent uniqueness). *Let* $(M^n_i, g_i, p_i) \xrightarrow{d_{GH}} (X^n, d, p)$ *satisfy* $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$ *and* $\mathrm{Ric}_{M^n_i} \geq -(n-1)$. *Then there exists a subset* $\tilde{S} \subseteq X$ *with* $\mathcal{H}^{n-2}(\tilde{S}) = 0$ *such that for each* $x \in X \setminus \tilde{S}$, *the tangent cones are unique and isometric to* $\mathbb{R}^{n-2} \times C(S^1_r)$ *for some* $0 < r \leq 1$.

1.6. *2-sided bounds on Ricci curvature.* In this subsection, we state a result for noncollapsed limit spaces with a 2-sided bound on Ricci curvature, Theorem 1.16. Recall that under the assumption of a 2-sided bound, the singular set S is closed and can be described as the set of points no neighborhood of which is diffeomorphic to an open subset of $\mathbb{R}^n$. Our result follows quickly by combining the quantitative stratification results of for limit spaces satisfying (1.1) with the ϵ-regularity theorem of [CN15]; see Section 4.11 for a review of this material. A stronger version of Theorem 1.16 was first proved in [JN21], where additionally L^2 bounds on the curvature were produced, but by using estimates and techniques which definitely require a 2-sided bound on Ricci curvature:

THEOREM 1.16 (Two sided Ricci). *Let* $(M^n_i, g_i, p_i) \xrightarrow{d_{GH}} (X^n, d, p)$ *satisfy* $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$ *and* $|\mathrm{Ric}_{M^n_i}| \leq (n-1)$. *Then* S *is* $(n-4)$*-rectifiable and there exists* $C = C(n, \mathrm{v})$ *such that*

$$\mathrm{Vol}\Big(B_r\big(S \cap B_1(p)\big)\Big) \leq C r^4. \tag{1.17}$$

In particular, $\mathcal{H}^{n-4}(S \cap B_1) \leq C$. *Furthermore, for* $\mathcal{H}^{n-4}$*-a.e.* $x \in S$, *the tangent cone at* x *is unique and isometric to* $\mathbb{R}^{n-4} \times C(S^3/\Gamma)$, *where* $\Gamma \subseteq O(4)$ *acts freely.*

1.7. *The remainder of the paper.* The paper can be viewed as having four parts. The first part consists of the present section and Section 2; the second consists of Sections 3–5; the third part consists of Sections 6–8; the fourth part consists of Sections 9 and 10.

Section 2 contains the definition and concept of "neck region," including an explanation of the role played by each of the conditions in the definition, the statements of the Neck Structure Theorem 2.9 and the Neck Decomposition Theorem 2.12, and some basic examples. In addition, this section contains the proofs of our main results on the quantitative stratification, under the assumption that the Neck Structure Theorem 2.9 and the Neck Decomposition Theorem 2.12 hold. Part three of the paper is devoted to developing the new tools which are needed for the proofs the neck theorems, while the proofs themselves are given in part four.

The second part of the paper begins with Section 3, in which we give some examples beyond those given in Section 2. One of these concerns neck regions. The remaining examples illustrate the sharpness of our results on the quantitative stratification.

In Section 4, we collect background results which are needed in parts three and four. Some of these results are by now rather standard in the smooth Riemannian geometry context (as opposed to the context of synthetic lower Ricci bounds). In such cases, we will just give references. For the more technical results which are less well known, we will give the proofs or at least outlines.

In Section 5, we give a brief outline of part three (Sections 6–8) and of part four (Sections 9 and 10).

In Sections 6–8, which form the third part of the paper, we prove *sharp estimates* on quantitative cone-splitting. The statements of these theorems involve the local pointed entropy. Like harmonic splitting maps and heat kernel estimates, the entropy can be viewed as analytical tool which, once it has been controlled by the geometry, enables one to draw additional (and in this case sharp) geometrical conclusions from the original geometric hypotheses. The results on necks, especially the Neck Structure Theorem 2.9, depend on the new sharp estimates. The estimates enable us to take full advantage of the behavior of the geometry over an *arbitrary number of consecutive scales*. This is crucial for the proofs of the neck theorems.

Sections 9 and Section 10 constitute the fourth part of the paper. In Section 9 we prove the Neck Structure Theorem 2.9. The proof depends on the results of Sections 6–8. In Section 10, via an induction argument, we prove the Neck Decomposition Theorem 2.12. Remarkably, for the most part the proof only involves (highly nontrivial) covering arguments, and only at a certain point is an appeal to Theorem 2.9 made.

Remark 1.17 (Future directions). Although in the present paper we have stated our results for fixed k, the complete description of the geometry should include *simultaneously* all $k = 0, 1, \dots, n-1$. In the general case, it should also involve behavior on multiple scales, thereby generalizing the bubble tree decompositions in [AC91], [Ban90] and Section 4 of [CN15].

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2. Proofs of the stratification theorems modulo results on neck regions

In this section we will begin by introducing the notion of neck regions and stating our main theorems for them: namely the Neck Structure Theorem 2.9 and the Decomposition Theorem 2.12. Proving these results will constitute the bulk of this paper. The proofs are outlined in Section 5. In the last subsection of this section, assuming that the Neck Structure Theorem and the Decomposition Theorem hold, we will prove all of the results on quantitative and classical stratifications. In a few places, we will appeal to results which are reviewed in Section 4.

2.1. *Background and motivation*. Let $\mathrm{Vol}^\kappa(B_r)$ denote the volume of an r-ball in a simply connected space M^n_κ of constant curvature $\equiv \kappa$. Define the *volume ratio* by

$$\mathcal{V}^\kappa_r(x) := \frac{\mathrm{Vol}(B_r(x))}{\mathrm{Vol}^\kappa(B_r)}. \tag{2.1}$$

The Bishop-Gromov inequality states that if $\mathrm{Ric}_{M^n} \geq -(n-1)\kappa$, then $\mathcal{V}^\kappa_r(x)$ is *monotone nonincreasing* in r:

$$\frac{d}{dr}\mathcal{V}^\kappa_r(x) \leq 0. \tag{2.2}$$

In addition to being monotone, the quantity $\mathcal{V}_r(x)$ *coercive* in the following sense. Given $\epsilon > 0$, there exists $0 < \delta = \delta(\epsilon, n, \kappa)$, such that if $r^2\kappa < \delta$ and

$$\left|\mathcal{V}^\kappa_r(x) - \mathcal{V}^\kappa_{r/2}(x)\right| < \delta, \tag{2.3}$$

then $B_r(x)$ is ϵ-Gromov Hausdorff close to a ball $B_r(y^*) \subset C(Y)$ for some metric cone with cross-section Y and vertex y^*. This statement is the "almost volume cone implies almost metric cone theorem" of [ChCo2]; see Section 4.1 for a more complete review.

Remark 2.1. Whenever we have specified a definite lower bound, say $\mathrm{Ric}_{M^n} \geq -(n-1)\kappa$, we will write $\mathcal{V}_r(x)$ for $\mathcal{V}^\kappa_r(x)$. Similarly, for a sequence $M^n_i \xrightarrow{d_{GH}} X^n$, if $\liminf_{i\to\infty} \mathrm{Ric}_{M^n_i} \geq 0$, we will write $\mathcal{V}_r(x)$ for $\mathcal{V}^0_r(x)$.

The noncollapsing assumption (1.2) and the monotonicity (2.2) of $\mathcal{V}_r(x)$ directly imply

$$\sum_i |\mathcal{V}_{\delta 2^{-i}}(x) - \mathcal{V}_{\delta^{-1}2^{-(i+1)}}(x)| \le C(n, \mathrm{v}, \delta). \tag{2.4}$$

As an immediate consequence, for any $\delta > 0$,

$$\lim_{r\to 0} |\mathcal{V}_{\delta r}(x) - \mathcal{V}_{\delta^{-1}r}(x)| = 0. \tag{2.5}$$

This, together with the "almost volume cone implies almost metric cone" theorem, was used in [CC97] to prove that for noncollapsed limit spaces satisfying (1.1), (1.2), *every tangent cone is a metric cone*.

For applications which concern S^k, the "cone-splitting principle" is also crucial. In abstract form, where we are using Definition 1.1, it can be stated as follows:

THE CONE-SPLITTING PRINCIPLE. *Let X be a metric space which is* 0*-symmetric with respect to two distinct points $x_0, x_1 \in X$. Then X is* 1*-symmetric with respect to these points.*

The estimate (2.4), together with the cone-splitting principle, was used in [CN13a] to prove the weak Minkowski estimate (1.12).

Notation. A *scale* is just a number of the form $r_j = 2^{-j}$. Note then that (2.4) actually yields the following:

Effective version of (2.5). Given $\epsilon > 0$, on all but a definite number N_ϵ of scales, relation (2.3) will hold and $B_r(x)$ will be $(0, \epsilon)$-symmetric.

Remark 2.2 (Lack of sharpness). The effective version of (2.5), together with a quantitative version of cone-splitting, was used in [CN13a] to obtain effective estimates on the sets S^k_ϵ, notably (1.12). Clearly this makes use of more information than the classical dimension reduction arguments of [CC97], which require only (2.5). Nonetheless, a lot of information is being disregarded when passing from (2.4) to the above effective version of (2.5). The ability to take full advantage of (2.4) eventually leads to the main volume and rectifiability estimates of this paper. However, this requires a number of new ideas in order to not lose any information, all of which turns out to be essential.

2.2. *Neck regions.* As explained in Section 1, our results on the classical stratification S^k follow from structural results for the quantitative stratification S^k_ϵ, and these results follow from results on neck regions and neck decompositions. Neck decompositions of the type employed here were first introduced in [JN21] and [NV19], where they played a key role in the proofs of the a priori L^2 curvature bound for spaces with a 2-sided bound on Ricci curvature and the energy identity, respectively;[3] compare also [NV17a]. As these papers illustrate, neck decompositions are of interest in their own right. In particular, their uses go beyond applications to structural results on singular sets, which are the main focus of the present paper.

We will need the following notion of a tubular neighborhood of variable radius.

Definition 2.3 (Tube of variable radius). If $\mathcal{D} \subseteq X$ is a closed subset and $x \to r_x$ (the radius function) is a nonnegative continuous function defined on $\mathcal{D}$, then the corresponding tube of variable radius is

$$\overline{B}_{r_x}(\mathcal{D}) := \bigcup_{x \in \mathcal{D}} \overline{B}_{r_x}(x).$$

Recall Definition 1.3 and Remark 1.4 the notion of (k, ϵ)-symmetry with respect to a subspace. We now give our definition of a neck region:

Definition 2.4 (Neck Regions). Let $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X, d, p)$ satisfy $\mathrm{Ric}_{M_i^n} \geq -(n-1)\delta^2$, $\mathrm{Vol}(B_1(p_i)) > \mathrm{v} > 0$, and let $\eta > 0$. Let $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_+ \subseteq B_2(p)$ denote a closed subset with $p \in \mathcal{C}$, and let $r_x : \mathcal{C} \to \mathbb{R}^+$ be continuous such that $r_x := 0$ on $\mathcal{C}_0$ and $r_x > 0$ on $\mathcal{C}_+$. The set $\mathcal{N} = B_2(p) \setminus \overline{B}_{r_x}(\mathcal{C})$ is a (k, δ, η)-neck region if for all $x \in \mathcal{C}$, the following hold:

(n1) $\{B_{\tau_n^2 r_x}(x)\} \subseteq B_2(p)$ are pairwise disjoint, where $\tau_n = 10^{-10n}\omega_n$;
(n2) $|\mathcal{V}_{\delta^{-1}}(x) - \mathcal{V}_{\delta r_x}(x)| < \delta^2$;
(n3) for each $r_x \leq r \leq \delta^{-1}$, the ball $B_r(x)$ is (k, δ^2)-symmetric, wrt $\mathcal{L}_{x,r}$, but not $(k+1, \eta)$-symmetric;
(n4) for each $r \geq r_x$ with $B_{2r}(x) \subseteq B_2(p)$, we have $\mathcal{L}_{x,r} \subseteq B_{\tau_n r}(\mathcal{C})$ and $\mathcal{C} \cap B_r(x) \subseteq B_{\tau_n r}(\mathcal{L}_{x,r})$.
(n5) $|\mathrm{Lip}\, r_x| \leq \delta$.

Remark 2.5 (Vitali covering terminology). Throughout the paper a covering as in (n1), but possibly with some other constant $\gamma < 1/6$ in place of τ_n, will be referred to as a *Vitali covering.*

Remark 2.6. The set $\mathcal{C}$ will be referred to as *the set of centers* of $\mathcal{N}$. Below we provide some explanation for the various conditions, (n1)–(n5), in Definition 2.4.

(1) The effective disjointness property of (n1) guarantees that we do not overly cover, which would prevent property (3) of the Neck Structure Theorem 2.12 from holding. The center set $\mathcal{C}$ is used not solely as approximation

[3] In those papers, only the top stratum of the neck regions could be controlled, and only under much more restrictive hypotheses.

to the singular set but also as an approximation to the relevant Hausdorff measure; see the packing measure defined in Definition 2.8. Without (n1) we would have no hope of controlling this packing measure; see Theorem 2.9. Another simple consequence is that the set $\mathcal{C}_+$ consists of a discrete set of points.

(2) Condition (n2) has the consequence that even if the neck region involves infinitely many scales, there is a summable energy condition over the whole region. This summable energy is key for both the rectifiability and measure estimates of Theorem 2.9.

(3) One consequence of (n3) is that if $x \in \mathcal{C}$, then $x \in S^k_{\eta,r_x}$; in particular, $\mathcal{C}_0 \subseteq S^k_\eta$. Both the assumed k-symmetry and the assumed lack of $(k+1)$-symmetry play a key role in the Geometric Transformation Theorem 5.6. These conditions act as a form of rigidity which stops harmonic splitting maps from degenerating in uncontrollable ways.

(4) Condition (n4) plays the role of a Reifenberg condition on the singular set. It is strong enough to prove bi-Hölder control on $\mathcal{C}$, but not bi-Lipschitz control, which requires in addition (n2), and is the main goal of this paper.

(5) Condition (n5) says that if $x \in \mathcal{C}$, then r_x looks roughly constant on $B_{10^4 r_x}(x)$. It turns out that constructing neck regions with this condition is quite painful, but it is especially important for the Nondegeneration Theorem 8.1. It allows us to take integral estimates on neck regions and use them to control the behavior of the center points themselves.

(6) If $\mathcal{N}$ is a neck region in a smooth Riemannian manifold M^n, then $\mathcal{C}_0 = \emptyset$.

(7) If $\mathcal{N} \subseteq B_2(p)$ is a (k,δ,η)-neck region and $B_{2s}(q) \subseteq B_2(p)$ with $q \in \mathcal{C}$, then $\mathcal{N} \cap B_{2s}(q) \subseteq B_{2s}(q)$ defines a (k,δ,η)-neck region.

Remark 2.7 (Important convention). Often throughout the paper we will state a result for balls of radius 1 and use it (often without comment) for balls of radius $r < 1$, where the more general case follows immediately from the special case by scaling.

We will want to view $\mathcal{C}$ as a discrete approximation of a k-dimensional set. Similarly, we want to associate to it a measure which is a discrete approximation of the k-dimensional Hausdorff measure on $\mathcal{C}$:

Definition 2.8. Let $\mathcal{N} := B_2(p) \setminus \overline{B}_{r_x}(\mathcal{C})$ denote a k-neck region. The associated packing measure is the measure

$$\mu := \mu_{\mathcal{N}} := \sum_{x \in \mathcal{C}_+} r_x^k \delta_x + \mathcal{H}^k|_{\mathcal{C}_0}, \tag{2.6}$$

where $\mathcal{H}^k|_{\mathcal{C}_0}$ denotes the k-dimensional Hausdorff measure restricted to $\mathcal{C}_0$.

Our main result on the structure of k-neck regions is the following. The proof, which will be outlined in Section 5, depends on several new ideas. It constitutes the bulk of the paper:

THEOREM 2.9 (Neck Structure Theorem). *Fix* $\eta > 0$ *and* $\delta \leq \delta(n, \mathrm{v}, \eta)$. *Then if* $\mathcal{N} = B_2(p) \setminus \overline{B}_{r_x}(\mathcal{C})$ *is a* (k, δ, η)*-neck region, the following hold*:

(1) *For each* $x \in \mathcal{C}$ *and* $B_{2r}(x) \subset B_2(p)$, *the induced packing measure* μ *is Ahlfors regular*:

$$A(n)^{-1} r^k < \mu(B_r(x)) < A(n) r^k. \tag{2.7}$$

(2) $\mathcal{C}_0$ *is* k*-rectifiable.*

Remark 2.10. One can view the Ahlfors regularity condition (2.7) as an effective consequence of rectifiability. Indeed, for simplicity, imagine that $u(\mathcal{C}_0)\cup \{B_{r_x}(u(x))\}$ is contained in $B_2(0^k)$ for a bi-Lipschitz map $u : B_2(p) \cap \mathcal{C} \to \mathbb{R}^k$. It is a simple but highly instructive exercise to see that (2.7) would follow immediately. Conversely, much of the work of this paper will be devoted to showing that if (2.7) holds, then such a mapping u exists. More precisely, the mapping u will be taken to be a harmonic splitting function. If (2.7) holds, then we will see that u is automatically bi-Lipschitz, at least on most of $\mathcal{C}$. One must do this carefully in order to close the loop. Thus, we will show essentially simultaneously through an inductive argument that (2.7) holds and that u is bi-Lipschitz. The proof of this, which is quite involved, takes up most of the paper; see Section 5 for a detailed outline.

Before continuing let us mention the simplest example of a k-neck region:

Example 2.11 (Simplest). Consider the metric cone space $X = \mathbb{R}^k \times C(S^1_r)$, where S^1_r is a circle of radius $r < 1$. Denote by $0 \in C(S^1_r)$ the cone point, so that $\mathcal{L} := \mathbb{R}^k \times \{0\}$ is the singular set of X. Choose any function $r_x : B_2(0^k) \subseteq \mathcal{L} \to \mathbb{R}^+$ such that $|\nabla r_x| \leq \delta$, and let $\mathcal{C} \subseteq B_2(0^k) \times \{0\}$ be any closed subset such that $\{B_{\tau_n^2 r_x}(x)\}$ is a maximal disjoint set. Then for $r < 1 - C(n)\eta$, it is an easy but instructive exercise to check that $B_2 \setminus \overline{B}_{r_x}(\mathcal{C})$ is a (k, δ, η)-neck region. Note that it is trivial that $\mathcal{C}_0$ is k-rectifiable, as $\mathcal{C}_0 \subseteq \mathbb{R}^k$ canonically. Similarly, the Ahlfors regularity condition (2.7) may be verified as $\{\overline{B}_{r_x}(x)\}$ forms a Vitali covering of $B_2(0^k)$.

For additional and more complicated examples, see Section 3.

2.3. *Neck decompositions.* In order to prove our theorems on stratifications, we also need to suitably control the part of X^n which does not consist of neck regions. This is provided by the following result.

THEOREM 2.12 (Neck Decomposition). *Let* $(M^n_i, g_i, p_i) \xrightarrow{d_{GH}} (X^n, d, p)$ *satisfy* $\mathrm{Vol}(B_1(p_i)) > \mathrm{v} > 0$ *and* $\mathrm{Ric}_{M^n_i} \geq -(n-1)$. *Then for each* $\eta > 0$

and $\delta \leq \delta(n, \mathrm{v}, \eta)$, we can write

$$B_1(p) \subseteq \bigcup_a \big(\mathcal{N}_a \cap B_{r_a}\big) \cup \bigcup_b B_{r_b}(x_b) \cup \mathcal{S}^{k,\delta,\eta}, \tag{2.8}$$

$$\mathcal{S}^{k,\delta,\eta} \subseteq \bigcup_a \big(\mathcal{C}_{0,a} \cap B_{r_a}\big) \cup \tilde{\mathcal{S}}^{k,\delta,\eta}, \tag{2.9}$$

where

(1) *for all a, the set $\mathcal{N}_a = B_{2r_a}(x_a) \setminus \overline{B}_{r_x}(\mathcal{C})$ is a (k, δ, η)-neck region;*
(2) *the balls $B_{2r_b}(x_b)$ are $(k+1, 2\eta)$-symmetric; hence $x_b \notin S^k_{2\eta, r_b}$;*
(3) $\sum_a r_a^k + \sum_b r_b^k + \mathcal{H}^k\big(\mathcal{S}^{k,\delta,\eta}\big) \leq C(n, \mathrm{v}, \delta, \eta)$;
(4) *$\mathcal{C}_{0,a} \subseteq B_{2r_a}(x_a)$ is the k-singular set associated to $\mathcal{N}_a$;*
(5) *$\tilde{\mathcal{S}}^{k,\delta,\eta}$ satisfies $\mathcal{H}^k\big(\tilde{\mathcal{S}}^{k,\delta,\eta}\big) = 0$;*
(6) *$\mathcal{S}^{k,\delta,\eta}$ is k-rectifiable;*
(7) *for any ϵ, if $\eta \leq \eta(n, \mathrm{v}, \epsilon)$ and $\delta \leq \delta(n, \mathrm{v}, \eta, \epsilon)$, we have $S^k_\epsilon \subset \mathcal{S}^{k,\delta,\eta}$.*

Remark 2.13. In the case of a smooth manifold M^n, we have $\mathcal{S}^{k,\delta,\eta} = \emptyset$; compare (6) of Remark 2.6. In that case, M^n decomposes into only two types of regions, k-neck regions and the $k+1$-symmetric balls B_{r_b}.

The following two examples illustrate the Decomposition Theorem.

Example 2.14 (k-Symmetric symmetric example). Let S^{n-s-1}/Γ denote a compact manifold of curvature $\equiv 1$. The space $X^n := C(S^{n-s-1}/\Gamma) \times \mathbb{R}^s$ is s-symmetric with $0 \leq s \leq n-2$. Consider $B_1(p) \subset X^n$, where $p = (y_c, 0^s)$ is the cone vertex. For each integer $0 \leq k \leq n-2$ and $\eta \leq \eta_0$ and $\delta = 0$, we are able to choose a decomposition as in Theorem 2.12. To see this, we will divide it into three cases:

Case 1: $0 \leq k \leq s-1$. We can choose our decomposition to be the single ball $B_{r_b}(x_b) = B_2(p)$, which is $k+1$-symmetric.

Case 2: $k = s$. We can choose $B_{r_a}(x_a) = B_2(p)$ with $\mathcal{N}_a = B_{r_a}(x_a) \setminus \{y_c\} \times \mathbb{R}^s$ and $\mathcal{S}^{k,0,\eta} = (\{y_c\} \times \mathbb{R}^s) \cap B_2(p)$. Then

$$B_1(p) \subseteq \mathcal{N}_a \cup \mathcal{S}^{k,0,\eta}, \tag{2.10}$$

and $\mathcal{S}^{k,0,\eta}$ is k-rectifiable. In this case, $\mathcal{N}_a$ is a $(k, 0, \eta)$-neck region with $\mathcal{C} = \mathcal{C}_0 = \mathcal{S}^{k,0,\eta}$.

Case 3: $k \geq s+1$. For each $r > 0$, let us consider a Vitali covering $\{B_{\epsilon_0 r}(x_{r,j}), j = 1, \ldots, N_r\}$ of $B_2(p) \cap B_{2r}(\{y_c\} \times \mathbb{R}^s) \setminus B_r(\{y_c\} \times \mathbb{R}^s)$, where $\epsilon_0 \leq \epsilon_0(n, \Gamma)$ so that $B_{2\epsilon_0 r}(x_{r,j})$ is n-symmetric. Then the cardinality satisfies $N_r \leq C(n, \Gamma) r^{-s}$. Each $B_{2\epsilon_0 r}(x_{r,j})$ is n-symmetric, and we will belong to the b-ball in the decomposition. Let us define $\mathcal{S}^{k,0,\eta} = \tilde{\mathcal{S}}^{k,0,\eta} = \{y_c\} \times \mathbb{R}^s$. Then we have

$$B_1(p) \subseteq \bigcup_{1\geq r_b=2^{-b}>0} \bigcup_{i=1}^{N_{r_b}} B_{\epsilon_0 r_b}(x_{r_b,i}) \cup \mathcal{S}^{k,0,\eta}. \tag{2.11}$$

We have $\mathcal{H}^k(\tilde{\mathcal{S}}^{k,0,\eta}) = 0$, and the k-content of b-balls satisfies

$$\sum_{1\geq r_b>0} \sum_{i=1}^{N_{r_b}} r_b^k \leq \sum_{1\geq r_b>0} C(n,\Gamma) r_b^{k-s} \leq C(n,\Gamma). \tag{2.12}$$

Hence (2.11) is the desired decomposition.

Example 2.15 (The boundary of a simplex). Let $X^n := \partial\sigma^{n+1}$ denote the boundary of the standard $(n+1)$-simplex in $\mathbb{R}^{n+1}$ normalized so that all edges have length 1. Let Σ^k denote the closed k-skeleton of X^n. By appropriately smoothing the sequence of boundaries $\partial B_{r_i}(\sigma^{n+1})$, of the tubular neighborhoods, $B_{r_i}(\sigma^{n+1})$, and letting $r_i \to 0$, one see that X^n is a limit space with $\mathrm{Ric}_{M_i^n} \geq 0$, indeed $\sec_{M_i^n} \geq 0$. Note that $S^k = \Sigma^k$ is k-rectifiable and $\mathcal{H}^k(S^k \cap B_1(p)) < c(n)$ for all $0 \leq k \leq n-2$.

For each $0 \leq k \leq n-2$ and $0 < \delta, \eta \leq \eta(n)$, we will build a decomposition for X^n as in Theorem 2.12. The idea is similar to Case 3 of Example 2.14. The decomposition consists of two parts, corresponding to the a-balls and b-balls of 2.12, respectively.

(1) *Neck regions.* We will construct neck regions with center in $S^k \setminus S^{k-1}$. For each $0 < r \leq 1$, consider a Vitali covering $\{B_{\delta^2 r}(x_{a,r}), x_{a,r} \in S^k \setminus S^{k-1}\}$ of the annuli $A_{r,2r}(S^{k-1}) \cap B_{\delta^3 r}(S^k)$. One checks that $\mathcal{N}_{a,r} = B_{\delta^2 r}(x_{a,r}) \setminus S^k$ is a (k,δ,η)-neck region for $\eta \leq \eta(n)$. The neck regions $\{\mathcal{N}_{a,r_i}, r_i = 2^{-i}, i = 1, \ldots\}$ are the desired neck regions of the decomposition. Moreover, by noting that $\mathcal{H}^k(S^k) \leq C(n)$, we obtain the k-content estimate

$$\sum_a \sum_i r_{a,i}^k \leq C(n,\delta). \tag{2.13}$$

(2) $(k+1)$-*symmetric balls.* Consider a Vitali covering $\{B_{\delta^4 r}(x_{d,r}), x_{d,r} \in A_{r,2r}(S^{k-1}) \setminus B_{\delta^3 r}(S^k)\}$ of $A_{r,2r}(S^{k-1}) \setminus B_{\delta^3 r}(S^k)$. The cardinality of this covering is less than $C(n,\delta) r^{-k+1}$. From the construction we have that $B_{\delta^4 r}(x_{d,r}) \cap S^k = \emptyset$, which implies that $B_{\delta^4 r}(x_{d,r})$ is $(k+1)$-splitting. For each $\eta > 0$, by the Almost Volume Cone Implies Almost Metric Cone Theorem, we have that for each $y \in B_{\delta^4 r}(x_{d,r})$, the ball $B_{\gamma\delta^4 r}(y)$ would be $(0,\eta^2)$-symmetric for some $\gamma(n,\delta,\eta) \leq 1$. Therefore, $B_{\gamma\delta^4 r}(y)$ is $(k+1,\eta/2)$-symmetric, which implies that each $B_{\delta^4 r}(x_{d,r})$ can be covered by finitely many $(k+1,\eta/2)$-symmetric balls. Hence, we can choose at most $N_r = C(n,\delta,\eta) r^{-k+1}$ $(k+1,\eta/2)$-symmetric balls $B_{\gamma\delta^4 r}$, whose union covers $A_{r,2r}(S^{k-1}) \setminus B_{\delta^3 r}(S^k)$. By combining them

all for $r = r_i = 2^{-i} \leq 1$, we get the desired b-balls in our decomposition which satisfy the following content estimate:

$$\sum_{0<r_i=2^{-i}\leq 1} \sum_{j=1}^{N_{r_i}} (\gamma\delta^4 r_i)^k \leq \sum_{0<r_i=2^{-i}\leq 1} C(n,\eta,\delta) r_i \leq C(n,\eta,\delta). \tag{2.14}$$

Define $\tilde{\mathcal{S}}^{k,\delta,\eta} := S^{k-1}$, then

$$X^n \subseteq \bigcup_a \big(\mathcal{N}_a \cap B_{r_a}\big) \cup \bigcup_b B_{r_b}(x_b) \cup \mathcal{S}^{k,\delta,\eta}, \tag{2.15}$$

$$\mathcal{S}^{k,\delta,\eta} \subseteq \bigcup_a \big(\mathcal{C}_{0,a} \cap B_{r_a}\big) \cup \tilde{\mathcal{S}}^{k,\delta,\eta}, \tag{2.16}$$

with $\mathcal{C}_{0,a} = B_{r_a} \cap S^k$. This completes the description of the decomposition for $X^n = \partial\sigma^{n+1}$.

Remark 2.16 (Role of the $\sum_b r_b^k$ bound). In light of the fact that the b-balls are approximately $(k+1)$-symmetric, the crucial role of the a priori bound on $\sum_b r_b^k$ in the Neck Decomposition Theorem 2.12 might not be immediately obvious if one thinks only of the application to $\mathcal{H}^k(S^k_\epsilon \cap B_1(x))$. Recall, however, that our volume bounds for the quantitative stratification pertain to tubes of fixed radius r, while the function r_x of the a-balls, goes to zero as $x \to S^{k-1}$. This suggests that it should not suffice to consider only a-balls in obtaining the applications to the volumes of the tubes around the quantitative strata on neck regions, particularly, the Neck Decomposition Theorem 2.12. This should be kept in mind when reading the details of the proofs which are given in the next subsection.

2.4. *Proofs of the stratification theorems assuming the neck theorems.* In this subsection we will prove the main stratification Theorems 1.7, 1.9 and the classical stratification Theorems 1.12, 1.14, 1.16, under the assumption that the Neck Structure and Decompositions of Theorems 2.9, 2.12 hold. We will outline the proof of the Neck Structure Theorem in Section 5.

The main result concerns the (ϵ, r)-stratification of Theorem 1.7. The other theorems follow fairly quickly from it and the Decomposition Theorem 2.12.

Proof of Theorem 1.7. From (6) and (7) of the Neck Decomposition Theorem it follows that S_ϵ is rectifiable. Thus, it remains to prove estimate (1.13) in Theorem 1.7, which states

$$\mathrm{Vol}\Big(B_r\Big(S^k_{\epsilon,r} \cap B_1(p)\Big)\Big) \leq C_\epsilon r^{n-k}. \tag{2.17}$$

By the Volume Convergence Theorem of [Col97], [Che01] and the definition of the sets $S^k_{\epsilon,r}$, to obtain the estimate in (2.17) for the case of limit spaces, it easily suffices to prove (2.17) for the case of manifolds M^n. We will now give the proof in that case.

Given $\epsilon > 0$, let $\eta \le \eta(n, \mathrm{v}, \epsilon)$ and $\delta \le \delta(n, \mathrm{v}, \epsilon, \eta)$ be chosen sufficiently small, to be fixed later. Recall that for the case of manifolds, the Decomposition Theorem 2.12 states

$$B_1(p) \subset \bigcup_a \Big(\mathcal{N}_a \cap B_{r_a}(x_a)\Big) \cup \bigcup_b B_{r_b}(x_b), \tag{2.18}$$

where $\mathcal{N}_a \subset B_{2r_a}(x_a)$ is a (k, δ, η)-neck and $B_{2r_b}(x_b)$ is $(k+1, 2\eta)$-symmetric. In addition, Theorem 2.12 provides the k-content estimate:

$$\sum_a r_a^k + \sum_b r_b^k \le C(n, \mathrm{v}, \delta, \eta). \tag{2.19}$$

The proof of Theorem 1.7 amounts to combining the estimates of Lemmas 2.17 and 2.19 below. The proof of Lemma 2.17 relies on the Ahlfors regularity of the packing measures μ_a on the balls $B_{r_a}(x_a)$; see (2.7) of Theorem 2.9.

Lemma 2.17. *Let $\eta \le \eta(n, \mathrm{v}, \epsilon)$, $\delta \le \delta(n, \mathrm{v}, \epsilon)$ and $\chi \le \chi(\epsilon, n, \mathrm{v})$. If the neck region $\mathcal{N}_a$ satisfies $r_a \ge \chi^{-1} r$, then*

$$\mathrm{Vol}\Big(B_r\left(S^k_{\epsilon,r} \cap \mathcal{N}_a\right)\Big) \le C(n, \mathrm{v}, \chi) r_a^k \cdot r^{n-k}, \tag{2.20}$$

$$\mathrm{Vol}\left(B_r\left(S^k_{\epsilon,r} \cap \bigcup_{r_a \ge \chi^{-1} r} \mathcal{N}_a\right)\right) \le C(n, \mathrm{v}, \delta, \eta, \chi) r^{n-k}. \tag{2.21}$$

Proof. First we will prove (2.20). Let $\mathcal{C}_a \subset B_{2r_a}(x_a)$ be the associated center points of the neck region $\mathcal{N}_a$, and let μ_a be the associated packing measure.

Claim. If $y \in S^k_{\epsilon,r} \cap \mathcal{N}_a$, then $d(y, \mathcal{C}_a) \le \chi^{-1} r$.

Let us prove the claim. We will show that if $y \in \mathcal{N}_a$ with $d(y, \mathcal{C}_a) \ge \chi^{-1} r$, then there exists $B_s(y)$ with $s \ge 2r$ such that $B_s(y)$ is $(k+1, \epsilon/2)$-symmetric, which implies that $y \notin S^k_{\epsilon,r}$. Hence it will prove the claim.

For $y \in \mathcal{N}_a$ with $d(y, \mathcal{C}_a) \ge \chi^{-1} r$ and for any $\epsilon' > 0$ if $\chi \le \chi(n, \epsilon', \mathrm{v})$, we have by the "almost volume cone implies almost metric cone" Theorem 4.1 that $B_s(y)$ is $(0, \epsilon')$-symmetric for some $s > 2r$. On the other hand, by the almost splitting Theorem 4.11 and almost splitting Theorem 9.25 in [Che01] along geodesic , if $\delta \le \delta(n, \mathrm{v}, \epsilon')$ and $\chi \le \chi(n, \epsilon', \mathrm{v})$, then $B_{2s}(y)$ is $\epsilon' s$-close to a product space $\mathbb{R}^{k+1} \times Z$. These imply that $B_s(y)$ is $(k+1, \epsilon/2)$-symmetric if $\epsilon' = \epsilon'(n, \mathrm{v}, \epsilon)$ is sufficiently small. Hence $y \notin S^k_{\epsilon,r}$. Thus, the proof of the claim is completed.

Now choose a maximal disjoint collection of balls $\{B_r(x_j),\ x_j \in \mathcal{C}_a, j = 1, \ldots, K_a\}$ with centers in $\mathcal{C}_a$. By the Ahlfors regularity for μ_a, (2.7) of Theorem 2.9, we have

$$K_a C(n,\chi) r^k \le \sum_{j=1}^{K_a} \mu_a(B_{2\chi^{-1}r}(x_j)) \le C(n,\chi) \sum_{j=1}^{K_a} \mu_a(B_r(x_j)) \le C(n,\chi)\mu_a(B_{2r_a}(x_a)) \le C(n,\chi) r_a^k. \tag{2.22}$$

Thus, $K_a \le C(n,\chi) r^{-k} r_a^k$, which, by using the claim, clearly implies (2.20).

Relation (2.21) follows by summing (2.20) over all neck regions and using (2.19). Namely,

$$\mathrm{Vol}\left(B_r\left(S^k_{\epsilon,r} \cap \bigcup_a \mathcal{N}_a\right)\right) \le \sum_a \mathrm{Vol}\Big(B_r\left(S^k_{\epsilon,r} \cap \mathcal{N}_a\right)\Big) \le C(n,\mathrm{v},\chi) \sum r_a^k r^{n-k} \le C(n,\mathrm{v},\delta,\eta,\chi) r^{n-k}. \tag{2.23}$$

This completes the proof of of Lemma 2.17. □

LEMMA 2.18. *Let $\gamma \le \gamma(n,\mathrm{v},\epsilon)$, $\eta \le \eta(n,\mathrm{v},\epsilon)$. If the b-ball $B_{r_b}(x_b)$ satisfies $r \le \gamma \cdot r_b$, then*

$$S^k_{\epsilon,r} \cap B_{3r_b/2}(x_b) = \emptyset. \tag{2.24}$$

Proof. It suffices to show that for $y \in B_{3r_b/2}(x_b)$, the ball $B_s(y)$ is $(k+1, \epsilon/2)$-symmetric for some $s \ge \gamma r_b$. To see this, fix $\eta' = \eta'(n,\mathrm{v},\epsilon) > 0$ and $\epsilon' = \epsilon'(n,\epsilon,\mathrm{v})$ to be chosen below. If $\eta \le \eta(\eta', n, \mathrm{v})$, then since $B_{2r_b}(x_b)$ is $(k+1, 2\eta)$-symmetric, it follows that $B_{r_b/4}(y)$ is $(k+1,\eta')$-splitting. Also, by the Almost Volume Cone Implies Almost Metric Cone Theorem and (2.3), (2.4), (2.5), it follows that for some $\gamma = \gamma(n,\mathrm{v},\epsilon')$, the ball $B_{\gamma r_b}(y)$ is $(0,\epsilon')$-symmetric. For $\eta'(n,\mathrm{v},\epsilon)$ and $\epsilon'(n,\mathrm{v},\epsilon)$ sufficiently small, this implies that $B_{\gamma r_b}(y)$ is $(k+1,\epsilon/2)$-symmetric. This completes the proof of (2.24) and thus of Lemma 2.18. □

LEMMA 2.19. *Let $\Omega := \{x_1, \ldots, x_N\}$ denote a minimal $r/4$-dense subset of $S^k_{\epsilon,r} \setminus \cup_{r_a \ge \chi^{-1} r} B_r\left(S^k_{\epsilon,r} \cap \mathcal{N}_a\right)$ for χ the constant in Lemma 2.17. Then for γ the constant in Lemma 2.18, the following hold:*

(1) *Any ball $B_{r/4}(x_j)$ satisfies*

$$\sum_{B_{r_a} \subset B_{4\chi^{-1}r}(x_j)} r_a^k + \sum_{B_{r_b} \subset B_{4\gamma^{-1}r}(x_j)} r_b^k \ge C(n,\mathrm{v},\gamma,\chi) r^k. \tag{2.25}$$

(2) *The cardinality of Ω satisfies $N \le r^{-k} C(n,\mathrm{v},\delta,\eta,\gamma,\chi)$.*

(3) *The measure estimate*:

$$\mathrm{Vol}\Big(B_r\left(S^k_{\epsilon,r}\setminus\cup_{r_a\geq\chi^{-1}r}B_r\left(S^k_{\epsilon,r}\cap\mathcal{N}_a\right)\right)\Big)\leq C(n,\mathrm{v},\delta,\eta,\gamma,\chi)r^{n-k}.$$

Proof. First we will prove (1). Since $x_j\in S^k_{\epsilon,r}\setminus\cup_{r_a\geq\chi^{-1}r}B_r\left(S^k_{\epsilon,r}\cap\mathcal{N}_a\right)$, we have $B_r(x_j)\cap\mathcal{N}_a=\emptyset$ for any $r_a\geq\chi^{-1}r$. In addition, for any $r_a<r\chi^{-1}$, if $B_{r/4}(x_j)\cap\mathcal{N}_a\neq\emptyset$, then we have $B_{r_a}(x_a)\subset B_{4\chi^{-1}r}(x_j)$. If $B_{r/4}(x_j)\cap B_{r_b}\neq\emptyset$, by Lemma 2.18 we have $r_b\leq\gamma^{-1}r$, which implies $B_{r_b}(x_b)\subset B_{4\gamma^{-1}r}(x_j)$. Therefore, by (2.18) of the Decomposition Theorem, we have

$$B_{r/4}(x_j)\subset\Big(\bigcup_{B_{r_a}\subset B_{4\chi^{-1}r}(x_j)}B_{r_a}(x_a)\Big)\cup\Big(\bigcup_{B_{r_b}\subset B_{4\gamma^{-1}r}(x_j)}B_{r_b}(x_b)\Big).\tag{2.26}$$

Thus,

$$\begin{aligned}C(n,\mathrm{v})r^n&\leq\mathrm{Vol}(B_{r/4}(x_j))\\&\leq\sum_{B_{r_a}\subset B_{4\chi^{-1}r}(x_j)}\mathrm{Vol}(B_{r_a}(x_a))\ +\sum_{B_{r_b}\subset B_{4\gamma^{-1}r}(x_j)}\mathrm{Vol}(B_{r_b}(x_b))\\&\leq C(n,\mathrm{v})\left(\sum_{B_{r_a}\subset B_{4\chi^{-1}r}(x_j)}r_a^n+\sum_{B_{r_b}\subset B_{4\gamma^{-1}r}(x_j)}r_b^n\right)\\&\leq C(n,\mathrm{v},\gamma,\chi)r^{n-k}\left(\sum_{B_{r_a}\subset B_{4\chi^{-1}r}(x_j)}r_a^k+\sum_{B_{r_b}\subset B_{4\gamma^{-1}r}(x_j)}r_b^k\right).\end{aligned}\tag{2.27}$$

This implies (2.25), i.e., (1). Furthermore, from (2.19) and the fact that the balls $B_{r/10}(x_j)$ are disjoint, we have

$$\begin{aligned}NC(n,\mathrm{v})r^n&\leq\sum_{j=1}^N\mathrm{Vol}(B_{r/4}(x_j))\\&\leq C(n,\mathrm{v},\gamma,\chi)r^{n-k}\sum_{j=1}^N\left(\sum_{B_{r_a}\subset B_{4\chi^{-1}r}(x_j)}r_a^k+\sum_{B_{r_b}\subset B_{4\gamma^{-1}r}(x_j)}r_b^k\right)\\&\leq C(n,\mathrm{v},\gamma,\chi)r^{n-k}\left(\sum_a r_a^k+\sum_b r_b^k\right)\leq C(n,\mathrm{v},\delta,\gamma,\chi)r^{n-k},\end{aligned}\tag{2.28}$$

which implies (2).

For (3), let us consider the covering $\{B_{2r}(x_j), j=1,\dots,N\}$ of

$$S^k_{\epsilon,r}\setminus \bigcup_{r_a\geq \chi^{-1}r} B_r\left(S^k_{\epsilon,r}\cap \mathcal{N}_a\right).$$

By the definition of Ω this is also a covering of

$$B_r\left(S^k_{\epsilon,r}\setminus \bigcup_{r_a\geq r} B_r\left(S^k_{\epsilon,r}\cap \mathcal{N}_a\right)\right).$$

Thus, we have

$$\begin{aligned}&\mathrm{Vol}\Big(B_r\left(S^k_{\epsilon,r}\setminus \cup_{r_a\geq \chi^{-1}r}B_r\left(S^k_{\epsilon,r}\cap \mathcal{N}_a\right)\right)\Big)\\ &\leq \sum_{j=1}^N \mathrm{Vol}\Big(B_{2r}(x_j)\Big)\leq C(n,\mathrm{v})Nr^n\leq C(n,\mathrm{v},\delta,\eta,\gamma,\chi)r^{n-k}.\end{aligned}\tag{2.29}$$

This completes the proof Lemma 2.19. □

Now we can complete the proof of Theorem 1.7 as follows. Fix $\gamma=\gamma(n,\mathrm{v},\epsilon),\eta=\eta(n,\mathrm{v},\epsilon)$ and $\delta=\delta(n,\mathrm{v},\epsilon,\eta),\chi=\chi(n,\mathrm{v},\epsilon)$ as in the previous lemmas. Combining the estimates in (2.21) and (2.29) gives the volume estimate (2.17), which completes the proof of Theorem 1.7. □

Proof of ϵ-Stratification Theorem. Since $S^k_\epsilon\subset S^k_{\epsilon,r}$, the estimate for S^k_ϵ follows directly from Theorem 1.7. On the other hand, by the Decomposition Theorem 2.12, for $\eta\leq\eta(n,\mathrm{v},\epsilon)$ and $\delta\leq\delta(n,\mathrm{v},\epsilon,\eta)$, we have $S^k_\epsilon\subset \mathcal{S}^{k,\delta,\eta}$, where by Theorem 2.12 the set $\mathcal{S}^{k,\delta,\eta}$ is k-rectifiable.

For $\mathcal{H}^k$-a.e. $x\in S^k_\epsilon$, let us show that every tangent cone at x is k-symmetric. In fact we will show that for any δ, there exists a subset $\tilde S_\delta\subset S^k_\epsilon$ with $\mathcal{H}^k(\tilde S_\delta)=0$ such that every tangent cone of $x\in S^k_\epsilon\setminus\tilde S_\delta$ is (k,δ)-symmetric. Indeed, we can choose $\tilde S_\delta=\tilde{\mathcal{S}}^{k,\delta,\eta}$ as in Theorem 2.12, which satisfies the desired estimate as a consequence of the definition of a neck region. Now we consider $\tilde S=\cup_{i=1}^\infty \tilde S_{2^{-i}}$, where $\mathcal{H}^k(\tilde S)=0$. For any $x\in S^k_\epsilon\setminus\tilde S$, we have that every tangent cone of x is (k,δ)-symmetric for any δ which, in particular, implies that every tangent cone of x is k-symmetric. This completes the proof of Theorem 1.9. □

Proof of Theorem 1.12. The theorem follows directly from Theorem 1.9 and the fact that $S^k(X)=\cup_{j\geq1}S^k_{2^{-j}}(X)$, which is a countable union of rectifiable sets. □

Proof of Theorem 1.14. Let us choose $S_\epsilon=S^{n-2}_\epsilon$. Then for any $x\in X^n\setminus S_\epsilon$, we have for some $r_x>0$ that $B_{2r_x}(x)$ is $(n-1,\epsilon)$symmetric and hence $B_{r_x}(x)$ is (n,ϵ')-symmetric for $\epsilon\leq\epsilon(\epsilon',n,\mathrm{v})$. By the Reifenberg Theorem 7.10 (see also [CC97]) it follows that $B_{r_x/2}(x)$ is bi-Hölder to $B_{r_2/2}(0^n)\subset\mathbb{R}^n$ for ϵ' small. This suffices to complete the proof. □

Proof of Theorem 1.15. As was shown in [CC97], $S = S^{n-2}$. From Theorem 1.12 we now know that for $\mathcal{H}^{n-2}$-a.e. $x \in S^{n-2}$, every tangent cone is $(n-2)$-symmetric. For such x, any tangent cone is isometric to $\mathbb{R}^{n-2} \times C(S^1_\beta)$, where S^1_β denotes the circle of length $\beta < 2\pi$. By Theorem 4.2, β is determined by the limiting volume ratio, $\lim_{r\to 0} \mathcal{V}_r(x)$. This suffices to complete the proof. □

Proof of Theorem 1.16. The theorem follows from the ϵ-regularity theorem, Theorem 4.35 and the stratification of Theorem 1.9.

To see this, note that if $y \notin S^{n-4}_\epsilon(X)$, then there exists some $r_y > 0$ such that $B_{r_y}(y)$ is $(n-3,\epsilon)$-symmetric. According to Theorem 4.35 we then have the harmonic radius bound $r_h(y) \geq c(n) r_y > 0$ which, in particular, implies $y \notin S(X)$. Thus, we have shown that $S \subseteq S^{n-4} \subset S^{n-4}_\epsilon$. The volume estimates of $B_r\big(S\big) \cap B_1(p)$ now follow from Theorem 1.9.

The proof of the tangent cone uniqueness result is similar to that of Theorem 1.15. By Theorem 1.12, there exists an $(n-4)$-Hausdorff measure zero set $\tilde{S} \subset S^{n-4} = S(X)$ such that every tangent cone at $x \in X \setminus \tilde{S}$ is $(n-4)$-symmetric. In particular, this means that every tangent cone is isometric to $\mathbb{R}^{n-4} \times C(Y^3)$ for some metric space Y^3. By the main result of [CN15], which states that the singular set of a noncollapsed limit space with a 2-sided bound on Ricci curvature has codimension 4, it follows that Y^3 is a 3-dimensional smooth manifold with $\mathrm{Ric}_Y = 2g^Y$. This implies that Y^3 is a space form S^3/Γ for some discrete subgroup Γ of $O(4)$ acting freely. By Theorem 4.2, the order of subgroup Γ is determined by the volume ratio at x. Since the space of cross-sections of tangent cones at one point is connected (see Theorem 4.2), it follows that Γ is unique. Thus, the tangent cone at x is unique. This completes the proof of Theorem 1.16. □

3. **Additional examples**

This is the first of three sections which constitute the second part of the paper.

Basic examples of neck regions and the neck decomposition were given in Examples 2.11, 2.14, and 2.20. In the present brief section, we will provide some additional examples. They show the sharpness of our results and illustrate how more naive versions of the statements can fail to hold.

3.1. *Example* 1: *Conical neck region.* A key result in this paper states that the packing measure of a neck region $\mathcal{N} = B_2(p) \setminus \overline{B}_{r_x}(\mathcal{C})$ is uniformly Ahlfors regular; see Example 2.11 and Theorem 2.9. The key technical result needed to be proved is the statement that if $u : B_2(p) \to \mathbb{R}^k$ is a harmonic splitting function, then for $\mathcal{H}^k$-most points of $\mathcal{C}_\epsilon \subseteq \mathcal{C}$, u is a $(1+\epsilon)$-bi-Lipschitz map onto its image; see Proposition 9.3. In the simplest example of a neck region, Example 2.11, we could take $\mathcal{C}_\epsilon = \mathcal{C}$. The present example shows that in general, this is not the case.

In fact, the map u can degenerate on parts of a neck region. This explains the statement of the structural result given in Proposition 9.3. Although it deals with what at first glance might seem like a relatively minor technical point, this example is useful to remember when one is faced with traversing the maze of technical results which come later in the paper. In particular, it demonstrates why simpler sounding statements just do not hold.

Let $Y_r := \mathrm{Susp}(S^1_r)$ denote the suspension of a circle of radius r. Note that if $r = 1$, then $Y_1 = S^2$. For $r < 1$, the space Y_r will have two singular points $p, q \in Y$ at antipodal points. It will look like an American football. By using a warped product construction, one can easily check that Y_r can be smoothed to obtain $Y_{\epsilon,r}$, which is diffeomorphic to S^2, satisfies $Y_\epsilon = Y_{\epsilon,r}$ outside of $B_\epsilon(p) \cup B_\epsilon(q)$ and has sectional curvature ≥ 1.

Let $X^3 = C(Y_{\epsilon,r})$ denote the cone over $Y_{\epsilon,r}$. Note that X^3 has a unique singular point at $0 \in X^3$. Using the techniques of [CN13], one can check that X^3 itself arises as a Ricci limit space. Let γ_p, γ_q denote the rays in X^3 through the cross-section points $p, q \in Y_{\epsilon,r}$. Though X^3 is smooth along these rays, for ϵ very small X^3 is looking increasingly singular. For each $x \in \gamma_p \cup \gamma_q$, let $r_x = r_0 \cdot d(x, 0)$, where $r_0 \gg \epsilon$ is fixed and small. Finally let $\mathcal{C} = \{0\} \cup \{x_i\} \subseteq (\gamma_p \cup \gamma_q) \cap B_2(p)$ be a maximal subset such that $B_{\tau^2 r_i}(x_i)$ are disjoint. Note then that for any $\delta, \eta > 0$, one can check for $\epsilon \ll \delta$ that $\mathcal{N} := B_2(p) \setminus \overline{B}_{r_x}(\mathcal{C})$ defines a $(1, \delta, \eta)$-neck region.

Now let $u : B_2(p) \to \mathbb{R}$ denote a harmonic ϵ-splitting map. By using separation of variables one can check that $|\nabla u(x)| \to 0$ as $x \to 0$ approaches the vertex of the cone. Indeed, this holds for *any* harmonic function on $B_2(p)$. In particular, it is certainly not possible that u defines a $(1+\delta)$-bi-Lipschitz map on all of $\mathcal{C}$. One can check that as $\epsilon \to 0$, u remains bi-Lipschitz on $\mathcal{C}$ away from an increasingly small ball around 0. This shows the sharpness of the bi-Lipschitz structure of Proposition 9.3.

3.2. *Example* 2: *Sharpness of k-rectifiable structure.* One of the primary results of this paper is to show that the kth-stratum $S^k \setminus S^{k-1}$ of the singular set is k-rectifiable. The following example from [LN20] shows that this statement is sharp in the sense that there need not exist any points in the singular set S in a neighborhood of which S is a manifold.

In [LN20] the following examples are produced: For each real number $s \in [0, n-2]$, there exists $(M^n_j, g_j) \xrightarrow{d_{GH}} (X, d)$, with $\mathrm{diam}\,(M^n_j) \leq 1$, $\mathrm{Vol}\,(M^n_j) > v > 0$, such that the singular set, S satisfies

$$\begin{aligned} &\dim S = s, \\ &S \text{ is a } s\text{-Cantor set.} \end{aligned} \tag{3.1}$$

If $s = k \in \mathbb{N}$ is an integer, one can further arrange it so that $S = S^k = S^k_\epsilon$ satisfies the case in which $0 < \mathcal{H}^k(S) < \infty$ is both k-rectifiable and a k-Cantor set. In particular, we see from these examples that the structure theory of Theorem 1.12 is sharp, and one cannot hope to do better.

We will briefly explain the example above from [LN20] for the case, $0 \le s \le 1$. Higher dimensional examples are built in an analogous manner.

Let $Z = \overline{B}_1(0^2) \times [0,1] \subseteq \mathbb{R}^3$ denote the closed 3-cylinder. Observe that Z is an Alexandrov space with boundary, and that its singular set is the codimension 2 circles $S(Z) = \partial B_1(0^2) \times \{0,1\}$.

Double Z to obtain an Alexandrov space without boundary $\tilde{Z}$ with codimension 2 singular sets $S(\tilde{Z}) = S^1 \times \mathbb{Z}_2 \subseteq \tilde{Z}$. It is not difficult to see that $\tilde{Z}$ may be smoothed to obtain a manifold by rounding off the doubled boundary points. Intuitively, the key point of this example is that these circle singular sets are sets of infinite positive sectional curvature in every direction, as opposed to the easier construction of a codimension 2 singular set built by looking at $\mathbb{R} \times C(S^1_r)$ as in the neck example. Note that because of this, the singular set $S(\tilde{Z})$ is not totally geodesic. However, the regular set of $\tilde{Z}$ is convex. In fact, by [CN12], this must be the case.

Now choose an arbitrary open set $U = \bigcup (a_i, b_i) \subseteq S(Z)$. By using the fact that $S(Z)$ consists of completely convex points of $Z \subseteq \mathbb{R}^3$, one can construct a subset $Y \subseteq \mathbb{R}^3$ by (informally speaking) "sanding off" each interval (a_i, b_i) to obtain smooth boundary points such that Y is still convex. Hence, after this procedure has been carried out, Z is still an Alexandrov space.

At this point, we have $S(Y) = S(Z) \setminus U$, and we can again double $\tilde{Y}$ to obtain Alexandrov space without boundary such that $S(\tilde{Y})$ is isometric to $S(Y)$. By choosing U as in the standard Cantor constructions, we can make $S(\tilde{Y})$ a s-Cantor set for $0 \le s \le 1$, as claimed. Since $S(Y)$ is contained in a circle, this set is 1-rectifiable.

3.3. *Example* 3: *Sharpness of k-symmetries of tangent cones.* One of the main statements in Theorem 1.12 is that for $\mathcal{H}^k$-a.e. $x \in S^k$, all tangent cones are k-symmetric. Recall however that we do not assert that tangent cones are unique. In this example, we show that indeed tangent cones need *not* be unique for $\mathcal{H}^k$-a.e. $x \in S^k$.

The examples are rather straight forward. For instance, let Y be a noncollapsed limit space such that there is an isolated singularity at $p \in Y$. Assume $p \in S^0_\epsilon(Y)$ is such that the tangent cone at p is not unique; see, for instance, [CC97], [CN13] for such examples. Nonetheless, every tangent cone is 0-symmetric as this is a noncollapsed limit. Put $X = \mathbb{R}^k \times Y$. Then the singular set of X satisfies $S = S^k(X) = S^k_\epsilon = \mathbb{R}^k \times \{p\}$. In this case, as claimed, every tangent cone in $S^k(X)$ is k-symmetric, but none are unique.

3.4. *Example* 4: *Sharpness of S^k_ϵ-finiteness.* Theorem 1.12 states that S^k is k-rectifiable. Theorem 1.9 states that the quantitative stratification S^k_ϵ has uniformly bounded k-dimensional Hausdorff measure. Well-known examples demonstrate that this need not hold for S^k. Thus, the best one can say is that $S^k(Y)$ is a countable union of finite measure rectifiable sets, as stated in Theorem 1.12.

Start with solid regular tetrahedron Z^3_0, centered at the origin. Attach to each face F^2_i a tetrahedron with very small altitude and base F^2_i. Call the resulting convex polytope Z^3_1. Proceed inductively in this fashion to obtain a sequence of convex polytopes Z^3_2, Z^3_3 …, in such a way that the sequence of altitudes goes sufficiently rapidly to zero so that the following will hold. The sequence of convex polytopes converges in the Hausdorff sense to a convex subset Z^3_∞ and $\partial Z^3_i \xrightarrow{GH} \partial Z^3_\infty$ as well. Moreover, ∂Z^3 has a dense set of singular points, although for all $\epsilon > 0$, only a finite number fail to have a neighborhood which is ϵ-regular. The polytopes Z^3_i can be "sanded" to produce a sequence of smooth convex surfaces $M^2_i \xrightarrow{GH} \partial Z^3_\infty$. Thus, ∂Z^3_∞ is the GH-limit of a sequence of smooth manifolds with nonnegative curvature. Of course, higher dimensional examples can be constructed similarly.

4. **Preliminaries**

In this section, we will review the technical background material which is required for the proofs of our main theorems. For the less standard material, particularly the results concerning entropy, we will give detailed indications of the proofs. In all cases we will give complete references.

4.1. *Almost volume cones are almost metric cones.* For any $\kappa \in \mathbb{R}$, let the metric on the unique simply connect n-manifold M^n_κ with constant curvature $\equiv \kappa$ be written in geodesic polar coordinates as $dr^2 + f^2_\kappa g^{S^{n-1}}$, where $g^{S^{n-1}}$ denotes the metric on the unit $(n-1)$-sphere. Let $\mathrm{Ric}_{M^n_i} \geq (n-1)\kappa$, and assume $M^n_i \xrightarrow{d_{GH}} X^n$, where X^n is noncollapsed. Then if X^n is equipped with n-dimensional Hausdorff measure, the convergence is actually in the measured Gromov-Hausdorff sense ([CC97]). If we extend the definition of the volume ratio $\mathcal{V}_r(x) = \mathcal{V}^\kappa_r(x)$ to points $x \in X^n$, then as in (2.1), we have $\frac{d}{dr}\mathcal{V}^\kappa_r(x) \leq 0$.

Suppose $\frac{d}{dr}\mathcal{V}^\kappa_r(x) = 0$ for some fixed r and suppose the metric on X^n is smooth in a neighborhood of $Y := \partial B_r(x)$, with g^Y the induced metric on Y. Then in geodesic polar coordinates on $B_r(x)$ the Riemannian metric is given by

$$dr^2 + f^2_\kappa g^Y. \tag{4.1}$$

The proof of this fact given in [CC96] uses the characterization of the warped product metrics as those for which there is a potential function whose Hessian is a multiple of the metric.

Notation. Below, the Hessian of f is sometimes denoted by Hess_f and sometimes by $\nabla^2 f$.

If $\kappa = 0$, then the warped product is a metric cone, in which case,

$$\mathrm{Hess}\, r^2 - 2g = 0. \tag{4.2}$$

This should be compared to the corresponding formula for the time derivative of the entropy given in (4.20).

The discussion can be extended to the case in which the smoothness assumption on $\partial B_r(x) := Y$ is dropped, provided the expression in (4.1) is replaced by the expression for the distance function d on $B_r(x)$. Let d^Y denote the distance function on Y. Then for the case $\kappa = 0$, $(B_r(x), d)$ is isometric to a ball in the metric cone with cross-section (Y, d^Y) with $\mathrm{diam}(Y) \le \pi$ and d is given by the *law of cosines* formula

$$d^2((r_1, y_1), (r_2, y_2)) = r_1^2 + r_2^2 - 2r_1 r_2 \cdot \cos d^Y(y_1, y_2). \tag{4.3}$$

By using Gromov's compactness theorem, the following "almost volume cone implies almost metric cone theorem" is easily seen to be equivalent to what has just been discussed.

THEOREM 4.1 ([CC96]). *Let (M^n, g, p) denote a Riemannian manifold with $\mathrm{Ric}_{M^n} \ge -(n-1)\delta$. Given $\epsilon > 0$, if $\delta \le \delta(n, \epsilon)$ and $\mathcal{V}_2(p) \ge (1-\delta)\mathcal{V}_1(p)$, then $B_1(p)$ is $(0, \epsilon)$-symmetric.*

In the proof of tangent cone uniqueness we also used the following well-known result in which relation (2) below follows from volume convergence; compare Theorems 1.15 and 4.2.

THEOREM 4.2. *Let $(M_i^n, g_i, p_i) \to (X, d, p)$ satisfy $\mathrm{Ric}_{M_i^n} \ge -(n-1)$ and $\mathrm{Vol}(B_1(p_i)) \ge \mathrm{v} > 0$. Then the cross-section space $C_x := \{(Y, d_Y) : C(Y)$ is a tangent cone at $x\}$ of tangent cones at $x \in X$ satisfies*

(1) *(C_x, d_{GH}) is connected*;

(2) *for every $Y \in C_x$, we have $\mathrm{Vol}(Y) = \lim_{r\to 0} \frac{n\mathrm{Vol}(B_r(x))}{r^n}$.*

Proof. Let us give a brief proof of Theorem 4.2. The second statement follows directly by volume convergence. It will suffice to prove the first one. Consider the space $\mathcal{M}_x = \{\bar{B}_1(y_c) : y_c \in C(Y)$ is the vertex and $Y \in C_x\}$. To prove (1), it suffices to prove $(\mathcal{M}_x, d_{GH})$ is connected. In fact, we will show $(\mathcal{M}_x, d_{GH})$ is a compact, connected space. Let $L_x := \{(\bar{B}_r(x), r^{-1}d), 0 < r \le 1\}$. By the definition of tangent cone, for any $\bar{B}_1(z_c) \in \mathcal{M}_x$, there exists $r_i \to 0$ such that

$$(\bar{B}_{r_i}(x), r_i^{-1}d) \to \bar{B}_1(z_c). \tag{4.4}$$

By a diagonal argument this implies that any point in the closure of $\mathcal{M}_x$ is a limit of $(\bar{B}_{r_i}(x), r_i^{-1}d)$ for some sequence $r_i \to 0$. Therefore, we have proved $(\mathcal{M}_x, d_{GH})$ is a closed subset of $(\mathcal{M}, d_{GH})$, where $\mathcal{M}$ is the space of all compact metric spaces. On the other hand, by volume doubling and by Gromov's precompactness theorem (see Chapter 10 in [Pet16] or [Che01]), $(\mathcal{M}_x, d_{GH})$ is a compact subset of $(\mathcal{M}, d_{GH})$. To prove $(\mathcal{M}_x, d_{GH})$ is connected, we will require the following claim for compact metric spaces, which is an easy exercise.

Claim. A compact metric space (Y, d) is connected if for any $y_1, y_2 \in Y$ and $\epsilon > 0$, there exists an ϵ-curve $\{z_1 = y_1, \ldots, z_N = y_2\} \subset Y$ connecting y_1, y_2, i.e., for any i that $d(z_i, z_{i+1}) \le \epsilon$.

Therefore, by the claim it suffices to show that for any $\epsilon > 0$ and any two balls $\bar{B}_1(z_c), \bar{B}_1(w_c) \in \mathcal{M}_x$, there exists an ϵ-curve connecting them. Assume $(\bar{B}_{r_i}(x), r_i^{-1}d) \to \bar{B}_1(z_c)$ and $(\bar{B}_{s_i}(x), s_i^{-1}d) \to \bar{B}_1(w_c)$ with $r_i < s_i$. For any $\epsilon > 0$, since $\gamma(t) = (\bar{B}_t(x), t^{-1}d)_{r_i \le t \le s_i}$ is a continuous curve connecting $(\bar{B}_{r_i}(x), r_i^{-1}d)$ and $(\bar{B}_{s_i}(x), s_i^{-1}d)$, choose a subset $\{\gamma(t_i^1), \ldots, \gamma(t_i^{N_i(\epsilon)}), t_i^\alpha \in [r_i, s_i]\}$ such that $t_i^1 = r_i, t_i^{N_i(\epsilon)} = s_i$ and

$$d_{GH}(\gamma(t_i^\alpha), \gamma(t_i^{\alpha+1})) \le \epsilon/2, \tag{4.5}$$

and such that for any $\alpha \ne \beta$,

$$d_{GH}(\gamma(t_i^\alpha), \gamma(t_i^\beta)) \ge \epsilon/4. \tag{4.6}$$

By (4.6) and the fact that the closure of L_x is compact, we have $N_i(\epsilon) \le C(n, \epsilon, \mathrm{v})$, which is independent of i. Denote $N(\epsilon) := \limsup N_i(\epsilon)$. Then by Gromov's precompactness theorem (see [Pet16, Ch. 10] or [Che01]), we have $\{\gamma_1, \ldots, \gamma_{N(\epsilon)}\} \subset \mathcal{M}_x$ such that $\gamma_1 = \bar{B}_1(z_c), \gamma_{N(\epsilon)} = \bar{B}_1(w_c)$ and

$$d_{GH}(\gamma_\alpha, \gamma_{\alpha+1}) \le \epsilon/2. \tag{4.7}$$

Therefore, $\{\gamma_1, \ldots, \gamma_{N(\epsilon)}\} \subset \mathcal{M}_x$ is an ϵ-curve connecting $\gamma_1 = \bar{B}_1(z_c)$ and $\gamma_{N(\epsilon)} = \bar{B}_1(w_c)$. The claim now implies $\mathcal{M}_x$ is connected. This completes the proof. □

The following result was proved in [Che01], [CC97], [Col97]. It implies, in particular, that at a point in the regular set, $\mathcal{R} := X^n \setminus S$, the tangent cone is unique and isometric to $\mathbb{R}^n$. In fact, since the conclusion applies to all balls $B_r(x) \subset B_3(p)$, it is actually a kind of quantitative ϵ-regularity theorem.

THEOREM 4.3. *Let $(M_i^n, g_i, p_i) \to (X, d, p)$ satisfy $\mathrm{Ric}_{M_i^n} \ge -(n-1)\delta$ and $\mathrm{Vol}(B_1(p_i)) \ge \mathrm{v} > 0$. Let $\epsilon > 0$, $\delta \le \delta(n, \mathrm{v}, \epsilon)$, and assume $B_4(p)$ is (n, δ)-symmetric. Then each $B_r(x) \subset B_3(p)$ is also (n, ϵ)-symmetric.*

4.2. *Quantitative cone-splitting.* As recalled in Section 2, if a metric cone has two distinct vertices, then the cone isometrically splits off a line which contains these two vertices. If there are several independent such cone vertices, then this statement can be iterated to produce further splittings. A quantitative version of cone-splitting was introduced in [CN13a]. Prior to stating this theorem, it is convenient to introduce a quantitative notion of $k+1$ points $x_0, \ldots, x_k$ being k-independent.

In $\mathbb{R}^n$ we say that as set of points $\{x_0, \ldots, x_k\}$ is *k-independent* if the $\{x_i\}_0^k$ is not contained in any $(k-1)$-plane. Here is a quantitative version of this notion.

Definition 4.4 ((k,α)-independence). In a metric space (X,d), a set of points $U = \{x_0, \ldots, x_k\} \subset B_{2r}(x)$ is (k,α)-independent if for any subset $U' = \{x'_0, \ldots, x'_k\} \subset \mathbb{R}^{k-1}$, we have

$$d_{GH}(U, U') \geq \alpha \cdot r. \tag{4.8}$$

Remark 4.5. Let $X \subset \mathbb{R}^n$. If there exists no (k,α)-independent set in $B_r(x) \cap X$, then $B_r(x) \cap X \subset B_{4\alpha r}(\mathbb{R}^{k-1})$ for some $(k-1)$-plane $\mathbb{R}^{k-1} \subset \mathbb{R}^n$. To see this, if $B_r(x) \cap X$ is not a subset of $B_{3\alpha r}(\mathbb{R}^{k-1})$ for any $(k-1)$-plane then, by induction, one can find a (k,α)-independent set in $B_r(x) \cap X$.

The following Quantitative Cone-Splitting Theorem was introduced in [CN13a].

THEOREM 4.6 (Cone-Splitting). *Let (M^n, g, p) satisfy $\mathrm{Ric}_{M^n} \geq -(n-1)\delta$. Let $\epsilon, \tau > 0$ and $\delta \leq \delta(n, \epsilon, \tau)$, and assume the following*:

(1) *$B_2(p)$ is (k,δ)-symmetric with respect to $\mathcal{L}^k_\delta \subseteq B_2(p)$ as in Remark 1.4*;
(2) *there exists $x \in B_1(p) \setminus B_\tau(\mathcal{L}^k_\delta)$ such that $B_2(x)$ is $(0,\delta)$-symmetric.*

Then $B_1(p)$ is $(k+1,\epsilon)$-symmetric.

Remark 4.7. We can rephrase the above as follows: If $U = \{x_0, \ldots, x_k\} \subset B_{2r}(x)$ is (k,α)-independent and each x_i is $(0,\delta)$-symmetric, by the Cone-Splitting Theorem 4.6, the ball $B_{2r}(x_0)$ is (k,ϵ)-symmetric for $\delta \leq \delta(n,\alpha,k,\epsilon)$.

A second version of quantitative cone-splitting theorem is implicit in [CN13a]. It is a direct consequence of Theorem 4.6. To define it let us define the notion of the pinching set:

Definition 4.8 (Points with small volume pinching). Let (M^n, g, p) satisfy $\mathrm{Ric}_{M^n} \geq -(n-1)\xi$, and put

$$\bar{V} := \inf_{x \in B_1(p)} \mathcal{V}_{\xi^{-1}}(x). \tag{4.9}$$

The set with small volume pinching is

$$\mathcal{P}_{r,\xi}(x) := \{y \in B_{4r}(x) : \ \mathcal{V}_{\xi r}(y) \leq \bar{V} + \xi\}. \tag{4.10}$$

Note that if ξ^{-1} is large, then the point in $\mathcal{P}_{r,\xi}(x)$ is an "almost cone vertex" for each scale between r and ξ^{-1}. By Theorem 4.1, each point $y \in \mathcal{P}_{r,\xi}(x)$ is an "almost cone vertex." Thus, with Theorem 4.6 we immediately have the following:

THEOREM 4.9 (Cone-Splitting based on k-content). *Let (M^n, g, p) satisfy $\mathrm{Vol}(B_1(p)) \geq \mathrm{v} > 0$ with $\mathrm{Ric}_{M^n} \geq -(n-1)\xi$. Assume that $0 < \delta, \epsilon \leq \delta(n, \mathrm{v})$, $\gamma \leq \gamma(n, \mathrm{v}, \epsilon)$, $\xi \leq \xi(\delta, \epsilon, \gamma, n, \mathrm{v})$ and*

$$\mathrm{Vol}(B_\gamma(\mathcal{P}_{1,\xi}(p))) \geq \epsilon \gamma^{n-k}. \tag{4.11}$$

Then there exists $q \in B_4(p)$ such that $B_{\delta^{-1}}(q)$ is (k, δ^2)-symmetric.

The import of Theorem 4.9 is that if the set of pinched points $\mathcal{P}_{1,\xi}$ has a definite amount of k-content, then the ball must be k-symmetric. The scale invariant version states that if

$$\mathrm{Vol}(B_{\gamma r}(\mathcal{P}_{r,\xi}(p))) \geq \epsilon \gamma^{n-k} r^n, \tag{4.12}$$

then $B_{\delta^{-1} r}(q)$ is (k, δ^2)-symmetric for some $q \in B_r(p)$.

4.3. *Harmonic ϵ-splitting functions.* The following definition, which encapsulates the technique of [CC96] for obtaining approximate splittings, is essentially the one formalized in [CN15].

Definition 4.10 (Harmonic δ-splitting map). The map $u : B_r(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map if

(1) $\Delta u = 0$;
(2) $⨍_{B_r(p)} \left|\langle \nabla u^i, \nabla u^j \rangle - \delta^{ij}\right| < \delta$;
(3) $\sup_{B_r(p)} |\nabla u| \leq 1 + \delta$;
(4) $r^2 ⨍_{B_r(p)} \left|\nabla^2 u\right|^2 < \delta^2$.

For the case of limit spaces, we can define δ-splitting maps as follows. If $B_r(p_i) \subset M_i \to B_r(p) \subset X$ and δ_i-splitting maps $u_i : B_r(p_i) \to \mathbb{R}^k$ converge uniformly to $u : B_r(p) \to \mathbb{R}^k$ with $\delta_i \to \delta$, we say u is δ-splitting on $B_r(p) \subset X$. By the $W^{1,2}$-convergence in Proposition 4.29, we have that the δ-splitting u satisfies (1)–(4) in the limit space.

The following is a slight extension of the result in [CC96].

THEOREM 4.11. *Let (M^n, g, p) satisfy $\mathrm{Ric}_{M^n} \geq -(n-1)\delta$. For any $\epsilon > 0$, if $\delta \leq \delta(n, \epsilon)$, then the following hold*:

(1) *If there exists a δ-splitting function $u : B_2(p) \to \mathbb{R}^k$, then $B_1(p)$ is ϵ-GH close to $\mathbb{R}^k \times X$.*

(2) *If $B_2(p)$ is δ-GH close to $\mathbb{R}^k \times X$, then there exists an ϵ-splitting function $u : B_1(p) \to \mathbb{R}^k$.*

Remark 4.12. In Cheeger-Colding [CC96], the second result is proved under the assumption that the ball $B_{\delta^{-1}}(p)$ is δ-GH close to $\mathbb{R}^k \times X$. This suffices for the purposes of the present paper. However, the closeness assumption can actually be weakened to $B_2(p)$ (or indeed $B_{1+\epsilon}(p)$) by using a contradiction argument combined with what is now understood about continuity of limiting harmonic functions and $W^{1,2}$-convergence of harmonic functions, in the context of GH-convergence; see also some related discussion in Section 4.9.

4.4. *A cutoff function with bounded Laplacian.* The existence of a cutoff function which satisfies the standard estimates and has a definite pointwise bound on its Laplacian is important technical tool. In particular, such a cutoff function is required for the discussion of the local pointed entropy; see Section 4.6.[4]

THEOREM 4.13 ([CC96]). *Let (M^n, g, p) be a Riemannian manifold with $\mathrm{Ric}_{M^n} \geq -(n-1)r^2$. Then there exists cutoff function $\phi_r : M^n \to [0,1]$ with support in $B_r(p)$ such that $\phi_r := 1$ in $B_{r/2}(p)$. Moreover,*

$$r^2|\nabla\phi_r|^2 + r^2|\Delta\phi_r| \leq C(n). \tag{4.13}$$

4.5. *Heat kernel estimates and heat kernel convergence.* Let $\rho_t(x,y)$ denote the heat kernel on M^n. For each x, we have

$$\int_{M^n} \rho_t(x,y)\, d\mu(y) = 1.$$

Define the function $f_t(x,y)$ by

$$\rho_t(x,y) = (4\pi t)^{-n/2} e^{-f_t(x,y)}. \tag{4.14}$$

Next we recall some classical heat kernel estimates for manifolds with lower Ricci curvature bounds, as well as the heat kernel convergence result for Gromov-Hausdorff convergence. We summarize the heat kernel estimates in the following theorem; see [LY86], [SZ06], [SY94], [Ham93], [Kot07], [CY81].

THEOREM 4.14 (Heat Kernel Estimates). *Let (M^n, g, p) satisfy $\mathrm{Ric}_{M^n_i} \geq -(n-1)\delta^2$ and $\mathrm{Vol}(B_r(p)) \geq \mathrm{v}\cdot r^n > 0$ for $r \leq \delta^{-1}$. Then for any $0 < t \leq 10\delta^{-2}$ and $\epsilon > 0$ with $x, y \in B_{10\delta^{-1}}(p)$,*

(1) $-C(n,\mathrm{v},\epsilon) + \frac{d^2(x,y)}{(4+\epsilon)t} \leq f_t \leq C(n,\mathrm{v},\epsilon) + \frac{d^2(x,y)}{(4-\epsilon)t}$;

[4] The original proof of the existence of the required cutoff function employed solutions of the Poisson equation, $\Delta u = 1$ and a delicate argument based on the quantitative maximum principle. One can also give a proof by using heat flow as in [MN19].

(2) $t|\nabla f_t|^2 \le C(n,\mathrm{v},\epsilon) + \frac{d^2(x,y)}{(4-\epsilon)t}$;

(3) $-C(n,\mathrm{v},\epsilon) - \frac{d^2(x,y)}{(4+\epsilon)t} \le t\Delta f_t \le C(n,\mathrm{v},\epsilon) + \frac{d^2(x,y)}{(4-\epsilon)t}$.

The estimates in (1) are Li-Yau heat kernel upper and lower bound estimates; see, for instance, [LY86], [CY81]. (2) follows from (1) and a local gradient estimate; see, for instance, [SZ06]. (3) follows from the Li-Yau Harnack inequality, (2) and [Ham93], [Kot07].

The following result is well known in the context of Ricci limit spaces and even for RCD spaces. One direct proof is obtained by using gradient flow convergence of the Cheeger energy in [AGS14a]; see also [AH18], [GMS15]. See [AGS14a], [AH18], [AHT18], [ZZ19] for more general results in the RCD setting. In our application the limit space X is a metric cone, in which case the heat kernel convergence was proved in [Din02].

PROPOSITION 4.15 (Heat kernel convergence). *Suppose* $(M_i, g_i, x_i, \mu_i) \to (X, d, x_\infty, \mu)$ *with* $\mathrm{Ric}_{M_i^n} \ge -(n-1)$ *and* $\mu_i = \mathrm{Vol}(B_1(x_i))^{-1}\mathrm{Vol}(\,\cdot\,)$. *Then the heat kernel* $\rho^i_t(x,y)$ *converges uniformly to the heat kernel* $\rho^\infty_t(x,y)$ *on any compact subset of* $\mathbb{R}_+ \times X \times X$.

Remark 4.16. By the heat kernel Laplacian estimate in Theorem 4.14 and $W^{1,2}$-convergence in Proposition 4.29, it follows that for any fixed t, we have $\rho^i_t(x_i,\cdot) \to \rho^\infty_t(x_\infty,\cdot)$ in the $W^{1,2}$-sense as in Definition 4.27.

Remark 4.17. If the limit space is a noncollapsed metric cone $Y = C(X)$ with cone vertex x_∞, then the heat kernel on Y is

$$\rho^\infty_t(x_\infty, y) = \frac{\mathrm{Vol}(S^{n-1})}{\mathrm{Vol}(X)} \cdot \frac{e^{-d^2(x_\infty,y)/4t}}{(4\pi t)^{n/2}},$$

where $\mathrm{Vol}(X)$ is the $(n-1)$-Hausdorff measure of X with respect to the metric (X, d_X). This follows easily by computing the s-derivative of

(4.15)
$$\eta(t,s,x) := \int_{C(X)} \rho^\infty_{t-s}(x,y) \cdot \Big(\rho^\infty_s(x_\infty,y) - \frac{\mathrm{Vol}(S^{n-1})}{\mathrm{Vol}(X)} \cdot \frac{e^{-d^2(x_\infty,y)/4t}}{(4\pi t)^{n/2}}\Big)\, dy$$

to conclude that $\eta(t,t,x) = \eta(t,0,x) = 0$.

4.6. *The local pointed entropy,* $\mathcal{W}_t(x)$ *and its relation to cone structure.* As discussed in Section 4.1, "almost volume cones are almost metric cones," previously known results on quantitative cone-splitting were stated in Theorems 4.6 and 4.9. As with the definition of neck regions, the hypotheses of these results, as well as the definition of neck regions, involve the volume ratio $\mathcal{V}_r(x)$. For our purposes, it is crucial to have a sharp version of quantitative cone-splitting. As mentioned in previous sections, it turns out that many technical details are simpler if in place of $\mathcal{V}_r(x)$, we use a less elementary monotone quantity, the *local pointed entropy* $\mathcal{W}_t(x)$. Therefore, it is necessary to have a result stating that (with suitable interpretation) $\mathcal{W}_t(x)$ and $\mathcal{V}_r(x)$ have essentially the same behavior. This is the content of Theorem 4.22, which also includes the fact that $\mathcal{W}_t(x)$ is monotone in t. The sharp cone-splitting estimate, the statement of which involves entropy, is given in Theorem 6.1.

In the present subsection, we derive the needed background results on the local pointed entropy. This quantity is a local version of Perelman's $\mathcal{W}$-entropy, generalized in [Ni04] to smooth manifolds. In order to emphasize the basics, we will first discuss the technically simpler concept of the *pointed entropy*.

If as in (4.14) we write $\rho_t(x,dy) = (4\pi t)^{-n/2} \cdot e^{-f_{x,t}(y)}$, then by definition the *weighted Laplacian* Δ_f is the second order operator associated to the weighted Dirichlet energy

$$\int_{M^n} (4\pi t)^{-n/2} |\nabla f|^2 \, e^{-f} \, dv_g(y) = \int |\nabla f|^2 \, \rho_t(x,dy).$$

Then

$$\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle.$$

Set

$$W_t = 2t\,\Delta_f f + t|\nabla f|^2 + f - n. \tag{4.16}$$

The *pointed entropy*, $\mathcal{W}_t(x)$, is for each x a *global quantity* defined as follows.

Definition 4.18 (Pointed entropy).

$$\mathcal{W}_t(x) := \int_{M^n} W_t \cdot \rho_t(x,dy). \tag{4.17}$$

Bochner's formula states for $u \in C^\infty(M)$ that

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla \Delta u, \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u). \tag{4.18}$$

The following lemma is proved by direct computation. It shows, in particular, that $\mathcal{W}_t(x)$ is monotone decreasing if $\mathrm{Ric}_{M^n} \geq 0$. Moreover, if in addition $\mathcal{W}_t(x)$ is constant on $[0,r]$, then the ball $B_r(x)$ is isometric to $B_r(0) \subset \mathbb{R}^n$; see (4.2).

Lemma 4.19.

$$\partial_t \mathcal{W}_t(x) = -2t \int_M \left(\left| \nabla^2 f - \frac{1}{2t} g \right|^2 + \mathrm{Ric}(\nabla f, \nabla f) \right) \rho_t(x,dy) \leq 0. \tag{4.19}$$

Proof. Equation (4.19) is easily implied by the following computation (compare (4.2)):

$$\frac{d}{dt} W_t = \Delta_f W_t - \langle \nabla f, \nabla W_t\rangle - 2t\left(\left| \nabla^2 f - \frac{1}{2t} g \right|^2 + \mathrm{Ric}_{M^n}(\nabla f, \nabla f) \right). \tag{4.20}$$ $\square$

Next assume (M^n, g, p) satisfies $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$ and $\mathrm{Vol}(B_r(p)) \geq \mathrm{v}r^n > 0$ for $r \leq \delta^{-1}$. In this case we will define a local monotone quantity which will play a role analogous to the one played by pointed entropy.

Let $\varphi : M^n \to [0,1]$ be a cutoff function as in (4.13), with support in $B_{2\delta^{-1}}(p)$, satisfying $\varphi := 1$ in $B_{\delta^{-1}}(p)$ and $|\Delta\varphi| + |\nabla\varphi|^2 \leq C(n)\delta^2$.

Set

$$\mathcal{W}_{t,\varphi}(x) := \int_{M^n} W_t \varphi\, \rho_t(x, dy) - \int_0^t \Big(4s \int_M (n-1)\delta^2 |\nabla f|^2 \varphi \rho_s(x, dy) \Big)\, ds. \tag{4.21}$$

Then, by direct computation,

$$\begin{aligned} \partial_t \mathcal{W}_{t,\varphi}(x) = -2t \int_M \Big(\Big|\nabla^2 f - \frac{1}{2t} g\Big|^2 + \mathrm{Ric}(\nabla f, \nabla f) \\ + 2(n-1)\delta^2 |\nabla f|^2 \Big) \varphi \cdot \rho_t(x, dy) + \int_M W_t \Delta\varphi\, \rho_t(x, dy). \end{aligned} \tag{4.22}$$

By using the heat kernel estimate in Theorem 4.14, we can control the last term on the right-hand side of (4.22). Namely, for any $x \in B_{\delta^{-1}/2}(p)$ and $t \leq \delta^{-2}$, we have

$$\begin{aligned} \Big| \int_M W_t \Delta\varphi \cdot \rho_t(x, dy) \Big| &\leq \int_M |W_t|\, |\Delta\varphi|\, \rho_t(x, dy) \\ &\leq C(n, \mathrm{v})\delta^2 \int_{A_{\delta^{-1}, 2\delta^{-1}}(p)} \Big(1 + \frac{d^2(x, y)}{4t} \Big)\, \rho_t(x, dy) \\ &\leq C(n, \mathrm{v})\delta^2 \cdot e^{-1/100\delta^2 t}. \end{aligned} \tag{4.23}$$

This motivates the following definition of the local $\mathcal{W}_t^\delta$ pointed entropy.

Definition 4.20 (Local $\mathcal{W}_t^\delta$ pointed entropy). Let (M^n, g, p) satisfy $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$ and $\mathrm{Vol}(B_r(p)) \geq \mathrm{v}r^n > 0$ for $r \leq \delta^{-1}$. For any $t \leq \delta^{-2}$ and $x \in B_{\delta^{-1}/2}(p)$, the local $\mathcal{W}_t^\delta$ pointed entropy is defined by

$$\mathcal{W}_t^\delta(x) := \mathcal{W}_{t,\varphi}^\delta(x) := \mathcal{W}_{t,\varphi}(x) - C(n, \mathrm{v})\delta^2 \int_0^t e^{-1/100\delta^2 s}\, ds. \tag{4.24}$$

Remark 4.21 (Scaling). Put $\tilde{g} = r^{-2} g$. If $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$, then $\mathrm{Ric}_{\tilde{M}^n} \geq -(n-1)\delta^2 r^2$. Let $\widetilde{\mathcal{W}}_t^{\delta r}(x)$ denote the local $\mathcal{W}$-entropy associated with $\tilde{g}$. Then

$$\mathcal{W}_{tr^2}^\delta(x) = \widetilde{\mathcal{W}}_t^{\delta r}(x).$$

The following theorem is the main result of this subsection. According to relation (1), the local $\mathcal{W}_t^\delta$ pointed entropy is monotone. By relation (2), it has essentially the same behavior as the volume ratio $\mathcal{V}_r(x)$.

THEOREM 4.22. *Let (M^n, g, p) denote a pointed Riemannian manifold with $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$ and $\mathrm{Vol}(B_r(p)) \geq \mathrm{v} \cdot r^n > 0$ for $r \leq \delta^{-1}$. Then for all $x \in B_{\delta^{-1}/2}(p)$ and $t \leq \delta^{-2}$, the local $\mathcal{W}_t^\delta$-entropy satisfies the following:*

(1) $\partial_t \mathcal{W}_t^\delta(x) \leq -2t \int_M \Big(|\nabla^2 f - \frac{1}{2t} g|^2 + \mathrm{Ric}(\nabla f, \nabla f) + 2(n-1)\delta^2 |\nabla f|^2\Big) \varphi\, \rho_t(x, dy)$ ≤ 0.

(2) *Given $\epsilon > 0$, assume that $\delta \leq \delta(n, \mathrm{v}, \epsilon)$, $0 < t \leq 10$, and*

$$|\mathcal{V}_{\sqrt{t}\delta^{-1}}(x) - \mathcal{V}_{\sqrt{t}\delta}(x)| \leq \delta. \tag{4.25}$$

Then

$$|\mathcal{W}_t^\delta(x) - \log \mathcal{V}^{\delta^2}_{\sqrt{t}}(x)| \leq \epsilon.$$

Proof. It suffices to prove (2). Assume (2) does not hold for some $\epsilon_0 > 0$. Then there exists $\delta_i \to 0$ and there exists (M_i^n, g_i, p_i) satisfying $\mathrm{Vol}(B_r(p_i)) \geq \mathrm{v} r^n > 0$ for $r \leq \delta_i^{-1}$, $\mathrm{Ric}_{M_i^n} \geq -(n-1)\delta_i^2$ and such that for some $x_i \in B_{\delta^{-1}/2}(p_i)$, we have

$$|\mathcal{V}_{\sqrt{t_i}\delta_i}(x_i) - \mathcal{V}_{\delta^{-1}\sqrt{t_i}}(x_i)| \leq \delta_i$$

with $0 < t_i \leq 10$, but

$$|\mathcal{W}_{t_i}^{\delta_i}(x_i) - \log \mathcal{V}_{\sqrt{t_i}}(x_i)| \geq \epsilon_0.$$

The rescaled spaces, $({M^n}_i, \tilde{g}_i, x_i) = (M_i^n, t_i^{-1} g_i, x_i)$, satisfy $\mathrm{Ric}_{M_i^n} \geq -(n-1)\delta_i^2$, and

$$|\tilde{\mathcal{V}}_{\delta_i}(x_i) - \tilde{\mathcal{V}}_{\delta_i^{-1}}(x_i)| \leq \delta_i,$$
$$|\tilde{\mathcal{W}}_1^{\delta_i\sqrt{t_i}}(x_i) - \log \tilde{\mathcal{V}}_1(x_i)| \geq \epsilon_0.$$

Denote the heat kernel at time $t = 1$ of $({M^n}_i, x_i, \tilde{g}_i)$ by

$$\tilde{\rho}_1(x_i, y) = (4\pi)^{-n/2} e^{-\tilde{f}}.$$

By the heat kernel estimate in Theorem 4.14, it follows that for δ_i sufficiently small, we have

$$|\tilde{\mathcal{W}}_1(x_i) - \tilde{\mathcal{W}}_1^{\delta_i\sqrt{t_i}}(x_i)| < \epsilon_0/4,$$

where

$$\tilde{\mathcal{W}}_1(x_i) = \int_{B_{\delta_i^{-1}/2}(x_i)} \Big(|\nabla \tilde{f}|^2 + \tilde{f} - n\Big)\, \tilde{\rho}_1(x_i, dy). \tag{4.26}$$

Therefore, for δ_i sufficiently small,

$$|\tilde{\mathcal{W}}_1(x_i) - \log \tilde{\mathcal{V}}_1(x_i)| \geq \epsilon_0/2. \tag{4.27}$$

We will deduce a contradiction to this estimate by letting $i \to \infty$.

Thus by Gromov's compactness theorem, there exists a subsequence of $(M_i^n, t_i^{-1}g_i, x_i)$ converging to some metric cone $(C(X^{n-1}), d, x_\infty)$. By the volume convergence result in [Col97], [Che01] (see also Theorem 4.2) we have

$$\frac{\mathrm{Vol}(X)}{\mathrm{Vol}(S^{n-1})} = \lim_{i\to\infty} \tilde{\mathcal{V}}_1(x_i).$$

By using the heat kernel convergence in Proposition 4.15, together with Remarks 4.17, 4.16 and the heat kernel estimate in Theorem 4.14, we conclude that

$$\lim_{i\to\infty} \tilde{\mathcal{W}}_1(x_i) = \int_{C(X)} \Big(|\nabla f_\infty|^2 + f_\infty - n\Big)\, \rho_1(x_\infty, dy), \tag{4.28}$$

where

$$\rho_1(x_\infty, y) = (4\pi)^{-n/2} e^{-f_\infty} = \frac{\mathrm{Vol}(S^{n-1})}{\mathrm{Vol}(X)} (4\pi)^{-n/2} e^{-d^2(x_\infty, y)/4}.$$

A simple computation gives

$$\begin{aligned}
&\int_{C(X)} \Big(|\nabla f_\infty|^2 + f_\infty - n\Big)\, \rho_1(x_\infty, dy)\\
&\quad = \int_{C(X)} \Big(\frac{d^2(x_\infty, y)}{2} + \log\frac{\mathrm{Vol}(X)}{\mathrm{Vol}(S^{n-1})} - n\Big)\, \rho_1(x_\infty, dy)\\
&\quad = \int_0^\infty \frac{r^{n+1}}{2} \mathrm{Vol}(S^{n-1})(4\pi)^{-n/2} e^{-r^2/4} dr + \log\frac{\mathrm{Vol}(X)}{\mathrm{Vol}(S^{n-1})} - n\\
&\quad = \Gamma(1+\frac{n}{2})\mathrm{Vol}(S^{n-1})\pi^{-n/2} + \log\frac{\mathrm{Vol}(X)}{\mathrm{Vol}(S^{n-1})} - n\\
&\quad = \log\frac{\mathrm{Vol}(X)}{\mathrm{Vol}(S^{n-1})}.
\end{aligned} \tag{4.29}$$

Since $\tilde{\mathcal{W}}_1(x_i)$ and $\log \tilde{\mathcal{V}}_1(x_i)$ have the same limit, this is a contradiction. □

4.7. (k, α, δ)-*entropy pinching*. Recall that in Definition 4.4, we introduced the notion of a collection of a (k, α)-independent set of points $x_0, \ldots, x_k$. We will use a refinement of this notion to define the pinching of the local pointed entropy $\mathcal{W}_t(x)$. This will be used in the Sharp Cone-Splitting Theorem 6.1.

Definition 4.23. The (k, α, δ)-entropy pinching, $\mathcal{E}_r^{k,\alpha,\delta}(x)$, is

$$\mathcal{E}_r^{k,\alpha,\delta}(x) := \inf_{\{x_i\}_0^k} \sum \big|\mathcal{W}^\delta_{2r^2}(x_i) - \mathcal{W}^\delta_{r^2}(x_i)\big|, \tag{4.30}$$

where the infimum is taken over all (k, α)-independent subsets and the parameter δ is corresponding to Ricci curvature lower bound.

From the discussion above, it follows that if $\mathcal{E}_1^{k,\alpha,\delta}(p) < \delta = \delta(\epsilon, \alpha)$, then there exists a (k, ϵ)-splitting map $u : B_1(p) \to \mathbb{R}^k$. The sharp version of this relationship is the content of Theorem 6.1, the Sharp Cone-Splitting Theorem. This theorem states that there exists $C(n, \mathrm{v}, \alpha)$ and a splitting map u for which the integral of the norm squared of the Hessian has the following sharp *linear* bound in terms of the k-pinching:[5]

$$\fint_{B_1(p)} |\nabla^2 u|^2 \leq C(n, \mathrm{v}, \alpha) \cdot \mathcal{E}_1^{k,\alpha,\delta}(p). \tag{4.31}$$

4.8. *Poincaré inequalities.* We recall various Poincaré inequalities which hold on manifolds with Ricci lower bound; see also [Bus82], [Che99], [Che01], [CC00]. We will need the ones which follow:

THEOREM 4.24 (Poincaré Inequalities). *Let* (M^n, g, x) *satisfy* $\mathrm{Ric}_{M^n} \geq -(n-1)$. *Then for any* $0 < r \leq 10$, *the following Poincaré inequalities hold*:

(1) $\fint_{B_r(x)} f^2 \leq C(n) \cdot r^2 \fint_{B_r(x)} |\nabla f|^2 \qquad (\text{for all } f \in C_0^\infty(B_r(x)))$;

(2) $\fint_{B_r(x)} \left| f - \fint_{B_r(x)} f \right|^2 \leq C(n) \cdot r^2 \fint_{B_r(x)} |\nabla f|^2 \qquad (\text{for all } f \in C^\infty(B_r(x))$.

The Dirichlet Poincaré inequality (1) follows directly from *segment inequality* in [CC96] and [Che01]. For the Neumann Poincaré inequality (2), by using segment inequality, we have a weak Poincaré inequality [CC00]. By the volume doubling and a covering argument in [HK95] or [Jer86], we can obtain the Neumann Poincaré inequality (2).

4.9. $W^{1,2}$*-convergence.* Below, the notation $(Z_i, d_i, z_i) \xrightarrow{d_{GH}} (Z, d, z)$ should always be understood as convergence in the measured Gromov-Hausdorff sense. In this subsection, we will assume without explicit mention that the metric measure space (Z, d, μ) is separable and complete and that μ is a Borel measure which is finite on bounded subsets of Z.

Definition 4.25 (Uniform convergence). Let $(Z_i, d_i, z_i) \xrightarrow{d_{GH}} (Z, d, z)$. If f_i are Borel functions on Z_i, then we say $f_i \to f : Z \to \mathbb{R}$ *uniformly* if for any compact subset $K_i \subset Z_i \to K \subset Z$ and ϵ_i-GH approximation $\Psi_i : K \to K_i$ with $\epsilon_i \to 0$, the function $f_i \circ \Psi_i$ converges to f uniformly on K.

As motivation for what follows, recall that on a fixed metric measure space, for $1 < p < \infty$, weak convergence together with convergence of norms implies strong convergence.

[5]The proof of (5.2) is one instance in which choosing to use the pointed entropy as our monotone quantity helps to make the argument run more smoothly than if we had chosen to work with the volume ratio $\mathcal{V}_r(x)$.

Definition 4.26 (Weak L^p convergence). Let $(Z_i, d_i, z_i, \mu_i) \xrightarrow{d_{GH}} (Z, d, z, \mu)$. If f_i are Borel functions on Z_i, we say $f_i \to f : Z \to \mathbb{R}$ in the *weak sense* if for any uniformly converging sequence of compactly supported Lipschitz functions $\varphi_i \to \varphi$, we have

$$\lim_{i\to\infty} \int f_i \varphi_i d\mu_i = \int f \varphi d\mu. \tag{4.32}$$

Moreover if f_i, f have uniformly bounded L^p integrals, then we say $f_i \to f$ in the weak L^p-sense.

Any uniformly bounded L^p sequence f_i has a weak limit f. See also [GMS15] for a definition of the weak convergence by embedding Z_i, Z to a common metric space Y.

Definition 4.27 (L^p and $W^{1,p}$ convergence). Let f_i denote Borel functions on Z_i, and let $(Z_i, d_i, z_i, \mu_i) \xrightarrow{d_{GH}} (Z, d, z, \mu)$. For $p < \infty$, we say $f_i \to f : Z \to \mathbb{R}$ *in the L^p-sense* if $f_i \to f$ in the weak L^p-sense and

$$\int_{Z_i} |f_i|^p \to \int_Z |f|^p.$$

If $f_i \to f$ in the L^p-sense and

$$\int_{Z_i} |\nabla f_i|^p \to \int_Z |\nabla f|^p,$$

we say $f_i \to f$ in the *$W^{1,p}$-sense*.

The following can easily be checked. Thus, the proof will be omitted.

PROPOSITION 4.28.

(1) *If f_i converges to a constant A in the L^2-sense, then $f_i^2 - A$ converges in L^1 to zero.*
(2) *If f_i and g_i converge to f and g in the L^2-sense respectively, then $f_i g_i \to fg$ in the L^1-sense.*
(3) *Uniform convergence implies L^p convergence for any $0 < p < \infty$.*

The proof of the following Proposition 4.29, on $W^{1,2}$-convergence for functions with L^2 Laplacian bound, depends on the Mosco convergence of the Cheeger energy; see Theorem 4.4 of [AH18]. In our application the limit X is a metric cone and u_i is Lipschitz, in which case the proposition is simply proved by using the result in [Din02] without involving RCD notions; for related discussions in the metric measure space context, see [AHT18], [Che99], [GMS15], [MN19], [ZZ19].

PROPOSITION 4.29 ($W^{1,2}$-convergence). *Let $(M_i^n, g_i, x_i, \mu_i) \to (X, d, x_\infty, \mu)$ with $\mathrm{Ric}_{M_i^n} \geq -(n-1)$ and $\mu_i = \mathrm{Vol}(B_1(x_i))^{-1}\mathrm{Vol}$. Let $u_i : B_R(x_i) \to \mathbb{R}$ be* *smooth functions satisfying*

$$\fint_{B_R(x_i)} |u_i|^2 + \fint_{B_R(x_i)} |\nabla u_i|^2 + \fint_{B_R(x_i)} |\Delta u_i|^2 \le C$$

for some C. *If* u_i *converge in the* L^2*-sense to a* $W^{1,2}$*-function* $u_\infty : B_R(x_\infty) \to \mathbb{R}$, *then*

(1) $u_i \to u_\infty$ *in the* $W^{1,2}$*-sense over* $B_R(x_\infty)$;
(2) $\Delta u_i \to \Delta u_\infty$ *in the weak* L^2*-sense*;
(3) *if* $\sup_{B_R(x_i)} |\nabla u_i| \le L$ *for some uniform constant, then* $u_i \to u_\infty$ *in the* $W^{1,p}$*-sense for any* $0 < p < \infty$.

Proof (*outline following* [AH18], [Din02]). We will argue under a uniform Lipschitz assumption; the general case is similar but a bit more technical.

In view of the uniform Lipschitz condition $\sup_{B_R(x_i)} |\nabla u_i| \le L$ it follows by an Ascoli type argument that we have uniform convergence, $u_i \to u_\infty$. Also, $f_i := \Delta u_i$ converges in the weak L^2-sense to some L^2 function f_∞. Consider the energy

$$E_i(u_i) := \int_{B_R(x_i)} \Big(\frac{1}{2}|\nabla u_i|^2 + u_i f_i\Big).$$

By the lower semicontinuity of the Cheeger energy, we have

$$\lim_{i\to\infty} \inf E_i(u_i) \ge E_\infty(u_\infty).$$

Moreover, using Lemma 10.7 of [Che99] one can construct *some* Lipschitz sequence v_i in $B_R(x_i)$ which converges uniformly to u_∞ with $v_i = u_i$ on $\partial B_R(x_i)$ and

$$\lim_{i\to\infty} \sup \int_{B_R(x_i)} |\nabla v_i|^2 \le \int_{B_R(x_\infty)} |\nabla u_\infty|^2.$$

From the fact that u_i minimizes the energy E_i, it then follows that $E_i(u_i) \to E_\infty(u_\infty)$, which gives us the $W^{1,2}$-convergence. The weak convergence of Δu_i also follows from the energy convergence. That is, we need to show $\Delta u_\infty = f_\infty$, i.e., for any Lipschitz h with $h = u_\infty$ on $\partial B_R(x_\infty)$, that $E_\infty(h) \ge E_\infty(u_\infty)$. Assume there exists Lipschitz h_∞ with $h_\infty = u_\infty$ on $\partial B_R(x_\infty)$ and $\epsilon_0 > 0$ such that $E_\infty(h_\infty) < E_\infty(u_\infty) - \epsilon_0$. Then we can construct by using Lemma 10.7 of [Che99] a sequence of Lipschitz function h_i in $B_R(x_i)$ with $h_i = u_i$ on $\partial B_R(x_i)$ and

$$\lim_{i\to\infty} \sup \int_{B_R(x_i)} |\nabla h_i|^2 \le \int_{B_R(x_\infty)} |\nabla h_\infty|^2.$$

Since $E_i(u_i) \to E_\infty(u_\infty)$, this implies for large i that

$$E_i(h_i) < E_i(u_i) - \epsilon_0/2, \tag{4.33}$$

which contradicts the fact that u_i minimizes $E_i(u_i)$ over all Lipschitz functions with the same boundary condition. Hence, we conclude that $\Delta u_\infty = f_\infty$. This completes the (outline of the) proof of Proposition 4.29. $\square$

The following lemma was proved in [Din04] for metric cone limits and in [AH18] for the general case.

LEMMA 4.30. *Let $(M_i, g_i, x_i, \mu_i) \to (X, d, x_\infty, \mu)$ with $\mathrm{Ric}_{M_i^n} \geq -(n-1)$ and $\mu_i = \mathrm{Vol}(B_1(x_i))^{-1}\mathrm{Vol}(\cdot)$. Let $f, F \in L^2(X)$ have compact support, and assume $\Delta F = f$ and that f is Lipschitz. Then for any $R > 0$, there exist solutions $\Delta F_i = f_i$ on $B_{R_i}(x_i)$ with $R_i \to R$ such that F_i and f_i converge uniformly to F and f in any compact subset of $B_R(x_\infty)$ respectively .*

Outline of the proof. From a generalized Bochner formula in [EKS15] and the standard elliptic estimate, it follows that F is Lipschitz. By Lemma 10.7 of [Che99] one can construct Lipschitz functions $\hat{F}_i, f_i$ on $B_R(x_i)$ converging uniformly and in the $W^{1,2}$-sense to F and f respectively.

For $\epsilon > 0$, define $F_{i,\epsilon}$ on $B_R(x_i)$ such that $\Delta F_{i,\epsilon} = f_i$ on $B_{R-\epsilon}(x_i)$ with $F_{i,\epsilon} = \hat{F}_i$ on $\partial B_{R-\epsilon}(x_i)$, and $F_{i,\epsilon} = \hat{F}_i$ on $B_R \setminus B_{R-\epsilon}(x_i)$. By the definition of $F_{i,\epsilon}$ we have

$$\int_{B_R(x_i)} |\nabla F_{i,\epsilon}|^2 + 2f_i F_{i,\epsilon} \leq \int_{B_R(x_i)} |\nabla \hat{F}_i|^2 + 2f_i \hat{F}_i. \tag{4.34}$$

Assume the limit of $F_{i,\epsilon}$ is $F_{\infty,\epsilon}$ whose existence is asserted by Proposition 4.29. Moreover, $F_{\infty,\epsilon} - F \in W_0^{1,2}(B_R)$. By applying the lower semicontinuity of Cheeger energy to $\int_{B_R(x_i)} |\nabla F_{i,\epsilon}|^2$, we have

$$\liminf_{i\to\infty} \int_{B_R(x_i)} |\nabla F_{i,\epsilon}|^2 + 2f_i F_{i,\epsilon} \geq \int_{B_R} |\nabla F_{\infty,\epsilon}|^2 + 2fF_{\infty,\epsilon}. \tag{4.35}$$

Since $F_{\infty,\epsilon} - F \in W_0^{1,2}(B_R)$ and $\Delta F = f$ on B_R, we have that

$$\int_{B_R} |\nabla F_{\infty,\epsilon}|^2 + 2fF_{\infty,\epsilon} \geq \int_{B_R} |\nabla F|^2 + 2fF. \tag{4.36}$$

On the other hand, noting that (4.34) and $\int_{B_R(x_i)} |\nabla \hat{F}_i|^2 \to \int_{B_R(x_\infty)} |\nabla F|^2$, we get

$$\limsup_{i\to\infty} \int_{B_R(x_i)} |\nabla F_{i,\epsilon}|^2 + 2f_i F_{i,\epsilon} \leq \int_{B_R} |\nabla F|^2 + 2fF. \tag{4.37}$$

Combining (4.35), (4.36) and (4.37), we have that

$$\int_{B_R} |\nabla F|^2 + 2fF = \int_{B_R} |\nabla F_{\infty,\epsilon}|^2 + 2fF_{\infty,\epsilon}. \tag{4.38}$$

Since $\Delta F = f$ on B_R and $F - F_{\infty,\epsilon} \in W_0^{1,2}(B_R)$, this implies that $F_{\infty,\epsilon} = F$. Let us choose $\epsilon_i \to 0$ and define $F_i = F_{i,\epsilon_i}$. Therefore, the convergence $F_i \to F_{\infty,\epsilon}$

is pointwise and is in the $W^{1,2}$-sense. The uniform convergence in any compact subset of $B_R(x_\infty)$ follows from the standard interior gradient estimate for equation $\Delta F_i = f_i$ in $B_{R-\epsilon_i}$. Hence the proof of Lemma 4.30 is completed. □

4.10. *The Laplacian on a metric cone.* Next, we will recall the existence of the Laplacian operator on metric cones with suitable cross-sections. The explicit formulas, (4.40), (4.41), in Theorem 4.32, were initially derived in the context of spaces with iterated conical singularities [Che79], [Che83]. This context is in certain ways more special and in other ways more general than that of the present subsection. Theorem 4.32 below was originally proved for metric measure spaces satisfying a doubling condition and Poincaré inequality in [Che99] and for Gromov-Hausdorff limits of smooth manifolds in [CC00] and [Din02]. It is also understood in the context of RCD spaces [AGS14b], [AGS14a].

THEOREM 4.31. *Let* $(M^n_i, g_i, p_i) \to (X, d_X, p) := (C(Y), d_X, p)$ *satisfy* $\mathrm{Ric}_{M^n_i} \geq -\delta_i \to 0$ *and* $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$. *Then*

(1) *there exists a nonpositive, linear, self-adjoint, Laplacian operator*

$$\Delta_X : \mathrm{Dom}(X) \subset L^2(X) \to L^2(X)$$

with $\mathrm{Dom}\sqrt{-\Delta_X} = W^{1,2}(X)$;

(2) *for compact supported Lipschitz functions* f *on* X, $|\nabla f| = |\mathrm{Lip} f|$,

$$\int_X |\nabla f|^2 d\mathcal{H}^n = \langle \sqrt{-\Delta_X} f, \sqrt{-\Delta_X} f \rangle;$$

(3) *there exists a nonpositive, linear, self-adjoint, Laplacian operator* Δ_Y : $\mathrm{Dom}(Y) \subset L^2(Y) \to L^2(Y)$ *with* $\mathrm{Dom}\sqrt{-\Delta_Y} = W^{1,2}(Y)$;

(4) *in geodesic polar coordinate* $x = (r, y)$, *the Laplace operator* Δ_X *and* Δ_Y *satisfy, in the* $W^{1,2}(X)$ *distribution sense,*

$$\Delta_X = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_Y. \tag{4.39}$$

Originally, relations (1) and (2) were proved in [CC00] and [Che99]. Relations (3) and (4) were proved in [Din02].

The Sobolev space $W^{1,2}(X)$ is the closure of Lipschitz functions under a $W^{1,2}$-norm defined in [Che99]; see Section 2 of [Che99] for the precise definition, which ensures that the $W^{1,2}$-norm behaves lower semicontinuously under L^2 convergence. It then becomes a highly nontrivial theorem that in actuality, $|\nabla f| = \mathrm{Lip}\, f$, the pointwise Lipschitz constant almost everywhere. These results were proved in [Che99] under the assumption that the measure is doubling and a Poincaré inequality holds.

The cross-section Y may itself be viewed as a space which satisfies the lower Ricci curvature bound $\mathrm{Ric}_Y \geq n-2$ in a generalized sense. The consequences were initially established directly for cross-sections of limit cones. Subsequently, it was shown that in the precise formal sense, Y is an RCD space with $\mathrm{Ric} \geq n-2$; see [Ket15], [BS14].

THEOREM 4.32. *Let* $(M_i^n, g_i, p_i) \to (X, d_X, p) = (C(Y), d_X, p)$ *satisfy* $\mathrm{Ric}_{M_i^n} \geq -\delta_i \to 0$ *and* $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$. *Then*

(1) $(I - \Delta_Y)^{-1} : L^2(Y) \to L^2(Y)$ *is a compact operator*;
(2) *Laplacian* Δ_Y *has discrete spectrum* $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$;
(3) *if* ϕ_i *is an eigenfunction associated to* λ_i, *then* ϕ_i *is Lipschitz*;
(4) *the following functions are harmonic on* X:

$$u(r, y) = r^{\alpha_i} \phi_i, \tag{4.40}$$

where

$$\alpha_i = -\frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_i}; \tag{4.41}$$

(5) *the harmonic functions* $r^{\alpha_i}\phi_i$ *are Lipschitz on* $C(Y)$;
(6) *the first nonzero eigenvalue satisfies* $\lambda_1 \geq n-1$.

Remark 4.33. (1) follows from a Neumann Poincaré inequality on Y, which is induced from the Neumann Poincare inequality on X. See a proof of Lemma 4.3 in [Din02] and see also [CC00].

(2) follows from (1). See also Theorem 1.8 of [CC00], which only uses Neumann Poincare inequality and volume doubling.

(3) follows from the fact that the harmonic function $u(r, y) = r^{\alpha_i}\phi_i$ is locally Lipschitz, which was proved in [Din02].

(4) follows from the statement (4) of Theorem 4.31.

(5) and (6) were proved in [Din02].

4.11. *ϵ-regularity for 2-sided Ricci bounds.*

Definition 4.34. For $x \in M^n$, we define the harmonic radius $r_h(x) > 0$ to be the maximum over all $r > 0$ such that there exists a mapping $\psi = (\psi_1, \ldots, \psi_n) : B_r(x) \to \mathbb{R}^n$ with the following properties:

(1) $\Delta \psi_i = 0$ for $i = 1, \ldots, n$;
(2) ψ is a diffeomorphism onto its image with $B_r(0^n) \subseteq \psi(B_r(x))$, and hence defines a coordinate chart;
(3) the coordinate metric $g_{ij} = \langle \nabla \psi_i, \nabla \psi_j \rangle$ on $B_r(x)$ satisfies $||g_{ij} - \delta_{ij}||_{C^0(B_r(x))} + r \cdot ||\partial\, g_{ij}||_{C_0(B_r(x))} < 10^{-n}$.

The formula for the Ricci tensor in harmonic coordinates can be viewed as a (nonlinear) elliptic equation on the metric g_{ij} in which the Ricci tensor is the inhomogeneous term. If the Ricci curvature is bounded, $|\mathrm{Ric}_{M^n}| \leq n-1$, then via elliptic regularity we obtain for any $p < \infty$ and $0 < \alpha < 1$ the a priori estimates for the case $r = 1$:

$$||g_{ij} - \delta_{ij}||_{C^{1,\alpha}(B_{1/2}(x))} \leq C(n, \alpha), \tag{4.42}$$

$$||g_{ij} - \delta_{ij}||_{W^{2,p}(B_{1/2}(x))} < C(n, p). \tag{4.43}$$

The following ϵ-regularity theorem from [CN15] can be viewed as a consequence of the proof of the codimension 4 conjecture proved in that paper. It states in quantitative form that if a ball has a sufficient amount of symmetry, then the ball is in the domain of a harmonic coordinate system in which the metric satisfies definite bounds.

THEOREM 4.35 ([CN15]). *There exists* $\epsilon(n, \mathrm{v}) > 0$ *such that if* $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, $|\mathrm{Ric}_{M^n}| \leq n-1$ *and* $B_2(p)$ *is* $(n-3, \epsilon)$*-symmetric, then* $r_h(p) > 1$.

5. Outline of proof of Neck Structure Theorem

The idea of a neck region is derived primarily from [JN21] and is motivated by ideas from [NV17a]. Given the Neck Structure Theorem 2.9, the proof of the Neck Decomposition of Theorem 2.12 follows along lines similar to what was done in a more restricted context in [JN21]. More precisely, much of the proof of the Neck Decomposition of Theorem 2.12 involves an elaborate and highly nontrivial covering argument. At a few places, an appeal is made to Theorem 2.9 to provide sharp estimates; however none of the technology which goes into the proof of Theorem 2.9 plays a role in the proof of Theorem 2.12. Thus, the bulk of this paper is focused on proving the Neck Structure Theorem 2.9. This requires a completely new set of ideas and tools, quite distinct from those of the abovementioned citations. Our purpose in this section is to introduce these new ideas in order to sketch a clean picture of the proof of Theorem 2.9. Some of our explanations will be repeated in subsequent sections.

The proof of Theorem 2.9 involves a nonlinear induction scheme. In it, we will assume a weaker version of the Ahlfors regularity condition (2.7) already holds, and we will use it to prove the stronger version. Precisely, our main inductive lemma is the following:

LEMMA 5.1 (Inductive Lemma). *Fix* $\eta, B > 0$ *and* $\delta \leq \delta(n, \mathrm{v}, \eta, B)$. *Let* $\mathcal{N} = B_2(p) \setminus \overline{B}_{r_x}(\mathcal{C})$ *denote a* (k, δ, η)*-neck region, and assume for each* $x \in \mathcal{C}$ *and* $B_{2r}(x) \subset B_2(p)$ *that*

$$B^{-1} r^k < \mu(B_r(x)) < B\, r^k. \tag{5.1}$$

Then

(1) *for each* $x \in \mathcal{C}$ *and* $B_{2r}(x) \subset B_2(p)$, *we have the improved estimate* $A(n)^{-1} r^k < \mu(B_r(x)) < A(n) r^k$;
(2) $\mathcal{C}_0$ *is* k*-rectifiable.*

Outlining the proof of the inductive lemma will be the main goal of this section. In Section 9 we rigorously prove Theorem 2.9 from the Inductive Lemma.

5.1. *Harmonic splittings on neck regions.* In order to prove the Inductive Lemma 5.1, and hence Theorem 2.9, let us first make the following observation. Let $\mathcal{C}' \subseteq B_2 \subseteq \mathbb{R}^k$ be a closed subset with $r'_x : \mathcal{C}' \to \mathbb{R}$ a radius function such that $\{\overline{B}_{\tau_n r'_x}(x)\}$ are all disjoint and $B_2 \subseteq \mathcal{C}'_0 \cup \bigcup B_{r'_x}(\mathcal{C}'_+)$, where as usual $\mathcal{C}'_0 = \{r'_x = 0\}$ and $\mathcal{C}'_+ = \{r'_x > 0\}$. Consider the packing measure $\mu' = \mathcal{H}^k \cap \mathcal{C}'_0 + \omega_k \sum_{\mathcal{C}'_+} r_i^k \delta_{x_i}$. It is a straightforward though instructive exercise to see that μ' automatically satisfies the Ahflors regularity condition (2.7). For this, one notes that the Lebesgue measure on $\mathbb{R}^k$ coincides with the Hausdorff measure. Therefore, the strategy to prove the Inductive Lemma 5.1 will be to find a mapping $u : \mathcal{C} \to \mathbb{R}^k$ which is bi-Hölder onto its image and $(1+\epsilon)$-bi-Lipschitz on most of $\mathcal{C}$. Then, with $\mathcal{C}' := u(\mathcal{C})$ and $r'_x := r_x$, we can turn the covering $\{B_{r'_x}(\mathcal{C})\}$ into a well-behaved covering of $B_2(0^k) \subseteq \mathbb{R}^k$, and therefore conclude the asserted Ahlfors regularity. For further discussion of the role of the Ahlfors regularity of the packing measure, see Remark 2.10.

Remark 5.2 (Digression). At this point, we will digress in order to explain what will not work in the present context. This will motivate the strategy used here and relate it to that in the the previous literature. In [NV17a] a quite similar strategy was implemented in order to study the singular sets of nonlinear harmonic maps. In that case, the map u was built by hand, using a Reifenberg construction. Showing that the construction worked required new estimates on nonlinear harmonic maps and a new rectifiable Reifenberg theorem. It is natural to examine the possibility of implementing a similar approach in the present context, by using metric Reifenberg constructions in the spirit of [CC97]. However, these ideas break down in the context of lower Ricci curvature bounds. Essentially, this is because the underlying space itself is curved. This gives rise to error terms which are quantitatively worse than those which arise in connection the bi-Lipschitz Reifenberg techniques of [NV17a]. As a result, those techniques fail in the present context. Therefore, of necessity, our construction of the map u will be completely different from that of [NV17a]. Instead of relying on a Reifenberg type construction, our mapping u will be more canonical in nature. It will solve an equation.

To make the above more precise, recall from Definition 4.10, the notion of a harmonic splitting function.

It follows from Theorem 4.11 that if $B_{2r}(p)$ is (k,δ)-symmetric, then there exists an ϵ-splitting map $u : B_r(p) \to \mathbb{R}^k$. In particular, splitting maps exist on neck regions. In general, splitting functions can degenerate on sets of infinite codimension 2 content. In particular, the degeneration set of u may in general be much larger than the center point set $\mathcal{C}$ of a neck region. However, as we will see, something rather miraculous takes place. Namely, if we are on a neck region, then the map u can degenerate in at most a weak sense on all of $\mathcal{C}$, and on most of $\mathcal{C}$, it cannot degenerate at all. Precisely, we will prove the following:

THEOREM 5.3 (Harmonic splittings on neck regions). *Let $B, \epsilon, \eta > 0$ with $\delta' \le \delta'(n, \mathrm{v}, B, \epsilon, \eta)$ and $\delta \le \delta(n, \mathrm{v}, \eta, B, \epsilon)$. Let $\mathcal{N} = B_2 \setminus \overline{B}_{r_x}(\mathcal{C})$ be a (k, δ, η)-neck region satisfying (5.1) with $u : B_4 \to \mathbb{R}^k$ a δ'-splitting map. Then there exists $\mathcal{C}_\epsilon \subset \mathcal{C} \cap B_{15/8}(p)$ such that*

(1) $\mu\big(\mathcal{C}_\epsilon \cap B_{15/8}(p)\big) \ge (1-\epsilon)\mu\big(\mathcal{C} \cap B_{15/8}(p)\big)$;
(2) *u is $(1+\epsilon)$-bi-Lipschitz on $\mathcal{C}_\epsilon$, i.e.,* $(1+\epsilon)^{-1} \cdot d(x,y) \le |u(x) - u(y)| \le (1+\epsilon) \cdot d(x,y)$ *for any* $x, y \in \mathcal{C}_\epsilon$;
(3) *u is $(1+\epsilon)$-bi-Hölder on $\mathcal{C}$, i.e.,* $(1+\epsilon)^{-1} \cdot d(x,y)^{1-\epsilon} \le |u(x) - u(y)| \le (1+\epsilon) \cdot d(x,y)$ *for any* $x, y \in \mathcal{C}$.

Theorem 5.3 is an abbreviated version of Proposition 9.3, which is the result which will be proved in the body of the paper.

The proof of Theorem 5.3 relies on three main new points: The Sharp Splitting Theorem 6.1, the Geometric Transformation Theorem 7.2, and the Nondegeneration Theorem of 8.1. The remainder of this outline will discuss these results and explain how they lead to the proof of Theorem 5.3. For convenience, we restate these results below as Theorems 5.4, 5.6, 5.7, respectively.

5.2. *Sharp cone-splitting.* It is a now classical point that if $B_2(p)$ is (k,δ)-symmetric, then there exists a harmonic (k,ϵ)-splitting function $u : B_1(p) \to \mathbb{R}^k$; see Theorem 4.11. In this paper, it will be crucial to have a quantitatively sharp understanding of how good a splitting exists.

Recall that in Definition 4.4 we introduced the notion of a (k,α)-independent set of points $x_0, \ldots, x_k$. Also, in Definition 4.23 we defined the notion of (k,α,δ)-entropy pinching. The following is a slight specialization of Theorem 6.1. The crucial point is the precise linear relationship between the k-pinching of a ball and the squared Hessian of a splitting map. This is what, under appropriate circumstances, eventually allows the result to be summed over an arbitrary number of scales without having the resulting estimate blow up uncontrollably.

THEOREM 5.4 (Sharp Cone-Splitting). *Given $\epsilon, \alpha > 0$ there exist positive constants $\delta(n, \mathrm{v}, \alpha, \epsilon)$ and $C(n, \mathrm{v}, \alpha) > 0$ with the following properties. Let (M^n, g, p) satisfy $\mathrm{Ric}_{M^n} \ge -(n-1)\delta^2$ and $\mathrm{Vol}(B_{\delta^{-1}}(p)) > \mathrm{v}\delta^{-n} > 0$, and* *let $B_{4\delta^{-1}}(p)$ be (k,δ^2)-symmetric. Then there exists a (k,ϵ)-splitting map $u : B_2(p) \to \mathbb{R}^k$ satisfying*

$$\fint_{B_2(p)} \left(|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + 2(n-1)\,\delta^2 |\nabla u|^2\right) \le C\,\mathcal{E}^k_1(p). \tag{5.2}$$

5.3. *Sharp Transformation Theorem.* The results of the last subsection tell us, in terms of the k-pinching, how good we can expect the *best* splitting map to be on a typical ball. The proof of Theorem 5.3 depends on fixing a single splitting map on the original ball $B_2(p)$ and seeing how it behaves on smaller balls.

To this end, let us begin by describing a simple situation. If $u : B_2(0^n) \to \mathbb{R}^k$ is a k-splitting map in $\mathbb{R}^n$, then as with any solution of an elliptic PDE, u has *pointwise* bounds on the Hessian. Among other things, this implies that if we restrict to some sub-ball, $B_r(x) \subseteq B_1$, then $u : B_r(x) \to \mathbb{R}^k$ is still a splitting map. More than that, we know by the smoothness estimates that the matrix $T^{-1} := \langle \nabla u_i, \nabla u_j\rangle(x)$ is close to δ_{ij}. Thus, if we look at the map $T \circ u$, so that $\langle \nabla T u_i, \nabla T u_j\rangle(x) = \delta_{ij}$, then we even know that $Tu|_{B_r}$ is becoming an increasing improved splitting map, as Tu is scaled invariantly converging to an isometric linear map at a polynomial rate. Unfortunately, on spaces with only lower Ricci curvature bounds, such statements are highly false. For instance there could be points where $|\nabla u| = 0$, so that $u|_{B_r}$ is not even a splitting map on small balls, much less a better one; see, for instance, Example 3.1.

However, it turns out that although the restriction of $u : B_r(x) \to \mathbb{R}^k$ to a sub-ball may not be well behaved, if we are on a neck region and $x \in \mathcal{C}$, then u may only degenerate in a very special way. Namely, though $u|_{B_r}$ may not be a splitting map, there is a $k \times k$-matrix T such that $Tu = T^j_i u_j : B_r(x) \to \mathbb{R}^k$ is a splitting map. What is more important, and as it turns out a lot harder to prove, is that after transformation Tu is the *best* splitting map on the ball, in that it satisfies the estimates from the Sharp Cone-Splitting Theorem 6.1.

Remark 5.5. Note that in comparison to the $\mathbb{R}^n$ case above the matrix T depends on the scale, and not just the point, as $T = T_{x,r}$ may blow up in norm. Additionally, we of course cannot expect that Tu is converging polynomially to a splitting map, since no such splitting map may exist. All we can hope for is that Tu is the best splitting map which does exist.

Our precise result is the following, which is a slight specialization of Theorem 7.2.

THEOREM 5.6 (Geometric Transformation). *Given $\alpha, \eta, \epsilon > 0$, there exists $C = C(n, \mathrm{v}, \eta, \alpha)$ and $\gamma = \gamma(n, \mathrm{v}, \eta) > 0$ such that if $\delta < \delta(n, \mathrm{v}, \eta)$, then the following holds: Let (M^n, g, p) satisfy $\mathrm{Ric}_{M^n} \ge -(n-1)\delta^2$, $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, and assume the following:*

(i) $u : B_2(p) \to \mathbb{R}^k$ *is a* (k, δ)*-splitting function;*
(ii) *for all* $r \le s \le \delta^{-1}$*, the ball* $B_s(p)$ *is* (k, δ^2)*-symmetric but not* $(k+1, \eta)$*-symmetric.*

Then for all $s \in [r, 1]$*, there exists a* $k \times k$*-matrix* $T = T_{p,s}$ *such that*

(1) (*weak estimate*) $Tu : B_s(p) \to \mathbb{R}^k$ *is a* (k, ϵ)*-splitting map;*
(2) (*strong estimate*) *for* $r_j := 2^{-j}$,

$$s^2 \fint_{B_s(p)} |\nabla^2 Tu|^2 \le C \sum_{s \le r_j \le 1} \left(\frac{s}{r_j}\right)^\gamma \mathcal{E}^k_{r_j}(p) + C\delta^2 s^\gamma. \tag{5.3}$$

First, note that the weak estimate above is the main ingredient in the proof of the bi-Hölder estimate, (3), of Theorem 5.3. To see this, observe that since the transformation exists on every scale, one can see it must change slowly. In particular, $|T_{2r}^{-1} \circ T_r| \le 1 + \epsilon$ and hence $|T_r| \le r^{-\epsilon}$. On the other hand, if one takes $x, y \in \mathcal{C}$ and considers $r = d(x, y)$, then by the weak estimate we have

$$|d(T_r u(x), T_r u(y)) - d(x, y)| < \epsilon r. \tag{5.4}$$

By using the norm control on T_r stated above, this exactly gives the bi-Hölder estimate; for the details, see Section 8.

The proof of the weak estimate itself is given by a contradiction argument in the spirit of [CN15]. Roughly, if the result fails at some $x \in \mathcal{C}$, then one looks for the first radius $s > r_x$ for which it fails. By blowing up $B_s(x)$ to a ball of radius 1 and passing to the limit, $T_{2s}u \to v : \mathbb{R}^k \times C(Y) \to \mathbb{R}^k$, one obtains a harmonic map v which is a (k, ϵ)-splitting map on $B_2(x)$, but for which by assumption, there is, in particular, no transformation so that Tv is a $(k, \epsilon/2)$-splitting map on $B_1(x)$. By using the transformation estimates of the previous paragraph, one gets that $\sup_{B_r(x)} |\nabla v| \le r^\epsilon$ for all $r \ge 1$. Therefore, v has slightly faster than linear growth. Then, using that X^n is not $(k+1, \eta)$-symmetric one can prove a Liouville type theorem stating that the map, v, must be exactly linear from one of the factors. In that case, it is clear that after a transformation, v is precisely $(k, 0)$-symmetric on $B_1(x)$. Therefore, we get a contradiction. For the precise details, see Section 7.

The proof of the strong estimate in Theorem 5.6 is much more involved. One again uses a contradiction argument, but this time to prove a more refined estimate. Roughly, if $\ell_r : B_r(x) \to \mathbb{R}^k$ is the *best* k-splitting on $B_r(x)$, in the sense of the Sharp Splitting of Theorem 5.4, then one shows that $r^2 \fint_{B_r(x)} |\nabla^2(T_r u - \ell_r)|^2$ is decaying polynomially. This involves a careful analysis and blow up argument; for details, see Section 7.

5.4. *Nondegeneration theorem.* As was discussed, the weak estimate of Theorem 5.6 is sufficient to prove the bi-Hölder estimate, (3) of Theorem 5.3. Next we want to see that the strong estimate of Theorem 5.6 suffices to prove the bi-Lipschitz estimate, (2) of Theorem 5.3. However, this takes a bit more work since there are a couple of additional points to address. To accomplish this we want to show that at most points $x \in \mathcal{C}$, we have for any $r_x < r < 1$ that $|T_{x,r} - I| < \epsilon$. At such points, $u : B_r(x) \to \mathbb{R}^k$ remains a $(k, 2\epsilon)$-splitting on all scales, even without transformation. By using (5.4) as in the bi-Hölder estimate, we conclude that u is a bi-Lipschitz map at such points. It is worth noting that this estimate does *not* hold at all points. This can be seen from Example 3.1. Thus showing that it holds at most points is the best we can hope for.

To accomplish this, we introduce our Nondegeneration Theorem:

THEOREM 5.7 (Nondegeneration of k-Splittings). *Given $\epsilon, \eta, \alpha > 0$ there exists $\delta(n, \mathrm{v}, \eta, \alpha, \epsilon) > 0$ such that the following holds. Let $\delta < \delta(n, \mathrm{v}, \eta, \alpha, \epsilon)$ with (M^n, g, p) satisfying $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$, $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$. Let $u : B_2(p) \to \mathbb{R}^k$ denote a (k, δ)-splitting function. Assume for $B_r(x) \subseteq B_1(p)$ that*

(1) *$B_{\delta^{-1}s}(x)$ is (k, δ^2)-symmetric but $B_s(x)$ is not $(k+1, \eta)$-symmetric for all $r \leq s \leq 1$;*

(2) *$\sum_{r_j \geq r} \mathcal{E}^{k,\alpha}_{r_j}(p) < \delta$, where $r_j = 2^{-j}$.*

Then $u : B_s(x) \to \mathbb{R}^k$ is an ϵ-splitting function for every $r \leq s \leq 1$.

The proof of the above comes down to showing that the assumptions imply that $|T_{x,r} - I| < \epsilon$. It turns out that the implication

$$\sum_{r_j > r_x} \mathcal{E}^{k,\alpha}(x, r_j) < \delta \implies |T_{x,r} - I| < \epsilon \tag{5.5}$$

is fairly subtle. It is much easier to show $\sum_{r_j > r_x} \sqrt{\mathcal{E}^{k,\alpha}}(x, r_j) < \delta \implies |T_{x,r} - I| < \epsilon$. However, for our applications, the square gain is crucial. The square gain depends heavily on the fact that u is harmonic; it does not hold for a general (nonharmonic) splitting function. The proof of (5.5) depends on the more local estimate:

$$|T_{2r} \circ T_r^{-1} - I| < Cr^2 \fint_{B_{2r}(x)} |\nabla^2 T_{2r} u|^2 \leq C\, \mathcal{E}^k(x, 2r), \tag{5.6}$$

where as previously discussed, the last inequality is the main result of the Transformation Theorem 5.6.

Remark 5.8. The first inequality is where the square gain occurs. As above, if the right-hand side was the L^2-norm instead of the *squared* L^2-norm, the inequality would be much more standard and would follow from a typical telescope type argument. That one can control $T_{2r} \circ T_r^{-1}$ by the *squared* Hessian is a point very much special to harmonic functions. It is crucial to the whole paper.

The key point is the following monotonicity formula, which holds for any harmonic function:

$$\frac{d}{dt} \int \langle \nabla u_i, \nabla u_j \rangle \rho_t(x, dy) = 2 \int \big(\langle \nabla^2 u_i, \nabla^2 u_j \rangle + \mathrm{Ric}(\nabla u_i, \nabla u_j) \big) \cdot \rho_t(x, dy). \tag{5.7}$$

Roughly speaking, since ρ_t is a probability measure which is essentially supported on $B_{\sqrt{t}}(x)$, the left-hand side of (5.7) measures the rate of change of $T_{ij}(x, \sqrt{t})$. Given that we want to use this when $|\nabla u| \approx 1$ and $|\nabla^2 u| \approx 0$, we find that the left-hand side behaves as a linear quantity, while the right-hand side behaves as a quadratic quantity. This leads to a a crucial gain in the analysis. For additional details, see Section 8.

5.5. *Completing the outline proof of Theorem* 5.3. Completing the outline proof of Theorem 5.3 requires a brief discussion of why the assumption of (5.5) holds for most $x \in \mathcal{C}$.

Recall that in Theorem 5.3 we are assuming the Ahlfors regularity of (5.1), so a key point is that one has the estimate

$$\mathcal{E}^k(x,r) \leq C r^{-k} \int_{B_r(x)} |\mathcal{W}_{2r}(y) - \mathcal{W}_r(y)|\, d\mu. \tag{5.8}$$

This is because the Ahlfors regularity allows us to find $k+1$ independently spaced points, $x_0, \ldots, x_k$, for which the quantities $|\mathcal{W}_{2r}(x_j) - \mathcal{W}_r(x_j)|$ are all roughly the same as the average drop $r^{-k} \int_{B_r(x)} |\mathcal{W}_{2r}(y) - \mathcal{W}_r(y)|\, d\mu$.

To see this, recall from the definition of a neck region that for *every* $x \in \mathcal{C}$, we have

$$|\mathcal{W}_1(x) - \mathcal{W}_{r_x}(x)| = \sum_{r_j = 2^{-j} \geq r_x} |\mathcal{W}_{2r_j}(x) - \mathcal{W}_{r_j}(x)| < \delta. \tag{5.9}$$

Then one has

$$\begin{aligned}
&\int_{B_1(p)} \Big(\sum_{r_j = 2^{-j} > r_x} r_j^{-k} \int_{B_{r_j}(x)} |\mathcal{W}_{2r_j}(y) - \mathcal{W}_{r_j}(y)| d\mu(y) \Big)\, d\mu(x) \\
&= \int_{B_1} \int_{B_1} \sum_{r_j = 2^{-j} > r_x} r_j^{-k} |\mathcal{W}_{2r_j}(y) - \mathcal{W}_{r_j}(y)|\, 1_{B_{r_j}(x)}(y)\, d\mu(y) d\mu(x) \\
&\leq C \int_{B_1} \int_{B_1} \sum_{r_j = 2^{-j} > r_y} r_j^{-k} |\mathcal{W}_{2r_j} - \mathcal{W}_{r_j}|(y) 1_{B_{r_j}(y)}(x)\, d\mu(x) d\mu(y) \\
&\leq C \int_{B_1} \sum_{r_j = 2^{-j} > r_y} |\mathcal{W}_{2r_j}(y) - \mathcal{W}_{r_j}(y)| \cdot \frac{\mu(B_{r_j}(y))}{r_j^k}\, d\mu(y) \\
&\leq C\, B \int_{B_1} \sum_{r_j = 2^{-j} > r_y} |\mathcal{W}_{2r_j}(y) - \mathcal{W}_{r_j}(y)|\, d\mu(y) \\
&\leq C\, B \int_{B_1} |\mathcal{W}_2(y) - \mathcal{W}_{r_y}(y)|\, d\mu(y) \\
&\leq C\, B^2\, \delta.
\end{aligned} \tag{5.10}$$

It follows from this and (5.8) that *most* of $\mathcal{C}$ satisfies (5.5), as claimed.

6. Sharp cone-splitting

This is the first of three sections which constitute the third part of the paper.

In this section, we prove Theorem 6.1, the *sharp* existence theorem for ϵ-splitting functions. The hypothesis involves the k-pinching of the local pointwise entropy; see Definition 4.23. The key point, the one which presents the real difficulty in the proof, is the *linear bound* of the squared L^2-norm of the Hessian of the splitting function in terms of the entropy pinching. This is what we mean by *sharp*. The main argument is given in the proof of Proposition 6.4. As explained in Section 5, the form of the bound is crucial for the proof of the Transformation Theorem 7.2.

THEOREM 6.1 (Sharp Cone-Splitting). *Given $\epsilon, \alpha > 0$, there exist positive constants $\delta(n, \mathrm{v}, \alpha, \epsilon)$ and $C(n, \mathrm{v}, \alpha) > 0$ with the following properties. Let (M^n, g, p) satisfy $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2 r^2$ and $\mathrm{Vol}(B_{\delta^{-1}}(p)) > \mathrm{v}\delta^{-n} > 0$ with $r \leq 1$, and let $B_{4\delta^{-1}}(p)$ be (k, δ^2)-symmetric. Then there exists a (k, ϵ)-splitting map $u : B_2(p) \to \mathbb{R}^k$ satisfying*

$$\fint_{B_2(q)} \left(|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + 2(n-1)\delta^2 r^2 |\nabla u|^2\right) \leq C(n, \mathrm{v}, \alpha) \cdot \mathcal{E}_1^{(k,\alpha,\delta)}(q). \tag{6.1}$$

Remark 6.2 (Sharpness). The example of the 2-dimensional cone $C(S^1_\beta)$ shows that the estimate in Theorem 6.1 is actually sharp. In checking this, it is useful to employ Theorem 4.22, which states the equivalence between the volume pinching and the entropy pinching.

6.1. *Approximation of the squared radius with sharp Hessian estimates.* The first step in the proof of Theorem 6.1 is to construct a regularization h of the squared distance function d^2. As in [CC96], the function h will be taken to satisfy the Poisson equation $\Delta h = 2n$. Note that for the case of metric cones, we have precisely $h = d^2$, $\nabla^2 h = 2g$. We will obtain sharp Hessian bounds for h in terms of the entropy drop. The splitting map u will be constructed explicitly using functions h_i as above corresponding to independent approximate cone vertices; see Example 6.6 and (6.22). This will lead to an estimate on $\nabla^2 u$ in terms of the entropy pinching. Recall that the k-pinching $\mathcal{E}_r^{k,\alpha,\delta}(x)$ is defined to be the minimal entropy pinching over all (k, α)-independent points; see Definition 4.23.

THEOREM 6.3 (Sharp Poisson regularization of d^2). *Let (M^n, g, p) satisfy $\mathrm{Ric} \geq -(n-1)\delta^2$ with $\mathrm{Vol}(B_{\delta^{-1}}(p)) > \mathrm{v}\delta^{-n} > 0$. For any $\epsilon > 0$ and $B_r(x) \subseteq B_5(p)$, if $\delta \leq \delta(n, \mathrm{v}, \epsilon)$ is such that $B_{r\delta^{-1}}(x)$ is $(0, \delta^2)$-symmetric, then there exists a function $h : B_{2r}(x) \to \mathbb{R}$ such that*

(1) $\Delta h = 2n$;

(2) $\fint_{B_{2r}(x)} \Big(|\nabla^2 h - 2g|^2 + \mathrm{Ric}(\nabla h, \nabla h) + 2(n-1)\delta^2 \cdot |\nabla h|^2\Big)$
$\leq C(n, \mathrm{v}) \cdot |\mathcal{W}^\delta_{r^2}(x) - \mathcal{W}^\delta_{2r^2}(x)|$;

(3) $\fint_{B_{2r}(x)} \Big||\nabla h|^2 - 4h\Big|^2 \leq C(n,\mathrm{v}) r^4 \cdot |\mathcal{W}^\delta_{r^2}(x) - \mathcal{W}^\delta_{2r^2}(x)|$;

(4) $|\nabla h| \leq C(n,\mathrm{v}) r$;

(5) $\sup_{B_{2r}(x)} |h - d_x^2| \leq \epsilon r^2$.

Proof. Set $t = r^2$, and as usual write $\rho_t(x,y) = (4\pi t)^{-n/2} e^{-f}$ for the heat kernel. The Hessian estimates on h will follow from the Hessian estimates on the function $4tf$, which is in turn given by the local $\mathcal{W}$-entropy pinching. Thus, we will begin by deriving the relevant estimates on $\nabla^2 f$.

By Theorem 4.22, we have

(6.2)
$$2\int_t^{2t} s \int_{M^n} \Big(|\nabla^2 f - \frac{1}{2s} g|^2 + \mathrm{Ric}(\nabla f, \nabla f) + 2(n-1)\delta^2 |\nabla f|^2\Big)\, \varphi\, \rho_s(x, dy)$$
$$\leq |\mathcal{W}^\delta_t(x) - \mathcal{W}^\delta_{2t}(x)| := \eta.$$

Hence, there exists $t \leq s \leq 2t$ such that

(6.3)
$$2ts \int_M \Big(|\nabla^2 f - \frac{1}{2s} g|^2 + \mathrm{Ric}(\nabla f, \nabla f) + 2\delta^2(n-1)|\nabla f|^2\Big)\varphi \rho_s(x, dy) \leq \eta.$$

In particular,

$$\int_{M^n} \Big(|\nabla^2(4sf) - 2g|^2 + \mathrm{Ric}(\nabla(4sf), \nabla(4sf)) + 2\delta^2(n-1)|\nabla(4sf)|^2\Big)\varphi\rho_s(x,dy) \leq 8\eta. \tag{6.4}$$

By using the heat kernel lower bound estimates in Theorem 4.14 and the volume noncollapsing assumption, we get

$$\fint_{B_{5\sqrt{s}}(x)} \Big(|\nabla^2(4sf) - 2g|^2 + \mathrm{Ric}(\nabla(4sf), \nabla(4sf)) + 2(n-1)\delta^2|\nabla(4sf)|^2\Big) \leq C(n,\mathrm{v})\eta. \tag{6.5}$$

Set $\tilde f := 4sf$, and consider the 1-form

$$\begin{aligned} \nabla|\nabla \tilde f|^2 - 4\nabla \tilde f &= 2\nabla^2 \tilde f(\nabla \tilde f, \cdot\,) - 4\nabla \tilde f \\ &= 2(\nabla^2 \tilde f - 2g)(\nabla \tilde f, \cdot\,). \end{aligned} \tag{6.6}$$

By using the Poincaré inequality in Theorem 4.24 and the gradient estimate $|\nabla\tilde f|^2=16s^2|\nabla f|^2$, we get

$$
\begin{aligned}
&\fint_{B_{4\sqrt{s}}(x)}\Big||\nabla\tilde f|^2-4\tilde f-\fint_{B_{4\sqrt{s}}(x)}(|\nabla\tilde f|^2-4\tilde f)\Big|^2\\
&\qquad\le C(n)s\fint_{B_{4\sqrt{s}}(x)}|\nabla^2(4sf)-2g|^2|\nabla\tilde f|^2\le C(n,\mathrm{v})\eta s^2.
\end{aligned}\tag{6.7}
$$

Put

$$\hat f:=\tilde f+\frac14\fint_{B_{4\sqrt{s}}(x)}(|\nabla\tilde f|^2-4\tilde f).$$

Then

$$\fint_{B_{4\sqrt{s}}(x)}\Big||\nabla\hat f|^2-4\hat f\Big|^2\le C(n,\mathrm{v})\eta s^2,\tag{6.8}$$

$$\fint_{B_{4\sqrt{s}}(x)}\Big(|\nabla^2\hat f-2g|^2+\mathrm{Ric}(\nabla\hat f,\nabla\hat f)+2\delta^2(n-1)|\nabla\hat f|^2\Big)\le C(n,\mathrm{v})\eta.\tag{6.9}$$

We now define the function h to be the solution of the Poisson equation,

$$
\begin{aligned}
\Delta h&=2n\qquad(\text{on } B_{4\sqrt{s}}(x)),\\
h&=\hat f\qquad(\text{on } \partial B_{4\sqrt{s}}(x)).
\end{aligned}\tag{6.10}
$$

We will show that h satisfies the desired estimates.[6]

By integrating by parts, we have

$$
\begin{aligned}
&\fint_{B_{4\sqrt{s}}(x)}|\nabla h-\nabla\hat f|^2\\
&=\fint_{B_{4\sqrt{s}}(x)}(h-\hat f)(\Delta\hat f-2n)\\
&\le\Big(\fint_{B_{4\sqrt{s}}(x)}|h-\hat f|^2\Big)^{1/2}\cdot\Big(\fint_{B_{4\sqrt{s}}(x)}|\Delta\hat f-2n|^2\Big)^{1/2}\\
&\le C(n)\Big(\fint_{B_{4\sqrt{s}}(x)}|h-\hat f|^2\Big)^{1/2}\cdot\Big(\fint_{B_{4\sqrt{s}}(x)}|\nabla^2\hat f-2g|^2\Big)^{1/2}.
\end{aligned}\tag{6.11}
$$

Since $h-\hat f=0$ on $\partial B_{4\sqrt{s}}(x)$, by the Poincaré inequality in Theorem 4.24 we have

$$\fint_{B_{4\sqrt{s}}(x)}|h-\hat f|^2\le C(n)s\fint_{B_{4\sqrt{s}}(x)}|\nabla h-\nabla\hat f|^2.\tag{6.12}$$

[6]To be precise, here we might have to change the domain by an arbitrarily small amount such that the boundary is smooth and, in particular, satisfies an exterior sphere condition. This does not affect the argument which follows.

By combining (6.12) with (6.11), we get

$$s \fint_{B_{4\sqrt{s}}(x)} |\nabla h - \nabla \hat{f}|^2 + \fint_{B_{4\sqrt{s}}(x)} |h - \hat{f}|^2 \le C(n)s^2 \fint_{B_{4\sqrt{s}}(x)} |\nabla^2 \hat{f} - 2g|^2. \tag{6.13}$$

Choose a cutoff function ϕ as in (4.13) with support in $B_{4\sqrt{s}}(x)$ and $\phi \equiv 1$ in $B_{3\sqrt{s}}(x)$ such that $s|\Delta\phi| + s|\nabla\phi|^2 \le C(n)$. Then we have

$$\begin{aligned}
&\fint_{B_{4\sqrt{s}}(x)} |\Delta\phi||\nabla h - \nabla \hat{f}|^2 \\
&\qquad \ge \fint_{B_{4\sqrt{s}}(x)} \Delta\phi|\nabla h - \nabla \hat{f}|^2 \\
&\qquad \ge \fint_{B_{4\sqrt{s}}(x)} \phi\Delta|\nabla h - \nabla \hat{f}|^2 \\
&\qquad \ge \fint_{B_{4\sqrt{s}}(x)} 2\phi\Big(|\nabla^2 h - \nabla^2 \hat{f}|^2 + \mathrm{Ric}(\nabla(h-\hat{f}), \nabla(h-\hat{f})) \\
&\qquad\qquad + \langle \nabla(\Delta h - \Delta \hat{f}), \nabla(h - \hat{f})\rangle\Big).
\end{aligned} \tag{6.14}$$

Therefore we have

$$\begin{aligned}
&\fint_{B_{4\sqrt{s}}(x)} \phi\left(|\nabla^2 h - \nabla^2 \hat{f}|^2 + \mathrm{Ric}(\nabla(h-\hat{f}), \nabla(h-\hat{f}))\right) \\
&\quad \le \frac{1}{2}\fint_{B_{4\sqrt{s}}(x)} |\Delta\phi||\nabla h - \nabla \hat{f}|^2 - \fint_{B_{4\sqrt{s}}(x)} \phi\langle \nabla(\Delta h - \Delta \hat{f}), \nabla(h-\hat{f})\rangle \\
&\quad \le C(n)\left(\fint_{B_{4\sqrt{s}}(x)} |\Delta\phi| \cdot |\nabla h - \nabla \hat{f}|^2 + \fint_{B_{4\sqrt{s}}(x)} \phi|\Delta \hat{f} - 2n|^2 \right.\\
&\qquad\qquad \left. + \fint_{B_{4\sqrt{s}}(x)} |\Delta \hat{f} - 2n| \cdot |\nabla h - \nabla \hat{f}| \cdot |\nabla\phi| \right) \\
&\quad \le C(n)\left(\fint_{B_{4\sqrt{s}}(x)} (|\Delta\phi| + |\nabla\phi|^2) \cdot |\nabla h - \nabla \hat{f}|^2 \right.\\
&\qquad\qquad \left. + \fint_{B_{4\sqrt{s}}(x)} (\phi+1)|\Delta \hat{f} - 2n|^2 \right) \\
&\quad \le C(n)\fint_{B_{4\sqrt{s}}(x)} |\nabla^2 \hat{f} - 2g|^2,
\end{aligned} \tag{6.15}$$

where we have used (6.13) in the last inequality and $|\Delta\hat{f}-2n|^2\le n|\nabla^2\hat{f}-2g|^2$. By using (6.13) and $s\le 10^2$, we have

(6.16)
$$\begin{aligned}
&\fint_{B_{3\sqrt{s}}(x)}\Big(|\nabla^2 h-\nabla^2\hat{f}|^2+\mathrm{Ric}(\nabla(h-\hat{f}),\nabla(h-\hat{f}))+2\delta^2(n-1)|\nabla h-\nabla\hat{f}|^2\Big)\\
&\qquad\le\fint_{B_{4\sqrt{s}}(x)}\phi\Big(|\nabla^2 h-\nabla^2\hat{f}|^2+\mathrm{Ric}(\nabla(h-\hat{f}),\nabla(h-\hat{f}))\\
&\qquad\qquad\qquad+2\delta^2(n-1)|\nabla h-\nabla\hat{f}|^2\Big)\\
&\qquad\le C(n)\fint_{B_{4\sqrt{s}}(x)}|\nabla^2\hat{f}-2g|^2.
\end{aligned}$$

By the Schwarz inequality and (6.8) we get

(6.17)
$$\begin{aligned}
&\fint_{B_{3\sqrt{s}}(x)}|\nabla^2 h-2g|^2+\mathrm{Ric}(\nabla h,\nabla h)+2\delta^2(n-1)|\nabla h|^2\\
&\le 2\fint_{B_{3\sqrt{s}}(x)}\Big(|\nabla^2 h-\nabla^2\hat{f}|^2+\mathrm{Ric}(\nabla(h-\hat{f}),\nabla(h-\hat{f}))+2\delta^2(n-1)|\nabla h-\nabla\hat{f}|^2\Big)\\
&\quad+2\fint_{B_{3\sqrt{s}}(x)}\Big(|\nabla^2\hat{f}-2g|^2+\mathrm{Ric}(\nabla\hat{f},\nabla\hat{f})+2\delta^2(n-1)|\nabla\hat{f}|^2\Big)\\
&\le C(n,\mathrm{v})\eta.
\end{aligned}$$

This gives (2).

To see (4), note that $2t\ge s\ge t=r^2$ and

$$\fint_{B_{4\sqrt{s}}(x)}|\nabla h|^2\le 2\sup_{B_{4\sqrt{s}(x)}}|\nabla f|^2+2\fint_{B_{4\sqrt{s}}(x)}|\nabla h-\nabla\hat{f}|^2\le C(n,\mathrm{v})s. \tag{6.18}$$

From this, the gradient estimate on h in (4) follows by a standard Moser iteration argument.

To prove (3), since $2t\ge s\ge t=r^2$, we can use estimates for $\hat{f}$ in (6.8) and the gradient estimates $|\nabla h|+|\nabla\hat{f}|\le C(n,\mathrm{v})\sqrt{s}$ in $B_{3\sqrt{s}}(x)$. By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\fint_{B_{3\sqrt{s}}(x)} \Big||\nabla h|^2-4h\Big|^2 \\
&\quad\le C(n)\cdot\Bigg(\fint_{B_{3\sqrt{s}}(x)} \Big||\nabla \hat f|^2-4\hat f\Big|^2 \\
&\qquad\qquad + \fint_{B_{3\sqrt{s}}(x)} \Big||\nabla h|^2-|\nabla \hat f|^2\Big|^2+\fint_{B_{3\sqrt{s}}(x)}|h-\hat f|^2\Bigg) \\
&\quad\le C(n,\mathrm{v})\eta s^2+\fint_{B_{3\sqrt{s}}(x)}|\nabla h-\nabla \hat f|^2\cdot|\nabla h+\nabla\hat f|^2 \\
&\quad\le C(n,\mathrm{v})\eta s^2+C(n,\mathrm{v})s\fint_{B_{3\sqrt{s}}(x)}|\nabla h-\nabla\hat f|^2 \\
&\quad\le C(n,\mathrm{v})\eta s^2.
\end{aligned}
\tag{6.19}
$$

This gives (3).

To complete the proof, we need to show (5). First, by the gradient estimates of $h, \hat f, d_x^2$ and (6.13), if $\delta\le\delta(n,\mathrm{v},\epsilon)$, then we get $\sup_{B_{2r}(x)}|\hat f-h|\le \epsilon r^2/4$. To get (5) it suffices to prove $\sup_{B_{2r}(x)}|\hat f-d_x^2|\le\epsilon r^2/4$. For this, we will use the heat kernel convergence and the $W^{1,2}$-convergence of functions as in Proposition 4.15 and argue by contradiction.

By scaling, we can assume $r=1$ and $\mathrm{Ric}\ge-(n-1)\delta^2$. Therefore assume there exist $\epsilon_0>0$, $\delta_i\to 0$ and a sequences of (M_i,g_i,x_i) such that $\mathrm{Vol}(B_{\delta_i^{-1}}(x_i))\ge \mathrm{v}\delta_i^{-n}$, $\mathrm{Ric}\ge-(n-1)\delta_i^2\to 0$ and the ball $B_{\delta_i^{-1}}(x_i)$ is $(0,\delta_i^2)$-symmetric. However the function $\hat f_i$ defined as above satisfies

$$
\sup_{B_{10}(x_i)}|\hat f_i-d_{x_i}^2|\ge\epsilon_0/4. \tag{6.20}
$$

Now let $i\to\infty$. By Gromov's compactness theorem, there exists a metric cone, $(C(Y),d,x_\infty)$, which is the Gromov-Hausdorff-limit of (M_i,g_i,x_i). By the heat kernel convergence in Proposition 4.15 and Remark 4.17, the heat kernel $\rho_1(x_i,\cdot)=(4\pi)^{-n/2}e^{-f_i}$ converges to the heat kernel $\rho_1(x_\infty,\cdot)=(4\pi)^{-n/2}e^{-d_{x_\infty}^2/4+A_X}$ uniformly on any compact subset, where

$$
A_X=\log\frac{\mathrm{Vol}(S^{n-1})}{\mathrm{Vol}(X)}.
$$

From the heat kernel Laplacian estimate and the $W^{1,2}$-convergence in Proposition 4.29, it follows that the sequence f_i converges to $f_\infty=d_{x_\infty}^2/4-A_X$ uniformly and in the local $W^{1,2}$-sense. Thus, $\hat f_i$ converges uniformly to a limit function

$$
\tilde f_\infty:=4f_\infty+4\fint_{B_{10}(x_\infty)}(|\nabla f_\infty|^2-f_\infty)=d_{x_\infty}^2.
$$

Since $d^2_{x_i}$ converges to $d^2_{x_\infty}$ uniformly in any compact set, while $\sup_{B_{10}(x_i)}|\hat{f}_i - d^2_{x_i}| \geq \epsilon_0/4$ for any i, this gives a contradiction. This completes the proof of Theorem 6.3. □

6.2. *The k-splitting associated to k independent points.* In this subsection, we construct a k-splitting map from k-independent points which satisfy the estimates of the splitting Theorem 6.1. By rescaling and taking the infimum over all (k,α)-independent sets of points we will see that the proof of Theorem 6.1 is a direct consequence of the following main result, Proposition 6.4. The proof of this proposition will occupy the remainder of this section.

PROPOSITION 6.4. *Let (M^n, g, p) satisfy $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$ with*

$$\mathrm{Vol}(B_{\delta^{-1}}(p)) \geq \mathrm{v}\delta^{-n} > 0.$$

For $\epsilon, \alpha > 0$ and $\delta \leq \delta(n, \mathrm{v}, \alpha, \epsilon)$, let $\{x_0, x_1, \dots, x_k\} \subset B_r(x) \subset B_{10}(p)$ be (k,α)-independent points with

$$\mathcal{E}^{k,\delta}_r(\{x_i\}) := \sum_{i=0}^{k} |\mathcal{W}^\delta_{r^2/2}(x_i) - \mathcal{W}^\delta_{2r^2}(x_i)| < \delta.$$

Then there exist $C(n, \mathrm{v}, \alpha) > 0$ and a (k,ϵ)-splitting map $u = (u^1, \dots, u^k) : B_{8r}(x) \to \mathbb{R}^k$ such that

(1) $r^2 \fint_{B_{8r}(x)} \Big(|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + 2(n-1)\delta^2 |\nabla u|^2\Big) \leq C \cdot \mathcal{E}^k_r(\{x_i\})$;

(2) $\fint_{B_{8r}(x)} \Big|\langle \nabla u_i, \nabla u_j\rangle - \delta_{ij}\Big|^2 \leq C \cdot \mathcal{E}^k_r(\{x_i\})$;

(3) $|\nabla u| \leq 1 + \epsilon$.

Remark 6.5. For the estimate (3), we will only prove $|\nabla u| \leq C(n)$. Once we get $|\nabla u| \leq C(n)$, the argument in [CN15] will imply (3).

Before giving the proof of Proposition 6.4, let us look at the following example to see how to build a splitting function from squared distance functions to distinct vertices of a cone.

Example 6.6 (Cone-splitting; the case $\mathbb{R}^2 = C(S^1)$). Cone-splitting, and more specifically the relation between squared distance functions $h_\pm$ from distinct cone points and a splitting function u, is perhaps most easily illustrated by the case of $\mathbb{R}^2$. Denote the square of the distance functions from the points $(\pm 1, 0)$ by $h_\pm(x,y) = (x \pm 1)^2 + y^2$. Then the linear function (splitting function) $u = x$ satisfies

$$u = \frac{1}{4} \cdot (h_+ - h_-). \tag{6.21}$$

The expression in (6.21), which builds a linear splitting function from squared distance functions, will reappear in the general quantitative context in (6.22) in the proof of Proposition 6.4.

Proof of Proposition 6.4 . It follows from Theorem 6.3 that for any $\epsilon > 0$, there exists $\delta_0(n, \mathrm{v}, \epsilon)$ such that for $\delta \leq \delta_0$ and each point x_i, there is a map $h_i : B_{20r}(x_i) \to \mathbb{R}$ such that

(1) $\Delta h_i = 2n$;

(2) $⨍_{B_{20r}(x_i)} \Big(|\nabla^2 h_i - 2g|^2 + \mathrm{Ric}(\nabla h_i, \nabla h_i) + 2(n-1)\delta^2 \, |\nabla h_i|^2\Big)$
$\leq C(n, \mathrm{v})|\mathcal{W}^\delta_{r^2/2}(x_i) - \mathcal{W}^\delta_{2r^2}|(x_i)|$;

(3) $⨍_{B_{20r}(x_i)} \Big||\nabla h_i|^2 - 4h_i\Big|^2 \leq C(n, \mathrm{v})r^4|\mathcal{W}^\delta_{r^2/2}(x_i) - \mathcal{W}^\delta_{2r^2}(x_i)|$;

(4) $|\nabla h_i| \leq C(n, \mathrm{v})r$ on $B_{20r}(x_i)$;

(5) $\sup_{B_{20r}(x_i)} |h_i - d^2_{x_i}| \leq \epsilon r^2$.

Note that $B_{10r}(x) \subset B_{20r}(x_i)$. We define the k-splitting functions as in Example 6.6:

$$\tilde{u}^i := \frac{h_i - h_0 - d(x_0, x_i)^2}{2d(x_i, x_0)}. \tag{6.22}$$

Note that by (1), we have $\Delta \tilde{u}^i = 0$ in $B_{10r}(x)$. By the Cauchy-Schwartz inequality we also have:

(a) $r^2 ⨍_{B_{10r}(x)} \Big(|\nabla^2 \tilde{u}^i|^2 + \mathrm{Ric}(\nabla \tilde{u}^i, \nabla \tilde{u}^i) + 2(n-1)\, |\nabla \tilde{u}^i|^2\Big) \leq C(n, \mathrm{v}, \alpha) \cdot \mathcal{E}^k_r(\{x_i\})$;

(b) $\sup_{B_{10r}(x)} |\nabla \tilde{u}^i| \leq C(n, \mathrm{v}, \alpha)$;

(c) $\sup_{B_{10r}(x)} \left|\tilde{u}^i - \frac{d^2_{x_i} - d^2_{x_0} - d(x_0, x_i)^2}{2d(x_i, x_0)}\right| \leq C(\alpha, n) \cdot \epsilon r$.

LEMMA 6.7. *There exists a $k \times k$ lower triangle matrix A with $|A| \leq C(n, \mathrm{v}, \alpha)$ such that $u := (u^1, \ldots, u^k) := A(\tilde{u}^1, \ldots, \tilde{u}^k)$ satisfies*

$$⨍_{B_{8r}(x)} \langle \nabla u^i, \nabla u^j \rangle = \delta^{ij}.$$

Assume provisionally that the lemma holds. Then since $|A| \leq C(n, \mathrm{v}, \alpha)$, by using estimates (a) and (b) and the Poincaré inequality, it follows easily that u satisfies (1) and (2) of the proposition. Estimate (3) follows exactly as in [CN15]. Therefore, to complete the proof of Theorem 6.1 it suffices to prove Lemma 6.7.

Proof of Lemma 6.7. We will argue by contradiction. By rescaling $B_r(x)$ to $B_1(x)$ we can take $r = 1$. Then we can assume there exist $(M^n_\beta, g_\beta, x_\beta)$ and (k, α)-independent points $\{x_{\beta,0}, x_{\beta,1}, \ldots, x_{\beta,k}\} \subset B_1(x_\beta)$ with $\delta_\beta \to 0$ as $\beta \to \infty$. Also, for each β, we can construct regularized maps $h_{\beta,i}$ and harmonic functions $\tilde{u}_{\beta,i}$ on $B_{10}(x_\beta)$ as in (6.22), satisfying (a), (b) and (c), with $\epsilon_\beta \to 0$ in (c).

Now, assume that either there exists no $k\times k$ lower triangle matrix A_β such that

$$u_\beta := A_\beta(\tilde{u}^1_\beta, \dots, \tilde{u}^k_\beta)$$

satisfies

$$\fint_{B_8(x_\beta)} \langle \nabla u^i_\beta, \nabla u^j_\beta\rangle = \delta^{ij}$$

or, if there exist such matrices A_β, then $|A_\beta| \to \infty$.

By the definition of independent points and the Cone-Splitting Theorem 4.6, there is a Gromov-Hausdorff limit space of the sequence M^n_β which is a metric cone $\mathbb{R}^k \times C(X)$. Moreover, the set $\{x_{\beta,i}\}$ converges to a set of (k,α) independent points $\{x_{\infty,i}\} \subset \mathbb{R}^k \times \{v\}$ where v is the vertex of $C(X)$.

By (c) above, the $\tilde{u}^i_\beta$ converge to the linear functions

$$\tilde{u}^i_\infty = \frac{d^2_{x_{\infty,i}} - d^2_{x_{\infty,0}} - d(x_{\infty,0}, x_{\infty,i})^2}{2d(x_{\infty,i}, x_{\infty,0})}.$$

Recall that $\{x_{\infty,i}\} \subset \mathbb{R}^k \times \{v\}$ is a collection of (k,α)-independent points. Thus, the linear functions $\{\tilde{u}^i_\infty, i = 1, \dots, k\}$ form a basis of linear space of $\mathbb{R}^k$, and there exists a lower triangular matrix, A_∞ with $|A_\infty| \le C(n, \mathrm{v}, \alpha)$, such that

$$u_\infty := (u^1_\infty, \dots, u^k_\infty) := A_\infty(\tilde{u}^1_\infty, \dots, \tilde{u}^k_\infty)$$

satisfies

$$\fint_{B_8(x_\infty)} \langle \nabla u^i_\infty, \nabla u^j_\infty\rangle = \delta^{ij}.$$

For β large enough, the $W^{1,2}$-convergence of harmonic functions stated in Proposition 4.29 implies for some A_β with $|A_\beta - A_\infty| \to 0$, the set of functions

$$\hat{u}_\beta := (\hat{u}^1_\beta, \dots, \hat{u}^k_\beta) := A_\beta(\tilde{u}^1_\beta, \dots, \tilde{u}^k_\beta)$$

is orthogonal in the integral sense over $B_8(x_\beta)$, as in Lemma 6.7. This leads to a contradiction. This completes the proof of Lemma 6.7. □

As we have seen, this also completes the proof of Proposition 6.4 and hence, of Theorem 6.1. □

7. The Geometric Transformation Theorem

We begin with some motivation. The results of the last section specify how good the *best* splitting will be on a sufficiently entropy pinched ball. However, in the eventual application to the Neck Structure Theorem 2.9, the proof will depend on *fixing a single splitting map on the original ball* $B_2(p)$ and showing that it behaves sufficiently well on most smaller balls. Recall from Section 5.3 the following motivating example:

Example 7.1. If $u : B_2(0^n) \to \mathbb{R}^k$ is a k-splitting map in $\mathbb{R}^n$, then as with any solution of an elliptic PDE, u has *pointwise* bounds on the Hessian. Among other things, this implies that if we restrict to some sub-ball $B_r(x) \subseteq B_1(p)$, then $u : B_r(x) \to \mathbb{R}^k$ is still a splitting map. In fact, if $T^{-1} := \langle \nabla u^i, \nabla u^j \rangle$, then $T \circ u|_{B_r(x)}$ becomes an increasingly good splitting map, since u converges to a linear map at a polynomial rate.

As discussed in Section 5.3 we wish to generalize, to the extent possible, the above example to spaces with lower Ricci curvature bounds. In this case we cannot hope that $u|_{B_r(x)}$ remains a splitting map, but we will see that we can choose a matrix $T = T(x, r)$ such that $T \circ u|_{B_r(x)}$ is comparable to the best splitting map on $B_r(x)$, in the sense of the last section.

7.1. *Statement of the Geometric Transformation Theorem.* The result referred to in [CN15] as the Transformation Theorem is a key component of the proof of the Codimension 4 Conjecture in that paper. For given $\epsilon > 0$, the statement of the Transformation Theorem 7.2 concerns an $(n-2, \delta(\epsilon))$-splitting function $u : B_1(x) \to \mathbb{R}^{n-2}$. Namely, although the restriction of u to a smaller ball $B_r(x)$ might not be an $(n-2, \epsilon)$-splitting function, the Transformation Theorem 7.2 gives conditions guaranteeing the existence of a suitable upper triangular $(n-2) \times (n-2)$ matrix T, with positive diagonal entries, such that $Tu : B_r(x) \to \mathbb{R}^{n-2}$ is an $(n-2, \epsilon)$-splitting function. The conditions of [CN15] are special to the codimension two stratum.

In the present long and somewhat technical section, we show that *with a different hypothesis*, the conclusion of the Transformation Theorem of [CN15] can be sharpened. In particular, our conditions and criteria will hold for any stratum. Given a (k, ϵ)-splitting function u, we will see that so long as $B_r(x)$ remains k-symmetric, there is a transformed function Tu satisfying the Hessian estimates given by Theorem 6.1. More precisely, the main result of this section is the following.[7]

THEOREM 7.2 (Geometric Transformation). *Given $\alpha, \eta, \epsilon, \delta > 0$, there exists $C = C(n, \mathrm{v}, \eta, \alpha)$ and $\gamma = \gamma(n, \mathrm{v}, \eta) > 0, \delta(n, \mathrm{v}, \eta) > 0$, such that if $\delta < \delta(n, \mathrm{v}, \eta)$, then the following holds: Let (M^n, g, p) satisfy* $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$, $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$, *and assume*

(i) *$u : B_2(p) \to \mathbb{R}^k$ is a (k, δ)-splitting function;*
(ii) *for all $r \leq s \leq \delta^{-1}$, the ball $B_s(p)$ is (k, δ^2)-symmetric but not $(k+1, \eta)$-symmetric.*

[7] As usual, $\mathcal{E}^k_s(p) = \mathcal{E}^{k,\alpha,\delta}_s(p)$ denotes the k-pinching; see Definition 4.23.

Then for all $s \in [r, 1]$, there exists a $k \times k$-matrix $T = T_{p,s}$ such that
(1) *$Tu : B_s(p) \to \mathbb{R}^k$ is a (k, ϵ)-splitting map;*
(2) *for $r_j := 2^{-j}$,*

$$\begin{aligned} s^2 \fint_{B_s(p)} \big(|\nabla^2 Tu|^2 + \mathrm{Ric}(\nabla Tu, \nabla Tu) + 2\delta^2(n-1)|\nabla Tu|^2\big) \\ \le C \cdot \sum_{s \le r_j \le 1} \left(\frac{s}{r_j}\right)^{\gamma} \mathcal{E}^k_{r_j}(p) + C\,\delta^2. \end{aligned} \tag{7.1}$$

7.2. *Outline of the proof.* Essentially, the $k \times k$ matrix $T_{p,s}$ is obtained from gradients on the given scale. The point is to show that this procedure produces an ϵ-splitting as in (1) and, what is more challenging, that this splitting satisfies the sharp estimate in (2).

The statements (1) and (2) are both proved by contradiction arguments, in which the assumption that the conclusion fails is shown to lead to a statement about metric cones which, by explicit computation, can be shown to be false. Before giving a brief description of the arguments, we mention that there are three technical points which will have to be taken into account when the arguments are carried out.

The first technical point concerns our being able to pass the assumption that the conclusion of the theorem fails to a statement about limit cones. For this, we use $W^{1,2}$ convergence result in Proposition 4.29.

The second technical point pertains to checking that the resulting statement which concerns limit cones is actually false. At the formal level, one can do explicit calculations which employ separation of variables. If we could assume that the cross section Y^{n-1} of the limit cone $C(Y^{n-1})$ were smooth, then the relevant computations would be straightforward exercises, using that Y^{n-1} is a space with $\mathrm{Ric}_{Y^{n-1}} \ge (n-2)$. In our context, making this rigorous will take a fair amount of technical work.

The third technical point concerns the fact that the Hessian of the norm squared of a harmonic function need not be well defined on a limit cone. However, the Laplacian *is* well defined, and it will suffice to state all of our estimates on limit cones which correspond to Hessian estimates on manifolds in weak form using Bochner's formula (4.18).

The proof of conclusion (1) of Theorem 7.2 is similar to the proof of the Transformation Theorem of [CN15]. It is a quantitative implementation of the following fact. On a metric cone $\mathbb{R}^k \times C(Z)$, which is a definite amount away from splitting off $\mathbb{R}^{k+1}$, a harmonic function which is assumed to grow only slightly more than linearly must in fact, be linear and have linear growth. The reason is the following.

Consider a metric cone $C(Y)$ which is a Gromov-Hausdorff limit with the lower bound on Ricci going to zero. The Laplacian Δ_Y on the cross-section has a discrete spectrum and an orthonormal basis of eigenfunctions, ϕ_i, with corresponding eigenvalues $-\Delta_Y \phi_i = \lambda_i \phi_i$, satisfying $\lambda_0 = 0$, $\lambda_i \geq n-1$, for $i \geq 1$. It follows from (i) of Theorem 7.2 that in our case, $n-1 = \lambda_1 = \cdots = \lambda_k$ and from (ii) that there exists $\tau(n, \mathrm{v}, \eta) > 0$ such that $\lambda_{k+1} \geq (n-1)+\tau(n, \mathrm{v}, \eta)$; i.e., there is a definite gap in the spectrum.

Let $u(r,x)$ denote a harmonic function on $C(Y) = \mathbb{R}^k \times C(Z)$ which is normalized to satisfy $u(0,x) = 0$. Then there is an expansion in terms of homogeneous harmonic functions (see Proposition 7.4)

$$u(r,x) = \sum_i c_i r^{a_i} \cdot \phi_i(y), \tag{7.2}$$

where $a_0 = 0$, $a_1 = \cdots = a_k = 1$ and $a_{k+1} \geq 1+\theta(n, \mathrm{v}, \eta)$ for some $\theta(n, \mathrm{v}, \eta) > 0$. From this, it follows that in our case, a harmonic function on $\mathbb{R}^k \times C(Z)$ which grows only a bit more than linearly is actually linear. This is the fact about cones which enables us to prove (1) via a contradiction argument.

Although the idea behind the proof of (2) is equally simple, finding the right sharp quantitative estimate on cones is more subtle. Intuitively, in this case we consider the behavior of an arbitrary harmonic function, $u(r,z)$ as in (7.2). Note that as $r \to 0$, the nonlinear terms in the expansion decay faster than the linear terms. Thus $u(r,z)$ becomes increasingly linear as $r \to 0$. The technically precise version of this decay estimate on limit cones is given in (7.39) of Proposition 7.12. The corresponding decay estimate for manifolds is given in (7.50) of Proposition 7.15. The latter contains a pinching term on the right-hand side which compensates for the fact that we are not dealing with an actual metric cone. In particular, the best we can hope for in general is that $u|_{B_r(x)}$ looks increasingly like the "best" linear function on $B_r(x)$, in the sense of Theorem 6.1.

The remainder of this section can be viewed as consisting of five parts.

In Section 7.3, we derive the results on cones needed to prove (1) of Theorem 7.2. The section is essentially technical and routine.

In Section 7.4 we give the proof of (1).

In Section 7.5, which is brief, we digress to prove a Reifenberg theorem for which the map is canonical. The proof is an easy consequence of the arguments in Sections 7.3 and 7.4. While this result is not used elsewhere in the paper, it is of some interest in and of itself. Moreover, it provides motivation for the arguments which are used in Section 10 to prove rectifiability of the strata S^k for all k.

In Section 7.6 we state and prove the key decay estimate for cones, (7.39) of Proposition 7.12.

In Section 7.7 we prove the corresponding decay estimate (7.50) of Proposition 7.15.

In Sections 7.7.3 and 7.8, we complete the proof of (2) of Theorem 7.2.

7.3. *Harmonic functions and eigenvalue estimates on limit cones.* Let

$$(M_i^n, g_i, x_i) \xrightarrow{d_{GH}} (C(Y), d, x_\infty) = \mathbb{R}^k \times C(Z)$$

with $\mathrm{Ric}_{M_i^n} \geq -\delta_i \to 0$ and $\mathrm{Vol}(B_1(x_i)) \geq \mathrm{v} > 0$. As in Section 4.10 there exist Laplacians $\Delta_{C(Y)}$, Δ_Y on the cone and its cross-section.

The cross-section Y is an RCD space with positive Ricci curvature $\mathrm{Ric}_{Y^{n-1}} \geq n-2$; see [Ket15], [BS14]. Spectral results which hold for smooth spaces with this lower Ricci bound are known to hold for Y^{n-1}. In particular, the spectrum of Δ_Y is discrete; see Section 4.10 and Theorem 4.32. Denote the spectrum of Δ_Y by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ with an associated orthonormal basis of eigenfunctions $\phi_0 = \frac{1}{\sqrt{\mathrm{Vol}(Y)}}, \phi_1, \phi_2, \ldots$.

The main results of this subsection are Propositions 7.3 and 7.4.[8]

PROPOSITION 7.3 (Eigenvalue estimates on limit cone). *Let*

$$(M_i^n, g_i, x_i) \xrightarrow{d_{GH}} (X, d, x_\infty) = (C(Y), d, x_\infty) = (\mathbb{R}^k \times C(Z), d, x_\infty)$$

satisfy $\mathrm{Ric}_{M_i^n} \geq -\delta_i \to 0$ *and* $\mathrm{Vol}(B_1(x_i)) \geq \mathrm{v} > 0$. *If* $B_1(x_i)$ *is not* $(k+1, \eta)$-*symmetric, then*

$$0 = \lambda_0 < n-1 = \lambda_1 = \cdots = \lambda_k < \lambda_{k+1} \leq \lambda_{k+2} \leq \cdots . \tag{7.3}$$

Moreover, there exists $\tau(\eta, n, \mathrm{v}) > 0$ *such that*

$$\lambda_{k+1} > \lambda_k + \tau .$$

PROPOSITION 7.4. *Let* $(M_i^n, g_i, x_i) \xrightarrow{d_{GH}} (C(Y), d, x_\infty)$ *satisfy* $\mathrm{Ric}_{M_i^n} \geq -\delta_i \to 0$ *and* $\mathrm{Vol}(B_1(x_i)) \geq \mathrm{v} > 0$. *Then* $r^{\alpha_i}\phi_i$ *is harmonic where* $\lambda_i = \alpha_i(n-2+\alpha_i)$ *with* $\alpha_i \geq 0$ *and* $-\Delta_Y \phi_i = \lambda_i \phi_i$. *Moreover, any harmonic function* $u(r, Y) : B_1(x_\infty) \to \mathbb{R}$ *satisfies*[9]

$$u = \sum_{i=0}^{\infty} b_i r^{\alpha_i} \phi_i,$$

where the convergence is in the $W^{1,2}$-*sense on* $B_1(x_\infty)$.

Proof of Proposition 7.4. By Theorem 4.31 and Remark 4.33 the function $r^{\alpha_i}\phi_i$ is harmonic. So let us begin the proof of the second part of the proposition. Since u is bounded, in particular $u \in L^2(\partial B_1(x_\infty))$. Then we have the

[8] In the case in which the cross-section is smooth, the second of these results is derived from the first; see [Che79]. Under our assumptions, it will be convenient to derive the first from the second.

[9] We mention that on any RCD space, which includes this context, a harmonic function is automatically Lipschitz; see, for instance ,[AGS14a], [AGS14b]

expansion in $L^2(Y)$,

$$u(1,y) = \sum_{i=0}^{\infty} b_i \phi_i, \tag{7.4}$$

where $b_i = \int_Y \phi_i(y)u(1,y)$. On $B_1(x_\infty)$, define the function

$$v_k(r,y) := \sum_{i=0}^{k} b_i r^{\alpha_i}\phi_i(y).$$

Denote the limit in $L^2(B_1(x_\infty))$ as $k \to \infty$ of v_k by v. Since the operator Δ is closed, it follows that v is also harmonic. We have

$$v = \sum_{i=0}^{\infty} b_i r^{\alpha_i}\phi_i(y) \in L^2(B_1). \tag{7.5}$$

To finish the whole proof, we need to show the above convergence (7.5) is in the $W^{1,2}$-sense and $v = u$. Denote the annulus $A_{r,1}(x_\infty)$ by

$$A_{r,1}(x_\infty) := B_1(x_\infty) \setminus \bar{B}_r(x_\infty).$$

The following Lemma 7.5 will suffice to complete the proof of Proposition 7.4. The argument will be given after the proof of Lemma 7.5 is completed.

LEMMA 7.5. *With the notation above, we have $v_k \to v$ in $W^{1,2}(B_1(x_\infty))$ and*

$$\lim_{r\to 1} \frac{1}{(1-r)^2}\int_{A_{r,1}(x_\infty)} |v - u(1,y)|^2 = 0. \tag{7.6}$$

Proof. To begin with, we will show that v_k converges to v in $W^{1,2}(B_1(x_\infty))$. From the fact that u is Lipschitz it follows that $\sum_i b_i^2\lambda_i < \infty$. Namely,

$$\begin{aligned}
&\int_Y |\nabla u(1,y) - \nabla v_k(1,y)|^2 \\
&\quad = \int_Y |\nabla u(1,y)|^2 + \int_Y |\nabla v_k(1,y)|^2 - 2\int_Y \langle \nabla u(1,y), \nabla v_k(1,y)\rangle \\
&\quad = \int_Y |\nabla u(1,y)|^2 + \int_Y |\nabla v_k(1,y)|^2 + 2\int_Y u(1,y)\Delta v_k(1,y) \\
&\quad = \int_Y |\nabla u(1,y)|^2 - \sum_{i=0}^{k} \lambda_i b_i^2.
\end{aligned} \tag{7.7}$$

This implies

$$\sum_{i=0}^{\infty} \lambda_i b_i^2 \le \int_Y |\nabla u(1,y)|^2.$$

Since $\alpha_i^2 \le \lambda_i$, we have

(7.8)
$$\int_{B_1} |\nabla v_k(r,y)|^2 = \sum_{i=0}^k b_i^2 \int_{B_1} |\nabla (r^{\alpha_i}\phi_i)|^2 = \sum_{i=0}^k b_i^2 \frac{\lambda_i + \alpha_i^2}{n+2\alpha_i - 2} \le C(n) \sum_{i=0}^k \lambda_i b_i^2.$$

By applying the same computation to $v_k - v_\ell$ we get

$$\int_{B_1} |\nabla (v_k - v_\ell)(r,y)|^2 \le C(n) \sum_{i=\ell}^k \lambda_i b_i^2. \tag{7.9}$$

Therefore, $\{v_k\}$ is a Cauchy sequence in $W^{1,2}(B_1(x_\infty))$. Since the space $W^{1,2}(B_1(x_\infty)$ is complete, it follows that $v_k \to v \in W^{1,2}(B_1(x_\infty))$. This concludes the proof of the first part of Lemma 7.5.

To complete the proof of Lemma 7.5, we need to prove (7.6). We will begin by showing that $v = \sum_{i=0}^\infty b_i r^{\alpha_i}\phi_i(y)$ is also in $L^2(\partial B_r(x_\infty))$ for $0 < r < 1$.

Since $\sum_i b_i^2 < \infty$, it follows that $\{v_k(r,y)\}$ is a Cauchy sequence in $L^2(\partial B_r(x_\infty))$. Denote the limit of $v_k(r,y)$ in $L^2(\partial B_r(x_\infty))$ by $\tilde v(r,y)$. By Fubini's theorem we have that

(7.10)
$$\begin{aligned}\int_{B_1(x_\infty)} |v_k(r,y) - \tilde v(r,y)|^2 d\mathcal{H}^n &= \int_0^1 \int_Y |v_k(r,y) - \tilde v(r,y)|^2 dY\,dr\\ &= \int_0^1 \lim_{j\to\infty} \int_Y |v_k(r,y) - v_j(r,y)|^2 dY\,dr\\ &\le \int_0^1 \sum_{\ell=k}^\infty b_\ell^2 dr \le \sum_{\ell=k}^\infty b_\ell^2.\end{aligned}$$

Letting $k \to \infty$ we get

$$\int_{B_1(x_\infty)} |v(r,y) - \tilde v(r,y)|^2 d\mathcal{H}^n = 0.$$

In particular, this implies that $v(r,y) = \sum_{i=0}^\infty b_i r^{\alpha_i}\phi_i(y)$ is in $L^2(\partial B_r)$. By Fubini's theorem we can compute

$$\frac{1}{(1-r)^2} \int_{A_{r,1}(x_\infty)} |v(s,y) - u(1,y)|^2 = \frac{\int_r^1 s^{n-1} \int_Y |v(s,y) - u(1,y)|^2}{(1-r)^2}. \tag{7.11}$$

Since $v(r,y)$ is the L^2 limit of $v_k(r,y)$ on ∂B_r, we have

$$\begin{aligned}\frac{1}{(1-r)^2}\int_{A_{r,1}(x_\infty)}|v(r,y)-u(1,y)|^2 &= \frac{\int_r^1 s^{n-1}\sum_{i=0}^\infty b_i^2|s^{\alpha_i}-1|^2 ds}{(1-r)^2}\\ &\le \sum_{i=1}^\infty b_i^2\cdot\frac{(1-r^{\alpha_i})^2}{1-r}\\ &\le C(n)\sum_{i=1}^\infty(\alpha_i+1)^2 b_i^2(1-r),\end{aligned} \tag{7.12}$$

where we have used $\alpha_i \ge 0$ to deduce $(1-r^{\alpha_i}) \le (\alpha_i+1)(1-r)$. Since $\sum_{i=0}^\infty b_i^2\alpha_i^2 < \infty$, this implies $\lim_{r\to 1}\frac{1}{(1-r)^2}\int_{A_{r,1}(x_\infty)}|v-u(1,y)|^2 = 0$. This completes the proof of Lemma 7.5. □

Now, we can complete the proof of Proposition 7.4. For u, v, as in (7.4), (7.5), it suffices to prove that $u=v$. Since u is Lipschitz, Lemma 7.5 implies

$$\lim_{r\to 0}\frac{1}{(1-r)^2}\int_{A_{r,1}(x_\infty)}|v-u|^2 \to 0.$$

Choose a cutoff function φ_r with support in $B_1(x_\infty)$ and $\varphi_r := 1$ in $B_r(x_\infty)$ such that $|\nabla\varphi_r| \le C(n)/(1-r)$. Then

$$\begin{aligned}\int_{B_1(x_\infty)}|\nabla(u-v)|^2\varphi_r^2 &= -2\int_{B_1}(u-v)\varphi_r\langle\nabla(u-v),\nabla\varphi_r\rangle\\ &\le \frac{1}{2}\int_{B_1}|\nabla(u-v)|^2\varphi_r^2 + C(n)\int_{A_{r,1}}|u-v|^2|\nabla\varphi_r|^2.\end{aligned} \tag{7.13}$$

By letting $r\to 1$ we have that $\int_{B_1}|\nabla(u-v)|^2 = 0$, which implies $u-v$ is a constant. Moreover, since $\frac{1}{(1-r)^2}\int_{A_{r,1}(x_\infty)}|v-u|^2 \to 0$ as $r\to 1$, we have that $u=v$. This completes the proof of Proposition 7.4. □

Next we will prove Proposition 7.3. As explained at the beginning of this section, the idea is the following:

By Theorem 4.32, we know that $\lambda_0 = 0$ and $\lambda_1 \ge n-1$. Consider a harmonic function $u = r^{\alpha_i}\phi_i$ on $X = C(Y) = \mathbb{R}^k \times C(Z)$, where ϕ_i is an eigenfunction of Y with eigenvalue λ_i and $\alpha_i \ge 0$ satisfies $\lambda_i = \alpha_i(n-2+\alpha_i)$. If u is a linear function on the $\mathbb{R}^k$ component, then we have $\alpha_i = 1$, or equivalently $\lambda_i = n-1$. Therefore, we have $\lambda_0 = 0$ and $\lambda_1 = \lambda_2 = \cdots\lambda_k = n-1$. To finish the proof, we will need to show that

$$\lambda_{k+1} > n-1+\tau(n,\mathrm{v},\eta) > n-1.$$

Consider the harmonic function $u = r^{\alpha_{k+1}}\phi_{k+1}$ where $-\Delta\phi_{k+1} = \lambda_{k+1}\phi_{k+1}$. We will use a contradiction argument to show that $\alpha_{k+1} > 1+\alpha(n,\mathrm{v},\eta) > 1$, which implies $\lambda_{k+1} > \lambda_k + \tau(n,\mathrm{v},\eta)$. The moral is simple. We will show that if α_{k+1} is close to 1, then $u = r^{\alpha_{k+1}}\phi_{k+1}$ is close to a new linear splitting function. Then if α_{k+1} is too close to 1, this contradicts the assumption that $B_1(x_i)$ is not $(k+1,\eta)$-symmetric.

Proof of Proposition 7.3. Let $u = r^{\alpha_m}\phi_m$ denote a harmonic function in X. By scaling invariance, we have

$$t^{1-\alpha_m}\int |\nabla u|^2 \rho_t(x_\infty, dx) = s^{1-\alpha_m}\int |\nabla u|^2 \rho_s(x_\infty, dx). \tag{7.14}$$

LEMMA 7.6. *For any $\epsilon > 0$, we will show that if $|\alpha_m - 1| \le \delta \le \delta(n, \mathrm{v}, \eta, \epsilon)$, then there exist harmonic functions $u_i : B_1(x_i) \subset M_i \to \mathbb{R}$ converging in the $W^{1,2}$-sense* (*see Definition 4.27*) *to u with*

$$\fint_{B_1(x_i)} |\nabla^2 u_i|^2 \le \epsilon \quad \text{for } i \ge i(n, \mathrm{v}, \epsilon, \eta). \tag{7.15}$$

Let us assume Lemma 7.6 and finish the proof of Proposition 7.3.

Note that $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1 < \alpha_{k+1}$ for any $\epsilon > 0$, if $|\alpha_{k+1} - 1| \le \delta \le \delta(n, \mathrm{v}, \eta, \epsilon)$. Then by Lemma 7.6, we have $k+1$ harmonic functions $u_i^1, u_i^2, \ldots, u_i^{k+1} : B_1(x_i) \to \mathbb{R}$ which converge in the $W^{1,2}$-sense to $u^1 = x^1, u^2 = x^2, \ldots, u^k = x^k, u^{k+1} = r^{\alpha_{k+1}}\phi_{k+1}$. Here $x^1, \ldots, x^k$ are the coordinate functions of $\mathbb{R}^k \subset \mathbb{R}^k \times C(Z)$, and $u^1, u^2, \ldots, u^{k+1}$ are mutually perpendicular with respect to the inner product

$$(u, v) := \fint_{B_1(x_\infty)} \langle \nabla u, \nabla v\rangle \text{ for all } u, v \in W^{1,2}(B_1). \tag{7.16}$$

Moreover, since $(u^\ell, u^\ell) = 1$ for $\ell = 1, \ldots, k$ and $|(u^{k+1}, u^{k+1}) - 1| \le C(n)\delta$, and $u_i^\ell \to u^\ell$ in the $W^{1,2}$-sense, we have for $i \ge i(n, \mathrm{v}, \epsilon, \eta)$ that

$$\left|\fint_{B_1(x_i)} \langle \nabla u_i^a, \nabla u_i^b\rangle - \delta^{ab}\right| \le \epsilon, \text{ for all } a, b = 1, \ldots, k+1. \tag{7.17}$$

On the other hand, by Lemma 7.6 we have the Hessian estimate

$$\fint_{B_1(x_i)} |\nabla^2 u_i^a|^2 \le \epsilon. \tag{7.18}$$

It follows that the map

$$u := (u_i^1, \ldots, u_i^{k+1}) : B_1(x_i) \to \mathbb{R}^{k+1} \tag{7.19}$$

is a $(k+1, C(n)\epsilon)$-splitting map. If $\epsilon \le \epsilon(n, \mathrm{v}, \eta)$, this contradicts the assumption that $B_1(x_i)$ is not $(k+1,\eta)$-symmetric. This concludes the proof of Proposition 7.3 under the assumption that Lemma 7.6 holds.

Proof of Lemma 7.6. This proof requires the result on the heat kernel convergence of Proposition 4.15 and the harmonic function convergence in Lemma 4.30.

By (7.14), we have for $|\alpha_m - 1| \leq \delta$ that

$$\begin{aligned} \Big| \int |\nabla u|^2(x)\, \rho_1(x_\infty, x)dx - \int |\nabla u|^2(x)\, \rho_2(x_\infty, x)dx \Big| \\ \leq 2\delta \int |\nabla u|^2(x)\, \rho_2(x_\infty, x)dx. \end{aligned} \tag{7.20}$$

Since u has polynomial growth and the heat kernel ρ_t is exponentially decaying as in Theorem 4.14, we can choose a big $R \geq R(n, \mathrm{v}, \delta)$ and a cutoff function $\varphi = \psi(r^2)$, with support in B_R and $\varphi \equiv 1$ in $B_{R/2}$, such that $|\nabla\varphi|^2 + |\Delta\varphi| \leq C(n)R^{-2}$ and

$$\begin{aligned} \Big| \int_{B_R(x_\infty)} \varphi^2|\nabla u|^2(x)\rho_1(x_\infty, x)dx - \int_{B_R(x_\infty)} \varphi^2|\nabla u|^2(x)\rho_2(x_\infty, x)dx \Big| \\ \leq 4\delta \int_{B_R(x_\infty)} \varphi^2|\nabla u|^2(x)\rho_2(x_\infty, x)dx. \end{aligned} \tag{7.21}$$

By using Lemma 4.30 and Proposition 4.29, we can now construct a sequence of harmonic functions, $u_i : B_R(x_i) \to \mathbb{R}$, which converge in the $W^{1,2}$-sense to $u : B_R(x_\infty) \to \mathbb{R}$.

Let $\varphi = \psi(h_i)$ with $\Delta h_i = 2n$, where h_i approximates d^2 pointwise (see [Che01]).[10] By the heat kernel convergence in Proposition 4.15, for $i \geq i(n, \mathrm{v}, \delta)$, we have

$$\begin{aligned} \Big| \int_{B_R(x_i)} \varphi^2|\nabla u_i|^2(x)\rho_1(x_i, x)dx - \int_{B_R(x_i)} \varphi^2|\nabla u_i|^2(x)\rho_2(x_i, x)dx \Big| \\ \leq 8\delta \int_{B_R(x_i)} \varphi^2|\nabla u_i|^2(x)\rho_2(x_i, x)dx. \end{aligned} \tag{7.22}$$

Since ρ_t is the heat kernel, this gives

$$\Big| \int_1^2 \int \Delta(\varphi^2|\nabla u_i|^2)\rho_t(x_i, dx) \Big| \leq 8\delta \int_{B_R(x_i)} \varphi^2|\nabla u_i|^2(x)\rho_2(x_i, dx). \tag{7.23}$$

From Bochner's formula and the Schwartz inequality, we get

$$\int_1^2 \int \varphi^2|\nabla^2 u_i|^2\rho_t(x_i, dx)dt \leq C(\delta + R^{-2}) \int_{B_R(x_i)} |\nabla u_i|^2(x)\rho_2(x_i, dx). \tag{7.24}$$

[10] Note that we are not just applying Theorem 4.13 to produce a cutoff function but are specifying its construction. This is to ensure $\psi(h_i)$ converge to the cutoff function $\psi(r^2)$ in the limit space, which will be important.

By the mean value theorem and the heat kernel lower bound estimate Theorem 4.14, we have

(7.25)
$$\fint_{B_1(x_i)} |\nabla^2 u_i|^2 \leq C(n,\mathrm{v})(\delta + R^{-2}) \int_{B_R(x_i)} |\nabla u_i|^2 \rho_2(x_i, dx) \leq C(\delta + R^{-2}).$$

By fixing $R = R(\epsilon, n, \mathrm{v})$ we conclude that

$$\fint_{B_1(x_i)} |\nabla^2 u_i|^2 \leq \epsilon.$$

This completes the proof of Lemma 7.6. Hence, the proof of Proposition 7.3 is complete as well. □

7.4. *Part* (1) *of the Geometric Transformation Theorem.* In this subsection, we will prove estimate (1) of Theorem 7.2. We will see in subsequent subsections that the transformation T satisfies the vastly improved estimate (2). As we explained, the proof of (1) is based on a contradiction argument:

Proposition 7.7 (Transformation). *Let* (M^n, g, x) *satisfy* $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$ *and* $\mathrm{Vol}(B_1(x)) \geq \mathrm{v} > 0$. *Let* $\epsilon > 0$ *and* $\delta \leq \delta(n, \mathrm{v}, \eta, \epsilon)$. *Assume that*

(1) $B_s(x)$ *is* (k, δ^2)*-symmetric but not* $(k+1, \eta)$ *symmetric for each scale* $r_0 \leq s \leq 1$;
(2) $u : B_2(x) \to \mathbb{R}^k$ *is a* δ*-splitting map.*

Then for each scale $r_0 \leq s \leq 1$, *there exists a* $k \times k$ *lower triangle matrix* T_s *such that*

(1) $T_s u : B_s(x) \to \mathbb{R}^k$ *is a* (k, ϵ)*-splitting map on* $B_s(x)$;
(2) $\fint_{B_s(x)} \langle \nabla (T_s u)^a, \nabla (T_s u)^b \rangle = \delta^{ab}$;
(3) $|T_s \circ T_{2s}^{-1} - I| \leq \epsilon$.

The proof of Proposition 7.7 will rely on the eigenvalue estimate (7.3) of Proposition 7.3. The key point is that almost linear growth harmonic function on the limit cone must be linear. We begin with the following:

Lemma 7.8 (Harmonic function with almost linear growth). *Let*

$$(M^n_i, g_i, x_i) \to (C(Y), d, x_\infty) = (\mathbb{R}^k \times C(Z), d, x_\infty)$$

satisfy $\mathrm{Ric} \geq -\delta_i \to 0$ *and* $\mathrm{Vol}(B_1(x_i)) \geq \mathrm{v} > 0$. *Assume* $B_{10}(x_i)$ *is not* $(k+1, \eta)$*-symmetric. Then there exists* $\epsilon(n, \mathrm{v}, \eta) > 0$ *such that any harmonic function* u *on* $C(Y)$ *with almost linear growth* $|u|(y) \leq C d(y, x_\infty)^{1+\epsilon} + C$ *is a linear function induced from an* $\mathbb{R}$ *factor.*

Proof. To begin with, it follows from Proposition 7.4 that a harmonic function on $C(Y)$ has the form

$$u = \sum_{i=0}^{\infty} b_i \cdot r^{\alpha_i}\phi_i, \tag{7.26}$$

where the convergence is in $W^{1,2}$ on compact subsets.

By using the eigenvalue estimate in Proposition 7.3 and noting that $\alpha_0 = 0$, we have

$$1 = \alpha_1 = \cdots = \alpha_k < 1 + \beta(n, \eta, \mathrm{Vol}(Z)) \leq \alpha_{k+1}.$$

If we put $u_0 = u - \sum_{i=0}^{k} b_i r^{\alpha_i}\phi_i$, then we still have $|u_0|(y) \leq C d(y, x_\infty)^{1+\epsilon} + C$.

To finish the proof, it suffices to show that $\epsilon \leq \beta/2$ implies $u_0 = 0$. For this, we consider the L^2 integral of u_0 over $B_R(x_\infty)$. Since $r^{\alpha_i}\phi_i$ are orthogonal in $L^2(\partial B_r(x_\infty))$, for each r we have

$$\sum_{i=k+1}^{\infty} b_i^2 \mathrm{Vol}(Y)^{-1} \frac{n}{n+2\alpha_i} R^{2\alpha_i} = ⨍_{B_R(x_\infty)} |u_0|^2 \leq C + CR^{2+2\epsilon}. \tag{7.27}$$

Since R is arbitrary, it follows that $b_i = 0$ for all $i \geq k+1$ if $\epsilon \leq \beta/2$. Indeed, since $\alpha_i > 1+\epsilon$ for $i \geq k+1$ and $b_i^2 \mathrm{Vol}(Y)^{-1} \frac{n}{n+2\alpha_i} R^{2\alpha_i} \leq C + CR^{2+2\epsilon}$ for any R, we have $b_i = 0$ for $i \geq k+1$. This implies $u_0 = 0$, which completes the proof of Lemma 7.8. □

Proof of Proposition 7.7. We will argue by contradiction. Make the following assumptions:

- There exists $\epsilon_0 \ll 1$ and (M_i, g_i, x_i) such that $B_{\delta_i^{-1} r}(x_i)$ is (k, δ_i^2) splitting but $B_r(x_i)$ is not $(k+1, \eta)$-splitting for all $r_i \leq r \leq 1$. Let $u_i : B_2(x_i) \to \mathbb{R}^k$ be a (k, δ_i)-splitting map with $\delta_i \to 0$.
- There exists $s_i > r_i \to 0$ such that for all $1 \geq r \geq s_i$, there exists a lower triangle matrix $T_{x_i,r}$ such that $T_{x_i,r} u_i$ is a (k, ϵ_0) splitting on $B_r(x_i)$ with $⨍_{B_r(x_i)} \langle \nabla (T_{x_i,r} u)^a, \nabla (T_{x_i,r} u)^b \rangle = \delta^{ab}$.
- No such mapping $T_i = T_{x_i, s_i/10}$ exists on $B_{s_i/10}(x_i)$. (Note that since $\delta_i \to 0$, we have trivially that $s_i \to 0$.)

We will contradict the assumption that $s_i > r_i$.

To complete the proof of Proposition 7.7, we will need the following lemma. It states that as long as they exist, the transformation matrices, T_s, change slowly.

Let $|\cdot|$ denote the L^∞-norm on matrices.

LEMMA 7.9. *There exists $C(n)$ such that for all $1 \geq r \geq s_i$,*

$$|T_{x_i,r} \circ T_{x_i,2r}^{-1} - I| \leq C\sqrt{\epsilon_0}.$$

Proof. By volume doubling and noting that $T_{x_i,2r}u : B_{2r}(x_i) \to \mathbb{R}^k$ is (k,ϵ_0)-splitting, we have

$$\begin{aligned}(7.28)\qquad &\fint_{B_r(x_i)} \Big|\langle \nabla (T_{x_i,2r}u)^a, \nabla (T_{x_i,2r}u)^b\rangle - \delta^{ab}\Big| \\ &\quad \le C(n) \fint_{B_{2r}(x_i)} \Big|\langle \nabla (T_{x_i,2r}u)^a, \nabla (T_{x_i,2r}u)^b\rangle - \delta^{ab}\Big| \le C(n)\sqrt{\epsilon_0}.\end{aligned}$$

Thus, there exists a lower triangular matrix A_{2r} with $|A_{2r} - I| \le C(n)\sqrt{\epsilon_0}$ such that $\tilde{T}_{x_i,2r} := A_{2r}T_{x_i,2r}$ satisfies

$$\fint_{B_r(x_i)} \langle \nabla (\tilde{T}_{x_i,2r}u)^a, \nabla (\tilde{T}_{x_i,2r}u)^b\rangle = \delta^{ab}.$$

By the normalization, we have $\fint_{B_r(x_i)} \langle \nabla (T_{x_i,r}u)^a, \nabla (T_{x_i,r}u)^b\rangle = \delta^{ab}$.

Define a symmetric bilinear form $B(f,h)$, on $C^\infty(B_{2r}(x_i))$ by

$$B(f,h) := \fint_{B_r(x_i)} \langle \nabla f, \nabla h\rangle.$$

Denote the associated positive definite symmetric $k\times k$ matrix by $B := (B_{ab}) := (B(u^a,u^b))$. Thus, we have

$$T_{x_i,r} B T^*_{x_i,r} = I = \tilde{T}_{x_i,2r} B \tilde{T}^*_{x_i,2r}.$$

In particular,

$$T^{-1}_{x_i,r}(T^{-1}_{x_i,r})^* = B = \tilde{T}^{-1}_{x_i,2r}(\tilde{T}^{-1}_{x_i,2r})^*.$$

Since $T_{x_i,r}$ and $\tilde{T}_{x_i,2r}$ are lower triangle matrices with positive diagonal entries, the uniqueness of Cholesky decomposition (see [GVL96]) implies that $\tilde{T}^{-1}_{x_i,2r} = T^{-1}_{x_i,r}$. Therefore, we have $A_{2r}T_{x_i,2r} = T_{x_i,r}$. In particular,

$$|T_{x_i,r}T^{-1}_{x_i,2r} - I| = |A_{2r} - I| \le C(n)\sqrt{\epsilon_0}.$$

This completes the proof of Lemma 7.9. □

Now we can complete the proof of Proposition 7.7. For $k\times k$ matrices A_1, A_2 and the L^∞-norm for matrices, we have by a simple triangle inequality that

$$(7.29)\qquad |A_1A_2 - I| \le |A_1 - I| + |A_2 - I| + k|A_1 - I|\cdot|A_2 - I|.$$

By Lemma 7.9, (7.29) and an induction argument, we have

$$(7.30)\qquad |T^{-1}_{x_i,r}\circ T_{x_i,r/2^\ell} - I| \le \Big(1 + (k+1)C\sqrt{\epsilon_0}\Big)^\ell - 1.$$

Therefore

$$(7.31)\qquad |T^{-1}_{x_i,r}\circ T_{x_i,r/2^\ell}| \le \Big(1 + (k+1)C\sqrt{\epsilon_0}\Big)^\ell.$$

For simplicity we still denote $(k+1)C$ by C. Hence for all $r \geq s_i$, we have

$$|T_{x_i,r}^{-1} \circ T_{x_i,s_i}| \leq \left(\frac{r}{s_i}\right)^{\log(1+C\sqrt{\epsilon_0})/\log 2} \leq \left(\frac{r}{s_i}\right)^{C\sqrt{\epsilon_0}}. \tag{7.32}$$

Define $v_i = s_i^{-1} T_{x_i,s_i}(u_i - u_i(x_i))$ on the rescaled space $(M_i, s_i^{-2} g_i, x_i)$. Since $\delta_i \to 0$ and $B_{\delta_i^{-1} s_i}(x_i)$ is (k, δ_i^2)-symmetric, we know that $(M_i, s_i^{-2} g_i, x_i)$ converges to a cone $C(Y) = \mathbb{R}^k \times C(Z)$. By the Hölder growth estimate on $T_{x_i,r}^{-1}$ as in (7.32) and noting that $T_{x_i,r}(u_i - u_i(x_i))$ is a (k, ϵ_0) splitting map at scale r, for all x with $s_i^{-1} \geq d(x, x_i) = R > 1$, we have

$$|\nabla v_i(x)| \leq C \cdot R^{C\sqrt{\epsilon_0}} \implies |\nabla v_i(x)| \leq C \cdot d(x, x_i)^{C\sqrt{\epsilon_0}} + C.$$

Also, by Proposition 4.29, the sequence v_i converges in the local $W^{1,2}$-sense to a harmonic function v in $C(Y)$ with Hölder growth on the gradient, i.e., $|\nabla v|(x) \leq C R^{C\sqrt{\epsilon_0}}$ for $|x| \leq R$. Therefore, if the ϵ_0 is small as in Lemma 7.8, then we have that $v : C(Y) \to \mathbb{R}^k$ is actually linear. Moreover, by using the $W^{1,2}$ convergence in Proposition 4.29 and noting that the energy is quadratic, we have

$$⨍_{B_1(x_\infty)} \langle \nabla v^a, \nabla v^b \rangle = \delta^{ab}. \tag{7.33}$$

Hence $v = (v^1, \ldots, v^k)$ forms a basis of linear functions on $\mathbb{R}^k$. Without loss of generality we can assume $v = (x^1, \ldots, x^k)$ are the standard coordinates. By the $W^{1,2}$-convergence of v_i as in Propositions 4.29 and 4.28, we have

$$\begin{aligned}
&\lim_{i\to\infty} 4 ⨍_{B_1(x_i)} |\langle \nabla v_i^a, \nabla v_i^b \rangle - \delta^{ab}| \\
&\quad = \lim_{i\to\infty} ⨍_{B_1(x_i)} \Big| |\nabla v_i^a + \nabla v_i^b|^2 - |\nabla v_i^a - \nabla v_i^b|^2 - 4\delta^{ab} \Big| \\
&\quad = \lim_{i\to\infty} ⨍_{B_1(x_\infty)} \Big| |\nabla x^a + \nabla x^b|^2 - |\nabla x^a - \nabla x^b|^2 - 4\delta^{ab} \Big| \\
&\quad = 0.
\end{aligned} \tag{7.34}$$

Here we have used $|\nabla x^a + \nabla x^b|^2 = |\mathrm{Lip}(x^a + x^b)|^2 = 2 = |\mathrm{Lip}(x^a - x^b)|^2 = |\nabla x^a - \nabla x^b|^2$. Hence, v_i satisfies

$$\lim_{i\to\infty} ⨍_{B_1(x_i)} |\langle \nabla v_i^a, \nabla v_i^b \rangle - \delta^{ab}| = 0. \tag{7.35}$$

Thus, by Bochner's formula (4.18), the function v_i is a (k, ϵ_i)-splitting function on $B_1(x_i)$ with $\epsilon_i \to 0$. Hence for each $1/10 \leq r \leq 1$ and sufficiently large i we have a rotation $A_{r,i}$ such that $|A_{r,i} - I| \leq \epsilon_i$ and

$$⨍_{B_r(x_i)} \langle \nabla (A_{r,i} v_i)^a, \nabla (A_{r,i} v_i)^b \rangle = \delta^{ab}. \tag{7.36}$$

In particular, this implies for large i that $A_{r,i}v_i : B_r(x_i) \to \mathbb{R}^k$ is a $(k, \epsilon_0/100)$-splitting for $1/10 \le r \le 1$ and satisfies the orthonormal condition (2), which contradicts the existence of a minimal $s_i > r_i$. This finishes the proof of the existence of transformation matrices. By choosing δ small, the matrix estimate (3) comes from the transformation estimates in Lemma 7.9. This completes the proof of Proposition 7.7. $\square$

7.5. *A canonical Reifenberg theorem.* Prior to giving the proof of (2) of Theorem 7.2, we will make a brief digression to give a non-metric proof of the Reifenberg Theorem, which was first proved by Cheeger-Colding in [CC97]. Although this result is not used elsewhere in the paper, it seems to be of independent interest. There is also second reason for including it. Namely, it is a (much) easier instance of the sort of argument we will give when we eventually study the singular strata S^k; see Theorem 9.12.

THEOREM 7.10 (Canonical Reifenberg Theorem). *Let (M^n, g, p) satisfy $\mathrm{Ric}_{M^n} \ge -(n-1)\delta$ and $d_{GH}(B_4(p), B_4(0^n)) \le \delta$ with $0^n \in \mathbb{R}^n$. For any $\epsilon > 0$, if $\delta \le \delta(n, \epsilon)$, then there exists a harmonic map $u : B_1(p) \to \mathbb{R}^n$ such that*

(1) *for any $x, y \in B_1(p)$, we have*

$$(1-\epsilon)d(x,y)^{1+\epsilon} \le |u(x) - u(y)| \le (1+\epsilon)d(x,y);$$

(2) *for any $x \in B_1(p)$, we have that $du : T_xM \to \mathbb{R}^n$ is nondegenerate.*

In particular, u is a diffeomorphism which is uniformly bi-Hölder onto its image $u(B_1(p))$.

Remark 7.11. Consider $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X, d, p)$ with $d_{GH}(B_4(p), B_4(0^n)) \le \delta$, and a converging sequence of harmonic maps $u_i : B_1(p_i) \to \mathbb{R}^n$. Then by Theorem 7.10, we get that $B_1(p)$ is bi-Hölder to $\mathbb{R}^n$.

Proof of Theorem 7.10. Let $\delta' > 0$. By Theorem 4.3, if $\delta \le \delta(n, \delta')$, then every sub-ball $B_r(x) \subset B_{15/4}(p)$ is (n, δ')-symmetric. Moreover, there exists a δ'-splitting map $u : B_3(p) \to \mathbb{R}^n$.

By the Transformation Proposition 7.7, for any $\epsilon' > 0$, $x \in B_3(p)$ and $r \le 1/2$, if $\delta' \le \delta'(\epsilon', n)$, then there exists an $n \times n$ lower triangle matrix $T_{x,r}$, such that $T_{x,r}u : B_r(x) \to \mathbb{R}^n$ is an ϵ'-splitting map. Moreover, by the transformation estimate (3), $|T_{x,r}| \le r^{-\epsilon'}$. We will see that these estimates imply Theorem 7.10.

First, we will prove a Hölder estimate on u. Let $x, y \in B_{3/2}$ with $d(x,y) = r$. Since $T_{x,r}u : B_r(x) \to \mathbb{R}^n$ is an ϵ'-splitting map and, in particular, $T_{x,r}u$ is an ϵr-GH map if $\epsilon' \le \epsilon'(\epsilon, n)$, we have

$$|T_{x,r}u(x) - T_{x,r}u(y)| \ge (1-\epsilon)d(x,y). \tag{7.37}$$

By the matrix growth estimate $|T_{x,r}| \le r^{-\epsilon'}$ we then have

$$|u(x) - u(y)| \ge (1-\epsilon)d(x,y)^{1+\epsilon} \qquad \text{for } d(x,y) = r.$$

Since r is arbitrary, by using the gradient bound $|\nabla u| \le 1 + \delta'$ for splitting maps u, we conclude for that any $x, y \in B_{3/2}(p)$,

$$(1-\epsilon)d(x,y)^{1+\epsilon} \le |u(x) - u(y)| \le (1+\epsilon)d(x,y). \tag{7.38}$$

Therefore u is an injective map. In particular, this implies u is bi-Hölder to its image.

Next we show that $du : T_xM \to \mathbb{R}^n$ is nondegenerate, from which it follows that u is a diffeomorphism. Essentially, this is because $du(x) = T_{x,0}^{-1}$. In more detail, let $2r = r_h(x)$ be the harmonic radius at x; see Definition 4.34. Then by smooth elliptic estimates the splitting map $T_{x,r}u$ satisfies the pointwise bound $|\langle \nabla T_{x,r}u^a, \nabla T_{x,r}u^b\rangle - \delta^{ab}| < \epsilon$. In particular, this gives that $\det(du)(x) \ne 0$, as claimed. □

7.6. *Hessian decay estimates on limit cones.* The main result of this subsection is Proposition 7.12, the key Hessian decay estimate for harmonic functions on limit cones. In the next subsection, it will be promoted to the Hessian decay estimate on manifolds, and after that, to statement (2) of Theorem 7.2. Since a priori we cannot define the Hessian directly, we employ Bochner's formula (4.18). This will allow us to work with a weak version.

Notation. Let $\varphi : \mathbb{R} \to \mathbb{R}$ denote a smooth cutoff function such that $\varphi \equiv 1$ if $r \le 1$ and $\varphi \equiv 0$ if $r \ge 2$. In Proposition 7.12, we will consider a limit cone $(C(Y), d, x_\infty)$. We put $r = d(x, x_\infty)$ and $\psi_s(x) = \varphi(r^2/s^2)$.

PROPOSITION 7.12 (Main decay estimate for cones). *There exists $\beta = \beta(n, \eta, \mathrm{v}) > 0$ with the following property. Let $(M_i^n, g_i, x_i) \to (C(Y), d, x_\infty) = (\mathbb{R}^k \times C(Y), d, x_\infty)$ satisfy $\mathrm{Ric}_{M_i^n} \ge -\delta_i \to 0$ and $\mathrm{Vol}(B_1(x_i)) \ge \mathrm{v} > 0$. Let $u : B_{10}(x_\infty) \subset C(Y) \to \mathbb{R}$ be a harmonic function, and assume $B_{10}(x_\infty)$ is not $(k+1, \eta)$-symmetric. Then for all $0 < s \le t \le 2$,*

$$s^{2-n} \int_{\mathbb{R}^k \times C(Y)} |\nabla u|^2 \Delta \psi_s \le \left(\frac{t}{s}\right)^{-\beta} t^{2-n} \int_{\mathbb{R}^k \times C(Y)} |\nabla u|^2 \Delta \psi_t. \tag{7.39}$$

The proof of Proposition 7.12 is given at the end of this subsection. Ultimately, it is a consequence of the eigenvalue estimates in Section 7.3. We will begin with some preliminary computations.

According to Proposition 7.4, any harmonic function u can be written as $u = \sum b_i r^{\alpha_i} \phi_i$, where the convergence is in the $W^{1,2}$-sense. By Theorem 4.31, we have

(7.40)
$$|\nabla\phi_i|^2(r,y) = |\mathrm{Lip}\phi_i|^2(r,y) = r^{-2}|\mathrm{Lip}\phi_i|^2(y) = r^{-2}|\nabla\phi_i|^2_Y,$$
$$|\nabla u|^2 = \sum_{i,j} b_ib_j\alpha_i\alpha_j r^{\alpha_i+\alpha_j-2}\phi_i\phi_j + \sum_{i,j} b_ib_j r^{\alpha_i+\alpha_j-2}\langle\nabla\phi_i,\nabla\phi_j\rangle_Y.$$

Let $\varphi:\mathbb{R}\to[0,1]$ be a standard cutoff function such that the function $\psi: B_{10}(x_\infty)\to[0,1]$ defined by $\psi(x)=\varphi(d^2(x,x_\infty))$ satisfies $\operatorname{supp}\psi\subset B_{10}(x_\infty)$. Then

$$\Delta\psi = \varphi'(r^2)\Delta r^2 + \varphi''(r^2)|\nabla r^2|^2 = 2n\varphi'(r^2) + 4r^2\varphi''(r^2). \tag{7.41}$$

In particular, $|\Delta\psi|\le C(n)$.

LEMMA 7.13. *Let* $(M_i^n, g_i, x_i)\to(C(Y),d,x_\infty)=(\mathbb{R}^k\times C(Z),d,x_\infty)$ *satisfy* $\mathrm{Ric}_{M_i^n}\ge-\delta_i\to 0$ *and* $\mathrm{Vol}(B_1(x_i))\ge \mathrm{v}>0$. *Assume* $u=\sum b_ir^{\alpha_i}\phi_i$ *is a harmonic function on* $B_{10}(x_\infty)\subset C(Y)$ *where the convergence is in the* $W^{1,2}$*-sense. Then*

(7.42)
$$\int_{C(Y)}|\nabla u|^2\Delta\psi$$
$$= \sum_{\alpha_i>1}\Big(b_i^2\alpha_i^2+b_i^2\lambda_i\Big)(2\alpha_i-2)(n+2\alpha_i-4)\int_0^\infty \varphi(r^2)r^{n+2\alpha_i-5}dr.$$

Proof. Consider $u_\ell := \sum_{i=0}^{\ell} b_ir^{\alpha_i}\phi_i$. By Proposition 7.4, u_ℓ converges in the $W^{1,2}$-sense to u. Also, since $|\Delta\psi|\le C(n)$, we have

$$\int|\nabla u|^2\Delta\psi = \lim_{\ell\to\infty}\int|\nabla u_\ell|^2\Delta\psi. \tag{7.43}$$

It now suffices to compute $\int|\nabla u_\ell|^2\Delta\psi$. We have

(7.44)
$$\begin{aligned}\int|\nabla u_\ell|^2\Delta\psi &= \int_0^\infty r^{n-1}\int_Y|\nabla u_\ell|^2\Delta\psi d\mu_Y dr\\ &= \int_0^\infty r^{n-1}\Big(2n\varphi'(r^2)+4r^2\varphi''(r^2)\Big)\int_Y|\nabla u_\ell|^2 d\mu_Y dr\\ &= \int_0^\infty r^{n-1}\Big(2n\varphi'(r^2)+4r^2\varphi''(r^2)\Big)\sum_{i=0}^{\ell}\Big(b_i^2\alpha_i^2+b_i^2\lambda_i\Big)r^{2\alpha_i-2}dr\\ &= \int_0^\infty\Big(2n\varphi'(r^2)+4r^2\varphi''(r^2)\Big)\sum_{i=0}^{\ell}\Big(b_i^2\alpha_i^2+b_i^2\lambda_i\Big)r^{n+2\alpha_i-3}dr.\end{aligned}$$

Since $\alpha_0 = \lambda_0 = 0$, we can integrate by parts to get

(7.45)

$$\begin{aligned}
\int |\nabla u_\ell|^2 \Delta\psi &= \int_0^\infty \sum_{i=1}^{\ell} \Big(b_i^2\alpha_i^2 + b_i^2\lambda_i\Big) 2n\varphi'(r^2) r^{n+2\alpha_i-3} dr \\
&\quad + \int_0^\infty \sum_{i=1}^{\ell} \Big(b_i^2\alpha_i^2 + b_i^2\lambda_i\Big) 4\varphi''(r^2) r^{n+2\alpha_i-1} dr \\
&= \int_0^\infty \sum_{i=1}^{\ell} \Big(b_i^2\alpha_i^2 + b_i^2\lambda_i\Big) 2n\varphi'(r^2) r^{n+2\alpha_i-3} dr \\
&\quad - \int_0^\infty \sum_{i=1}^{\ell} \Big(b_i^2\alpha_i^2 + b_i^2\lambda_i\Big) 2(n+2\alpha_i-2)\varphi'(r^2) r^{n+2\alpha_i-3} dr \\
&= \sum_{i=1}^{\ell} \Big(b_i^2\alpha_i^2 + b_i^2\lambda_i\Big)(2\alpha_i - 2) \int_0^\infty -2\varphi'(r^2) r^{n+2\alpha_i-3} dr \\
&= \sum_{\alpha_i>1}^{\ell} \Big(b_i^2\alpha_i^2 + b_i^2\lambda_i\Big)(2\alpha_i - 2) \int_0^\infty -2\varphi'(r^2) r^{n+2\alpha_i-3} dr \\
&= \sum_{\alpha_i>1}^{\ell} \Big(b_i^2\alpha_i^2 + b_i^2\lambda_i\Big)(2\alpha_i - 2)(n+2\alpha_i-4) \int_0^\infty \varphi(r^2) r^{n+2\alpha_i-5} dr.
\end{aligned}$$

In the last integration by parts, we have used the fact that $\alpha_i > 1$ and $n \geq 2$ to deduce that $\lim_{r\to 0} r^{n+2\alpha_i-4} = 0$. □

Now we can complete the proof of Proposition 7.12.

Proof of Proposition 7.12. Let $\varphi : \mathbb{R} \to [0,1]$ be such that $\varphi \equiv 1$ if $r \leq 1$, $\varphi \equiv 0$ if $r \geq 2$, and $|\varphi'| + |\varphi''| \leq 100$. For any scale $s \leq 1$, define $\psi_s(x) := \varphi_s(r^2) := \varphi(r^2/s^2)$ with $r = d(x, x_\infty)$. Thus ψ_s has support contained in $B_{2s}(x_\infty)$.

By Proposition 7.4 we can write the harmonic function $u = \sum b_i r^{\alpha_i}\phi_i$, where the convergence is $W^{1,2}$. Applying Lemma 7.13 gives

$$\begin{aligned}
\int |\nabla u|^2 \Delta\psi_s &= \sum_i \Big(b_i^2\alpha_i^2 + b_i^2\lambda_i\Big)(2\alpha_i - 2)(n+2\alpha_i-4) \\
&\qquad \cdot \int_0^\infty \varphi(r^2/s^2) r^{n+2\alpha_i-5} dr \\
&= \sum_{\alpha_i>1} \Big(b_i^2\alpha_i^2 + b_i^2\lambda_i\Big)(2\alpha_i - 2)(n+2\alpha_i-4) s^{n+2\alpha_i-4} \\
&\qquad \cdot \int_0^\infty \varphi(r^2) r^{n+2\alpha_i-5} dr.
\end{aligned} \tag{7.46}$$

Therefore, for any $0 < s \le t \le 2$, we have

$$\begin{aligned}s^{2-n}\int|\nabla u|^2\Delta\psi_s = \sum_{\alpha_i>1}\Big(b_i^2\alpha_i^2+b_i^2\lambda_i\Big)(2\alpha_i-2)(n+2\alpha_i-4)s^{2\alpha_i-2}\\ \cdot\int_0^\infty\varphi(r^2)r^{n+2\alpha_i-5}dr,\end{aligned}\tag{7.47}$$

$$\begin{aligned}t^{2-n}\int|\nabla u|^2\Delta\psi_t = \sum_{\alpha_i>1}\Big(b_i^2\alpha_i^2+b_i^2\lambda_i\Big)(2\alpha_i-2)(n+2\alpha_i-4)t^{2\alpha_i-2}\\ \cdot\int_0^\infty\varphi(r^2)r^{n+2\alpha_i-5}dr.\end{aligned}\tag{7.48}$$

By the eigenvalue estimates in Proposition 7.3 we have $\alpha_i > 1+\beta(n,\mathrm{v},\eta) > 1$ for $\alpha_i \neq 1$. Hence, each of the terms in the sums (7.47) and (7.48) are nonnegative. It follows that

(7.49)

$$\begin{aligned}s^{2\alpha_i-2}\cdot\Big(b_i^2\alpha_i^2+b_i^2\lambda_i\Big)\cdot(2\alpha_i-2)\cdot(n+2\alpha_i-4)\cdot\int_0^\infty\varphi(r^2)r^{n+2\alpha_i-5}dr\\ =\Big(\frac{t}{s}\Big)^{2-2\alpha_i}\cdot t^{2\alpha_i-2}\cdot\Big(b_i^2\alpha_i^2+b_i^2\lambda_i\Big)\cdot(2\alpha_i-2)\cdot(n+2\alpha_i-4)\\ \cdot\int_0^\infty\varphi(r^2)r^{n+2\alpha_i-5}dr\\ \le\Big(\frac{t}{s}\Big)^{-2\beta}\cdot t^{2\alpha_i-2}\cdot\Big(b_i^2\alpha_i^2+b_i^2\lambda_i\Big)\cdot(2\alpha_i-2)\cdot(n+2\alpha_i-4)\\ \cdot\int_0^\infty\varphi(r^2)r^{n+2\alpha_i-5}dr.\end{aligned}$$

This gives (7.39); i.e., the conclusion of Proposition 7.12:

$$s^{2-n}\int|\nabla u|^2\Delta\psi_s\le\Big(\frac{t}{s}\Big)^{-\beta}t^{2-n}\int|\nabla u|^2\Delta\psi_t. \qquad \square$$

7.7. *The Hessian decay estimate on manifolds.* In this subsection, we will prove Proposition 7.15, which is a Hessian decay estimate for splitting maps. As explained at the beginning of this section, the proof is obtained by showing that if the conclusion were to fail, then Proposition 7.12 would be contradicted. The proof of Proposition 7.15 will be given at the end of this subsection. It depends on the decay estimates in Sections 7.7.1 and 7.7.2.

Remark 7.14. The constant α in Proposition 7.15 below appears in Definition 4.23 of $\mathcal{E}_s^k(x) = \mathcal{E}_s^{k,\alpha,\delta}(x)$.

PROPOSITION 7.15. *Let* (M^n, g, x) *satisfy* $\mathrm{Ric}_{M^n} \ge -(n-1)\delta^2$, $\mathrm{Vol}(B_1(x)) \ge \mathrm{v} > 0$. *Let* $\eta, \alpha > 0$. *Let* $u : B_2(x) \to \mathbb{R}^k$ *be a* (k,δ) *splitting map. Assume*

(1) $B_{\delta^{-1}r}(x)$ *is* (k,δ^2)*-symmetric for all* $r_0 \le r \le 1$;

(2) *$B_r(x)$ is not $(k+1,\eta)$-symmetric for all $r_0 \le r \le 1$.*

Then for all, $\epsilon > 0$, there exists $\delta(n,\mathrm{v},\epsilon,\eta,\alpha) > 0$ such that the following holds. If $\delta \le \delta(n,\mathrm{v},\epsilon,\eta,\alpha)$, then there exists $0 < c(n,\mathrm{v},\eta) < 1$, $C(n,\mathrm{v}) > 0$ and a $k\times k$ lower triangular matrix T_r such that $T_r u : B_r(x) \to \mathbb{R}^k$ is a (k,ϵ)-splitting map. If $r_0 \le r \le 1$ with $cs/2 \le r \le cs$, then

$$\begin{aligned} &r^{2-n}\int_{B_r(x)}\Big(|\nabla^2 T_r u|^2 + \mathrm{Ric}(\nabla T_r u, \nabla T_r u) + 2\delta^2(n-1)|\nabla T_r u|^2\Big)\\ &\le \frac{1}{2}s^{2-n}\int_{B_s(x)}\Big(|\nabla^2 T_s u|^2 + \mathrm{Ric}(\nabla T_s u, \nabla T_s u)\\ &\qquad + 2\delta^2(n-1)\,|\nabla T_s u|^2\Big) + C\mathcal{E}^k_s(x). \end{aligned} \tag{7.50}$$

7.7.1. *The Hessian decay for general harmonic functions.* In this subsubsection, as an essential step in the proof of Proposition 7.15, we will prove a decay estimate for general harmonic functions. It states that after subtracting off the linear terms, the L^2 Hessian has Hölder decay. Before giving the result, we will need some terminology.

Notation. Let $v = (v^1,\dots,v^k) : B_{10}(x) \to \mathbb{R}^k$ be a (k,δ)-splitting map which was constructed in Theorem 6.1. For harmonic function $u : B_{10}(x) \to \mathbb{R}$, we define

$$\tilde{u} = u - \sum_{\ell=1}^{k} a_\ell v^\ell \tag{7.51}$$

by stipulating that the coefficients are chosen to minimize

$$⨍_{B_1(x)} |\nabla\tilde{u}|^2 = \min_{(b_\ell)\in\mathbb{R}^k} ⨍_{B_1(x)} |\nabla u - \sum_{\ell=1}^{k} b_\ell \nabla v^\ell|^2. \tag{7.52}$$

After having subtracted off the "linear" term we can prove the following decay estimate for the harmonic function $\tilde{u}$.

Lemma 7.16. *There exists $0 < c(n,\mathrm{v},\eta) < 1$ such that the following holds. Let $\delta < \delta(n,\mathrm{v},\eta)$, and let (M^n,g,x) satisfy $\mathrm{Ric}_{M^n} \ge -(n-1)\delta^2$ and $\mathrm{Vol}(B_1(x)) \ge \mathrm{v} > 0$. Assume $B_{\delta^{-1}}$ is (k,δ^2)-symmetric but that $B_1(x)$ is not $(k+1,\eta)$-symmetric. Then if $u : B_2(x) \to \mathbb{R}$ denotes a harmonic function with $\tilde{u}$ defined as in (7.51) and $c/2 \le r \le c$, the following holds*:

$$\begin{aligned} &r^{2-n}\int_{B_r(x)}\Big(|\nabla^2\tilde{u}|^2 + \mathrm{Ric}(\nabla\tilde{u},\nabla\tilde{u}) + 2\delta^2(n-1)|\nabla\tilde{u}|^2\Big)\\ &\qquad\le \frac{1}{4}\int_{B_1(x)}\Big(|\nabla^2\tilde{u}|^2 + \mathrm{Ric}(\nabla\tilde{u},\nabla\tilde{u}) + 2\delta^2(n-1)\,|\nabla\tilde{u}|^2\Big). \end{aligned} \tag{7.53}$$

Proof. The constant $c(n,\mathrm{v},\eta)$ will be fixed at the end of the proof. The existence of $\delta(n,\mathrm{v},\eta)>0$ will be shown by arguing by contradiction. Therefore, assume there exist $\delta_i\to 0$ and (M_i^n,g_i,x_i) with $\mathrm{Ric}_{M_i^n}\ge -(n-1)\delta_i^2$ and $\mathrm{Vol}(B_1(x_i))\ge \mathrm{v}>0$. Assume further that the ball $B_{\delta_i^{-1}}(x_i)$ is (k,δ_i^2)-symmetric, $B_1(x_i)$ is not $(k+1,\eta)$-symmetric, and $u_i: B_2(x_i)\to\mathbb{R}$ is a harmonic function with corresponding $\tilde u_i$ defined in (7.51) such that for some $c/2\le r\le c$,

$$
\begin{aligned}
&r^{2-n}\int_{B_r(x_i)}\Big(|\nabla^2\tilde u_i|^2+\mathrm{Ric}(\nabla\tilde u_i,\nabla\tilde u_i)+2\delta_i^2(n-1)|\nabla\tilde u_i|^2\Big)\\
&\qquad>\frac14\int_{B_1(x_i)}\Big(|\nabla^2\tilde u_i|^2+\mathrm{Ric}(\nabla\tilde u_i,\nabla\tilde u_i)+2\delta_i^2(n-1)\,|\nabla\tilde u_i|^2\Big).
\end{aligned}
\tag{7.54}
$$

Normalize $\tilde u_i$ such that $⨍_{B_1(x_i)}|\nabla\tilde u_i|^2=1$ and $⨍_{B_1(x_i)}\tilde u_i=0$. Then by the Poincaré inequality, we have

$$
⨍_{B_1(x_i)}\tilde u_i^2\le C(n).
\tag{7.55}
$$

By the definition of $\tilde u_i$, we have $⨍_{B_1(x_i)}\langle\nabla v_{i,\alpha},\nabla\tilde u_i\rangle=0$ for any $\alpha=1,\dots,k$ and that the $v_{i,\alpha}$ are the k splitting maps for $B_2(x_i)$. Since $B_1(x_i)$ is not $(k+1,\eta)$-symmetric, we have

$$
⨍_{B_1(x_i)}|\nabla^2\tilde u_i|^2\ge\eta'(n,\mathrm{v},\eta).
\tag{7.56}
$$

Choose a cutoff function φ_i as in Theorem 4.13 with $\varphi_i:=1$ on $B_{1/4}(x_i)$ and $\varphi_i:=0$ away from $B_{1/2}(x_i)$. By the Bochner formula we have

$$
\begin{aligned}
&\int_{B_{1/4}(x_i)}\Big(|\nabla^2\tilde u_i|^2+\mathrm{Ric}(\nabla\tilde u_i,\nabla\tilde u_i)+2\delta_i^2(n-1)|\nabla\tilde u_i|^2\Big)\\
&\quad\le\int\Big(|\nabla^2\tilde u_i|^2+\mathrm{Ric}(\nabla\tilde u_i,\nabla\tilde u_i)+2\delta_i^2(n-1)|\nabla\tilde u_i|^2\Big)\varphi_i\\
&\quad=\frac12\int\Big(\Delta|\nabla\tilde u_i|^2+4\delta_i^2(n-1)|\nabla\tilde u_i|^2\Big)\varphi_i\\
&\quad\le 2\delta_i^2(n-1)\int_{B_1(x_i)}|\nabla\tilde u_i|^2+\int_{B_1(x_i)}|\nabla\tilde u_i|^2|\Delta\varphi_i|\\
&\quad\le C(n)\int_{B_1(x_i)}|\nabla\tilde u_i|^2\le C(n).
\end{aligned}
\tag{7.57}
$$

Therefore, from (7.54), we get

$$\fint_{B_1(x_i)} |\nabla \tilde{u}_i|^2 = 1, \tag{7.58}$$

$$\fint_{B_1(x_i)} \tilde{u}_i^2 \leq C(n), \tag{7.59}$$

$$\int_{B_{1/4}(x_i)} \Big(|\nabla^2 \tilde{u}_i|^2 + \mathrm{Ric}(\nabla \tilde{u}_i, \nabla \tilde{u}_i) + 2\delta_i^2(n-1)|\nabla \tilde{u}_i|^2\Big) \leq C(n), \tag{7.60}$$

$$\frac{\eta'(n, \mathrm{v}, \eta)}{4} \leq r^{2-n} \int_{B_r(x_i)} \Big(|\nabla^2 \tilde{u}_i|^2 + \mathrm{Ric}(\nabla \tilde{u}_i, \nabla \tilde{u}_i) + 2\delta_i^2(n-1)\,|\nabla \tilde{u}_i|^2\Big) \quad \text{(for some } r \text{ with } c/2 \leq r \leq c). \tag{7.61}$$

To complete the contradiction argument, we will show that one can pass to the limit and get a contradiction to the decay estimate Proposition 7.12 in the limit cone.

Choose a cutoff function $\varphi : \mathbb{R} \to [0,1]$ such that $\varphi := 1$ if $t \leq 1$ and $\varphi := 0$ if $t \geq 2$, and $|\varphi'| + |\varphi''| \leq 100$. For any scale $c/2 \leq s \leq 1/8$, define $\psi_{s,i}(x) := \varphi(h_i/s^2)$, where $\Delta h_i = 2n$ such that h approximates $d(x_i, x)^2$ as in Theorem 6.3 or from [CC96]. Thus $\psi_{s,i}(x)$ has support contained in $B_{2s}(x_i) \subset B_{1/4}(x_i)$ and $\psi_{s,i} \equiv 1$ on $B_{s/2}(x_i)$. Moreover, by the gradient estimates for h_i, we have that $s^2|\Delta\psi_{s,i}| + s^2|\nabla\psi_{s,i}|^2 \leq C(n, \mathrm{v})$.

Consider the quantity

$$\begin{aligned} s^{2-n} \int |\nabla \tilde{u}_i|^2 \Delta\psi_{s,i} &= s^{2-n} \int \Delta|\nabla \tilde{u}_i|^2 \psi_{s,i} \\ &= s^{2-n} \int 2\Big(|\nabla^2 \tilde{u}_i|^2 + \mathrm{Ric}(\nabla \tilde{u}_i, \nabla \tilde{u}_i)\Big)\psi_{s,i}. \end{aligned} \tag{7.62}$$

For δ_i small enough, by using (7.58) we can conclude that

$$C(n)^{-1}\eta' \leq r^{2-n} \int |\nabla \tilde{u}_i|^2 \Delta\psi_{r,i} \qquad \text{for some } c/2 \leq r \leq c, \tag{7.63}$$

$$s^{2-n} \int |\nabla \tilde{u}_i|^2 \Delta\psi_{s,i} \leq C(n) \qquad \text{for all } 1/16 \leq s \leq 1/8. \tag{7.64}$$

By letting $i \to \infty$, we obtain a limit cone $(C(Y), d, x_\infty) = \mathbb{R}^k \times C(Z)$ and a harmonic function u in $B_1(x_\infty)$. Moreover, by Proposition 4.29, $\tilde{u}_i \to u$ in the $W^{1,2}$-sense on $B_{9/10}(x_\infty)$. By Proposition 4.29,

$$\Delta\psi_{s,i} = \varphi' \frac{2n}{s^2} + \varphi'' \frac{|\nabla h_i|^2}{s^4}.$$

Also, both uniformly and in $W^{1,2}$ we have

$$h_i \to d(x, x_\infty)^2 := d(x)^2.$$

On the limit cone, put $\psi_s(x) = \varphi(d(x)^2/s^2)$. Then by Proposition 4.29, for any $c/2 \leq s \leq 1/8$, we get

$$\begin{aligned}
&\lim_{i\to\infty} s^{2-n}\int |\nabla \tilde u_i|^2 \Delta\psi_{s,i} \\
&\quad = \lim_{i\to\infty} s^{2-n}\int |\nabla \tilde u_i|^2\Big(\varphi'(h_i/s^2)\frac{2n}{s^2} + \varphi''(h_i/s^2)\frac{|\nabla h_i|^2}{s^4}\Big) \\
&\quad = s^{2-n}\int |\nabla u|^2\Big(\varphi'(d(x)^2/s^2)\frac{2n}{s^2} + \varphi''(d(x)^2/s^2)\frac{|\nabla d(x)^2|^2}{s^4}\Big) \\
&\quad = s^{2-n}\int |\nabla u|^2\Delta\psi_s.
\end{aligned} \tag{7.65}$$

In particular, we have

$$C(n)^{-1}\eta'(n,\mathrm{v},\eta) \leq r^{2-n}\int |\nabla u|^2\Delta\psi_r \qquad \text{for some } c/2 \leq r \leq c, \tag{7.66}$$

$$s^{2-n}\int |\nabla u|^2\Delta\psi_s \leq C(n) \qquad \text{for all } 1/16 \leq s \leq 1/8. \tag{7.67}$$

Now we can fix the value of $c = c(n,\mathrm{v},\eta)$ by choosing $c = c(n,\mathrm{v},\eta) = \frac{1}{10}\left(\frac{\eta'}{C(n)^2}\right)^{1/\beta}$, where β is the constant in Proposition 7.12 and η', $C(n)$ are in (7.66). Then by the decay estimates in Proposition 7.12, we obtain a contradiction. In fact, applying Proposition 7.12 to $s = r \in [c/2, c]$ and $t = 1/8$ gives

$$\begin{aligned}
C(n)^{-1}\eta'(n,\mathrm{v},\eta) &\leq r^{2-n}\int |\nabla u|^2\Delta\psi_r \\
&\leq (8r)^\beta 8^{n-2}\int |\nabla u|^2\Delta\psi_{1/8} \leq C(n)(8c)^\beta,
\end{aligned} \tag{7.68}$$

which contradicts $c = \frac{1}{10}\left(\frac{\eta'}{C(n)^2}\right)^{1/\beta}$. This completes the proof of Lemma 7.16. □

7.7.2. *Hessian decay with k-Pinching.* In this subsubsection, by combining the sharp cone-splitting estimates of Theorem 6.1 with the Hessian decay estimate in Lemma 7.16, we will prove a decay estimate for harmonic functions which does not require that we subtract off the k-splitting map. For this, we need to include an error term which is measured by $\mathcal{E}^k_s(x)$. The main result is the following proposition.

PROPOSITION 7.17. *Let (M^n, g, x) satisfy $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$, $\mathrm{Vol}(B_1(x)) \geq \mathrm{v} > 0$, and let $\alpha, \eta > 0$. Assume $B_{\delta^{-1}s}$ is (k,δ^2)-symmetric but $B_s(x)$ is not $(k+1,\eta)$-symmetric for some fixed $s \leq 1$. Let $u : B_{2s}(x) \to \mathbb{R}$ be a harmonic function with $⨍_{B_s(x)} |\nabla u|^2 = 1$, and let $\delta \leq \delta(n,\mathrm{v},\eta,\alpha)$. Then there exist constants $0 < c(n,\mathrm{v},\eta) < 1$ and $C(n,\mathrm{v}) > 0$ such that for any $cs/2 \leq r \leq cs$,*

$$\begin{aligned}(7.69)\quad & r^{2-n}\int_{B_r(x)}\Big(|\nabla^2 u|^2+\mathrm{Ric}(\nabla u,\nabla u)+2\delta^2(n-1)|\nabla u|^2\Big)\\ &\le \frac{1}{3}s^{2-n}\int_{B_s(x)}\Big(|\nabla^2 u|^2+\mathrm{Ric}(\nabla u,\nabla u)+2\delta^2(n-1)|\nabla u|^2\Big)+C\mathcal{E}^k_s(x).\end{aligned}$$

Proof. By scaling, it suffices to prove the result for $s=1$. Let $\tilde u = u-\sum a_i v^i := u-u_k$ as in (7.51). By Lemma 7.16 for $\delta\le\delta_0(n,\mathrm{v},\eta)$ and $c(n,\mathrm{v},\eta)$ small, we have that for any $c/2\le r\le c$,

$$\begin{aligned}& r^{2-n}\int_{B_r(x)}\Big(|\nabla^2 \tilde u|^2+\mathrm{Ric}(\nabla \tilde u,\nabla \tilde u)+2\delta^2(n-1)|\nabla \tilde u|^2\Big)\\ &\quad\le \frac{1}{4}\int_{B_1(x)}\Big(|\nabla^2 \tilde u|^2+\mathrm{Ric}(\nabla \tilde u,\nabla \tilde u)+2\delta^2(n-1)|\nabla \tilde u|^2\Big).\end{aligned}\tag{7.70}$$

By using the Schwartz inequality on the nonnegative inner product $\mathrm{Ric}+(n-1)\delta^2 g$, we get

$$\begin{aligned}& r^{2-n}\int_{B_r(x)}\Big(|\nabla^2 u|^2+\mathrm{Ric}(\nabla u,\nabla u)+2\delta^2(n-1)|\nabla u|^2\Big)\\ &\le \frac{1001}{1000}r^{2-n}\int_{B_r(x)}\Big(|\nabla^2 \tilde u|^2+\mathrm{Ric}(\nabla \tilde u,\nabla \tilde u)+2\delta^2(n-1)|\nabla \tilde u|^2\Big)\\ &\quad + Cr^{2-n}\int_{B_r(x)}\Big(|\nabla^2 u_k|^2+\mathrm{Ric}(\nabla u_k,\nabla u_k)+2\delta^2(n-1)|\nabla u_k|^2\Big)\\ &\le \frac{1001}{1000}r^{2-n}\int_{B_r(x)}\Big(|\nabla^2 \tilde u|^2+\mathrm{Ric}(\nabla \tilde u,\nabla \tilde u)+2\delta^2(n-1)|\nabla \tilde u|^2\Big)\\ &\quad + Cr^{2-n}\int_{B_r(x)}\Big(|\nabla^2 v|^2+\mathrm{Ric}(\nabla v,\nabla v)+2\delta^2(n-1)|\nabla u_k|^2\Big),\end{aligned}\tag{7.71}$$

where we have used the fact that $|a_i|\le C(n)$ from the definition of $\tilde u$ in (7.51). Similarly, we have

$$\begin{aligned}& \int_{B_1(x)}\Big(|\nabla^2 \tilde u|^2+\mathrm{Ric}(\nabla \tilde u,\nabla \tilde u)+2\delta^2(n-1)|\nabla \tilde u|^2\Big)\\ &\quad\le \frac{1001}{1000}\int_{B_1(x)}\Big(|\nabla^2 u|^2+\mathrm{Ric}(\nabla u,\nabla u)+2\delta^2(n-1)|\nabla u|^2\Big)\\ &\quad\quad + C\int_{B_1(x)}\Big(|\nabla^2 u_k|^2+\mathrm{Ric}(\nabla u_k,\nabla u_k)+2\delta^2(n-1)|\nabla u_k|^2\Big)\\ &\quad\le \frac{1001}{1000}\int_{B_1(x)}\Big(|\nabla^2 u|^2+\mathrm{Ric}(\nabla u,\nabla u)+2\delta^2(n-1)|\nabla u|^2\Big)\\ &\quad\quad + C\int_{B_1(x)}\Big(|\nabla^2 v|^2+\mathrm{Ric}(\nabla v,\nabla v)+2\delta^2(n-1)|\nabla v|^2\Big).\end{aligned}\tag{7.72}$$

By combining the above with (7.70) we get

$$\begin{aligned}
&r^{2-n}\int_{B_r(x)}\Big(|\nabla^2 u|^2+\mathrm{Ric}(\nabla u,\nabla u)+2\delta^2(n-1)|\nabla u|^2\Big)\\
&\quad\le \frac{1}{3}\int_{B_1(x)}\Big(|\nabla^2 u|^2+\mathrm{Ric}(\nabla u,\nabla u)+2\delta^2(n-1)|\nabla u|^2\Big)\\
&\qquad+Cr^{2-n}\int_{B_r(x)}\Big(|\nabla^2 v|^2+\mathrm{Ric}(\nabla v,\nabla v)+2\delta^2(n-1)|\nabla v|^2\Big)\\
&\qquad+C\int_{B_1(x)}\Big(|\nabla^2 v|^2+\mathrm{Ric}(\nabla v,\nabla v)+2\delta^2(n-1)|\nabla v|^2\Big).
\end{aligned}\tag{7.73}$$

Since $r\ge c(n,\mathrm{v},\eta)>0$, we have

$$\begin{aligned}
&r^{2-n}\int_{B_r(x)}\Big(|\nabla^2 u|^2+\mathrm{Ric}(\nabla u,\nabla u)+2\delta^2(n-1)|\nabla u|^2\Big)\\
&\quad\le \frac{1}{3}\int_{B_1(x)}\Big(|\nabla^2 u|^2+\mathrm{Ric}(\nabla u,\nabla u)+2\delta^2(n-1)|\nabla u|^2\Big)\\
&\qquad+C\int_{B_1(x)}\Big(|\nabla^2 v|^2+\mathrm{Ric}(\nabla v,\nabla v)+2\delta^2(n-1)|\nabla v|^2\Big).
\end{aligned}\tag{7.74}$$

On the other hand, the Sharp Cone-splitting Theorem 6.1 gives

$$\int_{B_1(x)}\Big(|\nabla^2 v|^2+\mathrm{Ric}(\nabla v,\nabla v)+2\delta^2(n-1)|\nabla v|^2\Big)\le C(n,\mathrm{v},\alpha)\mathcal{E}^k_1(x).\tag{7.75}$$

Therefore,

$$\begin{aligned}
&r^{2-n}\int_{B_r(x)}\Big(|\nabla^2 u|^2+\mathrm{Ric}(\nabla u,\nabla u)+2\delta^2(n-1)|\nabla u|^2\Big)\\
&\le \frac{1}{3}\int_{B_1(x)}\Big(|\nabla^2 u|^2+\mathrm{Ric}(\nabla u,\nabla u)+2\delta^2(n-1)|\nabla u|^2\Big)+C\mathcal{E}^k_1(x).
\end{aligned}\tag{7.76}$$

This completes the proof of Proposition 7.17 □

7.7.3. *The proof of Proposition* 7.15. Let $\epsilon>0$ small be fixed later. By Proposition 7.7(1), which has been proven at this stage, if $\delta\le\delta(n,\mathrm{v},\eta,\epsilon)$, then for each $r_0\le r\le 1$, we have a $k\times k$ lower triangle matrix T_r such that $T_r u$ is a (k,ϵ)-splitting map on $B_r(x)$ with $|T_{r/2}\circ T_r^{-1}-I|\le\epsilon$. Applying Proposition 7.17 to $T_s u$, we get that for all $cs/2\le r\le cs$,

$$\begin{aligned}
&r^{2-n}\int_{B_r(x)}\Big(|\nabla^2 T_s u|^2+\mathrm{Ric}(\nabla T_s u,\nabla T_s u)+2\delta^2(n-1)|\nabla T_s u|^2\Big)\\
&\le \frac{1}{3}s^{2-n}\int_{B_s(x)}\Big(|\nabla^2 T_s u|^2+\mathrm{Ric}(\nabla T_s u,\nabla T_s u)+2\delta^2(n-1)|\nabla T_s u|^2\Big)\\
&\quad+C\mathcal{E}^{\alpha,\delta,k}_s(x).
\end{aligned}\tag{7.77}$$

Fix $\epsilon \le \epsilon(n, \mathrm{v}, \eta)$ such that $|T_r \circ T_s^{-1} - I| \le 10^{-10n}$. We have

$$
\begin{aligned}
& r^{2-n} \int_{B_r(x)} \Big(|\nabla^2 T_r u|^2 + \mathrm{Ric}(\nabla T_r u, \nabla T_r u) + 2\delta^2(n-1)|\nabla T_r u|^2\Big) \\
& \le \frac{3}{2} r^{2-n} \int_{B_r(x)} \Big(|\nabla^2 T_s u|^2 + \mathrm{Ric}(\nabla T_s u, \nabla T_s u) + 2\delta^2(n-1)|\nabla T_s u|^2\Big) \\
& \le \frac{1}{2} s^{2-n} \int_{B_s(x)} \Big(|\nabla^2 T_s u|^2 + \mathrm{Ric}(\nabla T_s u, \nabla T_s u) + 2\delta^2(n-1)|\nabla T_s u|^2\Big) \\
& \quad + C\mathcal{E}_s^k(x).
\end{aligned} \tag{7.78}
$$

This completes the proof of Proposition 7.15. $\square$

7.8. *Proof of the Geometric Transformation Theorem.* For any $0 < \delta' < \epsilon$, if $\delta \le \delta(n, \mathrm{v}, \eta, \delta')$, then by the Transformation Proposition 7.7 we have for each scale $r \le s \le 1$ a lower triangular matrix T_s such that $T_s u : B_s(x) \to \mathbb{R}^k$ is a (k, δ')-splitting map. In particular, $T_s u : B_s(x) \to \mathbb{R}^k$ is (k, ϵ)-splitting. Therefore, it suffices to estimate the Hessian of $T_s u$.

First we choose $\delta' \le \delta'(n, \mathrm{v}, \eta, \epsilon) < \epsilon$ small such that Proposition 7.15 holds. Therefore, by (1) of Proposition 7.15, for any $r \le s \le 1$, we have

$$
\begin{aligned}
& (cs)^{2-n} \int_{B_{cs}(x)} \Big(|\nabla^2 T_{cs} u|^2 + \mathrm{Ric}(\nabla T_{cs} u, \nabla T_{cs} u) + 2\delta^2(n-1)|\nabla T_{cs} u|^2\Big) \\
& \le \frac{1}{2} s^{2-n} \int_{B_s(x)} \Big(|\nabla^2 T_s u|^2 + \mathrm{Ric}(\nabla T_s u, \nabla T_s u) + 2\delta^2(n-1)|\nabla T_s u|^2\Big) + C\mathcal{E}_s^k(x) \\
& \le c^\gamma s^{2-n} \int_{B_s(x)} \Big(|\nabla^2 T_s u|^2 + \mathrm{Ric}(\nabla T_s u, \nabla T_s u) + 2\delta^2(n-1)|\nabla T_s u|^2\Big) + C\mathcal{E}_s^k(x),
\end{aligned} \tag{7.79}
$$

where we can take $c = 2^{-i_0}$ for some integer $i_0(n, \mathrm{v}, \eta)$ and $\gamma = i_0^{-1}$. Thus, for $s_\ell = c^\ell$, we have

$$
\begin{aligned}
& s_\ell^{2-n} \int_{B_{s_\ell}(x)} \Big(|\nabla^2 T_{s_\ell} u|^2 + \mathrm{Ric}(\nabla T_{s_\ell} u, \nabla T_{s_\ell} u) + 2\delta^2(n-1)|\nabla T_{s_\ell} u|^2\Big) \\
& \le \Big(\frac{s_0}{s_\ell}\Big)^{-\gamma} s_0^{2-n} \int_{B_{s_0}(x)} \Big(|\nabla^2 T_{s_0} u|^2 + \mathrm{Ric}(\nabla T_{s_0} u, \nabla T_{s_0} u) + 2\delta^2(n-1)|\nabla T_{s_0} u|^2\Big) \\
& \quad + C \sum_{j=0}^{\ell-1} \Big(\frac{s_{j+1}}{s_\ell}\Big)^{-\gamma} \mathcal{E}_{s_j}^k(x) \\
& \le C \sum_{j=0}^{\ell-1} c^{\gamma(\ell-j))} \Big(\mathcal{E}_{s_j}^k(x) + s_j^2\delta^2\Big) := \tilde{\mathcal{E}}_{s_\ell}^k(x),
\end{aligned} \tag{7.80}
$$

where in the last inequality we have used the fact that

$$s_0^{2-n}\int_{B_{s_0}(x)}\Big(|\nabla^2 T_{s_0}u|^2+\mathrm{Ric}(\nabla T_{s_0}u,\nabla T_{s_0}u)+2\delta^2(n-1)|\nabla T_{s_0}u|^2\Big)\le\delta^2\,.$$

For general $s>r$ with $c^{\ell+1}<s\le c^\ell$, we have

(7.81)

$$\begin{aligned}
&s^{2-n}\int_{B_s(x)}\Big(|\nabla^2 T_s u|^2+\mathrm{Ric}(\nabla T_s u,\nabla T_s u)+2\delta^2(n-1)|\nabla T_s u|^2\Big)\\
&\quad\le Cs^{2-n}\int_{B_s(x)}\Big(|\nabla^2 T_{s_\ell}u|^2+\mathrm{Ric}(\nabla T_{s_\ell}u,\nabla T_{s_\ell}u)+2\delta^2(n-1)|\nabla T_{s_\ell}u|^2\Big)\\
&\quad\le s_\ell^{2-n}\int_{B_{s_\ell}(x)}\Big(|\nabla^2 T_{s_\ell}u|^2+\mathrm{Ric}(\nabla T_{s_\ell}u,\nabla T_{s_\ell}u)+2\delta^2(n-1)|\nabla T_{s_\ell}u|^2\Big)\\
&\quad\le C\tilde{\mathcal{E}}^k_{s_\ell}(x),
\end{aligned}$$

where we use the estimate $|T_s\circ T_{s_\ell}^{-1}-I|\le\epsilon$ in the first inequality. This completes the proof of Theorem 7.2, the Geometric Transformation Theorem. □

8. Nondegeneration of k-Splittings

In this section we state and prove Theorem 8.1, which is our our main result for k-splitting maps $u: B_2(p)\to\mathbb{R}^k$. Theorem 8.1 is a crucial ingredient in the proof of Theorem 2.9.

Essentially Theorem 8.1 is obtained by combining the Sharp Cone-Splitting Theorem 6.1, the Transformation Theorem 7.2, Proposition 8.4, and a telescope estimate for harmonic functions which is based on a monotonicity property. This estimate is much sharper than the corresponding more general telescope estimate for $W^{1,p}$ functions. In the proof of Theorem 8.1, this is essential. It allows us to adequately control the sum over arbitrarily many scales of the Hessian estimates in Theorem 6.1 and Theorem 7.2.

Recall that $\mathcal{E}^{k,\alpha,\delta}$ is the entropy pinching defined in Definition 4.23.

THEOREM 8.1 (Nondegeneration of k-splittings). *Given $\epsilon,\eta,\alpha>0$ and $\delta<\delta(n,\mathrm{v},\eta,\alpha,\epsilon)$, we have the following. Let (M^n,g,p) satisfy* $\mathrm{Ric}_{M^n}\ge-(n-1)\delta^2$, $\mathrm{Vol}(B_1(p))>v>0$, *and let $u:B_2(p)\to\mathbb{R}^k$ denote a (k,δ)-splitting function. Assume*

(1) *$B_{\delta^{-1}s}(p)$, is (k,δ^2)-symmetric but $B_s(p)$ is not $(k+1,\eta)$-symmetric for all $r\le s\le 1$;*

(2) *$\sum_{r_j\ge r}\mathcal{E}^{k,\delta,\alpha}_{r_j}(p)<\delta$, where $r_j=2^{-j}$.*

Then $u:B_s(p)\to\mathbb{R}^k$ is an ϵ-splitting function for every $r\le s\le 1$.

From the Transformation Theorem 7.2, we know that for some lower triangular matrix $T_r=T(p,r)$, the composition $T_r u: B_r(p)\to\mathbb{R}^k$ is a δ-splitting function. Our goal then is to show that under the above hypotheses,

T_r remains close to the identity. Proposition 8.4 below provides suitable control of the difference $|T_r \circ T_{2r}^{-1} - I|$. From this the Nondegeneration Theorem 8.1 will easily follow.

8.1. *Hessian estimates with respect to the heat kernel density.* The purpose of this subsection is to prove some technical results which convert ball-average estimates on Tu into estimates with respect to the heat kernel measure, which is important due to our use of entropy as the monotone quantity.

Notation. Throughout this section φ denotes a cutoff function as in (4.13), with support in $B_1(x)$ with $\varphi \equiv 1$ on $B_{1/2}(x)$ and such that $|\Delta\varphi|+|\nabla\varphi|^2 \leq C(n)$.

The main result of this subsection is the following technical proposition.

PROPOSITION 8.2. *Given $\alpha, \eta > 0$ and $\epsilon > 0$ there exist $\delta \leq \delta(n, \mathrm{v}, \eta, \alpha, \epsilon)$, $\gamma = \gamma(n, \mathrm{v}, \eta) > 0$, $C(n, \mathrm{v}, \eta, \alpha)$, $C(n, \mathrm{v})$, with the following properties. Let (M^n, g, x) satisfy $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$, $\mathrm{Vol}(B_1(x)) \geq \mathrm{v} > 0$, and let $u : B_2(x) \to \mathbb{R}^k$ be a (k, δ)- splitting map. Assume*

$B_{\delta^{-1}s}(x)$ is (k, δ^2)-symmetric and $B_s(x)$ is not $(k+1, \eta)$-symmetric for all $r \leq s \leq 1$.

Then for each $r \leq s_i \leq 1$, there exists a $k \times k$ lower triangular matrix T_{s_i} such that $T_{s_i}u : B_{s_i}(x) \to \mathbb{R}^k$ is a (k, ϵ)-splitting map such that

$$\int_{M^n} \langle \nabla (T_{s_i}u)^a, \nabla (T_{s_i}u)^b \rangle \varphi^2 \rho_{s_i^2}(x, dy) = \delta^{ab}. \tag{8.1}$$

Additionally, there is the following Hessian estimate on $T_{s_i}u$:

$$\begin{aligned} s_i^2 \int_M \Big(|\nabla^2 T_{s_i}u|^2 + \mathrm{Ric}(\nabla T_{s_i}u, \nabla T_{s_i}u) + 2\delta^2(n-1)|\nabla T_{s_i}u|^2 \Big) \varphi^2 \rho_{4s_i^2}(x, dy) \\ \leq C(n, \mathrm{v}) \sum_{j=0}^{i} \epsilon_j 2^{j-i}, \end{aligned} \tag{8.2}$$

where

$$\epsilon_i = C(n, \mathrm{v}, \eta, \alpha) \cdot \sum_{j=0}^{i} 2^{-\gamma(i-j)} \Big(\mathcal{E}^k_{s_j}(x) + \delta s_j^2 \Big). \tag{8.3}$$

Proof of Proposition 8.2. Note that by Theorem 7.2, for any ϵ', if $\delta \leq \delta(n, \mathrm{v}, \eta, \epsilon')$, then there exists $\tilde{T}_{s_i}$ such that $\tilde{T}_{s_i} \circ u : B_{s_i}(x) \to \mathbb{R}^k$ is a (k, ϵ')-splitting map whose Hessian satisfies

$$\begin{aligned} &s_i^{2-n} \int_{B_{s_i}(x)} \Big(|\nabla^2 \tilde{T}_{s_i}u|^2 + \mathrm{Ric}(\nabla \tilde{T}_{s_i}u, \nabla \tilde{T}_{s_i}u) + 2\delta^2(n-1)|\nabla \tilde{T}_{s_i}u|^2 \Big) \\ &\quad \leq C(n, \mathrm{v}, \eta, \alpha) \sum_{j=0}^{i} 2^{-\gamma(i-j)} \Big(\mathcal{E}^k_{s_j}(x) + \delta s_j^2 \Big). \end{aligned} \tag{8.4}$$

Set

$$\epsilon_i := C(n,\mathrm{v},\eta,\alpha)\sum_{j=0}^{i} 2^{-\gamma(i-j)}\Big(\mathcal{E}^k_{s_j}(x)+\delta s_j^2\Big). \tag{8.5}$$

In order to make sure matrix $\tilde{T}_{s_i}$ satisfies (8.1), we may need to do a rotation as in the following Lemma 8.3. Then we can fix $\epsilon'=\epsilon'(n,\epsilon,v)$ so that $T_{s_i}u : B_{s_i}(x)\to\mathbb{R}^k$ is (k,ϵ)-splitting.

LEMMA 8.3. *For any $\epsilon>0$, if $\epsilon'\le\epsilon'(n,\epsilon,\mathrm{v})$ and $\delta\le\delta(n,\mathrm{v},\eta,\alpha,\epsilon)$, then there exists a lower triangle matrix A_i with $|A_i-I|\le C(n)\epsilon$ such that $T_{s_i}=A_i\circ\tilde{T}_{s_i}$ satisfies*

$$\int_{M^n}\langle\nabla(T_{s_i}u)^a,\nabla(T_{s_i}u)^b\rangle\varphi^2\rho_{s_i^2}(x,dy)=\delta^{ab}, \tag{8.6}$$

and $T_{s_i}u : B_{s_i}(x)\to\mathbb{R}^k$ is (k,ϵ)-splitting.

Proof. For any ϵ, by the exponential heat kernel decay estimate in Theorem 4.14 and the matrix estimate in Proposition 7.7, there exists $R(n,\mathrm{v},\epsilon)$ such that

$$\int_{B_1(x)\setminus B_{Rs_i}(x)}|\langle\nabla(\tilde{T}_{s_i}u)^a,\nabla(\tilde{T}_{s_i}u)^b\rangle-\delta^{ab}|\cdot\rho_{s_i^2}(x,dy)<\epsilon/2. \tag{8.7}$$

Also, by Proposition 7.7, for any $\epsilon'>0$, if $\delta\le\delta(\epsilon',n,\mathrm{v},\eta)$, then we have the matrix growth estimate

$$|\tilde{T}_{s_i}\tilde{T}_{s_j}^{-1}-I|\le\Big(\frac{s_j}{s_i}\Big)^{\epsilon'}-1$$

for any $s_i\le s_j\le 1$. Therefore, if $\delta\le\delta(\epsilon,n,\mathrm{v},\eta)$, we have

$$\fint_{B_{Rs_i}(x)}|\langle\nabla(\tilde{T}_{s_i}u)^a,\nabla(\tilde{T}_{s_i}u)^b\rangle-\delta^{ab}|<\epsilon/2. \tag{8.8}$$

These two estimates imply

$$\int_M\Big|\langle\nabla(\tilde{T}_{s_i}u)^a,\nabla(\tilde{T}_{s_i}u)^b\rangle-\delta^{ab}\Big|\varphi^2\cdot\rho_{s_i^2}(x,dy)\le\epsilon.$$

By using the Gram-Schmidt process, there exists a lower triangle matrix A_i satisfying (8.6). This completes the proof of Lemma 8.3. □

To finish the proof of Proposition 8.2, it suffices to prove (8.2). Since $T_{s_i}=A_i\circ\tilde{T}_{s_i}$ with bounded A_i, relation (8.4) implies

$$s_j^{2-n}\int_{B_{s_j}(x)}\Big(|\nabla^2T_{s_j}u|^2+\mathrm{Ric}(\nabla T_{s_j}u,\nabla T_{s_j}u)+2\delta^2(n-1)|\nabla T_{s_j}u|^2\Big)\le C(n)\epsilon_j. \tag{8.9}$$

To prove (8.2), we only need to use (8.9) for each scale $s_j \geq s_i$, and the heat kernel estimates in Theorem 4.14. By the Hölder growth estimate for transformation matrices in Proposition 7.7, if $\delta \leq \delta(n, \mathrm{v}, \eta)$ is small, then we have $|T_{s_i} T_{s_j}^{-1}| \leq 2^{(i-j)/100}$. Therefore, for $s_i \leq s_j \leq 1$,

$$\int_{A_{s_{j+1},s_j}(x)} \Big(|\nabla^2 T_{s_i} u|^2 + \mathrm{Ric}(\nabla T_{s_i} u, \nabla T_{s_i} u) + 2\delta^2(n-1)|\nabla T_{s_i} u|^2\Big) \leq s_j^{n-2} 2^{(i-j)/10} \epsilon_j. \tag{8.10}$$

In particular, by the heat kernel estimates of Theorem 4.14, we have

$$\int_{A_{s_{j+1},s_j}(x)} \Big(|\nabla^2 T_{s_i} u|^2 + \mathrm{Ric}(\nabla T_{s_i} u, \nabla T_{s_i} u) + 2\delta^2(n-1)|\nabla T_{s_i} u|^2\Big) \cdot \rho_{4s_i^2}(x, dy)$$
$$\leq C(n, \mathrm{v}) s_i^{-n} e^{-\frac{s_j^2}{20 s_i^2}} s_j^{n-2} 2^{(i-j)/10} \epsilon_j \leq C(n, \mathrm{v}) s_j^{-2} 2^{(i-j)/10} \epsilon_j.$$

Thus,

$$\begin{aligned}
&s_i^2 \int_{B_1(x)} \Big(|\nabla^2 T_{s_i} u|^2 + \mathrm{Ric}(\nabla T_{s_i} u, \nabla T_{s_i} u) + 2\delta^2(n-1)|\nabla T_{s_i} u|^2\Big) \cdot \rho_{4s_i^2}(x, dy) \\
&\quad \leq s_i^2 \left(\int_{B_{s_i}(x_i)} + \sum_{j=0}^{i-1} \int_{A_{s_{j+1},s_j}(x)} \right) \\
&\qquad \cdot \Big(|\nabla^2 T_{s_i} u|^2 + \mathrm{Ric}(\nabla T_{s_i} u, \nabla T_{s_i} u) + 2\delta^2(n-1)|\nabla T_{s_i} u|^2\Big) \rho_{4s_i^2}(x, dy) \\
&\quad \leq C(n, \mathrm{v}) \epsilon_i + C(n, \mathrm{v}) \sum_{j=0}^{i-1} 2^{2(j-i)} 2^{(i-j)/10} \epsilon_j \\
&\quad \leq C(n, \mathrm{v}) \sum_{j=0}^{i} \epsilon_j 2^{j-i}.
\end{aligned} \tag{8.11}$$

This implies (8.2), which completes the proof of Proposition 8.2. □

8.2. *A telescope estimate for harmonic functions.* In this subsection, we prove a telescope estimate, Proposition 8.4, for harmonic functions in which *the squared L^2-norm of the Hessian linearly controls the difference of the norms of the gradients on concentric balls*; see (8.12). For a function which is not harmonic, the squared L^2-norm would have to be replaced by the L^2-norm itself. This weaker estimate would not suffice for our purposes.

Let φ be a cutoff function with support in $B_1(x)$ and $\varphi \equiv 1$ in $B_{1/2}(x)$ such that $|\Delta\varphi| + |\nabla\varphi| \leq C(n)$.

Proposition 8.4. *Let (M^n, g, x) satisfy $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$, $\mathrm{Vol}(B_1(x)) \geq \mathrm{v} > 0$ and $0 < s < 1$. Assume $u_1, u_2 : B_2(x) \to \mathbb{R}$ are two harmonic functions satisfying polynomial growth condition*[11] $\sup_{B_r(x)} \Big(|\nabla u_1|(y) + |\nabla u_2|(y)\Big) \leq K(1 + s^{-1}r)$ *for all $0 < r \leq 2$. Then*

(8.12)
$$\begin{aligned}
&\Big|\int_M \langle \nabla u_1, \nabla u_2\rangle \varphi^2 \rho_{s^2}(x, dy) - \int_M \langle \nabla u_1, \nabla u_2\rangle \varphi^2 \rho_{4s^2}(x, dy)\Big| \\
&\leq C(n) \sum_{i=1}^2 s^2 \int_M \left(|\nabla^2 u_i|^2 + \mathrm{Ric}(\nabla u_i, \nabla u_i) + 2(n-1)\delta^2 |\nabla u_i|^2\right) \varphi^2 \rho_{8s^2}(x, dy) \\
&\quad + C(n, \mathrm{v}, K) e^{-\frac{1}{100 s^2}}.
\end{aligned}$$

Remark 8.5. We will apply Proposition 8.4 with u_1, u_2 different components of $T_s u, T_s u$ as in Proposition 8.2, which asserts that

$$\sup_{B_r(x)} |\nabla T_s u| \leq C(n)(1 + \frac{r}{s}), \quad \text{for all } 0 < r \leq 2. \tag{8.13}$$

Proof. From Bochner's formula, we get

(8.14)
$$\begin{aligned}
&\Big|\partial_t \int_M \langle \nabla u_1, \nabla u_2\rangle \varphi^2 \rho_t(x, dy)\Big| \\
&= \Big|\int_M \langle \nabla u_1, \nabla u_2\rangle \varphi^2 \Delta \rho_t(x, dy)\Big| \\
&= \Big|\int_M \left(\Delta\langle \nabla u_1, \nabla u_2\rangle \varphi^2 + \Delta\varphi^2 \langle \nabla u_1, \nabla u_2\rangle + 2\varphi\langle \nabla\varphi, \nabla\langle \nabla u_1, \nabla u_2\rangle\rangle\right) \rho_t(x, dy)\Big| \\
&\leq C(n) \sum_{i=1}^2 \int_M \left(|\nabla^2 u_i|^2 + \mathrm{Ric}(\nabla u_i, \nabla u_i) + 2(n-1)\delta^2 |\nabla u_i|^2\right) \varphi^2 \rho_t(x, dy) \\
&\quad + C(n) \sum_{i=1}^2 \int_{A_{1/2,1}(x)} |\nabla u_i|^2 \rho_t(x, dy),
\end{aligned}$$

where in the last inequality we used

$$|\mathrm{Ric}(\nabla u_1, \nabla u_2)| \leq C(n) \sum_{i=1}^2 \left(\mathrm{Ric}(\nabla u_i, \nabla u_i) + 2(n-1)\delta^2 |\nabla u_i|^2\right). \tag{8.15}$$

To see (8.15), since the estimate is pointwise, for each point $x \in M$, one can view $\mathrm{Ric} + \delta^2(n-1)g$ as a nonnegative inner product on $T_x M$. Then the estimate (8.15) follows directly by Cauchy-Schwarz inequality.

[11] After rescaling $B_s(x)$ to $B_1(x)$, this condition just means that $|\nabla u|$ has linear growth in $B_{2s^{-1}}(x)$.

The heat kernel estimate in Theorem 4.14 can be used to control the last term on the right-hand side of the last line of (8.14). Namely, for all $t \le 1$, we have

$$\sum_{i=1}^{2} \int_{A_{1/2,1}(x)} |\nabla u_i|^2 \rho_t(x, dy) \le C(n, \mathrm{v}, K) s^{-2} t^{-n/2} e^{-\frac{1}{20t}}. \tag{8.16}$$

Therefore,

$$\begin{aligned}
&\Big| \int_{M^n} \langle \nabla u_1, \nabla u_2 \rangle \varphi^2 \rho_{s^2}(x, dy) - \int_{M^n} \langle \nabla u_1, \nabla u_2 \rangle \varphi^2 \rho_{4s^2}(x, dy) \Big| \\
&= \Big| \int_{s^2}^{4s^2} \partial_t \int_{M^n} \langle \nabla u_1, \nabla u_2 \rangle \varphi^2 \rho_t(x, dy) dt \Big| \\
&\le C(n) \int_{s^2}^{4s^2} \sum_{i=1}^{2} \int_{M^n} \left(|\nabla^2 u_i|^2 + \mathrm{Ric}(\nabla u_i, \nabla u_i) + 2(n-1)\delta^2 |\nabla u_i|^2 \right) \varphi^2 \rho_t(x, dy) dt \\
&\quad + C(n, \mathrm{v}, K) \delta^2 \int_{s^2}^{4s^2} s^{-2} t^{-n/2} e^{-\frac{1}{20t}} dt.
\end{aligned}$$

Hence

$$\begin{aligned}
&\Big| \int_{M^n} \langle \nabla u_1, \nabla u_2 \rangle \varphi^2 \rho_{s^2}(x, dy) - \int_{M^n} \langle \nabla u_1, \nabla u_2 \rangle \varphi^2 \rho_{4s^2}(x, dy) \Big| \\
&\le C(n) \int_{s^2}^{4s^2} \sum_{i=1}^{2} \int_{M} \left(|\nabla^2 u_i|^2 + \mathrm{Ric}(\nabla u_i, \nabla u_i) + 2(n-1)\delta^2 |\nabla u_i|^2 \right) \varphi^2 \rho_t(x, dy) dt \\
&\quad + C(n, \mathrm{v}, K) s^{-n} e^{-\frac{1}{80s^2}} \\
&\le C(n) \sum_{i=1}^{2} s^2 \int_{M^n} \left(|\nabla^2 u_i|^2 + \mathrm{Ric}(\nabla u_i, \nabla u_i) + 2(n-1)\delta^2 |\nabla u_i|^2 \right) \varphi^2 \rho_{8s^2}(x, dy) \\
&\quad + C(n, \mathrm{v}, K) e^{-\frac{1}{100s^2}},
\end{aligned} \tag{8.17}$$

where we have used the heat kernel estimate in Theorem 4.14 to conclude that $\rho_t(x, y) \le C(n, \mathrm{v}) \cdot \rho_{8s^2}(x, y)$ for any $s^2 \le t \le 4s^2$ and $y \in B_1(x)$. This completes the proof of Proposition 8.4. $\square$

8.3. *Proof of Theorem* 8.1. By Proposition 8.2, for any ϵ', if $\delta \le \delta(n, \mathrm{v}, \eta, \alpha, \epsilon')$, then for each $s_i = 2^{-i}$, there exists a lower triangular $k \times k$ matrix T_{s_i} such that $T_{s_i} u : B_{s_i}(x) \to \mathbb{R}^k$ is a (k, ϵ')-splitting with

$$
\begin{aligned}
&s_i^2 \int_M \Big(|\nabla^2 T_{s_i} u|^2 + \mathrm{Ric}(\nabla T_{s_i} u, \nabla T_{s_i} u) \\
&\qquad\qquad + 2\delta^2(n-1)|\nabla T_{s_i} u|^2 \Big) \varphi^2 \rho_{4s_i^2}(x, dy) \\
&\qquad \le C(n, \mathrm{v}) \sum_{j=0}^{i} \epsilon_j 2^{j-i} := \chi_i,
\end{aligned} \tag{8.18}
$$

$$
\epsilon_i = C(n, \mathrm{v}, \eta) \sum_{j=0}^{i} 2^{-\gamma(i-j)} \Big(\mathcal{E}^k_{s_j}(x) + \delta s_j^2 \Big), \tag{8.19}
$$

$$
\int_M \langle \nabla (T_{s_i} u)^a, \nabla (T_{s_i} u)^b \rangle \varphi^2 \rho_{s_i^2}(x, dy) = \delta^{ab}. \tag{8.20}
$$

Here, $\gamma(n, \mathrm{v}, \eta) > 0$ and φ is the cutoff function with support in $B_1(x)$ and $\varphi \equiv 1$ on $B_{1/2}(x)$.

By the estimate (8.19), for ϵ_i we get

$$
\begin{aligned}
\sum_{i=0}^{m} \epsilon_i &\le C(n, \mathrm{v}, \eta) \sum_{i=0}^{m} \sum_{j=0}^{i} 2^{-\gamma(i-j)} \Big(\mathcal{E}^k_{s_j}(x) + \delta s_j^2 \Big) \\
&\le C(n, \mathrm{v}, \eta) \sum_{j=0}^{m} \Big(\mathcal{E}^k_{s_j}(x) + \delta s_j^2 \Big) \le C(n, \mathrm{v}, \eta) \delta,
\end{aligned} \tag{8.21}
$$

$$
\sum_{i=0}^{m} \chi_i \le C(n, \mathrm{v}) \sum_{i=0}^{m} \sum_{j=0}^{i} \epsilon_j 2^{j-i} \le C(n, \mathrm{v}) \sum_{j=0}^{m} \epsilon_j \le C(n, \mathrm{v}, \eta) \delta. \tag{8.22}
$$

LEMMA 8.6. *For any ϵ', let $\delta \le \delta(n, \mathrm{v}, \eta, \epsilon', \alpha)$. Then $|T_{s_m} - I| \le \epsilon'$ for any $m \ge 1$ such that $s_m \ge r$.*

Proof. First note that by Proposition 7.7, $|\nabla T_{s_i} u|$ satisfies Hölder growth estimates; see also (7.32). Thus, we can apply Proposition 8.4 to obtain

$$
\begin{aligned}
&\Big| \int_M \langle \nabla (T_{s_i} u)^a, \nabla (T_{s_i} u)^b \rangle \varphi^2 \rho_{s_{i+1}^2}(x, dy) - \delta^{ab} \Big| \\
&\qquad\qquad \le C(n) \chi_i + C(n, \mathrm{v}) e^{-\frac{1}{100 s_i^2}} := \tilde{\chi}_i.
\end{aligned} \tag{8.23}
$$

For any ϵ'', there exists an integer $N(\epsilon'', n, \mathrm{v})$ such that if $i \ge N$ and $\delta \le \delta(n, \mathrm{v}, \eta, \alpha, \epsilon')$, then we have

$$
\sum_{j=N}^{i} \tilde{\chi}_j \le \epsilon''. \tag{8.24}
$$

By using the Gram-Schmidt process, there exists a lower triangle matrix $\tilde{A}_i$ with $|\tilde{A}_i - I| \le C(n) \tilde{\chi}_i$ such that $\hat{T}_{s_i} = \tilde{A}_i \circ T_{s_i}$ satisfies

$$
\int_{M^n} \langle \nabla (\hat{T}_{s_i} u)^a, \nabla (\hat{T}_{s_i} u)^b \rangle \varphi^2 \rho_{s_{i+1}^2}(x, dy) = \delta^{ab}. \tag{8.25}
$$

Since

$$\int_M \langle \nabla (T_{s_{i+1}} u)^a, \nabla (T_{s_{i+1}} u)^b \rangle \varphi^2 \rho_{s_{i+1}^2}(x, dy) = \delta^{ab}, \tag{8.26}$$

the uniqueness of Cholesky decompositions (see also [GVL96]) for positive definite symmetric matrices implies that $\hat{T}_{s_i} = T_{s_{i+1}}$. In particular, we get $T_{s_{i+1}} \circ T_{s_i}^{-1} = \tilde{A}_i$. Thus

$$|T_{s_{i+1}} \circ T_{s_i}^{-1} - I| \le C(n)\tilde{\chi}_i. \tag{8.27}$$

Recall that T_{s_i} is a $k \times k$ matrix. Hence for all $i \ge N$, we have

$$\begin{aligned} |T_{s_{i+1}} \circ T_{s_N}^{-1} - I| &\le \prod_{j=N}^{i} (1 + (k+1)C(n)\tilde{\chi}_j) - 1 \\ &\le e^{\sum_{j=\ell}^{i} kC(n)\tilde{\chi}_j} - 1 \le C(n) \sum_{j=N}^{i} \tilde{\chi}_j \le C\epsilon''. \end{aligned} \tag{8.28}$$

If $\delta \le \delta(\epsilon', \mathrm{v}, n, \eta)$ and $\epsilon'' \le \epsilon''(n, \mathrm{v}, \epsilon')$, we have for all $i \le N$ that

$$|T_{s_i} - I| \le \epsilon'/10. \tag{8.29}$$

Therefore, by (8.28), for any $i \ge N$, we have

$$|T_{s_i} - I| \le \epsilon'. \tag{8.30}$$

This completes the proof of Lemma 8.6. □

Now we can complete the proof of Theorem 8.1 as follows. Since $T_{s_i} u : B_{s_i}(x) \to \mathbb{R}^k$ is ϵ'-splitting when $\delta \le \delta(n, \mathrm{v}, \epsilon', \eta, \alpha)$, to show $u : B_{s_i}(x) \to \mathbb{R}^k$ is ϵ-splitting, it suffices to prove T_{s_i} is bounded and then fix $\epsilon' = \epsilon'(n, \epsilon, \mathrm{v})$. The later has been proven in Lemma 8.6. Therefore we complete the proof of Theorem 8.1.

9. **Proof of the Neck Structure Theorem 2.9**

This is the first of the two sections which constitute the fourth and last part of the paper. In it we give the proof of the Neck Structure Theorem 2.9. For convenience, we have restated it below. Recall that neck regions are defined in Definition 2.4.

THEOREM 2.9 RESTATED. *Fix $\eta > 0$ and $\delta \le \delta(n, \mathrm{v}, \eta)$. Then if $\mathcal{N} = B_2(p) \setminus \overline{B}_{r_x}(\mathcal{C})$ is a (k, δ, η)-neck region, the following hold:*

(1) *For each $x \in \mathcal{C}$ and $B_{2r}(x) \subset B_2(p)$, the induced packing measure μ is Ahlfors regular:*

$$A(n)^{-1} r^k < \mu(B_r(x)) < A(n) r^k. \tag{9.1}$$

(2) *$\mathcal{C}_0$ is k-rectifiable.*

Results on rectifiability of singular sets obtained via cone-splitting were first introduced in [NV17a] in the context of nonlinear harmonic maps, and the notion of neck regions was first formally introduced and studied in [JN21]. As was discussed in Sections 2 and 5, in order to conclude the structural results we will need to build a map from the center points $\mathcal{C} \to \mathbb{R}^k$. In [NV17a], the relevant splitting map u was built by hand using a Reifenberg construction. This approach required new estimates on harmonic maps and a new bi-Lipschitz Reifenberg theorem. As we have emphasized, for the case of lower Ricci curvature bounds, *the bi-Lipschitz Reifenberg ideas of* [NV17a] *do not work.* Attempting to implement them gives rise to additional error terms which are not summable over scales. Essentially, this is because approximating a subset $W \subseteq X^n$ by k-dimensional subspace also involves approximating X^n itself by a splitting. Instead, we rely on the results of Sections 6–8, especially the Nondegeneration Theorem 8.1.

In [JN21], results on structure and existence of $(n-4)$-neck regions were proved under the assumption of a 2-sided bound on Ricci curvature. In order to prove the final estimates in [JN21] the authors introduced a new estimate, which was termed a *superconvexity estimate.* This estimate definitely requires a 2-sided bound on the Ricci tensor. The estimates of this paper are entirely different. As mentioned in the introduction, for limit spaces with $|\mathrm{Ric}_{M_i^n}| \leq (n-1)$, we give a new proof of the rectifiability of $S = S^{n-4}$ and the bound $\mathcal{H}^{n-4}(S \cap B_1(p) \leq c(n, \mathrm{v})$. This was conjectured in [CC97] and first proved in [JN21].

We refer the reader to Section 5 for an outline of the strategy for proving Theorem 2.9.

9.1. *The basic assumptions.* Below, we will refer to the following standard assumptions.

Fix $\delta, \delta', \eta, B > 0$. We will assume the following:

(S1) $\mathrm{Vol}(B_1(p)) > \mathrm{v} > 0$ and $\mathrm{Ric}_{M^n} \geq -(n-1)\delta^2$.

(S2) $\mathcal{N} = B_2(p) \setminus \bar{B}_{r_x}(\mathcal{C})$ is a (k, δ, η)-neck region with the associated packing measure μ.

(S3) For any $x \in \mathcal{C}$ and $B_{2r}(x) \subset B_2(p)$ with $r \geq r_x$, we have

$$B^{-1} r^k \leq \mu(B_r(x)) \leq B r^k. \tag{9.2}$$

(S4) $u : B_4(p) \to \mathbb{R}^k$ is a δ'-splitting map.

Remark 9.1. Recall from Section 5 that (S3) is connected to our strategy of proving the theorem by induction. In particular, with $B \gg A(n)$, we will eventually show that for δ sufficiently small, (S3) automatically implies the stronger Ahlfors regularity estimate (9.1).

Remark 9.2. By the definition of neck regions and the Cone-Splitting Theorem 4.11, we can and will assume that $\delta(n, \mathrm{v}, \eta, \delta') > 0$ has been chosen sufficiently small so that there exists a δ'-splitting map $u : B_4(p) \to \mathbb{R}^k$. Then in actuality, the existence of u as in (S4) is actually a consequence of (S2).

9.2. *Bi-Lipschitz structure of the set of centers of a neck region.* This subsection is devoted to proving Proposition 9.3. Given a (k, δ, η)-neck region $\mathcal{N} = B_2(p) \setminus \overline{B}_{r_x}(\mathcal{C})$, Proposition 9.3 implies the existence of a subset $\mathcal{C}_\epsilon \subset \mathcal{C}$, which is almost all of $\mathcal{C}$, such that a splitting map $u : B_{2s}(x) \to \mathbb{R}^k$ is $(1+\epsilon)$-bi-Lipschitz on $\mathcal{C}_\epsilon$. This is the key step which is used to improve the weak Ahlfors regularity estimate (S3) to the strong one, (9.1), and to show that the singular set is rectifiable. The results of the previous sections play a key role in the proof of Proposition 9.3; compare the outline in Section 5.

PROPOSITION 9.3. *For any given positive constants $B, \epsilon, \eta > 0$, if* (S1)–(S4) *hold with $\delta' \le \delta'(n, \mathrm{v}, \eta, B, \epsilon)$ and $\delta \le \delta(n, \mathrm{v}, \eta, \delta', B, \epsilon)$, then there exists $\mathcal{C}_\epsilon \subset \mathcal{C} \cap B_{15/8}(p)$ such that*

(1) $\mu\big(\mathcal{C}_\epsilon \cap B_{15/8}(p)\big) \ge (1-\epsilon)\mu\big(\mathcal{C} \cap B_{15/8}(p)\big)$;
(2) *u is $(1+\epsilon)$-bi-Lipschitz on $\mathcal{C}_\epsilon$, i.e., for any $x, y \in \mathcal{C}_\epsilon$,*
$$(1+\epsilon)^{-1} \cdot d(x,y) \le |u(x) - u(y)| \le (1+\epsilon) \cdot d(x,y);$$
(3) *for any $x \in \mathcal{C}_\epsilon$ and $r \ge r_x$ with $B_{2r}(x) \subset B_2(p)$, the map $u : B_r(x) \to \mathbb{R}^k$ is a (k, ϵ)-splitting function.*
(4) *For any $x \in \mathcal{C}_\epsilon$,*
$$\sum_{r_x \le r_i \le 2^{-5}} \fint_{B_{r_i}(x)} |\mathcal{W}^\delta_{r_i^2/2}(y) - \mathcal{W}^\delta_{2r_i^2}(y)|\, d\mu(y) \le \epsilon.$$
(5) *$u : \mathcal{C} \to \mathbb{R}^k$ is a bi-Hölder map onto its image, i.e., for all $x, y \in B_{15/8}(p) \cap \mathcal{C}$,*
$$(1-\epsilon) \cdot d(x,y)^{1+\epsilon} \le |u(x) - u(y)| \le (1+\epsilon) \cdot d(x,y).$$

Note. In (4), the integral average is taken with respect to μ.

Essentially, $\mathcal{C}_\epsilon \subset \mathcal{C}$ consists of those points which satisfy (4). We will see, as in (1), that most points of $\mathcal{C}$ have this property. Then using Theorem 8.1 we will conclude (3). The estimates (2) and (5) will follow almost exactly the same argument as the one given in Section 7.5

We begin with some technical lemmas which will be used in the proof of Proposition 9.3. The proof of the proposition will be given at the end of this subsection, after the proofs of the lemmas have been completed.

The first of these, Lemma 9.4 below, will enable us to conclude that if $\mathcal{C}_\epsilon \subset \mathcal{C}$ is defined as indicated above, then (4) holds.

LEMMA 9.4. *Let* (M^n, g, p) *satisfy* (S1)–(S4) *with* $\delta'' > 0$ *fixed. If* $\delta \leq \delta(n, \mathrm{v}, B, \delta'')$ *and* $\delta' \leq \delta'(n, \mathrm{v}, B, \eta)$, *then the local* $\mathcal{W}$*-entropy satisfies*

$$\fint_{B_{15/8}(p)} \Big(\sum_{r_x \leq r_i \leq 2^{-5}} \fint_{B_{r_i}(x)} |\mathcal{W}^{\delta}_{r_i^2/2}(y) - \mathcal{W}^{\delta}_{2r_i^2}(y)| \, d\mu(y) \Big) \, d\mu(x) \leq \delta''. \tag{9.3}$$

Proof. Recall that under the assumptions of Theorem 4.22, including $\delta \leq \delta(n, \mathrm{v}, \epsilon)$, we have the following relation between the volume ratio and the local pointed entropy:

$$|\mathcal{W}^{\delta}_t(x) - \log \mathcal{V}^{\delta^2}_{\sqrt{t}}(x)| \leq \epsilon.$$

The proof will utilize this relation together with a Fubini type argument.

Let $\chi_{|x-y| \leq r}(x, y)$ be the characteristic function of the set $\{(x, y) \in M^n \times M^n : d(x, y) \leq r\}$, and put $r_i = 2^{-i}$.

In the following argument we will use Fubini's theorem to exchange variables. In order to make the argument easier, let us define $\hat{\mathcal{W}}^{\delta}_t(x) = \mathcal{W}^{\delta}_{r_x^2/4}(x)$ for any $t \leq r_x^2/4$ and $\hat{\mathcal{W}}^{\delta}_t(x) = \mathcal{W}^{\delta}_t(x)$ for $t > r_x^2/4$. Furthermore, let us point out that in the following estimate, the term $\mu(B_{r_i}(x))$ with $r_i < r_x/4$ always multiplies with $|\hat{\mathcal{W}}^{\delta}_{r_i^2/2}(y) - \hat{\mathcal{W}}^{\delta}_{2r_i^2}(y)|$ for $y \in B_{r_i}(x)$, which is seen to be vanishing by noting that $\mathrm{Lip}\, r_x \leq \delta$:

(9.4)
$$\begin{aligned}
&\fint_{B_{15/8}(p)} \Big(\sum_{r_x \leq r_i \leq 2^{-5}} \fint_{B_{r_i}(x)} |\mathcal{W}^{\delta}_{r_i^2/2}(y) - \mathcal{W}^{\delta}_{2r_i^2}(y)| \, d\mu(y) \Big) \, d\mu(x) \\
&\quad \leq \fint_{B_{15/8}(p)} \Big(\sum_{r_i \leq 2^{-5}} \fint_{B_{r_i}(x)} |\hat{\mathcal{W}}^{\delta}_{r_i^2/2}(y) - \hat{\mathcal{W}}^{\delta}_{2r_i^2}(y)| \, d\mu(y) \Big) \, d\mu(x) \\
&\quad \leq \frac{1}{\mu(B_{15/8}(p))} \int_{B_{15/8}(p)} \Big(\sum_{r_i \leq 2^{-5}} \frac{1}{\mu(B_{r_i}(x))} \int_{B_{31/16}(p)} \chi_{\{|x-y| \leq r_i\}}(x, y) \\
&\qquad\qquad \cdot |\hat{\mathcal{W}}^{\delta}_{r_i^2/2}(y) - \hat{\mathcal{W}}^{\delta}_{2r_i^2}(y)| d\mu(y) \Big) d\mu(x) \\
&\quad \leq C(n) \cdot B^2 \int_{B_{31/16}(p)} \int_{B_{31/16}(p)} \sum_{r_i \leq 2^{-5}} r_i^{-k} \chi_{\{|x-y| \leq r_i\}}(x, y) \\
&\qquad \cdot |\hat{\mathcal{W}}^{\delta}_{r_i^2/2}(y) - \hat{\mathcal{W}}^{\delta}_{2r_i^2}(y)| \, d\mu(y) \, d\mu(x) \\
&\quad \leq C(n) \cdot B^2 \int_{B_{31/16}(p)} \int_{B_{31/16}(p)} \sum_{r_i \leq 2^{-5}} r_i^{-k} \chi_{\{|x-y| \leq r_i\}}(x, y) \\
&\qquad \cdot |\hat{\mathcal{W}}^{\delta}_{r_i^2/2}(y) - \hat{\mathcal{W}}^{\delta}_{2r_i^2}(y)| \, d\mu(y) \, d\mu(x).
\end{aligned}$$

Applying Fubini's Theorem gives

$$
\begin{aligned}
&\fint_{B_{15/8}(p)} \Big(\sum_{r_x \le r_i \le 2^{-5}} \fint_{B_{r_i}(x)} |\mathcal{W}^\delta_{r_i^2/2}(y) - \mathcal{W}^\delta_{2r_i^2}(y)|\, d\mu(y) \Big)\, d\mu(x) \\
&\quad \le C(n) \cdot B^2 \int_{B_{31/16}(p)} \sum_{r_i \le 2^{-5}} r_i^{-k} \mu(B_{r_i}(y)) \cdot |\hat{\mathcal{W}}^\delta_{r_i^2/2}(y) - \hat{\mathcal{W}}^\delta_{2r_i^2}(y)|\, d\mu(y) \\
&\quad \le C(n) \cdot B^3 \int_{B_{31/16}(p)} \sum_{r_i \le 1} |\hat{\mathcal{W}}^\delta_{r_i^2/2}(y) - \hat{\mathcal{W}}^\delta_{2r_i^2}(y)|\, d\mu(y) \\
&\quad \le C(n) \cdot B^3 \int_{B_{31/16}(p)} |\hat{\mathcal{W}}^\delta_0(y) - \hat{\mathcal{W}}^\delta_2(y)|\, d\mu(y) \\
&\quad = C(n) \cdot B^3 \int_{B_{31/16}(p)} |\mathcal{W}^\delta_{r_y^2/4}(y) - \mathcal{W}^\delta_2(y)|\, d\mu(y) \\
&\quad \le C(n) \cdot B^4 \epsilon'.
\end{aligned}
\tag{9.5}
$$

We also used the pointwise estimate $|\mathcal{W}^\delta_{r_y^2/4}(y) - \mathcal{W}^\delta_2(y)(y)| \le \epsilon'$ (see Theorem 4.22), which follows if we choose $\delta < \delta(n, \mathrm{v}, \epsilon', \eta)$ in condition (n2) of Definition 2.4, the definition of a (k, δ, η)-neck.

By fixing ϵ' sufficiently small, so that $C(n) B^3 \epsilon' \le \delta''$, the proof of Lemma 9.4 is completed. □

The following lemma is a direct consequence of the Nondegeneration Theorem 8.1 and the assumed Ahlfors regularity with constant B as in (S3).

Lemma 9.5. *Let (M^n, g, p) satisfy* (S1)–(S4) *with $\epsilon > 0$ fixed. Assume $\delta'' \le \delta''(n, \mathrm{v}, \eta, \epsilon)$, $\delta' \le \delta'(n, \mathrm{v}, \eta, \epsilon)$, $\delta \le \delta(n, \mathrm{v}, B, \eta, \epsilon)$, and for some $x \in \mathcal{C} \cap B_{15/8}(p)$,*

$$
\sum_{s \le r_i \le 2^{-5}} \fint_{B_{r_i}(x)} |\mathcal{W}^\delta_{r_i^2/2}(y) - \mathcal{W}^\delta_{2r_i^2}(y)|\, d\mu(y) \le \delta''. \tag{9.6}
$$

Then for any $s \le r \le 1$, the map $u : B_r(x) \to \mathbb{R}^k$ is an ϵ-splitting map.

Proof. By the Nondegeneration Theorem 8.1 it suffices to find a set of (k, α)-independent points $\{x_0, x_1, \dots, x_k\} \subset B_{r_i}(x) \cap \mathcal{C}$ for some $\alpha(n, B) > 0$ such that for each r_i, we have the k-pinching estimate

$$
\begin{aligned}
\mathcal{E}^{k,\alpha,\delta}_{r_i}(x) &\le \sum_{j=0}^{k} |\mathcal{W}^\delta_{r_i^2/2}(x_j) - \mathcal{W}^\delta_{2r_i^2}(x_j)| \\
&\le C(n, B) \fint_{B_{r_i}(x)} |\mathcal{W}^\delta_{r_i^2/2}(y) - \mathcal{W}^\delta_{2r_i^2}|(y)|\, d\mu(y).
\end{aligned}
\tag{9.7}
$$

In the following argument we will show that (9.7) holds. We will first show that the existence of such points follows from the assumed Ahlfors regularity of μ in (S3).

First, note that there exists a subset $\mathcal{C}_{r_i,x} \subset \mathcal{C} \cap B_{r_i}(x)$ with $\mu(\mathcal{C}_{r_i,x}) \geq r_i^k B/2$ such that for any $y \in \mathcal{C}_{r_i,x}$, we have

$$|\mathcal{W}^\delta_{r_i^2/2}(y) - \mathcal{W}^\delta_{2r_i^2}(y)| \leq C(n) \fint_{B_{r_i}(x)} |\mathcal{W}^\delta_{r_i^2/2}(z) - \mathcal{W}^\delta_{2r_i^2}(z)|\, d\mu(z). \tag{9.8}$$

By using the Ahlfors regularity of μ (see (S3)), we will now show that we can find (k,α)-independent points in $\mathcal{C}_{r_i,x}$ for some small $\alpha(n,B)$. First we note that for any $\epsilon' > 0$, if $\delta \leq \delta(n, \mathrm{v}, \eta, \epsilon')$, then $\mathcal{C}_{r_i,x} \subset B_{\epsilon' r_i}\big(\iota(\mathbb{R}^k \times \{(y_c)\})\big)$ where $\iota : \mathbb{R}^k \times C(X) \to B_{r_i}(x)$ is a δr_i-GH map.

Comparing the result in Remark 4.5 about the (k,α)-independent points in $\mathbb{R}^n$, if there exist no (k,α)-independent points in $\mathcal{C}_{r_i,x}$ as in Definition 4.4, the set $\mathcal{C}_{r_i,x}$ must be contained in $B_{4\alpha r_i}\big(\iota(\mathbb{R}^{k-1} \times \{(0,y_c)\})\big)$ for some $\mathbb{R}^{k-1}$ plane. Therefore, we have obtained at most $C(n)\alpha^{-k+1}$ many balls $\{B_{8\alpha r_i}(y_j)\}$, with $y_j \in \mathcal{C}_{r_i,x}$, which cover $\mathcal{C}_{r_i,x}$. Thus, by the Ahlfors regularity of μ we have

$$\mu(\mathcal{C}_{r_i,x}) \leq C(n)\alpha^{-k+1} B (8\alpha r_i)^k \leq C(n,B)\alpha r_i^k.$$

Since $\mu(\mathcal{C}_{r_i,x}) \geq B r_i^k/2$, by choosing $\alpha = \alpha(n,B)$ small we get a contradiction. Hence there exist (k,α)-independent points in $\mathcal{C}_{r_i,x} \subset B_r(x) \cap \mathcal{C}$. At this point, Lemma 9.5 follows directly from the Nondegeneration Theorem 8.1. □

The following Lemma 9.6 provides a Gromov-Hausdorff-approximation for ϵ-splitting maps which will be used to prove the bi-Lipschitz estimate for u. The proof of Lemma 9.6 depends on Lemma 9.7. Thus, it will not be completed until after Lemma 9.7 has been proved.

LEMMA 9.6. *Let* (M^n, g, p) *satisfy* (S1)–(S4). *Assume* $\delta'' \leq \delta''(n, \mathrm{v}, \eta, \epsilon)$, $\delta' \leq \delta'(n, \mathrm{v}, \eta, \epsilon)$ *and* $\delta \leq \delta(n, \mathrm{v}, B, \eta, \epsilon)$. *Let* $u : B_r(x) \to \mathbb{R}^k$ *be a* δ''*-splitting map for some* $x \in \mathcal{C}$ *and all* $r_x \leq r \leq 1$. *Then for any* $y \in \mathcal{C}$,

$$\Big| |u(x) - u(y)| - d(x,y) \Big| \leq \epsilon d(x,y). \tag{9.9}$$

Proof. Pick $r \geq r_x$ so that $r/2 \leq d(x,y) \leq r$. By the definition of a neck region, we know that $B_{10r}(x)$ is δr-Gromov Hausdorff close to a cone $\mathbb{R}^k \times C(Y)$. Moreover, by the splitting guaranteed by Theorem 4.11, if $\delta \leq \delta(n, \mathrm{v}, \eta, B, \epsilon')$, then there exists a (k,ϵ')-splitting map $\tilde{u} : B_r(x) \to \mathbb{R}^k := \mathbb{R}^k \times \{y_c\} \subset \mathbb{R}^k \times C(Y)$ such that $\tilde{u} \circ \iota : \mathbb{R}^k \times \{y_c\} \to \mathbb{R}^k \times \{y_c\}$ is $\epsilon' \cdot r$ close to the identity map. Here, $\iota : \mathbb{R}^k \times C(Y) \to B_r(x)$ is the δr-GH map in the definition of neck region. Since $B_r(x)$ is not $(k+1,\eta)$-symmetric, we must have $\mathcal{C} \cap B_r(x) \subset B_{\epsilon' r}(\iota(B_r(0^k) \times y_c))$. Therefore, for any $y \in \mathcal{C} \cap B_r(x)$, we have

$$\Big| |\tilde u(x) - \tilde u(y)| - d(x,y) \Big| \le 100\epsilon' r. \tag{9.10}$$

In order to use (9.10) (which holds for $\tilde u$) to prove (9.9) (which pertains to u), the following lemma is required.

LEMMA 9.7. *For any ϵ, if $\epsilon' \le \epsilon'(\epsilon, n, \mathrm{v}, \eta)$ and $\delta'' \le \delta''(n, \mathrm{v}, \eta, \epsilon)$, then there exists a rotation $O \in O(k)$ and a vector $Z \in \mathbb{R}^k$, such that $\sup_{B_r(x)} |O\tilde u - u - Z| \le \epsilon r/10$.*

Proof. The proof of Lemma 9.7 is via a contradiction argument. First we will show that after composing with a suitable orthogonal transformation if necessary, the L^2 gradients are close.

SUBLEMMA 9.8. *If $\epsilon' \le \epsilon'(\epsilon, n, \mathrm{v}, \eta)$ and $\delta'' \le \delta''(n, \mathrm{v}, \eta, \epsilon)$, then there exists $O \in O(k)$ such that*

$$\fint_{B_r(x)} |\nabla(O\tilde u - u)|^2 \le \epsilon^{2n}. \tag{9.11}$$

Proof. Without loss of generality, assume

$$\fint_{B_r(x)} \langle \nabla \tilde u^j, \nabla \tilde u^i \rangle = \delta^{ij} = \fint_{B_r(x)} \langle \nabla u^j, \nabla u^i \rangle.$$

Let us define a $k \times k$ matrix $A = (a_{ij})$ by

$$a_{ij} = \fint_{B_r(x)} \langle \nabla u^i, \nabla \tilde u^j \rangle. \tag{9.12}$$

We will see for $\epsilon' \le \epsilon'(\epsilon, n, \mathrm{v}, \eta)$ and $\delta'' \le \delta''(n, \mathrm{v}, \eta, \epsilon)$ that

$$\Big| \sum_{\ell=1}^{k} a_{i\ell} a_{j\ell} - \delta_{ij} \Big| \le \epsilon^{7n}, \ \ i, j = 1, \dots, k. \tag{9.13}$$

Let us first assume (9.13) and finish the proof of the sublemma. By (9.13) we have

$$\fint_{B_r(x)} |\nabla(A\tilde u - u)|^2 \le \epsilon^{3n}. \tag{9.14}$$

Moreover, by (9.13) we can use the Gram-Schmidt process to produce a matrix $O \in O(k)$ with $|O - A| \le C(k)\epsilon^{4n}$. Combining this with (9.14) implies (9.11), i.e., the sublemma.

Now we begin the proof of (9.13). Since

$$\fint_{B_r(x)} \langle \nabla \tilde u^j, \nabla \tilde u^i \rangle = \delta^{ij} = \fint_{B_r(x)} \langle \nabla u^j, \nabla u^i \rangle,$$

it will suffice to prove

$$\Big|\sum_{\ell=1}^{k} a_{i\ell}a_{i\ell}-1\Big|\leq \epsilon^{10n},\ i=1,\dots,k. \tag{9.15}$$

Assume (9.15) does not hold for some $i=i_0\leq k$ and $\epsilon=\epsilon_0>0$ with $\epsilon'\to 0$ and $\delta''\to 0$. Consider the following $k+1$ harmonic functions $v^0=u^{i_0}-\sum_{j=1}^{k}a_{i_0j}\tilde u^j$, $\tilde u^1,\dots,\tilde u^k$. From the definition of a_{ij} in (9.12), we have that $⨍_{B_r(x)}\langle\nabla v^0,\nabla\tilde u^j\rangle=0$. Moreover, by the contradiction assumption, we have

$$⨍_{B_r(x)}|\nabla v^0|^2=1-\sum_{j=1}^{k}a_{i_0j}^2\geq\epsilon_0^{10n}. \tag{9.16}$$

Normalize v^0 to $\tilde u^0$ such that $\tilde u^0$ has unit L^2 gradient norm. Therefore, for ϵ' and δ'' sufficiently small, the map $(\tilde u^0,\tilde u^1,\dots,\tilde u^k):B_r(x)\to\mathbb{R}^{k+1}$ is a $(k+1,\eta/10)$-splitting map, which contradicts the fact that $B_r(x)$ is not $(k+1,\eta)$-symmetric. This completes the proof of (9.15) and (9.13). Hence, the proof of Sublemma 9.8 is complete. □

Now by using the Poincaré inequality in Theorem 4.24 we get

$$⨍_{B_r(x)}\Big|O\tilde u-u-⨍_{B_r(x)}(O\tilde u-u)\Big|^2\leq C(n)r^2⨍_{B_r(x)}|\nabla(O\tilde u-u)|^2\leq C(n)r^2\epsilon^{2n}. \tag{9.17}$$

Set $Z=⨍_{B_r(x)}(O\tilde u-u)\in\mathbb{R}^k$. At this point, the proof of Lemma 9.7 follows now from (9.17) and the gradient estimate $\sup_{B_r(x)}|\nabla(O\tilde u-u)|\leq 1+\epsilon$. This completes the proof of Lemma 9.7. □

The proof of Lemma 9.6 can now be completed by observing that for any $y\in\mathcal{C}\cap B_r(x)$ with $d(x,y)\geq r/2$, we have

$$\begin{aligned}\Big||u(x)-u(y)|-d(x,y)\Big| &\leq \Big||O(\tilde u(x)-\tilde u(y))|-d(x,y)\Big|\\ &\quad+\Big|O\tilde u(x)-u(x)-Z\Big|+\Big|O\tilde u(y)-u(y)-Z\Big|\\ &\leq\Big||\tilde u(x)-\tilde u(y)|-d(x,y)\Big|+\epsilon r/5\leq\epsilon r/2\leq\epsilon d(x,y).\end{aligned} \tag{9.18}$$

This completes the proof of Lemma 9.6. □

Proof of Proposition 9.3. Now we can finish the proof of Proposition 9.3. For this, note that for all $\epsilon''>0$, there exist $\delta'(n,B,\mathrm{v},\eta,\epsilon'')$ and $\delta(n,B,\mathrm{v},\eta,\epsilon'')$ such that by Lemma 9.4,

$$⨍_{B_{15/8}(p)}\Big(\sum_{r_x\leq r_i\leq 2^{-5}}⨍_{B_{r_i}(x)}|\mathcal{W}^\delta_{r_i^2/2}-\mathcal{W}^\delta_{2r_i^2}|(y)d\mu(y)\Big)d\mu(x)\leq\epsilon''. \tag{9.19}$$

For all $\delta'' > 0$, define the set $\mathcal{C}_{\delta''} \subset \mathcal{C} \cap B_{15/8}(p)$ such that $x \in \mathcal{C}_{\delta''}$ if

$$\sum_{r_x \le r_i \le 2^{-5}} \fint_{B_{r_i}(x)} |\mathcal{W}^{\delta}_{r_i^2/2} - \mathcal{W}^{\delta}_{2r_i^2}|(y)d\mu(y) \le \delta''. \tag{9.20}$$

If $\epsilon'' \le \epsilon''(n, B, \delta'')$, then by the Ahlfors regularity estimate $(S3)$ for μ, we have

$$\mu(\mathcal{C}_{\delta''}) \ge (1 - \delta'')\mu(\mathcal{C} \cap B_{15/8}(p)).$$

Fix $\epsilon' > 0$. If $\delta'' \le \delta''(n, \epsilon')$, then by Lemma 9.5, for any $x \in \mathcal{C}_{\delta''}$ and $r_x \le r \le 1$, there exists a (k, ϵ')-splitting map $u : B_r(x) \to \mathbb{R}^k$. Thus, by fixing $\epsilon' \le \epsilon'(n, \mathrm{v}, B, \eta, \epsilon)$ and putting $\mathcal{C}_{\delta''} = \mathcal{C}_\epsilon$, we obtain (1), (3) and (4) of Proposition 9.3.

To prove the bi-Lipschitz estimate, (2), note that for any $x, y \in \mathcal{C}_{\delta''}$, if $\epsilon' \le \epsilon'(n, \mathrm{v}, \eta, B, \epsilon)$, then Lemma 9.6 gives (9.9):

$$\Big|\, |u(x) - u(y)| - d(x, y) \Big| \le \epsilon d(x, y).$$

This implies the bi-Lipschitz estimate (2) of Proposition 9.3. By using the Transformation Proposition 7.7, the proof of the bi-Hölder estimate for u can be completed in just the same manner as in the proof of the Canonical Reifenberg Theorem, 7.10. This completes the proof of Proposition 9.3. □

9.3. *Ahlfors regularity for the packing measure.* In this subsection, we will show that if a neck region satisfies a weak Ahlfors regularity estimate as in (S3), then for δ sufficiently small, the neck region automatically satisfies a stronger universal Ahlfors regularity estimate. This is based on the bi-Lipschitz structure proved in Proposition 9.3. It is the key to the inductive scheme.

PROPOSITION 9.9. *Let (M^n, g, p) satisfy* (S1)–(S4) *with $\delta \le \delta(n, \mathrm{v}, B, \eta)$ and $\delta' \le \delta'(n, \mathrm{v}, B, \eta)$. Then there exists $A(n)$ such that for any $x \in \mathcal{C} \cap B_2(p)$, with $r \ge r_x$ and $B_{2r}(x) \subset B_2(p)$, we have*

$$A(n)^{-1} r^k \le \mu(B_r(x)) \le A(n) r^k. \tag{9.21}$$

Proof. We can assume without loss of generality that $x = p$ and $r = 1$. We will show that $\mu(B_1(p))$ satisfies the upper and lower bound in (9.21).

Consider the map $u : B_2(p) \to \mathbb{R}^k$. Assume $0^k = u(p)$, and recall that $\tau = \tau_n := 10^{-10n}\omega_n$.

We will begin by proving the upper bound for $\mu(B_1(p))$. For this, note that for any ϵ, if $\delta \le \delta(n, \mathrm{v}, \epsilon, B, \eta)$, then by the bi-Lipschitz estimate in Proposition 9.3, the balls $\{B_{\tau^3 r_x}(u(x)) \subset \mathbb{R}^k\}$ are mutually disjoint for $x \in \mathcal{C}_\epsilon$. In addition, $\mathcal{C}_\epsilon \subset \mathcal{C} \cap B_{15/8}(p)$ satisfies

$$\mu(\mathcal{C}_\epsilon \cap B_{15/8}) \ge (1 - \epsilon) \cdot \mu(\mathcal{C} \cap B_{15/8}(p)).$$

By the Lipschitz bound on u, we have $|u(x)| = |u(x) - u(p)| \leq 4$. Let Vol_k denote the volume form of $\mathbb{R}^k$. Then

$$
\begin{aligned}
\mu(\mathcal{C}_\epsilon \cap B_{15/8}(p)) &\leq \sum_{x\in\mathcal{C}_\epsilon\cap B_{15/8}} r_x^k \\
&\leq C(n)\cdot \sum_{x\in\mathcal{C}_\epsilon\cap B_{15/8}} \mathrm{Vol}_k(B_{\tau^3 r_x}(u(x))) \\
&\leq C(n)\cdot \mathrm{Vol}_k(B_5(0^k)) \\
&\leq C(n).
\end{aligned}
\tag{9.22}
$$

By combining the above with the estimate

$$\mu(\mathcal{C}_\epsilon \cap B_{15/8}(p)) \geq (1-\epsilon)\mu(\mathcal{C}\cap B_{15/8}(p)),$$

this gives the upper bound of $\mu(B_1(p))$.

The lower bound for $\mu(B_1(p))$ will follow from a covering argument.

The Geometric Transformation Theorem 7.2 implies that for any $\epsilon > 0$ and $\delta \leq \delta(n, \mathrm{v}, \epsilon, \eta)$, there exists for $x \in \mathcal{C}$ and $r_x \leq s \leq 1$ a $k\times k$ matrix $T_{x,s}$ such that the map $T_{x,s}u : B_s(x) \to \mathbb{R}^k$ is a (k, ϵ)-splitting map. Since $|\nabla u| \leq 1 + \delta'$, we have $|T_{x,s}| \geq 1/2$. The lower bound estimate in (9.21) will follow from the next lemma.

Lemma 9.10. *Let*

$$T^{-1}_{x,r_x}(B_{r_x}(u(x))) := u(x) + T^{-1}_{x,r_x}\Big(B_{r_x}(0^k)\Big).$$

Then a covering of $B_{1/8}(0^k) \subset \mathbb{R}^k$ is provided by the collection of ellipsoids:

$$\big\{T^{-1}_{x,r_x}(B_{r_x}(u(x)))\,|\, x\in\mathcal{C}\cap B_1(p)\big\}.$$

Proof. Assume there exists $w \in B_{1/8}(0^k)$ not in the covering. For every $x \in \mathcal{C}\cap B_1(p)$, define

$$
\begin{aligned}
s_x &:= \inf\{s \geq r_x : w \in T^{-1}_{x,s}B_s(u(x))\} \\
\bar{s} &:= s_{\bar{x}} := \min_{x\in\mathcal{C}\cap B_1(p)} s_x.
\end{aligned}
\tag{9.23}
$$

Then $\bar{s} > r_{\bar{x}}$ and

$$w \in T^{-1}_{\bar{x},2\bar{s}}B_{2\bar{s}}(u(\bar{x})).$$

This implies

$$T_{\bar{x},2\bar{s}}w \in B_{2\bar{s}}(T_{\bar{x},2\bar{s}}u(x)).$$

On the other hand, the map, $T_{\bar{x},2\bar{s}}u : B_{2\bar{s}}(\bar{x}) \to \mathbb{R}^k$ is a (k,ϵ)-splitting. From the covering property (n5) in Definition 2.4 of neck regions, there exists some $y \in B_{2\bar{s}}(\bar{x})\cap\mathcal{C}$ such that

$$|T_{\bar{x},2\bar{s}}w - T_{\bar{x},2\bar{s}}u(y)| \leq 3\tau\bar{s}. \tag{9.24}$$

By the Hölder growth estimate for $T_{x,s}$, with respect to s in the Transformation Proposition 7.7, we have

$$|T_{y,2\bar{s}}T^{-1}_{\bar{x},2\bar{s}} - I| \leq C(n)\epsilon.$$

This follows since $|T_{\bar{x},2\bar{s}}T^{-1}_{\bar{x},5\bar{s}} - I| \leq \epsilon$ and $|T_{y,2\bar{s}}T^{-1}_{\bar{x},5\bar{s}} - I| \leq \epsilon$ due to the fact that $T_{\bar{x},5\bar{s}}u : B_{2\bar{s}}(y) \to \mathbb{R}^k$ is also a $C\epsilon$-splitting map. Therefore,

$$|T_{y,2\bar{s}}w - T_{y,2\bar{s}}u(y)| \leq 4\tau\bar{s}. \tag{9.25}$$

Again by the Hölder growth estimate for $T_{y,s}$, in the Transformation Proposition 7.7 we have

$$|T_{y,\bar{s}/2}w - T_{y,\bar{s}/2}u(y)| \leq 5\tau\bar{s}.$$

Since $w \in T^{-1}_{y,\bar{s}/2}B_{\bar{s}/2}(u(y))$, this contradicts the definition of $\bar{s}$. This concludes the proof of Lemma 9.10. □

From Lemma 9.10 we obtain

$$\begin{aligned} C(k) &\leq \sum_{x\in\mathcal{C}\cap B_1(p)} \mathrm{Vol}_k(T^{-1}_{x,r_x}B_{r_x}(u(x))) \\ &\leq \sum_{x\in\mathcal{C}\cap B_1(p)} C_k r_x^k |T^{-1}_{x,r_x}| \leq C_k \sum_{x\in\mathcal{C}\cap B_1(p)} r_x^k = C_k\,\mu(B_1(p)). \end{aligned} \tag{9.26}$$

By using the estimate $|T^{-1}_{x,r_x}| \leq 2$, this provides a lower bound for $\mu(B_1(p))$. This completes the proof of Proposition 9.9. □

9.4. *Proof of the Neck Structure Theorem for smooth manifolds.* In the present subsection, we will prove the Ahlfors regularity estimate for the case of smooth Riemannian manifolds. The Ahlfors regularity estimate in the general case will be reduced to this one via a careful approximation argument.

In the case of smooth Riemann manifolds, neck regions satisfy $\mathcal{C}_0 = \emptyset$ and $\inf r_x > 0$. Thus, it suffices to prove the following lemma

LEMMA 9.11 (The smooth case of Theorem 2.9). *For all $\eta > 0$, there exist $\delta = \delta(n, \mathrm{v}, \eta) > 0$ and $A(n)$ such that if $\mathcal{N} \subset B_2(p) \subset M^n$ is a (k, δ, η)-neck, then for all $s \geq r_x$ with $B_{2s}(x) \subset B_2(p)$,*

$$A(n)^{-1}s^k \leq \mu(B_s(x)) \leq A(n)s^k. \tag{9.27}$$

Proof. We begin by making the following convention.

Terminology. We will say *statement* (j) *holds* if the lemma holds for all neck regions which satisfy $\inf r_x \geq 2^{-j} > 0$. The proof will be by induction on j.

We begin with the base step. Note that if $j \leq 10$ and $\delta \leq 10^{-10n}$, then $\#\mathcal{C} \leq C(n)$. In particular, the statement (j) holds for some universal constant $A_0(n)$.

Denote the universal constant $A(n)$ in Proposition 9.9 by $A_1(n)$. We will show that (j) holds for all j when $A(n) := A_0(n) + A_1(n)$ and $\delta(n, \mathrm{v}, \eta) = \delta(n, \mathrm{v}, \eta, B)$, where $\delta(n, \mathrm{v}, \eta, B)$ is the constant in Proposition 9.9 with $B = A(n)C(n)$. Here $C(n)$ is given by $C(n) = C_0(n)16^k$, where $C_0(n)$ is the cardinality of the maximal number of disjoint balls $\{B_{2^{-5}}(x_i)\,|\,x_i \in B_2(0^k)\}$ with center in $B_2(0^k)$. Therefore, $B = A(n)C(n)$ is a universal constant.

Note that if we take $\delta \le \delta(n, \mathrm{v}, B, \eta, \delta')$ sufficiently small, then by the structure of the neck region and cone-splitting theorem 4.9, there exists a (k, δ')-splitting map $u : B_2(p) \to \mathbb{R}^k$; see also Remark 9.2. Therefore, the constant $\delta'(n, \mathrm{v}, B, \eta)$ in Proposition 9.9 automatically holds if we choose $\delta \le \delta(n, \mathrm{v}, B, \eta, \delta')$.

Now let us assume statement (j) holds. Then we need to see that $(j+1)$ holds. So let $\mathcal{N} \subset B_2(p)$ be a (k, δ, η)-neck region with $\min_x r_x \ge 2^{-j-1}$ and the associated center points $\mathcal{C}$. By Proposition 9.9, it suffices to obtain a weak Ahlfors regularity bound for μ with $B = A(n)C(n)$.

Let $B_{2s}(x) \subset B_2(p)$. If $s \le 1/2$, then after rescaling $B_{2s}(x)$ to $B_2(x)$, the region $\mathcal{N} \cap B_{2s}(x) \subset B_{2s}(x)$ is a new (k, δ, η)-neck which satisfies (j). In particular, by our inductive hypothesis, we have

$$A^{-1}(n) \le \mu(B_s(x)) \le A(n)s^k.$$

If $s > 1/2$ then, in particular, we have $x \in B_{3/2}(p) \cap \mathcal{C}$ and $B_s(x) \subset B_{7/4}(p)$.

Choose a Vitali covering $\{B_{1/16}(x_j), x_j \in \mathcal{C} \cap B_{7/4}(p)\}$ of $B_{7/4}(p)$ with cardinality at most $C_0(n)$. Since $B_{1/8}(x_j) \subset B_2(p)$, by using the inductive assumption again we have

$$16^{-k}A^{-1}(n) \le \mu(B_{1/16}(x_j)) \le A(n)16^{-k}. \tag{9.28}$$

From this, it follows easily that

$$16^{-k}A^{-1}(n)s^k \le \mu(B_s(x)) \le C_0(n)16^{-k}2^k A(n)s^k. \tag{9.29}$$

Thus, we have proved μ satisfies the weak Ahlfors regularity estimate with constant $B = C_0(n)16^k A(n)$. By Proposition 9.9, if $\delta \le \delta(n, \mathrm{v}, \eta, B) = \delta(n, \mathrm{v}, \eta)$, then in fact we have the stronger estimate $A_1(n)^{-1}s^k \le \mu(B_s(x)) \le A_1(n)s^k$. In particular,

$$A(n)^{-1}s^k \le \mu(B_s(x)) \le A(n)s^k.$$

This completes the proof of Lemma 9.11, i.e., Ahlfors regularity for the case of smooth manifolds. □

9.5. *Approximating limit neck regions by smooth neck regions.* As mentioned in the previous subsection, to prove the neck structure theorem for neck regions for which $\mathcal{C}_0 \neq \emptyset$, we will approximate general neck regions $\mathcal{N}$ by neck regions $\mathcal{N}_j$ for which $\inf r_{x,j} > 0$. This will be carried out in the present subsection. In the following subsection, we will complete the proof of the Neck Structure Theorem 2.9.

Our main result in this subsection is the following.

THEOREM 9.12. *Let* $(M_j^n, g_j, p_j) \xrightarrow{d_{GH}} (X^n, d, p)$ *satisfy* $\mathrm{Vol}(B_1(p_j)) > v > 0$ *and* $\mathrm{Ric}_i \geq -(n-1)\delta^2$ *and let* $\mathcal{N} = B_2(p) \setminus \overline{B}_{r_x}(\mathcal{C})$ *be a* (k, δ, η)*-neck region. Then there exists a* (k, δ_j, η_j)*-neck region* $\mathcal{N}_j = B_2(p_j) \setminus \overline{B}_{r_{x,j}}(\mathcal{C}_j)$ *such that the following hold*:

(1) $\delta_j \to \delta$ *and* $\eta_j \to \eta$.
(2) *If* $\phi_j : B_2(p_j) \to B_2(p)$ *are the approximating Gromov-Hausdorff maps, then* $\phi_j(\mathcal{C}_j) \to \mathcal{C}$ *in the Hausdorff sense.*
(3) $r_{x,j} \to r_x : \mathcal{C} \to \mathbb{R}_+$ *uniformly.*
(4) *Let* μ_j, μ *denote the packing measure of* $\mathcal{N}_j$, *and* $\mathcal{N}$, *respectively and let* $\mu_j \to \mu_\infty$ *in measure sense. Then* $\mu \leq C(n)\mu_\infty$.
(5) *If* $\mathcal{C}_0 \subset \mathcal{C}$ *is* k*-rectifiable, we have* $\mu_\infty \leq C(n)\mu$.

Proof. Consider first the case $\inf r_x > 0$. This implies that $\mathcal{C}_0 = \emptyset$ and, in addition, that $\mathcal{C}$ is a finite set.

Let $\psi_j : B_2(p) \to B_2(p_j)$ be the ϵ_j-Gromov Hausdorff maps. For j sufficiently large with $\epsilon_j \ll \inf r_x$, let $\mathcal{C}_j := \{\psi_j(x),\ x \in \mathcal{C}\}$ and $r_{x,j} := r_{\psi_j^{-1}(x)}$. Then it is easy to check that $\mathcal{N}_j := B_2(p_j) \setminus \bar{B}_{r_{x,j}}(\mathcal{C}_j)$ are (k, δ_j, η_j)-neck regions which satisfy the criteria of the theorem. Actually, in this case, we can prove $\mu_j \to \mu_\infty = \mu$.

Next, for the case in which $\inf r_x = 0$, we construct a (k, δ, η)-neck region

$$\tilde{\mathcal{N}}_s = B_2(p) \setminus \bar{B}_{\tilde{r}_x}(\tilde{\mathcal{C}}),$$

with $\inf \tilde{r}_x \geq s > 0$. Given $s > 0$, we define $\tilde{r}_x$ on $\mathcal{C}$ by $\tilde{r}_x := \max\{r_x, s\}$. Then $|\mathrm{Lip}\ \tilde{r}_x| \leq \delta$ and all of the remaining properties of a neck region are satisfied with $\mathcal{C}$ and $\tilde{r}_x$, apart from the Vitali condition (n1).

To fix this, choose a maximal subset $\tilde{\mathcal{C}}_s := \{x_i^s\} \subset \mathcal{C}$ such that the balls $\{B_{\tau^2 \tilde{r}_{x_i^s}}(x_i^s)\}$ are disjoint. It is easy to check that $\tilde{\mathcal{N}}_s := B_2(p) \setminus \bar{B}_{\tilde{r}_x}(\tilde{\mathcal{C}})$ is a (k, δ, η)-neck region for which $\inf \tilde{r}_x \geq s > 0$. If we let $s \to 0$, then $\tilde{\mathcal{N}}_s$ converges to $\mathcal{N}$ in the Hausdorff sense.

Consider the limit packing measure $\tilde{\mu}_s \to \tilde{\mu}_\infty$. On $\mathcal{C}_+$, we have $\tilde{\mu}_\infty = \mu$. If $y \in \mathcal{C}_0$, then for all $s < r$, by the Vitali covering property of $\tilde{\mathcal{N}}_s$, it will follow that

$$s^{k-n}\mathrm{Vol}\Big(\bar{B}_r(y) \cap B_s(\mathcal{C}_0)\Big) \leq C(n)\tilde{\mu}_s\Big(\bar{B}_{2s+r}(y) \cap B_{3s}(\mathcal{C}_0)\Big). \tag{9.30}$$

To see this, consider the covering $\{B_s(x_i^s), x_i^s \in \tilde{\mathcal{C}}_s \cap B_s(\mathcal{C}_0)\}$ of $\mathcal{C}_0 \cap B_r(y)$. Since $B_{\tau_n^2 s}(x_i^s)$ are disjoint and $\tilde{\mu}_s(B_s(x_i^s)) \geq s^k$, using the estimate of the cardinality of $\{B_s(x_i^s)\,|\,x_i^s\in\tilde{\mathcal{C}}_s\cap B_s(\mathcal{C}_0)\}$ by $s^{-k}\tilde{\mu}_s\Big(B_{2s+r}(y)\cap B_{3s}(\mathcal{C}_0)\Big)$ we can get (9.30).

By letting $s\to 0$ in (9.30), we get the upper Minkowski k content bound,

$$\mathcal{M}^k(\bar{B}_r(y)\cap\mathcal{C}_0)\le C(n)\tilde{\mu}_\infty(\bar{B}_r(y)\cap\mathcal{C}_0). \tag{9.31}$$

In particular, this implies

$$\begin{aligned}\mu(\bar{B}_r(y)\cap\mathcal{C}_0)&=\mathcal{H}^k(\bar{B}_r(y)\cap\mathcal{C}_0)\\&\le C(k)\mathcal{M}^k(\bar{B}_r(y)\cap\mathcal{C}_0)\le C(n)\tilde{\mu}_\infty(\bar{B}_r(y)\cap\mathcal{C}_0).\end{aligned} \tag{9.32}$$

Therefore, we get the weaker estimate $\mu\le C(n)\tilde{\mu}_\infty$.

On the other hand, we claim that

$$\tilde{\mu}_s\Big(B_r(y)\cap B_s(\mathcal{C}_0)\Big)\le C(n)s^{k-n}\mathrm{Vol}\Big(B_{r+2s}(y)\cap B_{3s}(\mathcal{C}_0)\Big). \tag{9.33}$$

To see this, consider the covering $\{B_s(x_i^s)\,|\,x_i^s\in\tilde{\mathcal{C}}_s\cap B_s(\mathcal{C}_0)\}$ of $\mathcal{C}_0\cap B_r(y)$. Since $B_{\tau_n^2 s}(x_i^s)$ are disjoint and $\tilde{\mu}_s(B_s(x_i^s))\le A(n)s^k$, the estimate (9.33) follows easily from the estimate of the cardinality of $\{B_s(x_i^s), x_i^s\in\tilde{\mathcal{C}}_s\cap B_s(\mathcal{C}_0)\}$ by $s^{-n}\mathrm{Vol}\big(B_{r+2s}(y)\cap B_{3s}(\mathcal{C}_0)\big)$. By letting $s\to 0$, it follows that the upper Minkowski k content satisfies

$$C(n)\mathcal{M}^k(B_r(y)\cap\mathcal{C}_0)\ge\tilde{\mu}_\infty(B_r(y)\cap\mathcal{C}_0). \tag{9.34}$$

To prove (5) of Theorem 9.12, we will initially make the assumption that $\mathcal{C}_0$ is k-rectifiable. This will be proved in Lemma 9.14, the proof of which is completely independent of (5).

By a standard geometric measure theory argument (see Theorem 3.2.39 of [Fed69]), Hausdorff measure and Minkowski content are equivalent. Thus,

$$C(n)\mu(B_r(y)\cap\mathcal{C}_0)\ge\tilde{\mu}_\infty(B_r(y)\cap\mathcal{C}_0). \tag{9.35}$$

In particular, $C(n)\mu\ge\tilde{\mu}_\infty$.

Finally, for each $\tilde{\mathcal{N}}_s$, we have the (k,δ_j,η_j)-neck regions $\tilde{\mathcal{N}}_{s,j}=B_2(p_j)\setminus\bar{B}_{\tilde{r}_{x,j}}(\tilde{\mathcal{C}}_j)$ approximating $\tilde{\mathcal{N}}_s$ with $\tilde{\mu}_{s,j}\to\tilde{\mu}_s$. By a standard diagonal argument, we can finish the proof by taking a diagonal subsequence of $\tilde{\mathcal{N}}_{s,j}$ to approximate $\mathcal{N}$. This completes the proof of Theorem 9.12. $\square$

9.6. *Proof of the Neck Structure Theorem* 2.9. Given a (k,δ,η)-neck region $\mathcal{N}=B_2(p)\setminus\bar{B}_{r_x}(\mathcal{C})$, we have by the approximation theorem 9.12, a sequence of (k,δ_j,η_j)-neck regions $\mathcal{N}_j=B_2(p_j)\setminus\bar{B}_{r_{x,j}}(\mathcal{C}_j)\subset M_j$. By the Ahlfors regularity estimates in Section 9.4 for smooth neck regions, we have for $\delta\le\delta(n,\mathrm{v},\eta)/10$ that if $B_{2r}(x_j)\subset B_2(p_j)$ and $x_j\in\mathcal{C}_j$, then for j sufficiently large,

$$A(n)^{-1}r^k\le\mu_j(B_r(x_j))\le A(n)r^k. \tag{9.36}$$

Thus, by Theorem 9.12 we have for all $B_{2r}(x) \subset B_2(p)$ with $x \in \mathcal{C}$ that the limit $\mu_j \to \mu_\infty$ satisfies

$$A(n)^{-1} r^k \leq \mu_\infty(B_r(x)) \leq A(n) r^k. \tag{9.37}$$

By Theorem 9.12, since $\mu \leq C(n)\mu_\infty$, we directly get the upper bound estimates of $\mu(B_r(x)) \leq \tilde{A}(n) r^k$ for a universal constant $\tilde{A}(n) = A(n)C(n)$.

In order to prove the lower measure bound, we will first prove $\mathcal{C}_0$ is k-rectifiable. Then we can use (5) from Theorem 9.12 to deduce the lower bound. The main lemma needed for this result is the following.

LEMMA 9.13. *For each $\epsilon > 0$, if $\delta \leq \delta(n, \mathrm{v}, \epsilon, \eta)$, then for any $x \in \mathcal{C}_0$ and $B_{2r}(x) \subset B_2(p)$, there exists a closed subset $\mathcal{R}_\epsilon(B_r(x)) \subset \mathcal{C}_0 \cap B_r(x)$ such that $\mathcal{R}_\epsilon$ is bi-Lipschitz to a subset of $\mathbb{R}^k$ and $\mu\Big(B_r(x) \cap (\mathcal{C}_0 \setminus \mathcal{R}_\epsilon)\Big) < \epsilon r^k$.*

Proof. For each $B_{2r}(x) \subset B_2(p)$ with $x \in \mathcal{C}_0$, the set

$$\mathcal{N}_r := B_{2r}(x) \setminus \bar{B}_{r_x}(\mathcal{C}_r)$$

is a (k, δ, η)-neck region with associated $\mathcal{C}_r = \mathcal{C} \cap B_{2r}(x)$ and packing measure $\mu_r := \mu|_{\mathcal{C}_r}$. By the approximation theorem 9.12, there exists a (k, δ_j, η_j)-neck region

$$\mathcal{N}_{r,j} := B_{2r}(x_j) \setminus \bar{B}_{r_{x,j}}(\mathcal{C}_{r,j}) \subset M_j$$

which approximates $\mathcal{N}_r$.

By Theorem 4.11, there exist δ'_j-splitting maps $u_{r,j} : B_{2r}(x_j) \to \mathbb{R}^k$ with $\delta'_j = \delta'_j(n, \mathrm{v}, \eta, \delta_j)$. Additionally, by the Ahlfors regularity estimate for the smooth neck $\mathcal{N}_{r,j}$ in Section 9.4, we have for any $B_{2s}(x_{r,j}) \subset B_{2r}(x_j)$ and $x_{r,j} \in \mathcal{C}_{r,j}$ that

$$A(n)^{-1} s^k \leq \mu(B_s(x_{r,j})) \leq A(n) s^k. \tag{9.38}$$

By applying Proposition 9.3 with $B = A(n)$ and $\delta \leq \delta(n, \mathrm{v}, \epsilon, \eta)$, there exists $\mathcal{C}_{r,j,\epsilon} \subset \mathcal{C}_{r,j}$ such that $u_{r,j} : \mathcal{C}_{r,j,\epsilon} \to \mathbb{R}^k$ is $(1+\epsilon)$-bi-Lipschitz and $\mu_{r,j}(B_r(x_j) \setminus \mathcal{C}_{r,j,\epsilon}) \leq \epsilon^2 r^k$. Let $j \to \infty$, and denote the Gromov-Hausdorff limit by $\mathcal{C}_{r,\epsilon} := \lim \mathcal{C}_{r,\epsilon,j}$. Let $\mu_{r,\infty}$ denote the limit measure $\mu_{r,j} \to \mu_{r,\infty}$.

On the other hand, since $B_r(x) \setminus \mathcal{C}_{r,\epsilon}$ is an open set, a standard measure convergence argument implies

$$\mu_{r,\infty}(B_r(x) \setminus \mathcal{C}_{r,\epsilon}) \leq \liminf \mu_{r,j}(B_r(x_j) \setminus \mathcal{C}_{r,j,\epsilon}) \leq \epsilon^2 r^k. \tag{9.39}$$

Indeed, for any closed set $D \subset \bar{B}_r(x) \subset X$ and $D_i \subset M_i$, with $D_i \xrightarrow{d_{GH}} D$, we have by the measure convergence that for any $t > 0$,

$$\limsup \mu_{r,j}(D_i) \leq \mu_{r,\infty}(B_t(D)). \tag{9.40}$$

Let $t \to 0$. By using the monotone convergence theorem for measures and the fact that D is a closed set, it follows that

$$\limsup \mu_{r,j}(D_i) \le \mu_{r,\infty}(D).$$

This implies (9.39). Hence we have $\mathcal{C}_{r,\epsilon} \subset \mathcal{C}_r \subset \mathcal{C}$ and the estimate

(9.41)
$$\mu(B_r(x) \setminus \mathcal{C}_{r,\epsilon}) = \mu_r(B_r(x) \setminus \mathcal{C}_{r,\epsilon}) \le C(n)\mu_{r,\infty}(B_r(x) \setminus \mathcal{C}_{r,\epsilon}) \le C(n)\epsilon^2 r^k \le \epsilon r^k.$$

Here, we have used Theorem 9.12 in the first inequality.

Moreover, since $u_{r,j}$ is Lipschitz, by Ascoli's theorem, we have a uniform limit $u_r : B_{2r}(x) \to \mathbb{R}^k$ such that $u_r : \mathcal{C}_{r,\epsilon} \to \mathbb{R}^k$ is $(1+\epsilon)$-bilipschitz. From the estimate (9.41), the set $\mathcal{R}_\epsilon(B_r(x)) := \mathcal{C}_0 \cap \mathcal{C}_{r,\epsilon}$ is our desired set. This finishes the proof of Lemma 9.13. $\square$

Now we can prove the rectifiability of $\mathcal{C}_0$.

LEMMA 9.14. *$\mathcal{C}_0$ is rectifiable.*

Proof. Let $\{x_i\} \subset \mathcal{C}_0$ be a countable dense subset of $\mathcal{C}_0$, and for any $\epsilon > 0$, consider the set

$$\mathcal{R} := \bigcup_{B_{2r}(x_i):1\ge r\in\mathbb{Q}} \mathcal{R}_\epsilon(B_r(x_i)). \tag{9.42}$$

By definition, we have $\mathcal{R} \subset \mathcal{C}_0$. In addition, since $\mathcal{R}$ is a countable union of rectifiable sets, it is rectifiable. To finish the proof, we only need to choose a small ϵ and show that $\mu(\mathcal{C}_0 \setminus \mathcal{R}) = \mathcal{H}^k(\mathcal{C}_0 \setminus \mathcal{R}) = 0$. So assume $\mathcal{H}^k(\mathcal{C}_0 \setminus \mathcal{R}) > 0$. Then by a standard geometric measure theory argument, there exist $x \in \mathcal{C}_0 \setminus \mathcal{R}$, $r_a \to 0$ and a dimensional constant $\epsilon_k > 0$ (see Theorem 3.6 of [Sim83]) such that

$$\lim_{r_a\to 0} \frac{\mathcal{H}^k\Big(B_{r_a}(x) \cap (\mathcal{C}_0 \setminus \mathcal{R})\Big)}{r_a^k} > \epsilon_k > 0. \tag{9.43}$$

In particular, there exists $s > 0$ such that $\mathcal{H}^k\big(B_s(x) \cap (\mathcal{C}_0 \setminus \mathcal{R})\big) \ge s^k\epsilon_k/2$. Since $\{x_i\}$ is dense, there exist some x_i and $r \in \mathbb{Q}$ such that $s \le r \le 2s$ and $B_s(x) \subset B_r(x_i)$. Therefore, we have $\mathcal{H}^k(B_r(x_i) \cap (\mathcal{C}_0 \setminus \mathcal{R})) \ge C(k)\epsilon_k r^k$. By choosing $\epsilon = \epsilon(n)$ small, we contradict the definition of $\mathcal{R}_\epsilon$ in Lemma 9.13. Thus, for $\delta \le \delta(n, \mathrm{v}, \eta, \epsilon) = \delta(n, \mathrm{v}, \eta)$, the set $\mathcal{R} \subset \mathcal{C}_0$ has full measure. This completes the proof of Lemma 9.14. $\square$

At this point we can obtain the lower bound for the packing measure μ, and hence, complete the proof of Theorem 2.9. Since $\mathcal{C}_0$ is k-rectifiable, by Theorem 9.12 we have $\mu \ge C(n)\cdot\mu_\infty$ in (9.37). Therefore, the Ahlfors regularity estimate for μ_∞ in (9.37) gives us the desired lower bound for μ. This completes the proof of the Neck Structure Theorem 2.9. $\square$

10. Proof of the Neck Decomposition Theorem 2.12

In this section we prove the Neck Decomposition Theorem 2.12. Neck regions and their associated decomposition theorems were introduced in [JN21], where the focus was on the top $(n-4)$-stratum of the singular set for limits with 2-sided Ricci curvature bounds. This was an important ingredient in the proof of the a priori L^2 curvature bound for such spaces. This section follows very closely the constructions of [JN21], relying on the estimates provided by the Neck Structure Theorem 2.9. The main result of this section is Theorem 2.12, which for convenience is recalled below.

THEOREM 2.12 RESTATED. *Let $(M_i^n, g_i, p_i) \to (X^n, d, p)$ satisfy $\mathrm{Vol}(B_1(p_i)) > \mathrm{v} > 0$ and $\mathrm{Ric}_i \geq -(n-1)$. Then for each $\eta > 0$ and $\delta \leq \delta(n, \mathrm{v}, \eta)$, we can write*

$$B_1(p) \subseteq \bigcup_a \left(\mathcal{N}_a \cap B_{r_a}\right) \cup \bigcup_b B_{r_b}(x_b) \cup \mathcal{S}^{k,\delta,\eta}, \tag{10.1}$$

$$\mathcal{S}^{k,\delta,\eta} \subseteq \bigcup_a \left(\mathcal{C}_{0,a} \cap B_{r_a}\right) \cup \tilde{\mathcal{S}}^{k,\delta,\eta}, \tag{10.2}$$

such that

(1) *for all a, the set $\mathcal{N}_a = B_{2r_a}(x_a) \setminus \overline{B}_{r_x}(\mathcal{C})$ is a (k, δ, η)-neck region;*
(2) *the balls $B_{2r_b}(x_b)$ are $(k+1, 2\eta)$-symmetric, and hence $x_b \notin S^k_{2\eta, r_b}$;*
(3) $\sum_a r_a^k + \sum_b r_b^k + \mathcal{H}^k\left(\mathcal{S}^{k,\delta,\eta}\right) \leq C(n, \mathrm{v}, \delta, \eta)$;
(4) *$\mathcal{C}_{0,a} \subseteq B_{2r_a}(x_a)$ is the k-singular set associated to $\mathcal{N}_a$;*
(5) *$\tilde{\mathcal{S}}^{k,\delta,\eta}$ satisfies $\mathcal{H}^k\left(\tilde{\mathcal{S}}^{k,\delta,\eta}\right) = 0$;*
(6) *$\mathcal{S}^{k,\delta,\eta}$ is k-rectifiable;*
(7) *for any ϵ, if $\eta \leq \eta(n, \mathrm{v}, \epsilon)$ and $\delta \leq \delta(n, \mathrm{v}, \eta, \epsilon)$, then we have $S^k_\epsilon \subset \mathcal{S}^{k,\delta,\eta}$.*

Remark 10.1. As previously mentioned, in the special case of smooth Riemannian manifolds M^n only (1)–(3) carry nontrivial information.

10.1. *Proof of Theorem* 2.12 *modulo Proposition* 10.2. The proof of Theorem 2.12 proceeds via an iterative recovering argument. In Proposition 10.2 of this subsection, we will introduce a rougher decomposition which also includes a third type of ball, indexed by a subscript denoted by v. By iterating Proposition 10.2 we obtain a definite decrease in the volume of the v-balls. Thus, after iterating this recovering argument a definite number of times, the v-balls will no longer present. This gives the decomposition in Theorem 2.12.

The remaining sections, 10.2–10.5, will be devoted to establishing Proposition 10.2. This is the primary work in the proof. Initially, we will introduce coverings in which additional specific types of balls indexed by c, d, e will appear. Additional iterative arguments eventually lead to Proposition 10.2 itself.

In more detail, in Section 10.2, we define balls of types c, d, e. In Section 10.3, we state Propositions 10.3, 10.5, which are concerned, respectively, with recovering d-balls and c-balls. We state and prove Lemma 10.7 in Section 10.4. Using it, we prove Proposition 10.2 modulo Propositions 10.3 and 10.5. In Section 10.5, we prove Proposition 10.3. In Section 10.6, we prove Proposition 10.5, thereby completing the proof of Theorem 2.12.

To avoid confusion, we recall that in (10.3) below, the subscript 1 indicates radius 1. Set

$$\bar{V} := \inf_{y \in B_4(p)} \mathcal{V}_1(y) \geq \mathrm{v} > 0. \tag{10.3}$$

PROPOSITION 10.2 (Induction step decomposition). *For all $\eta > 0$ and $\delta \leq \delta(n, \mathrm{v}, \eta)$, there exists*

$$v^0(n, \mathrm{v}, \delta, \eta) > 0$$

such that if $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X^n, d, p)$ satisfies $\mathrm{Ric}_{M_i^n} \geq -(n-1)$, $\mathrm{Vol}(B_1(p_i)) > v > 0$, then the following exists:

$$B_1(p) \subset \bigcup_a (\mathcal{C}_{0,a} \cup \mathcal{N}_a \cap B_{r_a}(x_a)) \cup \bigcup_b B_{r_b}(x_b) \cup \bigcup_v B_{r_v}(x_v) \cup \tilde{\mathcal{S}}^k, \tag{10.4}$$

such that the following hold:

(1) *$\mathcal{N}_a \subset B_{2r_a}(x_a)$ are (k, δ, η)-neck regions with the associated singular set of centers $\mathcal{C}_{0,a}$;*
(2) *each b-ball $B_{2r_b}(x_b)$ is $(k+1, 2\eta)$-symmetric;*
(3) *$\bar{V}_v \geq \bar{V} + v^0$ (where $\bar{V}_v := \inf_{y \in B_{4r_v}(x_v)} \mathcal{V}_{r_v}(y)$);*
(4) *$\tilde{\mathcal{S}}^k \subset S$ and $\mathcal{H}^k(\tilde{\mathcal{S}}^k) = 0$;*
(5) *$\sum_a r_a^k + \sum_b r_b^k + \sum_v r_v^k \leq C(n, \mathrm{v}, \delta, \eta)$.*

If we temporarily assume Proposition 10.2, the proof of Theorem 2.12 can be completed:

Proof of Theorem 2.12. Fix $\eta > 0$, $\delta \leq \delta(n, \mathrm{v}, \eta)$ as in Theorem 2.9 and $v^0(n, \mathrm{v}, \delta, \eta) > 0$ as in Proposition 10.2.

By applying Proposition 10.2 to the limit ball $B_1(p)$, we get the following decomposition in which the subscript 1 indicates the first step in the inductive argument below:

$$B_1(p) \subset \tilde{\mathcal{S}}_1^k \cup \bigcup_{a_1} (\mathcal{C}_{0,a_1} \cup \mathcal{N}_{a_1} \cap B_{r_{a_1}}(x_{a_1})) \cup \bigcup_{b_1} B_{r_{b_1}}(x_{b_1}) \cup \bigcup_{v_1} B_{r_{v_1}}(x_{v_1}), \tag{10.5}$$

where

$$\begin{aligned}
&\bar{V}_{v_1} := \inf_{y \in B_{4r_{v_1}}(x_{v_1})} \mathcal{V}_{r_{v_1}}(y) \geq \bar{V} + \mathrm{v}^0, \\
&\mathcal{H}^k(\tilde{\mathcal{S}}^k_1) = 0, \\
&\sum_{a_1} \mathcal{H}^k(\mathcal{C}_{0,a_1}) + \sum_{a_1} (r_{a_1})^k + \sum_{b_1} (r_{b_1})^k + \sum_{v_1} (r_{v_1})^k \leq C(n, \mathrm{v}, \eta, \delta).
\end{aligned} \tag{10.6}$$

Next, by applying Proposition 10.2 to each v_1-ball $B_{r_{v_1}}(x_{v_1})$ we arrive at

$$\begin{aligned}
B_1(p) \subset \bigcup_{j=1}^{2} &\left(\tilde{\mathcal{S}}^k_j \cup \bigcup_{a_j} (\mathcal{C}_{0,a_j} \cup \mathcal{N}_{a_2} \cap B_{r_{a_j}}(x_{a_j})) \bigcup_{b_2} B_{r_{b_j}}(x_{b_j}) \right) \\
&\cup \bigcup_{v_2} B_{r_{v_2}}(x_{v_2}),
\end{aligned} \tag{10.7}$$

where

$$\begin{aligned}
&\bar{V}_{v_2} := \inf_{y \in B_{4r_{v_2}}(x_{v_2})} \mathcal{V}_{r_{v_2}}(y) \geq \bar{V} + 2\mathrm{v}^0, \\
&\mathcal{H}^k(\tilde{\mathcal{S}}^k_1) + \mathcal{H}^k(\tilde{\mathcal{S}}^k_2) = 0, \\
&\sum_{j=1}^{2} \left(\sum_{a_2} \mathcal{H}^k(\mathcal{C}_{0,a_2}) + \sum_{a_2} (r_{a_2})^k + \sum_{b_2} (r_{b_2})^k \right) \\
&\qquad\qquad\qquad\qquad \leq C(n, \mathrm{v}, \eta, \delta) + C(n, \mathrm{v}, \eta, \delta)^2, \\
&\sum_{v_2} (r_{v_2})^k \leq C(n, \mathrm{v}, \eta, \delta)^2.
\end{aligned} \tag{10.8}$$

Note that $\bar{V} + \mathrm{v}^0$ in (10.6) has been replaced by $\bar{V} + 2\mathrm{v}^0$ in (10.8), where as in Proposition 10.2, $v^0 = v^0(n, \mathrm{v}, \delta, \eta)$. Therefore, this process of recovering the v-balls can be iterated at most $i = i(n, \mathrm{v}, \delta, \eta)$ times before no v-balls exist; otherwise, we would contradict the noncollapsing assumption (1.2). By doing the iteration the maximal number of times, we obtain the following decomposition in which the v-balls are no longer present:

$$B_1(p) \subset \bigcup_{j=1}^{i} \left(\tilde{\mathcal{S}}^k_j \cup \bigcup_{a_j} (\mathcal{C}_{0,a_j} \cup \mathcal{N}_{a_j} \cap B_{r_{a_j}}(x_{a_j})) \cup \bigcup_{b_j} B_{r_{b_j}}(x_{b_j}) \right), \tag{10.9}$$

where $i = i(n, \mathrm{v}, \delta, \eta)$ and

$$\mathcal{H}^k(\tilde{\mathcal{S}}^k_1) + \cdots \mathcal{H}^k(\tilde{\mathcal{S}}^k_i) = 0, \tag{10.10}$$

$$\sum_{j=1}^{i} \left(\sum_{a_j} \mathcal{H}^k(\mathcal{C}_{0,a_j}) + \sum_{a_j} (r_{a_j})^k + \sum_{b_j} (r_{b_j})^k \right) \leq C'(n, \mathrm{v}, \eta, \delta). \tag{10.11}$$

Set

$$\tilde{\mathcal{S}}^{k,\delta,\eta} := \bigcup_{j=1}^{i} \tilde{S}^k_j \cap B_1(p), \qquad \mathcal{S}^{k,\delta,\eta} := \bigcup_{j=1}^{i} \left(\tilde{\mathcal{S}}^k_j \cup \bigcup_{a_j} \mathcal{C}_{0,a_j} \right) \cap B_1(p). \tag{10.12}$$

Since by the Neck Structure Theorem 2.9, each set $\mathcal{C}_{0,a_j}$ is k-rectifiable, it follows that $\mathcal{S}^{k,\delta,\eta}$ is k-rectifiable and by (10.11) that $\mathcal{H}^k(\mathcal{S}^{k,\delta,\eta}) \le C(n, \mathrm{v}, \eta, \delta)$. This gives the decomposition whose existence is asserted in Theorem 2.12. Moreover, from our decomposition, conditions (1)–(6) of that theorem are satisfied, where the content estimate is in (10.11) and $\mathcal{H}^k(\tilde{\mathcal{S}}^{k,\delta,\eta}) = 0$.

Finally, we will show that if $\eta \le \eta(n, \mathrm{v}, \epsilon)$, $\delta \le \delta(n, \mathrm{v}, \eta, \epsilon)$, then $S^k_\epsilon \subset \mathcal{S}^{k,\delta,\eta}$, which is the last statement (7) in Theorem 2.12

First, note that if $y \in \mathcal{N}_a$, with $r = d(y, \mathcal{C}_a)$ and $\delta \le \delta(n, \mathrm{v}, \eta, \epsilon)$, then by the Cone-Splitting Theorem 4.9, the ball $B_{r/2}(y)$ has a $(k+1, 2\eta)$-splitting. For any $\epsilon > 0$, by the Almost Volume Cone implies Almost Metric Cone Theorem 4.1, it follows that for some, $s = s(\epsilon, \mathrm{v}) \cdot r$, the ball, $B_s(y)$ is $(0, \epsilon^3)$-symmetric. If in addition, $\eta \le \eta(n, \mathrm{v}, \epsilon)$, this implies that $B_s(y)$ is $(k+1, \epsilon^2)$-symmetric. Hence, $y \notin S^k_\epsilon$.

Similarly, suppose $y \in B_{r_b}(x_b)$ and $B_{2r_b}(x_b)$ is $(k+1, \eta)$-symmetric. If in addition, $\eta \le \eta(n, \mathrm{v}, \epsilon, \eta')$, then it clear that $B_{r_b}(x_b)$ has a $(k+1, \eta')$-splitting. Then the same argument as above shows that if $\eta' \le \eta'(n, \mathrm{v}, \epsilon)$, then $y \notin S^k_\epsilon$. Since S^k_ϵ is covered by the union of $\mathcal{N}_a, B_{r_b}$ and $\mathcal{S}^{k,\delta,\eta}$, we see that $S^k_\epsilon \subset \mathcal{S}^{k,\delta,\eta}$. This completes the proof of Theorem 2.12, modulo the proof of Proposition 10.2. □

The remainder of this section will now be devoted to proving Proposition 10.2.

10.2. *Notation: constants and types of balls.* Throughout the remainder of this section we will consider constants $\xi, \delta, \gamma, \epsilon$, which will in general satisfy

$$0 < \xi \ll \delta < \gamma < \epsilon < \epsilon(n). \tag{10.13}$$

We will assume throughout that $\mathrm{Ric}_{M^n} \ge -(n-1)\xi$. The general case can be achieved by a standard covering argument and rescaling.[12]

As in Definition 4.8, we define the set of points with small volume pinching by

$$\bar{V} := \inf_{x \in B_4(p)} \mathcal{V}_{\xi^{-1}}(x)). \tag{10.14}$$

[12]Given $\xi \ll \delta$, choose a Vitali covering, $\{B_\xi(y_f)\}$, of $B_1(p)$, such that $B_{\xi/5}(y_f)$ are disjoint. By relative volume comparison, the cardinality of such covering is less than $C(n, \mathrm{v}, \xi)$. Finding the desired decomposition for $B_1(p)$ is then reduced to finding the corresponding decomposition for each $B_\xi(y_f)$.

In what follows, the set with small volume pinching is defined to be

$$\mathcal{P}_{r,\xi}(x) := \{y \in B_{4r}(x) : \mathcal{V}_{\xi r}(y) \leq \bar{V} + \xi\}. \tag{10.15}$$

The constants $\epsilon, \gamma > 0$ will denote the constants in the Cone-Splitting Theorem 4.9 based on k-content. Recall that this theorem states the following:

> *If* $\mathrm{Vol}(B_\gamma(\mathcal{P}_{1,\xi}(p))) \geq \epsilon\gamma^{n-k}$ *with* $0 < \delta, \epsilon \leq \delta(n, \mathrm{v})$, $\gamma \leq \gamma(n, \mathrm{v}, \epsilon)$, $\xi \leq \xi(\delta, \epsilon, \gamma, n, \mathrm{v})$, *then there exists* $q \in B_4(p)$ *such that* $B_{\delta^{-1}}(q)$ *is* (k, δ^2)-*symmetric.*

Next we introduce the various ball types which appear in the proof. These are indexed by a, b, c, d, e. Every ball $B_r(x)$ is one (or more) of these types. The balls indexed by a, b are of the type as in Proposition 10.2.

(a) A ball $B_{r_a}(x_a)$ is associated to a (k, δ, η)-neck region $\mathcal{N}_a \subset B_{2r_a}(x_a)$.
(b) A ball $B_{r_b}(x_b)$ is $(k+1, 2\eta)$-symmetric.
(c) A ball $B_{r_c}(x_c)$ is not a b-ball and satisfies

$$\mathrm{Vol}\Big(B_{\gamma r_c}(\mathcal{P}_{r_c,\xi}(x_c))\Big) \geq \epsilon\gamma^{n-k} r_c^n.$$

(d) A ball $B_{r_d}(x_d)$ is any ball with $\mathcal{P}_{r_d,\xi}(x_d) \neq \emptyset$ satisfying

$$\mathrm{Vol}\Big(B_{\gamma r_d}(\mathcal{P}_{r_d,\xi}(x_d))\Big) < \epsilon\gamma^{n-k} r_d^n.$$

(e) A ball $B_{r_e}(x_e)$ satisfies $\mathcal{P}_{r_e,\xi}(x_e) = \emptyset$.

10.3. *Statements of Propositions* 10.3 *and* 10.5. The first proposition in this subsection asserts that a d-ball can be recovered using only balls of type b, c and e. A key point is that in this covering, the content of the c-balls in the collection can be taken to be small.

Proposition 10.3 (d-ball decomposition). *Fix* $\eta > 0$, $\epsilon \leq \epsilon(n, \mathrm{v})$, $\gamma \leq \gamma(n, \mathrm{v}, \epsilon)$, $\delta \leq \delta(n, \mathrm{v}, \eta)$ *and* $\xi \leq \xi(n, \mathrm{v}, \epsilon, \gamma, \delta, \eta)$. *Let* $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X^n, d, p)$ *satisfy* $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$ *and* $\bar{V} \leq \inf_{x \in B_4(p)} \mathcal{V}_{\xi^{-1}}(x)$. *Assume also* $\mathrm{Ric}_{M_i^n} \geq -(n-1)\xi$, $\mathrm{Vol}(B_\gamma(\mathcal{P}_{1,\xi}(p))) < \epsilon\gamma^{n-k}$. *Then there exists a decomposition*

$$B_1(p) \subseteq \tilde{\mathcal{S}}^k_d \cup \bigcup B_{r_b}(x_b) \cup \bigcup B_{r_c}(x_c) \cup \bigcup B_{r_e}(x_e), \tag{10.16}$$

where

(b) *each* b-*ball* $B_{2r_b}(x_b)$ *is* $(k+1, 2\eta)$-*symmetric*;
(c) *a* c-*ball* $B_{2r_c}(x_c)$ *is not a* b-*ball and satisfies* $\mathrm{Vol}(B_{\gamma r_c}\mathcal{P}_{r_c,\xi}(x_c)) \geq \epsilon\gamma^{n-k} r_c^n$;
(e) *each* e-*ball* $B_{2r_e}(x_e)$ *satisfies* $\mathcal{P}_{r_e,\xi}(x_e) = \emptyset$;
(s) $\tilde{\mathcal{S}}^k_d \subset S$ *and* $\mathcal{H}^k(\tilde{\mathcal{S}}^k_d) = 0$.

Furthermore, we have k-content estimates

$$\sum_b r_b^k + \sum_e r_e^k \le C(n,\gamma), \tag{10.17}$$

$$\sum_c r_c^k \le C(n,\mathrm{v})\epsilon. \tag{10.18}$$

Remark 10.4. In this proposition, the ball types and the pinching set $\mathcal{P}_{r,\xi}$ are with respect to the given $\bar V \le \inf_{x\in B_4(p)} \mathcal{V}_{\xi^{-1}}(x)$ above.

PROPOSITION 10.5 (c-ball decomposition). *Let $\epsilon \le \epsilon(n,\mathrm{v})$, $\gamma \le \gamma(n,\mathrm{v},\epsilon)$, $\delta \le \delta(n,\mathrm{v},\eta), \xi \le \xi(n,\mathrm{v},\epsilon,\gamma,\delta,\eta)$. Let $(M_i^n, g_i, p_i) \to (X,d,p)$ satisfy* $\mathrm{Vol}(B_1(p_i)) \ge \mathrm{v} > 0$, *and let $\eta > 0$ and $\bar V \le \inf_{x\in B_4(p)} \mathcal{V}_{\xi^{-1}}(x)$. Assume in addition that* $\mathrm{Ric}_{M_i^n} \ge -(n-1)\xi$, $\mathrm{Vol}(B_\gamma(\mathcal{P}_{1,\xi}(p))) \ge \epsilon\gamma^{n-k}$ *and $B_2(p)$ is not $(k+1,2\eta)$-symmetric. Then there exists a decomposition*

$$\begin{aligned} B_1(p) \subset &\Big(\mathcal{C}_0 \cup \mathcal{N} \cap B_1(p)\Big) \\ &\cup \bigcup_b B_{r_b}(x_b) \cup \bigcup_c B_{r_c}(x_c) \cup \bigcup_d B_{r_d}(x_d) \cup \bigcup_e B_{r_e}(x_e), \end{aligned} \tag{10.19}$$

where

(a) $\mathcal{N} = B_2(p) \setminus \Big(\mathcal{C}_0 \cup \bigcup_b B_{r_b}(x_b) \cup \bigcup_c B_{r_c}(x_c) \cup \bigcup_d B_{r_d}(x_d) \cup \bigcup_e B_{r_e}(x_e)\Big)$ *is a (k,δ,η)-neck region*;

(b) *each b-ball $B_{2r_b}(x_b)$ is $(k+1,2\eta)$-symmetric*;

(c) *each c-ball $B_{2r_c}(x_c)$ is not $(k+1,2\eta)$-symmetric and satisfies*

$$\mathrm{Vol}(B_{\gamma r_c}\mathcal{P}_{r_c,\xi}(x_c)) \ge \epsilon\gamma^{n-k} r_c^n;$$

(d) *each d-ball $B_{2r_d}(x_d)$ satisfies* $\mathrm{Vol}(B_{\gamma r_d}\mathcal{P}_{r_d,\xi}(x_d)) < \epsilon\gamma^{n-k} r_d^n$;

(e) *each e-ball $B_{2r_e}(x_e)$ satisfies $\mathcal{P}_{r_e,\xi}(x_e) = \emptyset$.*

Furthermore, the following k-content estimates hold:

$$\begin{aligned} &\sum_{x_b\in B_{3/2}(p)} r_b^k + \sum_{x_d\in B_{3/2}(p)} r_d^k \\ &\qquad + \sum_{x_e\in B_{3/2}(p)} r_e^k t + \mathcal{H}^k(\mathcal{C}_0 \cap B_{3/2}(p)) \le C(n,\mathrm{v}), \end{aligned} \tag{10.20}$$

$$\sum_{x_c\in B_{3/2}(p)} r_c^k \le C(n,\mathrm{v})\epsilon. \tag{10.21}$$

Remark 10.6. In this proposition the ball types and the pinching set $\mathcal{P}_{r,\xi}$ are defined with respect to the given $\bar V \le \inf_{x\in B_4(p)} \mathcal{V}_{\xi^{-1}}(x)$ above.

10.4. *Proof of Proposition* 10.2 *modulo Propositions* 10.3 *and* 10.5. In this subsection we will state and prove Lemma 10.7. The proof involves using iteratively the decompositions of Propositions 10.5 and 10.3. Then by using

Lemma 10.7 a definite number of times we are able to remove all the c-balls and d-balls, thereby proving Proposition 10.2. This proves Theorem 2.12 modulo the proofs of Propositions 10.5 and 10.3. These two propositions will be proved in the remaining two subsections.

LEMMA 10.7. *Let* $\eta > 0$, $\delta \le \delta(n, \mathrm{v}, \eta)$ *and* $\xi \le \xi(n, \mathrm{v}, \delta, \eta)$. *Let* $(M_i, g_i, p_i) \xrightarrow{d_{GH}} (X^n, d, p)$ *satisfy* $\mathrm{Vol}(B_1(p_i)) \ge \mathrm{v} > 0$, $\bar{V} := \inf_{x\in B_4(p)} \mathcal{V}_{\xi^{-1}}(x)$, $\mathrm{Ric}_{M_i^n} \ge -(n-1)\xi$. *Then*

$$B_1(p) \subset \bigcup_a \mathcal{C}_{0,a} \cup \mathcal{N}_a \cap B_{r_a}(x_a) \cup \bigcup_b B_{r_b}(x_b) \cup \bigcup_e B_{r_e}(x_e) \cup \tilde{\mathcal{S}}^k, \tag{10.22}$$

where

(1) $\mathcal{N}_a \subset B_{2r_a}(x_a)$ *are* (k, δ, η)*-neck regions with associated singular set* $\mathcal{C}_{0,a}$;
(2) *each* b*-ball* $B_{2r_b}(x_b)$ *is* $(k+1, 2\eta)$*-symmetric*;
(3) *for each* e*-ball* $B_{2r_e}(x_e)$, *we have* $\mathcal{P}_{r_e,\xi}(x_e) = \emptyset$ *where* $\mathcal{P}_{r_e,\xi}(x_e) := \{y \in B_{4r_e}(x_e) : \mathcal{V}_{\xi r_e}(y) \le \bar{V} + \xi\}$;
(4) $\tilde{\mathcal{S}}^k \subset S$ *and* $\mathcal{H}^k(\tilde{\mathcal{S}}^k) = 0$.

Moreover, the following content estimate holds:

$$\sum_a r_a^k + \sum_b r_b^k + \sum_e r_e^k \le C(n, \mathrm{v}). \tag{10.23}$$

Proof. Fix $\epsilon \le \epsilon(n, \mathrm{v})$, $\gamma \le \gamma(n, \mathrm{v}, \epsilon)$ and $\delta \le \delta(n, \mathrm{v}, \eta)$ such that Propositions 10.5 and 10.3 hold.

We can assume $B_2(p)$ is not a b-ball or e-ball. Otherwise, there is nothing to prove.

So assume one of the following two cases holds.

(1) $B_2(p)$ is a c-ball with $\mathrm{Vol}(B_\gamma \mathcal{P}_{1,\xi}(p)) \ge \epsilon\gamma^{n-k}$, and with $B_2(p)$ it is not $(k+1, 2\eta)$-symmetric;
(2) $B_2(p)$ is a d-ball with $\mathrm{Vol}(B_\gamma \mathcal{P}_{1,\xi}(p)) < \epsilon\gamma^{n-k}$.

It will be evident that up to reversing the order of which decomposition we apply first, the argument is the same in both cases. Therefore, without essential loss of generality, we will assume that $B_2(p)$ is a c-ball.

By the c-ball decomposition Proposition 10.5, if $\xi \le \xi(n, \mathrm{v}, \delta, \epsilon, \eta)$, then we have

(10.24)
$$B_1(p) \subseteq (\mathcal{C}_0 \cup \mathcal{N} \cap B_1(p)) \cup \bigcup_b B_{r_b}(x_b) \cup \bigcup_c B_{r_c}(x_c) \cup \bigcup_d B_{r_d}(x_d) \cup \bigcup_e B_{r_e}(x_e)$$

and, in addition, the following k-content estimates hold:

$$\sum_b r_b^k + \sum_d r_d^k + \sum_e r_e^k + \mathcal{H}^k(\mathcal{C}_0) \leq C(n), \tag{10.25}$$

$$\sum_c r_c^k \leq C(n, \mathrm{v})\epsilon. \tag{10.26}$$

By applying the d-ball decomposition of Proposition 10.3 to each d-ball $B_{2r_d}(x_d)$, we arrive at

$$B_1(p) \subseteq \tilde{\mathcal{S}}_1^k \cup (\mathcal{C}_0 \cup \mathcal{N} \cap B_1(p)) \cup \bigcup_b B_{r_b}(x_b) \cup \bigcup_c B_{r_c^1}(x_c^1) \cup \bigcup_e B_{r_e}(x_e), \tag{10.27}$$

where $\tilde{\mathcal{S}}_1^k = \bigcup_d \tilde{\mathcal{S}}_d^k$ is a countable union of k-Hausdorff measure zero sets, and thus $\mathcal{H}^k(\tilde{\mathcal{S}}_1^k) = 0$. Moreover, we have the following content estimates:

$$\sum_c (r_c^1)^k \leq C(n, \mathrm{v})\epsilon + C(n)C(n, \mathrm{v})\epsilon \leq \bar{C}(n, \mathrm{v})\epsilon, \tag{10.28}$$

$$\sum_b r_b^k + \sum_e r_e^k + \mathcal{H}^k(\mathcal{C}_0) \leq C(n) + C(n)C(n, \gamma) \leq \bar{C}(n, \gamma). \tag{10.29}$$

Next, we repeat the above process verbatim, except that we first apply the c-ball decomposition of Proposition 10.5 to each c-ball above and then apply the d-ball decomposition of Proposition 10.3 to each remaining d-ball. The result is

$$B_1(p) \subseteq \tilde{\mathcal{S}}_2^k \cup \bigcup_a (\mathcal{C}_{0,a} \cup \mathcal{N}_a \cap B_{r_a}(x_a)) \cup \bigcup_b B_{r_b}(x_b) \cup \bigcup_c B_{r_c^2}(x_c^2) \cup \bigcup_e B_{r_e}(x_e), \tag{10.30}$$

with content estimates $\mathcal{H}^k(\tilde{\mathcal{S}}_2^k) = 0$ and

$$\sum_a r_a^k \leq 1 + \bar{C}(n, \mathrm{v})\epsilon, \ \sum_c (r_c^2)^k \leq \Big(\bar{C}(n, \mathrm{v})\epsilon\Big)^2, \tag{10.31}$$

$$\sum_b r_b^k + \sum_e r_e^k + \sum_a \mathcal{H}^k(\mathcal{C}_{0,a}) \leq \bar{C}(n, \gamma)\Big(1 + \bar{C}(n, \mathrm{v})\epsilon\Big). \tag{10.32}$$

After repeating this process i times we arrive at

$$B_1(p) \subseteq \tilde{\mathcal{S}}_i^k \cup \bigcup_a (\mathcal{C}_{0,a} \cup \mathcal{N}_a \cap B_{r_a}(x_a)) \cup \bigcup_b B_{r_b}(x_b) \cup \bigcup_c B_{r_c^i}(x_c^i) \cup \bigcup_e B_{r_e}(x_e), \tag{10.33}$$

with content estimates $\mathcal{H}^k(\tilde{\mathcal{S}}^k_i) = 0$ and

$$\sum_a r_a^k \le \sum_{j=0}^{i} \Big(\bar{C}(n,\mathrm{v})\epsilon\Big)^j, \ \sum_c (r_c^i)^k \le \Big(\bar{C}(n,\mathrm{v})\epsilon\Big)^i, \tag{10.34}$$

$$\sum_b r_b^k + \sum_e r_e^k + \sum_a \mathcal{H}^k(\mathcal{C}_{0,a}) \le \bar{C}(n,\gamma)\sum_{j=0}^{i}\Big(\bar{C}(n,\mathrm{v})\epsilon\Big)^j. \tag{10.35}$$

Consider the discrete set $\tilde{\mathcal{S}}^i_c := \{x^i_c\}$. By the construction, we have

$$B_{2r_c^{i+1}}\tilde{\mathcal{S}}^{i+1}_c \subset B_{2r_c^i}(\tilde{\mathcal{S}}^i_c), \tag{10.36}$$

where

$$B_{2r_c^i}(\tilde{\mathcal{S}}^i_c) := \cup_c B_{2r_c^i}(x^i_c). \tag{10.37}$$

Define the set limit by

$$\tilde{\mathcal{S}}_c := \bigcap_{i\ge 1}\bigcup_{j\ge i} B_{2r_c^i}(\tilde{\mathcal{S}}^i_c). \tag{10.38}$$

It is clear from the construction that $\tilde{\mathcal{S}}_c \subset S(X^n)$. Set $\delta_i := 2\max_c r_c^i$. Since $\tilde{\mathcal{S}}_c \subset B_{2r_c^i}(\tilde{\mathcal{S}}^i_c)$, by the definition of Hausdorff measure, we have

$$\begin{aligned} H^k_{\delta_i}(\tilde{\mathcal{S}}_c) &:= \inf\Big\{\sum_\alpha r_\alpha^k, \quad \text{where } r_\alpha \le \delta_i \text{ and } \tilde{\mathcal{S}}_c \subset \cup_\alpha B_{r_\alpha}(y_\alpha)\Big\} \\ &\le 2^k \sum_c (r_c^i)^k \le 2^k\Big(\bar{C}(n,\mathrm{v})\epsilon\Big)^i, \end{aligned} \tag{10.39}$$

which implies $\mathcal{H}^k(\tilde{\mathcal{S}}_c) = 0$.

Set $\tilde{\mathcal{S}}^k := \tilde{\mathcal{S}}_c \cup \bigcup_{i\ge 1}\tilde{\mathcal{S}}^k_i$. Then $\mathcal{H}^k(\tilde{\mathcal{S}}^k) = 0$ and $\tilde{\mathcal{S}}^k \subset S(X)$.

Fix $\epsilon = \epsilon(n,\mathrm{v})$ and $\gamma = \gamma(n,\mathrm{v})$ such that $\bar{C}(n,\mathrm{v})\epsilon \le 1/10$. Then by taking the limit as $i\to\infty$, we will arrive at the decomposition

$$B_1(p) \subset \tilde{\mathcal{S}}^k \cup \bigcup_a (\mathcal{C}_{0,a}\cup \mathcal{N}_a \cap B_{r_a}(x_a)) \cup \bigcup_b B_{r_b}(x_b) \cup \bigcup_e B_{r_e}(x_e). \tag{10.40}$$

To see (10.40), if $y \in B_1(p)\setminus \tilde{\mathcal{S}}^k$, then by (10.38) we must have $y \notin B_{2r_c^i}(\tilde{\mathcal{S}}^i_c)$ for some i which, in particular, implies by (10.33) that y belongs to the set on the right-hand side of (10.40).

By letting $i\to\infty$, by (10.35) we have the following content estimates:

$$\sum_a r_a^k \le 2, \tag{10.41}$$

$$\sum_b r_b^k + \sum_e r_e^k + \sum_a \mathcal{H}^k(\mathcal{C}_{0,a}) \le C(n,\mathrm{v}). \tag{10.42}$$

This completes the proof of Lemma 10.7. □

Now we can prove the inductive decomposition of Proposition 10.2.

Proof of Proposition 10.2. For any η and $\delta \leq \delta(n, \mathrm{v}, \eta)$, fix $\xi = \xi(n, \mathrm{v}, \delta, \eta)$ as in Lemma 10.7. Consider a Vitali covering $\{B_{\xi^2}(x_f)\}$ of $B_2(p)$ such that $B_{\xi^2/5}(x_f)$ are disjoint. Thus, by volume comparison, the number of such balls is bounded by a constant $L(n, \mathrm{v}, \xi)$. By scaling the ball $B_{\xi^2}(x_f)$ to a unit ball, we arrive at a unit ball satisfying all the conditions of Lemma 10.7 with

$$\bar{V}_f := \inf_{y \in B_{4\xi^2}(x_f)}, \qquad \mathcal{V}_\xi(y) \geq \inf_{y \in B_4(p)} \mathcal{V}_1(y) := \bar{V}.$$

If we apply the decomposition of Lemma 10.7 to each ball $B_{\xi^2}(x_f)$ in order, we arrive at the covering

$$B_1(p) \subset \tilde{\mathcal{S}}^k \cup \bigcup_a (\mathcal{C}_{0,a} \cup \mathcal{N}_a \cap B_{r_a}(x_a)) \cup \bigcup_b B_{r_b}(x_b) \cup \bigcup_e B_{r_e}(x_e), \tag{10.43}$$

with $r_a, r_b, r_e \leq \xi^2$ and

$$\sum_a r_a^k + \sum_b r_b^k + \sum_e r_e^k \leq C(n, \mathrm{v}) L(n, \mathrm{v}, \xi) \leq C(n, \mathrm{v}, \delta),$$
$$\mathcal{H}^k(\tilde{\mathcal{S}}^k) = 0.$$

To finish the proof, it suffices to recover each e-ball by v-balls. In fact, for each e-ball $B_{r_e}(x_e) \subset B_{2\xi^2}(x_f)$, consider the Vitali covering $\{B_{\xi r_e}(x_e^j)\}$ of $B_{r_e}(x_e)$ with $x_e^j \in B_{r_e}(x_e)$ such that $B_{\xi r_e/5}(x_e^j)$ are disjoint. We will show that $B_{\xi r_e}(x_e^j)$ are v-ball for $v_0 = \xi$[13].

Since $\mathcal{P}_{r_e,\xi}(x_e) := \{y \in B_{4r_e}(x_e) : \mathcal{V}_{\xi r_e}(y) \leq \bar{V}_f + \xi\} = \emptyset$, we have for all $y \in B_{4r_e}(x_e)$ that $\mathcal{V}_{\xi r_e}(y) \geq \bar{V}_f + \xi \geq \bar{V} + \xi$. On the other hand, we have $B_{4\xi r_e}(x_e^j) \subset B_{2r_e}(x_e)$. Therefore, $\inf_{y \in B_{4\xi r_e}(x_e^j)} \mathcal{V}_{\xi r_e}(y) \geq \bar{V} + \xi$. Setting $v_0 := \xi$ we have that $B_{\xi r_e}(x_e^j)$ is a v-ball as in Proposition 10.2. The content estimate for v-balls follows easily from the content estimate of e-balls and the Vitali covering. This completes the proof of Proposition 10.2, modulo the proofs of Propositions 10.3 and 10.5. $\square$

10.5. *Proof of the d-ball covering Proposition* 10.3.

Proof of Proposition 10.3. For any $0 < \epsilon, \gamma \leq 1/10$, let us first consider a Vitali covering $\{B_\gamma(x_f^1),\ x_f^1 \in B_1(p)\}$ of $B_1(p)$ such that $B_{\gamma/5}(x_f^1)$ are disjoint. Let us separate $\{B_\gamma(x_f^1)\}$ into b-balls, c-balls, d-balls and e-ball's from Section 10.2:

$$B_1(p) \subseteq \bigcup_{b=1}^{N_b^1} B_\gamma(x_b^1) \cup \bigcup_{c=1}^{N_c^1} B_\gamma(x_c^1) \cup \bigcup_{d=1}^{N_d^1} B_\gamma(x_d^1) \cup \bigcup_{e=1}^{N_e^1} B_\gamma(x_e^1), \tag{10.44}$$

[13] Recall that v-balls are defined with respect to the background parameter v_0

where $B_{2\gamma}(x^1_b)$ is $(k+1,2\eta)$-symmetric, $B_{2\gamma}(x^1_c)$ is not $(k+1,2\eta)$-symmetric and satisfies $\mathrm{Vol}(B_{\gamma^2}\mathcal{P}_{\gamma,\xi}(x^1_c)) \geq \epsilon\gamma^{n-k}\gamma^n$, and $\mathrm{Vol}(B_{\gamma^2}\mathcal{P}_{\gamma,\xi}(x^1_d)) < \epsilon\gamma^{n-k}\gamma^n$, and $\mathcal{P}_{\gamma,\xi}(x^1_e) = \emptyset$ with $\mathcal{P}_{r,\xi}(x) := \{y \in B_{4r}(x) : \mathcal{V}_{r\xi}(y) \leq \bar{V} + \xi\}$ and $\bar{V} := \inf_{y\in B_4(p)} \mathcal{V}_{\xi^{-1}}(y)$. By volume doubling we have

$$\sum_{b=1}^{N^1_b} \gamma^k + \sum_{e=1}^{N^1_e} \gamma^k \leq C(n,\gamma)\gamma^k \leq C(n,\gamma). \tag{10.45}$$

Let us prove a slightly more refined content estimate for the c-balls and d-balls. Since $B_{2\gamma}(x^1_c), B_{2\gamma}(x^1_d) \subset B_2(p)$, we have $\mathcal{P}_{\gamma,\xi}(x_c), \mathcal{P}_{\gamma,\xi}(x_d) \subset \mathcal{P}_{1,\xi}(p)$, where we should notice that in our setting a d-ball is not an e-ball. The following content estimates for c-balls and d-balls depend only on the fact that $\mathcal{P}_{\gamma,\xi}(x_c)$ and $\mathcal{P}_{\gamma,\xi}(x_d)$ are nonempty. We will only discuss the content estimate for d-balls, since the case of c-balls is no different from this one. Indeed, for each ball $B_\gamma(x^1_d)$, there exists a point $y^1_d \in B_{2\gamma}(x^1_d) \cap \mathcal{P}_{1,\xi}(p)$ which, in particular, implies $B_\gamma(y^1_d) \subset B_\gamma\mathcal{P}_{1,\xi}(p)$. The ball $B_\gamma(y^1_d)$ may overlap with other balls $B_\gamma(y^1_{d'})$. Due to the Vitali covering property and volume doubling, the balls overlap at most $C(n)$ times. By a standard covering argument and noting $\mathrm{Vol}(B_\gamma(\mathcal{P}_{1,\xi}(p))) < \epsilon\gamma^{n-k}$, we can now conclude that

$$\sum_{c=1}^{N^1_c} \gamma^k + \sum_{d=1}^{N^1_d} \gamma^k \leq C(n,\mathrm{v})\epsilon. \tag{10.46}$$

For each d-ball $B_\gamma(x^1_d)$, let us repeat this decomposition. We get

$$\bigcup_{d=1}^{N^1_d} B_\gamma(x^1_d) \subset \bigcup_{b=1}^{N^2_b} B_{\gamma^2}(x^2_b) \cup \bigcup_{c=1}^{N^2_c} B_{\gamma^2}(x^2_c) \cup \bigcup_{d=1}^{N^2_d} B_{\gamma^2}(x^2_d) \cup \bigcup_{e=1}^{N^2_e} B_{\gamma^2}(x^2_e). \tag{10.47}$$

Furthermore, by the same arguments as above we have the content estimates

$$\sum_{b=1}^{N^2_b} \gamma^{2k} + \sum_{e=1}^{N^2_e} \gamma^{2k} \leq C(n,\gamma) \sum_{d=1}^{N^1_d} \gamma^k \leq C(n,\gamma)C(n,\mathrm{v})\epsilon, \tag{10.48}$$

$$\sum_{d=1}^{N^2_d} \gamma^{2k} + \sum_{c=1}^{N^2_c} \gamma^{2k} \leq C(n,\mathrm{v})\epsilon \sum_{d=1}^{N^1_d} \gamma^k \leq \Big(C(n,\mathrm{v})\epsilon\Big)^2. \tag{10.49}$$

Therefore, we arrive at the decomposition

(10.50)

$$B_1(p) \subset \bigcup_{d=1}^{N^2_d} B_{\gamma^2}(x^2_d) \cup \bigcup_{j=1}^{2}\bigcup_{b=1}^{N^j_b} B_{\gamma^j}(x^j_b) \cup \bigcup_{j=1}^{2}\bigcup_{c=1}^{N^j_c} B_{\gamma^j}(x^j_c) \cup \bigcup_{j=1}^{2}\bigcup_{e=1}^{N^j_e} B_{\gamma^j}(x^j_e),$$

with content estimates

$$\sum_{d=1}^{N_d^2} \gamma^{2k} \leq \Big(C(n,\mathrm{v})\epsilon\Big)^2, \tag{10.51}$$

$$\begin{aligned} \sum_{j=1}^{2}\sum_{b=1}^{N_b^j} \gamma^{jk} + \sum_{j=1}^{2}\sum_{e=1}^{N_e^j} \gamma^{jk} &\leq C(n,\gamma) + C(n,\gamma)C(n,\mathrm{v})\epsilon \\ &\leq C(n,\gamma)\Big(1 + C(n,\mathrm{v})\epsilon\Big), \end{aligned} \tag{10.52}$$

$$\sum_{j=1}^{2}\sum_{c=1}^{N_c^j} \gamma^{jk} \leq C(n,\mathrm{v})\epsilon + \Big(C(n,\mathrm{v})\epsilon)\Big)^2. \tag{10.53}$$

If we repeat this d-ball decomposition for each $B_{\gamma^2}(x_d^2)$, then after i iterations of the decomposition we get

$$B_1(p) \subset \bigcup_{d=1}^{N_d^i} B_{\gamma^2}(x_d^2) \cup \bigcup_{j=1}^{i}\bigcup_{b=1}^{N_b^j} B_{\gamma^j}(x_b^j) \cup \bigcup_{j=1}^{i}\bigcup_{c=1}^{N_c^j} B_{\gamma^j}(x_c^j) \cup \bigcup_{j=1}^{i}\bigcup_{e=1}^{N_e^j} B_{\gamma^j}(x_e^j), \tag{10.54}$$

with content estimates

$$\sum_{d=1}^{N_d^i} \gamma^{ik} \leq \Big(C(n,\mathrm{v})\epsilon\Big)^i, \tag{10.55}$$

$$\sum_{j=1}^{i}\sum_{b=1}^{N_b^j} \gamma^{jk} + \sum_{j=1}^{i}\sum_{e=1}^{N_e^j} \gamma^{jk} \leq C(n,\gamma)\sum_{j=0}^{i-1}\Big(C(n,\mathrm{v})\epsilon\Big)^j, \tag{10.56}$$

$$\sum_{j=1}^{i}\sum_{c=1}^{N_c^j} \gamma^{jk} \leq \sum_{j=1}^{i}\Big(C(n,\mathrm{v})\epsilon)\Big)^j. \tag{10.57}$$

Let $\epsilon \leq \epsilon(n,\mathrm{v})$ and $\gamma \leq \gamma(n,\mathrm{v},\epsilon)$ be such that γ and ϵ satisfies Theorem 4.9 and $C(n,\mathrm{v})\epsilon \leq 1/10$.

Consider the discrete set $\tilde{\mathcal{S}}_i^k := \{x_d^i\}$. By construction, we have $\tilde{\mathcal{S}}_{i+1}^k \subset B_{\gamma^i}\tilde{\mathcal{S}}_i^k$. Additionally,

$$\mathrm{Vol}(B_{\gamma^i}\tilde{\mathcal{S}}_i^k) \leq \sum_{d=1}^{N_d^i}\mathrm{Vol}(B_{\gamma^i}(x_d^i)) \leq C(n)\sum_{d=1}^{N_d^i}\gamma^{in} \leq C(n)\Big(C(n,\mathrm{v})\epsilon\Big)^i\gamma^{i(n-k)}. \tag{10.58}$$

Denote the Hausdorff limit of $\tilde{\mathcal{S}}^k_i$ by $\tilde{\mathcal{S}}^k_d := \lim_{i\to\infty} \tilde{\mathcal{S}}^k_i$. Then by (10.58) and $\tilde{\mathcal{S}}^k_{i+1} \subset B_{\gamma^i}\tilde{\mathcal{S}}^k_i$, for any $i \geq 1$, we have

$$\mathrm{Vol}(B_{\gamma^i}\tilde{\mathcal{S}}^k_d) \leq C(n)10^{-i}\gamma^{i(n-k)}. \tag{10.59}$$

This implies $\mathcal{H}^k(\tilde{\mathcal{S}}^k_d) = 0$.

We claim that $\tilde{\mathcal{S}}^k_d \subset S$. To see this, assume there exists $x \in \tilde{S}^k_d \setminus S$. This implies that for any $\epsilon' > 0$, there exists $r_{x,\epsilon'} > 0$ such that $d_{GH}(B_{r_{x,\epsilon'}}(x), B_{r_{x,\epsilon'}}(0^n)) \leq \epsilon' r_{x,\epsilon'}$. On the other hand, since $x \in \tilde{\mathcal{S}}^k_d$, we have that $\mathcal{P}_{r_{x,\epsilon'},\xi}(x)$ is nonempty. Hence, applying the volume convergence in [Col97] and [Che01] to $B_{r_{x,\epsilon'}}(x)$ gives $\bar{V} + \xi \geq 1 - \epsilon''$ providing $\epsilon' \leq \epsilon'(n, \mathrm{v}, \epsilon'')$. Therefore, we arrive at $\bar{V} \geq 1 - \xi$, which implies $B_2(p) \subset \mathcal{P}_{1,\xi}(p)$. In particular, $\mathrm{Vol}(B_\gamma(\mathcal{P}_{1,\xi}(p))) \geq \mathrm{Vol}(B_2(p)) \geq \mathrm{v} > 0$ which contradicts the d-ball assumption if $\epsilon \leq \mathrm{v}$.

On the other hand, since $C(n, \mathrm{v})\epsilon \leq 1/10$, the content estimate (10.57) holds. Therefore, we arrive at the desired decomposition. This completes the proof of Proposition 10.3. □

10.6. *Proof of the c-ball covering Proposition* 10.5. In this subsection we prove Proposition 10.5, which is concerned with the decomposition of a c-ball. We will construct a neck region on $B_1(p)$ which is GH-close to a ball in some cone $\mathbb{R}^k \times C(Y)$.

Proof of Proposition 10.5. Recall that in the definition of neck region we have $\tau = \tau_n = 10^{-10n}\omega_n$. Fix $\epsilon > 0$ and $\gamma \leq \gamma(n, \mathrm{v}, \epsilon)$ such that Theorem 4.9 (cone-splitting based on k-content) holds. By Theorem 4.9 we have that $B_{\delta'^{-1}}(q)$ is (k, δ'^2)-symmetric for some $q \in B_4(p)$. In particular, $B_{\delta'^{-1}}(q)$ is δ'^2-close to a metric cone $\mathbb{R}^k \times C(Y)$.

Consider the δ'^2-GH map $\iota_{q,1} : B_{\delta'^{-1}}(0^k, y_c) \to B_{\delta'^{-1}}(q)$ and the approximate singular set $\mathcal{L}_{q,1} := \iota_{q,1}(\mathbb{R}^k \times \{y_c\}) \cap B_4(p)$. Choose a Vitali covering $\{B_{\gamma\tau^2}(x^1_f),\ x^1_f \in \mathcal{L}_{q,1}\}$ of $\mathcal{L}_{q,1}$ such that $B_{\gamma\tau^3}(x^1_f)$ are disjoint.

We denote the different types of balls $B_{2\gamma}(x^1_f)$ as follows:

(1) $\tilde{b}$-balls if $B_{2\gamma}(x^1_f)$ is $(k+1, 3\eta/2)$-symmetric;
(2) $\tilde{c}$-balls if $B_{2\gamma}(x^1_f)$ is not $(k+1, 3\eta/2)$-symmetric and $\mathrm{Vol}(B_{\gamma\cdot\gamma}\mathcal{P}_{\gamma,\xi}(x^1_f)) \geq \epsilon\gamma^{n-k}\gamma^n$;
(3) $\tilde{d}$-balls if $\mathrm{Vol}(B_{\gamma\cdot\gamma}\mathcal{P}_{\gamma,\xi}(x^1_f)) < \epsilon\gamma^{n-k}\gamma^n$.

We have

$$\mathcal{L}_{q,1} \subset \bigcup_b B_{\tau^2\gamma}(\tilde{x}^1_b) \cup \bigcup_c B_{\tau^2\gamma}(\tilde{x}^1_c) \cup \bigcup_d B_{\tau^2\gamma}(\tilde{x}^1_d). \tag{10.60}$$

Therefore, we arrive at an approximate neck region $\tilde{\mathcal{N}}^1$:

$$\tilde{\mathcal{N}}^1 := B_2(p) \setminus \Big(\bigcup_b B_{\tau^2\gamma}(\tilde{x}^1_b) \cup \bigcup_c B_{\tau^2\gamma}(\tilde{x}^1_c) \cup \bigcup_d B_{\tau^2\gamma}(\tilde{x}^1_d)\Big). \tag{10.61}$$

The approximate neck $\tilde{\mathcal{N}}^1$ is not yet the one we are looking for, since c-ball content is not small. Therefore, we continue to refine the construction by redecomposing the $\tilde{c}$-balls in the decomposition. Once again, by applying the Content Cone-Splitting Theorem 4.9 to each $\tilde{c}$-ball, we have the approximate singular set $\mathcal{L}_{\tilde{x}^1_c,\gamma} := \iota_{\tilde{x}^1_c,\gamma}(\mathbb{R}^k \times \{y_c\}) \cap B_{4\gamma}(\tilde{x}^1_c)$ associated with a $\delta'^2\gamma$-GH map $\iota_{\tilde{x}^1_c,\gamma} : B_{\gamma\delta'^{-1}}(0^k, y_c) \to B_{\gamma\delta'^{-1}}(\tilde{x}^1_c)$.

Consider the Vitali covering $\{B_{\tau^2\gamma^2}(x^2_f)\}$ of

$$\bigcup_c \mathcal{L}_{\tilde{x}^1_c,\gamma} \setminus \Big(\bigcup_b B_{\tau^3\gamma}(\tilde{x}^1_b) \cup \bigcup_d B_{\tau^3\gamma}(\tilde{x}^1_d)\Big), \tag{10.62}$$

such that $B_{\tau^4\gamma^2}(x^2_f)$ are disjoint and

$$x^2_f \in \bigcup_c \mathcal{L}_{\tilde{x}^1_c,\gamma} \setminus \Big(\bigcup_b B_{\tau^3\gamma}(\tilde{x}^1_b) \cup \bigcup_d B_{\tau^3\gamma}(\tilde{x}^1_d)\Big). \tag{10.63}$$

In particular, if $\gamma \le 10^{-10}$, then the balls $B_{\tau^4\gamma^2}(x^2_f)$ are also mutually disjoint with $B_{\tau^4\gamma}(\tilde{x}^1_b)$ and $B_{\tau^4\gamma}(\tilde{x}^1_d)$.

We denote the ball $B_{2\gamma^2}(x^2_f)$ by $B_{2\gamma^2}(\tilde{x}^2_b)$, $B_{2\gamma^2}(\tilde{x}^2_c)$ and $B_{2\gamma^2}(\tilde{x}^2_d)$ according to the same scheme as above. Thus, we have

$$\tilde{\mathcal{N}}^2 := B_2(p) \setminus \left(\bigcup_c \bar{B}_{\gamma^i}(\tilde{x}^i_c) \cup \bigcup_{1\le j\le 2} \Big(\bigcup_b \bar{B}_{\gamma^j}(\tilde{x}^j_b) \cup \bigcup_d \bar{B}_{\gamma^j}(\tilde{x}^j_d)\Big)\right). \tag{10.64}$$

After applying this decomposition i times in succession to each $\tilde{c}$-ball, we get an approximate neck region given by

$$\tilde{\mathcal{N}}^i := B_2(p) \setminus \left(\bigcup_c \bar{B}_{\gamma^i}(\tilde{x}^i_c) \cup \bigcup_{1\le j\le i} \Big(\bigcup_b \bar{B}_{\gamma^j}(\tilde{x}^j_b) \cup \bigcup_d \bar{B}_{\gamma^j}(\tilde{x}^j_d)\Big)\right). \tag{10.65}$$

Set

$$\tilde{\mathcal{C}}^i_c := \{\tilde{x}^i_c\}.$$

By construction we have $\tilde{\mathcal{C}}^{i+1}_c \subset B_{\gamma^i}(\tilde{\mathcal{C}}^i_c)$. Therefore, we can define the Hausdorff limit:

$$\tilde{\mathcal{C}}_0 := \lim_{i\to\infty} \tilde{\mathcal{C}}^i_c. \tag{10.66}$$

By letting $i \to \infty$, we get

$$\tilde{\mathcal{N}} := B_2(p) \setminus \left(\tilde{\mathcal{C}}_0 \cup \bigcup_b \bar{B}_{\tilde{r}_b}(\tilde{x}_b) \cup \bigcup_d \bar{B}_{\tilde{r}_d}(\tilde{x}_d)\right). \tag{10.67}$$

Set

$$\tilde{\mathcal{C}}_+ := \{\tilde{x}_d, \tilde{x}_b\}.$$

By construction, the balls $B_{\tau^4\tilde r_{\tilde x}}(\tilde x)$ are disjoint for $\tilde x\in\tilde{\mathcal C}_+$ and in addition,

$$\tilde x\notin\bigcup_{\tilde y\in\tilde{\mathcal C}_+,\tilde r_{\tilde y}>\tilde r_{\tilde x}}B_{\tau^3\tilde r_{\tilde y}}(\tilde y).\tag{10.68}$$

Moreover, $\tilde{\mathcal C}:=\tilde{\mathcal C}_+\cup\tilde{\mathcal C}_0$ is a closed set.

It is easy to check that $\tilde{\mathcal N}$ satisfies all the conditions of a (k,δ',η)-neck region except (n5), i.e., $\operatorname{Lip}\tilde r_x\le\delta'$. Therefore, our construction requires some additional refinement.

In the following construction, in which we refine our covering in order to get the desired (k,δ,η)-neck, we will use $\tilde x_b,\tilde x_d\in\tilde{\mathcal C}_+$ to denote the center of a $\tilde b$-ball and $\tilde d$-ball of $\tilde{\mathcal N}$, and the associated radius $\tilde r_{\tilde x_b}$ and $\tilde r_{\tilde x_d}$, respectively.

By the construction of $B_{\tilde r_{\tilde x}}(\tilde x)$ with $\tilde x\in\tilde{\mathcal C}$, there exists some $\tilde c$-ball $B_{\gamma^{-1}\tilde r_{\tilde x}}(\tilde x_c)$ which is (k,δ'^2)-symmetric with respect to $\mathcal L_{\tilde x_c,\gamma^{-1}\tilde r_{\tilde x}}$ such that $B_{\tilde r_{\tilde x}}(\tilde x)\subset B_{2\gamma^{-1}\tilde r_{\tilde x}}(\tilde x_c)$ and $\tilde x\in\mathcal L_{\tilde x_c,\gamma^{-1}\tilde r_{\tilde x}}$. It is easy to see that for any $\gamma>r\ge\tilde r_{\tilde x}$, the ball $B_{\gamma^{-1}r}(\tilde x_c)$ is also (k,δ'^2)-symmetric with respect to a set $\mathcal L_{\tilde x_c,\gamma^{-1}r}$. This follows from the volume pinching estimate

$$|\mathcal V_{\xi\tilde r_{\tilde x}}(\tilde x_c)-\mathcal V_{\xi^{-1}}(\tilde x_c)|\le\xi$$

and the fact that $B_{\gamma^{-1}r}(\tilde x_c)$ is (k,δ'^3)-splitting since this ball is contained in a (k,δ'^3)-symmetric ball with comparable radius.

For convenience sake we will introduce the following notation.

(10.69) For any $\tilde x\in\tilde{\mathcal C}$, let $\tilde x_c$ denote the center of the $\tilde c$-ball satisfying the above properties.

To refine the approximate neck $\tilde{\mathcal N}$, let us build a good approximate singular set $\tilde{\mathcal S}$. We define $\tilde{\mathcal S}$ to be a subset of $\cup_{\tilde x\in\tilde{\mathcal C}}B_{\tau^3\tilde r_{\tilde x}}(\tilde x)$ such that $y\in\tilde{\mathcal S}$ if and only if one of the following holds:

(1) $y\in\mathcal L_{\tilde x_c,\gamma^{-1}\tilde r_{\tilde x}}$ with $d(y,\tilde{\mathcal C})=d(y,\tilde x)\le\tilde r_{\tilde x}$;
(2) $y\in\mathcal L_{\tilde x_c,\gamma^{-1}r}$ with $r:=d(y,\tilde{\mathcal C})=d(y,\tilde x)>\tilde r_{\tilde x}$.

Now we define a radius function on $\tilde{\mathcal S}$ such that

$$\begin{aligned}&r_x:=\delta^2\tau^4\tilde r_{\tilde x}\text{ if }d(x,\tilde{\mathcal C})=d(x,\tilde x)\le\tau^4\tilde r_{\tilde x},\\&r_x:=\delta^2d(x,\tilde{\mathcal C})\text{ otherwise.}\end{aligned}\tag{10.70}$$

It is obvious that $|\mathrm{Lip}r_x|\le\delta^2$ and $\tilde{\mathcal C}\subset\tilde{\mathcal S}$. Choose a maximal disjoint collection $\{B_{\tau^2r_x}(x),x\in\tilde{\mathcal S}\}$ such that the center set $\mathcal C_+\subset\tilde{\mathcal S}$ contains $\tilde{\mathcal C}$. This allows us to build a neck region

$$\mathcal N:=B_2(p)\setminus\Big(\tilde{\mathcal C}_0\cup\bigcup_{x\in\mathcal C_+}\bar B_{r_x}(x)\Big).\tag{10.71}$$

Notation. In order to make the various notations consistent, we put $\mathcal{C}_0 := \tilde{\mathcal{C}}_0$ and $\mathcal{C} := \mathcal{C}_+ \cup \mathcal{C}_0$.

Next, we will check that $\mathcal{N}$ is a (k, δ, η)-neck if $\delta' \le \delta'(\delta, \gamma, \eta)$ sufficiently small.

The Lipschitz condition (n5) and the Vitali condition (n1) in the neck region are satisfied by the construction. If $\delta' \le \delta'(\delta, \gamma, \eta)$, let us check the volume ratio condition (n2). In fact, for any $x \in \mathcal{C}$, let $\tilde{x} \in \tilde{\mathcal{C}}$ be such that $d(x, \tilde{\mathcal{C}}) = d(x, \tilde{x})$. Denote by $\tilde{x}_c$ the associated center point of $\tilde{c}$-ball such that $B_{\tilde{r}_{\tilde{x}}}(\tilde{x}) \subset B_{\gamma^{-1}\tilde{r}_{\tilde{x}}}(\tilde{x}_c)$. For $\delta' \le \delta'(n, \mathrm{v}, \delta)$ small and $y \in B_{\delta' r_x}\mathcal{L}_{\tilde{x}_c, \delta^{-3} r_x}$, we always have $|\mathcal{V}_{\delta r_x}(x) - \mathcal{V}_{\delta r_x}(y)| \le \delta^{100}$ since $B_{r_x}(x)$ and $B_{r_x}(y)$ are $\delta' r_x$-close to the same cone at scale r_x. On the other hand, by the definition of a $\tilde{c}$-ball, there exists

$$y \in B_{\delta' r_x}\mathcal{L}_{\tilde{x}_c, \delta^{-3} r_x} \cap \mathcal{P}_{\delta^{-3} r_x, \xi}(\tilde{x}_c).$$

This implies $|\mathcal{V}_{\xi\delta^{-3} r_x}(y) - \bar{V}| \le \xi$. Therefore, if $\xi \le \delta^{20}$, we conclude that $|\mathcal{V}_{\delta r_x}(x) - \bar{V}| \le \delta^{15}$. By (2.2), the monotonicity of the volume ratio, we finally get

$$|\mathcal{V}_{\delta r_x}(x) - \mathcal{V}_{\delta^{-1}}(x)| \le \delta^{10}.$$

Thus, the volume ratio condition (n2) is satisfied.

Next, note that if $\delta' \le \delta'(\delta, \gamma, \eta)$, it follows from the definition of $\tilde{\mathcal{S}}$ that if $x \in \mathcal{C}$, then $B_r(x)$ is not $(k+1, \eta)$-symmetric and $B_r(x)$ is (k, δ^2)-symmetric for all $\delta^{-1} \ge r \ge r_x$. To see this, first observe that $B_r(x)$ is $(0, \delta^3)$-symmetric by the volume pinching estimate (n2) for $\delta' \le \delta'(n, \mathrm{v}, \eta, \delta)$. On the other hand, $B_r(x) \subset B_{\gamma^{-1} r}(\tilde{x}_c)$ and these two balls are comparable. Moreover, the latter ball is (k, δ'^2)-symmetric with respect to $\mathcal{L}_{\tilde{x}_c, \gamma^{-1} r}$ but not $(k+1, 3\eta/2)$-symmetric. From this we conclude that $B_r(x)$ is not $(k+1, \eta)$-symmetric and $B_r(x)$ is (k, δ^2)-symmetric. Hence we prove the condition (n3).

The covering condition (n4), which states that the approximate singular set $\mathcal{L}_{x,r}$ with $r \ge r_x$ is covered by $B_{\tau r}(\mathcal{C})$, is satisfied by the construction of $\tilde{\mathcal{N}}$ and $\mathcal{N}$. To see this, for each $x \in \mathcal{C}$, denote the associated $\tilde{x} \in \tilde{\mathcal{C}}$ with $d(x, \tilde{\mathcal{C}}) = d(\tilde{x}, x)$. Let $\tilde{x}_c$ denote the associated center point of $\tilde{c}$-ball such that $B_{\tilde{r}_{\tilde{x}}}(\tilde{x}) \subset B_{\gamma^{-1}\tilde{r}_{\tilde{x}}}(\tilde{x}_c)$. Since $B_{\gamma^{-1} r}(\tilde{x}_c)$ is a $\tilde{c}$-ball, and since $B_{\gamma^{-1} r}(\tilde{x}_c)$ is not $(k+1, 3\eta/2)$-symmetric, by the Cone-Splitting Theorem 4.6, we have $\mathcal{L}_{x,r} \subset B_{\tau r/4}(\mathcal{L}_{\tilde{x}_c, \gamma^{-1} r})$.

On the other hand, by the construction of the approximate neck $\tilde{\mathcal{N}}$, we have that $\mathcal{L}_{\tilde{x}_c, \gamma^{-1} r} \subset B_{\tau r/4}(\tilde{\mathcal{C}})$. Noting from the construction of $\mathcal{C}$ that $\tilde{\mathcal{C}} \subset \mathcal{C}$, we arrive at

$$\mathcal{L}_{x,r} \subset B_{\tau r/4}\mathcal{L}_{\tilde{x}_c, \gamma^{-1} r} \subset B_{\tau r/2}(\tilde{\mathcal{C}}) \subset B_{\tau r/2}(\mathcal{C}), \tag{10.72}$$

which proves the condition (n4). Therefore, we have shown that if $\xi \le \xi(n, \mathrm{v}, \gamma, \delta, \eta)$, then $\mathcal{N}$ is a (k, δ, η)-neck.

We now focus on the content estimates of Proposition 10.5.

Notation. Denote the various types of ball $B_{r_x}(x)$ with $x \in \mathcal{C}$ by $B_{r_b}(x_b)$, $B_{r_c}(x_c)$, $B_{r_d}(x_d)$ and $B_{r_e}(x_e)$, where $B_{2r_b}(x_b)$ is $(k+1, 2\eta)$-symmetric, $B_{2r_c}(x_c)$ is not $(k+1, 2\eta)$-symmetric and $\mathrm{Vol}(B_{\gamma r_c}\mathcal{P}_{r_c,\xi}(x_c)) \geq \epsilon\gamma^{n-k}r_c^n$, $B_{2r_d}(x_d)$ satisfies $\mathrm{Vol}(B_{\gamma r_d}\mathcal{P}_{r_d,\xi}(x_d)) < \epsilon\gamma^{n-k}r_d^n$, and $B_{2r_e}(x_e)$ satisfies $\mathcal{P}_{r_e,\xi}(x_e) = \emptyset$.

The content estimates rely on the Neck Structure Theorem 2.9. As mentioned prior to the definitions of the various types of balls a, b, c, d, e, every ball is at least one of these types. Therefore, the following is a restatement of (2.7) of the Neck Structure Theorem (equivalently (9.1)):

$$\begin{aligned}\mathcal{H}^k(\mathcal{C}_0 \cap B_{15/8}(x)) + \sum_{x_b \in B_{15/8}(x)} r_b^k \\ + \sum_{x_c \in B_{15/8}(x)} r_c^k + \sum_{x_d \in B_{15/8}(x)} r_d^k + \sum_{x_e \in B_{15/8}(x)} r_e^k \leq C(n).\end{aligned} \tag{10.73}$$

In order to finish the proof, it suffices to show that the content of c-balls is small. This is reasonable since the approximate neck $\tilde{\mathcal{N}}$ does not contain any $\tilde{c}$-balls at all. Thus, we need to verify that our process of going from $\tilde{\mathcal{N}}$ to $\mathcal{N}$ did not create too many c-balls.

Denote the center of the c-balls $B_{r_c}(x_c)$ by a subset $\mathcal{C}_c \subset \mathcal{C} \cap B_{3/2}(p)$. From the construction of the approximate neck $\tilde{\mathcal{N}}$ and the definition of $\tilde{\mathcal{S}}$, it follows that $\tilde{\mathcal{S}} \subset \cup_{\tilde{x} \in \tilde{\mathcal{C}}} B_{\tau \tilde{r}_{\tilde{x}}}(\tilde{x})$. When considering $\mu(\mathcal{C}_c)$[14]we will restrict attention to each $B_{\tilde{r}_{\tilde{x}}}(\tilde{x})$ with $\tilde{x} \in \tilde{\mathcal{C}}$. Let us first consider the content of $\mathcal{C}_c$ in $B_{\tilde{r}_d}(\tilde{x}_d)$. Since $\mathcal{P}_{\tilde{r}_d,\xi}(\tilde{x}_d)$ has small volume, we have the following lemma.

LEMMA 10.8. $\mu\Big(\mathcal{C}_c \cap \bigcup_{\tilde{x}_d \in \tilde{\mathcal{C}}} B_{3\tilde{r}_d/2}(\tilde{x}_d)\Big) \leq C(n, \mathrm{v})\epsilon.$

Proof. We will see that it suffices to prove that for each $\tilde{d}$-ball $B_{\tilde{r}_d}(\tilde{x}_d)$ with $\tilde{x}_d \in \tilde{\mathcal{C}}$, we have

$$\mu\Big(\mathcal{C}_c \cap B_{3\tilde{r}_d/2}(\tilde{x}_d)\Big) \leq C(n, \mathrm{v})\epsilon\, \mu\Big(B_{\tau^4 \tilde{r}_d}(\tilde{x}_d)\Big). \tag{10.74}$$

In fact, since $B_{\tau^4\tilde{r}_d}(\tilde{x}_d)$ are disjoint and μ is a doubling measure with $\tau = \tau_n$, we have by (10.74) that

$$\begin{aligned}\mu\Big(\mathcal{C}_c \cap \bigcup_{\tilde{x}_d \in \tilde{\mathcal{C}}} B_{3\tilde{r}_d/2}(\tilde{x}_d)\Big) &\leq \sum_{\tilde{x}_d \in \tilde{\mathcal{C}}} \mu\Big(\mathcal{C}_c \cap B_{3\tilde{r}_d/2}(\tilde{x}_d)\Big) \\ \leq C(n, \mathrm{v})\epsilon \sum_{\tilde{x}_d \in \tilde{\mathcal{C}}} \mu\Big(B_{\tau^4\tilde{r}_d}(\tilde{x}_d)\Big) &\leq C(n, \mathrm{v})\epsilon\mu(B_{15/8}(p)) \leq C(n, \mathrm{v})\epsilon,\end{aligned} \tag{10.75}$$

[14]As usual, let $\mu = \sum_{x \in \mathcal{C}} r_x^k \delta_x + \mathcal{H}^k|_{\mathcal{C}_0}$ be the packing measure associated with the (k, δ, η)-neck region $\mathcal{N}$.

where in the last inequality, we have used the Neck Structure Theorem 2.9. Thus, we only need to prove (10.74).

By the definition of r_y in (10.70), we have for any $x_c \in \mathcal{C}_c \cap B_{3\tilde{r}_d/2}(\tilde{x}_d)$ that $r_c \le 10\delta^2 \tilde{r}_d$. Since $B_{r_c}(x_c)$ is a c-ball, there exists $y \in B_{4r_c}(x_c)$ such that $|\mathcal{V}_{\xi r_c}(y) - \bar{V}| \le \xi$. In particular, this implies $y \in \mathcal{P}_{\tilde{r}_d,\xi}(\tilde{x}_d) \cap B_{5\tilde{r}_d/3}(\tilde{x}_d)$ and $B_{\gamma\tilde{r}_d/10}(y) \subset B_{\gamma\tilde{r}_d}\mathcal{P}_{\tilde{r}_d,\xi}(\tilde{x}_d)$. On the other hand, since $r_c \le 10\delta^2\tilde{r}_d$ and $d(y,x_c) \le 4r_c$, we have $B_{\gamma\tilde{r}_d/20}(x_c) \subset B_{\gamma\tilde{r}_d/10}(y) \subset B_{\gamma\tilde{r}_d}\mathcal{P}_{\tilde{r}_d,\xi}(\tilde{x}_d)$.

Consider a maximal disjoint collection $\{B_{\gamma\tilde{r}_d/20}(x_i'),\ x_i' \in \mathcal{C}_c \cap B_{3\tilde{r}_d/2}(\tilde{x}_d)\}$ with cardinality N. We have

$$NC(n,\mathrm{v})\gamma^n\tilde{r}_d^n \le \sum_{x_i'} \mathrm{Vol}(B_{\gamma\tilde{r}_d/20}(x_i')) \le \mathrm{Vol}\Big(B_{\gamma\tilde{r}_d}\mathcal{P}_{\tilde{r}_d,\xi}(\tilde{x}_d)\Big) < \epsilon\gamma^{n-k}\tilde{r}_d^n. \tag{10.76}$$

Therefore, we have $N \le \epsilon C(n,\mathrm{v})\gamma^{-k}$, and so

$$\begin{aligned}\mu\Big(\mathcal{C}_c \cap B_{3\tilde{r}_d/2}(\tilde{x}_d)\Big) &\le \sum_{x_i'} \mu\Big(\mathcal{C}_c \cap B_{\gamma\tilde{r}_d/5}(x_i')\Big) \le C(n)N\gamma^k\tilde{r}_d^k \\ &\le \epsilon C(n,\mathrm{v})\tilde{r}_d^k \le \epsilon C(n,\mathrm{v})\mu(B_{\tau^4\tilde{r}_d}(\tilde{x}_d)).\end{aligned} \tag{10.77}$$

This finishes the proof of (10.74). Thus, the proof of Lemma 10.8 is complete. □

Having controlled the content of $\mathcal{C}_c$ in $\tilde{d}$-balls in Lemma 10.8, we will now consider the content of $\mathcal{C}_c$ in $\tilde{b}$-balls. However, unlike the case of $\tilde{d}$-balls, there exists no a priori small volume set. Thus, we will need to argue in a different way from in Lemma 10.8.

Remark 10.9. Prior to beginning the proof proper, we will give a brief indication of the argument.

Let $x_c \in B_{\tilde{r}_b}(\tilde{x}_b)$. In the definition of $\tilde{\mathcal{S}}$ (see (10.70)) we require that $\tilde{\mathcal{S}}$ is a subset of $\cup_{\tilde{x}\in\tilde{\mathcal{C}}} B_{\tau^3\tilde{r}_{\tilde{x}}}(\tilde{x})$. Therefore, we may assume $x_c \in B_{\tau^3\tilde{r}_b}(\tilde{x}_b)$. Since each $\tilde{b}$-ball is $(k+1, 3\eta/2)$-symmetric and each c-ball is not $(k+1, 2\eta)$-symmetric, this will force the c-ball $B_{r_c}(x_c) \subset B_{2\tilde{r}_b}(\tilde{x}_b)$ to have small radius $r_c \ll \tilde{r}_b$. From the definition of r_x in (10.70) there must exist some $\tilde{x} \in \tilde{\mathcal{C}}$ with $d(x_c, \tilde{x}) = d(x_c, \tilde{\mathcal{C}})$ such that $\tilde{r}_{\tilde{x}} \ll \tilde{r}_b$. By (10.68) we have that $\tilde{x} \notin B_{\tau^3\tilde{r}_b}(\tilde{x}_b)$. Thus, one sees from the definition of r_x in (10.70) that $x_c \in A_{\tau^3\tilde{r}_b(1-\delta),\tau^3\tilde{r}_b}(\tilde{x}_b)$; see (10.78) below for further details. Therefore, the content estimate of $\mathcal{C}_c \cap B_{\tilde{r}_b}(\tilde{x}_b)$ will be controlled by the content estimate of $\mathcal{C}_c \cap A_{\tau^3\tilde{r}_b(1-\delta),\tau^3\tilde{r}_b}(\tilde{x}_b)$ which, by a simple covering argument, is small.

Now we begin the actual proof the content estimate of $\mathcal{C}_c \cap B_{\tilde{r}_b}(\tilde{x}_b)$. For $0 < \tilde{\delta} < \epsilon^3$, we define a subset of $\mathcal{C}_c$ by those points with small radius compared with a $\tilde{b}$-ball by

$$\mathcal{C}_{\tilde{\delta}} := \bigcup_{\tilde{x}_b \in \tilde{\mathcal{C}}_+} \{y \in \mathcal{C}_c \cap B_{\tilde{r}_b}(\tilde{x}_b) :\ r_y \le \delta^4 \tilde{\delta} \tau^4 \tilde{r}_b\}. \tag{10.78}$$

We will see below that $\mathcal{C}_{\tilde{\delta}}$ contains all the centers of c-balls inside $\tilde{b}$-balls.

Indeed, note that if $\xi \le \xi(\tilde{\delta}, \delta, \epsilon, n, \mathrm{v}, \gamma)$ and $r_y \ge \delta^4 \tau^4 \tilde{\delta} \tilde{r}_b$ with $y \in \mathcal{C} \cap B_{\tilde{r}_b}(\tilde{x}_b)$, then the ball $B_{2r_y}(y)$ is $(k+1, 5\eta/3)$-symmetric which, in particular, implies that $B_{2r_y}(y)$ is not a c-ball. Therefore, to estimate the content of c-ball in $\tilde{b}$-balls, it will suffice to consider the set $\mathcal{C}_{\tilde{\delta}}$.

From the definition of $\mathcal{C}_{\tilde{\delta}}$, we can see that $\mathcal{C}_{\tilde{\delta}} \cap B_{(1-\tilde{\delta})\tau^3 \tilde{r}_b}(\tilde{x}_b) = \emptyset$. In fact, if $y \in \mathcal{C}_{\tilde{\delta}} \cap B_{(1-\tilde{\delta})\tau^3 \tilde{r}_b}(\tilde{x}_b)$, then by the definition of r_y there must exist $\tilde{x} \in \tilde{\mathcal{C}}$ such that $d(y, \tilde{x}) = d(y, \tilde{\mathcal{C}})$ with $\tilde{r}_{\tilde{x}} < \tilde{r}_b$. By (10.68), this implies $\tilde{x} \notin B_{\tau^3 \tilde{r}_b}(\tilde{x}_b)$. This contradicts $\delta^2 d(y, \tilde{x}) = r_y \le \delta^4 \tau^4 \tilde{\delta} \tilde{r}_b$ since $d(y, \tilde{x}) \ge \tilde{\delta} \tau^3 \tilde{r}_b$. Therefore, we have shown that

$$\mathcal{C}_{\tilde{\delta}} \cap B_{(1-\tilde{\delta})\tau^3 \tilde{r}_b}(\tilde{x}_b) = \emptyset. \tag{10.79}$$

Let $B_{\tilde{r}_d}(\tilde{x}_d)$ denote the $\tilde{d}$-balls with $\tilde{x}_d \in \tilde{\mathcal{C}}$ in the approximate neck region $\tilde{\mathcal{N}}$.

By removing the points in $\tilde{d}$-balls, we have the following content estimate of $\mathcal{C}_{\tilde{\delta}}$:

LEMMA 10.10. *Let* $\tilde{\delta} \le \tilde{\delta}(\gamma, \epsilon) \le \epsilon^3$ *and* $\xi \le \xi(\tilde{\delta}, \delta, n, \mathrm{v}, \gamma, \epsilon, \eta)$. *Then*

$$\mu(\mathcal{C}_{\tilde{\delta}} \setminus \Big(\bigcup_d B_{\tilde{r}_d}(\tilde{x}_d)\Big) \le \epsilon^2. \tag{10.80}$$

Proof. We will divide the proof into two steps. In the first step, we consider content of $\mathcal{C}_{\tilde{\delta}} \cap B_{\tau^3 \tilde{r}_b}(\tilde{x}_b)$ which is actually equal to the content of $\mathcal{C}_{\tilde{\delta}} \cap A_{(1-\tilde{\delta})\tau^3 \tilde{r}_b, \tau^3 \tilde{r}_b}(\tilde{x}_b)$ by (10.79). In the second step, we will consider content of the remainder $\mathcal{C}_{\tilde{\delta}} \cap A_{\tau^3 \tilde{r}_b, \tilde{r}_b}(\tilde{x}_b)$ which is zero after taking out all the points of $B_{\tilde{r}_d}(\tilde{x}_d)$.

The reason for this division of cases based on the radius $\tau^3 \tilde{r}_b$ is that the construction of the approximate neck $\tilde{\mathcal{N}}$ satisfies (10.62) and (10.68).

Step 1. Denote $\mathcal{C}_{\tilde{\delta},1} := \mathcal{C}_{\tilde{\delta}} \cap \Big(\bigcup_{\tilde{x}_b \in \tilde{\mathcal{C}}} B_{\tau^3 \tilde{r}_b}(\tilde{x}_b)\Big)$. We will show that $\mu\big(\mathcal{C}_{\tilde{\delta},1} \setminus \big(\bigcup_d B_{\tilde{r}_d}(\tilde{x}_d)\big)\big) \le C(n, \gamma)\tilde{\delta}$. By the Ahlfors regularity of measure μ, it will suffice to prove $\mu(\mathcal{C}_{\tilde{\delta}} \cap B_{\tau^3 \tilde{r}_b}(\tilde{x}_b)) \le C(n, \gamma) \tilde{\delta} \mu(B_{\tau^4 \tilde{r}_b}(\tilde{x}_b))$ for each $B_{\tilde{r}_b}(\tilde{x}_b)$. Since $\mathcal{C}_{\tilde{\delta}} \cap B_{(1-\tilde{\delta})\tau^3 \tilde{r}_b}(\tilde{x}_b) = \emptyset$ in (10.79), we will only need to prove

$$\mu\Big(\mathcal{C}_{\tilde{\delta}} \cap A_{(1-\tilde{\delta})\tau^3 \tilde{r}_b, (1+\tilde{\delta})\tau^3 \tilde{r}_b}(\tilde{x}_b)\Big) \le C(n, \gamma) \cdot \tilde{\delta} \cdot \mu(B_{\tau^4 \tilde{r}_b}(\tilde{x}_b)). \tag{10.81}$$

Let us prove (10.81). In fact, by the construction of $\tilde{\mathcal{N}}$, there exists a $\tilde{c}$-ball $B_{2\gamma^{-1}\tilde{r}_b}(\tilde{x}_c)$ as in (10.69) which is not $(k+1, 3\eta/2)$-symmetric such that $B_{\tilde{r}_b}(\tilde{x}_b)$

$\subset B_{2\gamma^{-1}\tilde r_b}(\tilde x_c)$. Assume the $\tilde c$-ball $B_{\gamma^{-1}\tilde r_b}(\tilde x_c)$ is (k,δ')-symmetric with respect to $\mathcal{L}_{\tilde x_c,\gamma^{-1}\tilde r_b}$. For $\xi\le\xi(\tilde\delta,\delta,\epsilon,\mathrm{v},n,\gamma,\eta)$, we must have

$$\begin{aligned}&\mathcal{C}_{\tilde\delta}\cap A_{(1-10\tilde\delta)\tau^3\tilde r_b,(1+10\tilde\delta)\tau^3\tilde r_b}(\tilde x_b)\\&\qquad\subset\Big(B_{\tilde\delta^2\gamma^{-1}\tau^3\tilde r_b}\mathcal{L}_{\tilde x_c,\gamma^{-1}\tilde r_b}\cap A_{(1-10\tilde\delta)\tau^3\tilde r_b,(1+10\tilde\delta)\tau^3\tilde r_b}(\tilde x_b)\Big).\end{aligned}\tag{10.82}$$

Otherwise, there will be another splitting factor for $B_{\gamma^{-1}\tilde r_b}(\tilde x_c)$ which would contradict the fact that $B_{\gamma^{-1}\tilde r_b}(\tilde x_c)$ is not $(k+1,3\eta/2)$-symmetric.

Now consider a collection of maximal disjoint balls

$$\{B_{\tilde\delta\tau^3\tilde r_b}(x_c),\ x_c\in\mathcal{C}_{\tilde\delta}\cap A_{(1-\tilde\delta)\tau^3\tilde r_b,(1+\tilde\delta)\tau^3\tilde r_b}(\tilde x_b)\}.\tag{10.83}$$

Denote this set by $\{B_{\tilde\delta\tau^3\tilde r_b}(x_i), i=1,\dots,K\}$ with cardinality K. By the covering property in (10.82), we have $K\le C(n,\gamma)\tilde\delta^{1-k}$. Therefore, we arrive at

$$\begin{aligned}\mu\Big(\mathcal{C}_{\tilde\delta}\cap A_{(1-\tilde\delta)\tau^3\tilde r_b,(1+\tilde\delta)\tau^3\tilde r_b}(\tilde x_b)\Big)&\le\sum_{i=1}^K\mu\Big(B_{3\tilde\delta\tau^3\tilde r_b}(x_i)\Big)\\&\le K\cdot A(n)\cdot\tau^{3k}\cdot\tilde\delta^k\cdot\tilde r_b^k\\&\le C(n,\gamma)\cdot\tilde\delta\cdot\tilde r_b^k\\&\le C(n,\gamma)\cdot\tilde\delta\cdot\mu(B_{\tau^4\tilde r_b}(\tilde x_b)),\end{aligned}\tag{10.84}$$

where we have used the Ahlfors regularity of μ for (k,δ,η)-neck regions in Theorem 2.9. Thus, we have proved (10.81). Since $B_{\tau^4\tilde r_b}(\tilde x_b)$ are disjoint and $\mathcal{C}_{\tilde\delta}\cap B_{(1-\tilde\delta)\tau^3\tilde r_b}(\tilde x_b)=\emptyset$, and noting the definition of $\mathcal{C}_{\tilde\delta,1}$, we have

$$\begin{aligned}\mu(\mathcal{C}_{\tilde\delta,1}\setminus\Big(\bigcup_d B_{\tilde r_d}(\tilde x_d)\Big))&\le\sum_{\tilde x_b\in\tilde{\mathcal{C}}}\mu(B_{\tau^3\tilde r_b}(\tilde x_b)\cap\mathcal{C}_{\tilde\delta}\le\sum_{\tilde x_b\in\tilde{\mathcal{C}}}\mu(A_{(1-\tilde\delta)\tau^3\tilde r_b,\tilde r_b}(\tilde x_b)\cap\mathcal{C}_{\tilde\delta})\\&\le C(n,\gamma)\tilde\delta\sum_{\tilde x_b\in\tilde{\mathcal{C}}}\mu(B_{\tau^4\tilde r_b}(\tilde x_b)))\\&\le C(n,\gamma)\tilde\delta\mu(B_2(p))\le C(n,\gamma)\tilde\delta.\end{aligned}\tag{10.85}$$

Step 2. Denote $\mathcal{C}_{\tilde\delta,2}:=\mathcal{C}_{\tilde\delta}\setminus\mathcal{C}_{\tilde\delta,1}$ to be the centers of c-balls outside $B_{\tau^3\tilde r_b}(\tilde x_b)$. We will show that $\mathcal{C}_{\tilde\delta,2}\setminus\Big(\bigcup_d B_{\tilde r_d}(\tilde x_d)\Big)=\emptyset$. One key ingredient in what follows is the construction of the approximate neck region satisfying (10.62) and (10.68). Roughly speaking, this implies there exits no approximating singular set outside $B_{\tau^3\tilde r_{\tilde x}}(\tilde x)$ with $\tilde x\in\tilde{\mathcal{C}}$.

For a given $\tilde b$-ball $B_{\tilde r_b}(\tilde x_b)$, consider

$$\mathcal{C}_{\tilde\delta,b}:=\Big(\mathcal{C}_{\tilde\delta,2}\cap B_{\tilde r_b}(\tilde x_b)\Big)\setminus\left(\bigcup_{x\in\tilde{\mathcal{C}},\tilde r_x<\tilde r_b}B_{\tilde r_x/2}(\tilde x)\right).$$

Let us see that it will suffice to prove $\mathcal{C}_{\tilde{\delta},b} = \emptyset$ for any $\tilde{x}_b \in \tilde{\mathcal{C}}$. Indeed, assume $\mathcal{C}_{\tilde{\delta},b} = \emptyset$ for any $\tilde{x}_b \in \tilde{\mathcal{C}}$. Then we will show that $\mathcal{C}_{\tilde{\delta},2} \setminus \left(\bigcup_d B_{\tilde{r}_d}(\tilde{x}_d)\right) = \emptyset$. Assume there exists $y \in \mathcal{C}_{\tilde{\delta},2} \setminus \left(\bigcup_d B_{\tilde{r}_d}(\tilde{x}_d)\right) \neq \emptyset$. Let $B_{\tilde{r}_b}(\tilde{x}_b)$ be the minimal sized ball such that $y \in B_{\tilde{r}_b}(\tilde{x}_b)$. Note that $\tilde{r}_b > 0$ since otherwise $y \in \tilde{\mathcal{C}}_0$, which is not a point in $\mathcal{C}_c$. Therefore, we have $y \in \mathcal{C}_{\tilde{\delta},b}$. But this is a contradiction as $\mathcal{C}_{\tilde{\delta},b} = \emptyset$.

We will now prove that $\mathcal{C}_{\tilde{\delta},b} = \emptyset$. For a given $\tilde{b}$-ball $B_{\tilde{r}}(\tilde{x}_b)$, if there exists $y \in \mathcal{C}_{\tilde{\delta},b}$, then by the definitions of r_y, $\mathcal{C}_{\tilde{\delta},b}$ and $\mathcal{C}_{\tilde{\delta}}$ there must exist $\tilde{y} \in \tilde{\mathcal{C}}$ such that

$$\delta\tilde{\delta}\tau^4\tilde{r}_b > s := d(y,\tilde{y}) = d(y,\tilde{\mathcal{C}}) \geq \tilde{r}_{\tilde{y}}/2. \tag{10.86}$$

Here, the last inequality follows from the definition of $\mathcal{C}_{\tilde{\delta},b}$, while the first inequality follows from the definitions of r_y and $\tilde{\mathcal{C}}_{\tilde{\delta}}$.

Let $B_{\gamma^{-1}\tilde{r}_{\tilde{y}}}(\tilde{y}_c)$ be the associated $\tilde{c}$-ball covering $B_{\tilde{r}_{\tilde{y}}}(\tilde{y})$ as in (10.69). Then by the definition of $\tilde{\mathcal{S}}$ we have $y \in \mathcal{L}_{\tilde{y}_c,\gamma^{-1}s}$. By the construction of the approximating neck $\tilde{\mathcal{N}}$ through (10.62) and $s \leq \delta\gamma^2\tilde{r}_b$, we have

$$\mathcal{L}_{\tilde{y}_c,\gamma^{-1}s} \subset \bigcup_{\tilde{x}\in\tilde{\mathcal{C}}} B_{\tau^3\tilde{r}_{\tilde{x}}}(\tilde{x}) \cup \bigcup_{\tilde{x}\in\tilde{\mathcal{C}},\tilde{r}_{\tilde{x}}<\tilde{r}_b} B_{\tau s}(\tilde{x}). \tag{10.87}$$

Since $y \notin B_{\tau^3\tilde{r}_{\tilde{x}}}(\tilde{x})$ by the definition of $\mathcal{C}_{\tilde{\delta},2}$, there exists $\tilde{x} \in \tilde{\mathcal{C}}$ such that $d(y,\tilde{x}) \leq \tau s < s$, which contradicts that $d(y,\tilde{\mathcal{C}}) = s$. Thus, we have finished Step 2.

Fix $\tilde{\delta} \leq \tilde{\delta}(n,\gamma,\epsilon)$. Since $\mathcal{C}_{\tilde{\delta}} = \mathcal{C}_{\tilde{\delta},1} \cup \mathcal{C}_{\tilde{\delta},2}$ by combining Steps 1 and 2, we complete the proof of Lemma 10.10. □

The bound on the content of c-balls follows easily from Lemmas 10.8 and 10.10. This completes the proof of Proposition 10.5 and hence, of Proposition 10.2 and Theorem 2.12 as well. □

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