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Lefschetz properties of some codimension three Artinian Gorenstein algebras

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Codimension two Artinian algebras have the strong and weak Lefschetz properties provided the characteristic is zero or greater than the socle degree. It is open to what extent such results might extend to codimension three Artinian Gorenstein algebras. Despite much work, the strong Lefschetz property for codimension three Artinian Gorenstein algebra has remained largely mysterious; our results build on and strengthen some of the previous results. We here show that every standard-graded codimension three Artinian Gorenstein algebra A having maximum value of the Hilbert function at most six has the strong Lefschetz property, provided that the characteristic is zero. When the characteristic is greater than the socle degree of A, we show that A is almost strong

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Lefschetz, they are strong Lefschetz except in the extremal pair of degrees.

Contents

1. Introduction

We consider standard graded Artinian Gorenstein (AG) algebras A over a field k. Recall that the Hilbert function of A is the function $T(A) : \mathbb{N} \to \mathbb{N}, T(A)_i = \dim_{\mathsf{k}} A_i$. The maximum value of the Hilbert function $T(A)$ of A is called the *Sperner number* of A. The *socle degree* of A is $j = \max\{i \mid T(A)_i \neq 0\}$.

Definition 1.1. Consider a standard-graded Artinian Gorenstein algebra A and an element $\ell \in A_1$. The pair (A, ℓ) is called *weak Lefschetz* (WL) if the multiplication maps $\ell : A_i \to A_{i+1}$ have full rank for each integer i.

The pair (A, ℓ) with A of socle degree j is called *strong Lefschetz* (SL) if the maps $\ell^{j-2i} : A_i \to A_{j-i}$ are isomorphisms for each $i \in [0, \lfloor j/2 \rfloor]$.

We call A *weak Lefschetz* (WL) or *strong Lefschetz* (SL), respectively, if there is a linear form ℓ for which (A, ℓ) is WL or SL, respectively.

The SL and WL properties can be defined equivalently in terms of the *Jordan type* $P_{A,\ell}$ of the map $\ell : A \to A$. The Jordan type is the partition of $\dim_{\mathsf{k}} A$ giving the Jordan block decomposition of this linear transformation. The study of the Jordan type of Artinian algebras was initiated in the context of Lefschetz properties by T. Harima and J. Watanabe and collaborators [18–21,29,22,23]. It was since developed by many, then extended to more general Jordan types (see [11,23,25]).

The pair (A, ℓ) is WL if and only if the number of parts of $P_{A,\ell}$ is the Sperner number of A. The pair is SL if and only if the partition $P_{A,\ell}$ is the conjugate $T(A)^\vee$ of the

Hilbert function $T(A)$, when $T(A)$ is reordered as a partition. For the equivalence of the definitions in the SL case see [23, Proposition 3.64] or [25, Proposition 2.10].

A consequence of J. Briançon's work [6] on standard bases is that all standard-graded or local Artinian algebras A in codimension two are SL, provided k is of characteristic zero or characteristic larger than j, the socle degree of A ([25, Proposition 2.15]). A main tool in studying codimension three AG algebras is the Pfaffian structure theorem of D. Buchsbaum and D. Eisenbud [9]. It has been key to prove that graded codimension three AG algebras of given Hilbert function T are parametrized by a smooth irreducible variety $\mathrm{Gor}(T)$ of known dimension (see [24, Chapter 5] and [12,10,19]). We say that a family of codimension three AG algebras A is *general* of Hilbert function T if it is parametrized by an open dense subset of $\mathrm{Gor}(T)$.

Work of T. Harima, J. Migliore, U. Nagel and J. Watanabe [22] showed that all codimension three graded *complete intersections* (CI) are WL when k is of characteristic zero. This result uses the Grauert Mülich theorem for vector bundles on $\mathbb{P}^2$. The question of whether all codimension three AG algebras A are WL, equivalently, whether for a general linear form ℓ the partition $P_{A,\ell}$ has the Sperner number of parts, has been reduced in arbitrary characteristic to the case of A compressed Gorenstein of odd socle degree in [4]. Here, a *compressed* Gorenstein algebra A of codimension r and socle degree j is one having the maximum possible symmetric Hilbert function $T(A)_i = \min\{r_i, r_{j-i}\}$ where $R = \mathsf{k}[x_1, \dots, x_r]$ and $r_i = \dim_{\mathsf{k}} R_i$.

The graded compressed AG algebras of embedding dimension r and socle degree j form an irreducible family parametrized by an open dense subset of $\mathbb{P}(S_j)$, where $S = \mathsf{k}[X_1, \dots, X_r]$; the algebras A in the family correspond to their Macaulay dual generators $F_A \in S_j$. A *general* compressed Gorenstein algebra of any codimension and socle degree was known to be SL by [31], [34, Proposition 3.4 and Corollary 3.5]), [23, Corollary 3.40]. Finally, N. Altafi, again in characteristic zero, has shown that for any codimension three Gorenstein sequence T, a general element of $\mathrm{Gor}(T)$ is SL.

The question of whether each codimension three AG algebra is SL - or even WL - remains open despite many contributions made to this topic, see for instance [20,22,29, 14,5,15,26]. Our result builds on and extends some of this previous work.

The *order* ν of an Artinian algebra $A = R/I$ given by $\nu = \min\{i \mid T(A)_i \neq \dim_{\mathsf{k}}(R_i)\}$ and we denote by k the multiplicity of the Sperner number in the Hilbert function $T(A)$.

Definition 1.2 *(Almost strong Lefschetz).* We say that an AG algebra A is *almost strong Lefschetz* (almost SL) if for a general form ℓ the multiplication maps $\ell^{j-2i} : A_i \to A_{j-i}$ are isomorphisms for $2 \leq i \leq \lfloor j/2 \rfloor$. That is, only the map $\ell^{j-2} : A_1 \to A_{j-1}$ among those in Definition 1.1 may not be an isomorphism if A is almost SL.

Next, we state our main result in this paper. The proof is carried out in Theorems 3.3, 3.7, and 3.9.

Main Theorem. *The following codimension three standard-graded AG algebras A over a field k of characteristic zero are SL. When k is an infinite field of characteristic $p > j$, the socle degree of A, then the AG algebras listed below are almost SL.*

i. All AG algebras with Sperner number less than or equal to six, and
ii. some AG algebras of order two whose Hilbert function is listed in Table 1.

Table 1
Hilbert functions of AG algebras covered by the Main Theorem.

$T = H(A)$, $k \geq 1$	SL char $\mathsf{k} = 0$
$(1, 3^k, 1)$ and $(1, 3, s^k, 3, 1)$, $4 \leq s \leq 6$	Theorem 3.3
$(1, 3, 4, 5, \dots, (s-1), s^k, (s-1), \dots, 3, 1)$, $5 \leq s$	Theorem 3.7
$(1, 3, 5, 6^k, 5, 3, 1)$	Theorem 3.9

Main tools upon which our results rely include a study of the relation between an AG ideal I and $I : x$, which is also an AG ideal, the theory of Macaulay inverse systems, and the connection between Hessian matrices and the Lefschetz properties developed in [29,14,15]. Moreover, we employ the morphism from $\operatorname{Gor}(T)$ to the punctual Hilbert scheme $\operatorname{Hilb}^H(\mathbb{P}^2)$ of projective space, where H is the first, non-decreasing portion of T, when T has a consecutive subsequence (s, s, s) (Theorem 2.6), a consequence of the D. Buchsbaum and D. Eisenbud Pfaffian structure theorem. See Section 2.3 for details regarding this morphism. The proof in some places is surprisingly subtle, particularly so for the case $T = (1, 3, 5, 6^k, 5, 3, 1)$. Our expectation is that for large enough Sperner number and socle degree, there should be many non-SL AG algebras of codimension three, but it is an open problem to find just one.

2. Tools

2.1. Macaulay inverse systems

Let $R = \mathsf{k}[x_1, \dots, x_r]$ be a polynomial ring and let $S = \mathsf{k}_{DP}[X_1, \dots, X_r]$ be a divided power algebra on which R acts by contraction (see [24, Appendix A]):

$$x_i \circ X_1^{a_1} \cdots X_i^{a_i} \cdots X_r^{a_r} = \begin{cases} X_1^{a_1} \cdots X_i^{a_i - 1} \cdots X_r^{a_r} & \text{if } a_i > 0 \\ 0 & \text{if } a_i = 0 \end{cases}.$$

In [28] F.H.S. Macaulay proved that a homogeneous ideal $I \subset R$ defines an AG quotient algebra $A = R/I$ of socle degree j if and only if there exists a homogeneous form $F \in S_j$ for which $I = \operatorname{Ann}(F) = \{r \in R \mid r \circ F = 0\}$.

Definition 2.1. Let $R = \mathsf{k}[x_1, \dots, x_r]$ and let $A = R/I$ be a graded AG algebra of socle degree j. A polynomial $F \in S_j$ satisfying $I = \mathrm{Ann}(F)$ is called a *Macaulay dual generator* of A.

A Macaulay dual generator of A is unique up to a non-zero constant multiple.

Remark 2.2. Suppose that $A = R/I$ is an AG algebra with $I = \mathrm{Ann}(F)$ and let $\omega \in A_{j-k}$ and $J = (I \colon \omega)$. Then by [33, Lemma 4] we have that $J = \mathrm{Ann}(G)$ where $G = \omega \circ F$. In particular R/J is AG.

2.2. Hessians and the strong Lefschetz property

Definition 2.3 *([29, Definition 3.1]).* Let F be a polynomial in S and $A = R/\mathrm{Ann}(F)$ be its associated AG algebra. Let $\mathcal{B}_i = \{\alpha_m^{(i)}\}_m$ be a k-basis of A_i. The entries of the i-th Hessian matrix of F with respect to $\mathcal{B}_i$ are given by

$$(\mathrm{Hess}^i(F))_{u,v} = (\alpha_u^{(i)} \alpha_v^{(i)} \circ F).$$

Up to a non-zero constant multiple the determinant $\det \mathrm{Hess}^i(F)$ is independent of the basis $\mathcal{B}_i$. We note that when $i = 1$ the form $\mathrm{Hess}^1(F)$ coincides with the usual Hessian.

T. Maeno and J. Watanabe provided a criterion for AG algebras to be SL and to identify the SL elements using the Hessians (see [23, §3.6]).

Theorem 2.4 *([29, Theorem 3.1]). Assume* char $\mathsf{k} = 0$. *A linear form $\ell = a_1x_1 + \cdots + a_rx_r \in A_1$ is a SL element of $A = R/\mathrm{Ann}(F)$ if and only if* $(\det \mathrm{Hess}^i(F))(a_1, \dots, a_r) \neq 0$*, for $i = 0, 1, \dots, \lfloor \frac{j}{2} \rfloor$. In particular, for $i = 0, 1, \dots, \lfloor \frac{j}{2} \rfloor$ the multiplication map $\times \ell^{j-2i} : A_i \longrightarrow A_{j-i}$ has maximal rank if and only if* $(\det \mathrm{Hess}^i(F))(a_1, \dots, a_r) \neq 0$.

The following theorem due to P. Gordan and M. Noether [16] was reproved in [27, Theorem 1.2], [35] and [7].

Theorem 2.5 *(Gordan-Noether). If F is a form expressed in at most four variables and the characteristic of k is zero, then the Hessian determinant* $\det \mathrm{Hess}(F)$ *is identically zero if and only if F is a cone, that is, if F is annihilated by a linear form in the Macaulay duality.*

In particular, this implies by Theorem 2.4 that if k is of characteristic zero, setting $j = \deg(F)$, $I = \mathrm{Ann}(F)$ with $I \subset \mathfrak{m}^2$ and $A = R/I$ for a polynomial ring R of codimension at most four, the multiplication map $\ell^{j-2} : A_1 \to A_{j-1}$ is an isomorphism for a general linear form $\ell \in A_1$.

2.3. The morphism $\pi : \mathrm{Gor}(T) \to \mathrm{Hilb}^s(\mathbb{P}^2)$

The Gorenstein sequences of codimension three in which the Sperner number occurs at least three times play a distinguished role in our results. This is because of a connection between the AG algebras whose Hilbert function is such a sequence to certain 0-dimensional projective schemes in $\mathbb{P}^2$. Recall that an ideal $J \subset R$ is *saturated* if $J : \mathfrak{m} = J$. We denote by $\underline{s}$ the infinite sequence $(s, s, \ldots)$.

Theorem 2.6 *([24, Theorem 5.31]). Assume* k *is an infinite field*[1] *and let* T *be the Hilbert function of a codimension three AG algebra* $A = R/I$ *of socle degree* j. *Suppose* $T \supset (s, s, s)$, *that is,* T *contains a consecutive subsequence of at least three* s *where* s *is the Sperner number of* T, *and let* τ *be the smallest integer such that* $T(A)_\tau = s$. *Let* $H = (T_{\leq j/2}, s, s, s, \underline{s})$. *Then*

- *i. The ideal* $J = (I_{\leq \tau+1}) \subset R$ *is a saturated ideal defining a subscheme* $\mathfrak{Z} = \mathrm{Proj}\,(R/J) \subset \mathbb{P}^2$, *having length* s, *Castelnuovo-Mumford regularity* $\tau+1$ *and Hilbert function* H.
- *ii. The scheme* $\mathfrak{Z}$ *is the unique degree-*s *scheme in* $\mathbb{P}^2$ *whose ideal is contained in* I.
- *iii. The map* $I \to J$ *defines a morphism:* $\pi : \mathrm{Gor}(T) \to \mathrm{Hilb}^H(\mathbb{P}^2) \subset \mathrm{Hilb}^s(\mathbb{P}^2)$ *whose image contains the open dense subscheme* $\mathrm{Sm}^H(\mathbb{P}^2)$ *parametrizing smooth length-*s *punctual subschemes of* $\mathbb{P}^2$ *having Hilbert function* H.

A similar result is valid when the Gorenstein sequence $T \supset (s, s)$ or just has Sperner number s, but is restricted to a proper scheme sublocus $\mathrm{Gor}_{sch}(T) \subset \mathrm{Gor}(T)$ [24, Theorems 5.39 and 5.46].

Definition 2.7. In the language of [24, §5.1], if the Macaulay dual generator for the AG algebra A is F (Definition 2.1) and if A has Sperner number s, then a length-s scheme $\mathfrak{Z} \in \mathbb{P}^2$ such that $J = I(\mathfrak{Z}) \subset I$ is termed a *tight annihilating scheme* of F.

In particular, the scheme $\mathfrak{Z}$ in Theorem 2.6 is a tight annihilating scheme of F. A tight annihilating scheme is unique under some further conditions [24, Theorem 5.3]. The role of tight annihilating scheme here is due to the following lemma.

Lemma 2.8. *Let* $F \in \mathsf{k}[X_1, \ldots, X_r]_j$ *be the Macaulay dual generator of the AG algebra* $A = \mathsf{k}[x_1, \ldots, x_r]/I$, $I = \mathrm{Ann}(F)$ *of Hilbert function* T. *Assume that* F *has a tight length-*s *punctual annihilating scheme* $\mathfrak{Z}$, *and suppose* $\tau = \tau(\mathfrak{Z}) = \min\{i \mid T_i = s\}$. *The dimension-one algebra* $B = \mathsf{k}[x_1, \ldots, x_r]/I(\mathfrak{Z})$, $I(\mathfrak{Z}) = (I_{\leq} j/2)$ *of Hilbert function* $H = (T_{\leq j/2}, \underline{s})$ *has a non-zero divisor* $\ell \in A_1$. *For each pair* (u, v) *with* $0 \leq u \leq v \leq j - \tau$

[1] In [24] there is an implicit assumption that k is algebraically closed, but the result holds for k infinite.

the multiplication map $\ell^{v-u}: A_u \to A_v$ is an injection. In particular, for $\tau \leq u \leq j-\tau$ this map is an isomorphism, and $\ell^{j-2i}: A_i \to A_{j-i}$ is an isomorphism for $\tau \leq i \leq j/2$.

2.4. Gorenstein sequences in codimension three

Recall that an *O-sequence* $T = (1, t_1, \dots, t_j, 0)$ is one that occurs as the Hilbert function of an Artinian algebra: it satisfies certain Macaulay conditions ([8, §4.2]). A *Gorenstein sequence* $T = (1, t_1, \dots, t_j = 1, 0)$ is a symmetric sequence that occurs as the Hilbert function of a standard-graded AG algebra. We need

Definition 2.9 *(SI sequence).* Let $T = (1, r, \dots t_i, \dots, r, 1_j)$ be a sequence symmetric about $j/2$. Let $j' = \lfloor j/2 \rfloor$ and $\delta_i(T) = t_i - t_{i-1}$. We say that T is an *SI-sequence* if we have

$$\Delta T = (1, \delta_1(T), \delta_2(T), \dots, \delta_{j'}(T)) \text{ is an O-sequence.} \tag{1}$$

The Pfaffian structure theorem of D. Buchsbaum and D. Eisenbud showed that a codimension three Gorenstein ideal I has a length 3 minimal free resolution where the middle map is an alternating matrix whose diagonal Pfaffians are the generators of I (see [9] and [24, Theorem B2, Appendix B.2]). A consequence of the Pfaffian structure theorem is a characterization of the Hilbert functions of AG algebras of codimension three.

Lemma 2.10. *([31, Theorem 4.2]) The symmetric sequence T with $r = 3$ is a Gorenstein sequence if and only if T is a SI-sequence.*

The *order* of T is the smallest integer ν such that $T_\nu \neq \dim_{\mathsf{k}} R_\nu$; we denote it by $\nu(T)$. In other terms $\nu(T)$ is the lowest degree of a generator of any ideal I defining a quotient $A = R/I$ of Hilbert function T. The necessary and sufficient condition for ΔT to be an O-sequence of an algebra of codimension two is

$$\delta_i(T) = i + 1 \text{ for } i < \nu(T) \text{ and } \delta_i(T) \geq \delta_{i+1}(T) \text{ if } i \geq \nu(T). \tag{2}$$

This allows us to determine the possible Gorenstein sequences for codimension three AG algebras having order two or Sperner number at most six.

Corollary 2.11. *The codimension three Gorenstein sequences of Sperner number at most six are $T = (1, 3^k, 1)$, $(1, 3, 4^k, 3, 1)$, $(1, 3, 4, 5^k, 4, 3, 1)$, $(1, 3, 4, 5, 6^k, 5, 4, 3, 1)$, $(1, 3, 5^k, 3, 1)$, $(1, 3, 5, 6^k, 5, 3, 1)$ and $(1, 3, 6^k, 3, 1)$ where $k \geq 1$. The additional Gorenstein sequences of order two are $T = (1, 3, 4, 5, \dots, s-1, s^k, s-1, \dots, 3, 1)$, $(1, 3, 5, 6, \dots, s-1, s^k, s-1, \dots, 3, 1)$ and $(1, 3, 5, 7, \dots, 2t+1, 2t+2, 2t+3 \dots, s^k, \dots 3, 1)$, where $s \geq 7$ and $k \geq 1$.*

Proof. From (2) we obtain under the assumption that the Sperner number does not exceed 6 that $\Delta(T) = (1, 2, \underline{0})$ or $\Delta(T) = (1, 2, 1, \underline{0})$ or $\Delta(T) = (1, 2, 1, 1, \underline{0})$ or $\Delta(T) = (1, 2, 1, 1, 1, \underline{0})$ or $\Delta(T) = (1, 2, 2, \underline{0})$. This is an exhaustive list because the Sperner number is the sum of the $\Delta(T)$ vector. This yields the first claim.

Under the assumption that the order is two the difference vector must have the form $\Delta(T) = (1, 2, \underline{0})$ or $\Delta(T) = (1, 2, 1^{s-3}, \underline{0})$ or $\Delta(T) = (1, 2^t, 1^{s-2t-1}, \underline{0})$. The first case was discussed above. The second yields $T = (1, 3, 4, 5, \ldots, s-1, s^k, s-1, \ldots, 3, 1)$. The third yields both of the remaining Hilbert functions according to whether $t = 2$ or $t > 2$. $\square$

We now elaborate on structural aspects for one of the above mentioned Hilbert sequences.

Lemma 2.12. *Assume that* k *is of characteristic zero, or infinite of characteristic* $p \geq 3$. *If a Gorenstein sequence has the form*

$$T = (1, 3, 5, 7, \ldots, 2t+1, 2t+2, 2t+3, \ldots, s^k, \ldots 3, 1_j), \tag{3}$$

with $\Delta(T) = (1, 2^t, 1^{s-2t-1}, \underline{0})$ *for* $t \geq 3$ *and* $s \geq 2t+1$, *then any AG algebra* $A = \mathsf{k}[x, y, z]/I$ *of Hilbert function* T *satisfies* $I_2 \cong (xy)$ *or* $I_2 \cong (z^2)$, *under the PGL(3) action.*

Proof. Assume by way of contradiction that $I \cong (f, g, \ldots)$, with $f \in R_2$ irreducible, and g the next minimal generator, of degree $a \geq 4$. Since f is irreducible, the ideal (f, g) is a complete intersection of generator degrees $(2, a)$ and its Hilbert function $T = T(A)$ begins $T = (1, 3, 5, 7, \ldots, 2a-1, \underline{2a})$. This yields $a = t+1$ and a componentwise inequality

$$T = T(A) \leq T(R/(f, g)) = (1, 3, 5, \ldots, 2t+1, 2t+2, 2t+2, \ldots),$$

contradicting the equation (3). $\square$

We say a symmetric unimodal sequence T is a *SL sequence* if there is at least one standard-graded Artinian Gorenstein algebra having Hilbert function T. N. Altafi showed the following.

Theorem 2.13. *[2, Theorem 3.2] Let* $R = \mathsf{k}[x_1, \ldots, x_r]$ *and suppose* char $\mathsf{k} = 0$.

(i). Let $A(\mathfrak{Z})$ *be the coordinate ring of a reduced set of points in* $\mathbb{P}^{r-1}$. *A general AG algebra quotient* A *of* $A(\mathfrak{Z})$ *is SL provided the socle degree* j *of* A *satisfies* $j \geq 2\tau(\mathfrak{Z}) - 1$.

(ii). A Gorenstein sequence is an SL sequence if and only if it is an SI-sequence.

Her method of proof of (i) uses the Hessian criterion; the proof of (ii) uses a result of P. Maroscia that $\mathrm{Hilb}^H(\mathbb{P}^{r-1})$ where $\Delta(H)$ is an O-sequence contains a smooth scheme (see [24, Thm 5.21]. L. J. Billera and C. W. Lee [3] and R. Stanley [32] showed that WL Gorenstein sequences are SI when k is of characteristic zero. T. Harima [18] generalized this WL statement to a field of arbitrary characteristic using an algebraic method (linkage) and a technique of J. Watanabe [34, proof of Theorem 3.8].[2]

3. Lefschetz properties

Recall that all Artinian algebras A of codimension two are SL provided the characteristic of the base field is zero or larger than the socle degree of A.

For graded Gorenstein algebras of codimension three, some important cases of WL or SL are already known; the most relevant for this paper are contained in [22,4]. The main pertinent results known before 2017 can be summarized as follows:

Proposition 3.1.

i. *If* char $\mathsf{k} = 0$*, all codimension three CI are WL ([22], [23, Theorem 3.48 and Lemma 3.49]).*
ii. *If* char $\mathsf{k} = 0$*, all codimension three AG algebras of socle degrees* $2e - 1$ *and* $2e$ *over an infinite field* k *are WL if and only if all compressed codimension three AG algebras of odd socle degree* $2e - 1$ *are WL [4, Corollary 2.5].*
iii. *If* char $\mathsf{k} = 0$ *or* char $\mathsf{k} = p > 3$ *and* k *is an infinite field then all codimension three compressed AG algebras of socle degree* 3 *and* 5 *are WL (for $j = 5$ see [4, Theorem 3.8] where $H = (1, 3, 6, 6, 3, 1)$).*[3,4]
iv. *If* char $\mathsf{k} = 0$*, all codimension three AG algebras of socle degree $j \leq 5$ are SL [4, Corollary 3.13].*[5]
v. *If* char $\mathsf{k} = 0$*, all codimension three algebras defined by powers of linear forms are WL [30].*

In addition, as a consequence of Theorem 2.13 we have

[2] Part (ii) of Theorem 2.13 also follows from [3] and [32]: the former shows the g-Theorem that an SI sequence is the h-vector of a simplicial polytope P; the latter shows that P defines an algebraic variety X_P whose cohomology ring is SL and has Hilbert function h.

Corollary 4.6 of [22] shows that over an infinite field the set of unimodal Hilbert functions T possible for WL Artinian algebras and the set possible for SL Artinian algebras are the same: they are those such that the positive part of the first differences ΔT is an O-sequence [22, Proposition 3.5]. This is different than identifying the sets of Hilbert functions that are possible for WL AG algebras and for SL AG algebras using Theorem 2.13 and the earlier results. The characterization of SL Gorenstein sequences remains open in characteristic p.

[3] They also show that when $j = 5$ and char $\mathsf{k} = 3$ then the only non-WL exception is when $I = (x^2y, x^2z, y^3, z^3, x^4 + y^2z^2)$, after a change of variables. The case $j = 3$ was known in char $\mathsf{k} = 0$ and is worked out below when char $\mathsf{k} = p$ (Lemma 3.10).

[4] A. Vraciu writes in her MathSciNet review of [4]: "The proof involves subtle geometric properties of certain classical configurations of known points and Hesse configurations [in $\mathbb{P}^2$]."

[5] The characteristic $\mathsf{k} = 0$ assumption comes from use of the Gordan-Noether theorem to show the map $\ell^{j-2} : A_1 \to A_{j-1}$ has rank 3.

Proposition 3.2. *Let T be a codimension three Gorenstein sequence and assume* char $\mathsf{k} = 0$. *Then an open dense subset of* $\mathrm{Gor}(T)$ *is comprised of SL algebras.*

Proof. By Lemma 2.10, T is an SI sequence. By Theorem 2.13 there is a SL AG algebra of Hilbert function T. Since in codimension three $\mathrm{Gor}(T)$ is irreducible [12], and since being SL is an open condition on a family of algebras having constant Hilbert function - it corresponds to maximal rank of the matrices for multiplication by $\ell^{j-2i} : A_i \to A_{j-i}$ - the conclusion follows. $\square$

In the following we begin a proof of our main result.

Main Theorem. *The following codimension three standard-graded AG algebras A over a field k of characteristic zero are SL. When k is an infinite field of characteristic $p > j$, the socle degree of A, then the AG algebras listed below are almost SL.*

i. All AG algebras with Sperner number less than or equal to six, and
ii. some AG algebras of order two whose Hilbert function is listed in the Table 1.

The proof proceeds by case analysis based on the Hilbert functions listed in the following table. The Hilbert functions of AG algebras with Sperner number less than or equal to six are enumerated in Corollary 2.11 and form a subset of those listed below.

$T = H(A)$, $k \geq 1$	SL char $\mathsf{k} = 0$
$(1, 3^k, 1)$ and $(1, 3, s^k, 3, 1)$, $4 \leq s \leq 6$	Theorem 3.3
$(1, 3, 4, 5, \ldots, (s-1), s^k, (s-1), \ldots, 3, 1)$, $5 \leq s$	Theorem 3.7
$(1, 3, 5, 6^k, 5, 3, 1)$	Theorem 3.9

We will assume throughout that k is an infinite field of characteristic zero, or characteristic $p > j$, where j is the socle degree of the AG algebras being considered. We discuss char $\mathsf{k} = p > j$ explicitly in Section 3.3. Unless otherwise specified we take $R = \mathsf{k}[x, y, z]$ and $S = \mathsf{k}_{DP}[X, Y, Z]$.

3.1. The case $T = (1, 3, s^k, 3, 1), 3 \leq s \leq 6$

We start by analyzing the codimension three AG algebras having least values of the invariant $\tau = \min\{i \mid T(A)_i = s\}$, where s is the Sperner number of the AG algebra A.

Theorem 3.3. *Let $T = (1, 3, s^k, 3, 1)$ with $3 \leq s \leq 6$ and $A \in \mathrm{Gor}(T)$ an AG algebra of socle degree $j = k + 3$. If* char $\mathsf{k} = 0$, *then A is SL. When k is an infinite field of* char $\mathsf{k} > j$, *then A is almost SL.*

Proof. Assume $T = (1, 3, s^k, 3, 1_j), 3 \leq s \leq 6$ and let F be a Macaulay dual generator for A, that is, $A = R/\mathrm{Ann}(F)$ (see Definition 2.1). We fix $l \in A_1$ a general linear form.

We first observe that when char $\mathsf{k} = 0$ Theorem 2.4 and the Gordan-Noether Theorem 2.5 assure that the multiplication map $\ell^{j-2} : A_1 \to A_{j-1}$ is an isomorphism. So, to prove our result it is enough to show that the multiplication map $\ell^{j-2i} : A_i \to A_{j-i}$ is an isomorphism for $2 \le i \le \lfloor \frac{j}{2} \rfloor$.

When $k = 1$ there is nothing more to show. When $k = 2$ the WL property for A follows from Proposition 3.1(ii),(iii) and implies that the map $\ell : A_2 \to A_3$ is an isomorphism.

Now, when $k \ge 3$, by Theorem 2.6 F has a (unique) tight annihilating scheme $\mathfrak{Z}$ defined by a saturated ideal J and $\operatorname{Ann}(F)_t = J_t, t \in [2, j-2]$. Since R/J is saturated, a general linear form $\ell \in R_1$ is a non zero-divisor on R/J, so the map $\ell^{j-2i} : A_i \to A_{j-i}$ is an isomorphism for $2 \le i \le j/2$ (Lemma 2.8). $\square$

3.2. AG algebras of order two

Recall that the order $\nu(A)$ of an Artinian algebra A is $\nu(A) = \min\{i \mid \dim_{\mathsf{k}} A_i \neq \dim_{\mathsf{k}} R_i\}$. In this section we treat the case of codimension three AG algebras of order two, that is, AG algebras A with $T(A)_1 = 3$ and $T(A)_2 \le 5$. We do not restrict $T(A)$ otherwise.

Definition 3.4. We say that an Artinian Gorenstein algebra B has *almost constant Hilbert function* if $T(B) = (1, s^k, 1)$ for some integers $s, k > 0$.

Lemma 3.5. *Suppose that an AG algebra B has almost constant Hilbert function T with Sperner number $s \le 3$ or with $k \ge s+1$. Assume additionally that* char $\mathsf{k} = 0$ *or* k *is infinite with* char $\mathsf{k} = p > k+1$. *Then B is SL.*

Proof. Codimension $s = 1$ is immediate and codimension $s = 2$ is due to J. Briançon ([25, Proposition 2.15]). When $s = 3$ and $k = 1$ there is nothing to show. When $s = 3$ and $k = 2$ the SL property follows from the WL property, shown in Proposition 3.1(iii).

Now assume $s = 3$ and $k \ge 3$ and set $B = R/I$. Theorem 2.6 guarantees the existence of a saturated ideal $J \subset I$ defining a scheme $\mathfrak{Z} \subset \mathbb{P}^2$ and $B_i = (R/J)_i$ for $i \le s$. Since J is saturated, a general linear form ℓ is a non zero-divisor on R/J and consequently the pair (B, ℓ) is SL by Lemma 2.8.

Now assume s is arbitrary and $k \ge s+1$. Then, since Macaulay growth of $T_s = s$ to $T_{s+1} = s$ is maximum, by the Gotzmann theorem $J = (I_{\le s+1})$ is the saturated ideal defining a length s punctual scheme $\mathfrak{Z} \subset \mathbb{P}^{s-1}$ [24, Proposition C.32]. The SL property follows from Lemma 2.8. $\square$

The following result is a crucial ingredient to our proofs in this section.

Proposition 3.6. *Let $R = \mathsf{k}[x_1, \dots, x_r]$ with $r \le 3$ and $A = R/I$ an AG algebra with socle degree j. Assume* char $\mathsf{k} = 0$ *or* k *is infinite with* char $\mathsf{k} > j$. *Assume that there is a linear form $x \in R_1$ such that the ring $B = R/(I : (x))$ has almost constant Hilbert* *function and the ring $A/(x)$ is SL. If* char $\mathsf{k} = 0$*, then A is SL. If* char $\mathsf{k} > j$*, then A is almost SL.*

Proof. Set $C = A/(x) = R/(I + (x)) = k[y, z]/\overline{I}$, where $\overline{I} = I + (x)/(x)$. By Remark 2.2, B is AG with Macaulay dual polynomial $G = x \circ F$ and socle degree $j - 1$. The following short exact sequence relates the three rings

$$0 \to B(-1) \xrightarrow{\cdot x} A \to C \to 0. \tag{4}$$

Let $\ell \in R_1$ be a linear form. Multiplication by powers of ℓ in (4) produce the following commutative diagram where K', K, K'' denote the kernels and L', L, L'' denote the cokernels of the vertical maps

$$\begin{tikzcd}
 & K' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K'' \arrow[d] & \\
0 \arrow[r] & B_{i-1} \arrow[r] \arrow[d, "{\ell'}^{j-2i}"] & A_i \arrow[r] \arrow[d, "\ell^{j-2i}"] & C_i \arrow[r] \arrow[d, "{\ell'}^{j-2i}"] & 0 \\
0 \arrow[r] & B_{j-i-1} \arrow[r] \arrow[d] & A_{j-i} \arrow[r] \arrow[d] & C_{j-i} \arrow[r] \arrow[d] & 0 \\
 & L' \arrow[r] & L \arrow[r] & L'' &
\end{tikzcd}$$

Since $T(B)$ is almost constant and B has codimension at most three, hence also Sperner number at most three, B is SL by Lemma 3.5. Consequently, the leftmost vertical maps in the above diagram are isomorphisms for $i > 1$. Thus for $i \geq 2$ we have $K' = L' = 0$ and the snake lemma yields that the middle vertical map has maximum rank if and only if the rightmost vertical map does. Since C is assumed SL, the multiplication map ${\ell'}^{j-2i}$: $C_i \to C_{j-i}$ has maximum rank on C for general ℓ. Hence for $i \geq 2$ the multiplication map $\ell^{j-2i} : A_i \to A_{j-i}$ has maximum rank. This shows that A is almost SL.

It remains to show that when char $\mathsf{k} = 0$ that for $i = 1$, the multiplication map $\ell^{j-2} : A_1 \to A_{j-1}$ is an isomorphism. Setting $I = \mathrm{Ann}(F)$, this follows from the Gordan-Noether Theorem 2.5. $\square$

Theorem 3.7. *Let A be an AG k-algebra of socle degree j with $H(A) = (1, 3, 4, \ldots)$. If* char $\mathsf{k} = 0$*, then A is SL. If* char $\mathsf{k} = p > j$*, then A is almost SL.*

Proof. From the characterization of codimension three Gorenstein sequences in Corollary 2.11 we see that for $T(A) = (1, 3, 4, \ldots)$ we have $T(A)_3 = 3$ if and only if $j = 4$, otherwise $T(A)_3 \in \{4, 5\}$. The case $T(A)_3 \in \{3, 4\}$ is treated in Theorem 3.3. Note that $T(A)_2 = T(A)_3 = 4$ implies by Lemma 2.10 that $T = (1, 3, 4^k, 3, 1)$.

Now assume $T(A)_3 = 5$ and write $A = R/I$. The precise form of the Hilbert function is $T = (1, 3, 4, 5, \ldots, s-1, s^k, s-1, \ldots, 4, 3, 1)$ according to Corollary 2.11. In this case there must be a linear relation among the two basis elements of I_2, else $T(A) = (1, 3, 4, 4, \ldots)$. So we have $I_2 \cong \langle xy, xz \rangle$ or $I_2 \cong \langle x^2, xy \rangle$, up to PGL(3) action. Then either $(y, z) \subseteq I :$

(x) or $(x, y) \subseteq I : (x)$, respectively, and thus the AG ring $B = \mathsf{k}[x, y, z]/(I : (x))$ has constant Hilbert function $T(B) = (1^j)$. Moreover $A/(x)$ is a codimension two algebra and hence is SL under the given assumption on characteristic. Proposition 3.6 now yields that A is SL if char $\mathsf{k} = 0$ and is almost SL if char $\mathsf{k} = p > j$. $\square$

Remark 3.8. An alternate proof of the SL property in the case when the defining ideal of A contains the forms xy, xz is to observe that this implies A can be written as a connected sum $A = U\#_{\mathsf{k}}V$ where U is an AG quotient of $\mathsf{k}[x]$ and V is an AG quotient of $\mathsf{k}[y, z]$. Since such U, V are SL under the assumptions on the characteristic of k in the statement of Theorem 3.7, it follows from [26, Proposition 5.7] or [23, Theorem 3.76, Proposition 3.77ii] that A is SL as well. This alternative proof is fine for char $\mathsf{k} = p > j$, also.

3.2.1. Case $T = (1, 3, 5, 6^k, 5, 3, 1)$

Theorem 3.9. *Let A be an AG algebra of socle degree $j \geq 5$ with $T(A) = (1, 3, 5, 6^{j-5}, 5, 3, 1)$. If* char $\mathsf{k} = 0$*, then A is SL. If* char $\mathsf{k} = p > j$*, then A is almost SL.*

Proof. Theorem 3.3 deals with $j = 5$, so we may assume $j \geq 6$. Let $A = R/I$ with $R = \mathsf{k}[x, y, z]$. Then the lowest graded components of I are given by $I_2 = \langle f \rangle$ for some $f \in R_2$ and $I_3 = \langle xf, yf, zf, g \rangle$ for some $g \in R_3 \setminus (x, y, z)f$.

We distinguish two cases depending on whether f, g form a regular sequence.

Case (i): If f, g do not form a regular sequence, then they have a common linear divisor. Without loss of generality assume x is this common divisor and thus $f = xf'$ and $g = xg'$ with $f' \in R_1, g' \in R_2$. Set $C = A/(x)$ and $B = A/(0 :_A x)$ and note that both B and C are AG algebras of codimension two; the latter is because $(0 :_A x)$ contains the linear form f'. By Remark 2.2, B is AG of socle degree $j - 1$.

Since $f', g' \in (0 :_A x)$, we have $T(B)_1 = T(B)_2 = 2$ and therefore by the properties of O-sequences and the symmetry of AG Hilbert functions one concludes that

$$T(B)_i = \begin{cases} 1 & \text{for } i = 0, i = j - 1 \\ 2 & \text{for } i \in [1, j - 2] \\ 0 & \text{for } i \geq j. \end{cases}$$

Thus B has almost constant Hilbert function and C is SL since it is a codimension two Artinian algebra; so Proposition 3.6 applies to conclude that A is SL when char $\mathsf{k} = 0$, and that A is almost SL when char $\mathsf{k} = p > j$.

Case (ii): Suppose f, g form a regular sequence. Then A is a complete intersection with defining ideal $I = (f, g, h)$ so that $\deg(h) = j - 2 \geq 4$. Indeed, on one hand (f, g) is a CI in R, so it has only one syzygy, in degree 5. This implies that I has only one minimal generator in degree $j - 2$. On the second hand, assume I has a minimal generator in degree $j - 1$ or j. Then, the symmetry of a minimal graded free resolution of the codimension three AG algebra A of socle degree j implies that (f, g) has at least one syzygy in degree 4 or 3, which yields a contradiction.

A general linear form $\ell \in R_1$ satisfies $\ell^j \neq 0$ and by Theorem 2.4 and the Gordan-Noether Theorem 2.5, when char $\mathsf{k} = 0$ it yields a bijection $\ell^{j-2} : A_1 \to A_{j-1}$. Set $A' = \mathsf{k}[x, y, z]/(f, g)$. Since depth$(A') = 1$, a general linear form ℓ is a non zero-divisor on A'. Observe that $A'_i = A_i$ for $i \leq j - 3$ and thus the maps $\ell^{j-2i} : A_i \to A_{j-i}$ are bijective for $i \geq 3$ since ℓ is a non zero-divisor on A'. It remains to confirm that the map $\ell^{j-4} : A_2 \to A_{j-2}$ is also bijective, or equivalently in this case ℓ^{j-4} is injective.

The following argument is inspired by [7, proof of Proposition 2.4]. Assume by way of contradiction that for every $L \in A_1$ there exists $q \in A_2$ so that $L^{j-4}q = 0$.

Claim: for each $L \in A_1$ that is regular (non zero-divisor) on A' there exists exactly one $q \in \mathbb{P}(A_2)$ so that $L^{j-4}q = 0$. Indeed, suppose that $L^{j-4}q = L^{j-4}q' = 0$ in $A = A'/(h)$. This yields $L^{j-4}q \in (h)A'$ and $L^{j-4}q' \in (h)A$ and, since $h \in A'_{j-2}$, gives that both $L^{j-4}q$ and $L^{j-4}q'$ are scalar multiples of h in A'. Since L is regular on A' this yields that q, q' are linearly dependent, which completes the proof of the claim.

Fix $v \in R_1$, consider the regular map $L : \mathbb{A}^1_{\mathsf{k}} \to \mathbb{P}(A_1)$ defined by $L(t) := \ell + tv$ and set $C = \{L(t) : t \in \mathsf{k}\}$ to be its image. The set of forms that are non zero-divisors on A' forms a nonempty Zariski open set. Denote by U' the intersection of the set of non zero-divisors on A' with C. Then U' is a nonempty Zariski open subset of C since it contains $L(0) = \ell$. Moreover the set $U = L^{-1}(U')$ is a nonempty Zariski open subset of $\mathbb{A}^1_k$ which contains 0. For each $L(t) \in U'$ the claim yields a unique $q(t) \in \mathbb{P}(A_2)$ so that $L(t)^{j-4}q(t) = 0$.

Consider more generally the incidence correspondence $\Gamma \subset \mathbb{P}(A_1) \times \mathbb{P}(A_2)$ where

$$\Gamma = \{(L(t), q) \mid L(t) = \ell + tv \in A_1, q \in A_2, L(t)^{j-4}q = 0\}$$

and define $\pi_2 : \Gamma \to \mathbb{P}(A_2), \pi_2(L(t), q) = q$. Let $U' = \pi_1(U)$. Based on the claim, there is a well defined regular map

$$\phi : U \to \mathbb{P}(A_2),\ \phi(t) = \pi_2(L(t)) = q(t) = (u_0(t) : \ldots : u_4(t))$$

which induces a homomorphism

$$\psi : \mathsf{k}[\mathbb{P}(A_2)] \to \mathcal{O}_{\mathbb{A}^1}(U),\ \text{given by } \theta \mapsto \theta(q(t)),\ \forall \theta \in \mathsf{k}[\mathbb{P}(A_2)] = \mathsf{k}[x_0, \ldots, x_4].$$

Composing ψ with the inclusions $\mathcal{O}_{\mathbb{A}^1}(U) \hookrightarrow \mathcal{O}_{0,\mathbb{A}^1} = \mathsf{k}[t]_{(t)} \hookrightarrow \widehat{\mathcal{O}_{0,\mathbb{A}^1}} = \mathsf{k}[[t]]$ and taking $\theta_i = x_i$ to be the coordinate functions on $\mathbb{P}(A_2)$ allow to express each coordinate $u_i(t)$ of $q(t)$ as a power series in $\mathsf{k}[[t]]$. Hence $q(t)$ itself can be written as

$$q(t) = \sum_{i=0}^{\infty} q_i t^i \text{ with } q_i \in A_2.$$

Rewrite $L(t)^{j-4}q(t) = 0$ as

$$(\ell+tv)^{j-4}\left(\sum_{i=0}^{\infty} q_i t^i\right) = 0, \forall v \in A_1$$

where $0 \neq q_0 = q(0) \in R_2$ is such that $\ell^{j-4}q_0 = 0$. The coefficient of t in the above displayed equation is $(j-4)\ell^{j-5}q_0 v + \ell^{j-4}q_1$, which must be 0 for all $v \in A_1$. Since $j-4 \neq 0$ and

$$\ell^{j-5}\left((j-4)q_0 v + \ell q_1\right) = 0, \forall v \in A_1$$

we have that the kernel K of the map $\ell^{j-5} : A_3 \to A_{j-2}$ has dimension

$$\dim_{\mathsf{k}} K \geq \dim_{\mathsf{k}} \operatorname{Span}\{(j-4)q_0 v + \ell q_1 \mid v \in A_1\} \geq \dim_{\mathsf{k}}\{q_0 v \mid v \in A_1\}.$$

As $A_{\leq j-3} = A'_{\leq j-3}$, we have that $\ell^{j-5} : A_2 \to A_{j-3}$ is injective and thus, by Gorenstein duality, the map $\ell^{j-5} : A_3 \to A_{j-2}$ is surjective for a general $\ell \in R_1$. This means $\dim_{\mathsf{k}} K = 1$ which yields $\dim_{\mathsf{k}}\{q_0 v \mid v \in A_1\} = 1$ (note that the latter cannot be 0 since $q_0\ell \neq 0$). Consequently, we get $\dim((q_0) \cap (f,g))_3 = 2$. Since $(f,g)_3 = \langle xf, yf, zf, g\rangle$, it follows that there exists a nonzero element in $((q_0) \cap (f))_3$ and thus $\gcd(q_0, f) \neq 1$ (otherwise $(q_0) \cap (f) = (q_0 f)$ is generated in degree 4).

Lastly, recall that $\ell^{j-4}q_0 = 0$ in A, but $\ell^{j-4}q_0 \neq 0$ in A' so $\ell^{j-4}q_0 \in (f,g,h) \setminus (f,g)$. Since $\deg(h) = \deg(\ell^{j-4}q_0)$ we see that $I = (f,g,h) = (f,g,\ell^{j-4}q_0)$. It was shown above that I must be generated by a regular sequence, but the sequence $f, g, \ell^{j-4}q_0$ is not a regular sequence since $\gcd(q_0, f) \neq 1$, a contradiction. This completes the proof. □

3.3. Residue fields of positive characteristic

Let A be a codimension three AG algebra of socle degree j. For those algebras A covered by our Main Theorem, we have proved that when k is of characteristic zero or an infinite field of characteristic p bigger than j, then A is almost SL. Namely, for $\ell \in A_1$ a general linear form, the multiplication maps $\ell^{j-2i} : A_i \to A_{j-i}$ are isomorphisms for $2 \leq i \leq \lfloor \frac{j}{2} \rfloor$. In characteristic zero, then Theorem 2.4 and the Gordan-Noether Theorem 2.5 assures that the multiplication map $\ell^{j-2} : A_i \to A_{j-i}$ is also an isomorphism, so in this case A is SL.

In the following case we are able to bypass the Gordan-Noether theorem and give a strengthening of Lemma 3.5 for codimension three AG algebras of almost constant Hilbert function.

Lemma 3.10. *Let A be a graded AG algebra over an algebraically closed field k of characteristic $p \neq 2$ or 3. Let $A = R/I$ with $I = \operatorname{Ann}(F)$ and assume $T(A) = (1, 3^k, 1), k \geq 2$. Then A is SL.*

Proof. When $k \geq 3$ (so $j \geq 4$) Theorem 2.6 shows that R/J, where $J = (I_{\leq 3})$, defines a tight annihilating scheme $\mathfrak{Z}$ of F; this suffices to show that A is SL as in the proof of Lemma 3.5.

Assume $T = (1,3,3,1)$. If (I_2) in not a CI, then the structure theorem for codimension three AG algebras implies that $A' = R/(I_2)$ is an algebra with Hilbert function $H = (1,3,3,\underline{3})$. Thus A' has a tight annihilating scheme, which suffices to show A is SL (Lemma 2.8). When $H = (1,3,3,1)$ and $I = (I_2)$ is a complete intersection, the classification of nets of conics in [1, Table 1], which is valid for char $\mathsf{k} = p$ ([1] p. 6, 80) shows that the dual generator $F \in S_3$ is one of those listed in #8b,c, #7c, or #6d of Table 1 there. It is straightforward to check that in each case $A = R/\mathrm{Ann}(f)$ is SL.[6] This completes the proof. □

Conjecture 3.11. Let A be an AG algebra of codimension three and socle degree j over an infinite field k of char $\mathsf{k} > j$. Then the map $\ell^{j-2} : A_1 \to A_{j-1}$ is an isomorphism for a general linear form $\ell \in A_1$.

We now give an example to show that the condition char $\mathsf{k} > j$ is necessary for our results about almost SL AG algebras.

Example 3.12 *(Counterexamples: AG algebras,* char $\mathsf{k} < j$*).*

(a) The following example shows that the hypothesis char $\mathsf{k} = p > j$ is necessary in our Conjecture 3.11. Take $p = 3$ and let $I = (x^3, y^3, z^2)$ define an algebra with Hilbert function $T(A) = (1,3,5,5,3,1)$ and socle degree $j = 5 > p$. Then multiplication by $(x+y+z)^3$ is not full rank from A_1 to A_4. Since I is a monomial ideal, there exists a change of coordinates on A which takes any general linear form to $x + y + z$.

(b) The following example shows that codimension two AG algebras may not be almost SL when char $\mathsf{k} = p \leq j$. Take $p = 2$ and let $I = (x^4, y^4)$ define an algebra with Hilbert function $T(A) = (1,2,3,4,3,2,1)$. Then $(x+y)^2$ is not full rank from A_2 to A_4, so arguing that $x + y$ is general as in part (a) yields that A is not almost SL.

(c) The following example shows that a codimension three AG algebra may not be almost SL when char $\mathsf{k} = p \leq j$. Take $p = 3$ and let $I = (x^3, y^3, z^4)$ define an algebra with Hilbert function $T(A) = (1,3,6,8,8,6,3,1)$. Then $(x+y+z)^3$ is not full rank from A_2 to A_5, so arguing that $x+y+z$ is general as in part (a) yields that A is not almost SL.

(d) In the previous examples, the multiplication map $\ell^j : A_0 \to A_j$ is the zero map. However, if we take char $\mathsf{k} = 13$, a CI ideal $I = (x^3, y^3, z^{14})$, and $A = R/I$ of socle degree 17, we will have $\ell = x + y + z$ satisfying $\ell^{13} : A_2 \to A_{15}$ is not of full rank (z^2 is in the kernel), but $\ell^{17} : A_0 \to A_{17}$ has the non-zero image $\binom{17}{2,2,13} x^2y^2z^{13}$. A check with Macaulay2 [13] shows $\ell^{2i+1} : A_{8-i} \to A_{9+i}$ does not have full rank for $i \in \{5,6,7\}$, so A is not almost SL.

[6] #8c,#7c in [1, Table 1] specialize to #6d where $I = (x^2, y^2, z^2)$ which is evidently SL; and SL in the smooth case #8b is immediate.

Some have conjectured that all codimension three graded AG algebras over fields of characteristic $\mathsf{k} = 0$ should be WL (as [4, §1]). We expect that not all AG codimension three algebras would be SL, even in characteristic zero. Given the substantial connection of the failure of the WL property in different contexts with geometry, for example [17,5], and the fact that an open dense subset in each $\mathrm{Gor}(T)$ for codimension three is SL (Proposition 3.2), we could expect a strong geometric flavor to finding a counterexample of a codimension three non-SL AG algebra, for char $\mathsf{k} = 0$ or char $\mathsf{k} = p > j$.

Data availability

No data was used for the research described in the article.

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