# ON FEYNMAN GRAPHS, MATROIDS, AND GKZ-SYSTEMS

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Abstract. We show in several important cases that the $A$-hypergeometric system attached to a Feynman diagram in Lee–Pomeransky form, obtained by viewing the coefficients of the integrand as indeterminates, has a normal underlying semigroup. This continues a quest initiated by Klausen, and studied by Helmer and Tellander. In the process we identify several relevant matroids related to the situation and explore their relationships.

Contents



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## 1. Introduction

Throughout, $G$ is a graph with edge set $E := E_G$ and vertex set $V := V_G$.[1] Denote by $\mathcal{T}_G^i$ its set of *i-forests*; then $F \in \mathcal{T}_G^i$ whenever it is circuit-free and the graph on the set of vertices of $G$ with the set of edges of $F$ has exactly $(i-1)$ more connected components than $G$ does. The nomenclature comes from the fact that an $i$-forest in a connected graph has exactly $i$ connected components. If $G$ is connected, a 1-forest is often called a *spanning tree.*

In the theory of Feynman integrals, edges correspond to particles, and vertices to particle interactions. Some of the vertices are labeled as "external"; the set of external vertices is denoted $V_{\mathrm{Ext}}$. An external vertex connects to an external edge (that is not part of $G$) and these external edges represent the externally measurable in- and output particles that interact according to the graph.

Throughout we consider a mass function

$$m\colon E \longrightarrow \mathbb{R}_{\geq 0},$$

and denote by $m_e := m(e)$ the mass of the particle corresponding to edge $e$. As a matter of general notation, we call *massive* the edges $e$ with $m_e \neq 0$; the other edges are *massless.*

There is a momentum function

$$p\colon V_{\mathrm{Ext}} \longrightarrow \mathbb{R}^{1,3}$$

on the external vertices of $G$, with values in the 4-dimensional Minkowski space $\mathbb{R}^{1,3}$ with indefinite "norm" $|(p_0, p_1, p_3, p_3)|^2 := -p_0^2 + (p_1^2 + p_2^2 + p_3^2)$. Momentum conservation dictates that the momenta of the external particles must sum to zero. We will assume (see Hypothesis 1.2 below) that the momenta do not satisfy any other constraints. In particular, when measurements of experiments are taken, the momenta can be seen as generic (subject to summing to zero on $V_{\mathrm{Ext}}$); this setup fits most QFTs.

No generality on the Feynman diagram is lost if one assumes that the underlying graph $G$ be connected, since disconnected graphs describe separate particle interactions. Slightly more generally, one may assume that the graph have no *cut vertex*: the removal of any single vertex of $G$ should not increase the number of connected components. This property is in the Feynman context referred to as (1VI), short for "one vertex irreducible"; see for example [Sch18]. Physically, the presence of a cut vertex means that the particle interaction can be interpreted as a two-stage process with independent parts.

A *bridge* is an edge whose removal increases the number of connected components. In the presence of bridges, as well as when the graph has edges linking some vertex to itself, the corresponding Feynman amplitude factors into amplitudes from simpler graphs. In physics, a connected graph without any edges linking a vertex to itself, and without bridges is called (1PI), short for "one particle irreducible". It implies in particular that no edge is part of every 1-forest.

**Definition 1.1.** We will say that the graph $G$ is *strongly 1-irreducible*, abbreviated as *(s1I)* if it is particle irreducible and one vertex irreducible. Equivalently, such graphs are connected, and have no bridges, no cut vertices, nor edges that link a vertex to itself. ♢

Mathematically, the (s1I) property is: "the graphical (or, equivalently, the co-graphical) matroid to $G$ is connected", see Subsection 2.3 below.

For $e \in E$ we denote the unit vector of $\mathbb{R}^E$ pointing in $e$-direction by $\mathbf{e}_e$; so

$$\mathbb{R}^E := \bigoplus_{e\in E} \mathbb{R} \cdot \mathbf{e}_e.$$

The graph $G$ induces several interesting functions on $\mathbb{R}^E$ that lie inside the polynomial ring $\mathbb{C}[\boldsymbol{x}_E]$ on variables $\boldsymbol{x}_E := \{x_e \mid e \in E\}$ indexed by $E$; relevant to us are the following. The *dual graph polynomial* is

$$\mathcal{U} := \sum_{T\in\mathcal{T}_G^1} (\boldsymbol{x}^E/\boldsymbol{x}^T),$$

[1] We will typically use $E$ and reserve $E_G$ for cases where extra clarity is needed, for example when several graphs are around.

where here and elsewhere, $\boldsymbol{x}^S := \prod_{e\in S} x_e$ for any $S \subseteq E$, and more generally $\boldsymbol{x}^{\mathbf{a}} := \prod_{e\in E} x_e^{a_e}$ for $\mathbf{a} \in \mathbb{Z}^E$.

Many QFT techniques take recourse to *Wick rotation*, the coordinate transformation in momenta space that multiplies the coordinate $p_0$ by $\sqrt{-1}$. We shall write $p_W$ for this Wick rotated momentum function. The effect is that the Minkowski norm turns into the Euclidean norm, but it also moves the study of Feynman amplitudes to the complex domain. For certain purposes, such as considering families of Feynman type integrals in the spirit discussed below, this is no actual disadvantage.

Given an external momenta function $p$, a second polynomial can be derived from $G$, namely

$$\mathcal{F}_0 := -\sum_{F\in\mathcal{T}_G^2} |p_W(F)|^2(\boldsymbol{x}^E/\boldsymbol{x}^F).$$

Here, $p_W(F)$ is the (Wick rotated) sum of the momenta of the external vertices of $G$ that belong to one [2] of the two components of $F$, compare the introduction of [HT22].

In contrast to the momenta, there is no genericity assumption on the masses, and in particular they can be zero. In the theory of Feynman integrals, in Lee–Pomeransky form, the function

$$\mathcal{G}_{m,p} := \mathcal{U}\cdot(1+\sum_{e\in E} m_e^2 x_e) + \mathcal{F}_0$$

and the integrals of its powers are relevant.

For fixed masses, special choices of the momentum function $p$ allow for the possibility of cancellation of coefficients in the sum $\mathcal{G}_{m,p}$, resulting in the disappearance of certain monomials (although for degree reasons no cancellation can occur between terms of $\mathcal{U}$ and terms of $\mathcal{G}_{m,p} - \mathcal{U}$). In order to avoid such pathologies we shall make the following assumptions.

**Hypothesis 1.2.** Throughout, we shall assume that

(1) the underlying graph $G$ is (s1I) and has at least one edge (hence actually at least two);
(2) the values of the momenta are sufficiently generic, so that
    (a) in the sum $\mathcal{U}\cdot(\sum_{e\in E} m_e^2 x_e) + \mathcal{F}_0$ no cancellation of terms occurs, and
    (b) no proper subset of $V_{\text{Ext}}$ has zero momentum sum.
(3) At least one 2-forest term appears in $\mathcal{G}_{m,p}$. ♢

**Remark 1.3.** (1) Hypothesis 1.2.(1) can be postulated since Feynman amplitudes to graphs that fail this condition can be decomposed into amplitudes that come from graphs that satisfy the condition.

(2) Hypothesis 1.2.(2) is known in physics as "general kinematics", and is sometimes assumed without the requisite advertisement. The desired consequence of non-cancellation of terms is always in force when the external momenta are in the *Euclidean region*. Moreover, for the purpose of studying Feynman integrals as a family (for example, via GKZ-systems), momenta are viewed as parameter variables (subject to the external momentum sum being zero), and then Hypothesis 1.2.(2) holds as well.

(3) If Hypothesis 1.2.(2) is satisfied but 1.2.(3) is violated, all masses must be zero and there can be no external vertices. ♢

Viewing the momenta and the nonzero masses as generic, and treating the resulting coefficients of $\mathcal{G}_{m,p}$ as indeterminates, one arrives at a differentiable family of integrals. One method to study Feynman integrals is by computing differential equations that govern this family, and then solving them with a power series Ansatz. After that, one may consider the specialization of certain variables to special values, or one can investigate geometric behavior (such as monodromy) of the family.

Already Regge et al. [dAJR65] realized that Feynman amplitudes satisfy rather special differential equations that resemble the classical hypergeometric ones. Later, Golubeva used Griffiths' results on the integrals

[2] Since the total momentum sum is zero, both 2-forest components give the same coefficient.

of rational differential forms to study the partial differential equations satisfied by the Feynman integral [Gol73]. By Bernstein's theory, solutions of such systems are multi-valued analytic, branched at the Landau variety. The true nature of these differential equations eventually found its final formulation in the theory $A$-hypergeometric systems of Gel'fand, Graev, Kapranov and Zelevinsky, introduced in the 1980s.

In order to provide the connection, let $\{\mathbf{a}_i \mid 1 \leq i \leq n\}$ be the exponents of the monomials $\boldsymbol{x}^{\mathbf{a}_i}$ appearing in $\mathcal{G}_{m,p}$ with nonzero coefficient. Then let

$$A_{\mathcal{G}} := \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n \end{pmatrix} \in \mathbb{Z}^{(1+|E|)\times n};$$

for any $\mathbf{a} \in \mathbb{Z}^E$ we shall refer to $(1, \mathbf{a}) \in \mathbb{Z} \times \mathbb{Z}^E$ as its *lift*.

For an arbitrary integer $(1 + |E|) \times n$ matrix $A$, the group $\mathbb{Z}A$ of integer linear combinations of its columns

$$\{\sum k_i \mathbf{a}_i \mid k_i \in \mathbb{Z}\} =: \mathbb{Z}A \supseteq \mathbb{N}A := \{\sum k_i \mathbf{a}_i \mid k_i \in \mathbb{N}\}$$

is the *lattice of* $A$, containing the semigroup $\mathbb{N}A$ of linear combinations with natural coefficients. In conjunction with any choice of a complex parameter vector $\beta \in \mathbb{C} \times \mathbb{C}^E$, such matrix $A$ induces a *GKZ-system* (or also called *$A$-hypergeometric system*) $H_A(\beta)$ of linear partial differential equations in $n$ new variables $y_1, \ldots, y_n$, as we explain in the next section.

One observes that a suitable choice of the parameter $\beta$ causes the $A_{\mathcal{G}}$-hypergeometric system $H_{A_{\mathcal{G}}}(\beta)$ to have among its solutions the family of Feynman integrals to the data $(G, m, p, V_{\text{Ext}})$. Algorithmic methods for general hypergeometric systems were worked out in [SST00], and for more than two decades there has been much activity applying the abstract theory to the Feynman context, see for example [dlC19, Kla20] (and the bibliography trees therein) for a down-to-earth discussion and more details on this.

In the construction of the hypergeometric system $H_A(\beta)$ enters a certain toric ideal

$$I_A \subseteq R_A := \mathbb{C}[\boldsymbol{\partial}]$$

in the (polynomial) ring of partial differentiation operators $\partial_1 := \frac{\partial}{\partial y_1}, \ldots, \partial_n := \frac{\partial}{\partial y_n}$. The ideal $I_A$ describes the closure of the image of $\mathbb{C}^* \times (\mathbb{C}^*)^E$ in $\mathbb{C}^n$ under the monomial map encoded by $A$. If the quotient

$$S_A := \mathbb{C}[\mathbb{N}A] \simeq R_A / I_A$$

enjoys a certain algebraic property known as *Cohen–Macaulay*, then various desirable simplifications regarding the solutions of $H_A(\beta)$ occur. As is discussed in [dlC19, Kla20, Kla22, HT22], of practical value in the theory of Feynman integrals are: access to integral representations of the solutions; suitable initial ideals of $H_A(\beta)$ become computable in elementary fashion without the need to look at Gröbner bases; classical combinatorial recipes for manufacturing solutions become much simpler, see [SST00] for background on hypergeometric differential equations.

The Cohen–Macaulayness of $S_A$ is implied by, but by no means equivalent to, the condition that the semigroup $\mathbb{N}A \subseteq \mathbb{R} \times \mathbb{R}^E$ be *saturated*, which means that the intersection of the non-negative rational cone $\mathbb{R}_{\geq 0}A$ spanned by the columns of $A$ over the origin with the lattice $\mathbb{Z}A$ contains no other lattice points than those in $\mathbb{N}A$; see [SST00, MMW05] for more details on Cohen–Macaulayness in this context. Saturatedness is an arithmetic condition that involves the study of the interior lattice points of the dilations of the polytope spanned by the columns of $A$.

For notation, let the *support* $\text{Supp}(f)$ of a Laurent polynomial $f = \sum c_{\mathbf{a}} \boldsymbol{x}^{\mathbf{a}}$ be the exponent vectors

$$\text{Supp}(f) := \{\mathbf{a} \mid c_{\mathbf{a}} \neq 0\}$$

of the monomials appearing with nonzero coefficient in $f$. Denoting the convex hull of a set $S \subseteq \mathbb{R}^E$ by $\overline{S}$, the *support polytope* of $f$ is $\overline{\text{Supp}(f)}$. Let $P_{m,p}$ be the support polytope of $\mathcal{G}_{m,p}$. Assuming general kinematics, Helmer and Tellander [HT22] showed in the following two extreme cases that the semigroup of $A_{\mathcal{G}}$ is saturated:

(HT1) in the *massive case* (*i.e.*, $m_e > 0$ for all $e \in E$);
(HT2) in the *massless case* (*i.e.*, $m_e = 0$ for all $e \in E$) assuming that *every vertex is external.*

In both cases, their result implies that $S_{A_\mathcal{G}}$ is Cohen–Macaulay. The tools they use include edge-unimodularity, flag matroid polytopes, Cayley and Minkowski sums, which they use to study IDP properties of polytopes.

In this note, we start with discussing the support vectors of $\mathcal{G}_{m,p}$ from the point of view of matroid theory. Of course, the support vectors of $\mathcal{U}$, interpreted as indicator functions, describe the co-graphical matroid of $G$. We prove here that the support vectors of $\mathcal{F}_0$ and those of the square-free terms in $\mathcal{U}\cdot(\sum_{e\in E} m_e^2 x_e)$ both describe matroids as well. We show further that, remarkably, their union also forms a matroid. Thus, for all Feynman graphs that satisfy Hypothesis 1.2, the support vectors of the square-free terms of $\mathcal{G}_{m,p}$ form a matroid.

We use these matroidal results and some ideas of [HT22] to show that, with general kinematics, the semigroup generated by $A_\mathcal{G}$ is saturated for (s1I) graphs $G$ in the following two cases:

(1) if every 2-forest of $G$ induces a nonzero term in $\mathcal{G}_{m,p}$ (Theorem 4.3);
(2) if $m_e = 0$ for all $e$ (Theorem 4.8);

these generalize the two corresponding cases in [HT22]. In consequence, $A_\mathcal{G}$ defines in these situations a hypergeometric system that enjoys the Cohen–Macaulay property.

In the next section we set up the necessary notation, and carefully describe the needed details about hypergeometric systems, as well as graphs, polytopes and matroids. In Section 3, we discuss the advertised matroids, and in Section 4 we state and prove the semigroup results. Under Condition (1) above, this follows from an inspection of the way that the cone over $A_\mathcal{G}$ behaves under specialization of a mass to zero. In the massless case we follow the route of [HT22] in the corresponding context. We also provide some partial results towards the general case. In the last section we discuss some examples of the failure of Hypothesis 1.2, and the ensuing consequences on matroids and the hypergeometric system. For the convenience of the reader, we provide a list of symbols at the end.

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## 2. Notation and basic concepts

2.1. **Hypergeometric systems.** We give here a minimal introduction to $A$-hypergeometric systems invented by Gel'fand, Graev, Kapranov and Zelevinsky in the mid-1980s. For details and literature on them and on parametric integrals that occur as their solutions we refer to the book [SST00, Sec. 5.4], and to the survey [RSSW21].

Take an integer matrix $A \in \mathbb{Z}^{(1+d)\times n}$, and a set of variables $\boldsymbol{y} = y_1, \ldots, y_n$. Denote the partial derivative operators $\partial/\partial y_j$ by $\partial_j$ and consider the *Weyl algebra* $D_A$ in variables $y_1, \ldots, y_n$. This is the non-commutative ring $\mathbb{C}[\boldsymbol{\partial}]\langle \boldsymbol{y}\rangle$ subject to the commutator rules $\partial_i y_j - y_j \partial_i = \delta_{i,j}$, the Kronecker delta. The elements of $D_A$ can be interpreted as linear differential operators in $\boldsymbol{y}$ with polynomial coefficients.

The matrix $A$ induces a monomial action

$$\begin{array}{rcl} (\mathbb{C}^*)^{1+d} \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^n, \\ (\mathfrak{t}, \mathfrak{y}) & \mapsto & (\mathfrak{t}^{\mathbf{a}_1}\mathfrak{y}_1, \ldots, \mathfrak{t}^{\mathbf{a}_n}\mathfrak{y}_n) \end{array}$$

of the $(1+d)$-torus on the affine space with coordinates $\partial_1, \ldots, \partial_n$. The usual closure of the orbit of the point $(1, \ldots, 1) \in \mathbb{C}^n$ is also Zariski closed, and defined by the *toric ideal* $I_A$ generated by the binomials

$\square_{\mathbf{u},\mathbf{v}} := \boldsymbol{\partial}^{\mathbf{u}} - \boldsymbol{\partial}^{\mathbf{v}}$, running over all $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$ with $A \cdot \mathbf{u} = A \cdot \mathbf{v}$. One may view $I_A$ as a subset of $D_A$ via the embedding of rings $\mathbb{C}[\boldsymbol{\partial}] \hookrightarrow D_A$.

The matrix $A$ also induces $(1+d)$ *Euler operators*

$$E_i := \sum_{j=1}^{n} a_{i,j} y_j \partial_j \in D_A \qquad \text{for } 0 \le i \le d.$$

Given a choice of $\beta \in \mathbb{C}^{1+d}$, the *hypergeometric ideal* to $A$ and $\beta$ is the left $D_A$-ideal

$$H_A(\beta) := D_A \cdot (I_A, \{E_i - \beta_i\}_{i=0}^{d}).$$

Any left ideal $H = \sum D_A Q_i$ of $D_A$ generated by operators $\{Q_i\}_i \subseteq D_A$ can be interpreted as a system of linear partial differential equations on a solution function $\phi(\boldsymbol{y})$, by asking that $Q \bullet (\phi(\boldsymbol{y})) = 0$ for all $Q \in H$ (or, equivalently, that $Q_i \bullet (\phi(\boldsymbol{y})) = 0$ for all $i$). As is explained in [HT22], if one reads the coefficients of $\mathcal{G}_{m,p}$ as parameters then the Feynman integrals corresponding to $A_{\mathcal{G}}$ appear as solutions of $H_{A_{\mathcal{G}}}(\beta)$ for the right choice of $\beta$. For the study of Feynman integrals, the entire family is useful; for some purposes even $\beta$ is viewed as a variable.

**Remark 2.1.** A frequent hypothesis in the theory of $A$-hypergeometric systems is that the group $\mathbb{Z}A$ generated by the columns of $A$ agrees with the ambient lattice $\mathbb{Z}^{1+d}$ inside $\mathbb{R}^{1+d}$. The hypothesis is not crucial to the majority of known results, but it usually allows a much simpler formulation. However, the question whether a semigroup ring is normal is only decided by the saturatedness of the semigroup in its own lattice, the group it generates. ♢

2.2. **Polytopes.** A polytope $P$ in $\mathbb{R}^{1+d}$ is a *lattice polytope* if its vertices belong to the lattice $\mathbb{Z} \times \mathbb{Z}^d$ inside $\mathbb{R}^{1+d}$. All polytopes we consider will be compact convex lattice polytopes.

Given two polytopes $P, P'$ in $\mathbb{R}^E$, their *Minkowski sum* is the set of points

$$P + P' := \{w = v + v' \in \mathbb{R}^E \mid v \in P, v' \in P'\}.$$

The edges of a Minkowski sum are parallel to edges of the input polytopes. The vertices of a Minkowski sum are always sums of vertices of the input polytopes (although some such sums might be interior points of the sum polytope). In contrast, the set of the lattice points in a Minkowski sum is often not equal to the sum of the sets of lattice points in the two input polytopes.

Let us set, for our Feynman diagrams,

$$E_m := \{e \in E \mid m_e \neq 0\} \qquad \text{and} \qquad E_0 := E \smallsetminus E_m.$$

Moreover, put

$$\Sigma_m := \sum_{e \in E_m} m_e^2 x_e \quad \text{and} \quad \Delta_m := \overline{\operatorname{Supp}(\Sigma_m)};$$

the latter is the simplex in $\mathbb{R}^E$ spanned by the unit vectors $\{\mathbf{e}_e\}_{e \in E_m}$.

We also set

$$\tilde{\Sigma}_m := 1 + \Sigma_m, \qquad \text{and} \qquad \tilde{\Delta}_m := \overline{\operatorname{Supp}(\tilde{\Sigma}_m)}.$$

If we already have a specific mass function $m$ in mind, we write

$$\Sigma_E := \Sigma_m + \sum_{e \in E_0} x_e, \qquad \Delta_E := \operatorname{Supp}(\Sigma_E), \tag{2.2.1}$$

$$\tilde{\Sigma}_E := 1 + \Sigma_E, \qquad \tilde{\Delta}_E := \operatorname{Supp}(\tilde{\Sigma}_E). \tag{2.2.2}$$

According to Hypothesis 1.2, the support polytope $P_{m,p}$ of

$$\mathcal{G}_{m,p} = \mathcal{U} \cdot \tilde{\Sigma}_m + \mathcal{F}_0$$

is the same as the polytope spanned by the union $\operatorname{Supp}(\mathcal{U}) \cup \operatorname{Supp}(\mathcal{U} \cdot \Sigma_m) \cup \{\operatorname{Supp}(|p_W(F)|^2 \cdot \boldsymbol{x}^F) | F \in \mathcal{T}_G^2\})$
.

The reason for which we introduce $\Sigma_E$ and its derivates is that it allows us, for general kinematics, to compare the hypergeometric system from the actual Feynman graph $(G, m, p, V_{\text{Ext}})$ to that from a massive one with the same $(G, p, V_{\text{Ext}})$. This idea sets the stage for the proof of Theorem 4.3.

2.3. **Graphs and their matroids.** We generally use the graph and matroid language as it prevails in mathematics. So, for us a *loop* is an edge that is incident to only one vertex; a *circuit* is a set of edges whose union in a realization of the graph is homeomorphic to a polygon (while in physics this is called a loop).

In each term of $\mathcal{U}$ and of $\mathcal{F}_0$, each variable appears (by definition) with degree at most one. On the other hand, $\mathcal{U} \cdot \Sigma_m$ can have some terms with some variable of degree two (and the other variables of degree one or zero). Such square terms can occur only for massive variables (and if a variable is in fact massive then it will occur in some term with degree two since the corresponding edge cannot not belong to every 1-forest in the (s1I) graph $G$).

A *matroid* $\mathsf{M}$ on the ground set $E$ is determined by a distinguished collection $\mathcal{B}_{\mathsf{M}} \subseteq 2^E$ of *bases*. From this angle, the defining property of a matroid is a version of the Exchange Axiom of Steiner from linear algebra: if $B, B'$ are two bases of a matroid, and $e \in B$, then there is $e' \in B'$ such that $(B \smallsetminus \{e\}) \cup \{e'\}$ is again a basis. In fact, there is an equivalent "strong" version where in the same notation the set $(B' \smallsetminus \{e'\}) \cup \{e\}$ can also be arranged to be a basis. The notion of a matroid generalizes the idea of linear independence of sets of vectors, and much of the nomenclature is borrowed from linear algebra. We refer to [Oxl11] for background and all facts that we use about matroids.

Matroids have a *rank function*

$$\mathrm{rk}_{\mathsf{M}} \colon 2^E \longrightarrow \mathbb{N},$$

and the bases are precisely the minimal sets (with respect to inclusion) of maximum possible rank in $\mathsf{M}$. The rank of a matroid is (by definition) the size of any of its bases (which is indeed a well-defined integer as one can see from iterated application of the Exchange Axiom). A *loop* of a matroid $\mathsf{M}$ is an element $e$ for which $\mathrm{rk}_{\mathsf{M}}(\{e\}) = 0$; loops are those elements of $E$ not contained in any basis.

If $S \subseteq E$ then we write $\mathbf{v}_S$ for the *indicator vector* of $S$ defined by $\mathbf{v}_S = \sum_{e \in S} \mathbf{e}_e \in \mathbb{N}^E$. To each basis $B$ one has an indicator vector

$$\mathbf{v}_B \in \{0,1\}^E, \qquad [\mathbf{v}_B(e) = 1] \iff [e \in B].$$

The entry sum of any $\mathbf{v}_B$ is the rank of $\mathsf{M}$. The convex hull of the lattice vectors $\{\mathbf{v}_B \mid B \in \mathcal{B}_{\mathsf{M}}\}$ is the *matroid polytope* of $\mathsf{M}$. Every $\mathbf{v}_B$ is a vertex of the matroid polytope, since it is even a vertex of the polytope spanned by all integer vectors that have only 0/1 entries and entry sum $\mathrm{rk}(\mathsf{M})$. Indeed, among such integer vectors, $\mathbf{v}_B$ realizes the unique maximum of the linear function that takes dot product against $\mathbf{v}_B$.

The Strong Exchange Axiom implies that the edges of the matroid polytope are precisely those that link (indicator vectors of) bases that agree in all but two positions. In particular, edges of the matroid polytope are parallel to the vectors $\mathbf{e}_e - \mathbf{e}_{e'}$, [GGMS87].

A *circuit* of a matroid is a set that is not contained in any basis, and minimal (with respect to inclusion) in this regard. Loops are circuits. An *independent* set is one that contains no circuit; independent sets are exactly those subsets of $E$ on which the rank function agrees with the cardinality function, and they can also described as the sets that are subsets of bases. Bases are maximal independent sets, and proper subsets of circuits are independent.

If $G$ is a graph, the collection $\mathcal{T}^1_G$ of 1-forests of $G$ forms the set of bases for a matroid $\mathsf{M}^1_G$ on the underlying set $E$ of edges. Circuits of the graph are then circuits of $\mathsf{M}^1_G$, and (graph-theoretic) loops correspond to (matroid-theoretic) loops. Matroids that arise this way are called *graphic*.

For a set of edges $S$ from $G$ (which we read as a subgraph of $G$ on the same vertex set $V_G$) we call their *span* the collection of all edges of $G$ that connect vertices of $G$ that belong to the same connected component in the subgraph $S$. In other words, the vertex partitions of $V_G$ by sets of connected components of $S$ and span $(S)$

are the same, and span $(S)$ is the largest subgraph of $G$ in this regard. Put differently again, $e \in \operatorname{span}(S)$ if and only if $e$ becomes a loop in the graph obtained from $G$ by contracting all edges of $S$. In particular, $\operatorname{rk}(S) = \operatorname{rk}(\operatorname{span}(S))$ is the difference of the number of components of $S$ (as graph on the vertex set of $G$) and $|V_G|$ (which one may view as the number of connected components of a graph with that many vertices and no edges). The rank function can also be interpreted as the size of the largest circuit-free subset, and the span of a set in a general matroid is the largest superset with the same rank as the given set.

The set of complements $\{E \smallsetminus T \mid T \in \mathcal{T}_G^1\}$ forms the set of bases for another matroid $\mathsf{M}_G^{1,\perp}$ on $E$ that turns out to be dual to $\mathsf{M}_G$ in a suitable sense. For this *cographic* matroid $\mathsf{M}_G^{\perp}$, a loop is an edge that is part of every 1-forest of $G$. Its removal thus disconnects the graph and such edge cannot occur in a (s1I) Feynman diagram. So, for an (s1I) graph, neither the graphic nor the cographic matroid has loops.

Similarly, the set of 2-forests $\mathcal{T}_G^2$, as well as the set of their complements, form matroids that we denote $\mathsf{M}_G^2$ and $\mathsf{M}_G^{2,\perp}$ respectively. Such statements apply also to $\mathcal{T}_G^k$ for all other $k \in \mathbb{N}$, but they will not be used here.

Any matroid can be written as a matroid sum of simple matroids; a matroid is *simple* if it is impossible to write the set of bases $\mathcal{B}_\mathsf{M}$ as the set of all unions of the bases of two submatroids on disjoint subsets $E_1, E_2$ whose union is $E$. A graph is (s1I) if and only if its graphic and cographic matroid are simple.

Let $\boldsymbol{x}_E = \{x_e \mid e \in E\}$ be a set of indeterminates that are in correspondence with the elements of the ground set of $\mathsf{M}$. There is an induced *matroid basis polynomial*

$$\Phi_\mathsf{M} = \sum_{B \in \mathcal{B}_\mathsf{M}} \boldsymbol{x}^{\mathbf{v}_B} \in \mathbb{C}[\boldsymbol{x}_E]$$

with very interesting combinatorial properties.[3] The polynomial $\mathcal{U}$ is the matroid basis polynomial $\Phi_{\mathsf{M}_G^{1,\perp}}$ of $\mathsf{M}_G^{1,\perp}$, and the induced polytope

$$P_G^{1,\perp} := \overline{\operatorname{Supp}(\mathcal{U})}$$

is the matroid polytope to $\mathsf{M}_G^{1,\perp}$. On the other hand, $\Delta_E$ is the matroid polytope to the cographic matroid on $E$ corresponding to a connected polygon with $|E|$ edges (or, alternatively, to the graphic matroid to the graph on $|E|$ edges with only two vertices and no loops; these latter ones are called *banana* or *sunset* graphs).

If $\mathsf{M}$ is any matroid on the set $E$, then the semigroup generated by the indicator vectors $\{\mathbf{v}_B \mid B \in \mathcal{B}_\mathsf{M}\}$ is saturated in its own lattice (*i.e.*, the group it generates), by [Whi77, Thms. 1, 2].

For any pointed (*i.e.*, no invertibles except for the neutral element) sub-semigroup $S$ of a free Abelian group of finite rank, the semigroup ring $\mathbb{C}[S]$ is normal if and only if $S$ is saturated in the group generated by $S$. All such semigroup rings are toric, and therefore their normality implies Cohen–Macaulayness, see [Hoc72].

## 3. Matroids in Feynman Theory

Recall that we assume that $G$ satisfies the conditions in Hypothesis 1.2, and that $\mathcal{T}_G^1$ and $\mathcal{T}_G^2$ denote the collections of spanning trees and 2-forests of $G$ respectively.

By Hypothesis 1.2, the monomials appearing in $\mathcal{G}_{m,p}$ are exactly those appearing in at least one of the polynomials $\mathcal{U}$ or $\mathcal{U} \cdot \Sigma_m$ or $\mathcal{F}_0$. The square-free ones in these last two polynomials are indexed, respectively, by a massive edge in a spanning tree for $G$, or a 2-forest with non-vanishing momentum coefficient. In this section we investigate the matroidal properties of these two sets. They form the tools for the main results in the next section.

In order to simplify the discussion we introduce some language.

[3] A more general class of polynomials arises from *realizations* of matroids, see for example [BEK06, Pat10, DSW21, DPSW21].

**Notation 3.1.** If $G', G''$ are subgraphs of $G$ then if $e \in E_G$ is an edge we say it *links* $G'$ *to* $G''$ if it involves one vertex from $G'$ and one vertex from $G''$. We further say that $e$ *is supported on* $G'$ if both vertices of $e$ are vertices of $G'$. The notion of $e$ being supported on $G'$ does not require that $e$ be an edge of $G'$. $\diamondsuit$

### 3.1. Momentous 2-forests.

**Definition 3.2.** A 2-forest $F \in \mathcal{T}_G^2$ is *momentum-free* if the momentum coefficient $|p_W(F)|^2$ of $\boldsymbol{x}^{E \smallsetminus F}$ in $\mathcal{F}_0$ is zero. We denote the set of momentum-free 2-forests of $G$ by $\mathcal{T}_{G,0}^2$.

We call the elements of the complementary set

$$\mathcal{T}_{G,\neq}^2 := \mathcal{T}_G^2 \smallsetminus \mathcal{T}_{G,0}^2$$

the *momentous 2-forests*. $\diamondsuit$

Note that, by Hypothesis 1.2, a 2-forest $F = F_1 \sqcup F_2$ with connected components $F_1, F_2$ is in $\mathcal{T}_{G,0}^2$ precisely when either $V_{\text{Ext}} \subseteq F_1$ or $V_{\text{Ext}} \subseteq F_2$. Therefore, by Hypothesis 1.2, $\mathcal{T}_G^2$ is in natural bijection with the nonzero terms in $\mathcal{F}_0$. For example, let $v$ be a non-external vertex and let $F$ be a spanning tree for the graph obtained by deleting $v$ and all incident edges from $G$. Then $F \cup \{v\}$ is a 2-forest for $G$ that lies in $\mathcal{T}_{G,0}^2$. More extremely, if $G$ were permitted to have only one external vertex, no momentous 2-forest would exist at all, and $\mathcal{F}_0$ would be zero altogether.

**Lemma 3.3.** *The set $\mathcal{T}_{G,\neq}^2$ is the set of bases of a matroid on the edge set $E$ of $G$.*

*Proof.* If $|V_{\text{Ext}}| = 1$, there are no momentous 2-forest, so there is nothing to show. So we assume that at least two external vertices exist.

If $\mathcal{T}_{G,\neq}^2$ is non-empty, we need to show that the set of momentous 2-forests satisfies the Exchange Axiom. So, choose $F \in \mathcal{T}_{G,\neq}^2$, and suppose $F'$ is an arbitrary second 2-forest. We shall show that the failure of the Exchange Axiom implies that $F' \notin \mathcal{T}_{G,\neq}^2$.

Choose $e \in F$; then $F \smallsetminus \{e\}$ is a 3-forest $F_1 \sqcup F_2 \sqcup F_3$ of $G$, where the $F_i$ are the connected components of $F \smallsetminus \{e\}$. Since the full collection $\mathcal{T}_G^2$ of 2-forests forms the set of bases of a matroid, some edges of $F'$, when added to $F \smallsetminus \{e\}$, produce again a 2-forest. These are precisely those edges of $F'$ that link $F_i$ to $F_j$, for $i \neq j$ in the set $\{1, 2, 3\}$.

Since $F$ is in $\mathcal{T}_{G,\neq}^2$, the external vertices do not lie entirely inside one of the components of $F$, and even less do they lie entirely inside a connected component $F_i$ of $F \smallsetminus \{e\}$. Thus, after a suitable relabeling, both $F_1$ and $F_2$, and possibly also $F_3$, will contain an external vertex. If $F_3$ does in fact contain an external vertex, then adding any edge $f \in F'$ to $F_1 \sqcup F_2 \sqcup F_3$ will leave the external vertices split between at least two different connected components. Combined with the previous paragraph and Hypothesis 1.2.(3) we can dispose of the case when $F_3$ also contains an external vertex.

Now suppose $F_3$ does not contain an external vertex, so $V_{\text{Ext}}$ is contained in the disjoint union $F_1 \sqcup F_2$. If $F'$ contains an edge $f$ that links $F_3$ either to $F_1$ or to $F_2$, we are done, since then $(F \smallsetminus \{e\}) \cup \{f\}$ is a 2-forest in $\mathcal{T}_{G,\neq}^2$. So consider the possibility that no edge of $F'$ links $F_3$ to $F_1 \cup F_2$. This disconnection shows that the 2-forest $F'$ has one connected component that uses the vertices of $F_3$, and one component that uses the vertices of $F_1 \cup F_2$. But then $F'$ has $V_{\text{Ext}}$ inside one of its components and thus cannot be in $\mathcal{T}_{G,\neq}^2$. The lemma follows. $\square$

**Definition 3.4.** We denote the matroid of Lemma 3.3 by $\mathsf{M}_{G,\neq}^2$. $\diamondsuit$

**Remark 3.5.** By matroid duality, the set of complements $\{E \smallsetminus F \mid F \in \mathcal{T}_{G,\neq}^2\}$ is the set of bases of another matroid that we denote $\mathcal{T}_{G,\neq}^{2,\perp}$ and the bases of which are labeled by $\mathrm{Supp}(\mathcal{F}_0)$. $\diamondsuit$

Recall that a matroid $\mathsf{M}'$ is a quotient of the matroid $\mathsf{M}$ if (they are matroids on the same ground set and) any circuit in $\mathsf{M}$ is a union of circuits in $\mathsf{M}'$. The quotient property was used by Helmer and Tellander in order to prove their main result in the massive case. The following lemma and Corollary 3.14 are not used in this note; however, it seems conceivable that they might be useful for the investigation of the support polytope $P_{m,p}$ in the general case of generic kinematics, especially if Question 3.19 has a positive answer.

**Proposition 3.6.** *The matroid* $\mathsf{M}^2_{G,\neq}$ *is a quotient of* $\mathsf{M}^1_G$.

*Proof.* The graphic matroid $\mathsf{M}^1_G$ of $G$ has as circuits the circuits of $G$. Suppose $C$ is one such circuit; it cannot be independent in $\mathsf{M}^2_{G,\neq}$ since it cannot be contained in any 2-forest. We will show that it is the union of circuits in $\mathsf{M}^2_{G,\neq}$.

If $\mathsf{M}^2_{G,\neq}$ is the trivial matroid, each singleton is a circuit, and the proposition follows. So, we can assume that $\mathsf{M}^2_{G,\neq}$ is not trivial.

For the moment assume that $C$ contains at least one, but not every, external vertex. Let $e$ be any edge of $C$. As $C \smallsetminus \{e\}$ is independent in $\mathsf{M}^1_G$, we can embed it into a spanning tree $T$ for $G$. Then let $v$ be an external vertex not in $C$. Since the set $C \smallsetminus \{e\}$ is connected and $T$ is a tree, there is a unique shortest path in $T$ that connects $v$ with $C \smallsetminus \{e\}$. Remove one of the edges $f$ in this shortest path to obtain from $T$ a 2-forest $F$ which contains $v$ and $C \smallsetminus \{e\}$, but in different connected components. It follows from Hypothesis 1.2 that $F$ is a basis in $\mathsf{M}^2_{G,\neq}$ and so $C \smallsetminus \{e\} \subseteq F$ is independent in $\mathsf{M}^2_{G,\neq}$. Since this is so for any $e \in C$, $C$ is a circuit in $\mathsf{M}^2_{G,\neq}$.

Now suppose $C$ contains no external vertex. Again, remove an arbitrary edge $e \in C$ and embed the resulting $C \smallsetminus \{e\}$ into a spanning tree $T$ for $G$. Choose any two external vertices $v, v'$. Within $T$ there is a unique minimal path from $v$ to $v'$. Since neither vertex is in $C$, there is at least one edge $f$ in this minimal path that does not belong to $C$. Remove $f$ from $T$ to arrive at a 2-forest containing $C \smallsetminus \{e\}$. It is momentous by Hypothesis 1.2 since the external vertices $v$ and $v'$ are not in the same connected component of $T \smallsetminus \{f\}$. It follows that removing any edge from $C$ makes it independent in $\mathsf{M}^2_{G,\neq}$ and thus $C$ is a circuit in $\mathsf{M}^2_{G,\neq}$.

Finally, suppose $C$ contains all $\ell \geq 2$ external vertices. Denote the vertices of $C$ by $v_1, \ldots, v_c$, written in such a way that $(v_j, v_{j+1})$ are the edges of the circuit (with the understanding that $v_{c+1} = v_1$). Let $1 \leq i_1 < \ldots < i_\ell \leq c$ be the labels that correspond to the $\ell = |V_{\mathrm{Ext}}|$ external vertices. Let $C_k$ be the result of removing from $C$ the (non-external) vertices $v_{i_k+1}, \ldots, v_{i_{k+1}-1}$ as well as the edges in $C$ incident to them. Then $C_k \subseteq C$ is a path with endpoints $v_{i_k}$ and $v_{i_{k+1}}$. (Again, we agree that by $v_{i_{\ell+1}}$ we mean $v_{i_1}$). Then in $\mathsf{M}^1_G$, these sets $C_k$ are independent, but in $\mathsf{M}^2_{G,\neq}$ they are still dependent since they contain all external vertices. We claim that $C_k$ is in fact a circuit in $\mathsf{M}^2_{G,\neq}$. Indeed, for any edge $e \in C_k$, the graph $C_k \smallsetminus \{e\}$ has two connected components and $V_{\mathrm{Ext}}$ is not contained in either one: one component contains $v_{i_k}$ and the other contains $v_{i_{k+1}}$. Thus, $C_k \smallsetminus \{e\}$ can be completed to a 2-forest such that neither of its components contains $V_{\mathrm{Ext}}$, and hence $C_k \smallsetminus \{e\}$ is independent in $\mathsf{M}^2_{G,\neq}$. To finish the proof, observe that $C$ is covered by the various $C_k$. □

### 3.2. **Massive truncations.**

**Definition 3.7.** A 2-forest $F$ that can can be written as $T \smallsetminus \{e\}$ for a spanning tree $T$ and a massive edge $e$ is called a *massive truncation (of $T$ by $e$)*. We denote by $\mathcal{T}^2_{G,\mathrm{m.t.}}$ the collection of massive truncations. ♢

The massively truncated 2-forests are those that label nonzero square-free terms in $\mathcal{U} \cdot \Sigma_m$.

**Lemma 3.8.** *The set* $\mathcal{T}^2_{G,\mathrm{m.t.}}$ *is the set of bases of a matroid on the edge set* $E$ *of* $G$.

*Proof.* We need to show that the set of massively truncated 2-forests, if non-empty, satisfies the Exchange Axiom.

Let $F, F'$ be massively truncated 2-forests and choose massive edges $e, e'$ such that $T = F \cup \{e\}$ and $T' = F' \cup \{e'\}$ are spanning trees. Let $f \in F$ and consider the 3-forest $F \smallsetminus \{f\}$ with connected components

$F_1, F_{2a}, F_{2b}$ where $F_1$ is one component of $F$ and $F_{2a} \sqcup F_{2b} \sqcup \{f\}$ is the other. We need to show that for a suitable $g \in F'$, the set $(F \smallsetminus \{f\}) \cup \{g\}$ is a massively truncated 2-forest.

Since the 2-forests of $G$ form a matroid $\mathsf{M}^2_G$, certain edges $g$ of $F'$ must combine with $(F \smallsetminus \{f\})$ to a 2-forest. Moreover, the edge $e = T \smallsetminus F$ links $F_1$ to either $F_{2a}$ or $F_{2b}$; without loss of generality we can and do assume that $e$ links in fact $F_1$ to $F_{2a}$.

If some edge $g$ of $F'$ links a vertex of $F_{2a}$ to a vertex of $F_{2b}$, then $(F \smallsetminus \{f\}) \cup \{g\}$ is a 2-forest on the same connected components as $F$ and thus can be completed by the massive edge $e$ to a spanning tree. Similarly, if any edge $g$ of $F'$ links $F_1$ to $F_{2b}$, then $(F \smallsetminus \{f\}) \cup \{g\}$ is a 2-forest in which $F_{2a}$ is a connected component and again the 2-forest $(F \smallsetminus \{f\}) \cup \{g\}$ can be completed by the massive edge $e$ to a spanning tree. So, assume from now on that $F'$ has no edges from $F_{2a}$ to $F_{2b}$, and no edges from $F_1$ to $F_{2b}$.

In that case, the vertices of $F_{2b}$ must be exactly the vertices in one of the two components of the 2-forest $F'$ and therefore the other component of $F'$ uses exactly the vertices of $F_1 \sqcup F_{2a}$. In particular there is guaranteed to be an edge $g$ in $F'$ from a vertex of $F_1$ to a vertex of $F_{2a}$. Note that $(F \smallsetminus \{f\}) \cup \{g\}$ is then a 2-forest. Now recall that $F' = T' \smallsetminus \{e'\}$ is a massive truncation. Clearly, $e'$ must connect the two components of $F'$ and so links $F_{2b}$ to either $F_1$ or $F_{2a}$. In that case, $(F \smallsetminus \{f\}) \cup \{g\}$ is a massive truncation by $e'$. □

**Definition 3.9.** We denote the matroid from Lemma 3.8 by $\mathsf{M}^2_{G,\mathrm{m.t.}}$. ◇

**Remark 3.10.** By matroid duality, the set of complements $\{E \smallsetminus F \mid F \in \mathcal{T}^2_{G,\mathrm{m.t.}}\}$ is the set of bases of another matroid that we denote $\mathsf{M}^{2,\perp}_{G,\mathrm{m.t.}}$, and the bases of $\mathsf{M}^{2,\perp}_{G,\mathrm{m.t.}}$ are in natural bijection with $\mathrm{Supp}(\mathcal{U} \cdot \Sigma_m)$. ◇

We show next that the matroid of massively truncated 2-forests is also a quotient of $\mathcal{T}^1_G$, but we use a different strategy than for the momentous 2-forests.

**Definition 3.11.** Suppose $\mathsf{M}$ is a matroid on the set $E$ and $E' \subseteq E$. Define $\mathcal{B}_{/E'}$ to be the set of subsets $B$ of $E$ that have the property that there is some $e' \in E' \smallsetminus B$ such that $B \cup \{e'\}$ is a basis in $\mathsf{M}$. ◇

**Lemma 3.12.** *The set $\mathcal{B}_{/E'}$ is the set of bases of a matroid that we denote $\mathsf{M}_{/E'}$.*

*Proof.* Let $B, B' \in \mathcal{B}_{/E'}$ and choose $f \in B$. Let $e \in E' \smallsetminus B$ and $e' \in E' \smallsetminus B'$ be such that $B \cup \{e\}, B' \cup \{e'\}$ are bases for $\mathsf{M}$. Then for some element $g$ of $B' \cup \{e'\}$ the Exchange Axiom in $\mathsf{M}$ guarantees that $((B \cup \{e\}) \smallsetminus \{f\}) \cup \{g\}$ is a basis for $\mathsf{M}$. Since necessarily $f \neq e \neq g$, $(B \smallsetminus \{f\}) \cup \{g\}$ is the new basis for $\mathsf{M}_{/E'}$ that we want. □

**Remark 3.13.** The independent sets of $\mathsf{M}_{/E'}$ are those contained in a basis of $\mathsf{M}_{/E'}$ and therefore are the subsets of $E$ that can be augmented to an independent set in $\mathsf{M}$ by an element of $E'$. It follows that if $e' \in E'$ is a loop, then $\mathsf{M}_{/E'} = \mathsf{M}_{/(E' \smallsetminus \{e'\})}$. In particular, when $E'$ has rank zero (so $E'$ contains only loops) then $\mathsf{M}_{/E'}$ is the trivial matroid. ◇

The reader might consult Remark 3.15 below for visualization of the proof of the following result.

**Proposition 3.14.** *The matroid $\mathsf{M}_{/E'}$ is a quotient of the matroid $\mathsf{M}$.*

*Proof.* Throughout this proof, the concepts of rank and span will be used relative to the matroid $\mathsf{M}$. Remark 3.13 allows to assume that $E'$ has positive rank and contains no loops. We will make use of the standard fact that if a matroid element $e$ is added to an independent set $I$ and the union $I \cup \{e\}$ is dependent, then $I \cup \{e\}$ contains a unique circuit, and that circuit uses $e$. Let $C$ be a circuit of $\mathsf{M}$; in particular, $|C| = \mathrm{rk}(C) + 1$.

Suppose first that there is $e' \in E' \smallsetminus \mathrm{span}\,(C)$. Select an arbitrary $c \in C$. Then $|(C \smallsetminus \{c\}) \cup \{e'\}| = |C| = \mathrm{rk}(C) + 1 = \mathrm{rk}(C \cup \{e'\}) = \mathrm{rk}((C \smallsetminus \{c\}) \cup \{e'\})$. It follows that $(C \smallsetminus \{c\}) \cup \{e'\}$ is independent in $\mathsf{M}$ and hence $C \smallsetminus \{c\}$ is independent in $\mathsf{M}_{/E'}$. Thus, $E'$ not being contained in $\mathrm{span}\,(C)$ assures that $C$ itself is a circuit in $\mathsf{M}_{/E'}$.

From now on, suppose $E' \subseteq \operatorname{span}(C)$. Fix $c \in C$ and consider the $\mathsf{M}$-independent set $C \smallsetminus \{c\}$. Since $C$ is an $\mathsf{M}$-circuit, $\operatorname{span}(C \smallsetminus \{c\}) = \operatorname{span}(C) \supseteq E'$ and it follows that $C \smallsetminus \{c\}$ is $\mathsf{M}_{/E'}$-dependent. This means that for any $e \in E'$, the set $(C \smallsetminus \{c\}) \cup \{e\}$ is $\mathsf{M}$-dependent and so contains a unique $\mathsf{M}$-circuit containing $e$. Suppose $c_1 \in C \smallsetminus \{c\}$ has the property that it is not used in such $\mathsf{M}$-circuit for any $e \in E'$. Erase from $C \smallsetminus \{c\}$ any such $c_1$ and let $C'$ be the resulting subset of $C$. Alternatively, the erased elements $c_1$ are exactly the ones that are bridges in $(C \smallsetminus \{c\}) \cup \{e\}$ for every $e \in E'$. We call this set $C'$ the "pruning of $(C, E')$ initiated by $c$" and denote it $P(C, E', c)$. Note that the uniqueness of the circuits created in $(C \smallsetminus \{c\}) \cup \{e\}$ forces that $c' \notin P(C, E', c)$ if and only if $c \notin P(C, E', c')$. This sets up an "pruning equivalence" relation: $c$ is equivalent to $c'$ if and only if $c \notin P(C, E', c')$, which happens if and only if $P(C, E', c) = P(C, E', c')$.

By construction, $P(C, E', c) \cup \{e\}$ is $\mathsf{M}$-dependent for any $e \in E'$, and $P(C, E', c) \cup E'$ is the union of all $\mathsf{M}$-circuits that result from adding a single element of $E'$ to $P(C, E', c)$. In particular, $P(C, E', c)$ is $\mathsf{M}_{/E'}$-dependent if it is non-empty. But since $E'$ contains no $\mathsf{M}$-loops (and so no edge of $E'$ is dependent by itself), the sets $P(C, E', c)$ cannot empty.

Suppose $c' \in P(C, E', c)$ and consider $P(C, E', c) \smallsetminus \{c'\}$. This removal breaks at least one circuit of the form $P(C, E', c) \cup \{e\}$ for suitable $e \in E'$ and so adding this $e$ to $P(C, E, c) \smallsetminus \{c'\}$ produces an $\mathsf{M}$-independent set. Hence $P(C, E, c) \smallsetminus \{c'\}$ is independent in $\mathsf{M}_{/E'}$. This being so for arbitrary $c'$ implies that $P(C, E', c)$ is an $\mathsf{M}_{/E'}$-circuit.

Choose $c \in C$ and $c' \in P(C, E', c)$. Then pruning equivalence dictates that $c \in P(C, E', c')$ and so $C$ is the union of all $P(C, E', c)$, $c$ running through $C$. $\square$

**Remark 3.15.** It is perhaps helpful to visualize the ideas of this proof in the case of a graphical matroid. The circuit $C$ can be viewed as a polygon, and in the main case $E' \subseteq \operatorname{span}(C)$ one may picture $E'$ as a set of diagonals in $C$. The set $P(C, E', c)$ for $c \in C$ is then the set of edges in the connected graph $(C \smallsetminus \{c\}) \cup E'$ that are contained in $C$ and also in at least one circuit of $(C \smallsetminus \{c\}) \cup E'$. The complement of $P(C, E', c)$ are therefore the edges of $C \smallsetminus \{c\}$ whose removal would disconnect $(C \smallsetminus \{c\}) \cup E'$, and the equivalence relation becomes transparent: the circuits in $(C \smallsetminus \{c\}) \cup E'$ are unchanged if one removes a bridge and then adds $c$. $\diamondsuit$

**Corollary 3.16.** *The matroid* $\mathsf{M}^2_{G,\mathrm{m.t.}}$ *is a quotient of* $\mathsf{M}^1_G$.

*Proof.* In the previous lemma, take $\mathsf{M} = \mathsf{M}^1_G$ and $E'$ to be the massive edges. Then the definition of $\mathsf{M}^2_{G,\mathrm{m.t.}}$ matches that of $(\mathsf{M}^2_G)_{/E'}$. $\square$

3.3. **2-forests of $\mathcal{G}_{m,p}$.** Given two matroids on the same ground set, their union is usually not a matroid (in the sense that the union of their individual sets of bases is usually not the set of bases of a new matroid). Nonetheless, we have the following fact.

**Theorem 3.17.** *The set of 2-forests in the Feynman graph $G$ that arises as the union of the momentous 2-forests and the massively truncated 2-forests forms the set of bases of a matroid.*

*Proof.* Let $F, F' \in \mathsf{M}^2_{G,\neq} \cup \mathsf{M}_{G,\mathrm{m.t.}}$. We need to show the validity of the simple Exchange Axiom. Since $\mathsf{M}^2_{G,\mathrm{m.t.}}$ and $\mathsf{M}^2_{G,\neq}$ are matroids by Lemmas 3.8 and 3.3, it suffices to consider the two cases listed below.

*Case 1: $F$ is momentous and $F'$ is massively truncated.* Let $e \in F$ be any edge; then $F \smallsetminus \{e\}$ is a 3-forest, with components denoted $F_1, F_{2a}, F_{2b}$ where $e$ links $F_{2a}$ to $F_{2b}$. Since the set of all 2-forests is in fact a matroid, there is at least one edge $g \in F'$ such that $(F \smallsetminus \{e\}) \cup \{g\}$ is a 2-forest. If this is a momentous 2-forest we are done with this case. So, in the sequel we assume that no edge of $F'$ combines with $(F \smallsetminus \{e\})$ to a momentous 2-forest.

Choose a $g \in F'$ that forms a (non-momentous) 2-forest $(F \smallsetminus \{e\}) \cup \{g\}$. Then $(F \smallsetminus \{e\}) \cup \{g\}$ contains no circuits; hence, $g$ cannot link $F_{2a}$ to $F_{2b}$ (or else $(F \smallsetminus \{e\}) \cup \{g\}$ would be momentous) and so $g$ will connect either $F_1$ to $F_{2a}$, or $F_1$ to $F_{2b}$. Depending on the case, the implication would be that the external vertices are either completely contained in $F_1 \cup F_{2a}$ or in $F_{2b}$, or in $F_1 \cup F_{2b}$ or in $F_{2a}$. In other words, the external vertices are either contained completely in $F_1 \cup F_{2a}$ or in $F_1 \cup F_{2b}$. Without loss of generality, let us assume they are all inside $F_1 \cup F_{2b}$ and so none is in $F_{2a}$. Note that momentousness of $F$ implies that some external vertices are in $F_1$ and some in $F_{2b}$. In particular then, the edge $g$ from the start of this paragraph that creates the non-momentous 2-forest $(F \smallsetminus \{e\}) \cup \{g\}$ connects a vertex of $F_1$ to a vertex of $F_{2b}$.

It follows that if for no edge $g \in F'$ the set $(F \smallsetminus \{e\}) \cup \{g\}$ is a momentous 2-forest, then all edges of $F'$ are either supported on one of $F_1, F_{2a}, F_{2b}$, or they must connect $F_1$ to $F_{2b}$. That means that all edges of $F'$ are supported either on $F_{2a}$, or on $F_1 \cup F_{2b}$, implying that the vertex sets of $F_{2a}$ and $F_1 \cup F_{2b}$ are the same as the vertex sets of the two components of $F'$.

Now recall that $F'$ is massively truncated, and let $f$ be a massive edge such that $F' \cup \{f\}$ is a spanning tree. By the previous paragraph, $f$ must link a vertex of $F_1 \cup F_{2b}$ to one of $F_{2a}$. It follows that $(F \smallsetminus \{e\}) \cup \{g\}$ is massively truncated via $f$.

*Case 2: $F$ is massively truncated and $F'$ is momentous.* Fix an edge $e \in F$, and a massive edge $f$ such that $F \cup \{f\}$ is a spanning tree. Then $F \smallsetminus \{e\}$ has three components $F_1, F_{2a}, F_{2b}$ with $e$ linking a vertex from $F_{2a}$ to one from $F_{2b}$, and $f$ linking $F_1$ to either $F_{2a}$ or $F_{2b}$. Without loss of generality, assume the latter case.

Since 2-forests form a matroid, at least one edge $g$ of $F'$ turns $F \smallsetminus \{e\}$ into a 2-forest. Suppose all edges $g$ of $F'$ are either supported on one of $F_1, F_{2a}$ or $F_{2b}$, or make it impossible to certify $(F \smallsetminus \{e\}) \cup \{g\}$ as massively truncated via $f$ (*i.e.*, $(F \smallsetminus \{e\}) \cup \{g\} \cup \{f\}$ contains a circuit). Then all edges of $F'$ are either supported on one of $\{F_1, F_{2a}, F_{2b}\}$, or link $F_1$ to $F_{2b}$. Note that therefore an edge $g \in F'$ linking $F_1$ to $F_{2b}$ must exist, as else $F'$ should have more than two components. Since $F'$ has exactly two components, these components must be supported on $F_1 \cup F_{2b}$ and $F_{2a}$ respectively. Since $F'$ is momentous, $F_{2a}$ contains some but not all external vertices. By Hypothesis 1.2, with the edge $g \in F'$ that links a vertex from $F_1$ to one of $F_{2b}$, we find that $(F \smallsetminus \{e\}) \cup \{g\}$ is momentous, finishing the second case and the proof. □

**Definition 3.18.** We denote the matroid from Theorem 3.17 by $\mathsf{M}^2_{G,\mathrm{Feyn}}$, and remark that the bases of the dual matroid $\mathsf{M}^{2,\perp}_{G,\mathrm{Feyn}}$ are the subsets of $E$ that are either a basis for $\mathsf{M}^{2,\perp}_{G,\neq}$ or for $\mathsf{M}^{2,\perp}_{G,\mathrm{m.t.}}$ (or both). ♢

A positive answer to the following problem might be useful for proving that saturatedness is always implied by general kinematics, compare the proof of Theorem 4.3.

**Question 3.19.** Is $\mathsf{M}^2_{G,\mathrm{Feyn}}$ a quotient of $\mathsf{M}^1_G$, just like $\mathsf{M}^2_{G,\neq}$ and $\mathsf{M}^2_{G,\mathrm{m.t.}}$? ♢

## 4. Main Theorems

4.1. **All 2-forests present.** We recall a result from [HT22] that will be used in the proof below.

**Theorem 4.1.** *In the massive case, with Hypothesis 1.2 in force, the semigroup spanned by the lifts of* $\mathrm{Supp}(\mathcal{G}_{m,p})$ *is normal.* □

In the massive case with Hypothesis 1.2, for a fixed set $V_{\mathrm{Ext}}$ of external vertices, the support of $\mathcal{G}_{m,p}$ is as large as it can possibly be for any mass and any momentum function. We shall prove here that the conclusion of Theorem 4.1 continues to hold as long as every 2-forest of $G$ contributes to the support of $\mathcal{G}_{m,p}$; it is immaterial which terms with squares appear.

For this, recall Equations (2.2.1), (2.2.2) and set

$$\mathcal{G}_G := \mathcal{U} \cdot \tilde{\Sigma}_E.$$

Always assuming general kinematics, all monomials that appear in $\mathcal{G}_{m,p}$ also appear in $\mathcal{G}_G$. But $\mathcal{G}_G$ can contain monomials that do not show in $\mathcal{G}_{m,p}$, and these might or might not be square-free.

**Remark 4.2.** An idea that will be used repeatedly is the obvious observation:

(1-forest complement) $\cup$ (element inside the 1-forest) = (2-forest complement).

By the (s1I) condition, any given edge $e$ is not a loop, and hence contained in a 1-forest $T$. If the 2-forest $F = T \smallsetminus \{e\}$ labels a nonzero term in $\mathcal{G}_{m,p}$ then the matrix $A_{\mathcal{G}}$ contains two columns, one from $\boldsymbol{x}^{E \smallsetminus T}$ and one from $\boldsymbol{x}^{E \smallsetminus F}$. Their difference is $\mathbf{e}_e$ and so $\mathbb{Z}A_{\mathcal{G}}$ contains $\mathbb{Z}^E$. Thus, when all 2-forests are present in $\mathrm{Supp}(\mathcal{G}_{m,p})$, and also in most other cases, the lattice of $A_{\mathcal{G}}$ agrees with the ambient lattice. $\diamondsuit$

Our strategy will be to show that as long as all 2-forests of $G$ contribute to the support of $\mathcal{G}_{m,p}$, then the semigroup to the lifts of $\mathrm{Supp}(\mathcal{G}_{m,p})$ can be obtained from the semigroup to the lifts of $\mathrm{Supp}(\mathcal{G}_G)$ by intersecting with suitable half-spaces of $\mathbb{C} \times \mathbb{C}^E$. The point is that half-spaces contain saturated semigroups, and intersections of saturated semigroups are saturated.

Let us denote by
$$\mu_e \colon \mathbb{C}^E \longrightarrow \mathbb{C}$$
the $e$-th coordinate function on $\mathbb{C}^E$; on $\mathbb{C} \times \mathbb{C}^E$ we include the coordinate function $\mu_0$ on the first factor into the notation.

We can now prove the following generalization of [HT22, Thm. 1.1, part 1]:

**Theorem 4.3.** *Let $G$ be a (s1I) Feynman graph with mass function $m \colon E \longrightarrow \mathbb{R}_{\geq 0}$ satisfying Hypothesis 1.2. If $\mathsf{M}^2_G = \mathsf{M}^2_{G,\mathrm{Feyn}}$, or equivalently if every 2-forest complement of $G$ contributes to* $\mathrm{Supp}(\mathcal{G}_{m,p})$*, then the semigroup $\mathbb{N}A_{\mathcal{G}}$ is saturated and thus the semigroup ring $\mathbb{K}[\mathbb{N}A_{\mathcal{G}}]$ is normal and Cohen–Macaulay for all fields $\mathbb{K}$.*

*Proof.* That the second statement follows from the first is contained in [Hoc72].

Comparing the terms in $\mathcal{G}_{m,p}$ and $\mathcal{G}_G$ in light of our assumptions, $\mathrm{Supp}(\mathcal{G}_{m,p})$ arises from $\mathrm{Supp}(\mathcal{G}_G)$ by canceling all terms that are divided by the square of a massless variable, and no others. In other words, the monomials $\boldsymbol{x}^{\mathbf{a}}$ in $\mathrm{Supp}(\mathcal{G}_{m,p})$ are precisely those in $\mathrm{Supp}(\mathcal{G}_G)$ whose lifted exponent $(1, \mathbf{a})$ satisfies $\mu_0((1, \mathbf{a})) \geq \mu_e((1, \mathbf{a}))$ for all massless $e \in E$.

Let $A_E$ denote any matrix whose columns are the lifted support exponents of $\mathcal{G}_G$; in particular, we could order its columns in such a way that $A_{\mathcal{G}}$ becomes a submatrix. For elements $(k, \mathbf{a})$ in $\mathbb{N}A_E$ or $\mathbb{N}A_{\mathcal{G}}$, we call $k = \mu_0((k, \mathbf{a}))$ their *degree*. We have noted above that, as subsets of $\mathbb{Z} \times \mathbb{Z}^E$,
$$A_{\mathcal{G}} = A_E \cap \bigcap_{m_e = 0} H_e$$
where
$$H_e := \{\alpha \in \mathbb{R} \times \mathbb{R}^E \mid (\mu_0 - \mu_e)(\alpha) \geq 0\}$$
is the half-space on which $\mu_0 - \mu_e$ is non-negative. It follows also that
$$\mathbb{N}A_{\mathcal{G}} \subseteq (\mathbb{N}A_E) \cap \bigcap_{m_e = 0} H_e,$$
and the remainder of the proof is devoted to showing that this is an equality, which would show that $\mathbb{N}A_{\mathcal{G}}$ is the intersection of saturated semigroups, hence saturated itself.

Take any lattice element $(k, \mathbf{a})$ in the cone $\mathbb{R}_{\geq 0}A_{\mathcal{G}}$ of degree $k$. Since $\mathbb{N}A_E$ is saturated according to Theorem 4.1, one has $(\mathbb{R}_{\geq 0}A_E) \cap (\mathbb{Z} \times \mathbb{Z}^E) = \mathbb{N}A_E$. Since $(\mathbb{R}_{\geq 0}A_{\mathcal{G}}) \subseteq (\mathbb{R}_{\geq 0}A_E)$, one can write
$$(k, \mathbf{a}) \quad = \quad (1, \mathbf{a}_1) + \ldots + (1, \mathbf{a}_k) \tag{4.1.1}$$
where each $(1, \mathbf{a}_i)$ is a column of $A_E$.

We have $(k, \mathbf{a}) \in (\mathbb{R}_{\geq 0}A_{\mathcal{G}}) \subseteq H_e$ for all massless $e \in E_0$. We will show that, given $e \in E_0$, the condition $(k, \mathbf{a}) \in H_e$ implies that one can rewrite the sum (4.1.1) in such a way that he following exchange rules hold:

- the new sum only uses summands that are columns of $A_E$;

- the number of summands is unchanged;
- each summand lies in $H_e$,

and that, moreover, it can be arranged that

- if all summands were originally in $\bigcap_{e'\in E'} H_{e'}$ for some set $E' \subseteq E$, then after the rewriting they are in $H_e \cap \bigcap_{e'\in E'} H_{e'}$.

Establishing this rewriting forms the main part of the proof. Indeed, given such rewriting result, fix a massless edge $e \in E_0$. Our exchange rules above allow to change the sum in (4.1.1) into one where each support vector is in $H_e$. Since no exchange operation introduces square terms that were not there before, we can treat (4.1.1) one $e \in E_0$ at the time and arrive at a sum as in (4.1.1) in which every term is in $H_e$ for each $e \in E_0$. But that implies that we have written $\mathbf{a}$ as a sum of $k$ exponent vectors that appear in $\mathcal{G}_{m,p}$, implying that $\mathbb{N}A_{\mathcal{G}}$ is saturated.

Before we engage in the rewriting, note that for $e \in E$, the monomials $\boldsymbol{x}^{\mathbf{a}_j}$ appearing in $\mathcal{G}_G = \tilde{\Sigma}_E \cdot \mathcal{U}$ fall into three categories, depending on whether $\mu_e(\mathbf{a}_j)$ is $0, 1$, or $2$. Alternatively, they are classified by the value of $(\mu_0 - \mu_e)((1, \mathbf{a}_j)) \in \{1, 0, -1\}$. Those with $(\mu_0 - \mu_e)((1, \mathbf{a}_j)) = 1$, which are those with $\mu_e(\mathbf{a}_j) = 0$, fall themselves into two types:

(1) square-free monomials without $x_e$;
(2) monomials without $x_e$ that contain some (other) square.

Now suppose that the sum decomposition (4.1.1) involves an element $(1, \mathbf{a}_i)$ that is not in the positive real cone of $A_{\mathcal{G}}$ and therefore satisfies $\mu_e(\mathbf{a}_i) = 2$ for some (necessarily unique) $e$ with $m_e = 0$. In particular, $\mathbf{a}_i$ does then not appear in $\mathrm{Supp}(\mathcal{U})$ and so we will have $|\mathbf{a}_i| = r + 1$, where $r$ is the rank of the cographic matroid $\mathsf{M}_G^{1;\perp}$ of support vectors of $\mathcal{U}$.

Since $\mathbf{a}_i$ is a support vector of $\mathcal{G}_G$ with $\mu_e(\mathbf{a}_i) = 2$, $\boldsymbol{x}^{\mathbf{a}_i}$ appears in $\mathcal{U} \cdot \Sigma_E$ and so

$$\boldsymbol{x}^{\mathbf{a}_i} = \boldsymbol{x}^{E\smallsetminus T} x_e \qquad \text{with } T \in \mathcal{T}_G^1 \text{ and } e \notin T. \tag{4.1.2}$$

Since $(\mu_0 - \mu_e)((1, \mathbf{a}_i)) < 0$ but $(\mu_0 - \mu_e)((k, \mathbf{a})) \geq 0$ there must appear a semigroup element $(1, \mathbf{a}_j)$ in (4.1.1) with $(\mu_0 - \mu_e)((1, \mathbf{a}_j)) > 0$; choose one such. As $\mu_0((1, \mathbf{a}_j)) = 1$, it follows that $\mu_e((1, \mathbf{a}_j)) = 0$ and so $(1, \mathbf{a}_j)$ must be of one of the types (1) or (2) above.

In the remainder of the proof, references to rank, circuits, and span will always be in the graphical matroid $\mathsf{M}_G^1$.

*Case 1:* Suppose $\mathbf{a}_j$ is of type (1); then $\boldsymbol{x}^{\mathbf{a}_j} = \boldsymbol{x}^{E\smallsetminus F}$ for some 2-forest $F \in \mathcal{T}_G^2$ with $e \in F$.

The union $T \cup \{e\}$ has exactly one circuit $C$, $C$ contains $e$, and $F \smallsetminus \{e\}$ is a 3-forest. Since $C$ is a circuit, $C \smallsetminus \{e\}$ has the same span as $C$, and so $\mathrm{span}\,((C \smallsetminus \{e\}) \cup (F \smallsetminus \{e\})) = \mathrm{span}\,(C \cup (F \smallsetminus \{e\})) = \mathrm{span}\,(C \cup F)$, which contains the 2-forest $F$. Thus, there is a suitable edge $f \in (C \smallsetminus \{e\}) = C \cap T$ that combines with the 3-forest $F \smallsetminus \{e\}$ to a set of rank greater than $\mathrm{rk}(F \smallsetminus \{e\})$. For such $f$, $(F \smallsetminus \{e\}) \cup \{f\}$ is therefore a 2-forest. However, so is $T \smallsetminus \{f\}$, and so by the assumptions of the theorem the monomials $\boldsymbol{x}^{\mathbf{a}'_i} := \boldsymbol{x}^{E\smallsetminus(T\smallsetminus\{f\})}$ and $\boldsymbol{x}^{\mathbf{a}'_j} := \boldsymbol{x}^{E\smallsetminus((F\smallsetminus\{e\})\cup\{f\})}$ appear in $\mathcal{G}_{m,p}$. Moreover, their product is $\boldsymbol{x}^{\mathbf{a}'_i}\boldsymbol{x}^{\mathbf{a}'_j} = \boldsymbol{x}^{E\smallsetminus T}\boldsymbol{x}^{E\smallsetminus F}x_e = \boldsymbol{x}^{\mathbf{a}_i}\boldsymbol{x}^{\mathbf{a}_j}$ and so $(1, \mathbf{a}_i) + (1, \mathbf{a}_j) = (1, \mathbf{a}'_i) + (1, \mathbf{a}'_j)$ in $\mathbb{N}A_E$. We can thus replace $\mathbf{a}_j$ by $\mathbf{a}'_j$ and $\mathbf{a}_i$ by $\mathbf{a}'_i$ while preserving (4.1.1) as a sum in $\mathbb{N}A_E$. Note that the replacement terms have no square terms and so no new terms with squares in any variable have been introduced while the overall number of square terms has in fact decreased.

*Case 2:* Suppose now that $\mathbf{a}_j$ is of type (2).

Then $\mathbf{a}_j$ is a support vector of a term in $\Sigma_E \cdot \mathcal{U}$ with $\mu_f(\mathbf{a}_j) = 2$ for some $f \in E$, while $\mu_e(\mathbf{a}_j) = 0$. Thus, (we still have $\mathbf{a}_i$ as in (4.1.2) and) $\boldsymbol{x}^{\mathbf{a}_j} = x_f \boldsymbol{x}^{E \smallsetminus S}$ for some 1-forest $S$ of $G$ that does not involve $f$ (since else $x_f \boldsymbol{x}^{E \smallsetminus S}$ would be linear in $x_f$) but does involve $e$ (so that $x_e$ does not appear in $x_f \boldsymbol{x}^{E \smallsetminus S}$).

Then $T \cup \{e\}$ contains a unique circuit $C \ni e$, and the span of $(C \smallsetminus \{e\}) \cup (S \smallsetminus \{e\})$ contains span $(C \cup (S \smallsetminus \{e\})) =$ span $(C \cup S) \supseteq$ span $(S) = E$. It follows that some element $g \in (C \smallsetminus \{e\}) = C \cap T$ different from $e$ turns the 2-forest $S \smallsetminus \{e\}$ back into a 1-forest. As removal of $g$ from $T \cup \{e\}$ breaks the unique circuit $C$ in $T \cup \{e\}$, $(T \cup \{e\}) \smallsetminus \{g\}$ is a 1-forest. Then, $(x_e \boldsymbol{x}^{E \smallsetminus T}) \cdot (x_f \boldsymbol{x}^{E \smallsetminus S}) = (x_e \boldsymbol{x}^{E \smallsetminus ((T \cup \{e\}) \smallsetminus \{g\})}) \cdot (x_f \boldsymbol{x}^{E \smallsetminus ((S \cup \{g\}) \smallsetminus \{e\})})$. In (4.1.1), replace $(1, \mathbf{a}_i) + (1, \mathbf{a}_j)$ by the sum of $(1, E \smallsetminus (T \smallsetminus \{g\})) = (1, \mathbf{a}_i + \mathbf{e}_g - \mathbf{e}_e)$ and $(1, E \smallsetminus (S \cup \{g\} \smallsetminus \{e\})) + (0, \mathbf{e}_f) = (1, \mathbf{a}_j + \mathbf{e}_e - \mathbf{e}_g)$. Both new terms are lifts of support vectors of $\mathcal{G}_G$, both are in $H_e$, and the only square factor in either one is $x_f^2$ in the second one, inherited from $\mathbf{a}_j$. Moreover, the number of summands with squares of massless edges in (4.1.1) has decreased by one.

This finishes the rewriting claim, and as explained above proves the theorem. □

In the light of Theorem 4.3, it is natural to ask under what conditions we have the equality $\mathsf{M}^2_G = \mathsf{M}^2_{G,\mathrm{Feyn}}$; we address this question next.

**Definition 4.4.** A path $v_0, v_1, \ldots, v_t$ of vertices in $G$ (with $\{v_i, v_{i+1}\}$ adjacent for all $0 \le i < t$) is called *massive* if all edges $\{v_i, v_{i+1}\}$ are massive. ♢

**Theorem 4.5.** *In an (s1I) graph $G$ that satisfies Hypothesis 1.2, the equality $\mathsf{M}^2_G = \mathsf{M}^2_{G,\mathrm{Feyn}}$ holds if and only if every vertex of $G$ permits a massive path to an external vertex of $G$.*

*Proof.* First, assume that $\mathsf{M}^2_G \neq \mathsf{M}^2_{G,\mathrm{Feyn}}$. Let $F = F_1 \sqcup F_2$ be a 2-forest in $\mathsf{M}^2_G \smallsetminus \mathsf{M}^2_{G,\mathrm{Feyn}}$. By Hypothesis 1.2, all momentous 2-forests label a nonzero term in $\mathcal{G}_{m,p}$. Thus, $F$ is not momentous and so one of the components of $F$ contains $V_{\mathrm{Ext}}$; say $V_{\mathrm{Ext}} \subseteq F_1$. As this $F$ does not label a nonzero term in $\mathcal{G}_{m,p}$, it cannot be a massive truncation. This means that no vertex of $F_1$ can be linked by a massive edge to any vertex of $F_2$. In particular, no massive path can exist from the vertices of $F_2$ to $V_{\mathrm{Ext}}$.

Conversely, suppose that some vertex $v$ cannot be linked to $V_{\mathrm{Ext}}$ by a massive path. We now delete from $G$ all massless edges and call the result $G'$. Then $v$ belongs to a connected component $U$ of $G'$ that does not include a single external vertex, and so all external vertices are in $G \smallsetminus U$. Take any 2-forest for $G$ that has one connected component supported in $U$, and the other on $G \smallsetminus U$. By our choices, this 2-forest is neither massively truncated nor momentous and hence does not contribute to $\mathcal{G}_{m,p}$. □

4.2. **The general massless case.** In [HT22], Helmer and Tellander proved that if every vertex of $G$ is an external vertex, then the semigroup $\mathbb{N}A_{\mathcal{G}}$ is normal for the mass function that is identically zero. The advantage of the condition on $V_{\mathrm{Ext}}$ is that it places us in a special case of Theorem 4.5 above, and guarantees that $\mathcal{G}_{m,p}$ involves a term from the complement of every 2-forest, $\mathsf{M}^2_G = \mathsf{M}^2_{G,\mathrm{Feyn}}$. As it turns out, this condition can be completely removed: we now use our results from Section 3 to dispose of the general massless case.

We need to review edge-unimodularity and IDP properties of polytopes.

**Definition 4.6.** An integer matrix is *unimodular* if all maximal minors are in the set $\{-1, 0, 1\}$.

A lattice polytope $P$ is *edge-unimodular* if there is an integer unimodular matrix $M$ such that all edges of $P$ are parallel to columns of $M$.

A lattice polytope $P \subseteq \mathbb{Z}^d$ is said to have *the IDP property* if the intersection $(kP) \cap \mathbb{Z}^d$ agrees with the sum $((k-1)P \cap \mathbb{Z}^d) + (P \cap \mathbb{Z}^d)$ for all $k \in 1 + \mathbb{N}$. ♢

The benefit of the IDP property to the present context is that it is equivalent to the equation

$$\mathbb{N}((1, P) \cap (\mathbb{Z} \times \mathbb{Z}^d)) = \mathbb{R}_{\ge 0}((1, P)) \cap (\mathbb{Z} \times \mathbb{Z}^d).$$

In other words, a polytope is IDP if and only if the semigroup generated by the lattice points in its lift is saturated in $\mathbb{Z} \times \mathbb{Z}^d$.

The following result is due to Howard.

**Theorem 4.7** ([How07b, Thm. 4.5])**.** *Suppose that $A \in \mathbb{Z}^{d\times n}$ is a unimodular matrix, and that $P$ and $Q$ are lattice polytopes with edges parallel to columns of $A$. Then, $(P \cap \mathbb{Z}^d) + (Q \cap \mathbb{Z}^d) = (P + Q) \cap \mathbb{Z}^d$.*

In fact, the theorem is stated in a much more constrained context (inside a lattice of weights of a Lie algebra) and in a more opaque way, but the proof works in the generality stated here (which is also the version Howard states in [How07a, Thm. 1]). As Howard points out, this implies that if $P$ is a lattice polytope with edges parallel to the columns of a unimodular matrix, then $P$ is IDP and in consequence the semigroup generated by the lattice points in the lifted polytope $(1, P)$ inside its own lattice is saturated.

**Theorem 4.8.** *Let $G$ be a (s1I) Feynman graph satisfying Hypothesis 1.2. Suppose the mass function $m\colon E \longrightarrow \mathbb{R}_{\geq 0}$ is identically zero, $m_e = 0$ for all $e$. Then the semigroup $A_{\mathcal{G}}$ is saturated and thus the semigroup ring $\mathbb{K}[\mathbb{N}A_{\mathcal{G}}]$ is normal and Cohen–Macaulay for all fields $\mathbb{K}$.*

*Proof.* The proof follows the one from [HT22], with appropriate modifications.

Since $m$ is zero, $\mathcal{G}_{m,p} = \mathcal{U} + \mathcal{F}_0$. Since the momentous 2-forests $\mathcal{T}^2_{G,\neq}$ form the set of bases of a matroid, the support vectors of $\mathcal{F}_0$ (the complements of the elements of $\mathcal{T}^2_{G,\neq}$ in $E$) are the indicator vectors of the bases for the dual matroid $\mathsf{M}^{2,\perp}_{G,\neq}$ on the edge set $E$. By [GGMS87], the support polytopes $P^{2,\perp}_{G,\neq}$ of $\mathcal{F}_0$ and $P^{1,\perp}_G$ of $\mathcal{U}$ have their edges within the set of vectors $\{\mathbf{e}_e - \mathbf{e}_{e'}\}_{e,e'\in E}$. The matrix with these vectors as columns is unimodular, so the support polytopes of $\mathcal{F}_0$ and $\mathcal{U}$ are edge-unimodular and in particular IDP.

Since edge directions are invariant under scaling, Howard's theorem implies for all dilations that $(k \cdot P^2_{G,\neq} + \ell \cdot P^1_G) \cap \mathbb{Z}^d = (k \cdot P^2_{G,\neq} \cap \mathbb{Z}^d) + (\ell \cdot P^1_G \cap \mathbb{Z}^d)$. Recall that the Cayley sum of the lattice polytopes $P$ and $Q$ is the convex hull of $(\{0\} \times P) \cup (\{1\} \times Q)$ in $\mathbb{R}^{1+d}$. With the IDP properties of $P^{2,\perp}_{G,\neq}$ and $P^{1,\perp}_G$ this implies by a theorem of Tsuchiya that the Cayley sum of $P^{2,\perp}_{G,\neq}$ and $P^{1,\perp}_G$ has the IDP property, [Tsu19, Thm 0.4].

Since the entry sums of the vertices of $P^{2,\perp}_{G,\neq}$ and $P^{1,\perp}_G$ differ by one, a suitable integer coordinate change shows that the Cayley sum of $P^{2,\perp}_{G,\neq}$ and $P^{1,\perp}_G$ can be identified with the convex hull of their union. It follows that the union of $P^{2,\perp}_{G,\neq}$ and $P^{1,\perp}_G$, which is the support polytope of $\mathcal{G}_{m,p}$ as $m = 0$, has the IDP property.

Both polytopes $P^{2,\perp}_{G,\neq}$ and $P^{1,\perp}_G$ are matroid polytopes, so their lattice points are their vertices. Moreover, the polytopes sit in parallel hyperplanes of distance one. Thus, the lattice points in the convex hull of their union are precisely the lattice points of the two polytopes, which are their vertices. Since the vertices are (by definition) support vectors of terms in $\mathcal{G}_{m,p}$, the semigroup generated by the lifted support vectors of $\mathcal{G}_{m,p}$ is saturated. ☐

### 4.3. Approaching the general case.

**Proposition 4.9.** *For all mass functions on a Feynman graph $G$ satisfying Hypothesis 1.2, the support vectors of $\mathcal{G}_{m,p}$ are exactly the lattice points inside the support polytope of $\mathcal{G}_{m,p}$. In other words, the difference of semigroups $\widetilde{\mathbb{N}A_{\mathcal{G}}} \smallsetminus \mathbb{N}A_{\mathcal{G}}$ has no elements of degree 1.*

*Proof.* We induce on the number of edges of the graph $G$.

Suppose $\mathbf{a} = \sum \alpha_i \mathbf{a}_i$ is a lattice point inside the support polytope of $\mathcal{G}_{m,p}$ that can be written as a linear combination of support vectors $\mathbf{a}_i$ of $\mathcal{G}_{m,p}$ with $\sum \alpha_i = 1$. We need to show that $\mathbf{a}$ is a support vector itself.

Each $\mathbf{a}_i$ is the support vector of a monomial $\boldsymbol{x}^{E\smallsetminus T} \cdot x_f$ for some 1-forest $T$ and a massive edge $f$, or of $\boldsymbol{x}^{E\smallsetminus F}$ where $F$ is a momentous 2-forest, or of $\boldsymbol{x}^{E\smallsetminus T}$ where $T$ is a 1-forest. In any event, the entries of $\mathbf{a}_i$ are in $\{0, 1, 2\}$. It follows that the same is true for every entry of $\mathbf{a}$.

Since the entry sums of all $\mathbf{a}_i$ and of $\mathbf{a}$ are integers equal to either the size of a 1-forest complement or that of a 2-forest complement, either all $\mathbf{a}_i$ with nonzero $\alpha_i$ come from $\mathcal{U}$, or none does. Since the support vectors of $\mathcal{U}$ form a matroid, we may concentrate on the case where $\mathbf{a}$ and all $\mathbf{a}_i$ have coefficient sum $|E| - \mathrm{rk}(\mathsf{M}^2_G)$.

If $\mathbf{a}$ has a zero entry for some edge $e$, then this must also be the case for all $\mathbf{a}_i$ with nonzero $\alpha_i$ in the linear combination. For such $\mathbf{a}_i$, the corresponding tree $T$ or 2-forest $F$ must contain $e$ (and $e \neq f$ in the tree case). Note that spanning trees and 2-forests of $G$ that contain a fixed edge $e$ are in bijection with the spanning trees and 2-forests of the graph $G_{/e}$ obtained from $G$ by contracting the edge $e$; the correspondence links the spanning tree (resp. 2-forest) $S \ni e$ of $G$ to the spanning tree (resp. 2-forest) $S \smallsetminus \{e\}$ of $G_{/e}$. Moreover, $F \ni e$ being momentous for $G$ is equivalent to $F \smallsetminus \{e\}$ being momentous for $G_{/e}$. It follows that we can replace $G$ by $G_{/e}$, and remove in $\mathbf{a}$ and each $\mathbf{a}_i$ the row corresponding to $e$. This turns the computation in a corresponding one about $G_{/e}$. By induction, the claim is already shown for $G_{/e}$, so the case of a zero entry in $\mathbf{a}$ follows.

We are left to deal with the case where no entry of $\mathbf{a}$ is 0. Note that the entry sum of each $\mathbf{a}_i$, and thus also of $\mathbf{a}$, is exactly $|E \smallsetminus T| + 1$ for any spanning tree $T$. But if all entries of $\mathbf{a}$ are equal to 1 or more, the entry sum must also be equal to $|E|$ or more. We are thus reduced to considering graphs with $|E \smallsetminus T| + 1 \geq |E|$, so that spanning trees must be of size 1 or 0. In the latter case, the graph has only loops and the proposition holds trivially. In the case $|T| = 1$, apart from possible isolated points that make no difference to our purposes, $G$ must be a banana graph.

Suppose $G$ is a banana graph with $n_m$ massive and $n_0$ massless edges. Let $e_1, \ldots, e_{n_m}$ be the massive edges, and suppose $\mathbf{a} = \sum \alpha_i \mathbf{a}_i$ with $\sum \alpha_i = 1$ has $\mu_e(\mathbf{a}) \geq 1$ for all $e \in E$. For each $\mathbf{a}_i$, the massless components of $\mathbf{a}_i$ add up to at most $n_0$ since for massless edges no second power can occur in any term of $\mathcal{G}_{m,p}$. On the other hand, the (supposedly nonzero) massless components of $\mathbf{a}$ add up to at least $n_0$. Hence, every massless entry of $\mathbf{a}$ and of each $\mathbf{a}_i$ with nonzero $\alpha_i$ must be 1.

The computation of the massless coordinates above allows to reduce the question to the case of a banana graph with only massive edges. However, we already know the proposition to be true not just for massive banana trees but in fact for all graphs with only massive edges, by Theorem 4.3. □

## 5. Normality vs Cohen–Macaulayness, and Hypothesis 1.2

Let $A$ be an integer $(1+|E|) \times n$ matrix with $\mathbb{Z}A = \mathbb{Z} \times \mathbb{Z}^E$. The semigroup $\mathbb{N}A$ has an associated *saturation*, the semigroup $\widetilde{\mathbb{N}A}$ given by the points in $(\mathbb{Z}A) \cap (\mathbb{R}_{\geq 0}A)$. Since $\mathbb{N}A \subseteq \widetilde{\mathbb{N}A}$ and the latter is a semigroup, one can consider $\widetilde{\mathbb{N}A}$ as a module over $\mathbb{N}A$ by restricting the semigroup operation $\widetilde{\mathbb{N}A} \times \widetilde{\mathbb{N}A} \longrightarrow \widetilde{\mathbb{N}A}$ to $\mathbb{N}A \times \widetilde{\mathbb{N}A}$. The resulting semigroup quotient module $\widetilde{\mathbb{N}A}/\mathbb{N}A$ is a measure of the non-saturatedness of $\mathbb{N}A$.

On the level of associated semigroup rings, $\tilde{S}_A := \mathbb{K}[\widetilde{\mathbb{N}A}]$ is by Hochster's work [Hoc72] a normal Cohen–Macaulay domain, and $S_A := \mathbb{K}[\mathbb{N}A]$ is a subring of $\tilde{S}_A$ over which $\tilde{S}_A$ is a finite integral extension. The quotient $Q_A := \mathbb{K}[\widetilde{\mathbb{N}A}]/\mathbb{K}[\mathbb{N}A]$ is an $S_A$-module.

While $Q_A \neq 0$ is a clear indication that $\mathbb{N}A$ is not saturated, it can easily happen that $Q_A \neq 0$ but $S_A$ is Cohen–Macaulay.

**Example 5.1.** We consider here the massive bubble, whose underlying graph is the 2-banana graph given as the loopless graph with two vertices (both external) and two edges. The only 2-forest has no edge, and there are two 1-forests. So $\mathcal{U} = x_1 + x_2$ and $\tilde{\Sigma}_m = 1 + m_1^2 x_1 + m_2^2 x_2$. Because of momentum conservation, the two external momenta are opposite to one another, and if $|p_W|^2$ denotes the norm at either vertex after Wick rotation then $\mathcal{F}_0 = |p_W|^2 x_1 x_2$. So,

$$\begin{aligned} \mathcal{G}_{m,p} &= (x_1 + x_2) \cdot (1 + m_1^2 x_1 + m_2^2 x_2) + |p_W|^2 x_1 x_2 \\ &= x_1 + x_2 + m_1^2 x_1^2 + m_2^2 x_2^2 + (|p_W|^2 + m_1^2 + m_2^2) x_1 x_2. \end{aligned}$$

If $|p_W|^2 + m_1^2 + m_2^2 = 0$, then $\mathrm{Supp}(\mathcal{G}_{m,p}) = \{\binom{1}{0}, \binom{2}{0}, \binom{0}{1}, \binom{0}{2}\}$. The semigroup to the lifted support vectors is not saturated since on one hand we have the lattice equation

$$2\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \begin{pmatrix}1\\0\\2\end{pmatrix},$$

and so 2 times $\begin{pmatrix}1\\1\\1\end{pmatrix}$ belongs to the semigroup of $A_{\mathcal{G}}$, while on the other hand

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix}$$

belongs to the lattice spanned by $A_{\mathcal{G}}$. However, since the toric ideal is a hypersurface, it is automatically Cohen–Macaulay.

The semigroup quotient $Q_A$ consists here of the lattice points

$$\begin{pmatrix}1\\1\\1\end{pmatrix} + \mathbb{N}\begin{pmatrix}1\\2\\0\end{pmatrix} + \mathbb{N}\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$\diamondsuit$

There are certain conditions that $Q_A$ must satisfy for $S_A$ to have the chance of being Cohen–Macaulay. One of the easiest to describe concerns the dimension of the $S_A$-module $\tilde{S}_A/S_A$, or more precisely the dimensions of its associated primes. Fortunately, all technical algebraic details can be expressed in terms of the semigroup quotient $Q_A$. Note the following easy observation:

**Lemma 5.2.** *If $Q_A$ contains an element $\mathbf{a} + \mathbb{N}A$ such that the elements of $(\mathbf{a} + \mathbb{N}A) \smallsetminus \mathbb{N}A$ are contained in a union of (shifted) faces of cone $\mathbb{R}_{\geq 0}A$ of dimension $\dim(\mathbb{N}A) - 2$ or less, then the ring $S_A$ is not Cohen–Macaulay.*

*Proof.* If $\mathbb{Q}_A$ contains an element as described in the lemma, then $\tilde{S}_A/S_A$ has an associated prime of dimension less than $\dim(S_A) - 1$ and thus has depth less than $\dim(S_A) - 1$. By standard results on depth, this makes $\mathrm{depth}(S_A) = \dim(S_A)$ impossible. $\square$

In order to get a feeling, consider the following example.

**Example 5.3.** Let $G$ be the massive triple sunset graph on two vertices with three edges and no loop, assuming both vertices to be external. Then $\mathcal{U} = x_1x_2 + x_2x_3 + x_3x_1$, $\tilde{\Sigma}_m = 1 + m_1^2x_1 + m_2^2x_2 + m_3^2x_3$. The only 2-forests is the empty set, so $\mathcal{F}_0 = |p_W|^2x_1x_2x_3$, where $|p_W|$ is the norm of the momentum at either vertex after Wick rotation. One computes that in the massive case

$$A_{\mathcal{G}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 \end{pmatrix}$$

plus the lift $\mathbf{a}_0$ of the support vector of $\underbrace{(|p_W|^2 + m_1^2 + m_2^2 + m_3^2)}_{:=c_0} x_1x_2x_3$ if the coefficient of this term is nonzero.

Let $\mathbf{a}_1, \ldots, \mathbf{a}_9$ denote the columns of $A_{\mathcal{G}}$ that are displayed above. If $c_0$ is nonzero then the semigroup generated by $\mathrm{Supp}(\mathcal{G}_{m,p})$ is saturated by Theorem 4.3, while otherwise $Q_{A_{\mathcal{G}}}$ is generated by $\mathbf{a}_0$.

In any case, one has the identities $\mathbf{a}_0 + \mathbf{a}_1 = \mathbf{a}_3 + \mathbf{a}_4 \in \mathbb{N}A_{\mathcal{G}}$ and $\mathbf{a}_0 + \mathbf{a}_4 = \mathbf{a}_5 + \mathbf{a}_6 \in \mathbb{N}A_{\mathcal{G}}$. It follows from symmetry that $\mathbf{a}_0 + \mathbf{a}_i \in \mathbb{N}A_{\mathcal{G}}$ for $1 \le i \le 9$ and so $Q_{A_{\mathcal{G}}}$ is the singleton $\{\mathbf{a}_0\}$. Equivalently, the $S_A$-module $\tilde{S}_A/S_A$ is a 1-dimensional vector space in multi-degree $\beta = (1,1,1,1)$.

Application of the long Euler–Koszul homology functor from [MMW05] to the short exact sequence $S_A \longrightarrow \tilde{S}_A \longrightarrow \tilde{S}_A/S_A$ now implies that the GKZ-system attached to $A_{\mathcal{G}}$ with parameter $\beta$ has a larger solution space (namely, of dimension $v+9-1$) than all other GKZ-systems attached to $A_{\mathcal{G}}$ (whose rank is always the volume $v$ of the convex hull of $A_{\mathcal{G}}$). In particular, $S_{A_{\mathcal{G}}}$ is not Cohen–Macaulay.

An alternative way using commutative algebra is to observe that $\tilde{S}_{A_{\mathcal{G}}}/S_{A_{\mathcal{G}}}$ being a finite dimensional vector space (that is, a zero-dimensional module) means that as $S_{A_{\mathcal{G}}}$-module it must have depth zero, which then forces $S_{A_{\mathcal{G}}}$ to have depth one. But as the dimension of $S_{A_{\mathcal{G}}}$ is equal to the dimension of the lattice spanned by $A_{\mathcal{G}}$ (namely, 4), $S_{A_{\mathcal{G}}}$ is far from satisfying the equality $\dim(S_{A_{\mathcal{G}}}) = \operatorname{depth}(S_{A_{\mathcal{G}}})$ that determines Cohen–Macaulayness. ♢

In the light of this discussion it seems unlikely that there are significantly large classes of Feynman diagrams that violate Hypothesis 1.2.(2) and yet produce GKZ-systems that have the Cohen–Macaulayness property.

## 6. List of symbols

- $(G, m, p, V_{\mathrm{Ext}})$ a Feynman graph with edge set $E$, mass function $m\colon E \longrightarrow \mathbb{R}$, momentum function $p$, and external vertices $V_{\mathrm{Ext}}$.
- $E_m, E_0 \subseteq E$ the sets of massive and of massless edges.
- $\mathcal{T}_G^i$ the set of $i$-forests of $G$.
- $\mathsf{M}_G^i$ the matroid whose bases are the $i$-forests of $G$.
- $\mathsf{M}^2_{G,\neq}$ the matroid whose bases are the momentous 2-forests of $G$.
- $\mathsf{M}^2_{G,\mathrm{m.t.}}$ the matroid whose bases are the massively truncated 2-forests of $G$.
- $\mathsf{M}^{2,\perp}_{G,\mathrm{Feyn}}$ the matroid whose bases label the square-free terms in $\mathcal{G}_{m,p}$.
- $\mathcal{U}$ the first Symanzik polynomial.
- $\mathcal{F}_0$ the sum over $\mathsf{M}^2_{G,\neq}$ weighted with their Wick rotated moments.
- $\tilde{\Sigma}_m = 1 + \Sigma_m = 1 + \sum m_e^2 x_e$.
- $\mathcal{G}_{m,p} = \tilde{\Sigma}_m \cdot \mathcal{U} + \mathcal{F}_0$ the (already Wick rotated) Feynman integrand to mass and momentum functions $m$ and $p$.
- $\tilde{\Sigma}_E = 1 + \Sigma_E = 1 + \Sigma_m + \sum_{m_e=0} x_e$.
- $\mathcal{G}_G = \mathcal{U} \cdot \tilde{\Sigma}_E$.
- $P_{m,p}$ the support polytope of $\mathcal{G}_{m,p}$.
- $A_{\mathcal{G}}$ a matrix whose columns are lifted support vectors of $\mathcal{G}_{m,p}$.
- $A_E$ a matrix whose columns are the lifted support vectors of $\mathcal{G}_G$.
- $P^{2,\perp}_{G,\neq}$ the support polytope of the matroid dual to $\mathsf{M}^2_{G,\neq}$.

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