Ann. Inst. H. Poincaré D
Comb. Phys. Interact. 9 (2022), 793–812
DOI 10.4171/AIHPD/154
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Configuration polynomials under contact equivalence

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Abstract. Configuration polynomials generalize the classical Kirchhoff polynomial defined by a graph. Their study sheds light on certain polynomials appearing in Feynman integrands. Contact equivalence provides a way to study the associated configuration hypersurface. In the contact equivalence class of any configuration polynomial we identify a polynomial with minimal number of variables; it is a configuration polynomial. This minimal number is bounded by $\binom{r+1}{2}$, where r is the rank of the underlying matroid. We show that the number of equivalence classes is finite exactly up to rank 3 and list explicit normal forms for these classes.

1. Introduction

The Matrix-Tree Theorem is a classical result in algebraic graph theory. It was found by German physicist Gustav Kirchhoff in the mid-19th century in the study of electrical circuits. It states that the number of spanning trees of a connected undirected graph G with edge set E agrees with any principal submaximal minor of its Laplacian. Putting weights on the edges $e \in E$ of G and considering them as variables x_e yields the *Kirchhoff polynomial*

$$\psi_G = \sum_{T \in \mathcal{T}_G} x^T,$$

where $\mathcal{T}_G$ is the set of all spanning trees of G, and $x^T = \prod_{e \in T} x_e$.

Kirchhoff polynomials are a crucial ingredient of the theory of Feynman integrals (see, for example, [1, 4, 9, 10] and the literature trees in these works). In short, the Kirchhoff polynomial of a graph appears in the denominator of the Feynman integral attached to the particle scattering encoded by the dual graph via Feynman's rule. In certain cases, the integrand is just a power of the Kirchhoff polynomial, but in general there is also another component, a *second Symanzik polynomial*. In this way, singularities of Kirchhoff polynomials influence the behavior of the corresponding Feynman integral.

2020 Mathematics Subject Classification. Primary 14N20; Secondary 05C31, 14M12, 81Q30.
Keywords. Configuration, matroid, contact equivalence, Feynman, Kirchhoff, Symanzik.

Considered as functions over $\mathbb{K} = \mathbb{C}$, Kirchhoff polynomials are never zero if all variables take values in a common open half-plane (defined by positivity of a non-trivial $\mathbb{R}$-linear form). Because Kirchhoff polynomials are homogeneous, this property is independent of the choice of half-plane. For the right half-plane (with positive real part) it is referred to as the *(Hurwitz) half-plane property*; the upper half-plane (with positive imaginary part) defines the class of *stable polynomials*. Generalizing beyond graphs, any matroid $\mathtt{M}$ with set of bases $\mathcal{B}_{\mathtt{M}}$ defines a *matroid basis polynomial* $\psi_{\mathtt{M}} = \sum_{B \in \mathcal{B}_{\mathtt{M}}} x^B$. In this way, $\psi_G = \psi_{\mathtt{M}_G}$ depends only on the *graphic matroid* $\mathtt{M}_G$ on E with set of bases $\mathcal{T}_G$. Conditions for the half-plane property of $\psi_{\mathtt{M}}$ in terms of $\mathtt{M}$ were formulated by Choe, Oxley, Sokal and Wagner (see [11, p. 92]). They consider general polynomials $\psi_{\mathtt{M},\boldsymbol{a}} = \sum_{B \in \mathcal{B}_{\mathtt{M}}} a_B x^B$ with matroid support and arbitrary coefficients $\boldsymbol{a} = (a_B)_{B \in \mathcal{B}_{\mathtt{M}}}$. The question whether the half-plane property of $\psi_{\mathtt{M},\boldsymbol{a}}$ for some coefficients $\boldsymbol{a}$ descends to $\psi_{\mathtt{M}}$ is studied for example by Brändén and González D'León (see [7, Theorem 2.3]), while Amini and Brändén (see [2]) consider interactions of the half-plane property, representability and the Lax conjecture.

Recently, Brändén and Huh [8] introduced the class of *Lorentzian polynomials*, which are defined by induction over the degree using partial derivatives, starting from quadratic forms satisfying a signature condition. Stable polynomials are Lorentzian. These polynomials have interesting negative dependence properties and close relations with matroids. For example, if a multiaffine polynomial (that is, a polynomial supported on squarefree monomials) is Lorentzian, then it has the form $\psi_{\mathtt{M},\boldsymbol{a}}$ for some matroid $\mathtt{M}$ and positive coefficients $\boldsymbol{a}$.

By the Matrix-Tree Theorem, however, the coefficients 1 of the Kirchhoff polynomial arise in a particular way: Pick any orientation on G and let A be an incidence matrix with one row deleted. Then $\psi_G = \det(AXA^{\mathsf{T}})$, where X the diagonal matrix of variables x_e for all $e \in E$. In more intrinsic terms, this is the determinant of the generic diagonal bilinear form, restricted to the span $W_G \subseteq \mathbb{Z}^E$ of all incidence vectors. Bloch, Esnault and Kreimer took this point of view for any linear subspace $W \subseteq \mathbb{K}^E$ over a field $\mathbb{K}$ (see [5, 15]). With respect to the basis of $\mathbb{K}^E$, this is a linear realization of a matroid $\mathtt{M}$, or a *configuration*. The dimension $\dim W$ equals the rank of $\mathtt{M}$, which we refer to as the *rank* of the configuration W (see Definition 2.1). The generic diagonal bilinear form on $\mathbb{K}^E$ restricts to a *configuration form* Q_W on W. Its determinant $\psi_W = \det(Q_W)$ is the *configuration polynomial* associated with W, a homogeneous polynomial of degree $\dim W$ in variables x_e for all $e \in E$ (see Definitions 2.3 and 2.5). Configuration polynomials over $\mathbb{K} = \mathbb{C}$ are stable, by a result of Borcea and Brändén (see [6, Proposition 2.4]). Notably, the above mentioned second Symanzik polynomial is a configuration polynomial, but not a Kirchhoff polynomial.

The configuration point of view has recently led to new insights on the affine and projective hypersurfaces defined by Kirchhoff polynomials (see [12, 13]). At present, the understanding of all the details of the singularity structure, as well as a satisfactory

general treatment of Feynman integrals, is highly incomplete. There is some evidence that this is due to built-in complications coming from complexity issues (see [3]). A natural problem is then to determine to what extent the formula for a configuration hypersurface is the most efficient way to encode the geometry: given a configuration polynomial, can it be rewritten in fewer variables, and can this even be done via another configuration?

In this article, we elaborate on this idea by studying configurations through the lens of *(linear) contact equivalence* of their corresponding polynomials. This is the equivalence relation on polynomials induced by permitting coordinate changes on the source and target of the polynomial (see Definition 4.1). Polynomials in the same equivalence class define the same affine hypersurfaces, up to a product with an affine space. While this approach is very natural from a geometric point of view, forgetting the matroid structure under the equivalence makes it difficult to navigate, and provides certain surprises discussed below.

The main vehicle of our investigations is that any matrix representation of Q_W consists of *Hadamard products* $v \star w$ of vectors $v, w \in W$, defined with respect to a basis of $\mathbb{K}^E$ (see Notation 2.2). After some preliminary discussion in earlier sections, we focus in Section 5 on the problem of finding "small" representatives within the contact equivalence class of a given configuration. This requires us to look in detail at the structure of the higher *Hadamard powers* $W^{\star s}$ of W (see Section 3). While such Hadamard powers usually do not form chains with increasing s, they nonetheless have some monotonicity properties with regard to suitable restrictions to subsets of E (see Lemma 3.3). We use this to minimize the number of variables of configuration polynomials under contact equivalence (see Proposition 5.3). As a result, we obtain the following:

Theorem 1.1. *Let $W \subseteq \mathbb{K}^E$ be a configuration over a field $\mathbb{K}$ of characteristic $\operatorname{ch}\mathbb{K} = 0$, or $\operatorname{ch}\mathbb{K} > \dim W$. Let $\nu(W)$ be the minimal number of variables appearing in any polynomial contact equivalent to ψ_W. Then*

$$\nu_W \leq \binom{\dim W + 1}{2}.$$

This minimum is realized within the set of configuration polynomials: there is a configuration $W' \subseteq \mathbb{K}^{\nu(W)}$, constructed from W by a suitable matroid restriction, with ψ_W and $\psi_{W'}$ contact equivalent.

In Sections 6, 7 and 8, we then consider the classification problem of determining all contact equivalence classes for configurations W of a given rank, and we prove the following:

Theorem 1.2. *For configurations of rank up to* 3*, there are only finitely many contact equivalence classes. For each rank at least* 4*, there is an infinite family of pairwise inequivalent configurations over* $\mathbb{K} = \mathbb{Q}$.

More precisely, we identify for $\dim W \leq 3$ all contact equivalence classes and write down a normal form for each class (see Table 1). This list is made of all possible products of generic determinants in up to 6 variables together with

$$\det \begin{pmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & 0 \\ y_5 & 0 & y_3 \end{pmatrix} \quad \text{and} \quad \det \begin{pmatrix} y_1 & y_4 & y_4 + y_5 \\ y_4 & y_2 & y_5 \\ y_4 + y_5 & y_5 & y_3 \end{pmatrix}.$$

For $\dim W = 4$, already for $|E| = 6$ variables, we exhibit an infinite family of contact equivalence classes of configurations (see Proposition 8.2).

Our computations show that the contact equivalence class of a configuration neither determines nor is determined by the underlying matroid. Thus, one is prompted to wonder what characteristics of the graph/matroid of a Kirchhoff/configuration polynomial determine its complexity. We hope that our investigations here will help to shed light on this problem.

2. Configuration forms and polynomials

Let $\mathbb{K}$ be a field. We denote the dual of a $\mathbb{K}$-vector space W by

$$W^\vee := \mathrm{Hom}_{\mathbb{K}}(W, \mathbb{K}).$$

Let E be a finite set. Whenever convenient, we order E and identify

$$E = \{e_1, \ldots, e_n\} = \{1, \ldots, n\}.$$

We identify E with the canonical basis of the based $\mathbb{K}$-vector space

$$\mathbb{K}^E := \bigoplus_{e \in E} \mathbb{K} \cdot e.$$

We denote by $E^\vee = (e^\vee)_{e \in E}$ the dual basis of

$$(\mathbb{K}^E)^\vee = \mathbb{K}^{E^\vee}.$$

We write $x_e := e^\vee$ to emphasize that $x := (x_e)_{e \in E}$ is a coordinate system on $\mathbb{K}^E$. For $F \subseteq E$ we denote by

$$x^F := \prod_{f \in F} x_f$$

the corresponding monomial. For $w \in \mathbb{K}^E$ and $e \in E$, denote by $w_e := e^\vee(w)$ the e-component of w.

Definition 2.1. Let E be a finite set. A *configuration* over $\mathbb{K}$ is a $\mathbb{K}$-vector space $W \subseteq \mathbb{K}^E$. It gives rise to an *associated matroid* $\mathsf{M} = \mathsf{M}_W$ with rank function $S \mapsto \dim_{\mathbb{K}} \langle S^\vee|_W \rangle$ and set of bases $\mathcal{B}_{\mathsf{M}}$. We refer to its rank

$$r_W := \dim_{\mathbb{K}} W$$

as the *rank* of the configuration. *Equivalent* configurations obtained by rescaling E or by applying a field automorphism have the same associated matroid.

Notation 2.2. We denote the *Hadamard product* of $u, v \in \mathbb{K}^E$ by

$$u \star v := \sum_{e \in E} u_e \cdot v_e \cdot e \in \mathbb{K}^E.$$

We suppress the dependency on E in this notation. We abbreviate

$$u^{\star s} := \underbrace{u \star \cdots \star u}_{s}.$$

Definition 2.3 ([13, Remark 3.21, Definition 3.20], [14, §2.2]). Denote by $\mu_{\mathbb{K}}$ the multiplication map of $\mathbb{K}$. Let $W \subseteq \mathbb{K}^E$ be a configuration of rank $r = r_W$. The associated *configuration form* is

$$Q_W = \sum_{e \in E} x_e \cdot \mu_{\mathbb{K}} \circ (e^\vee \times e^\vee) \colon W \times W \to \langle x \rangle_{\mathbb{K}}.$$

A choice of (ordered) basis $w = (w^1, \ldots, w^r)$ of $W \subseteq \mathbb{K}^E$ together with an ordering of E is equivalent to the choice of a *configuration matrix* $A = (w^i_j)_{i,j} \in \mathbb{K}^{r \times n}$ with row span $\langle A \rangle$ equal to W. With respect to these choices, Q_W is represented by the $r \times r$ matrix

$$Q_w := Q_A := (\langle x, w^i \star w^j \rangle)_{i,j} = \Big(\sum_{e \in E} x_e \cdot w^i_e \cdot w^j_e \Big)_{i,j}.$$

Different choices of bases w, w' and orderings (or, equivalently, of configuration matrices) yield conjugate matrix representatives for Q_W.

Judicious choices of the basis and the orderings lead to a *normalized* configuration matrix $A = (I_r | A')$, where I_r is the $r \times r$ unit matrix.

Remark 2.4. For fixed $e \in E$, $(w^i_e \cdot w^j_e)_{i \le j}$ is the image of $(w^i_e)_i$ under the second Veronese map $\mathbb{K}^r \to \mathbb{K}^{\binom{r}{2}}$. Thus, Q_w determines the vectors $(w^i_e)_i$ up to a common sign. In particular, Q_W determines the configuration W up to equivalence.

Definition 2.5 ([13, Definition 3.2, Remark 3.3 and Lemma 3.23]). Let $W \subseteq \mathbb{K}^E$ be a configuration. If A is a configuration matrix for W with corresponding basis w, then the associated *configuration polynomial* is defined by

$$\psi_W := \psi_w := \psi_A := \det(Q_A) \in \mathbb{K}[x].$$

It is determined by W up to a square factor in $\mathbb{K}^*$. One has the alternative description

$$\psi_A = \sum_{B \in \mathcal{B}_{\mathsf{M}}} \det(\mathbb{K}^B \xrightarrow{w} W \twoheadrightarrow \mathbb{K}^B)^2 \cdot x^B,$$

using the ordering corresponding to A on every basis $B \subseteq E$.

The *matroid (basis) polynomial*

$$\psi_{\mathsf{M}} = \sum_{B \in \mathcal{B}_{\mathsf{M}}} x^B \in \mathbb{Z}[x]$$

of $\mathsf{M} = \mathsf{M}_W$ has the same monomial support as ψ_W but the two can be significantly different (see [13, Example 5.2]).

Remark 2.6. If $G = (V, E)$ is a graph and $W \subseteq \mathbb{K}^E$ is the row span of the incidence matrix of G, then $\psi_W = \psi_G$ is the *Kirchhoff polynomial* of G (see [13, Proposition 3.16]).

3. Hadamard products of configurations

Let $W \subseteq \mathbb{K}^E$ be a configuration of rank

$$r = r_W = \dim_{\mathbb{K}} W \leq |E|.$$

For $s \in \mathbb{N}_{\geq 1}$, denote by

$$W^{\star s} := \underbrace{W \star \cdots \star W}_{s} := \langle w^1 \star \cdots \star w^s \mid w^1, \ldots, w^s \in W \rangle \subseteq \mathbb{K}^E$$

the s-fold Hadamard product of W and by

$$r_W^s := \dim_{\mathbb{K}} W^{\star s} \leq |E|$$

its dimension. Note that $r_W = r_W^1$. By multilinearity and symmetry of the Hadamard product, we have a surjection

$$\operatorname{Sym}^s_{\mathbb{K}} W \twoheadrightarrow W^{\star s}, \quad w^{i_1} \cdots w^{i_s} \mapsto w^{i_1} \star \cdots \star w^{i_s}.$$

In particular, for all $s, s' \in \mathbb{N}_{\geq 1}$, there is an estimate

$$r_W^s \leq \binom{r_W + s - 1}{s}, \tag{3.1}$$

and equations

$$(\mathbb{K}^E)^{\star s} = \mathbb{K}^E, \quad W^{\star s} \star W^{\star s'} = W^{\star(s+s')}.$$

Example 3.1. Consider the non-isomorphic rank 2 configurations in $\mathbb{K}^n$

$$W = \langle (1, \ldots, 1), (1, 2, 3, \ldots, n) \rangle, \quad W' = \langle (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0) \rangle.$$

Then $r_W^s = \min\{s, n\}$ follows from properties of Vandermonde determinants, whereas $r_{W'}^s = 2$.

Remark 3.2. Extending a configuration $W \subseteq \mathbb{K}^E$ by a direct summand $\mathbb{K}$ with basis f yields a new configuration $W' = W \oplus \mathbb{K}^{\{f\}} \subseteq \mathbb{K}^{E \sqcup \{f\}}$ with configuration matrix $A' = \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $r_{W'}^s = r_W^s + 1$ and $\psi_{W'} = \psi_W \cdot x_f$.

For $F \subseteq E$, denote by

$$\pi_F \colon \mathbb{K}^E \to \mathbb{K}^F$$

the corresponding $\mathbb{K}$-linear projection map. Abbreviate

$$w_F := \pi_F(w), \quad W_F := \pi_F(W).$$

By definition, $(w^1 \star \cdots \star w^s)_F = w_F^1 \star \cdots \star w_F^s$ and hence

$$(W^{\star s})_F = (W_F)^{\star s} =: W_F^{\star s}.$$

Lemma 3.3. *For every configuration $W \subseteq \mathbb{K}^E$ there is a filtration*

$$F_1 \subseteq \cdots \subseteq F_t \subseteq \cdots \subseteq E$$

on E such that, for all $s' \leq s$ in $\mathbb{N}_{\geq 1}$, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{K}^E & \xrightarrow{\pi_{F_s}} & \mathbb{K}^{F_s} \\ \cup\mathrm{I} & & \cup\mathrm{I} \\ W^{\star s'} & \xrightarrow[\cong]{} & W_{F_s}^{\star s'} \end{array} \tag{3.2}$$

in which the right-hand containment is an equality for $s' = s$. In particular, for $s' \leq s$,

$$r_W^{s'} \leq r_W^s. \tag{3.3}$$

Proof. Note that (3.3) is a direct consequence of (3.2) and the filtration property. We will construct the filtration inductively, starting with F_1. Let F_1 be any subset of E such that $r_{W_{F_1}} = |F_1|$ (in other words, a basis for the matroid $\mathtt{M}_W$ represented by W). Then (3.2) is clear.

Suppose that $F_1 \subseteq \cdots \subseteq F_t$ have been constructed, satisfying (3.2) whenever $s' \leq s \leq t$. We claim first that $W_{F_s}^{\star(t+1)} = \mathbb{K}^{F_s}$ for all $1 \leq s \leq t$. So, take a basis element $e \in F_s$. From the inductive hypothesis $W_{F_s}^{\star s} = \mathbb{K}^{F_s}$ we obtain a $v \in W^{\star s}$ such that $v_{F_s} = e$. By definition of $W^{\star s}$, there must be a $u \in W$ such that $u_e = 1$ as otherwise $W_e = 0$. But then $w := u^{\star(t+1-s)} \star v \in W^{\star(t+1)}$ satisfies $w_{F_s} = e$, so that $W_{F_s}^{\star(t+1)} = \mathbb{K}^{F_s}$ as claimed.

The just established equation $W_{F_t}^{\star(t+1)} = \mathbb{K}^{F_t}$ says that F_t is an independent set for the matroid associated to the configuration $W^{\star(t+1)} \subseteq \mathbb{K}^E$. Extend it to a basis F_{t+1}. Then (3.2) follows for $s' = s = t+1$ (including the equality of the right inclusion). On the other hand, for $s' \leq t$, the natural composite surjection

$$W^{\star s'} \twoheadrightarrow W_{F_{t+1}}^{\star s'} \twoheadrightarrow W_{F_t}^{\star s'}$$

is by the inductive hypothesis an isomorphism. Hence each of the two arrows in the display is an isomorphism as well, proving that (3.2) holds for $s' < s = t+1$. ∎

Definition 3.4. Let $W \subseteq \mathbb{K}^E$ be a configuration. By Lemma 3.3, there is a minimal index t_W such that $r_W^t = r_W^{t_W}$ for all $t \geq t_W$. We call t_W the *Hadamard exponent* and $r_W^{t_W}$ the *Hadamard dimension* of W.

4. Linear contact equivalence

Definition 4.1. We call two polynomials $\phi \in \mathbb{K}[x_1, \ldots, x_m]$ and $\psi \in \mathbb{K}[x_1, \ldots, x_n]$ *(linearly contact) equivalent* if for some $p \geq m, n$ there exists an $\ell \in \mathrm{GL}_p(\mathbb{K})$ and a $\lambda \in \mathbb{K}^*$ such that

$$\phi = \lambda \cdot \psi \circ \ell \tag{4.1}$$

in $\mathbb{K}[x_1, \ldots, x_p]$. We write $\phi \simeq \psi$ in this case.

Remarks 4.2.

(a) If $\mathbb{K}$ is a perfect field and ψ is homogeneous, then one can assume $\lambda = 1$ in (4.1) at the cost of scaling ℓ by $\lambda^{1/\deg(\psi)}$.

(b) By definition, both adding redundant variables and permuting variables yield equivalent polynomials. In particular, enumerating E and considering $E \subseteq \{1, \ldots, p\}$ as a subset for any $p \geq |E|$ gives sense to equivalence of configuration polynomials ψ_W.

Notation 4.3. For a fixed field $\mathbb{K}$, we set

$$\Psi := \{\psi_W \mid E \text{ finite set}, W \subseteq \mathbb{K}^E\}.$$

We aim to understand linear contact equivalence on Ψ.

5. Reduction of variables modulo equivalence

Lemma 5.1. *Let $W \subseteq \mathbb{K}^E$ be a configuration. Then there is a subset $F \subseteq E$ of size $|F| = r^2_{W_F} = r^2_W$ such that $\psi_W \simeq \psi_{W_F}$.*

Proof. Lemma 3.3 with $t = 2$ yields a subset $F \subseteq E$ such that

$$\pi_F|_W\colon W \xrightarrow{\cong} W_F \quad \text{and} \quad \pi_F|_{W^{\star 2}}\colon W^{\star 2} \xrightarrow{\cong} W_F^{\star 2} = \mathbb{K}^F. \tag{5.1}$$

Let ι_F be the section of π_F that factors through the inverse of $\pi_F|_{W^{\star 2}}$,

$$\iota_F\colon \mathbb{K}^F \xrightarrow{(\pi_F|_{W^{\star 2}})^{-1}} W^{\star 2} \hookrightarrow \mathbb{K}^E. \tag{5.2}$$

Consider the $\mathbb{K}$-linear isomorphism of based vector spaces

$$q\colon \mathbb{K}^E \to \mathbb{K}^{E^\vee}, \quad w \mapsto \sum_{e\in E} w_e \cdot x_e$$

inducing the configuration $q(W) \subseteq \mathbb{K}^{E^\vee}$. Set $F^\vee := q(F)$ and $\iota_{F^\vee} := q \circ \iota_F \circ q^{-1}$. Then $\pi_{F^\vee} = q \circ \pi_F \circ q^{-1}$, and (5.1) and (5.2) persist if F is replaced by $F^\vee$ and W by $q(W)$ throughout.

Now choose a basis $w = (w^1, \dots, w^r)$ of W. Then $w_F = (w^1_F, \dots, w^r_F)$ is a basis of W_F by (5.1) and

$$\begin{aligned} Q_W &= (q(w^i \star w^j))_{i,j} = (q(w^i) \star q(w^j))_{i,j} \\ &\overset{(5.2)}{=} \big(\iota_{F^\vee} \circ \pi_{F^\vee}(q(w^i) \star q(w^j))\big)_{i,j} \\ &= \iota_{F^\vee}(q(w^i)_{F^\vee} \star q(w^j)_{F^\vee})_{i,j} = \iota_{F^\vee}(q(w^i_F) \star q(w^j_F))_{i,j} \\ &= \iota_{F^\vee}(q(w^i_F \star w^j_F))_{i,j} = \iota_{F^\vee} Q_{W_F}. \end{aligned}$$

Since $\iota_{F^\vee}$ is a section of $\pi_{F^\vee}$, $\psi_W \simeq \psi_{W_F}$ by taking determinants. ∎

Lemma 5.2. *Let $W \subseteq \mathbb{K}^E$ be a configuration. Suppose that $\operatorname{ch}\mathbb{K} = 0$, or $\operatorname{ch}\mathbb{K} > r_W$. If $\psi_W \simeq \phi \in \mathbb{K}[y_1, \dots, y_{n-1}]$, where $n := |E|$, then $\psi_W \simeq \psi_{W_{E\setminus\{e\}}}$ for some $e \in E$.*

Proof. Let $\ell \in \mathrm{GL}_p(\mathbb{K})$ and $\lambda \in \mathbb{K}^*$ realize the equivalence $\phi \simeq \psi_W$, that is, $\phi = \lambda \cdot \psi_W \circ \ell$, where $E \subseteq \{1, \ldots, p\}$ (see Remark 4.2 (b)). Consider the $\mathbb{K}$-linearly independent $\mathbb{K}$-linear derivations of $\mathbb{K}[x_1, \ldots, x_p]$

$$\delta_i := \ell_*\Big(\frac{\partial}{\partial y_{n-1+i}}\Big) = \frac{\partial}{\partial y_{n-1+i}}(- \circ \ell) \circ \ell^{-1}, \quad i = 1, \ldots, p-n+1.$$

Since ϕ is independent of $y_n, \ldots, y_p$, we have

$$\delta_i(\psi_W) = \lambda^{-1} \cdot \frac{\partial \phi}{\partial y_{n-1+i}} \circ \ell^{-1} = 0, \quad i = 1, \ldots, p-n+1. \tag{5.3}$$

By suitably reordering $\{1, \ldots, p\}$, we may assume that the matrix $(\delta_i(x_j))$ with $1 \leq i, j \leq p-n+1$ is invertible. After replacing the δ_i by suitable linear combinations, we may further assume that $\delta_i(x_j) = \delta_{i,j}$ for all $i, j \in \{1, \ldots, p-n+1\}$. Then

$$\begin{aligned} x_i &= x_i', \quad i = 1, \ldots, p-n+1, \\ x_i &= x_i' + \sum_{j=1}^{p-n+1} \delta_j(x_i) \cdot x_j', \quad i = p-n+2, \ldots, p, \end{aligned}$$

defines a coordinate change such that

$$\delta_j = \sum_{i=1}^{p} \delta_j(x_i) \frac{\partial}{\partial x_i} = \sum_{i=1}^{p} \frac{\partial x_i}{\partial x_j'} \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_j'}, \quad j = 1, \ldots, p-n+1. \tag{5.4}$$

If $\operatorname{ch} \mathbb{K} > 0$, then $\operatorname{ch} \mathbb{K} > r_W = \deg(\psi_W)$ by hypothesis. By (5.3) and (5.4), ψ_W is thus independent of $x_1', \ldots, x_{p-n+1}'$. Setting $x_i = x_i' = 0$ for $i = 1, \ldots, p-n+1$ thus leaves ψ_W unchanged and makes $x_i = x_i'$ for $i = p-n+2, \ldots, p$. It follows that

$$\psi_W \simeq \psi_W|_{x_1'=\cdots=x_{p-n+1}'=0} = \psi_W|_{x_1=\cdots=x_{p-n+1}=0} = \psi_{W_{E \setminus \{1,\ldots,p-n+1\}}}.$$

Then any $e \in E \cap \{1, \ldots, p-n+1\}$ satisfies the claim. ∎

Proposition 5.3. *Let $W \subseteq \mathbb{K}^E$ be a configuration. Then there is a subset $F \subseteq E$ of size $|F| = r_{W_F}^2 \leq r_W^2$ such that $\psi_W \simeq \psi_{W_F}$. Suppose that* $\operatorname{ch} \mathbb{K} = 0$, *or* $\operatorname{ch} \mathbb{K} > r_W$. *Then any polynomial $\phi \simeq \psi_{W_F}$ depends on at least $|F|$ variables. In other words, among the polynomials equivalent to ψ_W with minimal number of variables is the configuration polynomial ψ_{W_F}.*

Proof. By Lemma 5.1, there is a subset $G \subseteq E$ such that $|G| = r_{W_G}^2 = r_W^2$ and $\psi_W \simeq \psi_{W_G}$. Note that $|G| = r_{W_G}^2$ means $W_G^{\star 2} = \mathbb{K}^G$ which for any subset $F \subseteq G$ implies that $W_F^{\star 2} = \mathbb{K}^F$ and hence $|F| = r_{W_F}^2 \leq r_W^2$. Pick such an F with $\psi_{W_F} \simeq \psi_{W_G}$ minimizing $|F|$. Note that $r_{W_F} \leq r_W$. By Lemma 5.2 applied to the configuration

$W_F \subseteq \mathbb{K}^F$, any $\phi \simeq \psi_{W_F}$ depending on fewer than $|F|$ variables yields an $e \in F$ such that $\psi_{W_F} \simeq \psi_{W_{F\setminus\{e\}}}$, contradicting the minimality of F. ∎

Remark 5.4. By Remark 2.4, Q_W determines r^2_W. By definition, (the equivalence class of) ψ_W determines $r^1_W = r_W = \deg \psi_W$. We do not know whether it also determines r^2_W.

6. Extremal cases of equivalence classes

Notation 6.1. For $r, d \in \mathbb{N}$, set

$$\Psi^d_r = \{\psi_W \mid E \text{ finite set}, \ W \subseteq \mathbb{K}^E, \ r_W = r, \ r^2_W = d\}.$$

Lemma 6.2. *Let $W \subseteq \mathbb{K}^E$ be a configuration of rank r with basis $(w^1, \dots, w^r)$. Let G be the graph on the vertices $v_1, \dots, v_r$ in which $\{v_i, v_j\}$ is an edge if and only if $w^i \star w^j \neq 0$. Let G^* be the cone graph over G. If $\{w^i \star w^j \mid i \leq j, w^i \star w^j \neq 0\}$ is linearly independent, then*

$$\psi_W \simeq \psi_{G^*}$$

is the Kirchhoff polynomial of G^*.

Proof. See [3, Theorem 3.2] and its proof. ∎

Proposition 6.3. *If $d = r$, then every element of Ψ^d_r is equivalent to $x_1 \cdots x_r$. If $d = \binom{r+1}{2}$, then every element of Ψ^d_r is equivalent to the elementary symmetric polynomial of degree r in the variables $x_1, \dots, x_d$.*

Proof. Let $W \subseteq \mathbb{K}^E$ be a configuration.

First, suppose that $r^2_W = r_W$. By Lemma 5.1, we may assume that $|E| = r^2_W$. Then $W = \mathbb{K}^E$ and hence $\psi_W = x^E$ is the matroid polynomial of the free matroid on r_W elements.

Now suppose that $r^2_W = \binom{r_W+1}{2}$. Then $\{w^i \star w^j \mid 1 \leq i \leq j \leq r\}$ is linearly independent for any basis $(w^1, \dots, w^r)$ of W. By Lemma 6.2, ψ_W is then equivalent to the Kirchhoff polynomial of the complete graph on $r_W + 1$ vertices. ∎

7. Finite number of classes for small rank matroids

The purpose of this section is to give a complete classification of configuration polynomials for matroids of rank at most 3 with respect to the equivalence relation of Definition 4.1. Due to Proposition 5.3, we may assume that $|E| = r^2_W$.

Definition 7.1 ([14, §2.2]). A choice of basis $(w^1, \dots, w^r)$ of $W \subseteq \mathbb{K}^E$ and order of E gives rise to a *configuration matrix* $A = (w^i_j)_{i,j} \in \mathbb{K}^{r\times n}$, whose row span recovers $W = \langle A\rangle$. Up to reordering E it can be assumed in *normalized* form $A = (I_r|A')$, where I_r is the $r \times r$ unit matrix.

Proposition 7.2. *Let W be a configuration of rank 2. If $r^2_W = 2$, then $\psi_W \simeq x_1x_2$, otherwise, $r^2_W = 3$ and $\psi_W \simeq x_1x_2 - x_3^2$.*

Proof. Most of this follows from the proof of Proposition 6.3. Apply $x_1 \mapsto x_1 + x_2$ to the Kirchhoff polynomial $x_1x_2 + x_2x_3 + x_3x_1$ of K_3; the result is $x_1^2 + x_1(x_2 + 2x_3) + x_2x_3$.

If $\operatorname{ch}\mathbb{K} = 2$, then this is $x_1^2 + x_2(x_1 + x_3)$. If $2 \in \mathbb{K}$ is a unit, complete the square and scale x_2 by 2 to arrive at $x_1^2 - x_2^2 + x_3^2$. In both cases the result is easily seen to be equivalent to $x_1x_2 - x_3^2$. ∎

Proposition 7.3. *The numbers of equivalence classes for rank 3 configurations W for different values of r^2_W are*

$$|\Psi^3_3/_\simeq| = 1, \quad |\Psi^4_3/_\simeq| = 2, \quad |\Psi^5_3/_\simeq| = 2, \quad |\Psi^6_3/_\simeq| = 1.$$

Table 1 lists the equivalence classes of ψ_W that arise from normalized configuration matrices A when $r_W = 3$ and $r^2_W = |E|$.

$\lvert E\rvert = r^2_W$	A	Conditions	$\psi_W \simeq \det(-)$
3	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	None	$\begin{pmatrix}y_1&0&0\\0&y_2&0\\0&0&y_3\end{pmatrix}$
4	$\begin{pmatrix}1&0&0&a_1\\0&1&0&a_2\\0&0&1&a_3\end{pmatrix}$	$a_i = 0$ for exactly one i	$\begin{pmatrix}y_1&y_4&0\\y_4&y_2&0\\0&0&y_3\end{pmatrix}$
	$\begin{pmatrix}1&0&0&a_1\\0&1&0&a_2\\0&0&1&a_3\end{pmatrix}$	$a_i \neq 0$ for all i	$\begin{pmatrix}y_1&y_4&y_4\\y_4&y_2&y_4\\y_4&y_4&y_3\end{pmatrix}$
5	$\begin{pmatrix}1&0&0&a_{1,1}&a_{1,2}\\0&1&0&a_{2,1}&a_{2,2}\\0&0&1&a_{3,1}&a_{3,2}\end{pmatrix}$	Exactly one pair of $\binom{a_{i,1}\cdot a_{j,1}}{a_{i,2}\cdot a_{j,2}}$, $i \neq j$, is linearly dependent	$\begin{pmatrix}y_1&y_4&y_5\\y_4&y_2&0\\y_5&0&y_3\end{pmatrix}$
	$\begin{pmatrix}1&0&0&a_{1,1}&a_{1,2}\\0&1&0&a_{2,1}&a_{2,2}\\0&0&1&a_{3,1}&a_{3,2}\end{pmatrix}$	All pairs of $\binom{a_{i,1}\cdot a_{j,1}}{a_{i,2}\cdot a_{j,2}}$, $i \neq j$, are linearly independent	$\begin{pmatrix}y_1&y_4&y_4+y_5\\y_4&y_2&y_5\\y_4+y_5&y_5&y_3\end{pmatrix}$
6	$\begin{pmatrix}1&0&0&a_{1,1}&a_{1,2}&a_{1,3}\\0&1&0&a_{2,1}&a_{2,2}&a_{2,3}\\0&0&1&a_{3,1}&a_{3,2}&a_{3,3}\end{pmatrix}$	None	$\begin{pmatrix}y_1&y_4&y_6\\y_4&y_2&y_5\\y_6&y_5&y_3\end{pmatrix}$

Table 1. Equivalence classes for rank $r_W = 3$ configurations.

Proof. Let $W \subseteq \mathbb{K}^E$ be a configuration of rank $r_W = 3$ with normalized configuration matrix A. By (3.1) and Lemma 5.1, we may assume that

$$3 = r_W \leq r_W^2 = |E| \leq \binom{r_W + 1}{2} = 6.$$

The cases where $r_W^2 \in \{3, 6\}$ are covered by Proposition 6.3.

Suppose now that $r_W^2 = 4$. Up to reordering rows and columns, A then has the form

$$A = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \end{pmatrix}, \quad a_1, a_2, a_3 \in \mathbb{K}, \quad a_1 a_2 \neq 0,$$

and hence

$$Q_A = \begin{pmatrix} x_1 + a_1^2 x_4 & a_1 a_2 x_4 & a_1 a_3 x_4 \\ a_1 a_2 x_4 & x_2 + a_2^2 x_4 & a_2 a_3 x_4 \\ a_1 a_3 x_4 & a_2 a_3 x_4 & x_3 + a_3^2 x_4 \end{pmatrix}.$$

If $a_3 = 0$, then we can write, in terms of suitable coordinates y_1, y_2, y_3, y_4,

$$Q_A = \begin{pmatrix} y_1 & y_4 & 0 \\ y_4 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix}, \quad \psi_A = \det(Q_A) = (y_1 y_2 - y_4^2) y_3. \tag{7.1}$$

On the other hand, if $a_3 \neq 0$, then we can write

$$Q_{\lambda,\mu} := Q_A = \begin{pmatrix} y_1 & y_4 & \mu y_4 \\ y_4 & y_2 & \lambda y_4 \\ \mu y_4 & \lambda y_4 & y_3 \end{pmatrix}, \quad \lambda := \frac{a_3}{a_1}, \quad \mu := \frac{a_3}{a_2}.$$

Applying the coordinate change $(y_1, y_2, y_3, y_4) \mapsto (\frac{y_1}{\lambda^2}, \frac{y_2}{\mu^2}, y_3, \frac{y_4}{\lambda\mu})$ yields

$$Q'_{\lambda,\mu} := \begin{pmatrix} \frac{y_1}{\lambda^2} & \frac{y_4}{\lambda\mu} & \frac{y_4}{\lambda} \\ \frac{y_4}{\lambda\mu} & \frac{y_2}{\mu^2} & \frac{y_4}{\mu} \\ \frac{y_4}{\lambda} & \frac{y_4}{\mu} & y_3 \end{pmatrix},$$

and hence by extracting factors from the first and second row and column

$$\det(Q_{\lambda,\mu}) \simeq \lambda^2 \mu^2 \det(Q'_{\lambda,\mu}) = \det(Q_{1,1}).$$

In contrast to ψ_A in (7.1), this cubic is irreducible since $\mathtt{M}_W = U_{3,4}$ is connected (see [13, Theorem 4.16]). In particular, the cases $a_3 = 0$ and $a_3 \neq 0$ belong to different equivalence classes.

Suppose now that $r_W^2 = 5$. Then A has the form

$$A = \begin{pmatrix} 1 & 0 & 0 & a_{1,1} & a_{1,2} \\ 0 & 1 & 0 & a_{2,1} & a_{2,2} \\ 0 & 0 & 1 & a_{3,1} & a_{3,2} \end{pmatrix}.$$

First, suppose that, after suitably reordering the rows and columns of A, $w^1 \star w^2$ and $w^2 \star w^3$ are linearly dependent, and hence $w^1 \star w^2$ and $w^1 \star w^3$ are linearly independent. In terms of suitable coordinates $y_1, \ldots, y_5$, we can write

$$Q_\lambda := Q_A = \begin{pmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & \lambda y_4 \\ y_5 & \lambda y_4 & y_3 \end{pmatrix}, \quad \lambda \in \mathbb{K}.$$

By symmetric row and column operations,

$$\det(Q_\lambda) = \det \begin{pmatrix} y_1 & y_4 & y_5 - \lambda y_1 \\ y_4 & y_2 & 0 \\ y_5 - \lambda y_1 & 0 & y_3 - 2\lambda y_5 + \lambda^2 y_1 \end{pmatrix} \simeq \det(Q_0).$$

One computes that the ideal of submaximal minors of Q_0 equals

$$I_2(Q_0) = \langle y_1 y_2 - y_4^2, y_3, y_5 \rangle \cap \langle y_1 y_3 - y_5^2, y_2, y_4 \rangle. \tag{7.2}$$

Suppose now that all pairs of $w^i \star w^j$ with $i < j$ are linearly independent. In terms of suitable coordinates, $y_1, \ldots, y_5$, we can write

$$Q_{\lambda,\mu} = \begin{pmatrix} y_1 & y_4 & \lambda y_4 + \mu y_5 \\ y_4 & y_2 & y_5 \\ \lambda y_4 + \mu y_5 & y_5 & y_3 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{K}^*.$$

Applying the coordinate change

$$(y_1, y_2, y_3, y_4) \mapsto (\mu^2 y_1, y_2, \lambda^2 y_3, \mu y_4, \lambda y_5)$$

yields

$$Q'_{\lambda,\mu} = \begin{pmatrix} \mu^2 y_1 & \mu y_4 & \lambda\mu(y_4 + y_5) \\ \mu y_4 & y_2 & \lambda y_5 \\ \lambda\mu(y_4 + y_5) & \lambda y_5 & \lambda^2 y_3 \end{pmatrix},$$

and hence by extracting factors from the first and last row and column

$$\det(Q_{\lambda,\mu}) \simeq \frac{1}{\lambda^2\mu^2} \det(Q'_{\lambda,\mu}) = \det(Q_{1,1}).$$

The linear independence of all pairs of $w^i \star w^j$ with $i < j$ implies that $\mathsf{M}_W = U_{3,5}$ which is 3-connected (see [14, Table 8.1]). In contrast to $I_2(Q_0)$ in (7.2), $I_2(Q_{1,1})$ must be a prime ideal (see [13, Theorem 4.37]). In particular, the two cases with $r_W^2 = 5$ belong to different equivalence classes. ∎

8. Infinite number of classes for rank 4 matroids

For rank 4 configurations there are infinitely many equivalence classes of configuration polynomials. For simplicity, we prove this over the rationals, so in this section we assume $\mathbb{K} = \mathbb{Q}$.

Consider the family of normalized configuration matrices

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & a_1 & b_1 \\ 0 & 0 & 1 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & b_2 \end{pmatrix},$$

depending on parameters $a_1, a_2, b_1, b_2 \in \mathbb{Q}$, where $a_1 a_2 b_1 b_2 \neq 0$. We will see that it gives rise to an infinite family of polynomials

$$\psi_m := \det(Q_m), \quad Q_m := \begin{pmatrix} y_1 & y_5 + y_6 & y_5 & m y_6 \\ y_5 + y_6 & y_2 & y_5 & y_6 \\ y_5 & y_5 & y_3 & 0 \\ m y_6 & y_6 & 0 & y_4 \end{pmatrix}, \quad m := \frac{a_1}{b_1} \in \mathbb{Q},$$

which are pairwise inequivalent for $|m| > 1$.

Lemma 8.1. *With the above notation, we have $\psi_A \simeq \psi_m$.*

Proof. The configuration form associated to A is given by

$$Q_A = \begin{pmatrix} x_1 + x_5 + x_6 & a_1 x_5 + b_1 x_6 & a_2 x_5 & b_2 x_6 \\ a_1 x_5 + b_1 x_6 & x_2 + a_1^2 x_5 + b_1^2 x_6 & a_1 a_2 x_5 & b_1 b_2 x_6 \\ a_2 x_5 & a_1 a_2 x_5 & x_3 + a_2^2 x_5 & 0 \\ b_2 x_6 & b_1 b_2 x_6 & 0 & x_4 + b_2^2 x_6 \end{pmatrix}.$$

The coordinate changes

$$\begin{aligned} (z_1, \dots, z_6) &:= (x_1 + x_5 + x_6, x_2 + a_1^2 x_5 + b_1^2 x_6, x_3 + a_2^2 x_5, \\ &\qquad x_4 + b_2^2 x_6, a_1 x_5, b_1 x_6), \\ (y_1, \dots, y_6) &:= \left(z_1, \frac{z_2}{a_1^2}, \frac{z_3}{a_2^2}, \frac{z_4}{b_2^2}, \frac{z_5}{a_1}, \frac{z_6}{a_1} \right) \end{aligned}$$

turn Q_A into

$$Q_A = \begin{pmatrix} z_1 & z_5+z_6 & \frac{a_2}{a_1}z_5 & \frac{b_2}{b_1}z_6 \\ z_5+z_6 & z_2 & a_2z_5 & b_2z_6 \\ \frac{a_2}{a_1}z_5 & a_2z_5 & z_3 & 0 \\ \frac{b_2}{b_1}z_6 & b_2z_6 & 0 & z_4 \end{pmatrix}$$
$$= \begin{pmatrix} y_1 & a_1(y_5+y_6) & a_2y_5 & \frac{a_1b_2}{b_1}y_6 \\ a_1(y_5+y_6) & a_1^2y_2 & a_1a_2y_5 & a_1b_2y_6 \\ a_2y_5 & a_1a_2y_5 & a_2^2y_3 & 0 \\ \frac{a_1b_2}{b_1}y_6 & a_1b_2y_6 & 0 & b_2^2y_4 \end{pmatrix},$$

so that $\det(Q_A) = a_1^2a_2^2b_2^2\det(Q_m)$ by extracting factors from the last three rows and columns. ∎

Proposition 8.2. *For $m, m' \in \mathbb{Q}^*$, $\psi_m \simeq \psi_{m'}$ if and only if $m = m'$ or $mm' = 1$.*

Proof. By a SINGULAR computation, the primary decomposition of the ideal of submaximal minors of Q_m reads

$$I_2(Q_m) = P_{m,1} \cap P_{m,2} \cap P_{m,3},$$

where

$$\begin{aligned} P_{m,1} = \big\langle & y_1 + my_2 - (m+1)y_5 - (m+1)y_6, \\ & y_2y_4 - y_4y_5 - y_4y_6 + (m-1)y_6^2, \\ & my_2y_3 - y_3y_5 + (1-m)y_5^2 - y_3y_6 \big\rangle, \\ P_{m,2} = \big\langle & y_6, y_4, y_1y_2y_3 - y_5^2(y_1+y_2+y_3-2y_5) \big\rangle, \\ P_{m,3} = \big\langle & y_5, y_3, y_1y_2y_4 - y_6^2(y_1+m^2y_2+y_4-2my_6) \big\rangle. \end{aligned}$$

Fix $m, m' \in \mathbb{K}^*$ with $\psi_m \simeq \psi_{m'}$. Then there is an $\ell \in \mathrm{GL}_6(\mathbb{K})$ such that

$$\{\ell^*(P_{m,i}) \mid i \in \{1,2,3\}\} = \{\ell^*(P_{m',i}) \mid i \in \{1,2,3\}\}.$$

Let us assume first that

$$\ell^*(P_{m,1}) = P_{m',1}, \quad \ell^*(P_{m,2}) = P_{m',2}, \quad \ell^*(P_{m,3}) = P_{m',3}. \tag{8.1}$$

Then ℓ^* stabilizes the vector spaces $\langle y_3, y_5\rangle$ and $\langle y_4, y_6\rangle$ and hence

$$\begin{aligned} \ell^*(y_3) &= \ell_{3,3}y_3 + \ell_{3,5}y_5, \quad \ell^*(y_4) = \ell_{4,4}y_4 + \ell_{4,6}y_6, \\ \ell^*(y_5) &= \ell_{5,3}y_3 + \ell_{5,5}y_5, \quad \ell^*(y_6) = \ell_{6,4}y_4 + \ell_{6,6}y_6, \end{aligned}$$

with non-vanishing determinants

$$\ell_{1,1}\ell_{2,2} - \ell_{1,2}\ell_{2,1} \neq 0, \quad \ell_{3,3}\ell_{5,5} - \ell_{3,5}\ell_{5,3} \neq 0, \quad \ell_{4,4}\ell_{6,6} - \ell_{4,6}\ell_{6,4} \neq 0. \tag{8.2}$$

In degree 3 the second equality in (8.1) yields

$$\begin{aligned}(\ell_{3,3}y_3 + \ell_{3,5}y_5)\sum_{i=1}^{6}\ell_{1,i}y_i \sum_{j=1}^{6}\ell_{2,j}y_j - (\ell_{5,3}y_3 + \ell_{5,5}y_5)^2 \\ \times\left(\sum_{i=1}^{6}(\ell_{1,i} + \ell_{2,i})y_i + (\ell_{3,3} - 2\ell_{5,3})y_3 + (\ell_{3,5} - 2\ell_{5,5})y_5\right) \\ \equiv \lambda(y_1y_2y_3 - y_5^2(y_1 + y_2 + y_3 - 2y_5)) \mod \langle y_4, y_6\rangle, \quad \lambda \in \mathbb{K}^*. \end{aligned} \tag{8.3}$$

By comparing coefficients of $y_1y_2y_5$ in (8.3), we find $(\ell_{1,1}\ell_{2,2} + \ell_{1,2}\ell_{2,1})\ell_{3,5} = 0$, which forces $\ell_{3,5} = 0$ by (8.2). Comparing next the coefficients of the monomials

$$y_1^2, \quad y_2^2, \quad y_1y_5^2, \quad y_2y_5^2$$

in (8.3), we then obtain

$$\begin{aligned}\ell_{1,1}\ell_{2,1} = 0, \quad \ell_{1,2}\ell_{2,2} = 0, \\ -\ell_{5,5}^2(\ell_{1,1} + \ell_{2,1}) = -\lambda, \quad -\ell_{5,5}^2(\ell_{1,2} + \ell_{2,2}) = -\lambda,\end{aligned} \tag{8.4}$$

which yields

$$\ell_{1,1} + \ell_{2,1} = \ell_{1,2} + \ell_{2,2}. \tag{8.5}$$

In degree 1 the first equality in (8.1) yields

$$\begin{aligned}\sum_{i=1}^{6}(\ell_{1,i} + m\ell_{2,i})y_i - (m+1)(\ell_{5,3}y_3 + \ell_{5,5}y_5) - (m+1)(\ell_{6,4}y_4 + \ell_{6,6}y_6) \\ = \mu(y_1 + m'y_2 - (m'+1)y_5 - (m'+1)y_6).\end{aligned}$$

Comparing coefficients of y_1 and y_2, we find

$$\ell_{1,1} + m\ell_{2,1} = \mu, \quad \ell_{1,2} + m\ell_{2,2} = m'\mu. \tag{8.6}$$

By equation (8.4), $\ell_{1,i}$ or $\ell_{2,i}$ must be zero for $i = 1, 2$. Thus, we consider the following cases:

- If $\ell_{1,1} = \ell_{1,2} = 0$, then $\ell_{2,1} = \frac{\mu}{m}$ and $\ell_{2,2} = \frac{m'\mu}{m}$ by (8.6), hence $\frac{\mu}{m} = \frac{m'\mu}{m}$ by (8.5), so $m' = 1$.
- If $\ell_{1,1} = \ell_{2,2} = 0$, then $\ell_{2,1} = \frac{\mu}{m}$ and $\ell_{1,2} = m'\mu$ by (8.6), hence $\frac{\mu}{m} = m'\mu$ by (8.5), so $m' = \frac{1}{m}$.

- If $\ell_{2,1} = \ell_{1,2} = 0$, then $\ell_{1,1} = \mu$ and $\ell_{2,2} = \frac{m'\mu}{m}$ by (8.6), hence $\mu = \frac{m'\mu}{m}$ by (8.5), so $m' = m$.
- If $\ell_{2,1} = \ell_{2,2} = 0$, then $\ell_{1,1} = \mu$ and $\ell_{1,2} = m'\mu$ by (8.6), hence $\mu = m'\mu$ by (8.5), so $m' = 1$.

A similar discussion applies, with the same consequences, to the case where

$$\ell(P_{m,1}) = P_{m',1}, \quad \ell(P_{m,2}) = P_{m',3}, \quad \ell(P_{m,3}) = P_{m',2}.$$

In conclusion and by replacing ℓ with ℓ^{-1}, we find

$$m' \in \left\{1, m, \frac{1}{m}\right\}, \quad m \in \left\{1, m', \frac{1}{m'}\right\}.$$

Unless $m' = m$, we have $m' = \frac{1}{m} = \frac{b_1}{a_1}$. In terms of the coordinates from the proof of Lemma 8.1, we can write

$$\psi_A = a_2^2 b_2^2 \det \begin{pmatrix} z_1 & z_5 + z_6 & \frac{z_5}{a_1} & \frac{z_6}{b_1} \\ z_5 + z_6 & z_2 & z_5 & z_6 \\ \frac{z_5}{a_1} & z_5 & \frac{z_3}{a_2^2} & 0 \\ \frac{z_6}{b_1} & z_6 & 0 & \frac{z_4}{b_2^2} \end{pmatrix}$$

$$\simeq \det \begin{pmatrix} z_1 & z_5 + z_6 & \frac{z_5}{a_1} & \frac{z_6}{b_1} \\ z_5 + z_6 & z_2 & z_5 & z_6 \\ \frac{z_5}{a_1} & z_5 & z_3 & 0 \\ \frac{z_6}{b_1} & z_6 & 0 & z_4 \end{pmatrix}.$$

One can see that the morphism that leaves z_1, z_2 fixed, and interchanges the pairs $z_3 \leftrightarrow z_4$, $z_5 \leftrightarrow z_6$, $a_1 \leftrightarrow b_1$ transforms this final matrix into a conjugate matrix. However, by Lemma 8.1 the determinants of these two matrices are equivalent to ψ_m and $\psi_{1/m}$ respectively, where $m = \frac{a_1}{b_1}$. It follows that ψ_m and $\psi_{1/m}$ are equivalent. ∎

Corollary 8.3. *For every $k \in \mathbb{N}$, we have $|\Psi_{4+k}^{6+k}/\simeq| = \infty$ over $\mathbb{K} = \mathbb{Q}$.*

Proof. Applying the construction from Remark 3.2 yields configurations W with $r_W = 4 + k$ and $r_W^2 = 6 + k$ which give rise to the infinite family of polynomials $\psi_{m,k} = \psi_m \cdot y_7 \cdots y_{7+k}$, contact equivalent to elements of Ψ_{4+k}^{6+k}. For reasons of degree, $\psi_{m,k} \simeq \psi_{m',k}$ is equivalent to $\psi_m \simeq \psi_{m'}$, so the claim follows from Proposition 8.2. ∎

Acknowledgments. We gratefully acknowledge support by the Bernoulli Center at EPFL during a "Bernoulli Brainstorm" in February 2019, and by the Centro de Giorgi in Pisa during a "Research in Pairs" in February 2020. We also thank June Huh for helpful comments.

Funding. Graham Denham was supported by NSERC of Canada. Delphine Pol was supported by a Humboldt Research Fellowship for Postdoctoral Researchers. Uli Walther was supported in part by Simons Foundation Collaboration Grant for Mathematicians #580839 and NSF grant DMS-2100288.

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Communicated by Toshitake Kohno

Received 2 June 2020; revised 10 April 2021.

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