Complete Norm Preserving Extensions of Holomorphic Functions

Jim Agler   Łukasz Kosiński [1]   John E. McCarthy [2]

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1 Introduction

By a *Cartan pair* we mean a pair (Ω, V) where Ω is a connected pseudo-convex set in $\mathbb{C}^n$ and V is an analytic subvariety of Ω. The name is homage to H. Cartan, who proved that every holomorphic function on V (i.e. a function that locally agrees with the restriction of a holomorphic function defined on an open set in $\mathbb{C}^n$) extends to a holomorphic function on all of Ω [3]. We say that a pair (Ω, V) is a *norm preserving pair* (np pair for short) if it is a Cartan pair with the additional property that every bounded holomorphic function on V extends isometrically to a bounded holomorphic function on Ω.

For a fixed domain Ω, several papers have studied what analytic subvarieties gave rise to np pairs [2, 4, **?**, 7, 6]. If Ω is suitably nice, the conclusion of these papers was that V had to be a holomorphic retract of Ω for (Ω, V) to be an np pair. However, this is not true in general. The simplest example is the np pair (Δ, T), where Δ is the diamond $\{z \in \mathbb{C}^2 : |z_1|+|z_2| < 1\}$, and $T = (\mathbb{D} \times \{0\}) \cup (\{0\} \times \mathbb{D})$.

In [**?**], the perspective was shifted, to start with V and try to find a pseudoconvex set G so that (G, V) forms an np pair. We showed this can always be done:

Theorem 1.1. [**?**] If (Ω, V) is a Cartan pair, then their exists G such that (G, V) is an np pair.

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The first goal of this note is to extend Theorem 1.1 to the matrix and operator-valued case.

Definition 1.2. Let G be a domain of holomorphy, and V an analytic subvariety of G. We say (G,V) is a *complete np pair* if for every separable Hilbert space H and every bounded holomorphic function $f : V \to B(\mathrm{H})$ there is a bounded holomorphic extension $F : G \to B(\mathrm{H})$ such that $\|F\|_G = \|f\|_V$.

We are working in settings of separable Hilbert spaces. It seems this assumption is not required (use subnets to prove a Montel-type theorem).

Theorem 1.3. If (Ω,V) is a Cartan pair, and V is connected, then their exists G such that (G,V) is a complete np-pair.

We prove Theorem 1.3 in Section 3. Since any Stein manifold embeds properly as a submanifold into $\mathbb{C}^n$ for some n, the theorem carries over to the case when V is a subvariety of a Stein manifold. Notice that if V is not connected, the characteristic function of any component cannot be isometrically extended to any connected domain containing it, so the connectedness condition is necessary.

We do not know if whenever (G,V) is an np-pair it is always a complete np-pair. In Section 8 we study this question for a particular type of V, namely one that looks like two crossed discs.

Theorem 1.4. Let T be the union of two analytic disks, which intersect at one point a.

$$D_1 = \psi_1(\mathbb{D}), D_2 = \psi_2(\mathbb{D}), \mathrm{T} = D_1 \cup D_2, D_1 \cap D_2 = a = \psi_1(0) = \psi_2(0). \quad (1.5)$$

Let (G,T) be a Cartan pair. Then the following are equivalent:

(i) There is a map $\alpha : \mathbb{T}^2 \to H^\infty_1(G)$ so that

$$\begin{aligned} \alpha(\tau_1,\tau_2)(\psi_1(z)) &= \tau_1 z \\ \alpha(\tau_1,\tau_2)(\psi_2(z)) &= \tau_2 z. \end{aligned}$$

(ii) (G,T) is an np pair.

(iii) (G,T) is a complete np pair.

We shall let $H^\infty(V)$ denote the algebra of bounded holomorphic functions on V equipped with the supremum norm.

Definition 1.6. A Cartan pair (G,V) is said to be a *linear np pair* if there is a linear and isometric map $H^\infty(V) \to H^\infty(G)$. It is a *linear np pair vanishing at a* if there is a linear and isometric map from the subspace of $H^\infty(V)$ that vanishes at a to $H^\infty(G)$.

The linear extension property was first studied by W. Rudin [9]. There is a natural connection between the linear and complete extension properties. We show in Proposition 3.7 that if (G,V) is a linear np pair vanishing at some point a, then (G,V) is a complete np pair.

Proposition 3.7. Let (Ω,V) be a Cartan pair, $a \in V$, and assume that there is an isometric linear operator

$$E : S_a(V) \to S_a(\Omega).$$

Then (Ω,V) is a complete np-pair.

In [1] Agler, Lykova and Young studied the symmetrized bidisc

$$\mathrm{G}_2 = \{(z + w, zw) : z,w \in \mathrm{D}\}.$$

This is C-convex, though not convex, and there are np sets that are not retracts. More precisely they showed that all algebraic sets V in the symmetrized bidisc that have the norm preserving extension property are either retracts or are the union of two analytic discs of the form

$$\{(2\lambda,\lambda^2) : \lambda \in \mathrm{D}\} \cup \{(\beta + \beta\lambda,\lambda^{-} \quad) : \lambda \in \mathrm{D}\}, \tag{1.7}$$

where $\beta \in \mathrm{D}$. It follows from Theorem 1.4 that for algebraic sets in G_2, the np property and the complete np property are the same. However this cannot be deduced using a linear extension, as we shall show in Theorem 5.1 that if V is as in 1.7, there is no linear isometric extension operator of the functions vanishing at a point to all of G_2.

Theorem 5.1. Let T be given by (1.7), and let $a \in \mathrm{T}$. There is no linear isometric extension operator from $S_a(\mathrm{T})$ to $S(\mathrm{G}_2)$.

To here section 1

Problem C is to find, as explicitly as possible, a domain G so that (G,V) is a (complete) norm preserving pair.

It follows from Zorn's lemma and Proposition 6.1 that maximal norm preserving envelopes exist for V ; we shall call such a maximal set a *(complete) norm preserving hull*. In general, np hulls are not unique. The object of the third part is to show that model theory can produce np envelopes for analytic sets that are covered by domains for which there is a realization formula (Neil parabola, Newton's nodal cubic and other pet

2 Notation

If V is any set on which we can define holomorphic functions, we define the *Schur class* $S(V)$ to be the holomorphic functions from V to $\mathbb{D}$. If $\mathcal{H}$ is a Hilbert space, we let $S(V,B(\mathcal{H}))$ denote the holomorphic functions from V to $B(\mathcal{H})$ that are bounded by 1 in norm. Finally, if $a \in V$, we let $S_a(V)$ (resp. $S_a(V,B(\mathcal{H}))$) denote the Schur functions that vanish at a.

We define the map $\pi : \mathbb{D}^2 \to \mathbb{G}$ by

$$\pi(z_1,z_2) = (s,p) = (z_1 + z_2, z_1z_2). \tag{2.1}$$

Define Δ and T by

$$\Delta = \{z \in \mathbb{C}^2 : |z_1| + |z_2| < 1\} \tag{2.2}$$

$$T = \mathbb{D} \times \{0\} \cup \{0\} \times \mathbb{D}. \tag{2.3}$$

3 Complete np pairs

Throughout this section we shall assume that (Ω,V) is a Cartan pair and that V is connected.

It was proved by Bishop [**?**] and Fujimoto [**?**] that if (Ω,V) is a Cartan pair, then every $B(\mathcal{H})$-valued holomorphic function on V extends to a $B(\mathcal{H})$valued holomorphic function on Ω. With this tool in hand one could try to prove Theorem 1.3 by repeating the proof from one dimensional case. The main problem that appears here is that Montel's theorem fails for holomorphic functions with values in infinite dimensional vector spaces. There are topologies on B(H) for which a Montel-type theorem does hold and even such that Hol(V,B(H)) is paracompact, but then the projection

$$\mathrm{Hol}(\Omega,B(\mathrm{H})) \to \mathrm{Hol}(V,B(\mathrm{H}))$$

is not not open, so Michael's selection theorem cannot be used. To attack Problem A we shall establish a link between complete and linear norm preserving extensions.

Recall that $f : \Omega \to B(\mathrm{H})$ is holomorphic if and only if it is weakly holomorphic, i.e. $\Lambda(f)$ is a holomorphic function for any $\Lambda \in B(\mathrm{H})^0$. If f is locally bounded, a weaker condition needs to be verified for a function to be holomorphic:

Lemma 3.1. If G is open, and $f : G \to B(\mathrm{H})$ is locally bounded, then f is holomorphic if and only if z 7→ h$f(z)h,k$i is a holomorphic function.

Proof. This can be proved in a similar way to the standard argument proving that weakly holomorphic functions are holomorphic . Put in reference □

The Montel theorem fails in Hol(Ω,B(H)). However, it is true if we equip B(H) with the WOT topology:

Lemma 3.2. Let $(f_n) \subset S$ (Ω,B(H)). Then there is a subsequence (f_{n_k}) and $f \in S$ (Ω,B(H)) such that h$f_{n_k}(z)h,k$i converges to h$f(z)h,k$i locally uniformly on Ω for each $h,k \in$ H.

Proof. For an orthonormal basis e_i we apply the regular Montel theorem to h$f_n(z)e_i,e_j$i, and then using a Cantor diagonal argument we end up with f. It is elementary to see that f satisfies desired properties. □

For a final proof we need a few preparatory results more.

Lemma 3.3. Fix a point $a \in V$. Then (Ω,V) is completely norm preserving if and only if each $f \in S_a(V,B(\mathrm{H}))$ has an extension to an element $F \in S$ (Ω,B(H)).

Proof. If $||f(a)|| < 1$, there exists an automorphism m of ball[B(H)] such that $(m \circ f)(a) = 0$ [**?**]. As $h = m \circ f \in S$ (V,B(H)), the assumption of the lemma implies that there exists $H \in S$ (Ω,B(H)) such that $H|V = h$. But then if we define $F = m^{-1} \circ H$, $F \in S$ (Ω,B(H)) and $F|V = f$.

If $||f(a)|| = 1$, we approximate f uniformly with $f_n \in S(\Omega, B(H))$ such that $||f_n(a)|| < 1$ (eg. $f_n = \frac{n-1}{n} f$). It follows from the previous case that there are $F_n \in S(\Omega, B(H))$ that extend f_n. Applying Lemma 3.2 to F_n we find $F \in S(\Omega, B(H))$ that clearly extends f. □

Lemma 3.4. [**?**, Lem. 3.3] If (Ω, V) is a Cartan pair and $a \in V$, then $S_a(V)$ is a compact subset of $O(V)$.

The following result was proved in [**?**, Thm. 3.5].

Lemma 3.5. If $a \in V$, there is a continuous function $S : \mathrm{Hol}(V) \to \mathrm{Hol}(\Omega)$ such that $S(f)|V = f$ for $f \in O(V)$. Moreover, for each $a \in V$ there is an open $G \subset \Omega$ such that (G, V) is a Cartan pair and $S(S_a(V)) \subset S_a(G)$.

With this tool in hand we can prove the following linear extension result. Let $L^2_h(G)$ denote the weighted Bergman space obtained from using the Gaussian measure. (If G has finite volume, we could just use the standard Bergman space).

Shrinking G we can assume that its volume is finite.

Why? What if V is so wild that its polynomial hull is $\mathbb{C}^n$?

Lemma 3.6. Fix $a \in V$. Then there is a pseudoconvex domain D, $V \subset D \subset \Omega$, and a linear isomorphic extension map $S_a(V)$

$$\to S_a(D).$$

Proof. Let G and $S : \mathrm{Hol}(V) \to \mathrm{Hol}(G)$ be as in Lemma 3.5.

Then the inclusion $\iota : S_a(G) \subset L^2_h(G)$ is continuous; composing with S we get a continuous extension operator

$$\iota \circ S : S_a(V) \to L^2_h(G).$$

Let P be the orthogonal projection from $L^2_h(G)$ onto $\{g \in L^2_h(\Omega) : g|_V = 0\}^\perp$. Then

$$E(f) = P[\iota \circ S(f)]$$

is the element in $L^2_h(G)$ that extends f and has minimal norm. It is straightforward to see that it is linear. By the Cauchy formulas the inclusion

$L^2_h(G) \subset O(G)$ is continuous. Thus, we can construct a continuous and linear extension operator (which we will also call E)

$$E : S_a(V) \to O(G).$$

Define

$$D_1 := \left(\bigcap\{z \in G : |E(f)(z)| < 1 \ \forall f \in \mathscr{S}_a(V)\}\right)^{\circ}.$$

To see $V \subset D_1$, suppose $b \in V \setminus D_1$. Then there exist sequences $b_n \in G$ converging to b, and f_n in $S_a(V)$, such that $|E(f_n)(b_n)| \geq 1$. By Lemma 3.4, some subsequence of (f_n) converges to a function $f \in S_a(V)$. Since E is continuous, $|E(f)(b)| \geq 1$. This would violate the maximum principle.

Define D to be the connected component of D_1 that contains V. By [**?**, Prop. 4.1.7], D is pseudoconvex. □

Proposition 3.7. Let (Ω, V) be a Cartan pair, $a \in V$, and assume that there is an isometric linear operator

$$E : S_a(V) \to S_a(\Omega).$$

Then (Ω, V) is a complete np-pair.

Proof. It follows from Lemma 3.3 that it is enough to show that any mapping in $S_a(V, B(H))$ has an extension to $S(\Omega, B(H))$. So fix $f \in S_a(V, B(H))$ and $z \in \Omega$. Applying the Riesz reprezentation theorem to the maps

$$H \ni k \mapsto E(\langle f(\cdot)h, k\rangle)(z),$$

where $h \in H$, we get for each $z \in \Omega$ and $h \in H$, a vector $\Psi(z,h) \in H$ such that

$$E(\langle f(\cdot)h, k\rangle)(z) = \langle \Psi(z,h), k\rangle.$$

Note that $h \mapsto \Psi(z,h)$ is linear since E is, so we can define $F(z) : H \to H$ by $F(z)h = \Psi(z,h)$. It is straightforward to check that $F(z) \in B(H))$ and $||F(z)|| \leq 1$. Since $z \mapsto F(z)$ is holomorphic by Lemma 3.1, we are done. □

Combining Proposition 3.7 and Lemma 3.6, we have proved Theorem 1.3.

4 Two crossed discs

In this section we shall proof Theorem 1.4. When H is one dimensional the result was essentially proved in [**?**]. The key argument used there relied on the Herglotz representation theorem. To go to infinite dimensions, we shall use realization formulas.

Proof of theorem 1.4. The implications (iii) ⇒ (ii) ⇒ (i) are trivial. Let us show (i) ⇒ (iii).

Let $\phi \in S(\mathrm{T}, B(\mathrm{H}))$. By Lemma 3.3, we can assume that $\phi(a) = 0$. By using the newtwork realization formula ([**?**, Thm. 3.16]) for the functions

$$\lambda \mapsto \frac{\varphi(\psi_1(\lambda))}{\lambda} \text{ and } \lambda \mapsto \frac{\varphi(\psi_2(\lambda))}{\lambda},$$

we get Hilbert spaces K_1, K_2 and unitary operators $U_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$: $\mathrm{H} \oplus K_1 \to \mathrm{H} \oplus K_1$ and $U_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} : \mathcal{H} \oplus \mathcal{K}_2 \to \mathcal{H} \oplus \mathcal{K}_2$ such that

$$\phi(\psi_1(\lambda)) = A_1\lambda + B_1\lambda(I - D_1\lambda)^{-1}C_1\lambda$$

and

$$\phi(\psi_2(\lambda)) = A_2\lambda + B_2\lambda(I - D_2\lambda)^{-1}C_2\lambda.$$

Replacing U_1 with $\begin{pmatrix} A_1 & B_1 & 0 \\ C_1 & D_1 & 0 \\ 0 & 0 & I_{\mathcal{K}_2} \end{pmatrix}$ and U_2 with $\begin{pmatrix} A_2 & 0 & B_2 \\ 0 & I_{\mathcal{K}_1} & 0 \\ C_2 & 0 & D_2 \end{pmatrix}$, we have that

$$U_1, U_2 : \mathrm{H} \oplus \mathrm{K} \to \mathrm{H} \oplus \mathrm{K},$$

where $\mathrm{K} = K_1 \oplus K_2$.

Let $\mathrm{M} = \mathrm{H} \oplus \mathrm{K}$. Consider the following holomorphic map $\mathrm{T} \to B(\mathrm{M})$

$$(\dagger) \begin{cases} \psi_1(\lambda) \mapsto \lambda U_1 \\ \psi_2(\lambda) \mapsto \lambda U_2. \end{cases},$$

Claim. There is a sequence $\Phi_n \in S(G, B(M))$ that approximates (†) in the following sense: $\Phi_n(\psi_1(\lambda)) = \lambda U_1$ and $\Phi_n(\psi_2(\lambda)) = \lambda W_n$, where W_n are unitary and converge to U_2 in norm.

Observe that the Claim implies the assertion. Indeed, with respect to the decomposition $M = H \oplus K$, write

$$\Phi_n = \begin{pmatrix} \Phi_{1,n} & \Phi_{2,n} \\ \Phi_{3,n} & \Phi_{4,n} \end{pmatrix}.$$

Then

$$f_n(z) := \Phi_{1,n}(z) + \Phi_{2,n}(z)(I - \Phi_{4,n}(z))^{-1}\Phi_{3,n}(z)$$

is an extension from which we can take a subsequence converging to the extension we are looking for.

Proof of the claim. Without loss of generality we can assume that U_1 is the identity. If M is finite dimensional, we use the fact that the eigenvalues of U_2 are unimodular, and by hypothesis we can extend the function $\tau\lambda$ for any unimodular τ. In the infinite dimensional case, choose unitaries V_n that are diagonalizable and converge to U_2. Each $V_n = W_n D_n W_n^*$ where W_n is unitary and D_n is diagonal. For each diagonal entry τ_k, let g_k be the Schur function on G that extends the function $\psi_1(\lambda) \mapsto \lambda$ and $\psi_2(\lambda) \mapsto \tau_k\lambda$. Then $\Phi_n = W_n D_{g_k} W_n^*$ where D_{g_k} is the diagonal operator with entries g_k. □

5 Linear vs. complete

Consider two particular examples:

1. The diamond $\Delta = \{z \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$ and the two crossed discs $T := (\mathbb{D} \times \{0\}) \cup (\{0\} \times \mathbb{D})$.

2. The symmetrized bidisk $\mathbb{G}_2$ and the set $\mathrm{T} = \{(2\lambda, \lambda^2) : \lambda \in \mathbb{D}\} \cup \{(\beta + \beta\lambda, \bar\lambda) : \lambda \in \mathbb{D}\}$ from (1.7).

It follows from Theorem 1.4 that both (Δ, T) and $(\mathbb{G}_2, \mathrm{T})$ are complete nppairs.

Another way to prove this for (Δ,T) is to observe that the map that sends f in $S_0(T,B(H))$ to the function $\{z \mapsto f(z_1,0) + f(0,z_2)\}$ in $S(\Delta,B(H))$ is linear, and then apply Proposition 3.7. We shall show that this argument cannot be used for (G_2,T).

Let us introduce some additional notation before proving this. Let $\Sigma = \{(2\lambda,\lambda^2) : \lambda \in D\}$ and $D_0 = \{0\} \times D$. Let $[f(\lambda),g(\lambda)]$ denote the function on $\Sigma \cup D_0$ that is equal to $f(\lambda)$ on $(2\lambda,\lambda^2)$ and $g(\lambda)$ on $(0,\lambda)$. For $b \in D$ let m_b be a Möbius map $m_b(\lambda) = \frac{b-\lambda}{1-\bar{b}\lambda}$.

Theorem 5.1. Let T be given by (1.7), and let $a \in T$. There is no linear isometric extension operator from $S_a(T)$ to $S(G_2)$.

Proof. Since all sets of the form (1.7) are holomorphically equivalent, it suffices to prove the assertion for $T = \Sigma \cup D_0$.

For unimodular α and β consider the function $f_{\alpha,\beta} : T \to D$ given by the formula:

$$f_{\alpha,\beta}(s,p) = \begin{cases} \alpha s/2, & \text{on } \Sigma, \\ \beta p, & \text{on } D_0. \end{cases}$$

So $f_{\alpha,\beta} = [\alpha\lambda,\beta\lambda]$. Let $\omega = \beta\alpha^{-1}$. It was shown in [1] that $\alpha\Phi_\omega$ extends $f_{\alpha,\beta}$ where

$$\Phi_\omega(s,p) := \frac{s/2 + \omega p}{1 + \omega s/2}.$$

Claim. We shall show that $\alpha\Phi_{\beta\alpha^{-1}}$ is the unique np extension of $f_{\alpha,\beta}$ to $G_2 \to D$:

To prove the claim let F be some extension of $f_{\alpha,\beta}$ that has norm 1. Let $\omega \in T$. Then

$$F(s,p) = \alpha s/2 + \beta(p - (s/2)^2) + O(s(s^2 - 4p)),$$

since F minus the first two terms vanishes on T. With π as in (2.1), we get

$$F(\pi(\lambda,\omega\lambda)) = \alpha\frac{1+\omega}{2}\lambda - \beta\left(\frac{1-\omega}{2}\right)^2\lambda^2 + O(\lambda^3).$$

Then the Schwarz lemma implies that the map

$$\lambda \mapsto \frac{1}{\lambda} F(\pi(\lambda, \omega\lambda))$$

is a M¨obius map from D to D. Thus if G is another extension, $F \circ \pi$ and $G \circ \pi$ coincide on $\{|\lambda| = |\mu| : (\lambda,\mu) \in D^2\}$, and the claim follows.

Suppose that there is $a \in T$ and a linear isometric operator $L : S_a(T) \to S(G_2)$. Let us consider two cases.

i) $a = (0,\lambda_0) \in D_0$. Note that $\lambda \mapsto [m_{\beta a}(\alpha\lambda), m_{\beta a}(\beta\lambda)]$ belongs to the Schur class $S_a(T)$. The crucial fact following from the Claim is that the equality

$$L[m_{\beta\lambda_0}(\alpha\lambda), m_{\beta\lambda_0}(\beta\lambda)] = m_{\beta\lambda_0}(\alpha\Phi_{\beta/\alpha}) \tag{5.2}$$

holds for any $\alpha,\beta \in T$. Writing out (5.2), and using $\omega = \bar{\alpha}\beta$, we get

$$L\left[\frac{\beta\lambda_0 - \alpha\lambda}{1-\bar{\beta}\bar{\lambda}_0\alpha\lambda}, \frac{\beta\lambda_0 - \beta\lambda}{1-\lambda\bar{\lambda}_0}\right] = \frac{\beta\lambda_0 - \alpha\Phi_\omega(s,p)}{1-\bar{\beta}\bar{\lambda}_0\alpha\Phi_\omega(s,p)}.$$

Dividing by β we get

$$L\left[\frac{\lambda_0 - \lambda\bar{\omega}}{1-\lambda\bar{\lambda}_0\bar{\omega}}, \frac{\lambda_0 - \lambda}{1-\lambda\bar{\lambda}_0}\right] = \frac{\lambda_0 - \bar{\omega}\Phi_\omega(s,p)}{1-\bar{\lambda}_0\bar{\omega}\Phi_\omega(s,p)} \tag{5.3}$$

Write

$$\Phi_\omega(s,p) = \frac{s/2+\omega p}{1+\omega s/2} = \frac{\bar{\omega}s/2+p}{\bar{\omega}+s/2},$$

and expand both sides of (5.3) in powers of $\bar{\omega}$. Expanding the left hand side we get

$$\begin{aligned}\left[\frac{\lambda_0 - \lambda\bar{\omega}}{1-\lambda\bar{\lambda}_0\bar{\omega}}, \frac{\lambda_0 - \lambda}{1-\lambda\bar{\lambda}_0}\right] &= \sum_{n\geq 0} \bar{\omega}^n f_n = \\ &= \left[\lambda_0, \frac{\lambda_0-\lambda}{1-\lambda\bar{\lambda}_0}\right] + \bar{\omega}\left[(|\lambda_0|^2-1)\lambda, 0\right] + \sum_{n\geq 2}\bar{\omega}^n f_n,\end{aligned}$$

where $f_n \in H^\infty(T)$, $f_n(a) = 0$, and the series converges uniformly, so L can be applied term by term. The right hand side gives

$$\frac{\lambda_0 - \bar{\omega}\Phi_\omega(s,p)}{1-\bar{\lambda}_0\bar{\omega}\Phi_\omega(s,p)} = \lambda_0 - \bar{\omega}\frac{2p}{s}(1-|\lambda_0|^2) + O(\bar{\omega}^2).$$

Comparing the constant terms, we would have

$$L\left[\lambda_0, \frac{\lambda_0 - \lambda}{1 - \lambda\bar{\lambda}_0}\right] = \lambda_0,$$

a contradiction.

ii) We are left with the case $a = (2\lambda_0, \lambda_0^2) \in \Sigma,\ \lambda_0 \neq 0$. We shall proceed as before starting with a function $[m_{\alpha\lambda_0}(\alpha\lambda), m_{\alpha\lambda_0}(\beta\lambda)]$ that clearly lies in $S_a(\mathbb{T})$. As before, we get that

$$L\left[\frac{\lambda_0 - \lambda}{1 - \bar{\lambda}_0\lambda}, \frac{\lambda_0 - \omega\lambda}{1 - \bar{\lambda}_0\omega\lambda}\right] = \frac{\lambda_0 - \Phi_\omega(s,p)}{1 - \bar{\lambda}_0\Phi_\omega(s,p)}, \quad \omega \in \mathbb{T}. \tag{5.4}$$

Expanding in powers of ω and looking at the coefficient of ω, we get

$$L[0,\lambda] = \frac{p - (s/2)^2}{(1 - \bar{\lambda}_0 s/2)^2}. \tag{5.5}$$

As $[0,\lambda]$ lies in $S_a(\mathbb{T})$, we must have that the function $(s,p) \mapsto \frac{p-(s/2)^2}{(1-\bar{\lambda}_0 s/2)^2}$ sends the symmetrized bidisc to the unit disc. In particular, putting $(s,p) = (\lambda + \mu, \lambda\mu)$ for λ,μ in the unit disc we would get the inequality

$$|(\lambda - \mu)/2| \leq |1 - \bar{\lambda}_0(\lambda + \mu)/2| \tag{5.6}$$

holds for $(\lambda,\mu) \in \mathbb{D}^2$. This however is not possible whenever $\lambda_0 \neq 0$. Indeed, let $t = |\lambda_0|$. Then (5.6) is equivalent to the claim that

$$|\lambda - \mu|^2 \leq |2 - t(\lambda + \mu)|^2 \qquad \forall(\lambda,\mu) \in \overline{\mathbb{D}^2},$$

since by continuity the inequality would extend to the boundary. Assume both λ and μ are unimodular, then this becomes

$$-2(1 + t^2)\Re(\lambda\bar{\mu}) + 4t\Re(\lambda + \mu) \leq 2 + 2t^2.$$

Let $\lambda = e^{i\theta}$ and $\mu = e^{-i\theta}$. We get the inequality

$$-2(1 + t^2)\cos(2\theta) + 4t\cos(\theta) \leq 2 + 2t^2. \tag{5.7}$$

By calculus, the maximum of the left hand side comes when we choose θ so that

$$\cos(\theta) = \frac{t}{1+t^2}, \quad \sin(\theta) = \sqrt{1 - \frac{t^2}{(1+t^2)^2}}.$$

Then (5.7) becomes

$$2\frac{1+4t^2+t^4}{1+t^2} \ \leq \ 2(1+t^2) \qquad .$$

This clearly fails unless $t = 0.$ □

6 Problem C

Proposition 6.1. Suppose I is a well-ordered set, for each $i \in I$ there is a domain of holomorphy G_i so that (G_i,V) is np, and if $i \leq j$ then $G_i \subseteq G_j$. Let $G = \cup_{i\in I} G_i$. Then (G,V) is np.

Proof. For each i there is a map $E_i : \mathrm{Hol}^\infty(V) \to \mathrm{Hol}^\infty(G_i)$ that preserves norms and satisfies $\forall z \in V, E_i(f)(z) = f(z)$. The issue is that the E_i's need not be consistent.

Observe first, however, that if $I = \mathbb{N}$, then we can proceed as follows. Fix $f \in \mathrm{Hol}^\infty(V)$. On G_1, some subsequence of $E_n f$ converges locally uniformly. Some subsequence of this subsequence in turn converges locally uniformly on G_2. By a diagonalization argument, we get a subsequence of $E_n f$ that converges locally uniformly on each G_k, and hence it converges locally uniformly on G to a holomorphic function that extends f and has the same norm.

For the general case, observe that G is an open set in $\mathbb{C}^d$. Therefore there is a countable increasing sequence of compact sets K_n whose union is G. Each K_n is contained in one of the $G_i' s$; say $K_n \subset G_{i_n}$. Now apply the previous argument to $\cup_{n\in\mathbb{N}} G_{i_n} = G$. □

Example 6.2. Let $V = \mathrm{D} \times \{0\}$ in C^2. For any $c \in \mathrm{C}$, let

$$G_c = \{(z,w) \in \mathrm{C}^2 : |z + cw| < 1\}.$$

Then V is a retract of G_c, so if $f \in \mathrm{Hol}^\infty(V)$, we can define an np extension F to G_c by $F(z,w) = f(z + cw,0)$. Therefore the union of all np hulls of V must contain

$$\Omega := \cup_{c\in\mathrm{C}} G_c = \mathrm{C}^2 \setminus [\{\mathrm{C} \setminus \mathrm{D}\} \times \{0\}].$$

By Liouville's theorem, any bounded holomorphic F on Ω must have $F(z,w)$ is constant in z whenever w 6= 0; by continuity it must therefore be constant in z when $w = 0$, and so cannot be the extension of any non-constant function on V.

7 Neil parabola

Let $R = \{(z,w) \in \mathrm{D}^2 : z^3 = w^2\}$ be the Neil parabola; this has been studied in [5, **?**]. Let $\pi : \mathrm{D} \to R$ be $\pi(\lambda) = (\lambda^2,\lambda^3)$; this is one-to-one and unramified except at 0.

We have g is in the Schur class of R iff $\varphi = g \circ \pi$ is in the Schur class of D and its derivative vanishes at 0.

Assuming $\varphi(0)$ is also 0, there is a model and realization formula for φ [**?**, Sec. 2.5]:

$$\begin{aligned} \varphi(\lambda) &= \lambda^2 \mathrm{h}(I - \lambda D)^{-1}\gamma,\beta\mathrm{i} \\ &= \lambda^2\mathrm{h}(I - \lambda^2 D^2)^{-1}\gamma,\beta\mathrm{i} + \lambda^3\mathrm{h}(I - \lambda^2D^2)^{-1}D\gamma,\beta\mathrm{i}, \end{aligned}$$

where

$$\begin{pmatrix} 0 & 1 \otimes \beta \\ \gamma \otimes 1 & D \end{pmatrix} \qquad (7.1)$$

is unitary. So one holomorphic extension to a neighborhood of R is

$$g(z,w) = \mathrm{h}(z + wD)(I - zD^2)^{-1}\gamma,\beta\mathrm{i}. \qquad (7.2)$$

Definition 7.3. Let U_1 be the set of unitaries U on decomposed Hilbert spaces $\mathrm{M}_1 \oplus \mathrm{M}_2$, where both M_1 and M_2 are non-zero, and such that $P_{\mathrm{M}_1} U P_{\mathrm{M}_1} = 0$, so

$$U = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

$$13 \quad . \tag{7.4}$$

Definition 7.5. Let

$$\mathrm{G}_1 = \{(z,w) \in \mathbb{D}^2 : \|B(z + wD)(1 - zD^2)^{-1}C\| < 1 \qquad \forall U \in \mathrm{U}_1\}. \tag{7.6}$$

Note that $\{(z,w) : |w| < 1 - 2|z|\} \subset \mathrm{G}_1$, just by crashing through with norms.

Theorem 7.7. (G_1,R) is an np pair.

Proof: We need to show that (i) $R \subset \mathrm{G}_1$ and (ii) G_1 is open. From (ii) we get that G_1 is pseudoconvex, since it is a giant intersection of sub-level sets of holomorphic functions, and the interior of the intersection of these is always pseudoconvex [8, Cor. II.3.19]. Formula (7.2) will give a norm preserving extension for every function that vanishes at 0, so by Lemma **??** we get that (G_1,R) is an np pair.

(i) If $(z,w) = (\lambda^2,\lambda^3)$, then the quantity that needs to be less than 1 in (7.13) becomes

$$\|B(\lambda^2 + \lambda^3 D)(1 - \lambda^2 D)^{-1}C\| = \|\lambda^2 B(1 - \lambda D)^{-1}C\|$$

which is less than 1 by the usual realization argument.

(ii) To see that G_1 is open, it is sufficient to show that for any $(z,w) \in \mathrm{G}_1$,

$$\sup_{U \in \mathrm{U}_1} \|B(z + wD)(1 - zD^2)^{-1}C\|$$

is attained. But this holds because for any sequence $U_n \in \mathrm{U}_1$, the direct sum $\bigoplus U_n$ is also in U_1. *2*

Is there a more concrete description of G_1?

Is there a larger set? This construction was a bit ad hoc. (In particular the choice of g).

We can improve Theorem 7.7 to show that (G_1,R) is a complete np pair.

Lemma 7.8. Let H be a Hilbert space. Then $f \in (R)$ iff there exists a function φ in $(\mathbb{D})$ with $\varphi^0(0) = 0$ that satisfies $\varphi = f \circ \pi$.

Proof: Note first that π is a bijection, so f and φ determine each other as functions. If we are given f, then $\varphi := f \circ \pi$ is obviously in $_{(}$D). Moreover $\varphi^0(0) = Df(0,0)\pi^0(0) = 0$.

Conversely, suppose we are given φ and define $f = \varphi \circ \pi^{-1}$. As the range of f equals the range of φ, we only need to prove that f is holomorphic. Away from the origin, this follows because π is locally a biholomorphism except at the origin.

Expand φ in a Taylor series at 0. We have

$$\varphi(\lambda) = A_0 + A_2\lambda^2 + A_3\lambda^3 + \ldots$$

where each A_n is a contraction in B(H). To prove f is holomorphic at (0,0) we must show that there is a convergent power series in a neighborhood Ω of (0,0) that agrees with f on $\Omega \cap R$. One such is

$$A_0 + \sum_{n=1}^{\infty} A_{2n} z^n + A_{2n+1} z^{n-1} w.$$

2

Theorem 7.9. (G_1,R) is a complete np pair vanishing at (0,0).

Proof: In light of Lemma**??**, we need to show that whenever H is a finite dimensional Hilbert space and $f \in_{(} R)$ with $f(0,0) = 0$ then there is an extension to a function F in $_{(}G_1)$. By Lemma 7.8, there is $\varphi \in_{(}$ D) with $\varphi = f \circ \pi$, and $\varphi^0(0) = 0$. Since $\varphi(0) = 0$ also, there is a realization formula for φ [**?**, Sec. 3.3] in terms of a decomposed Hilbert space $H \oplus M_2$, a unitary U so that

$$U = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix},$$

and
$$\phi(\lambda) = \lambda B(1 - \lambda D)^{-1} C.$$

Define

$$F(z,w) = A + B(z + wD)(1 - zD^2)^{-1}C.$$

Then F is holomorphic and in $_{(}G_1)$, and it extends f as required. 2

In passing, let us observe that the np extension problem for R has nothing to do with retracts.

Proposition 7.10. There is no open set $G \in \mathbb{C}^2$ such that R is a retract of G.

Proof: Suppose $f = (f_1, f_2)$ is a holomorphic map from G to R that is the identity on R. Then $f_1^3 = f_2^2$. Around 0, let the Taylor expansions be

$$f_1(z,w) = \sum c_{ij} z^i w^j \quad f_2(z,w) = \sum d_{ij} z^i w^j.$$

As $f(0) = 0$, we have $c_{00} = 0 = d_{00}$. Therefore

$$\begin{aligned} f_1(z,w)^3 &= (c_{10}z + c_{01}w)^3 + \text{Higher order} \\ f_2(z,w)^2 &= d_{10}^2 z^2 + 2d_{10}d_{01}zw + d_{01}^2 w^2 + \text{Higher order.} \end{aligned}$$

As $f_1^3 = f_2^2$, we conclude that $d_{10} = 0 = d_{01}$. This means there are no third order terms in f_2^2, so we also get $c_{10} = 0 = c_{01}$. But if R were a retract, we would have $f(\lambda^2,\lambda^3) = (f_1(\lambda^2,\lambda^3), f_2(\lambda^2,\lambda^3)) = (\lambda^2,\lambda^3)$,

and this can't be, because the lowest order terms in the middle are at least fourth order. 2

Definition 7.11. Let $\mathcal{U}_2$ be the set of unitaries U on decomposed Hilbert spaces $\mathcal{M}_1 \oplus \mathcal{M}_2$, where both $\mathcal{M}_1$ and $\mathcal{M}_2$ are non-zero, so

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and with the additional requirements that $BDC = 0$ and that $\|A\| < 1$.

Definition 7.12. Let

$$G_2 = \{(z,w) \in \mathbb{D}^2 : \|A + B(z + wD)(1 - zD^2)^{-1}C\| < 1 \quad \forall U \in \mathcal{U}_2\}. \quad (7.13)$$

I would like to prove the following theorem, but can't.

Theorem 7.14. (G_2, R) is a complete np pair.

Proof: We need to prove that (i) $R \subseteq G_2$, (ii) G_2 is a domain of holomorphy, and (iii) whenever $\mathcal{H}$ is a finite dimensional Hilbert space and $f \in (R)$ then there is an extension to a function F in (G_2).

To prove (i) we need to show that if $U \in \mathcal{U}_2$, then

$$\forall \lambda \in \mathbb{D} \ \|A + \lambda^2 B(1+\lambda D)(1-\lambda^2 D^2)^{-1} C\| < 1.$$

As

$$(1+\lambda D)(1-\lambda^2 D^2)^{-1} = (1-\lambda D)^{-1},$$

we wish to show that

$$\forall \lambda \in \mathbb{D} \ \|A + \lambda^2 B(1-\lambda D)^{-1} C\| < 1.$$

But $BDC = 0$ since $U \in \mathcal{U}_2$, so we can add λBDC and (7.15) becomes

$$\forall \lambda \in \mathbb{D} \ \|A + \lambda B(1-\lambda D)^{-1} C\| < 1? \tag{7.15}$$

We calculate

$$\begin{aligned} &1 - (A + \lambda B(1-\lambda D)^{-1} C)^* (A + \lambda B(1-\lambda D)^{-1} C) \\ &= C^*(1-\bar{\lambda} D^*)^{-1}(1-|\lambda|^2)(1-\lambda D)^{-1} C \end{aligned} \tag{7.16}$$

As $C^*C = 1 - A^*A$ and A is a strict contraction, we get that C is bounded below, and hence (7.16) is strictly positive for every $\lambda \in \mathbb{D}$. Therefore the inequality in (7.15) holds.

Proving (ii) is the big problem. If one tries to do it just as in Theorem 7.7, one needs for each point $(z,w) \in \mathbb{G}_2$,

$$\sup_{U \in \mathcal{U}_2} \|\|A + B(z+wD)(1-zD^2)^{-1} C\|$$

is attained. The problem is that for a sequence $U_n \in \mathcal{U}_2$, the direct sum $\bigoplus U_n$ need not be in $\mathcal{U}_2$, since $\bigoplus A_n$ may not have norm less than 1.

(iii) Let $f \in (R)$, and let $\varphi = f \circ \pi$ be in $(\mathbb{D})$ with $\varphi'(0) = 0$. Then there is a realization formula for φ [**?**, Sec. 3.3] in terms of a decomposed Hilbert space $\mathcal{H} \oplus \mathcal{M}_2$, a unitary U so that

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and

$$\varphi(\lambda) = A + \lambda B(1 - \lambda D)^{-1}C.$$

Since $\varphi^0(0) = 0$ we get $BDC = 0$, and by the definition of the Schur class, k$A\xi$k < kξk for all non-zero ξ. Therefore $U \in U_2$. Define

$$F(z,w) = A + B(z + wD)(1 - zD^2)^{-1}C.$$

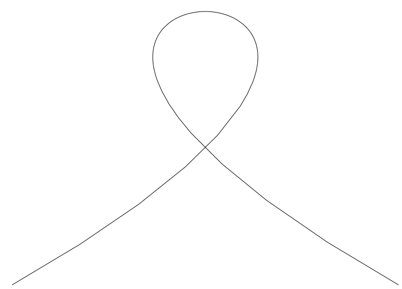

Figure 1: N, a hyperbolic subset of Newton's nodal cubic

Then F is holomorphic and in (G_2), and it extends f as required. 2

8888888888888888888888888888888888888888888888888888888888888888888888888 From (7.2) we get

$$\begin{aligned} 1 - |g|^2 &= \gamma^*(1 - \bar{z}D^{*2})^{-1}\big[(1 - \bar{z}D^{*2})(1 - zD^2) \\ &\quad -(\bar{z} + \bar{w}D^*)(1 - D^*D)(z + wD)\big](1 - zD^2)^{-1}\gamma. \quad (7.17) \end{aligned}$$

So if G_3 is the set of (z,w) for which (7.17) is positive for all D (where D is the (2,2) entry in a unitary like (7.1)), then every Schur function extends to G_3 with the same norm.

Note that $R \subset G_3$ since $g(\lambda^2,\lambda^3)$ has modulus less than 1. However, the expression in brackets in (7.17) is not positive if $D = 1 - \varepsilon$, $\lambda = 1 - \varepsilon$, and $(z,w) = (\lambda^2,\lambda^3)$, by calculation. Therefore $G_1 (G_3$.

The problem with G_3 is that I don't see how to prove it is open.

8 Newton's Nodal Cubic

The set $\{(z,w) \in \mathbb{C}^2 : z^2 = w^2(1 - w)\}$ is called Newton's nodal cubic. Let

$$\mathrm{N} = \{(z,w) \in \mathbb{C}^2 : z^2 = w^2(1 - w)\} \cap \{|w - 1| < 2\}.$$

The set N was studied in [**?**]. Let

$$\pi(\lambda) = (\sqrt{2}(\lambda - 2\lambda^3), 1 - 2\lambda^2).$$

Then N = π(D). The map π is one-to-one except that it identifies $\pm\frac{1}{\sqrt{2}}$. A function f is holomorphic on N if and only if there is a holomorphic function φ on D such that $f \circ \pi = \varphi$; this necessitates $\phi(\frac{1}{\sqrt{2}}) = \phi(-\frac{1}{\sqrt{2}})$.

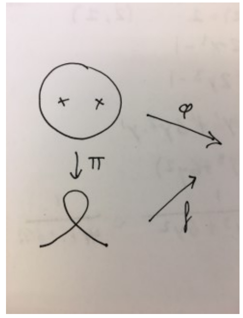

Figure 2: Lifting *f* to *φ*

Lemma 8.1. Let H be a Hilbert space. Then $f \in ($N) iff there exists a function φ in $($D) with $\phi(\frac{1}{\sqrt{2}}) = \phi(-\frac{1}{\sqrt{2}})$ that satisfies $\varphi = f \circ \pi$.

Proof: If f is given, the conclusion about $\varphi = f \circ \pi$ is immediate. Suppose now that φ is given. Then f is defined as a function by $f = \varphi \circ \pi^{-1}$; we need to show it is holomorphic. This is true at the origin (the only singular point) because N has only two sheets that meet there, so one can find a power series in two variables that matches φ in both directions. *2*

We wish to find a domain G so that $(G,$N) is an np pair. It is sufficient to show that every Schur function $f \in \mathrm{Hol}^\infty$(N) with $f(0) = 0$ extends to a Schur function on G. So we can assume that φ vanishes at $\pm\frac{1}{\sqrt{2}}$ and therefore has the form

$$\phi(\lambda) \;=\; \frac{\lambda^2 - \frac{1}{2}}{1 - \frac{1}{2}\lambda^2}\psi(\lambda), \tag{8.2}$$

where ψ is an arbitrary Schur function on D.

As

$$\frac{1}{2}(1 - \pi_2(\lambda)) \;=\; \lambda^2, \tag{8.3}$$

the even part of ψ can be written as $\beta \circ \frac{1-\pi_2(\lambda)}{2}$ where β is a Schur function on D. Similarly, the odd part can be written as $\left(\frac{\pi_1(\lambda)}{\sqrt{2}\pi_2(\lambda)}\right) \alpha \circ \frac{1-\pi_2(\lambda)}{2}$ where α is also a Schur function, and

$$\psi(\lambda) = \beta(\lambda^2) + \lambda\alpha(\lambda^2).$$

Therefore one extension of φ off N is the function

$$g(z,w) \;=\; \frac{-2}{3+w}\left(w\beta(\frac{1-w}{2}) + \frac{z}{\sqrt{2}}\alpha(\frac{1-w}{2})\right)$$

19 . (8.4)

As ψ is in the Schur class, we get that

$$|\beta(\lambda)|^2 + |\alpha(\lambda)|^2 \leq 1 \quad \forall \lambda \in \mathbb{D}. \tag{8.5}$$

In particular (8.5) gives the inequality

$$|g(z,w)| \;\leq\; \frac{2}{|3+w|}\sqrt{|w|^2 + |z|^2/2} \,. \tag{8.6}$$

Definition 8.7. Let

$$\mathcal{G}_{\mathcal{N}} \;=\; \{(z,w) \in \mathbb{D}(\sqrt{2}, 2\sqrt{2}) \times \mathbb{D}(1,2) :$$
$$\|\frac{2}{3+w}\left[wA + B\left(1 - \frac{(1-w)^2}{4}D^2\right)^{-1}\left(\frac{w(1-w)^2}{4}D + \frac{z}{\sqrt{2}}\right)C\right]\| < 1$$
$$\forall \text{ decomposed unitaries } \begin{pmatrix} A & B \\ C & D \end{pmatrix}\}.$$

Theorem 8.8. The pair (G_N,N) is a complete np pair.

Proof: The set G_N is open because if U_n is any sequence of decomposed unitaries, so is $\bigoplus U_n$. Therefore for any point (z,w), the supremum of the norms over all unitaries is attained. If this is less than 1, then it is also less than 1 on a neighborohood of (z,w).

To see $N \subset G_N$, let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be any decomposed unitary, and let $\psi \in_{(} \mathbb{D})$ be the function with realization

$$\psi(\lambda) = A + \lambda B(1 - \lambda D)^{-1}C. \tag{8.9}$$

Let

$$\phi(\lambda) = \frac{\lambda^2 - \frac{1}{2}}{1 - \frac{1}{2}\lambda^2}\psi(\lambda) \,. \tag{8.10}$$

Then from (8.3) we have

$$\phi(\lambda) = \frac{2}{3+\pi_2(\lambda)}\Big[\pi_2(\lambda)A + B\left(1-\frac{(1-\pi_2(\lambda))^2}{4}D^2\right)^{-1}\left(\frac{\pi_2(\lambda)(1-\pi_2(\lambda))^2}{4}D+\frac{\pi_1(\lambda)}{\sqrt{2}}\right)C\Big] \quad (8.11)$$

As ψ is in the Schur class, we have $\|\varphi(\lambda)\| < 1$ for every $\lambda \in \mathbb{D}$, so the norm of (8.11) is less than 1. Since the unitary was arbitrary, this proves that $\pi(\mathbb{D}) = \mathcal{N} \subset G_{\mathcal{N}}$.

Finally, to see that $(G_{\mathcal{N}}, \mathcal{N})$ is a complete np pair (and hence an np pair), we run this backwards. Start with $f \in (\mathcal{N})$ with $f(0) = 0$. By Lemma 8.1, we have $f = \varphi \circ \pi$ for some $\varphi \in (\mathbb{D})$ with $\phi(\pm\frac{1}{\sqrt{2}}) = 0$. Write φ as in (8.10), let (8.9) be a realization for ψ. Then

$$F(z,w) = \frac{2}{3+w}\left[wA + B\left(1-\frac{(1-w)^2}{4}D^2\right)^{-1}\left(\frac{w(1-w)^2}{4}D+\frac{z}{\sqrt{2}}\right)C\right]$$

is an extension of f which is in $(G_{\mathcal{N}})$ by definition of $G_{\mathcal{N}}$. 2

References

[1] Jim Agler, Zinaida Lykova, and Nicholas Young. Geodesics, retracts, and the norm-preserving extension property in the symmetrized bidisc. *Mem. Amer. Math. Soc.*, 258(1242):vii+108, 2019.

[2] Jim Agler and John E. McCarthy. Norm preserving extensions of holomorphic functions from subvarieties of the bidisk. *Ann. of Math. (2)*, 157(1):289–312, 2003.

[3] Henri Cartan. *Séminaire Henri Cartan 1951/2*. W.A. Benjamin, New York, 1967.

[4] Kunyu Guo, Hansong Huang, and Kai Wang. Retracts in polydisk and analytic varieties with the H^∞-extension property. *J. Geom. Anal.*, 18(1):148–171, 2008.

[5] Greg Knese. Function theory on the Neil parabola. *Michigan Math. J.*, 55:139–154, 2007.

[6] Łukasz Kosiński and John E. McCarthy. Extensions of bounded holomorphic functions on the tridisk. *Revista Math. Iber.* to appear.

[7] L ukasz Kosin´ski and John E. McCarthy. Norm preserving extensions of bounded holomorphic functions. *Trans. Amer. Math. Soc.*, 371(10):7243–7257, 2019.

[8] R. Michael Range. *Holomorphic functions and integral representations in several complex variables*, volume 108 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1986.

[9] Walter Rudin. *Function Theory in Polydiscs*. Benjamin, New York, 1969.