Monomial Operators

Jim Agler, John McCarthy

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1 The Hardy-Weyl Algebra

We define three operators M_x, V , and H on $L^2([0,1])$ by the formulas

$$M_x f\ (x) = xf(x), \quad Vf\ (x) = \int_0^x f(t)dt \text{ and } Hf(x) = \frac{1}{x}\int_0^x f(t)dt.$$

and let A, the *Hardy-Weyl algebra*, denote the algebra generated by these three operators. As $V = M_x H$ is actually generated by M_x and H, multiplication by x and the Hardy operator. Hence, our choice of terminology. Also, note the classic commutator relation of the Weyl algebra,

$$DM_x - M_x D = 1,$$

is replaced with the relations

$$V M_x - M_x V = -V^2 \text{ and } HM_x - V = -HV$$

in the Hardy-Weyl algebra. Our aim is to develop a structure theory for the Hardy-Weyl algebra.

2 The Szegö Transform

2.1 The Definition

We let k_α denote the Szegö kernel function for H^2, the classical Hardy space of square integrable functions on $\mathbb{D}$, i.e.,

$$k_\alpha(z) = \frac{1}{1-\bar{\alpha}z}, \qquad z \in \mathbb{D}.$$

As the monomials are linearly independant in $L^2([0,1])$, there is a well defined map L defined on polynomials in $L^2([0,1])$ into H^2 defined by the formula

$$L(\sum_{n=0}^N a_n x^n) = \sum_{n=0}^N a_n \frac{1}{n+1} k_{1-\frac{1}{n+1}}$$

Noting that $\langle L(x^i), L(x^j)\rangle_{H^2} = \langle x^i, x^j\rangle_{L^2([0,1])}$

for all nonnegative integers i and j, it follows that

$$\langle L(p), L(q)\rangle_{H^2} = \langle p, q\rangle_{L^2([0,1])}$$

for all polynomials p and q, i.e., L is isometric. Hence, as the polynomials are dense in $L^2([0,1])$, L has a unique extension to an isometry U defined on all of $L^2([0,1])$. Finally, noting that

$$\{1 - \frac{1}{n+1} \mid n \text{ is a nonegative integer}\}$$

is a set of uniqueness for H^2, it follows that the range of L is dense in H^2, which implies that U is a unitary transformation from $L^2([0,1])$ onto H^2.

Definition 2.1. In the sequel we let U denote the unique unitary transformation from $L^2([0,1])$ onto H^2 that satisfies

$$U(x^n) = \frac{1}{n+1} k_{1-\frac{1}{n+1}}$$

for all nonnegative integers n.

2.2 Moments in $L^2([0,1])$ and Interpolation in H^2

As the monomials are dense in $L^2([0,1])$, a function $f \in L^2([0,1])$ is uniquely determined by its moment sequence

$$\int_0^1 x^n \ f(x)dx, \qquad n = 0, 1, \ldots .$$

Similarly, as the sequence $\{1 - \frac{1}{n+1} \mid n = 0, 1, \ldots\}$ is a set of uniqueness for H^2, a function $h \in H^2$ is the unique solution g in H^2 to the interpolation problem

$$g(1 - \frac{1}{n+1}) = h(1 - \frac{1}{n+1}), \qquad n = 0, 1, \ldots$$

Proposition 2.2. Fix a sequence of complex numbers $w_0, w_1, w_2, \ldots$. If f in $L^2([0,1])$ solves the moment problem

$$\int_0^1 x^n \ f(x) \ dx = w_n, \qquad n = 0, 1, \ldots,$$

then $Uf \in H^2$ and solves the interpolation problem

$$Uf(1 - \frac{1}{n+1}) = (n+1)w_n, \qquad n = 0, 1, \ldots .$$

If $h \in H^2$ solves the interpolation problem

$$h(1 - \frac{1}{n+1}) = w_n, \qquad n = 0, 1, \ldots,$$

then $U^*h \in L^2([0,1])$ and solves the moment problem

$$\int_0^1 x^n \ U^*h(x) \ dx = \frac{1}{n+1} w_n, \qquad n = 0, 1, \ldots .$$

The correspondence between moments and interpolation described in the preceding proposition allows us to easily calculate the Szegö transform of many common functions. We illustrate this with the following two lemmas.

Lemma 2.3. If $\alpha \in \mathrm{D}$, then

$$U^*(k_\alpha)(x) = \frac{1}{1-\bar{\alpha}} x^{\frac{\bar{\alpha}}{1-\bar{\alpha}}}, \qquad x \in [0,1]. \tag{2.4}$$ $\boxed{40}$

If $Re\ \beta > -\frac{1}{2}$, then

$$U(x^\beta) = \frac{1}{\bar{\beta}+1} k_{\frac{\bar{\beta}}{\bar{\beta}+1}}.$$

Proof. We note that the two assertions of the lemma are equivalent. Therefore it suffices ~~40~~ 40 40 to prove (2.4). Since the left and right hand sides of (2.4) are in $\mathrm{L}^2([0,1])$, to show (2.4) it suffices to show that for each $n \geq 0$

$$\langle\, x^n \,, U^* k_\alpha \,\rangle_{\mathrm{L}^2([0,1])} = \langle\, x^n \,, \frac{1}{1-\bar{\alpha}} x^{\frac{\bar{\alpha}}{1-\bar{\alpha}}} \,\rangle_{\mathrm{L}^2([0,1])}.$$

But

$$\begin{aligned}
\langle\, x^n \,, \frac{1}{1-\bar{\alpha}} x^{\frac{\bar{\alpha}}{1-\bar{\alpha}}} \,\rangle_{\mathrm{L}^2([0,1])} &= \frac{1}{1-\alpha} \langle\, x^n \,, x^{\frac{\bar{\alpha}}{1-\bar{\alpha}}} \,\rangle_{\mathrm{L}^2([0,1])} \\
&= \frac{1}{1-\alpha} \int_0^1 x^n \,\overline{x^{\frac{\bar{\alpha}}{1-\bar{\alpha}}}}\, dx \\
&= \frac{1}{1-\alpha} \, \frac{1}{n + \frac{\alpha}{1-\alpha} + 1} \\
&= \frac{1}{n+1} \, \frac{1}{1 - \frac{n}{n+1}\alpha} \\
&= \frac{1}{n+1} k_{1-\frac{1}{n+1}}(\alpha) \\
&= (Ux^n)(\alpha) \\
&= \langle\, Ux^n \,, k_\alpha \,\rangle_{\mathrm{H}^2} \\
&= \langle\, x^n \,, U^* k_\alpha \,\rangle_{\mathrm{L}^2([0,1])}
\end{aligned}$$

QED

For S a measurable set in [0,1] let χ_S denote the characteristic function of S.

Lemma 2.5. If $s \in [0,1]$, then

$$U\chi_{[0,s]}(z) = \sqrt{s}\ e^{\frac{1}{2}\ln s\ \frac{1+z}{1-z}} \tag{2.6}$$ $\boxed{50}$

Proof. We first observe that

$$
\begin{aligned}
\frac{s^{n+1}}{n+1} &= \int_0^s x^n \ dx \\
&= \langle\, \chi_{[0,s]} \,, x^n \,\rangle_{\mathrm{L}^2([0,1])} \\
&= \langle\, U\chi_{[0,s]} \,, Ux^n \,\rangle_{\mathrm{H}^2} \\
&= \langle\, U\chi_{[0,s]} \,, \frac{1}{n+1} k_{1-\frac{1}{n+1}} \,\rangle_{\mathrm{H}^2} \\
&= \frac{1}{n+1} U\chi_{[0,s]}(1-\frac{1}{n+1}),
\end{aligned}
$$

so that

$$U\chi_{[0,s]}(1-\frac{1}{n+1}) = s^{n+1}$$

for all $n \geq 0$. On the other hand if for $w > 0$ we let E_w denote the singular inner function defined by

$$E_w(z) = e^{-w\frac{1+z}{1-z}} = e^w e^{-\frac{2w}{1-z}},$$

we have that

$$E_w(1-\frac{1}{n+1}) = e^w e^{-2w(n+1)} = e^w(e^{-2w})^{n+1}$$

for all $n \geq 0$. Hence if we choose $w = -\frac{1}{2}\ln s$,

$$U\chi_{[0,s]}(1-\frac{1}{n+1}) = e^{-w}E_w(1-\frac{1}{n+1})$$

for all $n \geq 0$. Since $\{1-\frac{1}{n+1} \mid n \geq 0\}$ is a set of uniqueness for H^2, it follows that

$$U\chi_{[0,s]}(z) = e^{-w}E_w(z)$$

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for all $z \in \mathbb{D}$, which implies (2.6). QED

2.3 A Formula for the Szegö Transform

Proposition 2.7. If $f \in \mathrm{L}^2([0,1])$, then

$$Uf(z) = \frac{1}{1-z}\int_0^1 f(x) x^{\frac{z}{1-z}} dx$$

for all $z \in \mathbb{D}$.

Proof.

$$
\begin{aligned}
Uf(z) &= \langle\, Uf \,, k_z \,\rangle_{\mathrm{H}^2} \\
&= \langle\, f \,, U^* k_z \,\rangle_{\mathrm{L}^2([0,1])} \\
& \langle\, f \,, \frac{1}{1-\bar{z}} x^{\frac{\bar{z}}{1-\bar{z}}} \,\rangle_{\mathrm{L}^2([0,1])} \\
&= \frac{1}{1-z}\int_0^1 f(x) x^{\frac{z}{1-z}} dx
\end{aligned}
$$

(*Lemma*(**??**)) =

QED

To obtain a nonrigorous, but highly interesting proof of the proposition, note that formally,

$$\frac{d}{ds}\chi_{[0,s]} = \delta_s.$$

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If (2.6) is written in the form

$$U\chi_{[0,s]}(z) = s^{\frac{1}{1-z}},$$

this suggests the formula

$$U\delta_s = \frac{1}{1-z}s^{\frac{1}{1-z}-1} = \frac{1}{1-z}s^{\frac{z}{1-z}}.$$

We then have

$$Uf = U\left(\int_0^1 f(x)\delta_x \; dx\right) = \int_0^1 f(x)U(\delta_x) \; dx = \int_0^1 f(x)\frac{1}{1-z}x^{\frac{z}{1-z}}dx,$$

the formula in Proposition 2.17.

Perhaps the argument in the previous paragraph could be made rigorous by extending the Szego Transform to more general objects; possibly measures or even distributions. For example, if *mu* is a measure on [0,1], we could define the *Szeg¨o Transform of* μ, S{μ}, to be the holomorphic function

$$\mathcal{S}\{\mu\}(z) = \frac{1}{1-z}\int x^{\frac{z}{1-z}}d\mu(x), \qquad |z - \frac{1}{2}| < \frac{1}{2}.$$

We would then have that S{fdx} would analytically continue to Uf when $f \in L^2([0,1])$.

Question 2.8. Is it possible that A transformed via the Szeg¨o transform has an elegant characterization if we change norms or work in a distribution setup? The *ltwo* − H^2 setup looks to get quite messy as compound operators are not built up from simple operators in a messy way.

2.4 The Szeg¨o Transform and the Laplace Transform

Recall that the Laplace Transform is defined by the formula

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st}f(t)dt.$$

Further, if for $f \in L^2([0,1])$ we define $f^\sim$ by the formula

$$f^\sim(t) = e^{-\frac{t}{2}}f(e^{-t}), \qquad t \in (o,\infty),$$

then the assignment $f \mapsto f^\sim$ is a Hilbert space isomorphism from $L^2([0,1])$ onto $L^2(0,\infty)$.
By making the substitution $x = e^{-t}$ we find that

$$\begin{aligned}\int_0^1 f(x)x^{\frac{z}{1-z}}dx &= \int_\infty^0 f(e^{-t})\ e^{-t\frac{z}{1-z}}\ (-e^{-t})\ dt \\ &= \int_0^\infty e^{-\frac{t}{2}}f(e^{-t})\ e^{-t(\frac{z}{1-z}+\frac{1}{2})}\ dt \\ &= \int_0^\infty f^\sim(t)\ e^{-t(\frac{1}{2}\frac{1+z}{1-z})}\ dt \\ &= \mathcal{L}\{f^\sim\}(\frac{1}{2}\frac{1+z}{1-z}).\end{aligned}$$

Hence,

$$Uf(z) = \frac{1}{1-z}\mathcal{L}\{f^\sim\}(\frac{1}{2}\frac{1+z}{1-z}).$$

Remark 2.9. Since U is unitary, this formula implies that L{} is a unitary transformation from $L^2(0,\infty)$ to the Hardy space of the right half plane. Presumably this fact is well known. So after changes of variable from $L^2([0,1])$ of the disc to $L^2(0,\infty)$ and from H^2 to H^2 of the right half plane, the Szego¨ Transform is simply the Laplace Transform!

2.5 The Szeg¨o Transform and Lat A

Lat A is the same as Lat (V) which is known to be the subspaces of $L^2([0,1])$ of the form

$$\{f \in L^2([0,1]) | t \in [0,s] \Longrightarrow f(t) = 0\}$$

for some $s \in [0,1]$. How do these spaces transform under the Szego¨ Transform? For $s \in (0,1]$ let Φ_s be the singular inner function defined by

$$\Phi_s(z) = e^{\frac{1}{2}\ln s\frac{1+z}{1-z}}, \qquad z \in \mathbb{D}.$$

For $s \in [0,1]$, define orthogonal projections $P_s^\pm$ on $L^2([0,1])$ by the formulas

$$P_s^- f = \chi_{[0,s]}f \quad \text{and } P_s^+ f = \chi_{[s,1]}f, \qquad f \in L^2([0,1]).$$

Lemma 2.10.

$$U \operatorname{ran} P_s^- = \Phi_s H^2 \text{ and } U \operatorname{ran} P_s^+ = \Phi_s {H^2}^\perp$$

3 The Hardy-Weyl Transform

Definition 3.1. If T is a bounded operator on $L^2([0,1])$, we define $T^\wedge$, the *Hardy-Weyl transform of T*, by the formula

$$T^\wedge = UTU^*.$$

Let $\gamma : \mathrm{D} \to \mathrm{D}$ be defined by

$$\gamma(s) = \frac{1}{2-s} \tag{3.2}$$ 2

and for $\varphi : \mathbb{D} \to \mathbb{D}$ a holomorphic mapping, let C_φ denote the bounded operator on H^2 defined by the formula

$$C_\varphi(f)(z) = f(\varphi(z)), \qquad f \in H^2,\ z \in \mathbb{D}.$$

Proposition 3.3. The following formulas hold:

$$M_x^\wedge = S^* C_\gamma^*, \tag{3.4}$$

$$V^\wedge = (1 - S^*) C_\gamma^*, \tag{3.5}$$

$$(M_x + V)^\wedge = C_\gamma^*, \tag{3.6}$$

$$H^\wedge = 1 - S^*. \tag{3.7}$$

Corollary 3.8. $\mathcal{A}^\wedge$ is generated by S^* and C_γ^*.

cor.2.20 **Corollary 3.9.** $S^* C_\gamma^*$ is a positive definite contraction, in particular is self-adjoint.

Corollary 3.10. $(1 - S)^* C_\gamma^*$ is compact.

Corollary 3.11. C_γ^* is essentially self-adjoint.

Corollary 3.12. $1 - H$ is unitarily equivalent to the backward shift.

4 Simple and Compound Operators

Let us agree to say that an operator T on $L^2([0,1])$ is a *simple operator* if there exists an integer $N \geq 0$ and scalars $c_0, c_1, c_2 \ldots$ such that

$$\forall_{n \geq 0}\ T x_n = c_n x_{n+N}. \tag{4.1}$$ 200

It T is a simple operator and (4.1) holds then we refer to the unique integer N as the *order of* T. More generally, we say T is a *compound operator* if T can be represented as a finite sum of simple operators, or equivalently, there exists an integer $N \geq 0$ such that

$$\forall_{n \geq 0}\ T x^n \in \operatorname{span}\{x^n, x^{n+1}, \ldots, x^{n+N}\}. \tag{4.2}$$ 210

It T is a compound operator, then we define the *order of T* to be the smallest integer N such that (4.2) holds. The interest of simple and compound operators stems from the following simple proposition.

Proposition 4.3. 1. H is a simple operator of order 0.

2. M_x and V are simple operators of order 1.

3. If $T \in \mathcal{A}$, then T is a compound operator.

4.1 Simple Operators of Order 0

These are exactly the operators that commute with H or equivalently, T is a simple operator of order 0 if and only if there exists $\varphi \in H^\infty(1 + \mathbb{D})$ such that $T^\wedge = \varphi(1 - S^*)$.

4.2 Simple Operators of Order 1

Let us agree to say that g *is an order one coefficient* if any of the following equivalent statements hold.

I. If μ is defined for Borel sets Δ in $\gamma(\mathbb{T})$ by

$$\mu(\Delta) = \int_{\gamma^{-1}(\Delta)} |g(e^{i\theta})|^2 \, \frac{d\theta}{2\pi} \qquad (4.4) \ \boxed{20}$$

then μ is a Carleson measure.

I(alt). If μ is as in (4.4)[20], then there exists ρ such that

$$\forall_{\lambda\in\mathbb{D}} \quad \int_0^{2\pi} \frac{1-|\lambda|^2}{|1-\bar{\lambda}\gamma(e^{i\theta})|^2} \, |g(e^{i\theta})|^2 \, \frac{d\theta}{2\pi} \le \rho \qquad (4.5) \ \boxed{25}$$

for all $\lambda \in \mathbb{D}$.

II. For all polynomials p,

$$\int |p(\gamma(e^{i\theta}))|^2 \, |g(e^{i\theta})|^2 \, \frac{d\theta}{2\pi} \le \int |p(e^{i\theta})|^2 \, \frac{d\theta}{2\pi}. \qquad (4.6) \ \boxed{30}$$

III. $M_g C_\gamma$ is a bounded operator on H^2.

IV. $C_\gamma^* M_g^*$ is a bounded operator on H^2.

V. There exists a constant C such that

$$\frac{\overline{g(w)}g(z)}{1-\overline{\gamma(w)}\gamma(z)} \le C \frac{1}{1-\bar{w}z}. \qquad (4.7) \ \boxed{30}$$

For μ a real Borel measure on $\mathbb{T}$ let $P[d\mu]$ denote the solution to the Dirichlet problem on $\mathbb{D}$ with boundary data μ. Concretely,

$$P[\mu](\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|\lambda|^2}{|1-\bar{\lambda}e^{i\theta}|^2}\, d\mu(\theta)$$

Proposition 4.8. g is an order 1 coefficient if and only if there exists ρ such that

$$\forall_{\lambda\in\mathbb{D}} \quad P[|g(e^{i\theta})|^2 d\theta](\gamma(\lambda)) \le \rho \frac{1-|\gamma(\lambda)|^2}{1-|\lambda|^2}. \tag{4.9}$$ $\boxed{100}$

Proof. Observe that

$$S^*C_\gamma^*\, k_\alpha(z) = \frac{\overline{\gamma(\alpha)}}{1-\overline{\gamma(\alpha)}z}$$

and

$$C_\gamma S\, k_\alpha(z) = \frac{\gamma(z)}{1-\overline{\alpha}\gamma(z)}$$

cor.2.20

Therefore, Corollary 3.9 implies that

$$\frac{\overline{\gamma(\alpha)}}{1-\overline{\gamma(\alpha)}z} = \frac{\gamma(z)}{1-\overline{\alpha}\gamma(z)}.$$

Letting $\alpha = \lambda$ and $z = e^{i\theta}$ yields that

$$\frac{1}{1-\bar{\lambda}\gamma(e^{i\theta})} = \frac{\overline{\gamma(\lambda)}}{\gamma(e^{i\theta})}\;\frac{1}{1-\overline{\gamma(\lambda)}e^{i\theta}}$$

Hence

$$\begin{aligned}
&\int_0^{2\pi} \frac{1-|\lambda|^2}{|1-\bar{\lambda}\gamma(e^{i\theta})|^2}\; |g(e^{i\theta})|^2\; \frac{d\theta}{2\pi}\\
&= \int_0^{2\pi} \left|\frac{\overline{\gamma(\lambda)}}{\gamma(e^{i\theta})}\right|^2 \frac{1-|\lambda|^2}{|1-\overline{\gamma(\lambda)}e^{i\theta}|^2}\; |g(e^{i\theta})|^2\; \frac{d\theta}{2\pi}\\
&= \frac{1-|\lambda|^2}{1-|\gamma(\lambda)|^2}\int_0^{2\pi} \left|\frac{\overline{\gamma(\lambda)}}{\gamma(e^{i\theta})}\right|^2 \frac{1-|\gamma(\lambda)|^2}{|1-\overline{\gamma(\lambda)}e^{i\theta}|^2}\; |g(e^{i\theta})|^2\; \frac{d\theta}{2\pi}.
\end{aligned}$$

Since

$$\left|\frac{\overline{\gamma(\lambda)}}{\gamma(e^{i\theta})}\right|^2$$

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is bounded above and below, it follows that (4.5) holds if and only if

$$\exists_\rho\ \forall_{\lambda\in\mathbb{D}} \quad \frac{1-|\lambda|^2}{1-|\gamma(\lambda)|^2}\int_0^{2\pi} \frac{1-|\gamma(\lambda)|^2}{|1-\overline{\gamma(\lambda)}e^{i\theta}|^2}\; |g(e^{i\theta})|^2\; \frac{d\theta}{2\pi} \le \rho,$$

100

or equivalently, (4.9) holds. QED

5 The Calkin Hardy Weyl Algebra

In this section we let A^- denote the closure of A (in operator norm). We let K denote the ideal of compact operators acting on $L^2([0,1])$ and set

$$K_0 = A^- \cap K.$$

Evidently, K_0 is a 2-sided ideal in A. Consequently, we may define an algebra C, the *Calkin Hardy Weyl algebra*, by

$$C = A^-/K_0$$

If $T \in A^-$ we let $[T]$ denote the coset of T in C, i.e.,

$$[T] = \{T + K \mid K \in K_0\}.$$

calk.prop.10 **Proposition 5.1.** C is an abelian Banach algebra.

Proof. That C is a Banach algebra follows from the fact that K_0 is closed in A. To see that C is abelian, observe that as M_x and H generate A, $[M_x]$ and $[H]$ generate C. Furthermore, as $M_xH = V \in K_0$,

$$[M_x][H] = 0. \qquad (5.2) \quad \text{calk.10}$$

Likewise, as $HM_x = (1 - H)V \in K_0$,

$$[H][M_x] = 0, \qquad (5.3) \quad \text{calk.20}$$

so that in particular we have that

$$[M_x][H] = [H][M_x].$$

As $[M_x]$ and $[H]$ commute and generate C, C is abelian. QED

5.1 A Uniform Algebra Homeomorphically Isomorphic to C

We begin by defining an algebra by gluing together two simpler algebras whose maximal ideal spaces overlap at a single point. Let

$P = \{ f = (f_-,f_+) : f_- \in C([-1,0]), f_+ \in P(1 + \mathbb{D}^-)$ and $f_-(0) = f_+(0) \}$ where we view P as an algebra with the operations

$$cf = (cf_-,cf_+),\ f + g = (f_- + g_-,f_+ + g_+), \text{and } fg = (f_-g_-,f_+g_+),$$

and the norm

$$\|f\| = \max\left\{ \max_{t\in[-1,0]} |f_-(t)|,\ \max_{z\in 1+\mathbb{D}^-} |f_+(z)| \right\}.$$

We abuse notation by letting

$$f(0) = f_-(0)$$

when $f \in P$.

We note that if $f \in C([-1,0])$, then as $-M_x$ is self-adjoint and has spectrum equal to $[-1,0]$, then we may form the operator $f(-M_x)$. Likewise,as H is cosubnormal and has spectrum equal to $1+\mathbb{D}^-$, if $g \in P(1+\mathbb{D}^-)$, then we may form the operator $g(H)$. Concretely,

$$f(-M_x) = M_{f(-x)} \text{ and } g(H)^\wedge = M_h^*,$$

where $h(z) = \overline{g(1-\bar{z})}$.

calk.lem.10 **Lemma 5.4.** If $f \in C([-1,0])$ and $g \in P(1+\mathbb{D}^-)$, then

$$[f(-M_x)][g(H)] = g(0)[f(-M_x)] + f(0)[g(H)] - f(0)g(0).$$

Proof. Since $[-1,0]$ is a spectral set for $-M_x$ and $1+\mathbb{D}^-$ is a spectral set for H it suffices to prove the lemma in the special case when f and g are polynomials. Let $f(x) = f(0)+xf_1(x)$ calk.10 calk.20 and $g(x) = g(0) + xg_1(x)$. Using (5.2) and (5.3) we see that

$$[f(-M_x)][g(H)] = \big([f(0)] + [-M_x][f_1(-M_x)]\big)\big([g(0)] + [H][g_1(H)]\big)$$

$$= f(0)g(0) + g(0)[-M_x][f_1(-M_x)] + f(0)[H][g_1(H)]$$

$$= f(0)g(0) + g(0)[f(-M_x) - f(0)] + f(0)[g(H) - g(0)]$$

$$= g(0)[f(-M_x)] + f(0)[g(H)] - f(0)g(0)$$

QED If $f \in P$ we define $\gamma(f) \in \mathcal{A}$ by the formula

$$\gamma(f) = f_-(-M_x) + f_+(H) - f(0).$$

We also define $\Gamma : P \to \mathcal{C}$ by the formula

$$\Gamma(f) = [\gamma(f)]$$

calk.prop.20 **Proposition 5.5.** Γ is a continuous unital homomorphism.

Proof. γ is linear and $\gamma(1) = 1$. Therefore, Γ is linear and $\Gamma(1) = 1$. Also,

$$\begin{aligned}\|\Gamma(f)\| &= \|[\gamma(f)]\| \\ &\le \|\gamma(f)\| \\ &\le \|f_-(-M_x) + f_+(H) - f(0)\| \\ &\le \|f_-(-M_x)\| + \|f_+(H)\| + |f(0)| = \max_{t\in[-1,0]} |f_-(t)| + \max_{z\in 1+\mathbb{D}^-} |f_+(z)| + |f(0)|\end{aligned}$$

$= \|f\| + |f(0)| \leq$
$2\|f\|,$

so Γ is continuous.

Finally, to see that Γ preserves products, fix $f,g \in P$.

$$
\begin{aligned}
\Gamma(f)\Gamma(g) &= [\gamma(f)]\,[\gamma(g)] \\
&= [f_-(-M_x) + f_+(H) - f(0)]\,[g_-(-M_x) + g_+(H) - g(0)] \\
&= \Big([f_-(-M_x)][g_-(-M_x)] + [f_+(H)][g_+(H)]\Big) \\
&\quad + \Big([f_-(-M_x)][g_+(H)] + [g_-(-M_x)][f_+(H)]\Big) \\
&\quad - \Big(f(0)[g_-(-M_x) + g_+(H)] + g(0)[f_-(-M_x) + f_+(H)]\Big) \\
&\quad + f(0)g(0) \\
&= A + B - C + f(0)g(0).
\end{aligned}
$$

But

$$
\begin{aligned}
A &= \Big([f_-(-M_x)][g_-(-M_x)] + [f_+(H)][g_+(H)]\Big) \\
&= \Big([f_-g_-(-M_x)] + [f_+g_+(H)] - f(0)g(0)\Big) + f(0)g(0) \\
&= [\gamma(fg)] + f(0)g(0) \\
&= \Gamma(fg) + f(0)g(0),
\end{aligned}
$$

calk.lem.10

and using Lemma 5.4, we see that

$$
\begin{aligned}
B &= \Big([f_-(-M_x)][g_+(H)]\Big) + \Big([g_-(-M_x)][f_+(H)]\Big) \\
&= \Big(g(0)[f_-(-M_x)] + [f(0)g_+(H)] - f(0)g(0)\Big) + \Big(f(0)[g_-(-M_x)] + [g(0)f_+(H)] - f(0)g(0)\Big) \\
&= C - 2f(0)g(0).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Gamma(f)\Gamma(g) &= A + B - C + f(0)g(0) \\
&= (\Gamma(fg) + f(0)g(0)) + (C - 2f(0)g(0)) - C + f(0)g(0) = \Gamma(fg).
\end{aligned}
$$

QED

calk.lem.20 **Lemma 5.6.** If p is a polynomial in two variables and we define $f \in P$ by letting

$$f_-(t) = p(t,0),\ t \in [-1,0] \qquad \text{and} \qquad f_+(z) = p(0,z),\ z \in 1 + \mathrm{D}^-,$$

then $p([-M_x],[H]) = \Gamma(f)$.

Proof. If $p = p(x,y)$ is a polynomial in two variables and we let

$$q(x,y) = p(x,y) - p(x,0) - p(0,y) + p(0,0),$$

then $p(x,y) = p(x,0) + p(0,y) - p(0,0) + q(x,y)$

and $q([-M_x],[H]) = 0$.

Therefore,

$$\begin{aligned} p([-M_x],[H]) &= p([-M_x],0) + p(0,[H]) - p(0,0) \\ &= f_(([M_x]) + f_+([h]) - f(0) \\ &= [\gamma(f)] = \\ &\Gamma(f). \end{aligned}$$

QED

alk.cor.10 **Corollary 5.7.** ranΓ is dense in C.

calk.lem.20

Proof. This follows immediately from Lemma 5.6 by recalling that $[-M_x]$ and $[H]$ generate

calk.prop.10

C (cf. proof of Proposition 5.1). QED

calk.lem.20

Lemma 5.6 suggests that we consider the subset P_0 of P defined by

$$P_0 = \{f \in P \mid f_- \text{ and } f_+ \text{ are polynomials}\}.$$

We note that it instantly follows from the facts that the polynomials are dense in both $C([-1,0])$ and $P(1 + \mathrm{D}^-)$ that P_0 is dense in P.

alk.lem.30 **Lemma 5.8.** If $s \in [-1,0]$, then

$$|f_-(s)| \leq \mathrm{k}\Gamma(f)\mathrm{k} \tag{5.9}$$ calk.30

for all $f \in P$.

Proof. As f is continuous, it suffices to prove the lemma under the assumption that $s \in (-1,0)$. For n satisfying $1/n < \min\{s,1-s\}$ we define a unit vector $\chi_n \in L^2([0,1])$ by the formula

$$\chi_n(t) = \begin{cases} \sqrt{2n} & \text{if } |t-s| \leq 1/n \\ 0 & \text{if } |t-s| > 1/n \end{cases}$$

We observe that the mean value theorem for integrals implies that

$$\lim_{n\to\infty} \mathrm{h}g\,\chi_n,\chi_n\,\mathrm{i} = g(s)$$

whenever $g \in C([0,1])$. Also, as $\chi_n \to 0$ weakly,

$$\lim_{n\to\infty} \|K\chi_n\| = 0$$

whenever K is a compact operator acting on $L^2([0,1])$. In particular, as V is compact and $V\chi_n(t) = 0$ when $t \in [0,s-1/n)$,

$$\lim_{n\to\infty} \|H\chi_n\| = \lim_{n\to\infty} \|M_{1/x}V\chi_n\| = 0.$$

More generally, if q is a polynomial and $q(0) = 0$,

$$\lim_{n\to\infty} \|q(H)\chi_n\| = \lim_{n\to\infty} \|\frac{q}{z}(H)\ H\chi_n\| = 0 .$$

Now fix $f \in P_0$ and a compact operator K acting on $L^2([0,1])$. Using the observations in the previous paragraph we have that

$$\begin{aligned}
&\langle(\gamma(f) + K)\chi_n,\chi_n\rangle \\
&= \langle(f_-(-M_x) + f_+(H) - f(0) + K)\chi_n,\chi_n\rangle \\
&= \langle f_-(-x)\chi_n,\chi_n\rangle + \langle(f_+ - f_+(0))(H)\chi_n,\chi_n\rangle + \langle K\chi_n,\chi_n\rangle \\
&\to \quad f_-(s) \quad + \quad 0 \quad + \quad 0 \\
&= \quad f_-(s).
\end{aligned}$$

Therefore, as $\|\chi_n\| = 1$,

$$|f_-(s)| \le \|\gamma(f) + K\|$$

for all $f \in P_0$ and K a compact operator acting on $L^2([0,1])$. Hence,

$$|f_-(s)| \le \inf_{K\in \mathcal{K}_0} \|\gamma(f) + K\| = \|\Gamma(f)\|$$

~~calk.30 f~~or all $f \in P_0$. As Γ is continuous and P_0 is dense in P, it follows that (5.9) holds for all $f \in P$. QED

alk.lem.40 **Lemma 5.10.** If $z \in 1 + \mathbb{D}^-$, then

$$|f_+(z)| \le \|\Gamma(f)\| \tag{5.11}$$ calk.40

for all $f \in P$.

Proof. We first observe that as $f_+ \in P(1 + \mathbb{D}^-)$, by the Maximum Modulus Theorem it suffices to prove the lemma under the assumption that $z = 1 + \tau$ where $\tau \in \mathbb{T} \setminus \{-1\}$. For $\alpha \in \mathbb{D}$, let

$$\chi_\alpha = U^* \frac{k_{-\bar\alpha}}{\|k_{-\bar\alpha}\|}.$$

Clearly, as $k_{-\bar\alpha}/\|k_{-\bar\alpha}\|$ is a unit vector and U^* is unitary, χ_α is a unit vector. Also, as

$$(1 - S^*)\frac{k_{-\bar\alpha}}{\|k_{-\bar\alpha}\|} = (1+\alpha)\frac{k_{-\bar\alpha}}{\|k_{-\bar\alpha}\|},$$

it follows that $H\chi_\alpha = (1 + \alpha)\chi_\alpha$, and more generally,

$$f_+(H)\chi_\alpha = f_+(1+\alpha)\chi_\alpha \tag{5.12}$$ calk.50

for all $f \in P$.

40

Now notice that (2.4) implies that

$$\chi_\alpha = \frac{\sqrt{1-|\alpha|^2}}{1+\alpha} x^{-\frac{\alpha}{1+\alpha}}.$$

k.claim.10 **Claim 5.13.** If $\rho > 0$ and $\tau \in \mathbb{T} \setminus \{-1\}$, then

$$\lim_{\alpha\to\tau} \langle x^\rho \chi_\alpha, \chi_\alpha \rangle = 0. \tag{5.14}$$ calk.60

Proof. First note that

$$\rho - \left(\frac{\alpha}{1+\alpha} + \frac{\bar\alpha}{1+\bar\alpha}\right) + 1 = \rho + \frac{1-|\alpha|^2}{|1+\alpha|^2},$$

so that

$$\int_0^1 x^{\rho - \left(\frac{\alpha}{1+\alpha} + \frac{\bar\alpha}{1+\bar\alpha}\right)} = \left(\rho + \frac{1-|\alpha|^2}{|1+\alpha|^2}\right)^{-1}.$$

Hence,

$$\begin{aligned}\langle x^\rho \chi_\alpha , \chi_\alpha \rangle &= \frac{1-|\alpha|^2}{|1+\alpha|^2} \int_0^1 x^{\rho - \left(\frac{\alpha}{1+\alpha} + \frac{\bar\alpha}{1+\bar\alpha}\right)} \\ &= \frac{1-|\alpha|^2}{|1+\alpha|^2} \left(\rho + \frac{1-|\alpha|^2}{|1+\alpha|^2}\right)^{-1} \\ &= \frac{1-|\alpha|^2}{|1-\alpha|^2\rho + 1 - |\alpha|^2}.\end{aligned}$$

calk.60

Therefore, if $\rho > 0$ and $\tau \in \mathbb{T} \setminus \{-1\}$, (5.14) holds. QED

calk.claim.10

Observe that if q is a polynomial and $\tau \in \mathbb{T} \setminus \{-1\}$, then Claim 5.13 implies that $\langle q\chi_\alpha, \chi_\alpha \rangle \to q(0)$ as $\alpha \to \tau$. In particular,

$$\lim_{\alpha\to\tau} \langle q(-M_x)\chi_\alpha, \chi_\alpha \rangle = 0 \tag{5.15}$$ calk.70

whenever $\tau \in \mathbb{T} \setminus \{-1\}$ and q is a polynomial satisfying $q(0) = 0$.

We now conclude the proof of the lemma. We need to show that if calk.40 $f \in P$ and $\tau \in \mathbb{T}\setminus\{-1\}$ then (5.11) holds with $z = 1+\tau$. First assume that $f \in P_0$ and fix $K \in \mathcal{K}_0$. Since $\chi_\alpha \to 0$
calk.50 calk.60
weakly as $\alpha \to \tau$, using (5.12) and (5.14) we have

$$\begin{aligned}&\langle (\gamma(f) + K)\chi_\alpha, \chi_\alpha \rangle \\ &\quad = \langle (f_- - f_-(0))(-M_x)\chi_\alpha, \chi_\alpha \rangle + \langle f_+(H)\chi_\alpha, \chi_\alpha \rangle + \langle K\chi_\alpha, \chi_\alpha \rangle\end{aligned}$$

$$\to \quad 0 \quad + \quad f_+(1+\tau) \quad + \quad 0$$
$$= f_+(1+\tau).$$

as $\alpha \to \tau$. Therefore, if $f \in P_0$ and $\tau \in \mathbb{T} \setminus \{-1\}$,

$$|f_+(1+\tau)| \le \|\gamma(f) + K\|.$$

Hence, if $f \in P_0$,

$$|f_+(1+\tau)| \le \inf_{K \in K_0} \|\gamma(f) + K\| = \|\Gamma(f)\|.$$

calk.40

As Γ is continuous and P_0 is dense in P, it follows that (5.11) holds with $z = 1 + \tau$ for all $f \in P$. QED calk.lem.50 **Lemma 5.16.** Γ is a homeomorphism.

calk.prop.20

Proof. In the proof of Proposition 5.5 we showed that

$$\|\Gamma(f)\| \le 2\|f\|$$

calk.lem.30

for all $f \in P$. On the other hand, Lemma 5.8 implies that

$$\max_{t \in [-1,0]} |f_-(t)| \le \|\Gamma(f)\|$$

calk.lem.40

for all $f \in P$ and Lemma 5.10 implies that

$$\max_{z \in 1+\mathbb{D}^-} |f_+(z)| \le \|\Gamma(f)\|$$

for all $f \in P$. Therefore,

$$\|f\| = \max\left\{ \max_{t \in [-1,0]} |f_-(t)|,\ \max_{z \in 1+\mathbb{D}^-} |f_+(z)| \right\} \le \|\Gamma(f)\|$$

for all $f \in P$. QED

5.2 Some Observations on the Gelfand Theory of C

Proposition 5.17. The map Γ is a homeomorphic unital isomorphism from P onto C. In particular, the assignment

$$\eta \mapsto \eta' \stackrel{\text{def}}{=} \Gamma \circ \eta$$

is a homeomorphism from the maximal ideal space of C onto the maximal ideal space of P.

Remark 5.18. If $E = [-1,0] \cup (1 + \mathbb{D}^-)$, then Mergelyon's Theorem implies that the assignment

$$P(E) \ni f \mapsto \left(f|[-1,0]\ ,\, f|(1+\mathbb{D}^-) \right)$$

is an isometric isomorphism from $P(E)$ onto P. So one could just as well state the previous proposition with P replaced by $P(E)$ and Γ replaced with the map $\tilde{\Gamma} : P(E) \to$ C defined by

$$\Gamma^{\sim}(f) = \Big[\, \big(f|[-1,0]\big)(-M_x) + \big(f|(1+\mathbb{D}^-)\big)(H) - f(0) \, \Big]$$

We note that since z generates $P(E)$, it follows that

$$\Gamma^{\sim}(z) = [-M_x + H]$$

generates C.

Proposition 5.19. If η is a complex homomorphism of C, then exactly one of the following statements is true.

$$\eta([-M_x] \neq 0 \text{ and } \eta([H]) = 0 \quad (5.20) \quad \boxed{\text{calk.80}}$$

$$\eta([-M_x] \neq 0 \text{ and } \eta([H]) = 0 \quad (5.21) \quad \boxed{\text{calk.90}}$$

$$\eta([-M_x] \neq 0 \text{ and } \eta([H]) = 0 \quad (5.22) \quad \boxed{\text{calk.100}}$$

Proof. This follows immediately from the observation that

$$\eta([-M_x])\eta([H]) = \eta([-M_xH]) = 0.$$

QED

Proposition 5.23. In the preceeding proposition,

calk.80]

η satisfies (5.20) $\Longleftrightarrow \exists_{t\in[-1,0)} \forall_{f\in P}\, \eta(f) = f_-(t)$, calk.90]

η satisfies (5.21) $\Longleftrightarrow \exists_{z\in(1+D-)\setminus\{0\}} \forall_{f\in P}\, \eta(f) = f_+(z)$, and calk.100

] η satisfies (5.22) $\Longleftrightarrow \forall_{f\in P}\ \eta(f) = f(0)$.