QUANTITATIVE STRAIGHTENING OF DISTANCE SPHERES

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ABSTRACT. We study "distance spheres": the set of points lying at constant distance from a fixed arbitrary subset K of $[0,1]^d$. We show that, away from the regions where K is "too dense" and a set of small volume, we can decompose $[0,1]^d$ into a finite number of sets on which the distance spheres can be "straightened" into subsets of parallel $(d-1)$-dimensional planes by a bi-Lipschitz map. Importantly, the number of sets and the bi-Lipschitz constants are independent of the set K.

1. INTRODUCTION

Let K be an arbitrary set in $\mathbb{R}^d$ and $r \geq 0$. The set of all points whose distance from K is equal to r forms a new set that we call a "distance sphere", and denote $S_K(r)$. (A precise definition is given below; in fact, we will focus our attention on the unit cube of $\mathbb{R}^d$ rather than the whole space.)

If K consists of a single point, then $S_K(r)$ is simply the sphere of radius r centered on K. If K is a general set, the distance spheres may be rather complicated objects, whose structure may change wildly as r varies. Figures 1 and 2 below depict some examples. These sets have been studied (under different names) by many authors, e.g., [2, 4, 5, 7].

This paper is concerned with the geometric structure of distance spheres from a quantitative perspective. Our goal is to find large subsets of $\mathbb{R}^d$ on which all the distance spheres can be simultaneously "straightened out" into (subsets of) parallel $(d-1)$-dimensional planes by a global mapping with controlled distortion. Moreover, we control the number of subsets and the distortion of the "straightening map" by constants that depend on the dimension d but are otherwise independent of the set K.

In order to accomplish this, we must "throw away" some pieces of the domain on which we cannot straighten the distance spheres. These pieces come in two types: one a piece of small d-dimensional volume, and one the union of all locations where the set K is "too dense". These are defined precisely below, and our main theorem is then stated as Theorem 1.5.

The main tools in our arguments are the results of [1] and [3] for general Lipschitz functions, combined with an analysis of the "mapping content" defined in [1] in the special case of the distance function $\operatorname{dist}(\cdot, K)$.

1.1. Main definitions and results.

Definition 1.1. Let $K \subseteq [0,1]^d$ be a set. For $r \geq 0$, the *distance spheres* for K are the sets

$$S_K(r) = \{x \in [0,1]^d : \operatorname{dist}(x,K) = r\}.$$

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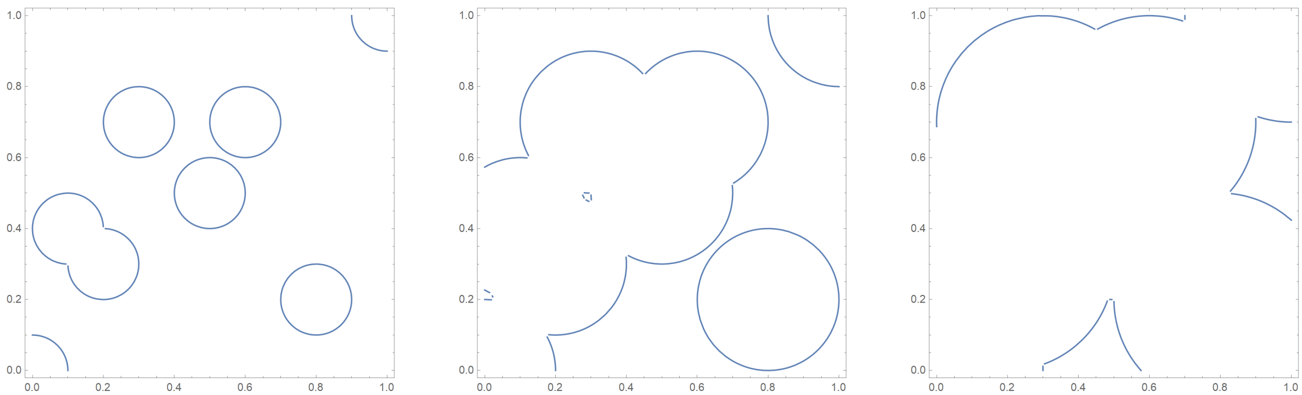


FIGURE 1. Examples of distance spheres $S_K(r)$ for a fixed finite set $K \subseteq [0,1]^2$ and three different values of r.

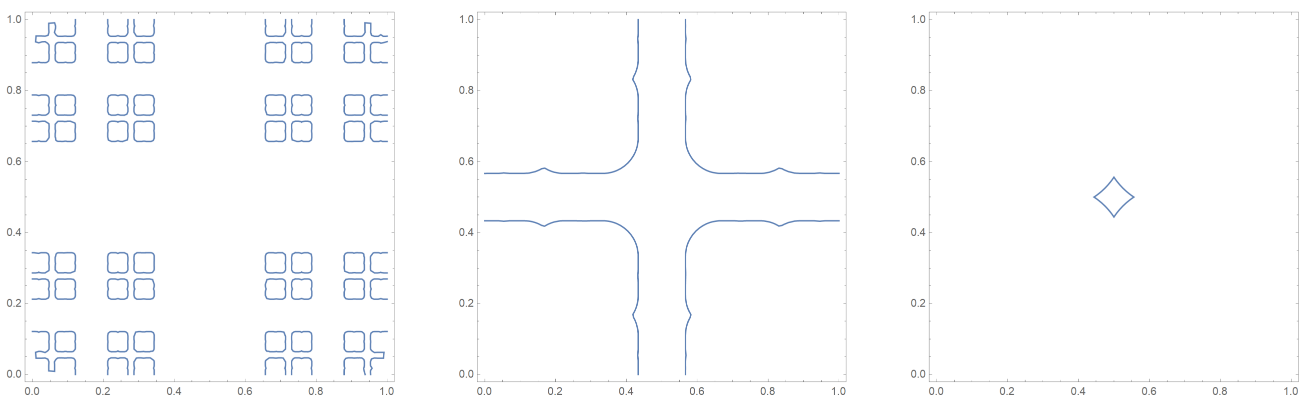


FIGURE 2. Examples of distance spheres $S_K(r)$ for three different values of r and a fixed set $K \subseteq [0,1]^2$ that is an approximation of a Cantor set.

Definition 1.2. Let $K \subseteq [0,1]^d$ be a set. A set $E \subseteq [0,1]^d$ is called K-*straightenable* if there is a bi-Lipschitz map

$$g \colon \mathbb{R}^d \to \mathbb{R}^d$$

and an injective function

$$\phi \colon \{r \geq 0 : S_K(r) \cap E \neq \emptyset\} \to \mathbb{R}$$

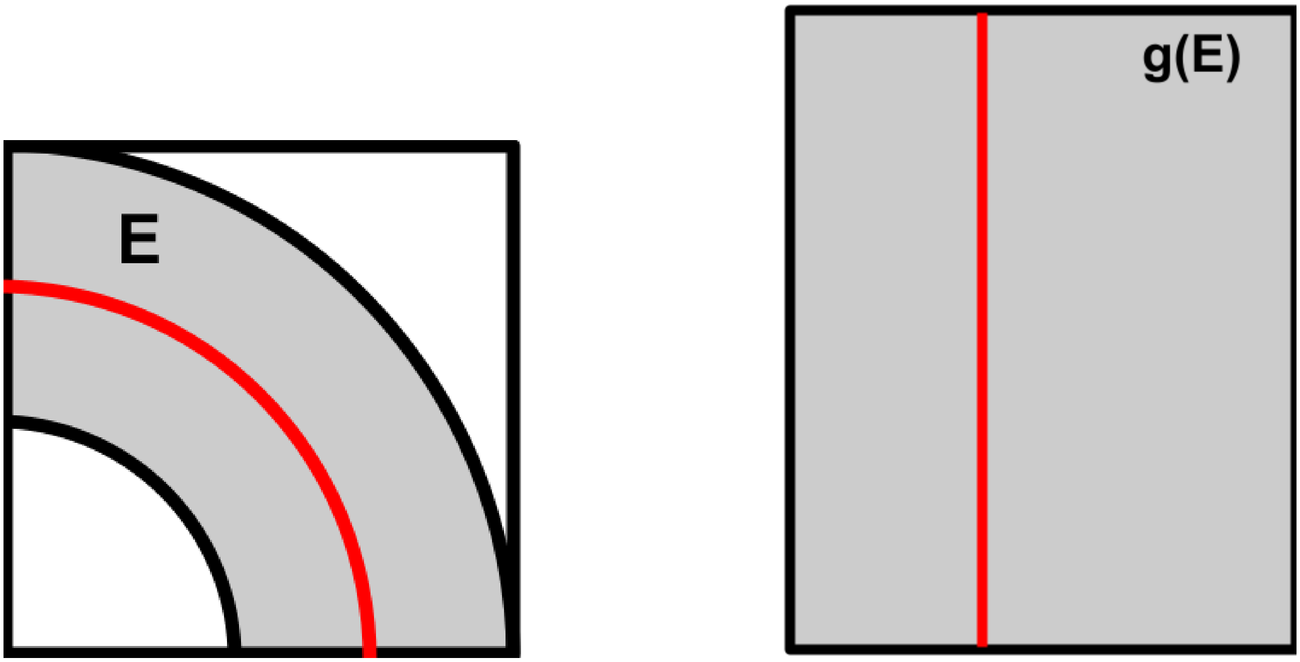
such that

$$g(S_K(r) \cap E) = \left(\{\phi(r)\} \times \mathbb{R}^{d-1}\right) \cap g(E) \text{ for all } r \text{ such that } S_K(r) \cap E \neq \emptyset. \tag{1.1}$$

In other words, g simultaneously "straightens" all the sets $S_K(r) \cap E$ into (subsets of) distinct vertical $(d-1)$-dimensional planes.

Example 1.3. If $K = \{(0,0)\} \subseteq [0,1]^2$, then the set

$$E = \{(x,y) \in [0,1]^2 : \frac{1}{2} \leq \sqrt{x^2+y^2} \leq 1\}$$



(A) The set E with a marked distance sphere (circle) $S_k(r)$ in red.

(B) The set $g(E)$ with straightened $g(S_k(r))$ in red.

FIGURE 3. A simple example of a straightenable set when K is the one-point set $\{(0,0)\}$.

is an example of a K-straightenable set. (See Figure 3.) Since K is a single point, the distance spheres $S_K(r)$ are simply arcs of circles. The map

$$g(x,y) = (\sqrt{x^2+y^2}, \arctan(y/x)),$$

i.e., the map that converts rectangular to polar coordinates, straightens out the distance spheres $S_K(r)\cap E$ into distinct vertical line segments $(\{\phi(r)\}\times\mathbb{R})\cap g(E)$, where we simply take $\phi(r)=r$. One can show that g is bi-Lipschitz on E and extends to a bi-Lipschitz map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Note that, while in this example E is the closure of a simple open domain, we do not require this in general.

Definition 1.4. Let $K\subseteq[0,1]^d$ be a set and $\epsilon>0$. We define

$$\mathcal{Q}(K,\epsilon) = \{\text{ dyadic cubes } Q : N_{\epsilon \text{side}(Q)}(K\cap Q)\supseteq Q\}$$

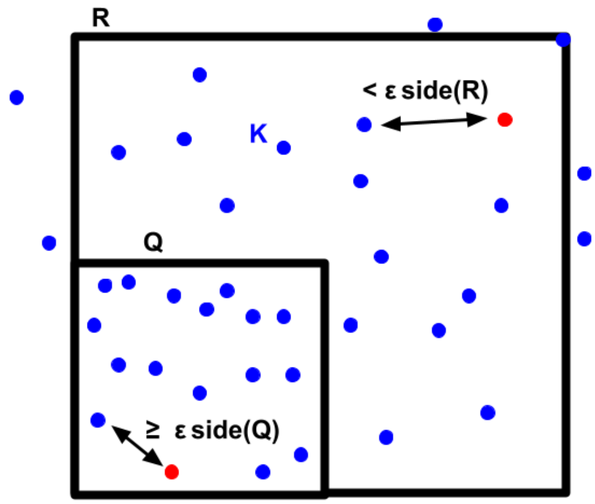
and

$$D_\epsilon(K) := \cup_{Q\in\mathcal{Q}(K,\epsilon)} Q.$$

Here $N_\eta(E)$ refers to the open η-neighborhood of a set E; see section 2. In other words, $D_\epsilon(K)$ is the union of all dyadic cubes Q in which $K\cap Q$ is ϵside(Q)-dense.

We illustrate Definition 1.4 by a simple picture, Figure 4.

FIGURE 4. The set K is in blue and two dyadic cubes Q and R are shown. For some choice of $\epsilon > 0$, the large cube R is in $\mathcal{Q}(K, \epsilon)$ but the small cube Q is not. Each point of R, like the two red examples, is less than ϵside(R) from the nearest point of $K \cap R$. The small cube Q is not in $\mathcal{Q}(K, \epsilon)$, because a (red) point in Q is farther than ϵside(Q) from the nearest point of $K \cap Q$.



Theorem 1.5. *Let $K \subseteq [0,1]^d$ be a set and $\epsilon > 0$. Then we can write*

$$[0,1]^d = E_1 \cup \cdots \cup E_M \cup D_\epsilon(K) \cup G,$$

where each E_i is K-straightenable and $|G| < \epsilon$.

Moreover, the number of straightenable sets M and the associated bi-Lipschitz constants depend only on ϵ and d. In particular, they do not depend on the set K.

In this result, $|G|$ refers to the d-dimensional volume (Lebesgue measure) of the set G; see section 2 for notation.

We emphasize that a large part of our interest in Theorem 1.5 lies in the fact that, in our decomposition, the number of straightenable sets and their associated constants are independent of the starting set K.

The proof of Theorem 1.5 relies on a recent result of Schul and the first named author; see Theorem 2.4 below. This result applies to a general Lipschitz mapping f from $[0,1]^d$ to a metric space. It shows that $[0,1]^d$ can be decomposed into a controlled number of sets on which the fibers of f can be straightened (so-called "Hard Sard" sets from Definition 2.3), and one additional piece which is "small" in a certain unusual sense, requiring the notion of "mapping content" from Definition 2.2.

Our proof of Theorem 1.5 proceeds by analyzing the results of Theorem 2.4 in the case where f is the distance function $\text{dist}(\cdot, K)$. We show (in Claim 4.1) that the "Hard Sard" sets for this f yield the

K-straightenable sets E_i from Theorem 1.5, and we carefully analyze the "mapping content" in this special case to yield the two sets $D_\epsilon(K)$ and G from Theorem 1.5 (in Claim 4.2).

While Theorem 1.5 applies to arbitrary sets $K \subseteq [0,1]^d$, we also prove a stronger corollary for a specific class of sets known as *porous sets*. A set $K \subseteq \mathbb{R}^d$ is *porous* if there is a constant $c > 0$ such that, for each $r > 0$ and $p \in \mathbb{R}^d$, the ball $B(p,r)$ contains a ball $B(q, cr)$ that is disjoint from K. Many classical fractals, such as the Cantor set and Sierpiński carpet, are porous. More discussion of porous sets can be found, e.g., in [6, Ch. 5].

If K is a porous set, then we can decompose the entirety of $[0,1]^d$, outside of a set of small measure, into K-straightenable sets:

Corollary 1.6. *Let $K \subseteq [0,1]^d$ be a porous set with constant c. Let $0 < \epsilon < c/2$. Then we can write*

$$[0,1]^d = E_1 \cup \cdots \cup E_M \cup G,$$

where each E_i is K-straightenable and $|G| < \epsilon$.

The number of straightenable sets M and the associated bi-Lipschitz constants depend only on ϵ and d, and not on the set K.

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2. Notation and preliminaries

2.1. **Basics.** We use the following basic definitions. A function f from a metric space (X, d_X) to a metric space (Y, d_Y) is called *Lipschitz* (or *L-Lipschitz* to emphasize the constant) if there is a constant L such that

$$d_Y(f(x), f(x')) \leq L d_X(x, x') \text{ for all } x, x' \in X.$$

It is called *bi-Lipschitz* (or *L-bi-Lipschitz*) if

$$L^{-1} d_X(x,x') \leq d_Y(f(x), f(x')) \leq L d_X(x,x') \text{ for all } x, x' \in X.$$

We use $B(x,r)$ to denote an open ball of radius r centered at x in a metric space, and $\overline{B}(x,r)$ for the corresponding closed ball.

The distance from a point p to a set K in $\mathbb{R}^d$ is defined as

$$\operatorname{dist}(p, K) := \inf\{|p-q| : q \in K\}.$$

If K is a set in $\mathbb{R}^d$ and $\eta > 0$, then $N_\eta(K)$ is the open η-neighborhood of K, defined as

$$N_\eta(K) = \{p \in \mathbb{R}^d : \operatorname{dist}(p,K) < \eta.\}$$

In $\mathbb{R}^d$, we will also use the collection of *dyadic cubes*. These consist of all cubes Q in $\mathbb{R}^d$ of the form

$$[a_1 2^n, (a_1+1)2^n] \times \cdots \times [a_d 2^n, (a_d+1)2^n],$$

where $a_1, \ldots, a_d$ and n are integers.

2.2. **Measure, Hausdorff content, and mapping content.** We use $|E|$ to denote the d-dimensional volume (Lebesgue measure) of a set in $\mathbb{R}^d$.

Definition 2.1. Let E be a subset of a metric space X, and $k \geq 0$. The *k-dimensional Hausdorff content of E* is defined by

$$\mathcal{H}^k_\infty(E) = \inf_{\mathcal{B}} \sum_{B\in\mathcal{B}} \mathrm{diam}(B)^k,$$

where the infimum is taken over all finite or countable collections of closed balls $\mathcal{B}$ whose union contains E.

The following definition appears first in [1].

Definition 2.2. Let $f\colon [0,1]^{n+m} \to Y$ be a function into a metric space, and let $A \subseteq [0,1]^{n+m}$. The *(n,m)-mapping content of f on A* is:

$$\mathcal{H}^{n,m}_\infty(f,A) = \inf_{\mathcal{Q}} \sum_{Q\in\mathcal{Q}} \mathcal{H}^n_\infty(f(Q))\mathrm{side}(Q)^m,$$

where the infimum is taken over all collections of dyadic cubes $\mathcal{Q}$ in $[0,1]^{n+m}$ whose union contains A.

2.3. **Hard Sard sets.** The following definition was first introduced in [1]. We present the slightly altered version from [3, Definition 1.3].

Definition 2.3. Let $n, m \geq 0$. Let $E \subseteq Q_0 = [0,1]^{n+m}$ and $f\colon Q_0 \to X$ a Lipschitz mapping into a metric space.

We call E a **Hard Sard set for** f if there is a constant C_{Lip} and a C_{Lip}-bi-Lipschitz mapping $g\colon \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ such that the following conditions hold. Write $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ in the standard way, and points of $\mathbb{R}^{n+m}$ as (x,y) with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Let $F = f \circ g^{-1}$. We require that:

(i) If (x,y) and (x',y') are in $g(E)$, then $F(x,y) = F(x',y')$ if and only if $x = x'$. Equivalently,

$$F^{-1}(F(x,y)) \cap g(E) = (\{x\} \times \mathbb{R}^m) \cap g(E)$$

(ii) The map

$$(x,y) \mapsto (F(x,y), y)$$

is C_{Lip}-bi-Lipschitz on the set $g(E)$.

Only condition (i) of the definition of a Hard Sard set will play a role in this paper.

A slightly simplified version of the main theorem of [3] is the following:

Theorem 2.4. *Let Q_0 be the unit cube in $\mathbb{R}^{n+m}$ and let $f\colon Q_0 \to \mathbb{R}^n$ be a 1-Lipschitz map.*

Given any $\gamma > 0$, we can write

$$Q_0 = E_1 \cup \cdots \cup E_M \cup G,$$

where E_i are Hard Sard sets and

$$\mathcal{H}^{n,m}_\infty(f,G) < \gamma.$$

The constant M and the constants C_{Lip} associated to the Hard Sard sets E_i depend only on n, m, and γ.

3. Lemmas

Lemma 3.1. *If K is any set in $\mathbb{R}^d$, the function*

$$f(x) = \operatorname{dist}(x, K)$$

is 1*-Lipschitz.*

Proof. Let $x, y \in \mathbb{R}^d$, and let $K \subseteq \mathbb{R}^d$. Without loss of generality, assume $f(x) \geq f(y)$. Let z_y be a point in the closure of K such that $\inf\{|y-z| : z \in K\} = |y-z_y|$. Then $f(y) = \operatorname{dist}(y, K) = |y-z_y|$. Then applying the triangle inequality, we have

$$\operatorname{dist}(x, K) = \inf\{|x-z| : z \in K\} \leq |x - z_y| \leq |x-y| + |y-z_y| = |x-y| + \operatorname{dist}(y, K).$$

Then

$$\operatorname{dist}(x, K) - \operatorname{dist}(y, K) \leq |x-y|.$$

Thus, since $f(x) \geq f(y)$,

$$|f(x) - f(y)| = |\operatorname{dist}(x, K) - \operatorname{dist}(y, K)| = \operatorname{dist}(x, K) - \operatorname{dist}(y, K) \leq |x-y|,$$

and so $f(x) = \operatorname{dist}(x, K)$ is 1-Lipschitz. □

Lemma 3.2. *If $[a, b]$ is a compact interval in $\mathbb{R}$, then $\mathcal{H}^1_\infty([a, b]) = b - a$.*

Proof. Notice that a closed ball in $\mathbb{R}$ is just a closed interval $[a_i, b_i]$. Then for an interval $[a, b]$, we have

$$\overline{B}\left(\frac{a+b}{2}, \frac{b-a}{2}\right) = [a, b],$$

which implies $\mathcal{H}^1_\infty([a, b]) \leq \operatorname{diam}(\overline{B}(\frac{a+b}{2}, \frac{b-a}{2})) = b - a$.

Now let $\{\overline{B}_i = [a_i, b_i]\}$ be a collection of closed balls that cover the interval $[a, b]$. Then

$$\sum_i \operatorname{diam}(\overline{B}_i) = \sum_i \operatorname{diam}([a_i, b_i]) \geq b - a,$$

where the inequality is a basic fact in measure theory. Taking the infimum of both sides we get $\mathcal{H}^1_\infty([a, b]) \geq b - a$. Hence, $\mathcal{H}^1_\infty([a, b]) = b - a$, as desired. □

Now fix $K \subseteq [0, 1]^d$. Let $f(x) = \operatorname{dist}(x, K)$.

Lemma 3.3. *Let $x \in [0, 1]^d$ and $z \in \overline{K}$ such that*

$$f(x) = |z - x|.$$

If y is a point on the line segment from x to z, then

$$|f(y) - f(x)| = |y - x|$$

Proof. By Lemma 3.1, we know $f(x) = \operatorname{dist}(x, K)$ is 1-Lipschitz. Then, $|f(x) - f(y)| \leq |x - y|$. However,

$$|f(x) - f(y)| = |\operatorname{dist}(x, K) - \operatorname{dist}(y, K)| = \operatorname{dist}(x, K) - \operatorname{dist}(y, K) \geq |x - z| - |y - z|,$$

as $\operatorname{dist}(y,K)=\inf\{|y-z| : z\in K\}$. Then,

$$\operatorname{dist}(x,z)-\operatorname{dist}(y,z)=|x-z|-|y-z|=|x-y|,$$

as y is on the line segment from x to z. Thus, $|f(x)-f(y)|=|f(y)-f(x)|=|y-x|$. □

Lemma 3.4. *Let $\delta>0$ and let Q be a dyadic cube in $\mathbb{R}^d$ such that*

$$\mathcal{H}^1_\infty(f(Q))<\delta\text{side}(Q).$$

Then $Q\in\mathcal{Q}(K,c_d\delta)$, where $c_d=\sqrt{d}+1$.

Proof. Let $\delta>0$ and let Q be a dyadic cube in $\mathbb{R}^d$ such that $\mathcal{H}^1_\infty(f(Q))<\delta\text{side}(Q)$. Let $Q'\subseteq Q$ be the set of points x in Q such that $\operatorname{dist}(x,\partial Q)\geq\delta\text{side}(Q)$, where ∂Q is the set of boundary points of Q.

Claim 3.5. *Let $x\in Q'$. Then there must be a point of K inside the ball $B(x,\delta\text{side}(Q))\subseteq Q$.*

Proof of Claim 3.5. Let $x\in Q'$, and let z' be a point in the closure of K such that

$$f(x)=\operatorname{dist}(x,K)=|x-z'|.$$

If z' is not in Q, then let S be the line segment from z' to x, and let y be the point on the boundary of Q such that $y\in S$. Then by Lemma 3.3,

$$|f(y)-f(x)|=|y-x|\geq\delta\text{side}(Q).$$

Now since Q is closed and bounded, it is compact. Also, since Q is convex, it is connected. Then since $f(x)=\operatorname{dist}(x,K)$ is continuous, $f(Q)\subseteq\mathbb{R}$ is also compact and connected. Then $f(Q)=[a,b]$ for some $a\leq b$. Then by Lemma 3.2,

$$\mathcal{H}^1_\infty(f(Q))=\mathcal{H}^1_\infty([a,b])=b-a.$$

Then we have

$$\mathcal{H}^1_\infty(f(Q))=b-a\geq|f(y)-f(x)|\geq\delta\text{side}(Q).$$

This contradicts the assumption that $\mathcal{H}^1_\infty(f(Q))<\delta\text{side}(Q)$. Thus it must be that z' is in Q. Then suppose for the sake of contradiction that z' is not contained in $B(x,\delta\text{side}(Q))$. Then

$$f(x)=\operatorname{dist}(x,K)=|x-z'|\geq\delta\text{side}(Q),$$

which leads us to the same contradiction as above. Thus it must be that z' is contained in $B(x,\delta\text{side}(Q))$. Since z' is in the closure of K, $B(x,\delta\text{side}(Q))$ must contain a point of K. □

Thus for any $x\in Q'$, there is a point z of K inside $B(x,\delta\text{side}(Q))$, and so

$$|x-z|<\delta\text{side}(Q)<c_d\delta\text{side}(Q)$$

.

Now consider $x\in Q$ such that $x\notin Q'$. Then there is some $x'\in Q'$ such that $|x-x'|\leq\sqrt{d}\delta\text{side}(Q)$. Since $x'\in Q'$, there is some $z\in K$ such that $z\in B(x',\delta\text{side}(Q))$. Then

$$|x-z|\leq|x-x'|+|x'-z|<\sqrt{d}\delta\text{side}(Q)+\delta\text{side}(Q)<c_d\delta\text{side}(Q).$$

Thus for any $x\in Q$, there exists $z\in K\cap Q$ such that $|x-z|<c_d\delta\text{side}(Q)$, and so $Q\in\mathcal{Q}(K,c_d\delta)$. □

The last lemma concerns the concept of mapping content $\mathcal{H}^{n,m}_\infty$ defined above.

Lemma 3.6. *Let $f : Q_0 \to X$ be 1-Lipschitz and $n, m \geq 1$. Let $A \subseteq Q_0$ and suppose*

$$\mathcal{H}^{n,m}_\infty(f, A) < \delta$$

Then we can write

$$A \subseteq A' \cup \bigcup_i Q_i,$$

where

(i) $|A'| < \sqrt{\delta}$*,*
(ii) Q_i *are dyadic cubes,*
(iii) $\mathcal{H}^n_\infty(f(Q_i)) < \sqrt{\delta}\text{side}(Q_i)^n$ *for each* i*.*

Proof. We have

$$\mathcal{H}^{n,m}_\infty(f, A) = \inf_{\mathcal{Q}} \sum_{Q \in \mathcal{Q}} \mathcal{H}^n_\infty(f(Q))\text{side}(Q)^m < \delta,$$

where the infimum is taken over all collections of dyadic cubes $\mathcal{Q}$ in Q_0 whose union contains A. By definition of the infimum, there exists a collection of dyadic cubes $\mathcal{R} = \{R_j\}_{j \in J}$, whose union contains A, such that

$$\sum_{j \in J} \mathcal{H}^n_\infty(f(R_j))\text{side}(R_j)^m < \delta. \tag{3.1}$$

We split these cubes R_j into two collections:

$$\mathcal{R}^1 = \{R_j \in \mathcal{R} : \mathcal{H}^n_\infty(f(R_j)) < \sqrt{\delta}\text{side}(R_j)^n\},$$

and

$$\mathcal{R}^2 = \{R_j \in \mathcal{R} : \mathcal{H}^n_\infty(f(R_j)) \geq \sqrt{\delta}\text{side}(R_j)^n\}.$$

$\mathcal{R}^1$ will become our collection of dyadic cubes $\{Q_i\}$. The union of cubes in $\mathcal{R}^2$ will be our set A', so we want to show that $\left|\bigcup_{R_j \in R^2} R_j\right| < \sqrt{\delta}$.

Let $J_2 := \{j \in J : R_j \in \mathcal{R}^2\}$. Then, using (3.1), we have

$$\delta > \sum_{j \in J} \mathcal{H}^n_\infty(f(R_j))\text{side}(R_j)^m \geq \sum_{j \in J_2} \mathcal{H}^n_\infty(f(R_j))\text{side}(R_j)^m.$$

Then by the definition of our set $\mathcal{R}^2$, we have

$$\delta > \sum_{j \in J_2} \mathcal{H}^n_\infty(f(R_j))\text{side}(R_j)^m \geq \sum_{j \in J_2} \sqrt{\delta}\text{side}(R_j)^n\text{side}(R_j)^m = \sqrt{\delta} \sum_{j \in J_2} |R_j| \geq \sqrt{\delta}\Big|\bigcup_{j \in J_2} R_j\Big|.$$

Thus we have

$$\Big|\bigcup_{j \in J_2} R_j\Big| < \sqrt{\delta}.$$

Therefore, if we define A' to be the union of the cubes in $\mathcal{R}^2$ and define $\{Q_i\}$ to be the collection of cubes in $\mathcal{R}^1$, then we can write

$$A \subseteq A' \cup \bigcup_i Q_i,$$

where properties (i)-(iii) hold for A' and each Q_i. □

4. Proofs of the main results

Proof of Theorem 1.5. Take $K \subseteq [0,1]^d$ and $\epsilon > 0$. Let $f(x) = \text{dist}(x, K)$, which is 1-Lipschitz by Lemma 3.1. Applying Theorem 2.4 to f with $n = 1$, $m = d - 1$, and $\gamma = \frac{\epsilon^2}{c_d^2}$ (where $c_d = \sqrt{d} + 1$ as in Lemma 3.4) we have that

$$[0,1]^d = E_1 \cup ... \cup E_M \cup G_0,$$

where E_i are Hard Sard sets for f and $\mathcal{H}^{1,d-1}_\infty(f, G_0) < \frac{\epsilon^2}{c_d^2}$.

The following two claims combine to complete the proof of Theorem 1.5.

Claim 4.1. *The Hard Sard sets E_i are K-straightenable sets.*

Proof of Claim 4.1. Throughout this proof, we write points of $\mathbb{R}^d$ as (x, y), where $x \in \mathbb{R}$ and $y \in \mathbb{R}^{d-1}$. Let $E = E_i$ for some $i \in \{1, \dots, M\}$.

By Definition 2.3, there is a bi-Lipschitz map $g : \mathbb{R}^d \to \mathbb{R}^d$ such that if $F = f \circ g^{-1}$, then for $(x, y), (x^{'}, y^{'}) \in g(E)$, $F(x, y) = F(x^{'}, y^{'})$ if and only if $x = x^{'}$.

Now, take any r such that $S_K(r)$ intersects E and consider any point $p \in S_K(r) \cap g(E)$. Then $g(p) = (x, y)$, and if any other point $q \in S_k(r) \cap g(E)$, then $f(p) = f(q) = r$, which implies that $F(g(q)) = F(g(p))$. Then, $g(q) \in (\{x\} \times \mathbb{R}^{d-1}) \cap g(E)$. Hence, $g(S_K(r)) \cap g(E) \subseteq (\{x\} \times \mathbb{R}^{d-1}) \cap g(E)$.

Now take any point $(x, y') \in (\{x\} \times \mathbb{R}^{d-1}) \cap g(E)$, where $(x, y^{'}) = g(p')$, for some $p' \in E$. Then $F(x, y') = F(x, y) = F(g(p)) = f(p) = r = f(p') = \text{dist}(p^{'}, K)$. Thus $p' \in S_K(r) \cap g(E)$, and it follows that $(x, y') = g(p') \in g(S_K(r)) \cap g(E)$. Thus $(\{x\} \times \mathbb{R}^{d-1}) \cap g(E) \subseteq g(S_K(r)) \cap g(E)$, yielding the desired equality.

Lastly, define

$$\phi : \{r \geq 0 : S_K(r) \cup E \neq \emptyset\} \to \mathbb{R}$$

so that $\phi(r)$ is equal to the first coordinate x of all points $(x, y) \in g(S_K(r) \cap E)$. (Note that all such points share a common first coordinate by our work above.)

By definition, $g(S_K(r) \cap E) = (\{\phi(r)\} \times \mathbb{R}^{d-1}) \cap g(E)$. Now suppose $\phi(r) = \phi(r^{'})$. Then there are points $p \in S_k(r) \cap E$ and $p^{'} \in S_K(r^{'}) \cap E$ such that $g(p) = (x, y) = g(p^{'})$. Thus $F(g(p)) = F(g(p^{'}))$, which implies that $f(p) = f(p^{'})$, and therefore $r = r^{'}$. Hence, ϕ is injective. □

Claim 4.2. *The set G_0 is contained in $G \cup D_\epsilon(K)$, where G is a subset of $[0,1]^d$ with $|G| < \epsilon$.*

Proof of Claim 4.2. Applying Lemma 3.6, we can write

$$G_0 \subseteq G \cup \bigcup_i Q_i,$$

where $|G| < \frac{\epsilon}{c_d}$ and Q_i are dyadic cubes with $\mathcal{H}^1_\infty(f(Q_i)) < \frac{\epsilon}{c_d}\text{side}(Q_i)$ for each i. Then by Lemma 3.4, we have $Q_i \in \mathcal{Q}(K, \epsilon)$ for every i, and so $Q_i \in \mathcal{Q}(K, \epsilon)$ for all i. Then we have

$$\bigcup_i Q_i \subseteq D_\epsilon(K),$$

and so

$$G_0 \subseteq G \cup \bigcup_i Q_i \subseteq G \cup D_\epsilon(K),$$

where $|G| < \frac{\epsilon}{c_d} < \epsilon$ □

□

Proof of Corollary 1.6. Let $K \subseteq [0,1]^d$ be a porous set with constant c, and let $0 < \epsilon < c/2$. Let Q be any dyadic cube in $[0,1]^d$. Let p be the point in the center of Q, and let $r := \frac{1}{2}\text{side}(Q)$. Consider the ball $B(p, r) \subseteq Q$. Since K is porous, there exists some point q in $B(p, r)$ such that

$$B(q, cr) \subseteq B(p, r)$$

and

$$B(q, cr) \cap K = \emptyset.$$

Then for every $z \in K$,

$$|q - z| \geq cr = \frac{c}{2}\text{side}(Q) > \epsilon\text{side}(Q).$$

Thus $Q \notin \mathcal{Q}(K, \epsilon)$. Since this is true for every dyadic cube in $[0,1]^d$,

$$\mathcal{Q}(K, \epsilon) = \emptyset,$$

and so

$$D_\epsilon(K) = \emptyset.$$

Then by Theorem 1.5, we can write

$$[0,1]^d = E_1 \cup ... \cup E_M \cup D_\epsilon(K) \cup G = E_1 \cup ... \cup E_M \cup G,$$

where each E_i is K-straightenable, $|G| < \epsilon$ and the number of straightenable sets M and the associated bi-Lipschitz constants depend only on ϵ and d, and not on the set K. □

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