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The canonical wall structure and intrinsic mirror symmetry

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Abstract As announced in Gross and Siebert (in Algebraic geometry: Salt Lake City 2015, Proceedings of Symposia in Pure Mathematics, vol 97, no 2. AMS, Providence, pp 199–230, 2018) in 2016, we construct and prove consistency of the *canonical wall structure*. This construction starts with a log Calabi–Yau pair (X, D) and produces a wall structure, as defined in Gross et al. (Mem. Amer. Math. Soc. 278(1376), 1376, 1–103, 2022). Roughly put, the canonical wall structure is a data structure which encodes an algebro-geometric analogue of counts of Maslov index zero disks. These enumerative invariants are defined in terms of the punctured invariants of Abramovich et al. (Punctured Gromov–Witten invariants, 2020. arXiv:2009.07720v2 [math.AG]). There are then two main theorems of the paper. First, we prove consistency of the canonical wall structure, so that, using the setup of Gross et al. (Mem. Amer. Math. Soc. 278(1376), 1376, 1–103, 2022), the canonical wall structure gives rise

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to a mirror family. Second, we prove that this mirror family coincides with the intrinsic mirror constructed in Gross and Siebert (Intrinsic mirror symmetry, 2019. arXiv:1909.07649v2 [math.AG]). While the setup of this paper is narrower than that of Gross and Siebert (Intrinsic mirror symmetry, 2019. arXiv:1909.07649v2 [math.AG]), it gives a more detailed description of the mirror.

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Introduction

The mathematical investigation of the mirror phenomenon observed in physics [25,26,33] has long been vexed by the basic question how broadly mirror pairs exist. Mirror symmetry is certainly too wide a phenomenon to even imagine an answer that is immediately satisfying from all perspectives. The authors have nevertheless long been convinced that there is a fundamental mechanism underlying all mirror phenomena that is both key to general constructions of mirror pairs and to establish various mirror correspondences between them.

Intrinsic mirror symmetry

Following a long line of developments and refinements around the SYZ philosophy of dual torus fibrations [1,2,10,30,34,37,42,44,59,60,78,81], the authors in [48] proposed an intrinsic mirror construction for logarithmic Calabi–Yau varieties.

The construction takes as input a pair (X, D) of a normal crossings divisor D on a smooth variety X, projective over a point or a curve, such that $X\setminus D$ supports a holomorphic volume form with at most logarithmic poles along D. We assume further that D has a zero-dimensional stratum. The mirror is constructed as the affine or projective spectrum of a newly defined degree zero relative quantum cohomology ring for the pair (X, D) that is expected to only depend on $X\setminus D$. The definition of the relative quantum cohomology ring has been made possible by the development of punctured Gromov–Witten invariants, a variant of logarithmic Gromov–Witten invariants [3,21,45] admitting negative contact orders, in joint work of the authors with Abramovich et al. [5]. The structure coefficients of relative quantum cohomology involve invariants with two positive and one negative or zero contact orders.

In our opinion this is a very clean and satisfying construction from the algebraic geometric point of view: It generalizes many of the known geometric mirror constructions to a large, *birationally distinguished* class of algebraic varieties.[1] For example, maximally unipotent degenerations of smooth proper Calabi–Yau varieties, the original object of study in mirror symmetry, and Fano manifolds with an anticanonical divisor with zero-dimensional stratum both fall in this class of varieties.

There is also a clear tentative interpretation of relative quantum cohomology in terms of symplectic geometry as a degree zero symplectic cohomology ring [31,71–73]. This link both points to possible generalizations of the construction to a purely symplectic framework and to higher degree relative quantum cohomology rings [41], as well as to a proof of homological mirror symmetry [58] in this framework [76].

The canonical wall structure and the SYZ interpretation of intrinsic mirror symmetry

The present paper puts the intrinsic mirror construction of [48] into the context of our long-term program aiming at an algebraic-geometric implementation of the SYZ picture of mirror symmetry. Our program is based on a tropical-

[1] It is a certain joke of history that a slight twist to one of the first objects arising in the context of mirror symmetry, namely quantum cohomology, turns out to hold an answer to the puzzling fundamental question in mirror symmetry in such great generality.

ization of the geometry, leading to a polyhedral manifold B with an integral affine structure with singularities (a *polyhedral affine manifold*) together with a wall structure. The polyhedral affine manifold provides the central fiber of the mirror family, the uncorrected mirror, while the wall structure carries the quantum corrections to build a consistent deformation of this uncorrected mirror. In [44] we gave an algorithmic inductive construction of the wall structure for cases with affine singularities that are locally rigid in some sense, purely in terms of affine geometry. We have long suspected, but could make precise only in very limited cases [37,40], that for the construction of the mirror of a pair (X, D) the wall structure should be constructible from the tropicalizations of rational curves on X virtually intersecting D at one point; the enumerative invariants carried by the wall structure should then be the corresponding punctured Gromov–Witten invariants, see e.g. [46, Sect. 4]. This picture is the algebraic-geometric analogue of the quantum corrections on the symplectic side from Maslov index zero pseudoholomorphic disks with boundaries on Lagrangian fibers that give rise to the bounding cochains in [28].

With punctured Gromov–Witten theory sufficiently developed [5], we have now been able to make this picture a reality. Moreover, using the general framework of constructing families from consistent wall structures worked out in [39], we give an alternative and technically simpler proof of the associativity of the degree zero relative quantum cohomology ring from [48] in the case of interest for mirror symmetry, where (X, D) has logarithmic Kodaira dimension zero and D has a zero-dimensional stratum. We emphasize that only the present paper makes the clear connection of the intrinsic mirror construction to the SYZ picture. The assumption that D has a zero-dimensional stratum indeed provides families of Lagrangian tori in $X\setminus D$ degenerating to the zero dimensional stratum. Thus this assumption can be both viewed as a replacement of the existence of Lagrangian torus fibrations or, in a degeneration situation, of the maximally unipotent monodromy assumption [66].

The present paper also provides the link to the algorithmic construction of walls [44], and to previous mirror constructions in two-dimensions [37] and for cluster varieties [38], also giving rise to rich combinatorial structures, see e.g. [22–24,62,63,74,75]. All these previous constructions generalize known and tested mirror constructions such as [12,13,35,51], and they have also been independently tested, see e.g. [16,36,43,49,61,64,77], thus providing further evidence that [48] really does produce mirror pairs. Wall structures also give powerful methods for explicit computations [9,40,44], and in fact, all examples of mirror pairs we have computed explicitly were first obtained via their wall structures.

Another motivation is that the wall structures contain a lot of information not directly accessible from the relative quantum cohomology ring in [48]. For example, our wall structures suggest to generalize Mikhalkin-style tropical correspondence theorems for curve counting from toric ambient geometries to logarithmic Calabi–Yau varieties. In this correspondence, straight lines are to be replaced by the *broken lines* reviewed in Sect. 4.1. Broken lines are certain piecewise straight lines with possible bends when crossing a wall due to interaction with the tropical disks carried by the wall. The present paper also gives a systematic treatment of the geometric information carried by the wall structure, which sometimes contains interesting enumerative information in its own right [9,20,32,40].

Statement of main results

We now describe the results of the paper in more detail. In Sect. 1 we define the relevant polyhedral affine manifold B as the dual intersection complex of D, study the relevant affine geometry and discuss the additional conditions compared to [48] to fulfill the assumptions on B in [39]. As in [48] we distinguish the *absolute* and *relative case*. In both cases X is a smooth variety over an algebraically closed field $\Bbbk$ of characteristic zero and $D \subset X$ is a normal crossing divisor. In the absolute case X is projective over $\Bbbk$ while in the relative case we have a projective morphism $g : X \to S$ with S an affine curve or spectrum of a DVR, g smooth away from a closed point $0 \in S$ and $g^{-1}(0) \subseteq D$. In other words, g induces a log smooth morphism $(X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ when endowing X, S with their respective divisorial log structures. Let $\Sigma(X)$ denote the tropicalization of (X, D) introduced in Sect. 1.1.

Assumption T *The pair* (X, D) *fulfills Assumptions* 1.1 and 1.2 *related to its tropicalization* $\Sigma(X)$ *in the absolute and relative cases, respectively.*

We show in Proposition 1.6 that Assumption T holds for resolutions of log Calabi–Yau minimal models with connected D having a zero-dimensional stratum, using a result of Kollár and Xu [56]. While the existence of log minimal models is not yet generally known in dimensions greater than three, a recent result of Birkar paraphrased in Theorem 1.7 shows their existence in the relevant situation under a technical assumption [15, Cor. 1.5]. The upshot is that we expect Assumption T to be fulfilled for all practical purposes for cases of interest in mirror symmetry.

Section 2 reviews some material concerning punctured maps, their moduli theory and their tropicalizations. A key result for this paper is Lemma 2.5 classifying those tropical punctured maps that later appear in walls and broken lines.

After these preparations, Sect. 3 introduces the two main players of this paper, the canonical wall structure $\mathscr{S}_{\mathrm{can}}$ on B and logarithmic broken lines, both defined in terms of certain punctured Gromov–Witten invariants. Walls are defined in terms of punctured Gromov–Witten invariants with one non-zero contact order. There are in fact two canonical wall structures $\mathscr{S}^{\mathrm{undec}}_{\mathrm{can}}$, $\mathscr{S}_{\mathrm{can}}$

(Constructions 3.8 and 3.13), where $\mathscr{S}_{\mathrm{can}}^{\mathrm{undec}}$ only fixes the total curve class while $\mathscr{S}_{\mathrm{can}}$ prescribes a curve class for each vertex of the tropical disk ("decorated wall type"). The refinement concerning curve classes is necessary for some proofs. The formulas defining the walls in terms of punctured Gromov–Witten invariants are stated in (3.9)–(3.11) and in (3.12) for the undecorated and the decorated cases, respectively. A similar definition, but with punctured invariants with two rather than one contact orders, one of which is negative, provides the notion of *logarithmic broken line*.

At this point we have in principle the same objects as in [39], but we neither know that the wall structure is consistent nor that our logarithmic broken lines have anything in common with the broken lines from [39], which are defined algebraic-combinatorially from the canonical wall structure $\mathscr{S}_{\mathrm{can}}$. Our first main result, covered in Sect. 4, is that these two notions of broken lines indeed agree.

Theorem A *Let (X, D) fulfill Assumption T. Then the broken lines for the canonical wall structures $\mathscr{S}_{\mathrm{can}}$ and $\mathscr{S}_{\mathrm{can}}^{\mathrm{undec}}$ are exactly the logarithmic broken lines.*

Theorem A follows from the bijection between logarithmic broken line types and families of broken lines of fixed combinatorial type (Proposition 4.13), with equality of the corresponding coefficients (Theorem 4.14).

The proof of the decisive Theorem 4.14 relies on a gluing formula for punctured Gromov–Witten invariants that gives the crucial interpretation of bending at a wall as attaching a number l of genus zero punctured maps with one puncture ("bubbles") to a genus zero punctured map with two punctures, by adding an irreducible component with trivial numerical information and $l+2$ punctures and identifying $l+1$ pairs of punctures to nodes. This gluing problem is simplified a lot since broken lines only interact with walls in a cell of codimension zero or one. The codimension zero case corresponds to gluing in a zero-dimensional stratum of D, with the added component to the stable map necessarily contracted. In codimension one the gluing happens along a one-dimensional stratum. Luckily, an argument by Kollár shows that Assumption T implies that a one-dimensional stratum is actually isomorphic to $\mathbb{P}^1$ with stratified structure given by two points (Proposition 1.3). Both cases can then be treated in a rather straightforward manner by the numerical gluing formula for punctured invariants proved by Wu [82], which applies when all gluing strata are toric. For the reader's convenience and to fix notations we recall this formula in Appendix A.

The essential property of wall structures needed for constructing deformations is a certain notion of "consistency". Consistency says that the schemes obtained by gluing local standard models for the deformation in codimension zero and one in a way prescribed by the walls containing any given codimension two subset $\mathfrak{j} \subset B$ has enough global regular functions. We prove this property by restricting to the wall structure on the star of $\mathfrak{j}$ given by the walls containing $\mathfrak{j}$, and then showing that the theta functions obtained from sums over logarithmic broken lines with fixed endpoint provide such functions. Thus consistency comes from invariance properties of certain sums of punctured invariants. This reverses the logic in [39] where we assume consistency to construct theta functions, due to a lack of an a priori interpretation of broken lines. We obtain our second main result, Theorem 5.2 in the body of the text.

Theorem B *Let* (X, D) *fulfill Assumption T. Then the canonical wall structures* $\mathscr{S}_{\mathrm{can}}$ *and* $\mathscr{S}_{\mathrm{can}}^{\mathrm{undec}}$ *are consistent in the sense of* [39].

Gross et al. [39] now provides a ring of theta functions that serves as the affine or projective coordinate ring of our mirror family, depending if we are in the absolute or relative case. The last section, Sect. 6, is devoted to proving that this ring agrees with the relative quantum cohomology of [48].

Theorem C *Let* (X, D) *fulfill Assumption T. The ring of theta functions associated to the canonical wall structure* $\mathscr{S}_{\mathrm{can}}$ *via* [39] *agrees with the degree zero relative quantum cohomology ring from* [48]. *In particular, the canonical wall structure in connection with* [39] *produces the intrinsic mirror family from* [48].

This theorem is stated as an equality of structure coefficients in Theorem 6.1. The key step in the proof is to split the structure coefficients from [48] according to types of tropical punctured maps with two unbounded and one bounded leg. A dimension count shows that such tropical punctured maps split into two connected components when removing the vertex containing the bounded leg, with each connected component leading to a tropical punctured map as they appear in broken lines. The relation to the structure coefficients of the rings defined by the theta functions then boils down to another application of Yixian Wu's gluing formula [82], in a particularly simple situation with a zero-dimensional gluing stratum and transversality of the tropical gluing situation, see Lemma 6.6.

Related work

Under the assumption that $X \setminus D$ is affine and contains a full-dimensional algebraic torus Sean Keel and Tony Yu gave an alternative construction of the degree zero relative quantum cohomology ring with Berkovich non-archimedean methods [52]. The presence of the algebraic torus makes it possible to avoid negative contact orders by tropically extending negative contact order legs out to infinity, so that contact orders become positive. [52]

also construct wall structures with the same assumptions, via a different, but presumably equivalent, approach to ours, by counting the effect on analytic cylinders interacting with the wall.

The paper [9] of the first author with Hülya Argüz has been written in parallel to the present work. Generalizing [40] to higher dimensions, this work treats the case that (X, D) is a blowing up of a toric variety in hypersurfaces contained in its toric boundary with D the strict transform of that toric boundary. The main result says that in this situation our canonical wall structure agrees with an algorithmically constructed consistent wall structure, following [44]. Thus [9] provides a rich source of explicit examples. Because of this, we refer the interested reader to that paper for examples more complicated than the ones given in this paper.

Honglu Fan, Longting Wu and Fenglong You, in the case of a smooth boundary divisor, and Hsian-Hua Tseng and Fenglong You in the case of a normal crossings divisor, made an alternative proposal for a relative quantum cohomology ring based on orbifold Gromov–Witten invariants [29,80]. As pointed out by Dhruv Ranganathan, their invariants do not have the correct invariance properties under log étale modifications to immediately agree with our relative quantum cohomology ring. However, since the first version of the current paper, [11] has achieved a comparison result between orbifold invariants and log invariants with non-negative contact orders. It remains to be seen if a modification of their definitions could also give negative contact orders. In any case, there are no wall structures in this picture since walls are genuinely tropical objects while orbifold Gromov–Witten theory is not known to be linked to tropical geometry.

In two dimensions, Bousseau [17–19] gave a higher genus interpretation for quantum versions of wall structures. It remains an interesting question as to whether there is a higher-dimensional generalization of this work.

Yoel Groman and Umut Varolgunes kindly informed us of their work in progress aiming at a construction of mirrors based on symplectic cohomology type invariants for compact subsets of symplectic manifolds. One of their geometric frameworks rely on decomposing a symplectic manifold into simpler pieces using a multiple cut configuration as in [79, Def. 13]. A multiple cut configuration gives rise to an SC symplectic degeneration, which is a symplectic analogue of a semi-stable degeneration. Therefore, this approach is a symplectic hybrid version of both the patching construction from [44] and the symplectic cohomology interpretation of [48]. It presently does not involve a wall structure.

1 The basic setup

1.1 The tropicalization of a log Calabi–Yau pair

We use the same setup for the log Calabi–Yau case as in [48], Sect. 1. Explicitly, we fix a non-singular variety $\underline{X}$ of dimension n and a simple normal crossings divisor $D \subseteq \underline{X}$ yielding a divisorial log structure on $\underline{X}$, log smooth over $\operatorname{Spec} \Bbbk$. We write X as this log scheme, and $D = D_1 + \cdots + D_s$ the decomposition of D into its irreducible components. We consider the *absolute case*, in which X is projective over $S = \operatorname{Spec} \Bbbk$, and the *relative case*, in which one has in addition a projective log smooth morphism $g : X \to S$ with S a regular one-dimensional scheme over $\operatorname{Spec} \Bbbk$ with a divisorial log structure coming from a single closed point $0 \in S$. We include here the case that S is the spectrum of a complete DVR, or is an affine curve. Necessarily $g^{-1}(0) \subseteq D$, but we do not require equality.

To avoid combinatorial complexities and to fit with the hypotheses of [39], we will assume that for any index set $I \subseteq \{1, \ldots, s\}$, the (possibly empty) stratum $X_I := \bigcap_{i \in I} D_i$ of D is connected. This can always be achieved via a log étale birational modification of X. In this case, the *tropicalization* $\Sigma(X)$ (see [4, Sect. 2.1.4]) has a simple description, following [48, Ex. 1.4], as a polyhedral cone complex in $\operatorname{Div}_D(X)^*_{\mathbb{R}}$. Here $\operatorname{Div}_D(X)$ is the group of divisors supported on D and $\operatorname{Div}_D(X)^*_{\mathbb{R}}$ is the dual vector space. If $D_1^*, \ldots, D_s^*$ is the basis dual to $D_1, \ldots, D_s$, then

$$\Sigma(X) := \left\{ \sum_{i \in I} \mathbb{R}_{\geq 0} D_i^* \mid I \subseteq \{1, \ldots, s\},\, X_I \neq \emptyset \right\}.$$

For $\rho = \sum_{i \in I} \mathbb{R}_{\geq 0} D_i^* \in \Sigma(X)$, we will often write X_ρ instead of X_I in the sequel, keeping with the convention of [4,5]. We usually view X_ρ as a log scheme, with the strict closed embedding $X_\rho \hookrightarrow X$. We write $|\Sigma(X)|$ for the support of this polyhedral cone complex.

For a stratum $X_I = X_\rho$, we write ∂X_I or ∂X_ρ for the reduced divisor on X_I given as

$$\sum_{j \notin I} D_j \cap X_I.$$

The pair (X, D) is *log Calabi–Yau* if the logarithmic canonical class $K_X + D$ is numerically equivalent to an effective $\mathbb{Q}$-divisor supported on D. We then fix once and for all an explicit representation

$$K_X + D \equiv_{\mathbb{Q}} \sum_i a_i D_i \tag{1.1}$$

with the $a_i \geq 0$. In general, our mirror construction depends on this choice in that it determines the set of good divisors below and hence the Kontsevich-Soibelman skeleton. For the situations considered in this paper, however, we expect this skeleton to be well-defined. This is known to be the case if $(X, D) \to S$ is a degeneration of Calabi–Yau manifolds with D the fibre over $0 \in S$, see [67]. More generally, see [48, Ex. 1.23] for further discussion on this point. Recall that $K_X + D$ is the first Chern class of the sheaf $\Omega^1_{\underline{X}} = \Omega^1_X(\log D)$ of logarithmic differential forms. We call D_i *good* if $a_i = 0$. We call a stratum X_I *good* if $a_i = 0$ for all $i \in I$. We write

$$\partial^{\text{good}} X_I := \sum_{j \notin I, a_j = 0} D_j \cap X_I, \qquad \partial^{\text{bad}} X_I := \sum_{j \notin I, a_j > 0} D_j \cap X_I,$$

so that $\partial X_I = \partial^{\text{good}} X_I + \partial^{\text{bad}} X_I$.

The *Kontsevich-Soibelman skeleton* of X is the pair $(B, \mathscr{P})$ where

$$\mathscr{P} := \{\sigma \in \Sigma(X) \mid X_\sigma \text{ is good}\}$$

and

$$B = \bigcup_{\sigma \in \mathscr{P}} \sigma.$$

Thus B is a topological space with polyhedral cone decomposition $\mathscr{P}$ with all cones standard simplicial cones.

We denote by $\mathscr{P}^{[i]}$ the set of i-dimensional cones of $\mathscr{P}$ and $\mathscr{P}_{\max}$ the set of maximal cones.

In [48], we were able to construct a ring from the above data with no further assumptions. However, for the setup of wall crossing structures in [39], we need to impose some additional hypotheses on the pair (X, D) and the map to S. We list these assumptions here, which will be in force for the remainder of the paper, and discuss in Sect. 1.2 the naturality of these assumptions.

Assumption 1.1 We assume that

(1) $\mathscr{P}$ contains an n-dimensional cone, where $n = \dim X$.
(2) Whenever $\rho \in \mathscr{P}^{[n-1]}$, $\sigma \in \Sigma(X)$ with $\dim \sigma = n$ and $\rho \subseteq \sigma$, we also have $\sigma \in \mathscr{P}$. Put another way, a good one-dimensional stratum X_ρ only intersects good divisors.
(3) Whenever $\rho \in \mathscr{P}$ with $\dim X_\rho > 1$, $\partial^{\text{good}} X_\rho$ is connected.

Assumption 1.2 We assume that, with $g : X \to S$ in the relative case and $\dim X = n$:

(1) Conditions (1)–(3) of Assumptions 1.1 hold for the pair (X, D).

(2) For any $\rho \in \mathscr{P}$, $g|_{X_\rho}$ has geometrically connected fibres. (Note $g|_{X_\rho}$ may be constant.)

Proposition 1.3 *If (X, D) satisfies Assumptions 1.1 or 1.2 in the absolute or relative cases respectively, then $(B, \mathscr{P})$ is a pseudomanifold in the sense of satisfying conditions (1)–(5) of* [39, Constr. 1.1]. *Further, for each one-dimensional good stratum X_ρ, either (a) $X_\rho \cong \mathbb{P}^1$ and ∂X_ρ consists of two points, or (b) we are in the relative case and $g|_{X_\rho}$ is an isomorphism, with ∂X_ρ a single point.*

Proof First, conditions (1) and (2) of [39, Constr. 1.1] follow immediately from the construction of $\mathscr{P}$ as a fan in $\mathrm{Div}_D(X)^*_{\mathbb{R}}$. Note this description is possible precisely because we have made the assumption that the X_I are connected.

Condition (3) states that every $\rho \in \mathscr{P}$ is contained in an n-dimensional cone $\sigma \in \mathscr{P}$. In other words:

$$\text{Every good stratum of } X \text{ contains a zero-dimensional good stratum.} \tag{1.2}$$

We will show (1.2) by adapting an argument of Kollár [55, Thm. 10]. First note that by adjunction, for $\rho \in \mathscr{P}$, we may inductively write

$$K_{X_\rho} + \partial X_\rho \equiv_{\mathbb{Q}} \sum_{i : D_i \not\supseteq X_\rho} a_i (D_i \cap X_\rho).$$

Thus the pair $(X_\rho, \partial X_\rho)$ may also be viewed as a log Calabi–Yau variety in our sense, with Assumptions 1.1, (2), (3) holding automatically for this pair.

We now prove (1.2) by induction on $\dim X$, with base case $\dim X = 1$. In this case, (1.2) is obvious, but we observe more. In the absolute case, X is a compact non-singular curve and D is non-empty. By Assumptions 1.1, (2), all components of D must be good, and thus $K_X + D \equiv 0$. Thus the only possibility is that $X \cong \mathbb{P}^1$ and D consists of two distinct points. In the relative case, by Assumptions 1.2, (2), $X \to S$ has connected fibres and hence is an isomorphism, and thus D consists of one point lying over $0 \in S$.

Now assume (1.2) is true when $\dim X < n$, and consider the case $\dim X = n$. Suppose that ρ is maximal in $\mathscr{P}$, i.e., is not contained in a larger cone of $\mathscr{P}$, but with $\dim \rho < n$. By assumption, there exists an n-dimensional cone $\sigma \in \mathscr{P}$. This gives strata X_ρ, X_σ with $\dim X_\rho > 0, \dim X_\sigma = 0$. After reordering the indices $1, \ldots, s$, we may assume $X_\sigma \subseteq D_1$ and $X_\rho \subseteq D_r$, and by connectivity of $\partial^{\mathrm{good}} X$ there exists a sequence of good divisors $D_2, \ldots, D_{r-1}$ such that $D_i \cap D_{i+1} \neq \emptyset$ for $1 \le i \le r-1$. Noting that $(D_1, \partial D_1)$ satisfies Assumptions 1.1, we see by the inductive hypothesis that necessarily the good stratum $D_1 \cap D_2$ of $(D_1, \partial D_1)$ contains a zero-dimensional good stratum. Thus the same is true of $(D_2, \partial D_2)$. Continuing in this way, we see the same

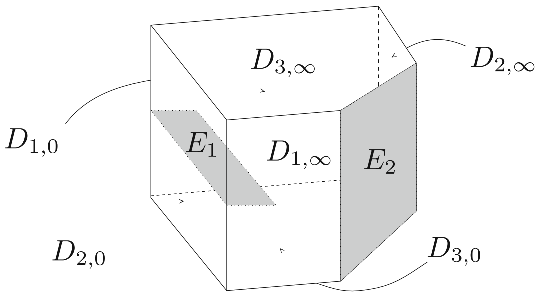

Fig. 1 Sketch of (X, D) in Example 1.4

is true of $(D_r, \partial D_r)$, and hence again by the inductive hypothesis X_ρ contains a zero-dimensional good stratum.

Note this argument also shows that all good one-dimensional strata satisfy conditions (a) or (b) in the statement of the proposition. Condition (4) of [39, Constr. 1.1], which states that every $n-1$-dimensional cone ρ of $\mathscr{P}$ is contained in one or two n-dimensional cones, thus follows immediately. In case (a), ρ is contained in two maximal cones and in case (b), ρ is contained in one maximal cone.

Finally, Condition (5) of [39, Constr. 1.1] follows immediately from the assumed connectedness of $\partial^{\text{good}} X_\tau$ for $\tau \in \mathscr{P}$ with $\dim \tau \leq n-2$. □

Example 1.4 We will illustrate some features of our construction via a simple example. This is a special case of the blow-ups of toric varieties considered more generally in [9], and we refer the reader to that paper for more details and other more interesting examples.

Write $(\overline{X}, \overline{D})$ for the toric pair given by $\overline{X} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\overline{D}$ its toric boundary. Label the six toric boundary divisors as $\overline{D}_{i,0}$, $\overline{D}_{i,\infty}$, where $\overline{D}_{i,0}$ indicates the product obtained by replacing the ith factor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with $\{0\}$, and $\overline{D}_{i,\infty}$ instead replaces the ith factor with $\{\infty\}$.

Let $Z_1, Z_2 \subseteq \overline{X}$ be the curves

$$Z_1 := \mathbb{P}^1 \times \{0\} \times \{1\} \subseteq \overline{D}_{2,0}, \quad Z_2 := \overline{D}_{1,\infty} \cap \overline{D}_{2,\infty} = \{\infty\} \times \{\infty\} \times \mathbb{P}^1.$$

Let $\pi : X \to \overline{X}$ be the blow-up with center $Z_1 \cup Z_2$, with exceptional divisors E_1, E_2 sitting over Z_1 and Z_2 respectively, see Fig. 1. Let $D_{i,0}$, $D_{i,\infty}$ be the strict transforms of $\overline{D}_{i,0}$, $\overline{D}_{i,\infty}$.

Then we take

$$D = D_{1,0} + D_{2,0} + D_{3,0} + D_{1,\infty} + D_{2,\infty} + D_{3,\infty} + E_2.$$

Note that the blow-up along Z_2 is a toric blow-up, so if π is factored as $X \to X' \to \overline{X}$ with X' this toric blow-up, D is the strict transform of the toric boundary of X', and $K_X + D = 0$. Thus all divisors are good, and (X, D) satisfies Assumptions 1.1.

In this case $(B, \mathscr{P})$ is piecewise linear isomorphic to the fan for X'.

1.2 Minimal log Calabi–Yau pairs

We now briefly discuss how restrictive Assumptions 1.1 and 1.2 might be. Certainly, they impose much stronger conditions than those imposed in [48]. Indeed, for a simple example, consider X a rational elliptic surface with $f : X \to \mathbb{P}^1$ the elliptic fibration, and assume the fibre of f over $p_1 \in \mathbb{P}^1$ is a Kodaira type I_n fibre for some $n \geq 3$. Take $D = f^*(p_1 + p_2 + p_3)$ where $p_2, p_3 \in \mathbb{P}^1$ are points over which the fibre of f is non-singular. Since $K_X = f^*(-p_1)$, we may in fact write

$$K_X + D \sim 2f^*(p_3).$$

Thus with this choice of representation (1.1), each irreducible component of $f^{-1}(p_1)$ and the fibre $f^{-1}(p_2)$ are good divisors. So $\partial^{\text{good}} X$ is not connected. Note also that $f^{-1}(p_2)$ does not contain a zero-dimensional stratum even though $\partial^{\text{good}} X$ does.

An obvious problem with this example is that the pair (X, D) is not log Kodaira dimension zero. In particular, if we assume (X, D) is log Kodaira dimension zero, we conjecturally obtain the desired hypotheses. The standard conjectures of the minimal model program in particular would imply the existence of a log Calabi–Yau minimal model:

Definition 1.5 A *log Calabi–Yau minimal model* is a $\mathbb{Q}$-factorial divisorial log terminal (dlt) pair (X', D') such that $K_{X'} + D' \equiv 0$.

Let (X', D') be a log Calabi–Yau minimal model. As necessarily each irreducible component of D' is $\mathbb{Q}$-Cartier, the argument of [68, Thm. 4.5] shows that if $C \subseteq D'$ is a one-dimensional stratum of D', then X' is non-singular and D' is simple normal crossings in a neighbourhood of C.[2] Hence, we may choose a resolution of singularities $\phi : X \to X'$ with exceptional divisor E, such that $D = \phi_*^{-1}(D') + E$ is simple normal crossings, and such that ϕ is an isomorphism in a neighbourhood of each one-dimensional stratum of D'. (Here ϕ_*^{-1} is the notation of [54] for the strict transform.) We may then write

$$K_X + \phi_*^{-1} D' = \phi^*(K_{X'} + D') + \sum_i a(E_i, X', D') E_i, \tag{1.3}$$

[2] The context of [68] is in the case of a degeneration of Calabi–Yau manifolds, but the argument still applies in our case.

where the E_i are the irreducible components of E, and $a(E_i, X', D')$ is, by definition, the log discrepancy. By the definition of dlt (see e.g., [54, Def. 2.8]), we have $a(E_i, X', D') > -1$ for all i. Since $K_{X'} + D' \equiv 0$, we may add E to both sides to obtain

$$K_X + D \equiv \sum_i a_i D_i \tag{1.4}$$

with $a_i \geq 0$ and $a_i = 0$ if and only if $D_i = \phi_*^{-1}(D'_j)$ for some j. Thus we see that good components of D are precisely the strict transforms of components of D'.

Proposition 1.6 *Let (X', D') be a log Calabi–Yau minimal model with X' projective, and let $\phi : (X, D) \to (X', D')$ be a resolution of singularities which is an isomorphism in neighbourhoods of one-dimensional strata of D'. Suppose that any intersection of components of D' is connected*[3] *and that D' has a zero-dimensional stratum. Then (X, D) satisfies Assumptions 1.1 and $(B, \mathscr{P})$ is the dual intersection cone complex of (X', D').*

Proof Since D' has a zero-dimensional stratum, it follows that every stratum of D' has a zero-dimensional stratum by [56, Thm. 2, (1)]. The assumption that ϕ is an isomorphism on neighbourhoods of one-dimensional strata of D' then implies that for any stratum Z' of D', there is a dense open subset $U' \subseteq Z'$ such that $\phi^{-1}(U') \to U'$ is an isomorphism. Let Z be the closure of $\phi^{-1}(U')$: this allows us to pass from a stratum of D' to a stratum of D. In particular, writing $D' = D'_1 + \cdots + D'_{s'}$, $D = D_1 + \cdots + D_s$ with $s \geq s'$, we may order the latter divisors so that $D_i = \phi_*^{-1} D'_i$ for $i \leq s'$. Thus if $Z' = \bigcap_{i \in I} D'_i$, then the corresponding stratum Z coincides with D_I. Since the good divisors amongst the D_i are precisely $D_1, \ldots, D_{s'}$, the claim concerning $(B, \mathscr{P})$ follows.

Now conditions (1) and (2) of Assumptions 1.1 hold by assumption, the first since D' is assumed to have a zero dimensional stratum and the second because ϕ is assumed to be an isomorphism in a neighbourhood of each one-dimensional stratum of D'. Finally (3) holds from results of Kollár and Xu [56]. Indeed, if Z' is a stratum of D' of dimension at least two, and $Z'' \subseteq Z'$ is the union of strata properly contained in Z', then Z'' is connected, see Sect. 2, paragraph 16 of [56], as well as the discussion immediately preceding [56, Sect. 2], which shows that (Z', Z'') is also a dlt log Calabi–Yau pair. □

In dimension larger than three, the existence of log Calabi–Yau minimal models of pairs (X, D) of log Kodaira dimension zero is expected but currently unknown. Caucher Birkar, however, has given a criterion for the existence of minimal models and the necessary connectedness statement of Assumption 1.1. We paraphrase [15, Thm. 1.4, Cor. 1.5]:

[3] This requirement is only necessary to ensure the assumptions on connectivity of strata made throughout this paper.

Theorem 1.7 *Let (X, D) be a pair as in Sect. 1.1 with $K_X + D \equiv_{\mathbb{R}} \sum_i a_i D_i$ with $a_i \geq 0$. Suppose $\mathscr{P}$ has an $n = \dim X$-dimensional cone corresponding to a zero dimensional stratum consisting of $x \in X$. Let $\phi : Y \to X$ be the blow-up of X at x with exceptional divisor E, and suppose that $\phi^*(K_X + D) - tE$ is not pseudo-effective*[4] *for every real number $t > 0$. Then (X, D) has a log Calabi–Yau minimal model, and the connectedness hypothesis Assumptions 1.1, (3) holds.*

In the relative case, there is a different kind of result to which one may appeal. With S as usual, write $S^\circ = S\setminus\{0\}$, and suppose given a $\mathbb{Q}$-factorial dlt pair (X°, D°) equipped with a flat morphism $g^\circ : X^\circ \to S^\circ$ which is a relatively minimal log Calabi–Yau, i.e., the intersection number of $K_{X^\circ} + D^\circ$ with any curve contracted by g° is zero. Then [57, Thm. 2], generalizing the statements of [14, Thm. 1.4] and [50, Thm. 1.1], states that there is a finite map $\tau : S' \to S$ totally ramified over $0 \in S$ and a morphism $g' : (X', D') \to S'$ such that:

(1) The restriction of g' to $\tau^{-1}(S^\circ)$ is isomorphic to the pull-back morphism

$$(X^\circ, D^\circ) \times_{S^\circ} \tau^{-1}(S^\circ) \to \tau^{-1}(S^\circ).$$

(2) $(X', D' + F')$ is dlt and relatively log Calabi–Yau, where F' is the fibre of g' over $0 \in S'$.

Given this, if the general fibre of g° has a zero-dimensional stratum then $(X', D' + F')$ has a zero-dimensional stratum over $0 \in S'$. In any event, as long as $(X', D' + F')$ has a zero-dimensional stratum, we may then resolve singularities to obtain $(X, D) \to (X', D' + F')$ and apply Proposition 1.6 to obtain the needed connectedness statement of Assumption 1.1, (3). Some additional care may be necessary, however, to guarantee that $(X, D) \to S'$ is log smooth.

In the classical case of a degeneration of genuine Calabi–Yau manifolds, the issue of log smoothness is not a concern, and the following proposition summarizes the above discussion in this case.

Proposition 1.8 *In the situation above, suppose $X^\circ \to S^\circ$ is a relatively minimal family of non-singular Calabi–Yau manifolds, i.e., $K_{X^\circ/S^\circ} \equiv 0$. Then:*

(1) *After a possible base-change $S' \to S$ branched at $0 \in S$, there is a dlt relatively minimal model $(X', F') \to S'$ where F' is the fibre over $0 \in S'$ and $X'\setminus F' \to S'\setminus\{0\}$ is the base-change of $X^\circ \to S^\circ$.*

[4] i.e., is not a limit of effective $\mathbb{R}$-divisors.

(2) *If the monodromy of the family $X^\circ \to S^\circ$ is maximally unipotent, then there is a resolution of singularities $\phi : (X, D) \to (X', F')$ and an expression*

$$K_X + D \equiv \sum a_i D_i$$

with $a_i \neq 0$ if and only if D_i is exceptional for ϕ. Further, possibly after first performing a further base-change, the data $(X, D) \to S'$ together with this description of $K_X + D$ satisfies Assumption 1.2.

Proof (1) follows from the discussion preceding the proposition.

For (2), first note that by [68, Thm. 4.5], X' is non-singular in some neighbourhood of each one-dimensional stratum of F' and D' is simple normal crossings in that neighbourhood. Thus we may find a resolution $\phi : (X, D) \to (X', F')$ with (1) D a normal crossings divisor which is the reduction of $\phi^{-1}(F')$; (2) ϕ is an isomorphism in neighbourhoods of one-dimensional strata of F'; and (3) ϕ induces an isomorphism $X \setminus D \cong X' \setminus F'$. After taking further base-change and resolving the resulting singularities in a standard crepant toric way appropriately (see [53, II, Sect. 4, III, Thm. 4.1]), we may further assume that any intersection of irreducible components of D is connected. We may then use (1.3) and (1.4) to obtain the desired expression for $K_X + D$.

The condition of Assumption 1.1, (1) is then implied by [67, Thm. 4.1.10] if the family $X \to S'$ is maximally unipotent. The remaining conditions of Assumptions 1.1 and 1.2 are then immediate by construction and Proposition 1.6. □

Of course the existence of a maximally unipotent degeneration for a type of Calabi–Yau manifold has been classically viewed as a prerequisite for the existence of a mirror.

The conclusion is that the hypotheses of Assumptions 1.1 and 1.2 should be expected to hold in the cases of interest for mirror symmetry.

1.3 The affine structure

We continue to assume that (X, D) satisfies Assumptions 1.1 or 1.2 in the absolute or relative cases, with $\dim X = n$.

We set

$$\Delta := \bigcup_{\substack{\sigma \in \mathscr{P} \\ \dim \sigma = n-2}} \sigma$$

and

$$B_0 := B \setminus \Delta.$$

We also denote by ∂B the union of those $\rho \in \mathscr{P}^{[n-1]}$ contained in only one top-dimensional cone, and write $\mathscr{P}_\partial \subseteq \mathscr{P}$ for the set of cones contained in ∂B. Set $\mathscr{P}_{\rm int} = \mathscr{P} \setminus \mathscr{P}_\partial$. By Proposition 1.3, in the absolute case $\partial B = \emptyset$.

We define an integral affine structure on B_0 as follows. For $\tau \in \mathscr{P}$, denote the *open star* of τ to be

$$\mathrm{Star}(\tau) := \bigcup_{\tau \subseteq \sigma \in \mathscr{P}} \mathrm{Int}(\sigma).$$

If $\sigma \in \mathscr{P}_{\max}$, then σ, hence $\mathrm{Int}(\sigma)$, is already endowed naturally with an affine coordinate chart arising from its linear embedding in $\mathrm{Div}_D(X)^*_{\mathbb{R}}$. If $\rho \in \mathscr{P}_\partial^{[n-1]}$, then $\mathrm{Star}(\rho)$ inherits an affine structure with boundary from the unique maximal cone σ containing ρ. On the other hand, if $\rho \in \mathscr{P}_{\rm int}^{[n-1]}$, then $\rho = \sigma \cap \sigma'$ with $\sigma, \sigma' \in \mathscr{P}^{[n]}$. We then define an embedding

$$\psi_\rho : \sigma \cup \sigma' \to \mathbb{R}^n, \tag{1.5}$$

well-defined up to an element of $\mathrm{GL}_n(\mathbb{Z})$, as follows. Let ρ be generated by $D^*_{i_1}, \ldots, D^*_{i_{n-1}}$, and assume that σ is generated by ρ and $D^*_{i_n}$, while σ' is generated by ρ and $D^*_{i'_n}$. Choose integral bases $e_1, \ldots, e_n$ and $e_1, \ldots, e_{n-1}, e'_n$ of $\mathbb{R}^n$ subject to the constraint that

$$e_n + e'_n = -\sum_{j=1}^{n-1} (D_{i_j} \cdot X_\rho) e_j. \tag{1.6}$$

We note the intersection numbers are defined: the fact that $\rho \in \mathscr{P}_{\rm int}$ implies X_ρ is proper. We then define ψ_ρ to be piecewise linear, linear on each cone, via

$$\psi_\rho(D^*_{i_j}) = e_j,\ 1 \le j \le n, \text{ and } \psi_\rho(D^*_{i'_n}) = e'_n.$$

This gives rise to affine charts $\psi_\rho : \mathrm{Star}(\rho) \to \mathbb{R}^n$, and hence an affine structure on B_0. In the language of [39, Const. 1.1], this gives $(B, \mathscr{P})$ the structure of a *polyhedral affine pseudomanifold*.

Remark 1.9 This affine structure was first given in [47] in the case that $K_X + D = 0$. More generally, in the relative case $g : X \to S$ with $D = g^{-1}(0)$, so that g is a degeneration of Calabi–Yau manifolds, [68] showed that the above affine structure was the correct affine structure on the base of the non-Archimedean SYZ fibration, at least in the case that (X, D) is a resolution of a dlt relatively minimal model (X', D') as in Sect. 1.2. In particular, they showed that if $Z \subseteq D$ was a one-dimensional stratum and $D_i \cdot Z < 0$ for all D_i containing Z, then the formal completion of X along Z is isomorphic to the formal completion of a one-dimensional stratum in a toric variety. The fan determining the toric variety then determines the affine structure above.

In our situation, the affine structure can be viewed as natural from the logarithmic point of view.

Lemma 1.10 *Let $\rho \in \mathscr{P}_{\mathrm{int}}^{[n-1]}$, $\rho \subseteq \sigma, \sigma' \in \mathscr{P}^{\max}$, with chart $\psi_\rho : \sigma \cup \sigma' \to \mathbb{R}^n$ constructed above. Let Σ_ρ be the fan consisting of the cones $\psi_\rho(\sigma)$, $\psi_\rho(\sigma')$ and their faces, and let X_{Σ_ρ} be the corresponding toric variety. For $\tau \in \Sigma_\rho$, write $X_{\Sigma_\rho,\tau}$ for the corresponding stratum of X_{Σ_ρ}. Let $X_\rho \subseteq X$, $X_{\Sigma_\rho,\psi_\rho(\rho)} \subseteq X_{\Sigma_\rho}$ be given the log structures making these inclusions strict, with X_{Σ_ρ} carrying the standard toric log structure. Then there is an isomorphism*

$$X_\rho \cong X_{\Sigma_\rho,\psi_\rho(\rho)}$$

as log schemes over Spec $\Bbbk$.

Proof By Proposition 1.3, $X_\rho \cong \mathbb{P}^1$, and certainly $X_{\Sigma_\rho,\psi_\rho(\rho)} \cong \mathbb{P}^1$, so the underlying schemes are isomorphic. Further, X_ρ contains precisely two zero-dimensional strata, X_σ and $X_{\sigma'}$. Let X'_ρ denote the log structure on $\underline{X}_\rho$ induced by the divisor $X_\sigma \cup X_{\sigma'}$. If X_ρ is contained in divisors $D_{i_1}, \ldots, D_{i_{n-1}}$ corresponding to the edges of ρ, let $\mathcal{M}_j$ denote the restriction to $\underline{X}_\rho$ of the divisorial log structure on X induced by the divisor D_{i_j}. This log structure is determined by the line bundle $\mathcal{O}_X(D_{i_j})|_{X_\rho}$. Finally,

$$\mathcal{M}_{X_\rho} = \mathcal{M}_{X'_\rho} \oplus_{\mathcal{O}^\times_{X_\rho}} \bigoplus_j \mathcal{M}_j,$$

where all pushouts are over $\mathcal{O}^\times_{X_\rho}$.

On the other hand, $\mathcal{M}_{X_{\Sigma_\rho,\psi_\rho(\rho)}}$ has a similar description, and if $D'_{i_1}, \ldots, D'_{i_{n-1}}$ are the toric divisors of X_{Σ_ρ} corresponding to the edges of $\psi_\rho(\rho)$, we have

$$\deg \mathcal{O}_X(D_{i_j})|_{X_\rho} = \deg \mathcal{O}_{X_{\Sigma_\rho}}(D'_{i_j})|_{X_{\Sigma_\rho,\psi_\rho(\rho)}} \qquad (1.7)$$

by [69], pg. 52 and the definition of ψ_ρ. Thus X_ρ and $X_{\Sigma_\rho,\psi_\rho(\rho)}$ are in fact isomorphic as log schemes. □

Example 1.11 We return to Example 1.4. We first remark that in general for a minimal log Calabi–Yau pair satisfying Assumptions 1.1, the affine structure described on B_0, the complement of the union of codimension two cones of $\mathscr{P}$, in fact extends across the interior of all cones $\rho \in \mathscr{P}$ such that $(X_\rho, \partial X_\rho)$ is a toric pair. See [9, Prop. 2.3] for details. In the case of Example 1.4, only the strict transforms of boundary divisors of $\overline{X}$ which meet the center Z_1 transversally fail to be toric: these are $D_{1,0}$ and $D_{1,\infty}$. Hence, after extending the affine structure where possible, the smaller discriminant locus Δ is homeomorphic to the line $\mathbb{R}_{\geq 0} D^*_{1,0} \cup \mathbb{R}_{\geq 0} D^*_{1,\infty}$. The monodromy about this singularity may be calculated, and takes the form $\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. See [9, Sect. 3.2] for related calculations carried out in a more general setting. This is the simplest kind of singularity appearing in affine manifolds in our program.

We recall some standard notation, e.g., from [39].

Definition 1.12 We write

$$B(\mathbb{Z}) := B \cap \mathrm{Div}_D(X)^*,$$

the set of integral points on B.

Definition 1.13 We take Λ to be the local system on $B_0 = B \setminus \Delta$ consisting of integral tangent vectors, and write $\check{\Lambda} := \mathrm{Hom}(\Lambda, \mathbb{Z})$.

Construction 1.14 We now fix a finitely generated abelian group $H_2(X)$ of possible degree data for curves on X, see, e.g., [48], Basic Setup 1.6 for a discussion on this. We may take, for example, $H_2(X) = H_2(X, \mathbb{Z})$ if working over $\Bbbk = \mathbb{C}$, or may take $H_2(X) = N_1(X)$, curve classes modulo numerical equivalence. We require that every proper curve $C \subseteq X$ gives a curve class $[C] \in H_2(X)$, and that for any stable map $f : C \to X$, there is a well-defined curve class $f_*[C] \in H_2(X)$. We also require use of intersection numbers with divisors, i.e., a pairing $H_2(X) \times \mathrm{Div}(X) \to \mathbb{Z}$. As in [48], Basic Setup 1.6, we assume that $f_*[C]$ is torsion if and only if f is a constant map, in which case $f_*[C] = 0$.

In the relative case, the inclusion $X_s \hookrightarrow X$ for $s \in S$ should induce a natural map

$$\iota : H_2(X_s) \to H_2(X). \tag{1.8}$$

We further choose a finitely generated monoid $Q \subseteq H_2(X)$ such that (1) Q contains the classes of all stable maps to $\underline{X}$; (2) Q is saturated; (3) the group of invertible elements of Q coincides with the torsion part of $H_2(X)$.

We may then define a multi-valued piecewise linear (MPL) function φ with values in $Q^{\mathrm{gp}}_{\mathbb{R}}$, see [39], Definition 1.8. Such a function consists of a choice, for each $\rho \in \mathscr{P}^{[n-1]}_{\mathrm{int}}$, of a single-valued PL function $\varphi_\rho : \mathrm{Star}(\rho) \to Q^{\mathrm{gp}}_{\mathbb{R}}$, well-defined up to linear functions. The kink of φ_ρ is defined as follows. Let $x \in \mathrm{Int}(\rho)$, and let $\rho \subseteq \sigma, \sigma' \in \mathscr{P}_{\max}$. Let $n, n' \in \check{\Lambda}_x$ be the slopes of $\varphi_\rho|_\sigma$ and $\varphi_\rho|_{\sigma'}$. Then we may write

$$n' - n = \delta \cdot \kappa_\rho \tag{1.9}$$

where $\kappa_\rho \in Q^{\mathrm{gp}}$ and $\delta : \Lambda_x \to \mathbb{Z}$ is the surjective map which vanishes on tangent vectors to ρ and is positive on tangent vectors pointing into σ'. Then κ_ρ is the kink of φ_ρ.

An MPL function is completely determined by giving its kinks,[5] see [39, Prop. 1.9]. Throughout this paper, we will work with φ such that for $\rho \in \mathscr{P}_{\mathrm{int}}^{[n-1]}$,

$$\kappa_\rho = [X_\rho] \in H_2(X).$$

Note that this curve class makes sense as $\rho \not\subseteq \partial B$ implies that X_ρ is proper. Further, the class $[X_\rho]$ is then not invertible in Q.

The data of the polyhedral affine manifold $(B, \mathscr{P})$ and the MPL function φ are the necessary ingredients for the setup of wall structures of [39, Sect. 2].

We end this subsection by examining, in the relative case, some additional structure. As tropicalization is functorial, we obtain a map

$$\Sigma(g) : \Sigma(X) \to \Sigma(S) = \mathbb{R}_{\geq 0},$$

which restricts to a map

$$g_{\mathrm{trop}} : B \to \mathbb{R}_{\geq 0}.$$

Explicitly, viewing $\Sigma(X)$ or B as a subset of $\mathrm{Div}_D(X)^*_{\mathbb{R}}$, the (not necessarily reduced) divisor $g^*(0)$ induces a linear function on the latter space, which restricts to $\Sigma(g)$ or g_{trop} in the two cases.

Proposition 1.15 *g_{trop} is an affine submersion, and ∂B is the union of all $(n-1)$-dimensional cones contained in $g_{\mathrm{trop}}^{-1}(0)$.*

Proof For the first statement, it is sufficient to show that $(g_{\mathrm{trop}}|_{\mathrm{Star}(\rho)}) \circ \psi_\rho^{-1}$ is a surjective linear function on the fan Σ_ρ for each $\rho \in \mathscr{P}_{\mathrm{int}}^{[n-1]}$. Here ψ_ρ is the local chart (1.5) and Σ_ρ is as defined in Lemma 1.10.

[5] In [39], we always worked with a monoid Q which was torsion free. Of course, if Q has torsion, then the kink as defined above only lies in $Q^{\mathrm{gp}}/Q^{\mathrm{gp}}_{\mathrm{tors}}$. However, if we wish to work with a curve class group $H_2(X)$ which has torsion, then it is more natural to view the MPL function simply as a collection of kinks in $Q^{\mathrm{gp}} = H_2(X)$. This will only affect the definition of the sheaf $\mathcal{P}$, discussed in Sect. 3.1.

If we write the divisor $g^*(0)$ as $\sum_i b_i D_i$, then g_{trop} takes the value b_i on D_i^*. In particular, $(g_{\mathrm{trop}}|_{\mathrm{Star}(\rho)}) \circ \psi_\rho^{-1}$ is the piecewise linear function on Σ_ρ corresponding to a toric divisor D' on X_{Σ_ρ} which may be written explicitly as follows. Denote by D'_j the toric divisor of X_{Σ_ρ} corresponding to the ray $\psi_\rho(\mathbb{R}_{\geq 0} D_j^*)$, for $j \in \{i_1, \ldots, i_n, i'_n\}$. Then

$$D' = b_{i'_n} D'_{i'_n} + \sum_{j=1}^{n} b_{i_j} D'_{i_j}.$$

However, by (1.7), $D' \cdot X_{\Sigma_\rho, \psi_\rho(\rho)} = g^*(0) \cdot X_\rho = 0$ as $g^*(0)$ is algebraically equivalent to the trivial divisor. Thus D' is linearly equivalent to zero, so $(g_{\mathrm{trop}}|_{\mathrm{Star}(\rho)}) \circ \psi_\rho^{-1}$ is linear, as claimed. This latter map is also non-zero, as X_ρ is contained in $g^{-1}(0)$, and hence it is surjective.

For the second statement, by definition ∂B is the union of those codimension one cones ρ only contained in one maximal cone. But these correspond to the non-compact one-dimensional good strata X_ρ, which are precisely the ones for which $g|_{X_\rho}$ is surjective. However by the definition of g_{trop}, these are in one-to-one correspondence with the set of those $\rho \in \mathscr{P}^{[n-1]}$ with $g_{\mathrm{trop}}(\rho) = 0$. □

Frequently, in the relative case $g : X \to S$, it is more intuitive to work not with $(B, \mathscr{P})$ but with the fibre of $g_{\mathrm{trop}} : B \to \mathbb{R}_{\geq 0}$ over 1. This fits more closely with the earlier point of view of affine manifolds associated with degenerations and wall structures in [42,44]. Set $B' := g_{\mathrm{trop}}^{-1}(1)$, equipped with the polyhedral decomposition

$$\mathscr{P}' := \{\sigma \cap B' \,|\, \sigma \in \mathscr{P}\}.$$

Note that the elements of $\mathscr{P}'$ are no longer, in general, cones. We then observe:

Proposition 1.16 *Suppose that all good divisors contained in $g^{-1}(0)$ have multiplicity one in $g^{-1}(0)$. Then $(B', \mathscr{P}')$ is a polyhedral affine pseudomanifold.*

Proof As g_{trop} is an affine submersion on B_0 by Proposition 1.15, $B' \cap B_0$ acquires an affine structure. Thus we just need to check that $(B', \mathscr{P}')$ satisfies the conditions of [39, Constr. 1.1]. Note that if $\rho \in \mathscr{P}$ is a ray with $g_{\mathrm{trop}}|_\rho : \rho \to \mathbb{R}_{\geq 0}$ surjective, the corresponding divisor X_ρ is contained in $g^{-1}(0)$ and is good, hence appears with multiplicity one in $g^{-1}(0)$ by assumption. Thus $g_{\mathrm{trop}}^{-1}(1) \cap \rho$ is an integral point of ρ, so all elements of $\mathscr{P}'$ are in fact lattice polytopes. Conditions (1)–(5) of [39, Constr. 1.1] now follow immediately from the same conditions for $\mathscr{P}$. □

Remark 1.17 In the above situation, given an MPL function φ on $(B, \mathscr{P})$, we obtain by restricting representatives of an MPL function $\varphi|_{B'}$. Note that if $\rho \in \mathscr{P}_{\mathrm{int}}^{[n-1]}$, then the kink κ_ρ of φ at ρ then agrees with the kink $\kappa_{\rho\cap B'}$ of $\varphi|_{B'}$ along $\rho \cap B'$. This follows immediately from the definition (1.9).

Remark 1.18 In [39, Sect. 4.2], given a polyhedral affine pseudomanifold with singularities B, we defined a polyhedral affine pseudomanifold $\mathbf{C}B$, the *cone* over B. In our current situation, B is a conical polyhedral affine pseudomanifold, and B' as defined above satisfies $\mathbf{C}B' = B$. Here we find it more natural to work with the cone.

We next consider a special case of the relative situation, satisfying Assumption 1.2, when S is an affine curve and $(X, D) \to S$ is a degeneration of log Calabi–Yau varieties which themselves satisfy Asssumption 1.1. Recall from Assumption 1.2 that in the relative case, for $s \in S$ a closed point, $s \neq 0$, we have $D_j \cap X_s$ is an irreducible divisor on X_s. We may then write

$$K_{X_s} + D_s \equiv_{\mathbb{Q}} \sum_j a_j (D_j \cap X_s).$$

Using this representative for the numerical equivalence class of $K_{X_s} + D_s$, it then makes sense to ask that the pair (X_s, D_s) satisfy Assumption 1.1. We write $B_{(X_s,D_s)}$ for the polyhedral affine pseudomanifold associated to the pair (X_s, D_s): this will be pure $(n-1)$-dimensional, still with $n = \dim X$. We wish to understand the relationship between $B_{(X_s,D_s)}$ and B, the polyhedral affine pseudomanifold associated to the pair (X, D). This is described by the following proposition, which will be of use in [9].

Proposition 1.19 *Assume we are in the relative case, with Assumption 1.2 holding. Suppose that as described above, for $s \in S$ a closed point, $s \neq 0$, (X_s, D_s) satisfies Assumption 1.1. Then:*

(1) *$\partial B = g_{\mathrm{trop}}^{-1}(0)$ is naturally identified with the polyhedral cone complex $B_{(X_s,D_s)}$.*
(2) *The structure of integral affine manifold with boundary on B_0 can be extended across the interiors of cells $\omega \in \mathscr{P}_\partial^{[n-2]}$. Further, the induced polyhedral affine pseudomanifold structure on ∂B then agrees with that determined by the pair (X_s, D_s).*
(3) *Let $\iota : H_2(X_s) \to H_2(X)$ be the map of groups of curve classes induced by the inclusion of the fibre $X_s \hookrightarrow X$. Then for each $\omega \in \mathscr{P}_\partial^{[n-2]}$, the MPL function φ of Construction 1.14 has a single-valued representative in* $\mathrm{Star}(\omega)$. *Further, the restriction of this single-valued representative to* $\mathrm{Star}(\omega) \cap \partial B$ *has kink $\iota([X_\omega \cap X_s])$. We write the MPL function on ∂B with such kinks as $\varphi|_{\partial B}$.*

Proof Step 1: Proof of (1). We call a stratum Z of X *vertical* if $g|_Z$ is constant and otherwise say Z is *horizontal*. Note that all vertical strata lie over $0 \in S$. The rays of $\mathscr{P}$ contained in $g_{\mathrm{trop}}^{-1}(0)$ correspond precisely to the good horizontal components of D. Thus there is a one-to-one correspondence between cones $\rho \in \mathscr{P}$ with $\rho \subseteq g_{\mathrm{trop}}^{-1}(0)$ and good horizontal strata of D. There is also a one-to-one correspondence between horizontal strata of X and strata of X_s, taking a horizontal stratum Z to the stratum $Z \cap X_s$. Note that by Assumption 1.2, (2), this is indeed an irreducible stratum. Conversely, every stratum of X_s is necessarily of this form by definition of D_s. This correspondence takes good strata to good strata. Thus $\mathscr{P}_{(X_s,D_s)}$ may be identified with the set of cones of $\mathscr{P}$ contained in $g_{\mathrm{trop}}^{-1}(0)$. Further, since (X_s, D_s) satisfies Assumptions 1.1, $B_{(X_s,D_s)}$ is pure $(n-1)$-dimensional, and hence coincides with ∂B. This gives (1).

Step 2: Analysis of horizontal good two-dimensional strata. For (2), we need to give, for $\omega \in \mathscr{P}_{\partial}^{[n-2]}$, a chart $\psi_\omega : \mathrm{Star}(\omega) \to \mathbb{R}^n$ which agrees, up to elements of $\mathrm{GL}_n(\mathbb{Z})$, with ψ_ρ on $\mathrm{Star}(\rho) \subseteq \mathrm{Star}(\omega)$ whenever $\omega \subset \rho \in \mathscr{P}^{[n-1]}$.

To do so, we first analyze the geometry of the surface X_ω. As X_ω is horizontal, we obtain a non-constant map $g_\omega : X_\omega \to S$. We first analyze ∂X_ω. Since $X_\omega \cap X_s$ is one-dimensional, $X_\omega \cap X_s$ is disjoint from any bad divisor on X_s by Assumptions 1.1, (2). Hence X_ω is disjoint from any horizontal bad divisor. On the other hand, $X_\omega \cap X_s$ necessarily contains precisely two zero-dimensional strata, and hence X_ω contains two horizontal one-dimensional strata X_ρ, $X_{\rho'}$ with $\omega \subseteq \rho, \rho'$. These are the only horizontal one-dimensional strata of X_ω. Note they are necessarily sections of g_ω by Assumptions 1.2, (2). However, since $\partial^{\mathrm{good}} X_\omega$ is connected, there must be a chain of vertical good one-dimensional strata connecting X_ρ and $X_{\rho'}$. Again, by Assumptions 1.1, (2), these are disjoint from any bad divisor. Now the union of vertical strata coincides with $g_\omega^{-1}(0)$ and $g_\omega^{-1}(0)$ is connected (again by Assumption 1.2, (2)), so it follows that all one-dimensional strata of X_ω are disjoint from bad divisors. Thus X_ω is disjoint from all bad divisors.

By Proposition 1.3, each vertical one-dimensional stratum in X_ω, being good, contains precisely two zero-dimensional strata. On the other hand, $g_\omega^{-1}(0)$ contains two zero-dimensional strata given by $X_\rho \cap g_\omega^{-1}(0)$ and $X_{\rho'} \cap g_\omega^{-1}(0)$, and all other zero-dimensional strata of X_ω are then intersections of two good vertical one-dimensional strata. Thus the only possibility for the set of one-dimensional strata of X_ω is as follows: this set of strata may be written as $\{C_0, \ldots, C_r\}$, with $C_0 = X_\rho$, $C_r = X_{\rho'}$, $C_1, \ldots, C_{r-1}$ vertical strata, and with $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Further, $C_i \cap C_{i+1}$ consists of one point. See Fig. 2.

Now let $(B_\omega, \mathscr{P}_\omega)$ be the two-dimensional polyhedral affine pseudomanifold associated to the log Calabi–Yau variety $(X_\omega, \partial X_\omega)$ defined over S. We

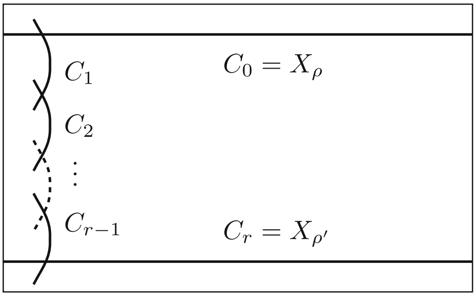

Fig. 2 The two-dimensional stratum X_ω

will describe B_ω and $\mathscr{P}_\omega$ explicitly. Let $\tau_0, \ldots, \tau_r$ be the rays of $\mathscr{P}_\omega$ corresponding to $C_0, \ldots, C_r$. Now define $n_{\tau_0} = (0,1) \in \mathbb{Z}^2$, $n_{\tau_1} = (1,0) \in \mathbb{Z}^2$, and inductively obtain $n_{\tau_2}, \ldots, n_{\tau_r} \in \mathbb{Z}^2$ by requiring that

$$n_{\tau_{\ell-1}} + n_{\tau_{\ell+1}} = -C_\ell^2 n_{\tau_\ell}, \quad 1 \le \ell \le r-1. \tag{1.10}$$

We may define a coordinate chart $\psi_{\tau_\ell} : \mathrm{Star}(\tau_\ell) \to \mathbb{R}^2$ by mapping primitive generators of $\tau_{\ell-1}$, τ_ℓ and $\tau_{\ell+1}$ to $n_{\tau_{\ell-1}}, n_{\tau_\ell}$ and $n_{\tau_{\ell+1}}$ respectively. By (1.6), this coordinate chart is compatible with the affine structure on $B_\omega \setminus \{0\}$. Indeed, in (1.6), we have $D_{i_j} = X_{\tau_\ell} = C_\ell$.

Now let $g_{\omega,\mathrm{trop}} : B_\omega \to \mathbb{R}_{\ge 0}$ be the tropicalization of $X_\omega \to S$. Note that $g_{\omega,\mathrm{trop}}|_{\mathrm{Star}(\tau_1)} \circ \psi_{\tau_1}^{-1}$ is given by $(1,0) \in (\mathbb{Z}^2)^*$. Indeed, as C_0 is not contained in $g^{-1}(0)$, $g_{\omega,\mathrm{trop}}$ vanishes on C_0. On the other hand, the multiplicity of C_1 in $g_\omega^{-1}(0)$ must be 1, as $C_0 \cdot g_\omega^{-1}(0) = 1$, C_0 being a section. Thus $g_{\omega,\mathrm{trop}}$ takes the value 1 on the primitive generator of τ_1.

Since ψ_{τ_ℓ} and $\psi_{\tau_{\ell+1}}$ agree on $\mathrm{Star}(\tau_\ell) \cap \mathrm{Star}(\tau_{\ell+1})$, in fact $g_{\omega,\mathrm{trop}}$ is given by $(0,1)$ on all charts. Since $g_{\omega,\mathrm{trop}}$ is positive on generators of $\tau_1, \ldots, \tau_{r-1}$ and takes the value 0 on τ_r, we see that all n_{τ_ℓ} lie in $\mathbb{R}_{\ge 0} \times \mathbb{R}$ and $n_{\tau_r} = (0,-1)$. Taking Σ_ω to be the fan in $\mathbb{R}^2$ with one-dimensional cones generated by n_{τ_ℓ}, $0 \le \ell \le r$ and two-dimensional cones spanned by $n_{\tau_\ell}, n_{\tau_{\ell+1}}$, $0 \le \ell \le r-1$, we see that the charts ψ_{ρ_ℓ} glue to give an identification of B_ω with $\mathbb{R}_{\ge 0} \times \mathbb{R}$ and $\mathscr{P}_\omega$ with Σ_ω, so that $B_\omega \setminus \{0\}$ is affine isomorphic to $(\mathbb{R}_{\ge 0} \times \mathbb{R}) \setminus \{(0,0)\}$. Further, under this identification, $g_{\omega,\mathrm{trop}}$ is given by projection onto $\mathbb{R}_{\ge 0}$.

Step 3: A chart on $\mathrm{Star}(\omega)$. We may now build a chart for $\mathrm{Star}(\omega)$ as follows. Let ω be generated by $D_{i_1}^*, \ldots, D_{i_{n-2}}^*$, and let ψ_j, $1 \le j \le n-2$, be the $\mathbb{R}$-valued piecewise linear function on B_ω with kink along τ_ℓ being $D_{i_j} \cdot C_\ell$. Explicitly, such ψ_j may be constructed by having it take the value 0 on some choice of two-dimensional cone in $\mathscr{P}_\omega$, and then using the relation

$$\psi_j(n_{\tau_{\ell-1}}) + \psi_j(n_{\tau_{\ell+1}}) = D_{i_j} \cdot C_\ell - C_\ell^2 \psi_j(n_{\tau_\ell}) \tag{1.11}$$

to determine the value of ψ_j on $n_{\tau_{\ell+1}}$ inductively. Indeed, the reader may easily check that the function defined on $\mathrm{Star}(\tau_\ell)$ satisfying (1.11) has kink $D_{i_j}\cdot C_\ell$ using (1.9) and (1.10).

Let $\rho_0,\ldots,\rho_r$ be the codimension one cones containing ω, with $X_{\rho_\ell}=C_\ell$, and let $D^*_{k_\ell}$ be such that ω and $D^*_{k_\ell}$ generate ρ_ℓ, $0\le\ell\le r$. We may then define $\psi_\omega:\mathrm{Star}(\omega)\to B_\omega\times\mathbb{R}^{n-2}$ via, with $e_1,\ldots,e_{n-2}$ the standard basis for $\mathbb{R}^{n-2}$,

$$\psi_\omega(D^*_{i_j})=(0,e_j),\qquad 1\le j\le n-2$$

and

$$\psi_\omega(D^*_{k_\ell})=\left(n_{\tau_\ell},-\sum_{j=1}^{n-2}\psi_j(n_{\tau_\ell})e_j\right),\qquad 0\le\ell\le r. \tag{1.12}$$

We then extend ψ_ω linearly on each cone containing ω. It is easy to see that ψ_ω induces an embedding of $\mathrm{Star}(\omega)$ into $B_\omega\times\mathbb{R}^{n-2}$. It is thus sufficient to check that ψ_ω is compatible with each ψ_{ρ_ℓ}, $1\le\ell\le r-1$, as defined in (1.5) and (1.6). For this, it is enough to verify the equality (1.6). We have for $1\le\ell\le r-1$,

$$\begin{aligned}
&\psi_\omega(D^*_{k_{\ell-1}})+\psi_\omega(D^*_{k_{\ell+1}})\\
&\overset{(1.12)}{=}\left(n_{\tau_{\ell-1}}+n_{\tau_{\ell+1}},-\sum_{j=1}^{n-2}(\psi_j(n_{\tau_{\ell-1}})+\psi_j(n_{\tau_{\ell+1}}))e_j\right)\\
&\overset{(1.10),(1.11)}{=}\left(-C_\ell^2 n_{\tau_\ell},-\sum_{j=1}^{n-2}(D_{i_j}\cdot C_\ell-C_\ell^2\psi_j(n_{\tau_\ell}))e_j\right)\\
&=-(D_{k_\ell}\cdot X_{\rho_\ell})\psi_\omega(D^*_{k_\ell})-\sum_{j=1}^{n-2}(D_{i_j}\cdot X_{\rho_\ell})\psi_\omega(D^*_{i_j}),
\end{aligned}$$

the last equality by $C_\ell^2=D_{k_\ell}\cdot X_{\rho_\ell}$. This verifies (1.6), as desired. Thus we obtain an affine chart on $\mathrm{Star}(\omega)$ compatible with the affine structure on B_0.

Step 4: Restricting the affine structure on $\mathrm{Star}(\omega)$ *to* ∂B. To complete the proof of (2), we compare the restriction of this chart to $\mathrm{Star}(\omega)\cap\partial B$ with the corresponding chart for ∂B arising from the pair (X_s,D_s). It is enough to verify that

$$\psi_\omega(D^*_{k_0})+\psi_\omega(D^*_{k_r})=-\sum_{j=1}^{n-2}\big((D_{i_j}\cap X_s)\cdot(X_\omega\cap X_s)\big)\psi_\omega(D^*_{i_j}), \tag{1.13}$$

where the intersection number is calculated on X_s. We first note that

$$\psi_j(n_{\tau_0}) + \psi_j(n_{\tau_r}) = D_{i_j} \cdot (X_\omega \cap g^{-1}(0)) = D_{i_j} \cdot (X_\omega \cap X_s). \tag{1.14}$$

Indeed, let $\sigma_{\ell,\ell+1} \in \mathscr{P}_\omega$ be the two-dimensional cone with boundary rays τ_ℓ and $\tau_{\ell+1}$, and let $n_{\tau_\ell} = (\alpha_\ell, \beta_\ell)$. Then by the definition of kink (1.9), for $1 \le \ell \le r-1$,

$$(d\psi_j|_{\sigma_{\ell,\ell+1}})(0,-1) + (d\psi_j|_{\sigma_{\ell-1,\ell}})(0,1) = \alpha_\ell(D_{i_j} \cdot X_{\rho_\ell}). \tag{1.15}$$

Since α_ℓ is the multiplicity of D_{k_ℓ} in $g^{-1}(0)$ by definition of g_{trop}, summing (1.15) for $1 \le \ell \le r-1$ gives (1.14). Thus we obtain

$$\begin{aligned}\psi_\omega(D^*_{k_0}) + \psi_\omega(D^*_{k_r}) = &\left(n_{\tau_0} + n_{\tau_r}, -\sum_{j=1}^{n-2}\big(\psi_j(n_{\tau_0}) + \psi_j(n_{\tau_r})\big)e_j\right)\\ &\overset{(1.14)}{=} \left(0, -\sum_{j=1}^{n-2}\big((D_{i_j} \cap X_s)\cdot(X_\omega \cap D_s)\big)e_j\right).\end{aligned}$$

Noting that $\psi_\omega(D^*_{i_j}) = (0, e_j)$, this gives (1.13).

Step 5: Proof of (3). For (3), note that we may obtain such a single-valued representative by constructing a single-valued piecewise linear function $\bar\varphi_\omega : B_\omega \to Q^{\mathrm{gp}}_{\mathbb{R}}$ with kink $[X_{\rho_\ell}]$ along τ_ℓ, just as we constructed ψ_j as above. Let $\mathrm{pr}_1 : B_\omega \times \mathbb{R}^{n-2} \to B_\omega$ be the first projection. We then obtain a single-valued function $\varphi_\omega = \bar\varphi_\omega \circ \mathrm{pr}_1 \circ \psi_\omega$ on $\mathrm{Star}(\omega)$ with the correct kinks along the ρ_ℓ. The kink of the restriction of φ_ω to $\mathrm{Star}(\omega) \cap \partial B$ is then easily calculated as $\bar\varphi_\omega(n_{\tau_0}) + \bar\varphi_\omega(n_{\tau_r}) = [X_\omega \cap g^{-1}(0)]$ as before, and the latter coincides with $\iota([X_\omega \cap X_s])$, as desired. □

2 Punctured log maps and tropical geometry

Throughout this section we will assume given $g : X \to S$ satisfying Assumptions 1.1 and 1.2 in the absolute and relative cases without further comment.

2.1 Review of notation

We briefly review notation from [4,5] for tropical maps to $\Sigma(X)$ and punctured maps to X. We first summarize the tropical language as developed in [4, Sect. 2.5] and [5, Sect. 2.2].

In what follows, **Cones** denotes the category of rational polyhedral cones with integral structure, i.e., objects are rational polyhedral cones $\omega \subseteq \Lambda_\omega \otimes_{\mathbb{Z}} \mathbb{R}$

for Λ_ω the lattice of integral tangent vectors[6] to ω. Morphisms are maps of cones induced by maps of the corresponding lattices. We write $\omega_{\mathbb{Z}} = \omega \cap \Lambda_\omega$ for the set of integral points of ω.

We consider graphs G, with sets of vertices $V(G)$, edges $E(G)$ and legs $L(G)$. In what follows, we will frequently confuse G with its topological realisation $|G|$. Legs will correspond to marked or punctured points of punctured curves, and are rays in the marked case and compact line segments in the punctured case. We view a compact leg as having only one vertex. An abstract tropical curve over $\omega \in \mathbf{Cones}$ is data $(G, \mathbf{g}, \ell)$ where $\mathbf{g} : V(G) \to \mathbb{N}$ is a genus function and $\ell : E(G) \to \mathrm{Hom}(\omega_{\mathbb{Z}}, \mathbb{N})\setminus\{0\}$ determines edge lengths. Since this paper will only deal with genus 0 curves, we omit $\mathbf{g}$ entirely from the sequel.

Associated to the data (G, ℓ) is a generalized cone complex (a diagram in the category of **Cones** with all morphisms being inclusions of faces induced by a morphism of corresponding lattices) $\Gamma(G, \ell)$ along with a morphism of cone complexes $\Gamma(G, \ell) \to \omega$ with fibre over $s \in \mathrm{Int}(\omega)$ being a tropical curve, i.e., a metric graph, with underlying graph G and affine edge length of $E \in E(G)$ being $\ell(E)(s) \in \mathbb{R}_{\geq 0}$. Associated to each vertex $v \in V(G)$ of G is a copy ω_v of ω in $\Gamma(G, \ell)$. Associated to each edge or leg $E \in E(G) \cup L(G)$ is a cone $\omega_E \in \Gamma(G, \ell)$ with $\omega_E \subseteq \omega \times \mathbb{R}_{\geq 0}$ and the map to ω given by projection onto the first coordinate. This projection fibres ω_E in compact intervals or rays over ω (rays for legs representing a marked point).

A *family of tropical maps* to $\Sigma(X)$ over $\omega \in \mathbf{Cones}$ is a morphism of cone complexes

$$h : \Gamma(G, \ell) \to \Sigma(X).$$

If $s \in \mathrm{Int}(\omega)$, we may view G as the fibre of $\Gamma(G, \ell) \to \omega$ over s as a metric graph, and write

$$h_s : G \to \Sigma(X)$$

for the corresponding tropical map with domain G. The *type* of such a family consists of the data $\tau := (G, \boldsymbol{\sigma}, \mathbf{u})$ where

$$\boldsymbol{\sigma} : V(G) \cup E(G) \cup L(G) \to \Sigma(X)$$

associates to $x \in V(G) \cup E(G) \cup L(G)$ the minimal cone of $\Sigma(X)$ containing $h(\omega_x)$. Further, $\mathbf{u}$ associates to each (oriented) edge or leg $E \in E(G) \cup L(G)$ the corresponding *contact order* $\mathbf{u}(E) \in \Lambda_{\boldsymbol{\sigma}(E)}$, the image of the tangent vector $(0, 1) \in \Lambda_{\omega_E} = \Lambda_\omega \oplus \mathbb{Z}$ under the map h.

[6] In [4,5], we write this lattice as N_ω; here we use the notation Λ_ω to fit the notation of [39].

As we shall only consider tropicalizations of pre-stable punctured curves (see [5, Def. 2.5]), following [5, Prop. 2.21] we may assume that for $L \in L(G)$ with adjacent vertex $v \in V(G)$ giving $\omega_L, \omega_v \in \Gamma(G, \ell)$, we have

$$h(\omega_L) = (h(\omega_v) + \mathbb{R}_{\geq 0}\mathbf{u}(L)) \cap \boldsymbol{\sigma}(L) \subseteq \Lambda_{\boldsymbol{\sigma}(L),\mathbb{R}}. \tag{2.1}$$

A *decorated type* is data $\boldsymbol{\tau} = (G, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A})$ where $\mathbf{A} : V(G) \to H_2(X)$ associates a curve class to each vertex of G. The *total curve class* of $\mathbf{A}$ is $A = \sum_{v \in V(G)} \mathbf{A}(v)$.

We also have a notion of a contraction morphism of types $\phi : \tau \to \tau'$, see [4, Def. 2.24]. This is a contraction of edges on the underlying graphs, and the additional data satisfies some relations as follows. If $x \in V(G) \cup E(G) \cup L(G)$, then $\boldsymbol{\sigma}'(\phi(x)) \subseteq \boldsymbol{\sigma}(x)$ (if x is an edge, it may be contracted to a vertex by ϕ). Further, if $E \in E(G) \cup L(G)$ then $\mathbf{u}(E) = \mathbf{u}'(\phi(E))$ under the inclusion $\Lambda_{\boldsymbol{\sigma}'(\phi(E))} \subseteq \Lambda_{\boldsymbol{\sigma}(E)}$, provided that E is not an edge contracted by ϕ.

We also recall the notion of *global contact order*, [5, Def. 2.29]. If $\sigma \in \Sigma(X)$, we set

$$\mathfrak{C}_\sigma(X) := \operatorname{colim}_{\sigma \subseteq \sigma'} \Lambda_{\sigma'},$$

where the colimit is taken in the category of sets, and define

$$\mathfrak{C}(X) := \coprod_{\sigma \in \Sigma(X)} \mathfrak{C}_\sigma(X).$$

A *global type of tropical or punctured map* is then $\bar{\tau} = (G, \boldsymbol{\sigma}, \bar{\mathbf{u}})$, where for each $E \in E(G) \cup L(G)$, $\bar{\mathbf{u}}(E) \in \mathfrak{C}_{\boldsymbol{\sigma}(E)}(X)$. Note that a type $\boldsymbol{\tau} = (G, \boldsymbol{\sigma}, \mathbf{u})$ determines a global type by replacing $\mathbf{u}(E)$ with its image under the natural map $\Lambda_{\boldsymbol{\sigma}(E)} \to \mathfrak{C}_{\boldsymbol{\sigma}(E)}(X)$.

We say a type (or global type) τ is *realizable* if there exists a family of tropical maps to $\Sigma(X)$ of type (or global type) τ. We also say $\boldsymbol{\tau} = (\tau, \mathbf{A})$ is realizable if τ is realizable. We note that if a global type $\bar{\tau}$ is realizable, there is only one type τ of tropical map realizing it (see [5, Lem. 3.5]). Most of the time in this paper, we will only deal with realizable types, and hence we will then not distinguish between a type and a global type in this case.

If a type τ is realizable, then there is a universal family of tropical maps of type τ, parameterized by an object of **Cones**. Hopefully without confusion, we will generally write this cone as τ. Hence we have a cone complex $\Gamma(G, \ell)$ equipped with a map to τ and a map of cone complexes $h = h_\tau : \Gamma(G, \ell) \to \Sigma(X)$. Generally we write h rather than h_τ when unambiguous. Note that for each $x \in E(G) \cup L(G) \cup V(G)$, we thus obtain $\tau_x \in \Gamma(G, \ell)$ the corresponding cone.

We write $\mathcal{X}$ for the Artin fan of X, see [6], as well as [4, Sect. 2.2] for a summary. We emphasize here that we always mean the absolute Artin fan, denoted as $\mathcal{A}_X$ in [5], even when we are working in the relative situation.

We refer to [5, Defs. 2.10, 2.13, 2.14] for the notion of a family $\pi : C^\circ \to W$ of punctured curves and pre-stable or stable punctured maps $f : C^\circ/W \to X$ or $f : C^\circ/W \to \mathcal{X}$. For the most part in this paper, we work with punctured maps with target X defined over $\operatorname{Spec} \Bbbk$, and in the relative case, only briefly work with punctured maps defined over S.

Given a punctured map with domain $C^\circ \to W$ and $W = \operatorname{Spec}(Q \to \kappa)$ for κ an algebraically closed field and target X or $\mathcal{X}$, we obtain by functoriality of tropicalizations a family of tropical maps

$$\begin{tikzcd} \Sigma(C) = \Gamma(G, \ell) \arrow[r] \arrow[d] & \Sigma(X) \\ \Sigma(W) = \omega = Q^\vee_{\mathbb{R}} & \end{tikzcd} \tag{2.2}$$

parameterized by W. The *type* of the punctured map is then the type $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$ of this family of tropical maps. We recall that the punctured map $f : C^\circ/W \to X$ is *basic* if (2.2) is the universal family of tropical maps of type τ.

Given a decorated global type $\bar{\boldsymbol{\tau}}$, [5, Def. 3.7] defines the notion of a marking of a punctured map by $\bar{\boldsymbol{\tau}}$. In particular, this gives rise to moduli spaces $\mathscr{M}(X, \bar{\boldsymbol{\tau}})$ (resp. $\mathfrak{M}(\mathcal{X}, \bar{\boldsymbol{\tau}})$) of stable (resp. pre-stable) $\bar{\boldsymbol{\tau}}$-marked punctured maps to X (resp. $\mathcal{X}$). When we are in the relative situation and wish to work with punctured maps to X defined over S, then we write $\mathscr{M}(X/S, \bar{\boldsymbol{\tau}})$ for the corresponding moduli space. We note that while curve classes in $\mathcal{X}$ are meaningless, the decoration on $\bar{\boldsymbol{\tau}}$ affects the notion of isomorphism in the category $\mathfrak{M}(\mathcal{X}, \bar{\boldsymbol{\tau}})$. By [5, Thm. 3.10], these are algebraic stacks, $\mathscr{M}(X, \bar{\boldsymbol{\tau}})$ is Deligne-Mumford and both stacks carry natural log structures, namely the basic log structure. Further, there is a natural morphism

$$\varepsilon : \mathscr{M}(X, \bar{\boldsymbol{\tau}}) \to \mathfrak{M}(\mathcal{X}, \bar{\boldsymbol{\tau}}) \tag{2.3}$$

given by composing a punctured map $C^\circ \to X$ with the canonical map $X \to \mathcal{X}$. [5, Sect. 4] then gives a perfect relative obstruction theory for ε. Most importantly, while the stacks $\mathfrak{M}(\mathcal{X}, \bar{\boldsymbol{\tau}})$ may be quite poorly behaved globally, in fact they have a simple local structure coming from the fact that they are idealized log smooth over S, see [5, Thm. 3.24, Rem. 3.25].

We sometimes work with moduli spaces $\mathfrak{M}(\mathcal{X}, \bar{\tau})$ of pre-stable $\bar{\tau}$-marked punctured maps to $\mathcal{X}$, forgetting the decorations. The forgetful morphism

$\mathfrak{M}(\mathcal{X}, \bar{\boldsymbol{\tau}}) \to \mathfrak{M}(\mathcal{X}, \bar{\tau})$ is strict étale, and we may also write $\varepsilon : \mathscr{M}(X, \bar{\boldsymbol{\tau}}) \to \mathfrak{M}(\mathcal{X}, \bar{\tau})$ for the composition of (2.3) with this forgetful morphism.

If there is a contraction morphism between decorated global types $\phi : \bar{\boldsymbol{\tau}} \to \bar{\boldsymbol{\tau}}'$, we obtain a forgetful map $\mathscr{M}(X, \bar{\boldsymbol{\tau}}) \to \mathscr{M}(X, \bar{\boldsymbol{\tau}}')$. This gives rise to a stratified description of these moduli spaces, see [5, Rem. 3.29].

There is one special choice of decorated type which imposes the minimal number of conditions on the curve. We always write β for a *class of punctured map*, which is a type where the underlying graph G has only one vertex and no edges, and $\boldsymbol{\sigma}$ takes the vertex and every leg to $\{0\} \in \Sigma(X)$. Put another way, a class of punctured map is the data $\beta = (\bar{\mathbf{u}}, A)$ where $\bar{\mathbf{u}}$ is a collection of global contact orders $\{\bar{u}_i\} \subseteq \mathfrak{C}_0(X)$ and $A \in H_2(X)$ is a curve class. Given a decorated global type $\bar{\boldsymbol{\tau}}$, there is always a unique choice of punctured curve class β with a contraction morphism $\bar{\boldsymbol{\tau}} \to \beta$. Here A is the total curve class of $\mathbf{A}$ if $\bar{\boldsymbol{\tau}} = (G, \boldsymbol{\sigma}, \bar{\mathbf{u}}, \mathbf{A})$. In this case we say the decorated type $\bar{\boldsymbol{\tau}}$ is *of class* β, and there is a canonical morphism $\mathscr{M}(X, \bar{\boldsymbol{\tau}}) \to \mathscr{M}(X, \beta)$, and an analogous morphism for maps to Artin fans. The moduli space $\mathscr{M}(X, \beta)$ is the moduli space of punctured maps where only the curve class and the contact orders of the marked and punctured points are fixed.

Suppose now given τ a realizable type equipped with a contraction morphism $\tau \to \tau'$ to a (not necessarily realizable) global type τ', and let $\boldsymbol{\tau}'$ be a decoration of τ'. As mentioned earlier, we do not distinguish between the type τ and its corresponding global type. We denote by $\mathfrak{M}_\tau(\mathcal{X}, \tau')$ the reduced closed substack of $\mathfrak{M}(\mathcal{X}, \tau')$ whose underlying closed subspace is the closure of the locus of geometric points whose corresponding punctured map is of type τ. We may then define

$$\mathscr{M}_\tau(X, \boldsymbol{\tau}') := \mathscr{M}(X, \boldsymbol{\tau}') \times_{\mathfrak{M}(\mathcal{X}, \tau')} \mathfrak{M}_\tau(\mathcal{X}, \tau').$$

While $\mathfrak{M}_\tau(\mathcal{X}, \tau')$, assuming non-empty, always carries points corresponding to curves of type τ, this may not be the case for $\mathscr{M}_\tau(X, \boldsymbol{\tau}')$. Nevertheless, the type τ'' of any curve over a geometric point in $\mathfrak{M}_\tau(\mathcal{X}, \tau')$ always has a (not necessarily unique) contraction map $\tau'' \to \tau$. We note that provided $\mathfrak{M}_\tau(\mathcal{X}, \tau')$ is non-empty, there is then a canonical morphism

$$\mathfrak{M}(\mathcal{X}, \tau) \to \mathfrak{M}_\tau(\mathcal{X}, \tau')$$

of degree $|\operatorname{Aut}(\tau/\tau')|$. Here $\operatorname{Aut}(\tau/\tau')$ denotes the group of automorphisms of the global type τ compatible with the contraction map $\tau \to \tau'$, i.e., automorphisms of the underlying graph G preserving $\boldsymbol{\sigma}$ and $\bar{\mathbf{u}}$ and the contraction map.

If $\bar{\boldsymbol{\tau}} = (G, \boldsymbol{\sigma}, \bar{\mathbf{u}}, \mathbf{A})$ denotes a choice of global type, and $I \subseteq E(G) \cup L(G)$ is a collection of edges and legs, then we write

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \bar{\boldsymbol{\tau}}) = \mathfrak{M}^{\mathrm{ev}(I)}(\mathcal{X}, \bar{\boldsymbol{\tau}}) := \mathfrak{M}(\mathcal{X}, \bar{\boldsymbol{\tau}}) \times_{\underline{\mathcal{X}}^I} \underline{X}^I.$$

Here $\underline{\mathcal{X}}^I$ denotes the product of $\# I$ copies of $\underline{\mathcal{X}}$ over $\underline{S}$, and similarly $\underline{X}^I$; the morphism $\mathfrak{M}(\mathcal{X}, \bar{\boldsymbol{\tau}}) \to \underline{\mathcal{X}}^I$ is given by evaluation at the nodes and punctured points indexed by elements of I, and $\underline{X}^I \to \underline{\mathcal{X}}^I$ is induced by the canonical smooth map $\underline{X} \to \underline{\mathcal{X}}$. The map ε then factors as

$$\mathscr{M}(X, \bar{\boldsymbol{\tau}}) \xrightarrow{\varepsilon^{\mathrm{ev}}} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \bar{\boldsymbol{\tau}}) \longrightarrow \mathfrak{M}(\mathcal{X}, \bar{\boldsymbol{\tau}}).$$

The second morphism is smooth, while $\varepsilon^{\mathrm{ev}}$ also possesses a relative obstruction theory compatible with the morphism ε of (2.3), see [5, Sect. 4.2].

2.2 Some tropical lemmas

We first observe the following result, which explains why the choice of affine structure on B given in Sect. 1.3 is the correct one.

Lemma 2.1 *Let $f : C^\circ/W \to X$ be a stable punctured map to X, with $W = \operatorname{Spec}(Q \to \kappa)$ a geometric log point. For $s \in \operatorname{Int}(Q^\vee_{\mathbb{R}})$, let $h_s : G \to \Sigma(X)$ be the corresponding tropical map. If $v \in V(G)$ satisfies $h_s(v) \in B_0$, then h_s satisfies the balancing condition at v. More precisely, if $E_1, \ldots, E_n$ are the legs and edges adjacent to v, oriented away from v, then the contact orders $\mathbf{u}(E_i)$ may be interpreted as elements of $\Lambda_{h_s(v)}$, the stalk of the local system Λ of Definition 1.13 at $h_s(v)$. In this group, the balancing condition*

$$\sum_{i=1}^{m} \mathbf{u}(E_i) = 0 \tag{2.4}$$

is satisfied. Further, in the relative case, the composition $\Sigma(g) \circ h_s : G \to \Sigma(S) = \mathbb{R}_{\geq 0}$ is a balanced tropical map.

Proof By definition of Δ, $h_s(v) \in B_0$ implies that $h_s(v) \in \operatorname{Int}(\sigma) \in \mathscr{P}$ for σ of codimension 0 or codimension 1. If $\sigma \in \mathscr{P}_{\max}$, then necessarily $\boldsymbol{\sigma}(E_i) = \sigma$ and $\mathbf{u}(E_i) \in \Lambda_\sigma$ is a tangent vector to σ, and hence can be viewed as an element of $\Lambda_{h_s(v)}$. Necessarily f contracts the corresponding component C_v of C to the zero-dimensional stratum X_σ, and hence the balancing condition holds from [5, Prop. 2.25], keeping in mind that τ_x given in the statement of that proposition vanishes because C_v is contracted.

If instead $\sigma \in \mathscr{P}^{[n-1]}$, note that by Assumptions 1.1, (2), σ is not contained in any cone of $\Sigma(X)$ which is not also in $\mathscr{P}$. Noting also that as $\boldsymbol{\sigma}(E_i)$ necessarily contains σ, we must have $\boldsymbol{\sigma}(E_i) \in \mathscr{P}$. In particular, $\mathbf{u}(E_i) \in \Lambda_{\boldsymbol{\sigma}(E_i)}$ may then be viewed, via parallel transport in Λ, as a tangent vector in $\Lambda_{h_s(v)}$.

We may now split the punctured map f at all nodes of C contained in C_v, and obtain by restriction a punctured log map $f_v : C_v^\circ \to X$ by [5, Prop. 5.2]. Note that the dual graph of C_v° consists of a single vertex with legs $E_1, \ldots, E_m$. Further, the contact order of f_v at the puncture corresponding to E_i agrees with the contact order $\mathbf{u}(E_i)$ for the map f of the corresponding edge or leg of G, oriented away from v. It is thus sufficient to show balancing for the tropical map induced by f_v.

Note f_v factors through the strict morphism $X_\sigma \hookrightarrow X$. Moreover, by Lemma 1.10, X_σ is isomorphic to a stratum of the toric variety X_{Σ_σ}, and hence we obtain a punctured map $f_v : C_v^\circ \to X_{\Sigma_\sigma}$. However, tropicalizations of punctured maps to toric varieties are always balanced when viewed as maps to the corresponding fans, see [5, Rem. 2.26]. The claimed balancing then follows from the construction of the affine chart ψ_σ.

Finally, in the relative case, necessarily the underlying map $\underline{g} \circ \underline{f}$ is constant on irreducible components of C since S is assumed to be affine. Thus balancing of $\Sigma(g) \circ h_s$ holds again from [5, Prop. 2.25]. □

Note that if $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$ is a type of tropical map to $\Sigma(X)$, the balancing condition of Lemma 2.1 still makes sense at vertices mapping to B_0. Indeed, for $v \in V(G)$ with $\boldsymbol{\sigma}(v)$ a codimension zero or one cone of $\mathscr{P}$, we may choose any point $x \in \mathrm{Int}(\boldsymbol{\sigma}(v))$ and (2.4) makes sense inside Λ_x. Further, in the relative case, by composing with $\Sigma(g)$, τ yields a type of tropical map to $\Sigma(S) = \mathbb{R}_{\geq 0}$, and again it makes sense to ask that such a type be balanced.

Thus we define:

Definition 2.2 A type τ of tropical map to $\Sigma(X)$ is *balanced* if:

(1) For each $v \in V(G)$ with $\boldsymbol{\sigma}(v) \in \mathscr{P}$ a codimension zero or one cone, the balancing condition (2.4) holds at v.
(2) In the relative case, τ induces a type of a balanced tropical map to $\Sigma(S)$.

The following observation shows that certain degree data of maps are determined by tropical data:

Lemma 2.3 *Let $\boldsymbol{\tau} = (\tau, \mathbf{A})$ be a decorated type, and suppose $\mathscr{M}(X, \boldsymbol{\tau})$ is non-empty. Then:*

(1) *For each $v \in V(G)$ with $\boldsymbol{\sigma}(v) \in \mathscr{P}_{\max}$, we have $\mathbf{A}(v) = 0$.*
(2) *Assume $v \in V(G)$ with $\rho = \boldsymbol{\sigma}(v) \in \mathscr{P}_{\mathrm{int}}^{[n-1]}$ contained in $\sigma \in \mathscr{P}_{\max}$. Let $E_1, \ldots, E_r$ be the edges adjacent to v with $\boldsymbol{\sigma}(E_i) = \sigma$ for $1 \leq i \leq r$. Let*

$\delta : \Lambda_\sigma \to \mathbb{Z}$ be the quotient map of Λ_σ by Λ_ρ positive on tangent vectors pointing from ρ into σ. Then $\mathbf{A}(v) = d[X_\rho]$, where

$$d = \sum_{i=1}^{r} \delta(\mathbf{u}(E_i)).$$

We note that this number is independent of the choice of σ containing ρ by the balancing condition.

Proof By assumption there exists a punctured map $f : C^\circ/W \to X$ over a geometric log point W marked by $\boldsymbol{\tau}$. Thus for a vertex $v \in V(G)$, we obtain via splitting as in the proof of Lemma 2.1 a subcurve C_v° and a punctured map $f_v : C_v^\circ \to X$.

In case (1), $\mathbf{A}(v) = 0$ is obvious because f_v maps C_v° to a zero-dimensional stratum.

In case (2), $\mathbf{A}(v)$ must be a multiple of the class $[X_\rho]$. If $\mathbb{R}_{\geq 0} D_i^*$ is the ray of σ not contained in ρ, then this multiple may be determined by intersecting the curve class of f_v with the divisor D_i. The claim now follows immediately from [48, Cor. 1.14]. □

Definition 2.4 Let G be a graph of genus 0. The *spine* of G is the smallest connected subgraph $G' \subseteq G$ containing all legs of G.

The following is the key tropical argument of the paper. In item (1) we consider the type of tropical map which will contribute to wall structures. These are tropical maps where the domain has only one (non-contracted) leg. Item (1) shows that this leg can sweep out at most a codimension one polyhedral cone: in the case of codimension one these will play the role of a wall in the canonical wall structure. Item (2), on the other hand, considers types which will correspond to broken lines with respect to the canonical wall structure, and the main point of (2) is to show that the spine of G, in this case, is mapped into $B_0 \subseteq |\Sigma(X)|$. This will be key for the logarithmic/broken line correspondence theorem, Theorem 4.14.

Lemma 2.5 *Fix a balanced realizable type $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$ of a genus zero tropical map to $\Sigma(X)$ with a distinguished leg $L_{\mathrm{out}} \in L(G)$ and $\mathbf{u}(L_{\mathrm{out}}) \neq 0$. Let $h : \Gamma(G, \ell) \to \Sigma(X)$ be the corresponding universal family of tropical maps, defined over the cone τ. For $s \in \tau$, we write $h_s : G \to \Sigma(X)$ for the induced map.*

(1) *Suppose G has only one leg, L_{out}, and let $\tau_{\mathrm{out}} \in \Gamma(G, \ell)$ be the corresponding cone. Suppose further that $\boldsymbol{\sigma}(L_{\mathrm{out}}) \in \mathscr{P}$. Then $\dim h(\tau_{\mathrm{out}}) \leq n - 1 = \dim X - 1$.*

(2) *Suppose G has precisely two legs, L_{in} and L_{out}, with $\boldsymbol{\sigma}(L_{\mathrm{in}}), \boldsymbol{\sigma}(L_{\mathrm{out}}) \in \mathscr{P}$. Suppose further that* $\dim \tau = n-1$ *and* $\dim h(\tau_{\mathrm{out}}) = n$. *Then with G' the spine of G and $s \in \mathrm{Int}(\tau)$, $h_s(G') \subseteq B \subseteq |\Sigma(X)|$, and $h_s(G')$ only intersects codimension zero and one cones of $\mathscr{P}$, except possibly for the non-vertex endpoints of L_{in} and L_{out}. Further,* $\dim h(\tau_v) = n-1$ *for every vertex v of the spine G'.*

Proof We prove both items simultaneously. We may assume in the first case that to the contrary $\dim h(\tau_{\mathrm{out}}) = n$, and thus in both cases we have $\dim h(\tau_{\mathrm{out}}) = n$. Further, in the first case we may also assume that $\dim \tau = n-1$. Indeed, if $\dim \tau > n-1$, there must be an $(n-1)$-dimensional face τ' of τ, necessarily corresponding to a type τ' which is a contraction of τ, such that $\dim h(\tau'_{\mathrm{out}}) = n$. Thus we may replace τ by τ' in this case to ultimately achieve a contradiction.

Write $\sigma_v := h(\tau_v)$. As $\dim \tau = n-1$, $\dim \sigma_v \leq n-1$.

We shall inductively find a sequence of distinct edges and legs $L_{\mathrm{out}} = E_1, \ldots, E_p$ and vertices $v_1, \ldots, v_{p-1}$ of G such that $E_1 = L_{\mathrm{out}}$ and v_i is a vertex of E_i and E_{i+1}. Further, this sequence will satisfy the following inductive properties, for $s \in \mathrm{Int}(\tau)$:

(1) The images of the edges $h_s(E_i)$ are all contained in B and only intersect codimension 0 and 1 cones of $\Sigma(X)$, except possibly for the non-vertex endpoint of L_{out} and, if $E_p = L_{\mathrm{in}}$, also the non-vertex endpoint of L_{in}.
(2) $\mathbf{u}(E_i)$ is not tangent to σ_{v_i}.
(3) $\dim \sigma_{v_i} = n-1$.

We will be able to continue the induction provided E_p is an edge. Thus in the first case of the lemma, the induction process would continue forever, a contradiction. In the second case, eventually $E_p = L_{\mathrm{in}}$, and G' is the union of $E_1, \ldots, E_p$, giving the desired result.

For the base case, we take $E_1 = L_{\mathrm{out}}$, v_1 the unique vertex of E_1. Note from (2.1) that

$$h(\tau_{\mathrm{out}}) = (\sigma_{v_1} + \mathbb{R}_{\geq 0}\mathbf{u}(L_{\mathrm{out}})) \cap \boldsymbol{\sigma}(L_{\mathrm{out}}). \tag{2.5}$$

As $\dim h(\tau_{\mathrm{out}}) = n$ by assumption, necessarily $\boldsymbol{\sigma}(L_{\mathrm{out}}) \in \mathscr{P}_{\max}$ and $\dim \sigma_{v_1} = n-1$. Thus $\boldsymbol{\sigma}(v_1) \in \mathscr{P}$ is either codimension 0 or 1, and in any case σ_{v_1} intersects the interior of $\boldsymbol{\sigma}(v_1)$. Further, $\boldsymbol{\sigma}(v_1)$ is a face of $\boldsymbol{\sigma}(L_{\mathrm{out}})$. It is then clear that for $s \in \mathrm{Int}(\tau)$, $h_s(v_1) \in \mathrm{Int}(\sigma_{v_1})$ and $h_s(L_{\mathrm{out}})$ is contained in B and only intersects codimension zero and one cones of $\Sigma(X)$ (except possibly for the non-vertex endpoint of L_{out}). Further, if $\mathbf{u}(L_{\mathrm{out}})$ were tangent to σ_{v_1}, then (2.5) implies $\dim h(\tau_{\mathrm{out}}) = n-1$, a contradiction. Thus inductive conditions (1)–(3) are satisfied.

We next observe that given a sequence of edges $L_{\mathrm{out}} = E_1, \ldots, E_p$ and vertices v_i of E_i and E_{i+1}, item (2) for $2 \leq i < p$ holds regardless of the details of the construction of this sequence of edges. Indeed, suppose that $\mathbf{u}(E_i)$ is tangent to σ_{v_i}. Split the graph G by detaching E_i from v_i to obtain two connected components G_1, G_2 such that E_i is a leg of G_1. Hence E_1 is also contained in G_1. Now, with $s \in \mathrm{Int}(\tau)$, necessarily $h_s(v_i) + \epsilon\mathbf{u}(E_i) \in \sigma_{v_i}$ for ϵ such that $|\epsilon|$ is sufficiently small. Thus there is an $s(\epsilon) \in \mathrm{Int}(\tau)$, depending on ϵ, with the property that $h_{s(\epsilon)}(v_i) = h_s(v_i) + \epsilon\mathbf{u}(E_i)$. In particular, by changing the length of the edge E_i, we may glue $h_s|_{G_1}$ to $h_{s(\epsilon)}|_{G_2}$ to obtain a tropical map $h_{s'} : G \to \Sigma(X)$ of type τ which does not coincide with h_s, but for which $h_{s'}(E_1) = h_s(E_1)$. Since $h_s(E_1)$ already varies in an $(n-1)$-dimensional family, necessarily $\dim\tau$ must be at least n, contradicting the assumption on the dimension.

Now assume further that $E_1, \ldots, E_p$ satisfy the inductive conditions (1)–(3). We wish to construct E_{p+1}. By assumption (1), σ_{v_p} is contained in a codimension one or codimension zero cone $\sigma = \boldsymbol{\sigma}(v_p)$ of $\mathscr{P}$, and thus it follows that for $s \in \mathrm{Int}(\tau)$, $h_s(v_p) \in B_0$. As τ is a balanced type, the balancing condition thus holds at v_p. Since $\mathbf{u}(E_p)$ is not tangent to σ_{v_p}, there must be at least one other edge E adjacent to v_p with $\mathbf{u}(E)$ not tangent to σ_{v_p} in order for balancing to hold. Choose one such edge to be E_{p+1}, and let v_{p+1} be the other vertex of E_{p+1} if E_{p+1} is an edge.

We check the inductive conditions (1) and (3). For (1), note that as $s \in \tau$ varies, $h_s(E_{p+1})$ varies in an $(n-1)$-dimensional family, and as $h_s(E_{p+1})$ is not tangent to σ_{v_p} by choice of E_{p+1}, $h_s(E_{p+1})$ fills out an n-dimensional subcone τ' of some $\sigma' \in \Sigma(X)$ containing σ. By Assumptions 1.1, (2), $\sigma' \in \mathscr{P}$. Further, for any $s \in \mathrm{Int}(\tau)$, $h_s(E_{p+1})$ may only intersect faces of σ' of codimension at most one, except possibly the non-vertex endpoint of E_{p+1} if $E_{p+1} = L_{\mathrm{in}}$. Indeed, this is clear from (2.1) in the latter case. Otherwise, E_{p+1} has another vertex v_{p+1}, and if $h_s(E_{p+1})$ meets a face $\sigma'' \subseteq \sigma'$ of codimension at least 2, then $h_s(v_{p+1}) \in \sigma''$ and $\boldsymbol{\sigma}(v_{p+1}) \subseteq \sigma''$. However, then $h_s(E_{p+1})$ can't vary in an $(n-1)$-dimensional family. Thus inductive condition (1) follows, and (3) is also clear for $\sigma_{v_{p+1}}$. □

Remark 2.6 In fact, the proof of the lemma tells us a bit more. In case (2), with notation as in the proof, all edges of G adjacent to v_i except for E_i and E_{i+1} are tangent to σ_{v_i}. Indeed, if not, there would be a choice for the edge E_{i+1} and the sequence of edges would not be unique. However, the edges $E_1, \ldots, E_p$ must be the unique sequence of edges from L_{out} to L_{in} in the spine G'.

3 The canonical wall structure and logarithmic broken lines

We continue with a pair (X, D) satisfying Assumptions 1.1 or 1.2 in the absolute and relative cases. We recall also we have fixed data:

(1) A group of degree data $H_2(X)$.

(2) A saturated finitely generated monoid $Q \subseteq H_2(X)$ such that $Q \cap (-Q) = H_2(X)_{\mathrm{tors}}$ which contains the classes of all effective curves on X. We write the monomial maximal ideal of Q

$$\mathfrak{m} := Q \setminus Q^\times.$$

Throughout this section, we also fix a monoid ideal $I \subseteq Q$ such that $\sqrt{I} = \mathfrak{m}$. Equivalently, we require $Q \setminus I$ to be finite.

3.1 Recall of monomials on *B*

We have constructed $(B, \mathscr{P}, \varphi)$ with φ a $Q^{\mathrm{gp}}_{\mathbb{R}}$-valued MPL function as given in Construction 1.14. This choice of function then yields a local system $\mathcal{P}$ [39, Def. 1.15] on $B_0 = B \setminus \Delta$ fitting into an exact sequence

$$0 \longrightarrow \underline{Q}^{\mathrm{gp}} \longrightarrow \mathcal{P} \longrightarrow \Lambda \longrightarrow 0,$$

where $\underline{Q}^{\mathrm{gp}}$ denotes the constant sheaf with stalk Q^{gp} on B_0.[7] We write the map $\mathcal{P}_x \to \Lambda_x$ as $p \mapsto \bar{p}$. Further, for each $x \in B_0$, [39, Def. 1.16] gives a submonoid $\mathcal{P}^+_x \subseteq \mathcal{P}_x$ of exponents of monomials defined at x.

For our purposes, rather than reviewing the definition of $\mathcal{P}$, it is easier to give explicit descriptions of the monoids $\mathcal{P}^+_x$ and the effects of parallel transport under these descriptions.

For $\sigma \in \mathscr{P}_{\max}$, $x \in \mathrm{Int}(\sigma)$, we have

$$\mathcal{P}^+_x = \Lambda_x \times Q. \tag{3.1}$$

For $\rho \in \mathscr{P}^{[n-1]}_{\partial}$, $x \in \mathrm{Int}(\rho)$, we have

$$\mathcal{P}^+_x = \Lambda_{\rho\sigma} \times Q \tag{3.2}$$

where $\Lambda_{\rho\sigma}$ is the monoid of integral tangent vectors contained in the tangent wedge $T_\rho \sigma$ of σ along the face ρ. If $\sigma \in \mathscr{P}_{\max}$ contains ρ, parallel transport in the local system $\mathcal{P}$ from x to $y \in \mathrm{Int}(\sigma)$ induces the inclusion $\mathcal{P}^+_x \hookrightarrow \mathcal{P}^+_y$ given by $(\lambda, q) \mapsto (\lambda, q)$.

For $\rho \in \mathscr{P}^{[n-1]}_{\mathrm{int}}$, $x \in \mathrm{Int}(\rho)$, we have

$$\mathcal{P}^+_x = (\Lambda_\rho \oplus \mathbb{N}Z_+ \oplus \mathbb{N}Z_- \oplus Q)/\langle Z_+ + Z_- = \kappa_\rho \rangle. \tag{3.3}$$

[7] In the case that Q^{gp} has torsion, [39, Def. 1.15] is not suitable. Rather, one may define $\mathcal{P}$ using the explicit description of parallel transport given below using the given kinks $\kappa_\rho \in Q^{\mathrm{gp}}$. In the formalism of [39], such torsion may be accommodated by incorporating it into the parameter ring A, see [39, Rem. 5.17]. However, we do not need such involved notions here.

This abstract description requires an ordering $\sigma, \sigma' \in \mathscr{P}_{\max}$ of the maximal cells containing ρ and a choice of vector $\xi \in \Lambda_x$ pointing into σ and representing a generator of $\Lambda_\sigma/\Lambda_\rho$. Then for $y \in \operatorname{Int}(\sigma)$, $y' \in \operatorname{Int}(\sigma')$, parallel transport in the local system $\mathcal{P}$ yields inclusions

$$\begin{aligned} \mathfrak{t}_{\rho\sigma} : \mathcal{P}_x^+ &\hookrightarrow \mathcal{P}_y^+ \\ (\lambda_\rho, aZ_+, bZ_-, q) &\mapsto \big(\lambda_\rho + (a-b)\xi, q + b\kappa_\rho\big) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \mathfrak{t}_{\rho\sigma'} : \mathcal{P}_x^+ &\hookrightarrow \mathcal{P}_{y'}^+ \\ (\lambda_\rho, aZ_+, bZ_-, q) &\mapsto \big(\lambda_\rho + (a-b)\xi, q + a\kappa_\rho\big) \end{aligned} \tag{3.5}$$

respectively. See the discussion of [39, Sect. 2.2].

Given a choice of monoid ideal $I \subseteq Q$, we also introduce the monoid ideal $I_x \subseteq \mathcal{P}_x^+$ when $x \in \operatorname{Int}(\sigma)$, $\sigma \in \mathscr{P}_{\max}$ defined in the description (3.1) as

$$I_x := \Lambda_x \times I. \tag{3.6}$$

Notation 3.1 For $x \in \operatorname{Int}(\sigma)$, $\sigma \in \mathscr{P}_{\max}$, we will often write a monomial in $\Bbbk[\mathcal{P}_x^+] = \Bbbk[Q][\Lambda_x]$ either as z^m for $m \in \mathcal{P}_x^+$ or as $t^q z^{\bar{m}}$ for $(\bar{m}, q) \in \Lambda_x \oplus Q$ via (3.1).

Similarly, if $x \in \operatorname{Int}(\rho)$ with $\rho \in \mathscr{P}^{[n-1]}$, we have a canonically defined submonoid $\Lambda_\rho \oplus Q \subseteq \mathcal{P}_x^+$ via (3.2) or (3.3), hence defining a subring $\Bbbk[Q][\Lambda_\rho] \subseteq \Bbbk[\mathcal{P}_x^+]$. We again write monomials in this ring as $t^q z^{\bar{m}}$ for $q \in Q$, $\bar{m} \in \Lambda_\rho$.

We will frequently need to use parallel transport in $\mathcal{P}^+$ from a cell $\sigma \in \mathscr{P}_{\max}$ to a cell $\sigma' \in \mathscr{P}_{\max}$, either with $\sigma = \sigma'$ or $\sigma \cap \sigma' = \rho \in \mathscr{P}^{[n-1]}$. Take any $x \in \operatorname{Int}(\sigma)$, $x' \in \operatorname{Int}(\sigma')$. If $\sigma = \sigma'$, we may define

$$\mathfrak{t}_{\sigma,\sigma'} : \mathcal{P}_x^+ \to \mathcal{P}_{x'}^+$$

to be given by parallel transport along a path contained in $\operatorname{Int}(\sigma)$; under the representation (3.1), this map is the identity. On the other hand, if $\sigma \cap \sigma' \in \mathscr{P}^{[n-1]}$, parallel transport along a path contained in $\operatorname{Star}(\rho)$ gives a map

$$\mathfrak{t}_{\sigma,\sigma'} : \mathcal{P}_x^+ \to \mathcal{P}_{x'}.$$

Noting that at the level of groups, $\mathfrak{t}_{\sigma,\sigma'} = \mathfrak{t}_{\rho\sigma'} \circ \mathfrak{t}_{\rho\sigma}^{-1}$, it follows that if $m \in \mathcal{P}_x^+$ and $\bar{m} \in T_\rho\sigma$, then $\mathfrak{t}_{\sigma,\sigma'}(m) \in \mathcal{P}_{x'}^+$. Thus, either in this case or the case $\sigma = \sigma'$, we may write $\mathfrak{t}_{\sigma,\sigma'}(m) \in \mathcal{P}^+_{x'}$ whenever

$$m \in \mathcal{P}^+_x, \quad \bar{m} \in T_{\sigma\cap\sigma'}\sigma.$$

If we let R'_I denote the subring of $(\Bbbk[Q]/I)[\Lambda_\sigma]$ generated by monomials of the form $t^q z^{\bar{m}}$ for $\bar{m} \in T_{\sigma\cap\sigma'}\sigma$, we obtain a ring homomorphism

$$\mathfrak{t}_{\sigma,\sigma'} : R'_I \to (\Bbbk[Q]/I)[\Lambda_{\sigma'}] \tag{3.7}$$

3.2 The canonical wall structure

3.2.1 Recall of wall structures

We recall the notion of *walls* and *wall structures* from [39, Def. 2.11]:

Definition 3.2 A *wall* on $(B, \mathscr{P})$ is a codimension one rational polyhedral subset $\mathfrak{p} \not\subseteq \partial B$ of some $\sigma \in \mathscr{P}_{\max}$, along with an element

$$f_{\mathfrak{p}} = \sum_{m\in\mathcal{P}^+_x, \bar{m}\in\Lambda_{\mathfrak{p}}} c_m z^m \in \Bbbk[\mathcal{P}^+_x],$$

for $x \in \mathrm{Int}(\mathfrak{p})$. Identifying $\mathcal{P}_y$ with $\mathcal{P}_x$ by parallel transport inside $\sigma\setminus\Delta$, we require that $m \in \mathcal{P}^+_y$ for all $y \in \mathfrak{p}\setminus\Delta$ when $c_m \neq 0$. We further require that $f_{\mathfrak{p}} \equiv 1 \mod \mathfrak{m}$.

Definition 3.3 A *wall structure* $\mathscr{S}$ on $(B, \mathscr{P})$ is a finite set of walls.

Remark 3.4 The above definitions differ in a couple of ways from that of [39, Def. 2.11]. First, we are less permissive with wall functions, insisting that $f_{\mathfrak{p}} \equiv 1 \mod \mathfrak{m}$. Second, we are more permissive with the notion of wall structure. In [39], we insist walls form the codimension one cells of a rational polyhedral decomposition of B refining $\mathscr{P}$. This is imposed there to make it easier to describe gluing. However, a wall structure in the above more liberal sense is equivalent (in the sense of Definition 3.5 below) to one in the sense of [39, Def. 2.11], and we ignore this issue in this section and the next, only returning to the convention of [39] in Sect. 5.

Definition 3.5 For a wall structure $\mathscr{S}$, we define

$$|\mathscr{S}| := \bigcup_{\mathfrak{p}\in\mathscr{S}} \mathfrak{p} \cup \bigcup_{\rho\in\mathscr{P}^{[n-1]}} \rho,$$

$$\mathrm{Sing}(\mathscr{S}) := \Delta \cup \bigcup_{\mathfrak{p}\in\mathscr{S}} \partial\mathfrak{p} \cup \bigcup_{\mathfrak{p},\mathfrak{p}'\in\mathscr{S}} (\mathfrak{p}\cap\mathfrak{p}')$$

where the last union is over all pairs of walls $\mathfrak{p}, \mathfrak{p}'$ with $\mathfrak{p} \cap \mathfrak{p}'$ codimension at least two.

If $x \in B\setminus \mathrm{Sing}(\mathscr{S})$, we define

$$f_x := \prod_{x\in\mathfrak{p}\in\mathscr{S}} f_{\mathfrak{p}}. \tag{3.8}$$

We say two wall structures are *equivalent (modulo I)* if $f_x = f'_x \mod I$ for all $x \in B\setminus(\mathrm{Sing}(\mathscr{S}) \cup \mathrm{Sing}(\mathscr{S}'))$. Generally we omit mention of I if clear from context.

3.2.2 The construction

Definition 3.6 A *wall type* is a type $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$ of tropical map to $\Sigma(X)$ such that:

(1) G is a genus zero graph with $L(G) = \{L_{\mathrm{out}}\}$ with $\boldsymbol{\sigma}(L_{\mathrm{out}}) \in \mathscr{P}$ and $u_\tau := \mathbf{u}(L_{\mathrm{out}}) \neq 0$.
(2) τ is realizable and balanced.
(3) Let $h : \Gamma(G, \ell) \to \Sigma(X)$ be the corresponding universal family of tropical maps, and $\tau_{\mathrm{out}} \in \Gamma(G, \ell)$ the cone corresponding to L_{out}. Then $\dim \tau = n - 2$ and $\dim h(\tau_{\mathrm{out}}) = n - 1$. Further, $h(\tau_{\mathrm{out}}) \not\subseteq \partial B$.

A *decorated wall type* is a decorated type $\boldsymbol{\tau} = (\tau, \mathbf{A})$ with τ a wall type.

Before using wall types to define the invariants we use in the canonical wall structure, we first make an observation in the relative case needed to show properness of the relevant moduli spaces. For most of the paper, we only work with the absolute moduli space $\mathscr{M}(X, \tau)$, but it turns out that in the relative case, we may also work with the relative moduli space $\mathscr{M}(X/S, \tau)$ when τ is a wall type. To make this precise, we note that a realizable type τ for a punctured map to X is a type for X/S if the universal tropical map over τ to $\Sigma(X)$ fits into a commutative diagram

$$\begin{tikzcd} \Gamma(G,\ell) \arrow[r, "h_\tau"] \arrow[d] & \Sigma(X) \arrow[d, "\Sigma(g)"] \\ \tau \arrow[r] & \Sigma(S) \end{tikzcd}$$

Proposition 3.7 *In the relative case, let τ be a wall type for X, and let β be the class of punctured map determined by the data u_τ and $A \in Q\setminus I$ a non-zero curve class. Then τ is a type of punctured map to X/S, and $\mathscr{M}(X, \beta) = \mathscr{M}(X/S, \beta)$.*

Proof We first show that the type $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$ is in fact a type for X/S. Since G is connected, it is sufficient to show that for each $E \in E(G) \cup L(G)$, $\Sigma(g)_*(\mathbf{u}(E)) = 0$. Recall that as a wall type, τ is realizable and balanced. Since τ is realizable, we obtain a family of tropical maps $h_s : G \to \Sigma(X), s \in \mathrm{Int}(\tau)$. Composing with $\Sigma(g)$ gives a tropical map $\Sigma(g) \circ h_s : G \to \Sigma(S) = \mathbb{R}_{\geq 0}$, which is balanced by Definition 2.2. But it is then immediate that this map must be constant, as any tropical map to $\mathbb{R}_{\geq 0}$ satisfying the balancing condition and with only one leg must be constant. Thus τ is a type defined over S.

The equalities of moduli spaces now follow from [5, Prop. 5.11]. □

Construction 3.8 Fix a wall type τ and a non-zero curve class $A \in Q \setminus I$. Let β be the class of punctured map determined by the data u_τ and A. Then we obtain a reduced closed stratum $\mathfrak{M}_\tau(\mathcal{X}, \beta) \subseteq \mathfrak{M}(\mathcal{X}, \beta)$, and a moduli space $\mathscr{M}_\tau(X, \beta)$ along with a morphism

$$\varepsilon : \mathscr{M}_\tau(X, \beta) \to \mathfrak{M}_\tau(\mathcal{X}, \beta).$$

Lemma 3.9 *$\mathscr{M}_\tau(X, \beta)$ is proper over* Spec $\Bbbk$ *and carries a virtual fundamental class of dimension* 0.

Proof In the absolute case, $\mathscr{M}_\tau(X, \beta)$ is closed substack of $\mathscr{M}(X, \beta)$, which is proper over Spec $\Bbbk$ by [5, Cor. 3.17].

We next consider the relative case. There is a morphism $\mathcal{X} \to \mathcal{S} = [\mathbb{A}^1/\mathbb{G}_m]$, induced by g, where $\mathcal{S}$ is the Artin fan of S. Write $\mathcal{X}_0 := \mathcal{X} \times_{\mathcal{S}} [0/\mathbb{G}_m]$; this is a closed substack of $\mathcal{X}$. As $\dim h(\tau_{\mathrm{out}}) = n - 1$ and $h(\tau_{\mathrm{out}}) \not\subseteq \partial B$, it follows that $\Sigma(g) : h(\tau_{\mathrm{out}}) \to \Sigma(S) = \mathbb{R}_{\geq 0}$ is surjective by Proposition 1.15. By Proposition 3.7, $\Sigma(g) \circ h_s$ is constant for each $s \in \tau$, and hence for any vertex $v \in V(G)$, $\Sigma(g) : h(\tau_v) \to \mathbb{R}_{\geq 0}$ is also surjective. From this it follows that any punctured map $C^\circ \to \mathcal{X}$ in $\mathfrak{M}_\tau(\mathcal{X}, \beta)$ has image lying set-theoretically in $|\mathcal{X}_0|$. Since $\mathfrak{M}_\tau(\mathcal{X}, \beta)$ is reduced by construction, any map must thus factor through $\mathcal{X}_0$ log-scheme-theoretically, and hence any punctured map in $\mathscr{M}_\tau(X, \beta)$ must factor through X_0 log-scheme-theoretically. Thus $\mathscr{M}_\tau(X, \beta)$ is a closed substack of $\mathscr{M}(X \times_S 0, \beta)$, which is again proper over Spec $\Bbbk$.

We now calculate the virtual dimension. By [5, Prop. 3.28], if $\bar{\tau}$ is the global type induced by τ, then $\mathfrak{M}(\mathcal{X}, \bar{\tau})$ is pure-dimensional and reduced of dimension

$$-3 + |L(G)| - \dim \tau = -3 + 1 - (n - 2) = -n.$$

The same is then true for $\mathfrak{M}_\tau(\mathcal{X}, \beta)$, as the forgetful map $\mathfrak{M}(\mathcal{X}, \bar{\tau}) \to \mathfrak{M}_\tau(\mathcal{X}, \beta)$ is finite of generic degree $|\mathrm{Aut}(\bar{\tau})|$. The virtual relative dimension of $\mathscr{M}_\tau(X, \beta)$ over $\mathfrak{M}_\tau(\mathcal{X}, \beta)$ at a punctured map $f : C^\circ \to X$ over a geometric point is $\chi(f^*\Theta_X) = A \cdot c_1(\Theta_X) + n$ by Riemann-Roch. Recall that $c_1(\Theta_X) = -(K_X + D) \equiv_{\mathbb{Q}} -\sum_i a_i D_i$ by assumption. Further, for any generator D_i^* of $\boldsymbol{\sigma}(L_{\mathrm{out}})$, it follows that $a_i = 0$ as $\boldsymbol{\sigma}(L_{\mathrm{out}}) \in \mathscr{P}$. Thus by [48, Cor. 1.14], $A \cdot D_i = 0$ whenever $a_i \neq 0$. Thus the total virtual dimension is 0 as claimed. $\square$

We now define

$$W_{\tau,A} := \deg[\mathscr{M}_\tau(X,\beta)]^{\mathrm{virt}}. \tag{3.9}$$

In addition, $h|_{\tau_{\mathrm{out}}} : \tau_{\mathrm{out}} \to \sigma$ induces a morphism

$$h_* : \Lambda_{\tau_{\mathrm{out}}} \to \Lambda_\sigma,$$

and we define

$$k_\tau := |\operatorname{coker}(h_*)_{\mathrm{tors}}| = |\Lambda_{h(\tau_{\mathrm{out}})}/h_*(\Lambda_{\tau_{\mathrm{out}}})|. \tag{3.10}$$

Finally, set

$$\boxed{\mathfrak{p}_{\tau,A} := \big(h(\tau_{\mathrm{out}}), \exp(k_\tau W_{\tau,A} t^A z^{-u_\tau})\big).} \tag{3.11}$$

Here we view $t^A z^{-u_\tau}$ as a monomial in $\Bbbk[\mathcal{P}_x^+]$ for $x \in \mathrm{Int}(h(\tau_{\mathrm{out}}))$ as in Notation 3.1. To view the exponential as a finite sum, note that $\Bbbk[Q][\Lambda_{h(\tau_{\mathrm{out}})}] \subseteq \Bbbk[\mathcal{P}_x^+]$, and we may truncate the infinite sum by removing all monomials which are zero in $(\Bbbk[Q]/I)[\Lambda_{h(\tau_{\mathrm{out}})}]$.

We then define:

Definition 3.10

$$\mathscr{S}_{\mathrm{can}}^{\mathrm{undec}} := \{\mathfrak{p}_{\tau,A} \mid \tau \text{ an isomorphism class of wall type, } A \in Q\setminus I,\ W_{\tau,A} \neq 0\}.$$

We note the superscript “undec” refers to the *undecorated* wall structure, in distinction with the *decorated* wall structure we will define in Construction 3.13, which will be equivalent to the above wall structure but which will be more useful in the proof of consistency.

Proposition 3.11 *$\mathscr{S}_{\mathrm{can}}^{\mathrm{undec}}$ is a wall structure.*

Proof We need to verify that (1) $\mathfrak{p}_{\tau,A}$ is always a wall and (2) $\mathscr{S}_{\mathrm{can}}^{\mathrm{undec}}$ is finite.

For the first item, fix τ, A. Write $u := u_\tau$. It is obvious that $h(\tau_{\mathrm{out}})$ is a rational polyhedral cone of codimension one by assumption. We need to check that the parallel transport of $(-u, A)$ to $\mathcal{P}_y$ lies in $\mathcal{P}_y^+$ for each point $y \in \mathfrak{p}_{\tau,A}\setminus\Delta$. Since Δ is the union of all codimension two cones of $\mathscr{P}$, this is only an issue if $y \in \rho \subseteq \sigma$ where ρ is codimension one and $\mathfrak{p}_{\tau,A} \nsubseteq \rho$. In this case, $\dim \boldsymbol{\sigma}(L_{\mathrm{out}}) = n$. Let $v \in V(G)$ be the vertex adjacent to L_{out},

$\tau_v \in \Gamma(G, \ell)$ the corresponding cone. We divide the analysis into three cases, depending on the relationship between y and $h(\tau_v)$.

Case 1: $y \notin h(\tau_v)$. From (2.1), necessarily $-u \in T_\rho\sigma$. If $y \in \partial B$, then it is immediate from (3.2) that $(-u, A) \in \mathcal{P}_y^+$. If instead $y \notin \partial B$, we may choose $\xi \in \Lambda_y$ pointing into σ as in the description of $\mathcal{P}_y^+$ of (3.3), and then write $-u = u' + a\xi$ for some $a > 0$ and $u' \in \Lambda_\rho$. Thus by (3.4), $(-u, A)$ is identified with the element $(u', aZ_+, 0, A)$ of $\mathcal{P}_y^+$.

Case 2: $y \in h(\tau_v) \cap \partial B$. We are thus in the relative case. By Proposition 3.7, β must be defined over S, which in particular means that u is tangent to the fibres of g_{trop}, and hence u is tangent to ∂B. However, by (3.2), it then follows that $(-u, A) \in \mathcal{P}_y^+$.

Case 3: $y \in h(\tau_v) \setminus \partial B$. Here we assume that $W_{\tau,A} \neq 0$, (as otherwise such a wall would not be included in $\mathscr{S}_{\mathrm{can}}^{\mathrm{undec}}$). Thus there is necessarily a punctured map $f : C^\circ/W \to X$ over a geometric log point with type τ' equipped with a contraction to τ. Thus we may mark f with τ, and this leads to a decorated type $\boldsymbol{\tau} = (\tau, \mathbf{A})$ with $\mathbf{A}(v)$ given by the curve class of the map $f_v : C_v^\circ \to X$, where C_v° is the subcurve of C° corresponding to $v \in V(G)$, as in the proof of Lemma 2.1. In this case, u points into σ, and in the notation δ of Lemma 2.3, (2), $\mathbf{A}(v) = d[X_\rho]$ with $d \geq \delta(u)$. Thus the total curve class A satisfies $A = \delta(u)[X_\rho] + A'$ for some $A' \in Q$.

We may now use the description (3.3) along with (3.4) to test whether $(-u, A) \in \mathcal{P}_x^+$ lies in the image of $\mathfrak{t}_{\rho\sigma} : \mathcal{P}_y^+ \to \mathcal{P}_x^+$. Choosing $\xi \in \Lambda_y$ as before, we may now write $u = u' + \delta(u)\xi$ for some $u' \in \Lambda_\rho$. Thus $(-u, A)$ equals $\mathfrak{t}_{\rho\sigma}$ applied to

$$(-u', 0, \delta(u)Z_-, A - \delta(u)\kappa_\rho) = (-u', 0, \delta(u)Z_-, A') \in \mathcal{P}_y^+.$$

We have now covered all possible cases for the location of y, and hence $\mathfrak{p}_{\tau,A}$ is a wall.

To show $\mathscr{S}_{\mathrm{can}}^{\mathrm{undec}}$ is finite, we first observe that as $Q \setminus I$ is finite by assumption on Q and I, there are only a finite number of choices for A. For determining τ, there are only a finite number of possibilities for $\boldsymbol{\sigma}(L_{\mathrm{out}}) \in \mathscr{P}$. Given a choice of A and wall type τ with given $\boldsymbol{\sigma}(L_{\mathrm{out}})$, $W_{\tau,A} \neq 0$ implies that $\mathscr{M}_\tau(X, \beta)$ is non-empty, and then [48, Cor. 1.14] shows u_τ is determined by the choice of A and $\boldsymbol{\sigma}(L_{\mathrm{out}})$. If β is the punctured curve class determined by A and a given u_τ, then, since $\mathscr{M}(X, \beta)$ is finite type, there are only a finite number of types of tropical maps τ appearing in the tropicalizations of curves in $\mathscr{M}(X, \beta)$, i.e., there are only a finite number of τ such that $\mathscr{M}_\tau(X, \beta)$ is non-empty. □

The main result of the paper, to be proved in Sect. 5, can then be stated:

Theorem 3.12 $\mathscr{S}_{\mathrm{can}}^{\mathrm{undec}}$ *is a consistent wall structure in the sense of* [39, Def. 3.9].

The above definition of the canonical wall structure is conceptually the simplest and most useful in practice (see [9] for some explicit examples). However, for the proofs of this paper, it is convenient to replace $\mathscr{S}_{\mathrm{can}}^{\mathrm{undec}}$ with an equivalent wall structure $\mathscr{S}_{\mathrm{can}}$ using decorated types as follows.

Construction 3.13 Fix a decorated wall type $\boldsymbol{\tau} = (\tau, \mathbf{A})$, and $A = \sum_{v\in V(G)} \mathbf{A}(v)$ the total curve class. As τ is realizable, we may view it equivalently as a global type. Hence we obtain a morphism of moduli spaces $\varepsilon : \mathscr{M}(X, \boldsymbol{\tau}) \to \mathfrak{M}(\mathcal{X}, \boldsymbol{\tau})$. From the proof of Lemma 3.9, one sees that $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau})$ is pure-dimensional and $[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}}$ is a zero dimensional cycle. Hence we may define

$$W_{\boldsymbol{\tau}} := \frac{\deg[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau})|}.$$

We may then define a wall

$$\boxed{\mathfrak{p}_{\boldsymbol{\tau}} := \big(h(\tau_{\mathrm{out}}), \exp(k_\tau W_{\boldsymbol{\tau}} t^A z^{-u_\tau})\big)} \tag{3.12}$$

and

$$\mathscr{S}_{\mathrm{can}} := \left\{ \mathfrak{p}_{\boldsymbol{\tau}} \;\middle|\; \begin{matrix} \boldsymbol{\tau} \text{ an isomorphism class of decorated wall} \\ \text{type with total curve class lying in } Q\setminus I \end{matrix} \right\}. \tag{3.13}$$

We note here we do not exclude walls with $W_{\boldsymbol{\tau}} = 0$, so this wall structure may include an infinite set of trivial walls, i.e., with attached function 1. So technically this is not a wall structure, but if we remove all trivial walls, it becomes a finite set as in Proposition 3.11. However, for bookkeeping purposes, it will prove useful to include these trivial walls.

To see that this gives a wall structure equivalent (in the sense of Definition 3.5) to the previous definition, fix a wall type τ and a curve class $A \in Q\setminus I$ such that $W_{\tau,A} \neq 0$; this data will give one wall in the earlier definition of $\mathscr{S}_{\mathrm{can}}$. With $\beta = (\{u_\tau\}, A)$ the associated class of punctured map, we have the canonical morphism $\mathfrak{M}(\mathcal{X}, \tau) \to \mathfrak{M}_\tau(\mathcal{X}, \beta)$, which is surjective and finite of degree $|\operatorname{Aut}(\tau)|$. Further, we have a Cartesian diagram in all categories [5, Prop. 5.19]

$$\begin{tikzcd}
\coprod_{\boldsymbol{\tau}=(\tau,\mathbf{A})} \mathscr{M}(X, \boldsymbol{\tau}) \arrow[r, "j'"] \arrow[d, "\varepsilon_\tau"'] & \mathscr{M}(X, \beta) \arrow[d, "\varepsilon"] \\
\mathfrak{M}(\mathcal{X}, \tau) \arrow[r, "j"'] & \mathfrak{M}(\mathcal{X}, \beta)
\end{tikzcd}$$

Here $\mathbf{A}$ runs over all decorations of τ with total curve class A. As j is a map of degree $|\operatorname{Aut}(\tau)|$ onto $\mathfrak{M}_\tau(\mathcal{X}, \beta)$, we see that

$$\begin{aligned} W_{\tau,A} &= \deg \varepsilon^![\mathfrak{M}_\tau(\mathcal{X}, \beta)] = \frac{\deg \varepsilon^! j_*[\mathfrak{M}(\mathcal{X}, \tau)]}{|\operatorname{Aut}(\tau)|} \\ &= \frac{\deg j'_* \varepsilon^!_\tau[\mathfrak{M}(\mathcal{X}, \tau)]}{|\operatorname{Aut}(\tau)|} = \sum_{\boldsymbol{\tau}=(\tau,\mathbf{A})} \frac{\deg[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\tau)|}. \end{aligned} \tag{3.14}$$

Here the third equality holds by push-pull of [65, Thm. 4.1], and the last summation runs over all choices of decorations $\mathbf{A}$ of τ with total curve class A. Note however that $\operatorname{Aut}(\tau)$ acts on the set of all choices of decoration $\mathbf{A}$, with orbits of this action giving isomorphism classes of decorated types $\boldsymbol{\tau}$. The stabilizer of a given $\boldsymbol{\tau}$ is the subgroup $\operatorname{Aut}(\boldsymbol{\tau}) \subseteq \operatorname{Aut}(\tau)$, and hence the orbit containing $\boldsymbol{\tau}$ is of size $|\operatorname{Aut}(\tau)|/|\operatorname{Aut}(\boldsymbol{\tau})|$. Thus the final expression of (3.14) can be now expressed as a sum over isomorphism classes of decorations $\boldsymbol{\tau}$ of τ with total class A:

$$W_{\tau,A} = \sum_{\boldsymbol{\tau}} \frac{\deg[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\tau)|} \cdot \frac{|\operatorname{Aut}(\tau)|}{|\operatorname{Aut}(\boldsymbol{\tau})|} = \sum_{\boldsymbol{\tau}} \frac{\deg[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau})|}. \tag{3.15}$$

From this, we see the collection of walls in the new definition of $\mathscr{S}_{\mathrm{can}}$ coming from all $\boldsymbol{\tau}$ with underlying type τ and total curve class A is equivalent to the corresponding single wall in the original definition of $\mathscr{S}_{\mathrm{can}}$.

Example 3.14 The construction of $\mathscr{S}_{\mathrm{can}}$ agrees (up to equivalence) with the construction of the canonical scattering diagram of [37] when $\dim X = 2$ and $K_X + D = 0$, so that $B = \Sigma(X)$. In other words, the construction given here is a generalization of the construction of [37].

To see this, we analyze decorated wall types $\boldsymbol{\tau}$ in dimension two. Since we require $\dim \tau = \dim X - 2$, in fact such types will be rigid. Thus there is a unique tropical map $h : G \to B$ of type τ, and all vertices of G are mapped to $0 \in B$; otherwise, rescaling B provides a non-trivial deformation of h. Further, we require $\dim h(\tau_{\mathrm{out}}) = 1$. So $h(\tau_{\mathrm{out}})$ is a ray in B with endpoint 0. Hence there is no choice but for $u_\tau \in B(\mathbb{Z})\setminus\{0\}$ and $h(\tau_{\mathrm{out}}) = \mathbb{R}_{\geq 0} u_\tau$. Further, any edge of G contracted by h has length a free parameter of the tropical curve, and hence again by rigidity, there are no edges. Thus the only possibility for $\boldsymbol{\tau}$ is that G has one vertex, with an attached curve class A, and one leg, L_{out}, with $\mathbf{u}(L_{\mathrm{out}}) = u_\tau \in B(\mathbb{Z})$. Note further that k_τ is then just the index of u_τ, i.e., the degree of divisibility of u_τ in $B(\mathbb{Z})$. Of course $|\operatorname{Aut}(\boldsymbol{\tau})| = 1$.

To make a comparison with the setup of [37], it is then convenient, for $u \in B(\mathbb{Z})$ primitive, to define

$$W_{A,u,k} := W_{\boldsymbol{\tau}},$$

where $\boldsymbol{\tau}$ is the decorated wall type as described above with curve class A and $u_\tau = ku$. In [37], only one wall for each ray $\mathbb{R}_{\geq 0}u$ of rational slope occurs, and hence it is then more useful to write the canonical wall structure in the equivalent form

$$\mathscr{S}'_{\mathrm{can}} := \left\{ \left(\mathbb{R}_{\geq 0}u, \exp\left(\sum_{A,k} k W_{A,u,k} t^A z^{-ku} \right) \right) \,\middle|\, u \in B(\mathbb{Z})\setminus\{0\} \text{ primitive} \right\}.$$

Here the sum is over all positive integers k and all curve classes A. However, if we write $u = aD_i^* + bD_j^*$ for some non-negative a, b with $a + b > 0$, then by the balancing condition as expressed in [48, Cor. 1.14], we may in fact sum over only those curve classes with $A \cdot D_i = a$, $A \cdot D_j = b$, $A \cdot D_k = 0$ for $k \neq i, j$.

In [37], a wall function is associated to each ray $\mathfrak{d} = \mathbb{R}_{\geq 0}u$ of rational slope using invariants defined as follows. Assuming $\mathfrak{d}$ does not coincide with a ray of $\mathscr{P}$, one performs a toric (log étale) blow-up $\pi : \widetilde{X} \to X$ by refining $(B, \mathscr{P})$ along the given ray $\mathfrak{d}$. Otherwise, if $\mathfrak{d}$ coincides with a ray of $\mathscr{P}$, we may take π to be the identity. Let $\widetilde{A}$ be a curve class on $\widetilde{X}$ with the following properties, depending on whether or not π is the identity. If π is not the identity, then we require that $\widetilde{A}$ has trivial intersection number with all boundary components of $\widetilde{X}$ except for the exceptional divisor E of π. If π is the identity, let E be the component of D corresponding to the ray $\mathfrak{d}$. We then require that $\widetilde{A}$ has zero intersection number with each component of D except for E. Then [37], following [40, Sect. 4], defines a number $N_{\widetilde{A}}$. This number is defined as a *relative* Gromov–Witten invariant for the non-compact pair $(\widetilde{X}^\circ, E^\circ)$ where $\widetilde{X}^\circ \subseteq \widetilde{X}$ is obtained by removing the closure of $\pi^{-1}(D)\setminus E$ from $\widetilde{X}$, and $E^\circ = E \cap \widetilde{X}^\circ$. This relative invariant counts rational curves of class $\widetilde{A}$ with one marked point, with contact order $k_{\widetilde{A}} := \widetilde{A} \cdot E$ with E°.

To compare $N_{\widetilde{A}}$ with the type of number considered in this paper, note that the data of $\widetilde{A}$ and the contact order $k_{\widetilde{A}}$ also specifies a type $\widetilde{\beta}$ of logarithmic map to the pair $(\widetilde{X}, \pi^{-1}(D))$, and we may compare the logarithmic Gromov–Witten invariant $\deg[\mathscr{M}(\widetilde{X}, \widetilde{\beta})]^{\mathrm{virt}}$ with $N_{\widetilde{A}}$. In fact, it follows from [7] that these two numbers will agree provided that every stable log map in the moduli space $\mathscr{M}(\widetilde{X}, \widetilde{\beta})$ factors through $\widetilde{X}^\circ$. However, it is an elementary exercise in tropical geometry to show that if given a stable log map $f : C/W \to \widetilde{X}$ of type $\widetilde{\beta}$ with W a log point, any tropical map in the corresponding family of tropical maps has image $\mathfrak{d}$. This may be proved using an argument similar to the argument given in Lemma 2.5, but a similar tropical argument in two dimensions has already appeared in the proof of [17, Lem. 12], which may also be viewed as a tropical interpretation of [40, Prop. 4.2]. On the other hand, any stable log map of type $\widetilde{\beta}$ which does not factor through $\widetilde{X}^\circ$ will necessarily have a tropicalization whose image does not coincide with $\mathfrak{d}$, as follows immediately from the construction of tropicalization.

Finally, let $A = \pi_* \widetilde{A}$. Note that in the case π is not the identity, the curve class $\widetilde{A}$ is uniquely determined by A and the constraint that $\widetilde{A}$ has zero intersection number with components of the strict transform of D. Then if u is the primitive generator of $\mathfrak{d} \cap B(\mathbb{Z})$, we may interpret $k_{\widetilde{A}} u \in B(\mathbb{Z})$ as a contact order, and the class A and the contact order $k_{\widetilde{A}} u$ determines a type of log map β to X. It then follows from the birational invariance of log Gromov–Witten theory of [8, Thm. 1.1.1] that $\deg[\mathscr{M}(\widetilde{X}, \widetilde{\beta})]^{\mathrm{virt}} = \deg[\mathscr{M}(X, \beta)]^{\mathrm{virt}}$. In particular, $N_{\widetilde{A}} = W_{A,u,k_{\widetilde{A}}}$.

Conversely, given the data of A, u and k giving a wall type τ with $\mathscr{M}(X, \tau)$ non-empty, we obtain via the above blow-up procedure a curve class $\widetilde{A}$. Thus we have a one-to-one correspondence between the set of data contributing to $\mathscr{S}'_{\mathrm{can}}$ and the data contributing to the canonical scattering diagram of [37]. Further, we have just observed an equality of the invariants. The equivalence of $\mathscr{S}'_{\mathrm{can}}$ with the canonical scattering diagram of [37] now follows by inspection of the definition of the canonical scattering diagram in [37].

Example 3.15 Returning to Examples 1.4 and 1.11, we first introduce notation for the relevant curve classes. Let e_i be the class of a fibre of $\pi|_{E_i} : E_i \to Z_i$. Let f be the class of a curve of the form $\{a\} \times \mathbb{P}^1 \times \{b\}$ disjoint from the centers Z_1, Z_2.

One can show that the only decorated wall types $\boldsymbol{\tau}$ with $W_{\boldsymbol{\tau}} \neq 0$ are of the following 5 types. In each case, the underlying graph G of $\boldsymbol{\tau}$ has one vertex v, no edges, and of course a unique leg L_{out}.

(1) For some $k > 0$, we have $u_\tau = kD^*_{2,0}$, $\boldsymbol{\sigma}(L_{\mathrm{out}}) = \mathfrak{p}_1 := \mathbb{R}_{\geq 0} D^*_{2,0} + \mathbb{R}_{\geq 0} D^*_{1,0}$, and $\mathbf{A}(v) = ke_1$.
(2) For some $k > 0$, we have $u_\tau = kD^*_{2,0}$, $\boldsymbol{\sigma}(L_{\mathrm{out}}) = \mathfrak{p}_2 := \mathbb{R}_{\geq 0} D^*_{2,0} + \mathbb{R}_{\geq 0} D^*_{1,\infty}$, and $\mathbf{A}(v) = ke_1$.
(3) For some $k > 0$, we have $u_\tau = k(E^*_2 - D^*_{1,\infty})$, $\boldsymbol{\sigma}(L_{\mathrm{out}}) = \mathfrak{p}_3 := \mathbb{R}_{\geq} E^*_2 + \mathbb{R}_{\geq 0} D^*_{1,\infty}$, and $\mathbf{A}(v) = k(f - e_1 - e_2)$.
(4) For some $k > 0$, we have $u_\tau = kD^*_{2,\infty}$, $\boldsymbol{\sigma}(L_{\mathrm{out}}) = \mathfrak{p}_4 := \mathbb{R}_{\geq 0} E^*_2 + \mathbb{R}_{\geq 0} D^*_{2,\infty}$, and $\mathbf{A}(v) = k(f - e_1)$.
(5) For some $k > 0$, we have $u_\tau = kD^*_{2,\infty}$, $\boldsymbol{\sigma}(L_{\mathrm{out}}) = \mathfrak{p}_5 := \mathbb{R}_{\geq 0} D^*_{1,0} + \mathbb{R}_{\geq 0} D^*_{2,\infty}$, and $\mathbf{A}(v) = k(f - e_1)$.

In all cases, $h(\tau_{\mathrm{out}}) = \boldsymbol{\sigma}(L_{\mathrm{out}})$, so walls only have five possible supports. Further, in every case, $k_\tau = k$ and $W_{\boldsymbol{\tau}} = (-1)^{k-1}/k^2$. After passing to an equivalent scattering diagram with only one wall with a given support, we see, for example, that we have a wall

$$\left(\mathfrak{p}_1, \exp\left(\sum_{k>0} k\frac{(-1)^{k-1}}{k^2} t^{ke_1} z^{-kD^*_{2,0}}\right)\right) = \left(\mathfrak{p}_1, 1 + t^{e_1} z^{-D^*_{2,0}}\right).$$

Similarly, we have four other walls:

$$\left(\mathfrak{p}_2, 1 + t^{e_1} z^{-D^*_{2,0}}\right), \left(\mathfrak{p}_3, 1 + t^{f-e_1-e_2} z^{D^*_{1,\infty}-E^*_2}\right),$$
$$\left(\mathfrak{p}_4, 1 + t^{f-e_1} z^{-D^*_{2,\infty}}\right), \left(\mathfrak{p}_5, 1 + t^{f-e_1} z^{-D^*_{2,\infty}}\right).$$

Together, these walls cover the affine plane contained in B which is the union of all two-dimensional cones of $\mathscr{P}$ not containing $D^*_{3,0}$ or $D^*_{3,\infty}$. We omit a derivation of these results. Showing (1)–(5) are the only possibilities is not difficult using [48, Cor. 1.14] to encode balancing requirements for punctured maps. A direct calculation of $W_{\boldsymbol{\tau}}$ has not been carried out, but presumably these multiple cover calculations can be carried out as in the two-dimensional case of [40, Prop. 5.2]. Instead, the above formulas for the wall functions are proved in a more general context in [9]. In fact, the formulas for these wall functions follows from the consistency of $\mathscr{S}_{\mathrm{can}}$ proved in the present paper.

The walls $\mathfrak{p}_i$ for $i \neq 3$ arise from zero-dimensional strata in one-dimensional moduli spaces of ordinary (non-punctured) stable log maps. For example, the inclusion of every fibre of $\pi|_{E_1} : E_1 \to Z_1$ into X may be viewed as a stable log map. The walls $\mathfrak{p}_1$ and $\mathfrak{p}_2$ capture the curves in this family which are degenerate with respect to the log structure on X, i.e., fall into $D_{1,0}$ or $D_{1,\infty}$. For $k > 1$, we count multiple covers of these fibres.

The same holds for $\mathfrak{p}_4$ and $\mathfrak{p}_5$, with the curves in question being, in general, strict transforms of curves of the form $\{a\} \times \mathbb{P}^1 \times \{1\} \subseteq \overline{X}$. Note as these curves intersect Z_1 at one point, the class of the strict transform is indeed $f - e_1$. However, as $a \to \infty$, this curve degenerates to a union $C = C_1 \cup C_2$ of two irreducible components, of class $f - e_1 - e_2$ (the strict transform C_1 of $\{\infty\} \times \mathbb{P}^1 \times \{1\}$) and of class e_2 (the curve $C_2 = \pi^{-1}(\infty, \infty, 1)$). It is this degenerate curve and its multiple covers which contribute to $\mathfrak{p}_4$. Meanwhile the strict transform of $\{0\} \times \mathbb{P}^1 \times \{1\}$ and its multiple covers contribute to the wall $\mathfrak{p}_5$.

Finally, $\mathfrak{p}_3$ arises not from a family of curves with one marked point of positive contact order with the boundary, but from the punctured log map whose image is C_1. This involves a negative contact order with the divisor $D_{1,\infty}$, which contains C_1. This curve is rigid, and if we did not include a wall for this curve, we would not get a consistent scattering diagram. This shows how it is essential, unlike in the case that $\dim X = 2$, to take into account punctured curves rather than just marked curves, as without $\mathfrak{p}_3$, $\mathscr{S}_{\mathrm{can}}$ would not be consistent.

Construction 3.16 (*The torus action*) As with the canonical wall structure in two dimensions in [37, Sect. 5], there is a natural torus action on the mirror family constructed using $\mathscr{S}_{\mathrm{can}}$ in all dimensions. We refer the reader to [39, Sect. 4.4] for the general setup for torus actions in the context of families of varieties built via wall structures, and do not review the notation here.

In this case, there is an action of a torus with character lattice

$$\Gamma := \mathrm{Div}_D(X),$$

i.e., the free abelian group generated by boundary divisors. In the language of [39, Sect. 4.4], the base ring A is taken here to be the ground field $\Bbbk$. To specify the torus action, we must specify the maps δ_Q and δ_B of the diagram [39, (4.8)]. Note that each irreducible component D_i of D defines an $\mathbb{R}$-valued PL function on B; indeed, by construction $B \subseteq \mathrm{Div}_D(X)^*_{\mathbb{R}}$, and the linear functional D_i on this vector space restricts to a piecewise linear function on B. We continue to denote this PL function as D_i. We may then define

$$\delta_Q : Q \longrightarrow \Gamma, \qquad \delta_Q(A) = \sum_i (D_i \cdot A) D_i$$

for $A \in Q$, and

$$\delta_B : \mathrm{PL}(B)^* \longrightarrow \Gamma, \qquad \delta_B(\beta) = \sum_i \beta(D_i) D_i$$

for $\beta \in \mathrm{PL}(B)^*$. Commutativity of [39, (4.8)] then follows by direct computation. Indeed, note that in our current context, Q_0 is the free monoid with generators e_ρ, $e_\rho \in \mathscr{P}^{[n-1]}$, and $h : Q_0 \to Q$ is defined by $h(e_\rho) = [X_\rho]$, the kink of our choice of MPL function φ along ρ. Then

$$\delta_Q(h(e_\rho)) = \sum_i (D_i \cdot X_\rho) D_i.$$

On the other hand, $g : Q_0 \to \mathrm{PL}(B)^*$ takes e_ρ to the linear functional on $\mathrm{PL}(B)$ taking $\psi \in \mathrm{PL}(B)$ to the kink of ψ along ρ. It is easy to check from toric geometry that the kink of the PL function induced by D_i along ρ is $D_i \cdot X_\rho$ (noting that if D_i is not good, it induces the zero function on B). The claimed equality $\delta_Q \circ h = \delta_B \circ g$ then follows.

Thus, by [39, Thm. 4.17], it follows there is an induced $\mathrm{Spec}\, \Bbbk[\Gamma]$-action on the flat family $\check{\mathfrak{X}} \to \mathrm{Spec}(\Bbbk[Q]/I)$ constructed from the wall structure $\mathscr{S}_{\mathrm{can}}$

provided that $\mathscr{S}_{\mathrm{can}}$ is homogeneous in the sense of [39, Def. 4.16]. For this, it is enough to check that if $\boldsymbol{\tau}$ is a decorated wall type with $W_{\boldsymbol{\tau}} \neq 0$ and total curve class A, then $\deg_\Gamma t^A z^{-u_\tau} = 0$. Here $\deg_\Gamma t^A = \delta_Q(A) = \sum_i (D_i \cdot A) D_i$. If we write $u_\tau = \sum u_i D_i^*$ as a tangent vector to $\boldsymbol{\sigma}(L_{\mathrm{out}})$ then $\deg_\Gamma(z^{-u_\tau})$ is computed from [39, (4.9)] as $-\sum_i u_i D_i$. It then follows from [48, Cor. 1.14] that necessarily $u_i = D_i \cdot A$, and hence $\deg_\Gamma t^A z^{-u_\tau} = 0$ as desired.

3.2.3 The relative case

Construction 3.17 Suppose we are in the situation of Proposition 1.19. It is useful (see [9]) to compare the wall structures for (X, D) and (X_s, D_s), which we do as follows.

Recall from that proposition that after extending the affine structure on B across the interior of cells $\omega \in \mathscr{P}_\partial^{[n-2]}$, ∂B may be viewed as the polyhedral affine pseudomanifold corresponding to the pair (X_s, D_s). As such, write $(\partial B)_0$ for the complement in ∂B of the union of cones of $\mathscr{P}_\partial^{[n-3]}$. Let $\Lambda_{\partial B}$ denote the sheaf on $(\partial B)_0$ of integral tangent vectors. We may view $\Lambda_{\partial B}$ as a subsheaf of $\Lambda|_{\partial B}$, where the latter sheaf is now extended across the interiors of cells $\omega \in \mathscr{P}_\partial^{[n-2]}$.

Further, Proposition 1.19, (3) then implies that if $\mathcal{P}_{\partial B}$ is defined using the MPL function $\varphi|_{\partial B}$ on ∂B, we obtain a natural inclusion $\mathcal{P}_{\partial B} \subseteq \mathcal{P}|_{\partial B}$. Again, the latter sheaf is viewed as extending across the interiors of $\omega \in \mathscr{P}_\partial^{[n-2]}$. Note that $\mathcal{P}_{\partial B}$ consists of those sections m of $\mathcal{P}|_{\partial B}$ such that $\bar{m}$ is tangent to ∂B.

On the other hand, we also have the MPL function $\varphi_{(X_s, D_s)}$ taking values in $H_2(X_s) \otimes_{\mathbb{Z}} \mathbb{R}$. This leads to a sheaf $\mathcal{P}_{(X_s, D_s)}$ on $(\partial B)_0$, and the map $\iota : H_2(X_s) \to H_2(X)$ then induces a map

$$\iota_* : \mathcal{P}_{(X_s, D_s)} \to \mathcal{P}_{\partial B}.$$

This allows us to define a map, for $x \in (\partial B)_0$,

$$\iota_* : \Bbbk[\mathcal{P}^+_{(X_s, D_s), x}] \to \Bbbk[\mathcal{P}^+_{\partial B, x}],$$

where $\mathcal{P}^+_{\partial B, x} := \mathcal{P}_{\partial B, x} \cap \mathcal{P}^+_x$. Finally, we may define

$$\iota(\mathscr{S}_{\mathrm{can}, s}) := \left\{ (\mathfrak{p}, \iota_*(f_{\mathfrak{p}})) \,|\, (\mathfrak{p}, f_{\mathfrak{p}}) \in \mathscr{S}_{(X_s, D_s)} \right\}.$$

Like $\mathscr{S}_{\mathrm{can}, s}$, this is a wall structure on ∂B.

On the other hand, we may define the *asymptotic wall structure* of the wall structure $\mathscr{S}_{\mathrm{can}}$ on B by

$$\mathscr{S}^{\mathrm{as}}_{\mathrm{can}} := \{(\mathfrak{p}\cap\partial B, f_{\mathfrak{p}})\,|\,(\mathfrak{p}, f_{\mathfrak{p}})\in\mathscr{S} \text{ with } \dim\mathfrak{p}\cap\partial B = n-2\}. \tag{3.16}$$

We note that for each such wall, $f_{\mathfrak{p}}\in \Bbbk[\mathcal{P}^+_{\partial B,x}]\subseteq \Bbbk[\mathcal{P}^+_x]$. Indeed, this follows from Proposition 3.7, which implies that for a wall type τ for X, u_τ is tangent to fibres of g_{trop}, and in particular tangent to ∂B. Hence $\mathscr{S}^{\mathrm{as}}_{\mathrm{can}}$ may be viewed as a wall structure on ∂B.

We then have:

Proposition 3.18 *Suppose that all good divisors contained in $g^{-1}(0)$ have multiplicity one in $g^{-1}(0)$. Then the wall structures $\iota(\mathscr{S}_{\mathrm{can},s})$ and $\mathscr{S}^{\mathrm{as}}_{\mathrm{can}}$ are equivalent.*

Proof Note $\iota(\mathscr{S}_{\mathrm{can},s})$ is equal to the canonical wall structure for (X_s, D_s) constructed using the curve class group $H_2(X)$ rather than $H_2(X_s)$. In the proof, we generally use $\boldsymbol{\tau}'$ to denote a decorated wall type for (X, D) and $\boldsymbol{\tau}$ for a decorated wall type for (X_s, D_s), with curve class decoration $\mathbf{A}$ taking values in $H_2(X)$.

Given a decorated wall type $\boldsymbol{\tau}'$, the corresponding cone τ' comes with a structure map $p:\tau'\to\mathbb{R}_{\ge 0}$ coming from the fact the type is defined over S. Note p is surjective as u_τ is tangent to fibres of g_{trop} by Proposition 3.7 and $h_{\tau'}(\tau'_{\mathrm{out}})\not\subseteq\partial B$ by Definition 3.6, (3). Then $p^{-1}(0)$ is a proper face of τ', and hence corresponds to a contraction morphism $\phi:\boldsymbol{\tau}'\to\boldsymbol{\tau}$ of decorated types. The type τ is realizable provided that the length of $L_{\mathrm{out}}\in L(G')$ isn't zero on the face $p^{-1}(0)$. Thus τ realizable and $\dim\tau = n-3$ is equivalent to $\dim\mathfrak{p}_{\boldsymbol{\tau}'}\cap\partial B = n-2$. From this it is immediate that τ is a wall type for (X_s, D_s) if and only if $\dim\mathfrak{p}_{\boldsymbol{\tau}'}\cap\partial B = n-2$.

Consider the sets

$$\begin{aligned} S' &:= \{\boldsymbol{\tau}'\,|\,\boldsymbol{\tau}' \text{ is a decorated wall type for } X \text{ with } \dim\mathfrak{p}_{\boldsymbol{\tau}'}\cap\partial B = n-2\},\\ S &:= \{\boldsymbol{\tau}\,|\,\boldsymbol{\tau} \text{ is a decorated wall type for } X_s\}. \end{aligned}$$

The discussion of the previous paragraph has constructed a map $\Phi: S'\to S$. To show the desired equivalence of wall structures, it is enough to show that for $\boldsymbol{\tau}\in S$ we have

$$k_\tau W_{\boldsymbol{\tau}} = \sum_{\boldsymbol{\tau}'\in\Phi^{-1}(\boldsymbol{\tau})} k_{\tau'} W_{\boldsymbol{\tau}'}. \tag{3.17}$$

To do so, we use the decomposition setup of [5, Thm. 5.21, Def. 5.22, Thm. 5.23]. Note that for $0\neq s\in S$, s is a trivial log point, so in fact

$\mathscr{M}(X_s, \boldsymbol{\tau}) = \mathscr{M}(X_s/s, \boldsymbol{\tau})$. Thus in particular we have

$$W_{\boldsymbol{\tau}} = \frac{\deg[\mathscr{M}(X_s/s, \boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau})|} = \frac{\deg[\mathscr{M}(X_0/0, \boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau})|},$$

the first equality by definition and the second equality by [5, Thm. 5.23, (1)]. Further, by [5, Thm. 5.23, (2)], we have

$$\deg[\mathscr{M}(X_0/0, \boldsymbol{\tau})]^{\mathrm{virt}} = \sum_{\boldsymbol{\tau}'} \frac{m_{\boldsymbol{\tau}'}}{|\operatorname{Aut}(\boldsymbol{\tau}'/\boldsymbol{\tau})|} \deg[\mathscr{M}(X_0/0, \boldsymbol{\tau}')]^{\mathrm{virt}}, \tag{3.18}$$

where the sum is over codimension one degenerations $\boldsymbol{\tau}'$ of $\boldsymbol{\tau}$ in the sense of [5, Def. 5.22, (2)], and $m_{\boldsymbol{\tau}'}$ is the order of the cokernel of $p_* : \Lambda_{\tau'} \to \mathbb{Z}$. It is immediate from that definition that if $\mathscr{M}(X_0/0, \boldsymbol{\tau}') \neq \emptyset$ then such a $\boldsymbol{\tau}'$ is a decorated wall type for X. (We need to assume non-emptiness to guarantee that τ' is balanced.) It also follows from Proposition 3.7 and the proof of Lemma 3.9 that $\mathscr{M}(X_0/0, \boldsymbol{\tau}') = \mathscr{M}(X/S, \boldsymbol{\tau}') = \mathscr{M}(X, \boldsymbol{\tau}')$. Thus we may rewrite (3.18) as

$$k_\tau \frac{\deg[\mathscr{M}(X_0/0, \boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau})|} = \sum_{\boldsymbol{\tau}'} \frac{m_{\boldsymbol{\tau}'} k_\tau}{|\operatorname{Aut}(\boldsymbol{\tau}'/\boldsymbol{\tau})||\operatorname{Aut}(\boldsymbol{\tau})|} \deg[\mathscr{M}(X, \boldsymbol{\tau}')]^{\mathrm{virt}}.$$

Certainly, $|\operatorname{Aut}(\boldsymbol{\tau}'/\boldsymbol{\tau})||\operatorname{Aut}(\boldsymbol{\tau})| = |\operatorname{Aut}(\boldsymbol{\tau}')|$. Further, $\Lambda_{\tau'_{\mathrm{out}}} = \Lambda_{\tau_{\mathrm{out}}} \oplus \mathbb{Z}$ and $p_*(\Lambda_{\tau'_{\mathrm{out}}}) = m_{\tau'}\mathbb{Z}$. Next $\Lambda_{\boldsymbol{\sigma}'(L_{\mathrm{out}})} = \Lambda_{\boldsymbol{\sigma}(L_{\mathrm{out}})} \oplus \mathbb{Z}$, with the induced map $g_{\mathrm{trop},*} : \Lambda_{\boldsymbol{\sigma}'(L_{\mathrm{out}})} \to \mathbb{Z}$ being surjective because of the assumption that $g^{-1}(0)$ is reduced. Thus an elementary diagram chase gives a short exact sequence

$$0 \to \operatorname{coker}(\Lambda_{\tau_{\mathrm{out}}} \to \Lambda_{\boldsymbol{\sigma}(L_{\mathrm{out}})}) \to \operatorname{coker}(\Lambda_{\tau'_{\mathrm{out}}} \to \Lambda_{\boldsymbol{\sigma}'(L_{\mathrm{out}})}) \to \mathbb{Z}/m_{\tau'}\mathbb{Z} \to 0,$$

so $m_{\tau'}k_\tau = k_{\tau'}$, giving (3.17). □

3.3 Logarithmic broken lines and theta functions

Wall types defined in the previous subsection allowed us to define the canonical wall structure. Consistency of a wall structure is defined partly via *broken lines* (reviewed in Definition 4.2). A key point of our proof of consistency is a correspondence result that associates counts of broken lines to certain punctured invariants. In this subsection, we shall define these punctured invariants. We proceed quite analogously with the notion of wall types.

Definition 3.19 A *(non-trivial) broken line type* is a type $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$ of tropical map to $\Sigma(X)$ such that:

(1) G is a genus zero graph with $L(G) = \{L_{\text{in}}, L_{\text{out}}\}$ with $\boldsymbol{\sigma}(L_{\text{out}}) \in \mathscr{P}$ and

$$u_\tau := \mathbf{u}(L_{\text{out}}) \neq 0, \quad p_\tau := \mathbf{u}(L_{\text{in}}) \in \boldsymbol{\sigma}(L_{\text{in}})\backslash\{0\}$$

(so that L_{in} represents a marked rather than punctured point).

(2) τ is realizable and balanced.

(3) Let $h : \Gamma(G, \ell) \to \Sigma(X)$ be the corresponding universal family of tropical maps, and let $\tau_{\text{out}} \in \Gamma(G, \ell)$ be the cone corresponding to L_{out}. Then $\dim \tau = n - 1$ and $\dim h(\tau_{\text{out}}) = n$.

We also consider the possibility of a *trivial broken line type* τ, which is not an actual type: the underlying graph G consists of just one leg $L_{\text{in}} = L_{\text{out}}$ and no vertices, with the convention that $u_\tau = -p_\tau$. Note this does not correspond to an actual punctured curve.[8]

A *decorated broken line type* is a decorated type $\boldsymbol{\tau} = (\tau, \mathbf{A})$ with τ a broken line type. In the trivial case, as there are no vertices, this does not involve any extra information, and in this case, the total curve class is taken to be 0.

A *degenerate broken line type* is a type τ which satisfies conditions (1) and (2) above and instead of (3),

(3′) $\dim \tau = n - 2$ and $\dim h(\tau_{\text{out}}) = n - 1$.

Lemma 3.20 *Let $\boldsymbol{\tau}$ be a non-trivial decorated broken line type. Then $\mathscr{M}(X, \boldsymbol{\tau})$ is proper over* $\operatorname{Spec} \Bbbk$ *and carries a virtual fundamental class of dimension zero.*

Proof The argument is essentially identical to that of the proof of Lemma 3.9, and we leave it to the reader to make the necessary modifications, except for one point. In that proof, we appealed to Proposition 3.7 to argue that in the relative case, for each $v \in V(G)$, we have $\Sigma(g) : h(\tau_v) \to \mathbb{R}_{\geq 0}$ surjective. Crucially, a broken line type is not in general a type of tropical map to X/S, so a different argument is necessary. However, note that by the assumption that τ is a balanced type, for any $s \in \text{Int}(\tau)$, $\Sigma(g) \circ h_s : G \to \mathbb{R}_{\geq 0}$ is a balanced tropical map. Necessarily $\Sigma(g)_*(\mathbf{u}(L_{\text{in}})) = \Sigma(g)(p_\tau) \in \mathbb{R}_{\geq 0}$, and hence by balancing, $\Sigma(g)_*(u_\tau) \in \mathbb{R}_{\leq 0}$. This implies, again by balancing, that if $v_{\text{in}}, v_{\text{out}}$ are the vertices adjacent to $L_{\text{in}}, L_{\text{out}}$ respectively, then $a = \Sigma(g) \circ h_s(v_{\text{out}}) \leq \Sigma(g) \circ h_s(v_{\text{in}}) = b$, and for any other vertex $v \in V(G)$, $\Sigma(g) \circ h_s(v) \in [a, b]$. Further, since $\dim h(\tau_{\text{out}}) = n$, necessarily $h(\tau_{\text{out}}) \not\subseteq \partial B$, and thus $\Sigma(g) : h(\tau_{\text{out}}) \to \mathbb{R}_{\geq 0}$ is surjective, and so $\Sigma(g) : h(\tau_{v_{\text{out}}}) \to \mathbb{R}_{\geq 0}$ is also surjective. Putting this together, we see that $\Sigma(g) : h(\tau_v) \to \mathbb{R}_{\geq 0}$ is surjective for any vertex $v \in V(G)$. This is sufficient to complete the argument of the proof of Proposition 3.9 for properness of the moduli spaces involved. □

[8] Such a broken line type will correspond, as discussed in Sect. 4, to a broken line contained entirely in $\boldsymbol{\sigma}(L_{\text{in}})$ and which does not bend.

We now define, for $\boldsymbol{\tau}$ a non-trivial decorated broken line type,

$$N_{\boldsymbol{\tau}} := \frac{\deg[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau})|}. \tag{3.19}$$

We also have a map $h_* : \Lambda_{\tau_{\mathrm{out}}} \to \Lambda_{\boldsymbol{\sigma}(L_{\mathrm{out}})}$, necessarily of finite index, and define

$$k_\tau := |\operatorname{coker} h_*| = |\Lambda_{\boldsymbol{\sigma}(L_{\mathrm{out}})} / h_*(\Lambda_{\tau_{\mathrm{out}}})|. \tag{3.20}$$

For a trivial decorated broken line type $\boldsymbol{\tau}$, we set

$$N_{\boldsymbol{\tau}} := 1, \quad k_{\boldsymbol{\tau}} := 1.$$

Definition 3.21 Fix $p \in B(\mathbb{Z})\backslash\{0\}$, $\sigma \in \mathscr{P}_{\max}$, $x \in \operatorname{Int}(\sigma)$ not contained in any rationally defined hyperplane in σ. We then define

$$\vartheta_p^{\log}(x) = \sum_{\boldsymbol{\tau}} k_\tau N_{\boldsymbol{\tau}} t^A z^{-u_\tau} \in \Bbbk[\mathcal{P}_x^+]/I_x$$

where the sum is over all isomorphism classes of decorated broken line types $\boldsymbol{\tau}$ with $p_\tau = p$, $x \in h(\tau_{\mathrm{out}})$. Here A is the total curve class of $\boldsymbol{\tau}$, and we require $A \in Q\backslash I$.

Lemma 3.22 *For $p \in B(\mathbb{Z})\backslash\{0\}$, the number of decorated broken line types $\boldsymbol{\tau}$ with $p_\tau = p$, total curve class in $Q\backslash I$, and $N_{\boldsymbol{\tau}} \neq 0$ is finite. In particular, the sum defining $\vartheta_p^{\log}(x)$ is finite.*

Proof As $Q\backslash I$ is finite, there are only a finite number of total curve classes. Further, there are only a finite number of possibilities for $\boldsymbol{\sigma}(L_{\mathrm{out}})$. For each curve class A and choice of $\boldsymbol{\sigma}(L_{\mathrm{out}})$, it follows from [48, Cor. 1.14] that u_τ is determined by p if $\mathscr{M}(X, \boldsymbol{\tau})$ is non-empty. If β is the punctured curve class given by contact orders p and u_τ and curve class A, then $\mathscr{M}(X, \beta)$ is of finite type, and there are only a finite number of decorated types $\boldsymbol{\tau}$ appearing as the types of tropicalizations of punctured maps in $\mathscr{M}(X, \beta)$. Thus there are only a finite number of possible choices of $\boldsymbol{\tau}$ with given total curve class A and $\boldsymbol{\sigma}(L_{\mathrm{out}})$ satisfying $\mathbf{u}(L_{\mathrm{in}}) = p$ and $\mathbf{u}(L_{\mathrm{out}}) = u_\tau$. □

4 A correspondence theorem for broken lines

4.1 Broken lines and statement of the correspondence theorem

We first recall the definition of *broken lines* [39, Def. 3.3] for a wall structure $\mathscr{S}$ on B. We continue to work with the specific choice of $(B, \mathscr{P})$ derived from

(X, D), a monoid $Q \subset H_2(X)$ and monoid ideal $I \subset Q$ with $Q \setminus I$ finite as usual.

Definition 4.1 Let $\gamma : (a, b) \to B_0$ be a path with $t \in (a, b)$ such that $\gamma(t) \in |\mathscr{S}| \setminus \mathrm{Sing}(\mathscr{S})$. Let $n \in \check{\Lambda}_{\gamma(t)}$ be a primitive cotangent vector annihilating the tangent space to $|\mathscr{S}|$ at $\gamma(t)$. Thinking of n as defining a linear function locally near $\gamma(t)$ which is zero along $|\mathscr{S}|$, we assume further that n is positive on $\gamma(t-\epsilon, t)$ and negative on $\gamma(t, t+\epsilon)$ for some $\epsilon > 0$. Suppose $\gamma(t-\epsilon) \in \sigma$, $\gamma(t+\epsilon) \in \sigma'$ with $\sigma, \sigma' \in \mathscr{P}_{\max}$. Given an expression az^m with $a \in \Bbbk[Q]/I$, $m \in \Lambda_\sigma$, $\langle m, n\rangle > 0$, we define a *result of transport along* γ of az^m to be $a_i z^{m_i}$ chosen as follows. Write

$$f_{\gamma(t)}^{\langle n,m\rangle} \cdot \mathfrak{t}_{\sigma,\sigma'}(az^m) = \sum_i a_i z^{m_i}$$

inside $(\Bbbk[Q]/I)[\Lambda_{\sigma'}]$, with the $m_i \in \Lambda_{\sigma'}$ mutually distinct. We then take $a_i z^{m_i}$ to be one of the terms in this sum. Here $\mathfrak{t}$ is as defined in (3.7) and $f_{\gamma(t)}$ is as defined in (3.8), which may be viewed via transport as an element of $(\Bbbk[Q]/I)[\Lambda_{\sigma'}]$.

Definition 4.2 A *broken line* for a wall structure $\mathscr{S}$ on $(B, \mathscr{P})$ is a proper continuous map

$$\beta : (-\infty, 0] \to B_0$$

with image disjoint from $\mathrm{Sing}(\mathscr{S})$, along with a sequence $-\infty = t_0 < t_1 < \cdots < t_r = 0$ for some $r \geq 1$ with $\beta(t_i) \in |\mathscr{S}|$ for $1 \leq i \leq r-1$, and for each $i = 1, \ldots, r$ an expression $a_i z^{m_i}$ with $a_i \in \Bbbk[Q]/I$, $m_i \in \Lambda_{\beta(t)}$ for any $t \in (t_{i-1}, t_i)$, subject to the following conditions:

(1) $\beta|_{(t_{i-1},t_i)}$ is a non-constant affine map with image disjoint from $|\mathscr{S}|$, hence contained in the interior of some $\sigma \in \mathscr{P}_{\max}$, and $\beta'(t) = -m_i$ for all $t \in (t_{i-1}, t_i)$.
(2) For each $i = 1, \ldots, r-1$ the expression $a_{i+1} z^{m_{i+1}}$ is a result of transport of $a_i z^{m_i}$ along $\beta|_{(t_{i-1},t_{i+1})}$.
(3) $a_1 = 1$.

The *asymptotic monomial* of β is m_1. Note that since $\beta((-\infty, t_1])$ is an affine half-line contained in a cone $\sigma \in \mathscr{P}$, we may identify m_1 with an integral point of σ, and hence view $m_1 \in B(\mathbb{Z})$.

Given a broken line β, we write $a_\beta = a_r$, $m_\beta = m_r$.

Remark 4.3 This only differs in one way from [39, Def. 3.3], namely the requirement that $a_1 = 1$. In [39], broken lines satisfying this additional condition are called *normalized* broken lines. As we never use any other type of broken line here, we include this in the definition.

Broken lines define theta functions:

Definition 4.4 Fix $p \in B(\mathbb{Z})\setminus\{0\}$, $\sigma \in \mathscr{P}_{\max}$, $x \in \mathrm{Int}(\sigma)$ not contained in any rationally defined hyperplane in σ. We then define

$$\vartheta_p(x) = \sum_\beta a_\beta z^{m_\beta} \in \Bbbk[\mathcal{P}_x^+]/I_x,$$

where the sum is over all broken lines β with asymptotic monomial p and $\beta(0) = x$.

We note this sum is finite, e.g., by [39, Lem. 3.7].

The main result of this section may now be stated as:

Theorem 4.5 *For* $p \in B(\mathbb{Z})\setminus\{0\}$, $\sigma \in \mathscr{P}_{\max}$, $x \in \mathrm{Int}(\sigma)$ *not contained in any rationally defined hyperplane, we have*

$$\vartheta_p(x) = \vartheta_p^{\log}(x).$$

Before proving this theorem, we first recast it as a tropical correspondence theorem between broken lines and punctured log curves. To do so, we need to deal with a certain amount of book-keeping. To this end, we first refine the notion of broken line. This refinement keeps track of which wall types cause the bending of the broken line.

Definition 4.6 A *decorated broken line* for $\mathscr{S}_{\mathrm{can}}$ is a proper continuous map

$$\beta : (-\infty, 0] \to B_0$$

with image disjoint from $\mathrm{Sing}(\mathscr{S}_{\mathrm{can}})$, along with a sequence $-\infty = t_0 < t_1 < \cdots < t_r = 0$ for some $r \geq 1$ with $\beta(t_i) \in |\mathscr{S}_{\mathrm{can}}|$ for $1 \leq i \leq r-1$, along with additional data:

(i) For each $i = 1, \ldots, r$ an expression $a_i z^{m_i}$ with $a_i \in \Bbbk[Q]/I$, $m_i \in \Lambda_{\beta(t)}$ for any $t \in (t_{i-1}, t_i)$.
(ii) For each $i = 1, \ldots, r-1$, let $\mathscr{S}_i := \{\mathfrak{p}_{\boldsymbol{\tau}} \in \mathscr{S}_{\mathrm{can}} \mid \beta(t_i) \in \mathfrak{p}_{\boldsymbol{\tau}}\}$. Then we are given in addition a function $\mu_i : \mathscr{S}_i \to \mathbb{N}$.

This data is subject to the following conditions:

(1) $\beta|_{(t_{i-1},t_i)}$ is a non-constant affine map, and $\beta'(t) = -m_i$ for all $t \in (t_{i-1}, t_i)$.
(2) Define the *support* of μ_i to be the subset of $\mathscr{S}_i$ on which μ_i takes non-zero value. Then the support of μ_i must be finite, and non-empty if $\beta(t_i)$ lies in the interior of a maximal cone of $\mathscr{P}$. Further, if $\beta(t) \in \rho \in \mathscr{P}^{[n-1]}$, then $t = t_i$ for some i.

(3) For each $i = 1, \ldots, r-1$, if $\beta(t_i - \epsilon) \in \sigma$, $\beta(t_i + \epsilon) \in \sigma'$, we have

$$a_{i+1} z^{m_{i+1}} = \mathfrak{t}_{\sigma,\sigma'}(a_i z^{m_i}) \prod_{\mathfrak{p}_{\boldsymbol{\tau}} \in \mathscr{S}_i} \frac{(\langle n_i, m_i \rangle k_\tau W_{\boldsymbol{\tau}} t^A z^{-u_\tau})^{\mu_i(\mathfrak{p}_{\boldsymbol{\tau}})}}{\mu_i(\mathfrak{p}_{\boldsymbol{\tau}})!}. \qquad (4.1)$$

Here $n_i \in \check{\Lambda}_{\beta(t_i)}$ is primitive, vanishing on the tangent space to $|\mathscr{S}_{\mathrm{can}}|$ at $\beta(t_i)$, and positive on m_i.

(4) $a_1 = 1$.

Remark 4.7 There are a couple of points of distinction between the above definition and Definition 4.2. First, it is possible that $\beta(t) \in |\mathscr{S}_{\mathrm{can}}|$ with $t \neq t_i$ for any i provided $\beta(t)$ lies in the interior of a maximal cell. Second, the data of the μ_i completely determines the change of monomial. To explain the expression (4.1), note that by the description (3.13) of $\mathscr{S}_{\mathrm{can}}$,

$$\begin{aligned} f_{\beta(t_i)}^{\langle n_i, m_i \rangle} &= \exp\left(\textstyle\sum_{\mathfrak{p}_{\boldsymbol{\tau}} \in \mathscr{S}_i} \langle n_i, m_i \rangle k_\tau W_{\boldsymbol{\tau}} t^A z^{-u_\tau} \right) \\ &= \textstyle\sum_{\mu_i} \prod_{\mathfrak{p}_{\boldsymbol{\tau}} \in \mathscr{S}_i} \frac{(\langle n_i, m_i \rangle k_\tau W_{\boldsymbol{\tau}} t^A z^{-u_\tau})^{\mu_i(\mathfrak{p}_{\boldsymbol{\tau}})}}{\mu_i(\mathfrak{p}_{\boldsymbol{\tau}})!} \end{aligned}$$

where the sum is over all functions $\mu_i : \mathscr{S}_i \to \mathbb{N}$ with finite support. Thus (4.1) arises from one of the summands, and if $\beta(t_i)$ falls in the interior of a maximal cell, we do not allow the trivial summand 1 (with $\mu_i \equiv 0$). This would correspond, in Definition 4.2, to a situation where $a_i z^{m_i} = a_{i+1} z^{m_{i+1}}$.

It is then clear that we have a refined description

$$\vartheta_p(x) = \sum_\beta a_\beta z^{m_\beta} \in \Bbbk[\mathcal{P}_x^+]/I_x$$

as a sum over decorated broken lines with asymptotic monomial $p \in B(\mathbb{Z})$ and $\beta(0) = x$. Each term in the summation of Definition 4.4 is now split up into a sum over a number of decorated broken lines.

Construction 4.8 We may now associate to a decorated broken line β an isomorphism class of a decorated broken line type $\boldsymbol{\tau}_\beta = (G_\beta, \boldsymbol{\sigma}_\beta, \mathbf{u}_\beta, \mathbf{A}_\beta)$. This is done as follows.

First, we adopt the following notation. For each $\mathfrak{p}_{\boldsymbol{\tau}} \in \mathscr{S}_i$, write $(G_{\boldsymbol{\tau}}, \boldsymbol{\sigma}_{\boldsymbol{\tau}}, \mathbf{u}_{\boldsymbol{\tau}})$ and $\mathbf{A}_{\boldsymbol{\tau}}$ for the data giving the decorated type $\boldsymbol{\tau}$. Let $L_{\mathrm{out},\boldsymbol{\tau}} \in L(G_{\boldsymbol{\tau}})$ be the unique leg, adjacent to a unique $v_{\mathrm{out},\boldsymbol{\tau}} \in V(G_{\boldsymbol{\tau}})$.

The graph G_β has a spine comprising of the legs L_{in} and L_{out} and the edges $E_1, \ldots, E_{r-2}$ and their vertices $v_1, \ldots, v_{r-1}$, with the vertices of E_i being v_i, v_{i+1} and the vertices of L_{in} and L_{out} being v_1 and v_{r-1} respectively. At the vertex v_i we glue $\mu_i(\mathfrak{p}_{\boldsymbol{\tau}})$ copies of $G_{\boldsymbol{\tau}}$ to v_i along the unique leg of $G_{\boldsymbol{\tau}}$.

Next, we define $\boldsymbol{\sigma}_\beta$ as follows. We define $\boldsymbol{\sigma}_\beta(v_i)$ or $\boldsymbol{\sigma}_\beta(E_i)$ to be the minimal cone of $\mathscr{P}$ containing $\beta(t_i)$ or $\beta((t_i, t_{i+1}))$ respectively, and $\boldsymbol{\sigma}_\beta(L_{\mathrm{in}})$ and $\boldsymbol{\sigma}_\beta(L_{\mathrm{out}})$ to be the minimal cones containing $\beta((-\infty, t_1))$ or $\beta((t_{r-1}, t_r))$ respectively. Further, for each i, $\mathfrak{p}_{\boldsymbol{\tau}} \in \mathscr{S}_i$, we may view each of the $\mu_i(\mathfrak{p}_{\boldsymbol{\tau}})$ copies of $G_{\boldsymbol{\tau}}$ as a subgraph of G_β, and we take $\boldsymbol{\sigma}_\beta$ to agree with $\boldsymbol{\sigma}_{\boldsymbol{\tau}}$ on each copy of $G_{\boldsymbol{\tau}}$.

To define $\mathbf{u}_\beta$, we again take it to agree with $\mathbf{u}_{\boldsymbol{\tau}}$ on any copy of $G_{\boldsymbol{\tau}}$, and take $\mathbf{u}_\beta(E_i) = -m_{i+1}$ with E_i oriented from v_i to v_{i+1}. We also set $\mathbf{u}_\beta(L_{\mathrm{in}}) = m_1$, $\mathbf{u}_\beta(L_{\mathrm{out}}) = -m_r$ (using the standard convention that legs are oriented away from their vertex).

This defines the global type τ_β. For the decoration $\mathbf{A}_\beta$, it similarly agrees with $\mathbf{A}_{\boldsymbol{\tau}}$ on each copy of $G_{\boldsymbol{\tau}}$. If $v = v_i$, we take $\mathbf{A}_\beta(v_i) = 0$ if $\beta(t_i)$ lies in the interior of a maximal cell of $\mathscr{P}$, but if $\beta(t_i) \in \mathrm{Int}(\rho)$ with $\rho \in \mathscr{P}^{[n-1]}$, then we take $\mathbf{A}_\beta(v_i) = \langle n_i, m_i \rangle [X_\rho] \in H_2(X)$.

Lemma 4.9 *Let β be a decorated broken line with $x = \beta(0) \in \mathrm{Int}(\sigma)$, $\sigma \in \mathscr{P}_{\max}$, x not contained in any rationally defined hyperplane. Then $\boldsymbol{\tau}_\beta$ is a decorated broken line type.*

Proof Condition (1) of Definition 3.19 is immediate.

For condition (2), we first show τ_β is realizable, i.e., there is a tropical map $h : G_\beta \to \Sigma(X)$ of type τ_β. On the spine G'_β of G_β, consisting of vertices $v_1, \ldots, v_{r-1}$, edges $E_1, \ldots, E_{r-2}$ and legs $L_{\mathrm{in}}, L_{\mathrm{out}}$, we define h to agree with β, extending β linearly on L_{out} if necessary. For each i and for each $\mathfrak{p}_{\boldsymbol{\tau}} \in \mathscr{S}_i$, $\boldsymbol{\tau}$ is a wall type, and hence is realizable with an $n-2$-dimensional universal family with output leg tracing out the wall $\mathfrak{p}_{\boldsymbol{\tau}}$. Further, $\beta(t_i) \in \mathrm{Int}(\mathfrak{p}_{\boldsymbol{\tau}})$. Thus there is a unique tropical map $h_{\boldsymbol{\tau}} : G_{\boldsymbol{\tau}} \to \Sigma(X)$ of type τ with $\beta(t_i) \in h_{\boldsymbol{\tau}}(L_{\mathrm{out},\boldsymbol{\tau}})$. Thus we may define h restricted to any copy of $G_{\boldsymbol{\tau}}$ in G_β to be given by $h_{\boldsymbol{\tau}}$, appropriately truncating the edge $L_{\mathrm{out},\boldsymbol{\tau}}$ as necessary. This shows realizability.

Next, we observe that τ_β is balanced. Because this is true for each wall type τ appearing in τ_β, this only needs to be checked at the vertices v_i, but such balancing follows from (4.1). Indeed, by considering exponents, that equation tells us that

$$m_{i+1} = m_i - \sum_{\mathfrak{p}_{\boldsymbol{\tau}} \in \mathscr{S}_i} \mu_i(\mathfrak{p}_{\boldsymbol{\tau}}) u_\tau,$$

which is the balancing condition at v_i given the specified contact orders in Construction 4.8 of edges adjacent to v_i. In the relative case, we also need to verify that $\Sigma(g) \circ h$ is balanced as a map to $\Sigma(S) = \mathbb{R}_{\geq 0}$. However, this again holds at all vertices of G_β except for the v_i's because the same is assumed true of the wall types τ. At a vertex v_i, the result follows from the balancing in B and the fact (Lemma 1.15) that g_{trop} is an affine submersion.

For condition (3), observe that, by elementary convex geometry, β varies in an n-dimensional family: each $\beta(t_i)$ is constrained to live in an $n-1$-dimensional subspace, and the location of $\beta(t_1)$ and the m_i completely determine the map β, up to the location of $\beta(0)$, which gives an extra parameter. In particular, the dimension of the universal family of tropical maps of type τ_β is then clearly $n-1$, and $h(\tau_{\beta,\mathrm{out}})$ is n-dimensional. This shows τ_β is a broken line type. $\square$

We note that the broken line type $\boldsymbol{\tau}_\beta$ had a very specific form for the curve classes associated to the vertices v_i. We codify this as follows:

Definition 4.10 Let $\boldsymbol{\tau} = (G, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A})$ be a decorated broken line type with spine $G' \subseteq G$ (see Definition 2.4), the latter having vertices $v_1, \ldots, v_{r-1}$ and edges E_i with vertices v_i, v_{i+1}. By Lemma 2.5, $\boldsymbol{\sigma}(v_i) \in \mathscr{P}^{[n]} \cup \mathscr{P}^{[n-1]}$. We say $\boldsymbol{\tau}$ is *admissible* if $\mathbf{A}(v_i) = 0$ whenever $\boldsymbol{\sigma}(v_i) \in \mathscr{P}^{[n]}$, and $\mathbf{A}(v_i) = |\langle n_i, \mathbf{u}(E_i)\rangle|[X_{\boldsymbol{\sigma}(v_i)}]$ whenever $\boldsymbol{\sigma}(v_i) \in \mathscr{P}^{[n-1]}$, where $n_i \in \check{\Lambda}_x$ for $x \in \mathrm{Int}(\boldsymbol{\sigma}(v_i))$ is a choice of primitive normal vector to $\boldsymbol{\sigma}(v_i)$.

By construction, if β is a decorated broken line, then $\boldsymbol{\tau}_\beta$ is admissible. The following is immediate from Lemma 2.3:

Lemma 4.11 *Let $\boldsymbol{\tau}$ be a decorated broken line type. Then $N_{\boldsymbol{\tau}} \neq 0$ implies that $\boldsymbol{\tau}$ is admissible.*

We now reverse the procedure of Construction 4.8 and go from decorated broken line types to decorated broken lines.

Construction 4.12 Let $\boldsymbol{\tau} = (G, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A})$ be a decorated broken line type, h the universal tropical map parameterized by the cone τ, and $x \in \mathrm{Int}(h(\tau_{\mathrm{out}}))$. We may construct a broken line $\beta_{\boldsymbol{\tau},x}$ as follows. There exists a unique $s \in \mathrm{Int}(\tau)$ such that $x \in h_s(L_{\mathrm{out}})$. Let G' be the spine of G. Label the vertices of G' as $v_1, \ldots, v_{r-1}$ with L_{in} adjacent to v_1, L_{out} adjacent to v_{r-1}, and edges $E_1, \ldots, E_{r-2}$, with E_i having vertices v_i and v_{i+1}. Shrink L_{out} so that the non-vertex endpoint of L_{out} maps to x under h_s. We may then choose an identification $\psi : \mathbb{R}_{\leq 0} \to G'$ with the following property. Let $t_i \in \mathbb{R}_{\leq 0}$ satisfy $\psi(t_i) = v_i$, $1 \leq i \leq r-1$, take $t_r = 0$, and set $\beta_{\boldsymbol{\tau},x} = h_s|_{G'} \circ \psi$. We require that condition (1) of Definition 4.6 is satisfied for $\beta_{\boldsymbol{\tau},x}$ with $m_i = -\mathbf{u}(E_i)$, with E_i oriented from v_i to v_{i+1}. Note by construction that $\beta_{\boldsymbol{\tau},x}(0) = x$.

We now specify the decorations, notably the functions μ_i. Let $G_{i1}, \ldots, G_{is_i}$ be the closures of the connected components of $G \setminus \{v_i\}$ not containing L_{in} or L_{out}. These are the graphs attached to v_i but only intersecting G' at v_i. We view the edge of G_{ij} adjacent to the vertex v_i as a leg $L_{ij,\mathrm{out}}$ of G_{ij}, so that v_i is not a vertex of G_{ij}. This gives rise to decorated types $\boldsymbol{\tau}_{i1}, \ldots, \boldsymbol{\tau}_{is_i}$ by restriction of $\boldsymbol{\sigma}, \mathbf{u}, \mathbf{A}$ to each graph G_{ij}.

We claim that each $\boldsymbol{\tau}_{ij}$ is a wall type. Indeed, condition (1) of Definition 3.6 is immediate except for the statement that $\mathbf{u}(L_{ij,\mathrm{out}}) \neq 0$, which we rule out below. For condition (2), note $\boldsymbol{\tau}_{ij}$ is realizable and balanced since $\boldsymbol{\tau}$ is realizable and balanced.

For condition (3), let $h_{ij} : \Gamma(G_{ij}, \ell_{ij}) \to \Sigma(X)$ be the universal tropical map of type $\boldsymbol{\tau}_{ij}$, defined over the cone τ_{ij}. First note that necessarily $h(\tau_{v_i}) \subseteq h_{ij}(\tau_{ij,\mathrm{out}})$, and by Lemma 2.5, (2), $\dim h(\tau_{v_i}) = n-1$. Thus $\dim h_{ij}(\tau_{ij,\mathrm{out}}) \geq n-1$. However by Lemma 2.5, (1), still assuming that $\mathbf{u}(L_{ij,\mathrm{out}}) \neq 0$, $\dim h_{ij}(\tau_{ij,\mathrm{out}}) \leq n-1$ and thus this dimension is $n-1$, as desired.

Suppose now that $\dim \tau_{ij} > n-2$. Thus for any fixed $s' \in \mathrm{Int}(\tau)$, there is a positive dimensional subset $\omega \subseteq \tau_{ij}$ such that for $s'' \in \omega$, $h_{s'}(v_i) \in h_{ij,s''}(L_{ij,\mathrm{out}})$, where $L_{ij,\mathrm{out}}$ is the unique leg of G_{ij}. Thus for each point $y \in h(\tau_{v_i})$, there is a positive dimensional family of tropical maps of type τ taking v_i to y. Again since $\dim h(\tau_{v_i}) = n-1$, this shows $\dim \tau \geq n$, a contradiction.

We now eliminate the case that $\mathbf{u}(L_{ij,\mathrm{out}}) = 0$. Indeed, if this is the case and v is the unique vertex of $L_{ij,\mathrm{out}}$ in G_{ij}, then $h_{ij}(\tau_{ij,v}) = h_{ij}(\tau_{ij,L_{ij,\mathrm{out}}})$, and this is again $n-1$-dimensional as above. Thus $\dim \tau_{ij} \geq n-1$. On the other hand, if E_{ij} is the edge of G corresponding to the leg $L_{ij,\mathrm{out}}$ of G_{ij}, we have $\mathbf{u}(E_{ij}) = \mathbf{u}(L_{ij,\mathrm{out}}) = 0$, so the length of E_{ij} is arbitrary, and thus we obtain again that $\dim \tau \geq n$. Thus we conclude that $\boldsymbol{\tau}_{ij}$ is a wall type.

We may now define $\mu_i : \mathscr{S}_i \to \mathbb{N}$ by defining, for $\mathfrak{p}_{\boldsymbol{\tau}'} \in \mathscr{S}_i$, $\mu_i(\mathfrak{p}_{\boldsymbol{\tau}'}) = |\{j \mid \boldsymbol{\tau}_{ij} \simeq \boldsymbol{\tau}'\}|$.

Having defined the μ_i, (4.1) defines the monomials $a_i z^{m_i}$. There is one thing to check: as we have already defined m_i in terms of $\mathbf{u}$, we need to check that (4.1) yields the same values for m_i. However, this is immediate inductively by balancing at the vertices v_i.

Hence we have constructed a decorated broken line $\beta_{\boldsymbol{\tau}}$.

Constructions 4.8 and 4.12 now immediately give:

Proposition 4.13 *Given $x \in B_0$ not contained in any rationally defined hyperplane and $p \in B(\mathbb{Z})\setminus\{0\}$, there is a one-to-one correspondence*

$$\begin{aligned}&\{\beta \mid \beta \textit{ is a decorated broken line with endpoint } x\\ &\quad \textit{and asymptotic monomial } p\}\\ &\quad \leftrightarrow \left\{\boldsymbol{\tau} \,\middle|\, \begin{array}{c}\tau \textit{ is an isomorphism class of admissible decorated}\\ \textit{broken line types with } x\in\mathrm{Int}(h(\tau_{\mathrm{out}})) \textit{ and } p_\tau = p\end{array}\right\}\end{aligned}$$

given by $\beta \mapsto \boldsymbol{\tau}_\beta$, $\boldsymbol{\tau} \mapsto \beta_{\boldsymbol{\tau},x}$.

The key result, to be proved via gluing of punctured curves in the next subsection, is then:

Theorem 4.14 *Let $\boldsymbol{\tau}$ be an admissible decorated broken line type, and let $\beta_{\boldsymbol{\tau},x}$ be the corresponding broken line with endpoint $x \in \mathrm{Int}(h(\tau_{\mathrm{out}}))$. Then we have*

$$a_{\beta_{\boldsymbol{\tau},x}} z^{m_{\beta_{\boldsymbol{\tau},x}}} = k_\tau N_{\boldsymbol{\tau}} t^A z^{-u_\tau},$$

where A is the total degree of $\boldsymbol{\tau}$.

Proof of Theorem 4.5 given Theorem 4.14. This follows immediately from the definitions Definition 4.4 and 3.21 of the two types of theta functions, Proposition 4.13, and Theorem 4.14. □

4.2 Proof of the correspondence theorem, Theorem 4.14

The proof is by induction on the length of $\beta_{\boldsymbol{\tau},x}$, that is, the number $r \geq 1$ in Definition 4.2. The base case $r = 1$ is the case of a straight broken line ending at some point x in the interior of a maximal cell. The corresponding broken line type is then a trivial one $\boldsymbol{\tau}_{\mathrm{in}}$ without a vertex and trivial curve class $A = 0$. Thus $N_{\boldsymbol{\tau}} = 1$, $k_{\boldsymbol{\tau}} = 1$ by definition and the claimed equality holds trivially.

For the inductive step, let $\boldsymbol{\tau} = (\tau, \mathbf{A})$, $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$ be an admissible decorated broken line type as in the theorem, with associated broken line $\beta_{\boldsymbol{\tau},x}$ of length r. Let $v \in V(G)$ be the vertex adjacent to L_{out}, $\tau_v \in \Gamma(G, \ell)$ the corresponding cone and $\sigma_{\mathfrak{p}} = \boldsymbol{\sigma}(v)$ the smallest cell containing $h(\tau_v)$. Here we denote by $\mathfrak{p}$ the set-theoretic intersection of all walls or codimension one cells of $\mathscr{P}$ containing $h(\tau_v)$. Thus $\mathfrak{p}$ is a polyhedral subset of B of dimension $n-1$. Note that $\dim \sigma_{\mathfrak{p}} \in \{n-1, n\}$, and we refer to the two cases as codimension one and codimension zero, respectively. We work in an affine chart of B_0 containing $\mathrm{Int}\, \sigma_{\mathfrak{p}}$, with Λ denoting the lattice of integral tangent vector fields by $\Lambda_{\mathfrak{p}} \subset \Lambda$ the corank one lattice of vectors tangent to $h(\tau_v)$. Let $E_{\mathrm{in}} \in E(G)$ be the unique edge adjacent to v and belonging to the spine of τ, and $E_1, \ldots, E_l$ the remaining edges adjacent to v, all oriented toward v. The case $l = 0$ means that no such edges are present and then $E_1, \ldots, E_l$ denotes the empty sequence. Denote further $u_{\mathrm{in}} = \mathbf{u}(E_{\mathrm{in}})$, $u_{\mathrm{out}} = \mathbf{u}(L_{\mathrm{out}}) = u_\tau$, $u_i = \mathbf{u}(E_i)$, and $\sigma_{\mathrm{in}} = \boldsymbol{\sigma}(E_{\mathrm{in}})$ the maximal cell containing $h_s(E_{\mathrm{in}})$ for all $s \in \tau$.

We now split G at all the edges $E_{\mathrm{in}}, E_1, \ldots, E_l$ adjacent to v. See [5, Sect. 5.1] for a formal treatment of splitting. Splitting an edge E with vertices v_1, v_2 leads to a pair of legs that we denote (E, v_1), (E, v_2). By Construction 4.12 the types obtained from $\boldsymbol{\tau}$ after splitting are as follows.

(1) A decorated broken line type $\boldsymbol{\tau}_{\mathrm{in}} = (G_{\mathrm{in}}, \boldsymbol{\sigma}_{\mathrm{in}}, \mathbf{u}_{\mathrm{in}}, \mathbf{A}_{\mathrm{in}})$, with the unique leg $(E_{\mathrm{in}}, v_{\mathrm{in}})$ obtained from splitting and $\mathbf{u}_{\mathrm{in}}(E_{\mathrm{in}}, v_{\mathrm{in}}) = u_{\mathrm{in}}$.
(2) Decorated wall types $\boldsymbol{\tau}_i = (G_i, \boldsymbol{\sigma}_i, \mathbf{u}_i, \mathbf{A}_i)$, $i = 1, \ldots, l$, with legs $(E_i, v_i) \in L(G_i)$ obtained from splitting, and $\mathbf{u}_i(E_i, v_i) = u_i$.

(3) $\boldsymbol{\tau}_0 = (G_0, \boldsymbol{\sigma}_0, \mathbf{u}_0, \mathbf{A}_0)$, the type of a decorated punctured map with only one vertex v with $\boldsymbol{\sigma}_0(v) = \sigma_{\mathfrak{p}} = \boldsymbol{\sigma}(v)$, no edges, and legs (E_{in}, v), $(E_1, v), \ldots, (E_l, v), (E_{\text{out}}, v) = L_{\text{out}}$ with $\mathbf{A}_0(v) = \mathbf{A}(v)$ and

$$\mathbf{u}_0(E_{\text{in}}, v) = -u_{\text{in}}, \quad \mathbf{u}_0(E_{\text{out}}, v) = u_{\text{out}}, \quad \mathbf{u}_0(E_i, v) = -u_i, \ i = 1, \ldots, l.$$

As usual the corresponding undecorated types are denoted τ_{in} and τ_i, $i = 0, 1, \ldots, l$. Grouping together identical $\boldsymbol{\tau}_i$ we may relabel and assume the $\boldsymbol{\tau}_i$ are pairwise non-isomorphic, but each $\boldsymbol{\tau}_i$ to occur μ_i-times in the splitting for some $\mu_i \in \mathbb{N}$.

The propagation rule (4.1) for monomials in the definition of decorated broken lines (Definition 4.6) together with the induction hypothesis shows that the claimed equality $a_{\beta_{\boldsymbol{\tau}}} z^{m_{\beta_{\boldsymbol{\tau}}}} = k_\tau N_{\boldsymbol{\tau}} t^A z^{-u_{\text{out}}}$ is equivalent to the following

$$k_\tau N_{\boldsymbol{\tau}} t^A z^{-u_{\text{out}}} = k_{\tau_{\text{in}}} N_{\boldsymbol{\tau}_{\text{in}}} t^{A_{\text{in}}} t^{d[X_{\sigma_{\mathfrak{p}}}]} z^{-u_{\text{in}}} \cdot \prod_{i=1}^{l} \frac{\left(d k_{\tau_i} W_{\boldsymbol{\tau}_i} t^{A_i} z^{-u_{\tau_i}}\right)^{\mu_i}}{\mu_i!}. \tag{4.2}$$

Note that $[X_{\sigma_{\mathfrak{p}}}] = 0$ in the codimension zero case and $[X_{\sigma_{\mathfrak{p}}}]$ is the class of the curve corresponding to the $(n-1)$-cell containing $\mathfrak{p}$ in the codimension one case. Furthermore,

$$d = |\delta(u_{\text{in}})| \tag{4.3}$$

for $\delta : \Lambda \to \mathbb{Z}$ the quotient by $\Lambda_{\mathfrak{p}}$, and $k_{\tau_{\text{in}}}, k_{\tau_i}$ are defined in (3.20) and (3.10), respectively.

To prove this equality, we employ the numerical splitting formula for punctured Gromov–Witten invariants proved by Wu [82] and recalled in the appendix. Assumption A.1 of toric gluing strata is trivially fulfilled in the codimension zero case, while in codimension one it follows from Lemma 1.10. In the following discussion, we freely use the notation introduced in the appendix, but note that we do not split all edges of τ, but just the subset $E(G, v) \subseteq E(G)$ of $l+1$ edges adjacent to v. Thus for any $E \in E(G, v)$ the vector space $(\Lambda_E)_{\mathbb{R}}$ holding the E-component of the displacement vector equals $\Lambda_{\mathbb{R}}$. For the time being we also revert to the original labelling of the $\boldsymbol{\tau}_i$ with possibly several identical wall types. The numerator $\mu_i!$ in (4.2) arises only at the very end when taking the automorphism group of the glued decorated type $\boldsymbol{\tau}$ into account.

To invoke Theorem A.4 we need to choose a general displacement vector $\nu = (\nu_E)_{E \in E(G,v)}$ and then determine the corresponding set $\Delta(\nu)$ of decorated transverse types

$$(\boldsymbol{\omega}_{\text{in}}, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_l)$$

for ν, with $\boldsymbol{\omega}_\bullet$ a decorated type marked by $\tau_\bullet$. Lemma 4.15 below shows that for a certain choice of ν there is a unique decorated transverse type with $\boldsymbol{\omega}_{\text{in}} = \boldsymbol{\tau}_{\text{in}}$,

$\boldsymbol{\omega}_i = \boldsymbol{\tau}_i$ for all i, and with

$$\boldsymbol{\omega}_0 = (\hat{G}_0, \hat{\boldsymbol{\sigma}}_0, \hat{\mathbf{u}}_0, \hat{\mathbf{A}}_0) = (\omega_0, \hat{\mathbf{A}}_0) \tag{4.4}$$

a certain maximal decorated global type marked by $\boldsymbol{\tau}_0$. If $l = 0$ we take $\boldsymbol{\omega}_0 = \boldsymbol{\tau}_0$. To define $\boldsymbol{\omega}_0$ for $l \geq 1$, assume first $\sigma_\mathfrak{p} = \sigma_{\text{in}}$, that is, we are in the situation of codimension zero. Take for $\hat{G}_0$ the graph with l trivalent vertices $v'_1, \ldots, v'_l$ connected in the given order by $l-1$ edges $E'_1, \ldots, E'_{l-1}$ to form a chain, and with legs distributed as

$$(E_{\text{in}}, v'_1), \quad (E_{\text{out}}, v'_l), \quad (E_i, v'_i),\ i = 1, \ldots, l,$$

see Fig. 3. The notation E_{in}, E_i, E_{out} indicates that we identify these legs with the legs of τ_0 via the contraction morphism $\omega_0 \to \tau_0$ contracting all edges of ω_0. All strata are given by σ_{in} and the contact orders for each edge are determined inductively from the contact orders $\mathbf{u}_0$ of τ_0 by enforcing the balancing condition Lemma 2.1. Necessarily $\hat{\mathbf{A}}_0 = 0$.

If $\sigma_\mathfrak{p}$ is of codimension one, we further split v'_l into two vertices v'_l and v'_{out} connected by another edge E'_l, with $\hat{\boldsymbol{\sigma}}_0(v'_{\text{out}}) = \sigma_\mathfrak{p}$, and the legs at v'_l and v'_{out} distributed as (E_l, v'_l), $(E_{\text{out}}, v'_{\text{out}})$. For $l = 1$ the leg $(E_{\text{in}}, v'_1) = (E_{\text{in}}, v'_l)$ is also adjacent to v'_l. The other strata are forced to be one of the two maximal cells $\sigma_{\text{in}} = \boldsymbol{\sigma}(E_{\text{in}}, v)$ or $\boldsymbol{\sigma}(L_{\text{out}})$ containing $\sigma_\mathfrak{p}$, namely $\hat{\boldsymbol{\sigma}}_0(E_{\text{out}}, v'_{\text{out}}) = \boldsymbol{\sigma}(L_{\text{out}})$ and all other strata equal to σ_{in}. The curve classes are determined by the contraction morphism as

$$\hat{\mathbf{A}}(v'_{\text{out}}) = \mathbf{A}(v) = d \cdot [X_{\sigma_\mathfrak{p}}]$$

and all other curves classes trivial. Note that in any case the curve classes are determined by the contraction morphism. It is therefore enough to drop the decoration in the following classification of transverse types according to Definition A.2.

Lemma 4.15 *There exists a displacement vector $\nu = (\nu_E)_E \in \prod_{E \in E(G)} \Lambda_\mathbb{R}$ that is general for τ such that any ν-transverse type ω for τ* (Definition A.2, 2) *with $\mathscr{M}(X, \omega'_0) \neq \emptyset$, for ω'_0 the part of ω marked by τ_0, is isomorphic to*

$$\omega = (\tau_{\text{in}}, \tau_1, \ldots, \tau_l, \omega_0),$$

where ω_0 is as described in (4.4).

Proof Take the quotient $\delta : \Lambda \to \mathbb{Z}$ by $\Lambda_\mathfrak{p}$ to evaluate non-negatively on vectors pointing from $\mathfrak{p}$ to the incoming direction, and let $m \in \Lambda$ be a vector

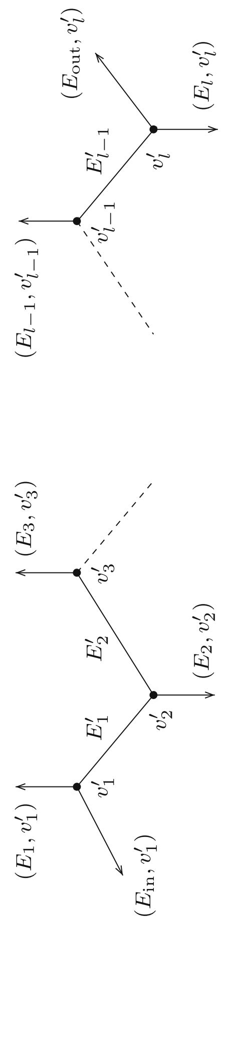

Fig. 3 The tropical type ω_0 (codimension zero case)

with $\delta(m) = 1$. Then in particular $\Lambda = \Lambda_{\mathfrak{p}} \oplus \mathbb{Z}m$. Choose $\nu_1, \ldots, \nu_l \in \mathbb{R}$ with

$$\nu_1 > \cdots > \nu_l > 0,$$

and define

$$\nu_{E_{\mathrm{in}}} = 0, \quad \nu_{E_i} = \nu_i \cdot m, \quad i = 1, \ldots, l.$$

Thus $\nu_{E_i} = (0, \nu_i)$ when identifying Λ with $\Lambda_{\mathfrak{p}} \oplus \mathbb{Z}$. Following the interpretation in Remark A.3 of $\Delta(\nu)$, we have to show that any ν-broken tropical punctured map marked by τ fulfilling $\mathscr{M}(X, \omega_0') \neq \emptyset$ has components of types

$$\omega_{\mathrm{in}} = \tau_{\mathrm{in}}, \ \omega_1 = \tau_1, \ldots, \omega_l = \tau_l, \ \omega_0.$$

We first consider the component marked by τ_0. Denote this component by $\omega_0' = (G_0', \boldsymbol{\sigma}_0', \mathbf{u}_0')$ and by

$$(E_{\mathrm{in}}, v_{\mathrm{in}}'), \ (E_{\mathrm{out}}, v_{\mathrm{out}}'), \ (E_1, v_1'), \ldots, (E_l, v_l')$$

the legs, with some of the vertices $v_{\mathrm{in}}', v_1', \ldots, v_l', v_{\mathrm{out}}'$ possibly coinciding. Denote further by $E_1', \ldots, E_r' \in E(G_0')$ the unique sequence of edges connecting v_{in}' to v_{out}', oriented away from v_{in}'. Note also that $d = -\delta(u_{\mathrm{in}})$ for d as introduced in (4.3) and that $E_{\mathrm{in}} \in E(G)$ is oriented away from v. Now since $\mathscr{M}(X, \omega_0') \neq \emptyset$, Lemma 2.1 shows that ω_0' fulfills the balancing condition. Thus $u_1, \ldots, u_l \in \Lambda_{\mathfrak{p}} = \ker(\delta)$ implies that the images under δ of the contact orders of all edges are determined as follows:

$$-\delta(\mathbf{u}_0'(E)) = \begin{cases} d, & E = E_1', \ldots, E_r' \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, by the perturbed matching condition (A.9), the fact that the outgoing vertex for each τ_i maps to $\mathfrak{p}$ and the choice of $\nu_1, \ldots, \nu_l$, it follows that δ is constant on the subtree connecting the leg (E_i, v_i) to the chain of edges $E_1', \ldots, E_r'$, with value ν_i. Since all ν_i are disjoint, it follows that all of these subtrees are disjoint. Note that there are $r+1$ vertices on the path connecting v_{in}' and v_{out}' (including these latter two vertices). Thus necessarily $r + 1 \geq l$. However, if $\dim \sigma_{\mathfrak{p}} = n - 1$, $\boldsymbol{\sigma}_0'(E_{\mathrm{in}}, v_{\mathrm{in}}')$ and $\boldsymbol{\sigma}_0'(E_{\mathrm{out}}, v_{\mathrm{out}}')$ are distinct maximal cones containing $\sigma_{\mathfrak{p}}$, and thus there must be at least one vertex v_j' on this path contained in $\sigma_{\mathfrak{p}}$. By the choice of displacement vectors, none of the subtrees connecting an edge (E_i, v_i) to the chain of edges $E_1', \ldots, E_r'$ is connected to v_j'. Thus in this case, $r \geq l$.

Now by the dimension formula (A.8) we require

$$\begin{aligned}&\dim\widetilde{\omega}_{\rm in}+\dim\widetilde{\omega}'_0+\textstyle\sum_{i=1}^l\dim\widetilde{\omega}_i \\ &\quad=\dim\widetilde{\tau}+\sum_E \operatorname{rk}\Lambda=(n+l)+(l+1)\cdot n=ln+2n+l.\end{aligned}\tag{4.5}$$

Note that if $\dim\sigma_{\mathfrak{p}}=n$, then $\dim\omega'_0\ge n+r$, as there is an n-dimensional choice of location for one vertex, and an additional choice of length for the edges $E'_1,\ldots,E'_r$. Thus, in this first case, $\dim\widetilde{\omega}'_0\ge n+r+l+1$. On the other hand, if $\dim\sigma_{\mathfrak{p}}=n-1$, then $\dim\omega'_0\ge n-1+r$, as there is an $(n-1)$-dimensional choice of location for the vertex v'_j. Thus, in this second case, $\dim\widetilde{\omega}'_0\ge n+r+l$. The left-hand side of (4.5) now has dimension at least

$$\begin{cases} n+(n+l+r+1)+l\cdot(n-1)=ln+2n+r+1, & \dim\sigma_{\mathfrak{p}}=n\\ n+(n+l+r)+l\cdot(n-1)=ln+2n+r, & \dim\sigma_{\mathfrak{p}}=n-1.\end{cases}$$

Note in both cases, this quantity is then $\ge ln+2n+l$, with equality with the right-hand side of (4.5) precisely if $r+1=l$ or $r=l$ in the two cases, $\omega_{\rm in}=\tau_{\rm in}$, $\omega_i=\tau_i$, $i=1,\ldots,l$, and the mentioned subtrees in ω'_0 trivial. Comparing with the right-hand side of (4.5) shows that we indeed have equality. In particular, G'_0 agrees with the graph G_0 in (4.4). Then also $\boldsymbol{\sigma}'_0=\hat{\boldsymbol{\sigma}}_0$ since each vertex, except $v'_{\rm out}$ in the codimension one case, maps to the maximal cell $\boldsymbol{\sigma}(E_{\rm in})$. Furthermore, the contact orders $\mathbf{u}'_0$ agree with $\hat{\mathbf{u}}_0$ due to the balancing condition.

Taken together this shows $\omega'_0=\omega_0$, and hence $(\tau_{\rm in},\tau_1,\ldots,\tau_l,\omega_0)$ is indeed the only transverse type for ν. □

It remains to compute the multiplicity $m(\boldsymbol{\omega})=m(\omega)$ occurring in Theorem A.4.

Lemma 4.16 *The multiplicity according to Definition* A.2, (3) *for the single element* $\omega=(\tau_{\rm in},\tau_1,\ldots,\tau_l,\omega_0)$ *of* $\Delta(\nu)$ *in Lemma* 4.15 *equals*

$$m(\omega)=\begin{cases} k_\tau^{-1}k_{\tau_{\rm in}}\prod_{i=1}^l(dk_{\tau_i}), & \dim\sigma_{\mathfrak{p}}=n\\ dk_\tau^{-1}k_{\tau_{\rm in}}\prod_{i=1}^l(dk_{\tau_i}), & \dim\sigma_{\mathfrak{p}}=n-1.\end{cases}$$

Proof The multiplicity in question is the index of $\operatorname{im}(\Phi^{\rm gp})$, where

$$\Phi:(\widetilde{\tau}_{\rm in})_{\mathbb{Z}}\times(\widetilde{\omega}_0)_{\mathbb{Z}}\times\textstyle\prod_{i=1}^l(\widetilde{\tau}_i)_{\mathbb{Z}}\to\Lambda\times\Lambda^l$$

is the map describing the matching at the gluing edges, denoted ε_{ω_0} in (A.7). In particular, $\Phi^{-1}(0)=\widetilde{\tau}$ is the enlarged basic cone for the glued type τ. For the case of a trivial incoming broken line type define $\widetilde{\tau}_{\rm in}$ as the maximal cone containing the incoming vector and Φ on this factor as the inclusion to the first copy of Λ.

We first treat the case $\dim \sigma_{\mathfrak{p}} = n$. In this case, by the description of the type ω_0, we can write

$$(\widetilde{\omega}_0^{\rm gp})_{\mathbb{Z}} = \Lambda \times \mathbb{Z}^l \times \mathbb{Z} \times \mathbb{Z}^{l-1} = \Lambda \times \mathbb{Z}^l \times \mathbb{Z}^l, \tag{4.6}$$

with the $\mathbb{Z}$-factors holding the lengths

$$\ell_1, \ldots, \ell_l, \ell_{\rm in}, \ell'_1, \ldots, \ell'_{l-1}$$

of the legs $(E_1, v'_1), \ldots, (E_l, v'_l)$, $(E_{\rm in}, v'_1)$ and edges $E'_1, \ldots, E'_{l-1}$, in this order, while the Λ-factor records the image of the point on the incoming leg. For uniformity of notation write $E'_0 = (E_{\rm in}, v'_1)$, $u'_0 = u_{\rm in}$, $\ell'_0 = \ell_{\rm in}$ and

$$u'_i = \hat{\mathbf{u}}_0(E'_i) = u_{\rm in} + u_1 + \cdots + u_i, \quad i = 1, \ldots, l-1.$$

Thus the image of $v'_i \in V(\hat{G}_0)$ under the tropical punctured map of type ω_0 defined by

$$(m, \ell_1, \ldots, \ell_l, \ell'_0, \ldots, \ell'_{l-1}) \in \Lambda \times \mathbb{Z}^l \times \mathbb{Z}^l$$

equals

$$m + \ell'_0 u'_0 + \ell'_1 u'_1 + \ldots + \ell'_{i-1} u'_{i-1}.$$

Denote further by $\mathrm{ev}_{\rm in} : (\widetilde{\tau}_{\rm in}^{\rm gp})_{\mathbb{Z}} \to \Lambda$ and $\mathrm{ev}_i : (\widetilde{\tau}_i^{\rm gp})_{\mathbb{Z}} \to \Lambda_{\mathfrak{p}} \subset \Lambda$ the linear extensions of the evaluation maps at the point on the respective unique leg.

We first compute the cokernel Q of the restriction of $\Phi^{\rm gp}$ to

$$(\widetilde{\tau}_{\rm in})^{\rm gp} \times (\{0\} \times \mathbb{Z}^l) \times \prod_{i=1}^{l} (\widetilde{\tau}_i)^{\rm gp}.$$

The further restriction to the subspace spanned by the last copy of $\mathbb{Z}$ in (4.6) and $(\widetilde{\tau}_l)^{\rm gp}$ leads to the map

$$\varepsilon_l : \mathbb{Z} \times (\widetilde{\tau}_l^{\rm gp})_{\mathbb{Z}} \to \Lambda, \quad (\ell'_{l-1}, h_l) \longmapsto \mathrm{ev}_l(h_l) - \ell'_{l-1} \cdot u'_{l-1}$$

to the last copy of Λ in the codomain of Φ^{gp}. Now since $|\delta(u'_{l-1})| = d$ we can fit ε_l into the diagram with exact rows and injective columns

$$
\begin{tikzcd}
0 \arrow[r] & (\widetilde{\tau}_l^{\mathrm{gp}})_{\mathbb{Z}} \arrow[r] \arrow[d, "\mathrm{ev}_l"] & \mathbb{Z} \times (\widetilde{\tau}_l^{\mathrm{gp}})_{\mathbb{Z}} \arrow[r] \arrow[d, "\varepsilon_l"] & \mathbb{Z} \arrow[r] \arrow[d, "\cdot d"] & 0 \\
0 \arrow[r] & \Lambda_{\mathfrak{p}} \arrow[r] & \Lambda \arrow[r] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}
\tag{4.7}
$$

The cokernel of ev_l having order k_{τ_l}, the snake lemma implies $|\operatorname{coker}(\varepsilon_l)| = d \cdot k_{\tau_l}$. An induction on l thus shows that $|Q|$ equals $\prod_{i=1}^{l} \left(dk_{\tau_i}\right)$ times the order of the cokernel of the case $l = 0$, which is $|\operatorname{coker}(\mathrm{ev}_{\mathrm{in}})| = k_{\tau_{\mathrm{in}}}$. We have thus shown

$$\big|Q\big| = k_{\tau_{\mathrm{in}}} \textstyle\prod_{i=1}^{l} (dk_{\tau_i}). \tag{4.8}$$

To finish the computation of $\big|\operatorname{coker} \Phi^{\mathrm{gp}}\big|$, consider the following diagram with exact rows and columns and with the obvious maps.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & {} \arrow[d, dashed] & \\
 & 0 \arrow[r] \arrow[d] & (\widetilde{\tau}_{\mathrm{in}}^{\mathrm{gp}})_{\mathbb{Z}} \times \{0\} \times \mathbb{Z}^l \times \prod_{i=1}^{l} (\widetilde{\tau}_i^{\mathrm{gp}})_{\mathbb{Z}} \arrow[r] \arrow[d] & \Lambda \times \Lambda^l \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{\tau}_{\mathbb{Z}}^{\mathrm{gp}} \arrow[r] \arrow[d, "\alpha"] & (\widetilde{\tau}_{\mathrm{in}}^{\mathrm{gp}})_{\mathbb{Z}} \times (\widetilde{\omega}_0^{\mathrm{gp}})_{\mathbb{Z}} \times \prod_{i=1}^{l} (\widetilde{\tau}_i^{\mathrm{gp}})_{\mathbb{Z}} \arrow[r, "\Phi^{\mathrm{gp}}"] \arrow[d] & \Lambda \times \Lambda^l \arrow[r] \arrow[d] & \operatorname{coker}(\Phi^{\mathrm{gp}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda \times \mathbb{Z}^l \arrow[r] \arrow[d, dash, dashed] & \Lambda \times \mathbb{Z}^l \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 & \\
 & {} & 0 & 0 & &
\end{tikzcd}
$$

The snake lemma yields the exact sequence

$$0 \longrightarrow \widetilde{\tau}_{\mathbb{Z}}^{\mathrm{gp}} \xrightarrow{\ \alpha\ } \Lambda \times \mathbb{Z}^l \longrightarrow Q \longrightarrow \operatorname{coker}(\Phi^{\mathrm{gp}}) \longrightarrow 0. \tag{4.9}$$

Note also that $\widetilde{\tau}_{\mathbb{Z}}^{\mathrm{gp}} = \tau_{\mathbb{Z}}^{\mathrm{gp}} \times \mathbb{Z} \times \mathbb{Z}^l$ with the $\mathbb{Z}$-factors from the additional points at the $l+1$ gluing edges, and the map α is a product of the injection of lattices

$$\bar{\alpha} : \tau_{\mathbb{Z}}^{\mathrm{gp}} \times \mathbb{Z} \to \Lambda, \quad (h, \lambda) \longmapsto V(h) - \lambda \cdot u_{\mathrm{in}}$$

and $\mathrm{id}_{\mathbb{Z}^l}$. Here $V : \tau_{\mathbb{Z}}^{\mathrm{gp}} \to \Lambda_{\mathfrak{p}}$ is the evaluation at the outgoing vertex. Replacing u_{in} in this formula by u_{out} yields $\mathrm{ev}_{\mathrm{out}} : \tau_{\mathbb{Z}}^{\mathrm{gp}} \times \mathbb{Z} \to \Lambda$ with index of the image defining k_τ. But $|\delta(u_{\mathrm{out}})| = |\delta(u_{\mathrm{in}})| = d$. By considering two diagrams similar to (4.7), it follows that

$$|\operatorname{coker}(\alpha)| = |\operatorname{coker}(\bar{\alpha})| = |\operatorname{coker}(\mathrm{ev}_{\mathrm{out}})| = k_\tau.$$

Together with (4.8) the exact sequence (4.9) thus yields

$$m(\omega) = \left|\operatorname{coker}(\Phi^{\mathrm{gp}})\right| = k_\tau^{-1} \cdot |Q| = k_\tau^{-1} k_{\tau_{\mathrm{in}}} \prod_{i=1}^{l} (d k_{\tau_i}),$$

as claimed.

If $\dim \sigma_{\mathfrak{p}} = n-1$ we have an additional vertex v'_{out} mapping to $\sigma_{\mathfrak{p}}$ and an additional edge E'_l connecting v'_l to v'_{out}, with contact order u'_l. Thus in this case we have

$$(\widetilde{\omega}_0)^{\mathrm{gp}}_{\mathbb{Z}} = \Lambda_{\mathfrak{p}} \times \mathbb{Z}^l \times \mathbb{Z} \times \mathbb{Z}^l,$$

with $\Lambda_{\mathfrak{p}}$ recording the position of v'_{out} in $\mathfrak{p}$. We can now exhibit $(\widetilde{\omega}_0)^{\mathrm{gp}}_{\mathbb{Z}}$ as a sublattice of $\Lambda \times \mathbb{Z}^l \times \mathbb{Z}^l$ from (4.6) by the map

$$\begin{aligned} \Psi : \Lambda_{\mathfrak{p}} \times \mathbb{Z}^l \times \mathbb{Z}^{l+1} &\longrightarrow \Lambda \times \mathbb{Z}^l \times \mathbb{Z}^l, \\ (m, \ell_1, \ldots, \ell_l, \ell'_0, \ldots, \ell'_l) &\longmapsto \big(m - \textstyle\sum_{i=0}^{l} \ell'_i u'_i, \ell_1, \ldots, \ell_l, \ell'_0, .., \ell'_{l-1}\big). \end{aligned}$$

Here $u'_0 = u_{\mathrm{in}}$, $\ell'_0 = \ell_{\mathrm{in}}$ are defined analogously to the codimension zero case. Since the images of u'_i in $\Lambda/\Lambda_{\mathfrak{p}} \simeq \mathbb{Z}$ generate $d\mathbb{Z}$, we see that Ψ is an inclusion of lattices of index d. The stated formula in codimension one now follows by observing that Φ^{gp} is the composition of Φ^{gp} in the codimension zero case with $\mathrm{id}_{(\tilde{\tau}_{\mathrm{in}})_{\mathbb{Z}}} \times \Psi \times \mathrm{id}_{\prod_i (\tilde{\tau}_i)_{\mathbb{Z}}}$. □

Finally we need to compute one simple punctured invariant.

Lemma 4.17

$$\deg \left[\mathscr{M}(X, \boldsymbol{\omega}_0)\right]^{\mathrm{virt}} = \begin{cases} 1, & \operatorname{codim} \sigma_{\mathfrak{p}} = n \\ 1/d, & \operatorname{codim} \sigma_{\mathfrak{p}} = n-1. \end{cases}$$

Proof The graph $\hat{G}_0$ given by $\boldsymbol{\omega}_0$ is a chain with l or $l+1$ trivalent vertices of genus 0, depending if $\dim \sigma_{\mathfrak{p}} = n$ or $\dim \sigma_{\mathfrak{p}} = n-1$. In any case, l vertices are marked by σ_{in}, the maximal cell containing the image of E_{in}. Thus if $\dim \sigma_{\mathfrak{p}} = n$, the underlying stable maps in $\mathscr{M}(X, \boldsymbol{\omega}_0)$ are contracted trees of l copies of $\mathbb{P}^1$ with three special points each, with $l-1$ pairs of such special points identified at the nodes. With the labelling of the $l+2$ remaining points, this is a rigid curve with trivial automorphism group. Moreover, the pull-back of Θ_X is trivial. Hence $\mathscr{M}(X, \boldsymbol{\omega}_0)$ is a reduced point, giving the result in the case $\dim \Lambda_{\mathfrak{p}} = n$.

In the codimension one case, $\mathfrak{p}$ is contained in the $(n-1)$-cell $\rho = \sigma_{\mathfrak{p}}$ with stratum $X_\rho \subset X$ isomorphic to $\mathbb{P}^1$ and $\mathcal{M}_X|_{X_\rho}$ admitting a lift of the torus action. In particular, it holds $\Theta_X|_{X_\rho} \simeq \mathcal{O}^n_{\mathbb{P}^1}$. The incoming and outgoing legs of G_0 are labelled by the maximal cells $\sigma_{\text{in}}, \boldsymbol{\sigma}(L_{\text{out}}) \in \Sigma(X)$ containing ρ. By the description of $\boldsymbol{\omega}_0$, only the outgoing vertex maps to ρ, all others map to σ_{in}. Thus the stable maps underlying objects in $\mathscr{M}(X, \boldsymbol{\omega}_0)$ are contracted trees of l copies of $\mathbb{P}^1$ with three special points each as before, joined at the outgoing point with another copy of $\mathbb{P}^1$ mapping to $X_\rho \simeq \mathbb{P}^1$. The tropical picture shows that the map $\mathbb{P}^1 \to \mathbb{P}^1$ is totally branched at the two zero-dimensional strata $X_{\sigma_{\text{in}}}, X_{\boldsymbol{\sigma}(L_{\text{out}})} \subset X_\rho$ with branching order $d = |\delta(u_{\text{in}})|$ the constant from (4.3). Hence the associated curve class is $d \cdot [X_\rho]$ (as follows also from Lemma 2.3). Thus there is a unique underlying stable map over $\underline{W} = \operatorname{Spec} \Bbbk$. Arguing similarly as in [48, Claim 3.22], there is then a unique enhancement, up to isomorphism, of this underlying stable map to a basic punctured map $f : C^\circ/W \to X$ of type $\boldsymbol{\omega}_0$ with $\underline{W} = \operatorname{Spec} \Bbbk$. Moreover, by the triviality of $\Theta_X|_{X_\rho}$ this is again a logarithmically unobstructed and rigid situation, now with automorphism group μ_d. Hence $\mathscr{M}(X, \boldsymbol{\omega}_0)$ is the quotient of a reduced point by μ_d, proving the stated result. □

Proof of (4.2) For applying the gluing formula we now use the labelling with pairwise different τ_i and multiplicities μ_i. Let

$$\delta : \mathscr{M}(X, \boldsymbol{\tau}) \to \mathscr{M}(X, \boldsymbol{\tau}_{\text{in}}) \times \mathscr{M}(X, \boldsymbol{\tau}_0) \times \prod_{i=1}^{l} \mathscr{M}(X, \boldsymbol{\tau}_i)^{\mu_i}$$

be the splitting map from (A.1) for $\boldsymbol{\tau}$. Applying Theorem A.4 with the general displacement vector ν from Lemma 4.15 and noting that $\operatorname{Aut}(\boldsymbol{\omega}/\boldsymbol{\tau}) = \{1\}$ yields

$$\begin{aligned} \delta_*\big[\mathscr{M}(X, \boldsymbol{\tau})\big]^{\text{virt}} &= m(\omega) \cdot \big[\mathscr{M}(X, \boldsymbol{\tau}_{\text{in}})^{\text{virt}}\big] \times (j_{\boldsymbol{\omega}_0})_*[\mathscr{M}(X, \boldsymbol{\omega}_0)]^{\text{virt}} \\ &\quad \times \prod_{i=1}^{l} \big[\mathscr{M}(X, \boldsymbol{\tau}_i)^{\mu_i}\big]^{\text{virt}}. \end{aligned} \tag{4.10}$$

Next observe

$$\operatorname{Aut}(\boldsymbol{\tau}) \simeq \operatorname{Aut}(\boldsymbol{\tau}_{\text{in}}) \times \prod_{i=1}^{l} \big(\operatorname{Aut}(\boldsymbol{\tau}_i) \rtimes S_{\mu_i}\big),$$

with the symmetric group S_{μ_i} acting by permutation of the μ_i copies of $\boldsymbol{\tau}_i$ attached at v. Taking degrees in (4.10) and applying Lemma 4.17, plugging in $m(\omega)$ from Lemma 4.16, multiplying by k_τ, and dividing by $|\operatorname{Aut}(\boldsymbol{\tau})|$ now leads to

$$\begin{aligned} k_\tau N_{\boldsymbol{\tau}} &= k_{\tau_{\text{in}}} \cdot \textstyle\prod_{i=1}^l (dk_{\tau_i})^{\mu_i} \cdot \dfrac{\deg[\mathscr{M}(X, \boldsymbol{\tau}_{\text{in}})]^{\text{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau}_{\text{in}})|} \cdot \displaystyle\prod_{i=1}^l \frac{\deg[\mathscr{M}(X, \boldsymbol{\tau}_i)^{\mu_i}]^{\text{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau}_i) \rtimes S_{\mu_i}|} \\ &= k_{\tau_{\text{in}}} N_{\tau_{\text{in}}} \cdot \prod_{i=1}^l \frac{(dk_{\tau_i} W_{\boldsymbol{\tau}_i})^{\mu_i}}{\mu_i!}. \end{aligned}$$

The total curve class and contact orders are computed from the decorations as

$$A = A_{\text{in}} + d[X_{\sigma_{\text{p}}}] + \sum_{i=1}^l \mu_i A_i,$$

and

$$u_{\text{out}} = u_{\text{in}} + \sum_{i=1}^l \mu_i u_i.$$

The last three equalities together imply the claimed scattering equation (4.2). □

5 Consistency of the canonical wall structure

We now prove the main result of the paper, namely that $\mathscr{S}_{\text{can}}$ is a consistent wall structure. The actual definition of consistency, while summarized in [39, Def. 3.9], is a bit complicated. However, the main point of the definition is to enable the construction of so-called theta functions. In other words, consistency guarantees that the local description $\vartheta_p(x)$ for theta functions patch together to give global functions on a scheme constructed from the wall structure.

Here, we proceed in the opposite direction. We will first show that the $\vartheta_p(x)$ satisfy the necessary patching criteria, and then show this patching implies consistency as defined in [39, Def. 3.9].

5.1 Patching of theta functions

As the discussion from now on largely follows the notions of [39], we now adopt the conventions and notation of that paper with regards to wall structures. In particular, we now work with a wall structure $\mathscr{S}$ satisfying precisely the properties of [39, Def. 2.11,2], by assuming that the walls of $\mathscr{S}$ comprise the codimension one cells of a polyhedral cone refinement $\mathscr{P}_{\mathscr{S}}$ of $\mathscr{P}$. As mentioned in Remark 3.4, this may be achieved for $\mathscr{S}_{\text{can}}$ by subdividing walls, combining walls with the same support, and adding walls with attached function 1 if necessary. In particular, as in [39], we use the notation $\mathfrak{u}$ for a maximal cell of $\mathscr{P}_{\mathscr{S}}$ and $\mathfrak{b}$ for a codimension one cell of $\mathscr{P}_{\mathscr{S}}$ contained in $\rho \in \mathscr{P}^{[n-1]}$. We call $\mathfrak{u}$ a *chamber* and $\mathfrak{b}$ a *slab* or *codimension one wall*, while we call those walls intersecting the interior of a maximal cell a *codimension zero wall*. A *joint* $\mathfrak{j}$ is a codimension two cell in $\mathscr{P}_{\mathscr{S}}$, and the codimension of $\mathfrak{j}$ is the codimension of the smallest cell of $\mathscr{P}$ containing $\mathfrak{j}$.

Recalling further notation from [39], if $\mathfrak{u}$ is a chamber contained in $\sigma \subseteq \mathscr{P}_{\max}$, we set

$$R_{\mathfrak{u}} = R_{\sigma} := \Bbbk[\mathcal{P}_x^+]/I_x = (\Bbbk[Q]/I)[\Lambda_\sigma]$$

for any point $x \in \text{Int}(\mathfrak{u})$.

Given a codimension zero wall $\mathfrak{p}$ separating chambers $\mathfrak{u}, \mathfrak{u}'$, we obtain a $\Bbbk[Q]/I$-algebra homomorphism

$$\theta_{\mathfrak{p}} : R_{\mathfrak{u}} \to R_{\mathfrak{u}'}, \quad z^m \mapsto f_{\mathfrak{p}}^{\langle n_{\mathfrak{p}}, m\rangle} z^m, \tag{5.1}$$

where $n_{\mathfrak{p}} \in \text{Hom}(\Lambda_\sigma, \mathbb{Z})$ is the primitive normal vector to $\mathfrak{p}$ positive on $\mathfrak{u}$. Since $f_{\mathfrak{p}} \equiv 1 \bmod \mathfrak{m}$ and $\sqrt{I} = \mathfrak{m}$, $f_{\mathfrak{p}}$ is in fact invertible so that $\theta_{\mathfrak{p}}$ is an automorphism.

For a slab $\mathfrak{b}$, after choosing ξ as chosen for the isomorphism (3.3), we define

$$R_{\mathfrak{b}} := (\Bbbk[Q]/I)[\Lambda_\rho][Z_+, Z_-]/(Z_+Z_- - f_{\mathfrak{b}} t^{\kappa_\rho}),$$

where $f_{\mathfrak{b}}$ is the function attached to $\mathfrak{b}$. If $\mathfrak{u}, \mathfrak{u}'$ are the adjacent chambers to $\mathfrak{b}$, there are natural localization maps at Z_+ and Z_- respectively

$$\chi_{\mathfrak{b},\mathfrak{u}} : R_{\mathfrak{b}} \to R_{\mathfrak{u}}, \quad \chi_{\mathfrak{b},\mathfrak{u}'} : R_{\mathfrak{b}} \to R_{\mathfrak{u}'}, \tag{5.2}$$

given by

$$\begin{aligned}
\chi_{\mathfrak{b},\mathfrak{u}}(t^A z^m Z_+^a Z_-^b) &= t^{A+b\kappa_\rho} z^{m+(a-b)\xi} f_{\mathfrak{b}}^b, \\
\chi_{\mathfrak{b},\mathfrak{u}'}(t^A z^m Z_+^a Z_-^b) &= t^{A+a\kappa_\rho} z^{m+(a-b)\xi} f_{\mathfrak{b}}^a,
\end{aligned}$$

where $A \in Q$ and $m \in \Lambda_\rho$. Note that these maps differ from those given in (3.4), (3.5) only in the powers of $f_{\mathfrak{b}}$ appearing here.

We may now make precise the notion that the $\vartheta_p(x)$ patch.

Theorem 5.1 *For the canonical wall structure $\mathscr{S}_{\text{can}}$ and any $p \in B(\mathbb{Z})\backslash\{0\}$, we have*

(1) *If $x, x' \in \text{Int}(\mathfrak{u})$ for a chamber $\mathfrak{u}$, then*

$$\vartheta_p(x) = \vartheta_p(x').$$

(2) *If $\mathfrak{p}$ is a codimension zero wall separating chambers $\mathfrak{u}, \mathfrak{u}'$, and $x \in \text{Int}(\mathfrak{u})$, $x' \in \text{Int}(\mathfrak{u}')$, then*

$$\vartheta_p(x') = \theta_{\mathfrak{p}}\big(\vartheta_p(x)\big).$$

(3) If $\mathfrak{b}$ is a slab with adjacent chambers $\mathfrak{u}, \mathfrak{u}'$, $x \in \text{Int}(\mathfrak{u})$, $x' \in \text{Int}(\mathfrak{u}')$, then there exists a unique $\vartheta_p(\mathfrak{b}) \in R_{\mathfrak{b}}$ such that

$$\chi_{\mathfrak{b},\mathfrak{u}}(\vartheta_p(\mathfrak{b})) = \vartheta_p(x), \quad \chi_{\mathfrak{b},\mathfrak{u}'}(\vartheta_p(\mathfrak{b})) = \vartheta_p(x').$$

Proof Step 1: Establishing the setup. We fix $p \in B(\mathbb{Z})\backslash\{0\}$ once and for all. We work throughout with $\vartheta_p^{\log}(x)$ using Theorem 4.5. Let $\mathscr{B}$ be the set of isomorphism classes of decorated broken line types given as

$$\begin{aligned}\mathscr{B} := \{\boldsymbol{\tau} = (\tau, \mathbf{A}) \mid\ & p_\tau = p, \text{ the total curve class of } \mathbf{A} \\ & \text{lies in } Q\backslash I \text{ and } N_{\boldsymbol{\tau}} \neq 0\}.\end{aligned}$$

$\mathscr{B}$ is a finite set by Lemma 3.22. Thus we may choose a polyhedral cone complex $\mathscr{P}_{\mathscr{B}}$ refining $\mathscr{P}_{\mathscr{S}}$ with the property that for each $\boldsymbol{\tau} \in \mathscr{B}$, $h(\boldsymbol{\tau}_{\text{out}})$ is a union of maximal cells $\mathfrak{u} \in \mathscr{P}_{\mathscr{B}}$. At the same time, we refine the wall structure $\mathscr{S}$ so that we may take $\mathscr{P}_{\mathscr{S}} = \mathscr{P}_{\mathscr{B}}$.

It is immediate from the definition of the log theta functions that if $\mathfrak{u} \in \mathscr{P}_{\mathscr{B}}$ is a maximal cone then $\vartheta_p^{\log}(x) = \vartheta_p^{\log}(x')$ for any $x, x' \in \text{Int}(\mathfrak{u})$. Thus, it is now sufficient to verify that if $\mathfrak{u}, \mathfrak{u}' \in \mathscr{P}_{\mathscr{B}}$ are two chambers separated by a codimension one cell $\mathfrak{p}$, then statements (2) or (3) of the theorem hold, depending on whether $\mathfrak{p}$ is a codimension zero wall or a slab.

Step 2: Broken lines transversal to $\mathfrak{p}$. For $y \in \text{Int}(\mathfrak{p})$, let $n_{\mathfrak{p}} \in \check{\Lambda}_y$ be a primitive normal vector to $\mathfrak{p}$, positive on $\mathfrak{u}$. We may then decompose, for $x \in \mathfrak{u}$, $x' \in \mathfrak{u}'$,

$$\begin{aligned}\vartheta_p^{\log}(x) &= \vartheta_+ + \vartheta_- + \vartheta_0 \\ \vartheta_p^{\log}(x') &= \vartheta'_+ + \vartheta'_- + \vartheta'_0\end{aligned}$$

where ϑ_+ (resp. ϑ_-, ϑ_0) consists of a sum of those monomials $at^A z^m$ appearing in $\vartheta_p^{\log}(x)$ with $\langle n_{\mathfrak{p}}, m\rangle > 0$ (resp. $\langle n_{\mathfrak{p}}, m\rangle < 0$ and $\langle n_{\mathfrak{p}}, m\rangle = 0$). The terms $\vartheta'_\pm$, ϑ'_0 are defined analogously.

Using the broken line description of these theta functions, the now standard argument of the proof of [36, Thm. 4.12] or [27, Lem. 4.9], shows immediately that if $\mathfrak{p}$ is a codimension zero wall, then

$$\theta_{\mathfrak{p}}(\vartheta_+) = \vartheta'_+, \quad \theta_{\mathfrak{p}}(\vartheta_-) = \vartheta'_-. \tag{5.3}$$

We briefly recall the argument. Given a decorated broken line contributing to ϑ_+ with endpoint in $\mathfrak{u}$ very close to $\mathfrak{p}$, we may perturb the broken line by moving its endpoint to a nearby point in $\mathfrak{p}$. We may then add an additional line segment (or simply extend the final line segment through $\mathfrak{p}$) following the definition of a decorated broken line to obtain a decorated broken line with endpoint nearby in $\mathfrak{u}'$. If this is done in all possible ways, it then follows from the propagation rule for monomials (4.1) that $\theta_{\mathfrak{p}}(\vartheta_+) = \vartheta'_+$. Finally, any monomial in ϑ_0 or ϑ'_0 has exponent tangent to $\mathfrak{p}$, and hence is left invariant under transport from $\mathfrak{u}$ to $\mathfrak{u}'$. Thus in this case, it is sufficient to show that $\vartheta_0 = \vartheta'_0$ under transport $\mathfrak{t}_{\sigma,\sigma'}$.

If $\mathfrak{p} = \mathfrak{b}$ is a slab contained in ρ, $\mathfrak{u} \subseteq \sigma$, $\mathfrak{u}' \subseteq \sigma'$, then we may write

$$\vartheta_+ = \sum_i a_i z^{m_i + \alpha_i \xi}, \quad \vartheta'_- = \sum_i a'_i z^{m'_i - \alpha'_i \xi}$$

with $a_i, a'_i \in \Bbbk[Q]/I$, $m_i, m'_i \in \Lambda_\rho$ and $\alpha_i, \alpha'_i > 0$. We may then define

$$\vartheta_+(\mathfrak{b}) := \sum_i a_i z^{m_i} Z_+^{\alpha_i}, \qquad \vartheta_-(\mathfrak{b}) := \sum_i a'_i z^{m'_i} Z_-^{\alpha'_i}.$$

By construction, $\chi_{\mathfrak{b},\mathfrak{u}}(\vartheta_+(\mathfrak{b})) = \vartheta_+$ and $\chi_{\mathfrak{b},\mathfrak{u}'}(\vartheta_-(\mathfrak{b})) = \vartheta'_-$. On the other hand, it follows from exactly the same argument as outlined above and the formulas for $\chi_{\mathfrak{b},\mathfrak{u}'}(Z_+)$ and $\chi_{\mathfrak{b},\mathfrak{u}}(Z_-)$ of (5.2) that

$$\chi_{\mathfrak{b},\mathfrak{u}'}(\vartheta_+(\mathfrak{b})) = \vartheta'_+, \qquad \chi_{\mathfrak{b},\mathfrak{u}}(\vartheta_-(\mathfrak{b})) = \vartheta_-. \tag{5.4}$$

As before, any monomial in ϑ_0 or ϑ'_0 is left invariant under transport from $\mathfrak{u}$ to $\mathfrak{u}'$. Thus again it is sufficient to show that $\vartheta_0 = \vartheta'_0$ under transport $\mathfrak{t}_{\sigma,\sigma'}$, as the expression

$$\vartheta_p(\mathfrak{b}) := \vartheta_+(\mathfrak{b}) + \vartheta_-(\mathfrak{b}) + \vartheta_0$$

then satisfies the desired properties.

Step 3: Reduction to a non-virtual linear equivalence relation on moduli spaces. Let $\mathscr{B}_{\mathfrak{u}}$ (resp. $\mathscr{B}_{\mathfrak{u}'}$) denote the collection of isomorphism classes of decorated broken line types $\boldsymbol{\tau}$ contributing to ϑ_0 (resp. ϑ'_0) such that $\mathfrak{u} \subseteq h(\tau_{\mathrm{out}})$, $\mathfrak{u}' \not\subseteq h(\tau_{\mathrm{out}})$ (resp. $\mathfrak{u}' \subseteq h(\tau_{\mathrm{out}})$, $\mathfrak{u} \not\subseteq h(\tau_{\mathrm{out}})$). Any other broken line type $\boldsymbol{\tau}$ contributing to either ϑ_0 or ϑ_0' must then have $\mathfrak{u} \cup \mathfrak{u}' \subseteq h(\tau_{\mathrm{out}})$, and hence contributes equally to both ϑ_0 and ϑ_0'. Hence it is sufficient to show that

$$\sum_{\boldsymbol{\tau} \in \mathscr{B}_{\mathfrak{u}}} k_\tau N_{\boldsymbol{\tau}} t^A z^{-u_\tau} = \sum_{\boldsymbol{\tau} \in \mathscr{B}_{\mathfrak{u}'}} k_\tau N_{\boldsymbol{\tau}} t^A z^{-u_\tau}.$$

Let $\mathscr{D}$ be the set of isomorphism classes of degenerate decorated broken line types (see Definition 3.19) $\boldsymbol{\tau}'$ for which there exists a $\boldsymbol{\tau} \in \mathscr{B}$ and a contraction morphism $\boldsymbol{\tau} \to \boldsymbol{\tau}'$. Recall that for a degenerate broken line type τ', $\dim \tau' = n - 2$ and $\dim h_{\tau'}(\tau'_{\mathrm{out}}) = n - 1$. Let

$$\mathscr{D}_{\mathfrak{p}} := \{\boldsymbol{\tau}' \in \mathscr{D} \,|\, \mathfrak{p} \subseteq h_{\tau'}(\tau'_{\mathrm{out}})\}.$$

Note by the defining assumption on $\mathscr{P}_{\mathscr{B}}$, if $\boldsymbol{\tau} \in \mathscr{B} \cup \mathscr{D}$ is a broken line type or a degenerate broken line type and $\dim \mathfrak{p} \cap h(\tau_{\mathrm{out}}) = n - 1$, then $\mathfrak{p} \subseteq h(\tau_{\mathrm{out}})$.

For any $\boldsymbol{\tau} \in \mathscr{B}_{\mathfrak{u}} \cup \mathscr{B}_{\mathfrak{u}'}$, since u_τ is tangent to $\mathfrak{p}$ and $\mathfrak{p}$ is contained in a codimension one face of $h(\tau_{\mathrm{out}})$, necessarily there is a unique choice of contraction morphism of decorated types $\phi : \boldsymbol{\tau} \to \boldsymbol{\tau}'$ with $\boldsymbol{\tau}' \in \mathscr{D}_{\mathfrak{p}}$. Note further the isomorphism class of $\boldsymbol{\tau}'$ only depends on the isomorphism class of $\boldsymbol{\tau}$. This gives maps $\Psi : \mathscr{B}_{\mathfrak{u}} \to \mathscr{D}_{\mathfrak{p}}$, $\Psi' : \mathscr{B}_{\mathfrak{u}'} \to \mathscr{D}_{\mathfrak{p}}$ taking $\boldsymbol{\tau}$ to the corresponding degenerate decorated broken line type $\boldsymbol{\tau}'$. It is now sufficient to show that for any $\boldsymbol{\tau}' \in \mathscr{D}_{\mathfrak{p}}$, we have

$$\sum_{\boldsymbol{\tau} \in \Psi^{-1}(\boldsymbol{\tau}')} k_\tau N_{\boldsymbol{\tau}} = \sum_{\boldsymbol{\tau} \in (\Psi')^{-1}(\boldsymbol{\tau}')} k_\tau N_{\boldsymbol{\tau}}. \tag{5.5}$$

To prove this, we now fix $\boldsymbol{\tau}' \in \mathscr{D}_{\mathfrak{p}}$ with underlying type τ'. We consider the moduli space

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau') := \mathfrak{M}(\mathcal{X}, \tau') \times_{\underline{\mathcal{X}}} \underline{X},$$

where the morphism $\mathfrak{M}(\mathcal{X}, \tau') \to \underline{\mathcal{X}}$ is given by schematic evaluation at the section of the universal curve corresponding to L_{out}. For any contraction morphism $\tau \to \tau'$, we obtain by [5, Prop. 5.19] a Cartesian diagram

$$\begin{tikzcd} \coprod_{\boldsymbol{\tau}=(\tau,\mathbf{A})} \mathscr{M}(X, \boldsymbol{\tau}) \arrow[r, "\iota'_\tau"] \arrow[d, "\varepsilon_\tau"'] & \mathscr{M}(X, \boldsymbol{\tau}') \arrow[d, "\varepsilon_{\tau'}"] \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau) \arrow[r, "\iota_\tau"'] & \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau') \end{tikzcd} \tag{5.6}$$

Here the disjoint union is over all decorations $\boldsymbol{\tau} = (\tau, \mathbf{A})$ of τ such that the contraction morphism $\tau \to \tau'$ induces a contraction morphism $\boldsymbol{\tau} \to \boldsymbol{\tau}'$. The maps ι_τ, ι'_τ are induced by the contraction morphisms $\tau \to \tau'$ and $\boldsymbol{\tau} \to \boldsymbol{\tau}'$. Note that

$$\begin{aligned}\sum_{\boldsymbol{\tau}\in\Psi^{-1}(\boldsymbol{\tau}')} k_\tau N_{\boldsymbol{\tau}} &= \sum_{\boldsymbol{\tau}\in\Psi^{-1}(\boldsymbol{\tau}')} k_\tau \frac{\deg[\mathscr{M}(X,\boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau})|}\\ &= \sum_{\substack{\tau\to\tau'\\ \boldsymbol{\tau}=(\tau,\mathbf{A})}} k_\tau \frac{\deg[\mathscr{M}(X,\boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau})||\operatorname{Aut}(\tau/\tau')||\operatorname{Aut}(\boldsymbol{\tau}/\boldsymbol{\tau}')|^{-1}} \qquad (5.7)\\ &= \sum_{\substack{\tau\to\tau'\\ \boldsymbol{\tau}=(\tau,\mathbf{A})}} k_\tau \frac{\deg[\mathscr{M}(X,\boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau}')||\operatorname{Aut}(\tau/\tau')|}.\end{aligned}$$

The last two sums are over (1) isomorphism classes of types τ which are the underlying type of some element of $\Psi^{-1}(\boldsymbol{\tau}')$ and (2) decorations $\boldsymbol{\tau} = (\tau, \mathbf{A})$ of τ yielding an induced contraction map $\boldsymbol{\tau} \to \boldsymbol{\tau}'$. The factor $|\operatorname{Aut}(\tau/\tau')||\operatorname{Aut}(\boldsymbol{\tau}/\boldsymbol{\tau}')|^{-1}$ arises in the third summation because we are now summing over multiple representatives for an isomorphism class in $\Psi^{-1}(\boldsymbol{\tau}')$. Indeed, giving a fixed representative τ equipped with its unique contraction morphism $\tau \to \tau'$, $\operatorname{Aut}(\tau/\tau')$ now acts on the set of decorations $\boldsymbol{\tau} = (\tau, \mathbf{A})$ of τ such that $\tau \to \tau'$ induces a contraction morphism $\boldsymbol{\tau} \to \boldsymbol{\tau}'$. The stabilizer of this action is the subgroup $\operatorname{Aut}(\boldsymbol{\tau}/\boldsymbol{\tau}') \subseteq \operatorname{Aut}(\tau/\tau')$, and hence each $\boldsymbol{\tau} \in \Psi^{-1}(\tau')$ appears $|\operatorname{Aut}(\tau/\tau')|/|\operatorname{Aut}(\boldsymbol{\tau}/\boldsymbol{\tau}')|$ times in the last two sums. For the last equality, we use $|\operatorname{Aut}(\boldsymbol{\tau})| = |\operatorname{Aut}(\boldsymbol{\tau}/\boldsymbol{\tau}')||\operatorname{Aut}(\boldsymbol{\tau}')|$.

Using (5.6), if we fix $\tau \to \tau'$, we may write

$$\sum_{\boldsymbol{\tau}=(\tau,\mathbf{A})} \deg[\mathscr{M}(X,\boldsymbol{\tau})]^{\mathrm{virt}} = \deg(\varepsilon_\tau)^![\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)].$$

Thus, with all sums below being over isomorphism classes of broken line types τ which are the underlying type of elements of $\Psi^{-1}(\boldsymbol{\tau}')$, the quantity of (5.7) may now be written as

$$\begin{aligned}\sum_\tau k_\tau \frac{\deg \iota'_{\tau,*}(\varepsilon_\tau)^![\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)]}{|\operatorname{Aut}(\boldsymbol{\tau}')||\operatorname{Aut}(\tau/\tau')|} &= \sum_\tau k_\tau \frac{\deg \varepsilon^!_{\tau'}\iota_{\tau,*}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)]}{|\operatorname{Aut}(\boldsymbol{\tau}')||\operatorname{Aut}(\tau/\tau')|}\\ &= \sum_\tau k_\tau \frac{|\operatorname{Aut}(\tau/\tau')| \deg \varepsilon^!_{\tau'}[\mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X},\tau')]}{|\operatorname{Aut}(\boldsymbol{\tau}')||\operatorname{Aut}(\tau/\tau')|}\\ &= \sum_\tau k_\tau \frac{\deg \varepsilon^!_{\tau'}[\mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X},\tau')]}{|\operatorname{Aut}(\boldsymbol{\tau}')|}.\end{aligned}$$

Here the first equality follows from [65, Thm. 4.1], while the second equality follows from the morphism $\iota_\tau : \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau) \to \mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X}, \tau')$ being of degree $|\operatorname{Aut}(\tau/\tau')|$. Thus it will be sufficient to show the following relation in the codimension one Chow group of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau')$

$$\sum_{\substack{\tau\to\tau' \\ \mathfrak{u}\subseteq h_\tau(\tau_{\mathrm{out}})}} k_\tau[\mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X}, \tau')] = \sum_{\substack{\tau\to\tau' \\ \mathfrak{u}'\subseteq h_\tau(\tau_{\mathrm{out}})}} k_\tau[\mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X}, \tau')], \tag{5.8}$$

where the sums are over isomorphism classes of broken line types τ with a contraction morphism $\tau \to \tau'$ and with the stated inclusion.

Step 4: Evaluation maps and local structure of moduli spaces. We continue with $\boldsymbol{\tau}' \in \mathscr{D}_{\mathfrak{p}}$ fixed as in the previous step. If $\mathfrak{C}^\circ \to \mathfrak{M}(\mathcal{X}, \tau')$ is the universal punctured curve over the moduli space $\mathfrak{M}(\mathcal{X}, \tau')$, with section x_{out} corresponding to the leg L_{out}, we write

$$\widetilde{\mathfrak{M}}(\mathcal{X}, \tau') := (\underline{\mathfrak{M}(\mathcal{X}, \tau')}, x^*_{\mathrm{out}}\mathcal{M}_{\mathfrak{C}^\circ})^{\mathrm{sat}},$$

the saturation of the log structure of $\mathfrak{C}^\circ$ pulled back via x_{out}: see [5, Sect. 5.2]. Denote the reduction by

$$\overline{\mathfrak{M}}(\mathcal{X}, \tau') := \widetilde{\mathfrak{M}}(\mathcal{X}, \tau')_{\mathrm{red}}.$$

By [5, Prop. 5.5], the composition $\overline{\mathfrak{M}}(\mathcal{X}, \tau') \to \widetilde{\mathfrak{M}}(\mathcal{X}, \tau') \to \mathfrak{M}(\mathcal{X}, \tau')$ induces an isomorphism on underlying stacks, as $\mathfrak{M}(\mathcal{X}, \tau')$ is already reduced by [5, Prop. 3.28].

By composing the universal morphism $f : \underline{\mathfrak{C}^\circ} \to \mathcal{X}$ with the section x_{out}, we obtain an evaluation map ev : $\overline{\mathfrak{M}}(\mathcal{X}, \tau') \to \mathcal{X}$. As the former space is reduced, this evaluation map factors through the stratum $\mathcal{X}_\sigma$ of $\mathcal{X}$ where $\sigma \in \mathscr{P}$ is the minimal cell containing $\mathfrak{p}$.

We recall from [5, Def. 3.22] that $\mathfrak{M}(\mathcal{X}, \tau')$ carries an idealized structure given by a coherent sheaf of monoid ideals $\mathcal{I} \subseteq \mathcal{M}_{\mathfrak{M}(\mathcal{X},\tau')}$. This sheaf may be described by giving the stalks of the image sheaf $\overline{\mathcal{I}}$ of $\mathcal{I}$ in the ghost sheaf $\overline{\mathcal{M}}_{\mathfrak{M}(\mathcal{X},\tau')}$. By [5, Prop. 3.23], because τ' is realizable, for a geometric point $\bar{w}$ of $\mathfrak{M}(\mathcal{X}, \tau')$, the monoid ideal $\overline{\mathcal{I}}_{\bar{w}}$ has the following description. Let $Q_{\bar{w}}$ be the stalk of the ghost sheaf at $\bar{w}$, and $Q_{\tau'}$ the stalk of the ghost sheaf at a generic point of $\mathfrak{M}(\mathcal{X}, \tau')$. Then there is a well-defined generization map $\chi_{\bar{w}} : Q_{\bar{w}} \to Q_{\tau'}$, and $\overline{\mathcal{I}}_{\bar{w}} = \chi^{-1}_{\bar{w}}(Q_{\tau'}\setminus\{0\})$.

Similarly, we may put an idealized structure on $\overline{\mathfrak{M}}(\mathcal{X}, \tau')$, with ideal sheaf $\mathcal{J}$. For a geometric point $\bar{w}$, let $Q^{\circ,\mathrm{sat}}_{\bar{w}}$ be the stalk of the ghost sheaf of this latter log structure; necessarily, this monoid is contained in $Q_{\bar{w}} \oplus \mathbb{Z}$ and contains $Q_{\bar{w}} \oplus \mathbb{N}$. Then we take the stalk of the corresponding monoid ideal $\overline{\mathcal{J}}_{\bar{w}}$ to be the monoid ideal generated by $\overline{\mathcal{I}}_{\bar{w}} \oplus 0$, $Q^{\circ,\mathrm{sat}}\setminus(Q \oplus \mathbb{N})$ and $(0, 1)$. It is straightforward to check that $\mathcal{J}$ yields an idealized structure on $\overline{\mathfrak{M}}(\mathcal{X}, \tau')$. Further, the natural morphism $\overline{\mathfrak{M}}(\mathcal{X}, \tau') \to \mathfrak{M}(\mathcal{X}, \tau')$ is idealized log étale.

Wu [82, Cor. 2.9] now shows that the map $\mathrm{ev} : \overline{\mathfrak{M}}(\mathcal{X}, \tau') \to \mathcal{X}$ is idealized log smooth. As this map factors through the idealized log étale strict closed embedding $\mathcal{X}_\sigma \to \mathcal{X}$, we write $\mathrm{ev} : \overline{\mathfrak{M}}(\mathcal{X}, \tau') \to \mathcal{X}_\sigma$, also idealized log smooth.

After a base-change, we also obtain

$$\overline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}, \tau') := \overline{\mathfrak{M}}(\mathcal{X}, \tau') \times_{\mathcal{X}_\sigma} X_\sigma \to X_\sigma$$

idealized log smooth, with the underlying stack of $\overline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}, \tau')$ isomorphic to the underlying stack of

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau') = \mathfrak{M}(\mathcal{X}, \tau') \times_{\underline{\mathcal{X}}_\sigma} \underline{X}_\sigma.$$

Step 5: Proof of the linear equivalence relation (5.8). Continuing with $\sigma \in \mathscr{P}$ the minimal cone containing $\mathfrak{p}$, denote by $\sigma_{\mathfrak{u}}, \sigma_{\mathfrak{u}'}$ the cones containing $\mathfrak{u}$ and $\mathfrak{u}'$ respectively. We write $X_{\mathfrak{u}}, X_{\mathfrak{u}'}$ for the corresponding zero-dimensional strata of X, and write $P_\sigma, P_{\mathfrak{u}}, P_{\mathfrak{u}'}$ for the stalks of the ghost sheaf of $X_\sigma, X_{\mathfrak{u}}$ and $X_{\mathfrak{u}'}$ at their generic points.

Our goal is to construct a rational function ψ on $\overline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}, \tau')$ and determine its divisor of zeros and poles. We note that as $\overline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}, \tau')$ is idealized log smooth over $\mathrm{Spec}\,\Bbbk$, it is isomorphic, locally in the smooth topology, to a subscheme of a toric variety defined by a monomial ideal, see [5, Prop. B.2]. Further, by the cited proposition and the explicit description of the idealized structure of $\overline{\mathfrak{M}}(\mathcal{X}, \tau')$, hence of $\overline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}, \tau')$, in fact $\overline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}, \tau')$ is smooth locally isomorphic to a toric stratum of a toric variety, and hence is normal.

Let $\bar{s} \in \Gamma(X_\sigma, \overline{\mathcal{M}}^{\mathrm{gp}}_{X_\sigma})$ be defined as follows. Note that giving such a section is equivalent to giving an integral piecewise linear function on $\mathrm{Star}(\sigma) \subseteq B$. Using the affine structure on $\mathrm{Star}(\sigma)$ induced by that on B, we take this function in fact to be linear in this affine structure, primitive, vanishing on $\mathfrak{p}$ and positive on $\mathfrak{u}$. We may then choose a lift of $\bar{s}$ to $s \in \Gamma(X_\sigma, \mathcal{M}^{\mathrm{gp}}_{X_\sigma})$. The existence of a lift is obvious when $\dim X_\sigma = 0$. When $\dim X_\sigma = 1$, one needs to check that the corresponding torsor is trivial. However, using the isomorphism of Lemma 1.10, we may identify the line bundle associated to the corresponding torsor with the restriction of a line bundle on X_{Σ_σ}. The line bundle on X_{Σ_σ} is defined by the linear function $\bar{s}$ on Σ_σ. However, it is a standard fact of toric geometry that the line bundle corresponding to a linear (as opposed to piecewise linear) function is in fact trivial.

Let $U \subseteq \overline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}, \tau')$ be the dense open stratum, i.e., the open substack whose geometric points are precisely those whose corresponding type is τ'. Note that the image of $\mathrm{ev}^\flat(s)$ in $\overline{\mathcal{M}}^{\mathrm{gp}}_{\overline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X},\tau')}$ vanishes on U. Indeed, this follows from the fact that $h_{\tau'}(\tau'_{\text{out}}) \subseteq \mathfrak{p}$ and $h_{\tau'}|_{\tau'_{\text{out}}}$ is dual to $\bar{\text{ev}}^\flat : P_\sigma \to Q^{\circ,\text{sat}}_{\tau'}$. Thus, viewing $\mathcal{O}^\times_{\overline{\mathfrak{M}}^{\text{ev}}(\mathcal{X},\tau')}$ as a subsheaf of $\mathcal{M}^{\text{gp}}_{\overline{\mathfrak{M}}^{\text{ev}}(\mathcal{X},\tau')}$ via the inverse of the structure map α, we see that $\text{ev}^\flat(s)$ restricts to an invertible function on U, and hence defines a rational function ψ on $\overline{\mathfrak{M}}^{\text{ev}}(\mathcal{X}, \tau')$. We complete the proof by determining the divisor of zeros and poles of ψ.

We now use the identification of underlying stacks of $\overline{\mathfrak{M}}^{\text{ev}}(\mathcal{X}, \tau')$ and $\mathfrak{M}^{\text{ev}}(\mathcal{X}, \tau')$. Since the complement of U is a union of strata of $\mathfrak{M}^{\text{ev}}(\mathcal{X}, \tau')$, it is enough to check the order of vanishing of ψ along each codimension one stratum of $\mathfrak{M}^{\text{ev}}(\mathcal{X}, \tau')$. These strata are given by $\mathfrak{M}^{\text{ev}}_\tau(\mathcal{X}, \tau')$ with $\phi : \tau \to \tau'$ a contraction morphism with $\dim \tau = n - 1$, see [5, Rem. 3.28]. We write $\overline{\mathfrak{M}}^{\text{ev}}_\tau(\mathcal{X}, \tau')$ for the corresponding stratum of $\overline{\mathfrak{M}}^{\text{ev}}(\mathcal{X}, \tau')$.

Given such a $\phi : \tau \to \tau'$, we have several cases:

Case 1: $\dim h_\tau(\tau_{\text{out}}) = n - 1$. In this case, necessarily $\bar{s}$ still vanishes on $h_\tau(\tau_{\text{out}})$, so as above, ψ extends as an invertible function across $\mathfrak{M}^{\text{ev}}_\tau(\mathcal{X}, \tau')$.

Case 2: $h_\tau(\tau_{\text{out}})$ *intersects the interior of* $\mathfrak{u}$. We adopt the notation from [5, App. B] that if P is a monoid, $K \subseteq P$ a monoid ideal, then $A_{P,K} := \operatorname{Spec} \Bbbk[P]/K$ and $\mathcal{A}_{P,K} := [A_{P,K}/\operatorname{Spec} \Bbbk[P^{\text{gp}}]]$.

Let $Q^{\circ,\text{sat}}_\tau$ be the stalk of the ghost sheaf of $\overline{\mathfrak{M}}^{\text{ev}}(\mathcal{X}, \tau')$ at a geometric generic point $\bar{w}$ of $\overline{\mathfrak{M}}^{\text{ev}}_\tau(\mathcal{X}, \tau')$, and let $J \subseteq Q^{\circ,\text{sat}}_\tau$ be the monoid ideal arising from the idealized structure. As ev is idealized log smooth, smooth locally in a neighbourhood of $\bar{w}$, ev concides with the morphism $\mathcal{A}_{Q^{\circ,\text{sat}},J} \to \mathcal{A}_{P_\mathfrak{u},K}$, by [5, Prop. B.4]. Here $K = P_\mathfrak{u} \setminus \chi^{-1}_{\sigma\mathfrak{u}}(0)$ where $\chi_{\sigma\mathfrak{u}} : P_\mathfrak{u} \to P_\sigma$ is the generization map. In particular, the stratum $X_\sigma \subseteq X$, smooth locally in a neighbourhood of the point $X_\mathfrak{u}$, is isomorphic to $A_{P_\mathfrak{u},K}$.

Let $\chi_{\tau'\tau} : Q^{\circ,\text{sat}}_\tau \to Q^{\circ,\text{sat}}_{\tau'}$ be the generization map, necessarily a localization along a face F of $Q^{\circ,\text{sat}}_\tau$. As $\dim \tau = \dim \tau' + 1$ we see that $F \cong \mathbb{N}$. Note that $\bar{s}$ is non-negative on $\mathfrak{u}$. Since $h_\tau(\tau_{\text{out}})$ contains $\mathfrak{u}$, on which $\bar{s}$ is non-negative, and $h_{\tau'}(\tau'_{\text{out}})$ is a face of $h_\tau(\tau_{\text{out}})$, on which $\bar{s}$ vanishes, it follows that $\bar{s}$ is non-negative on $h_\tau(\tau_{\text{out}})$. Dually, it follows that the stalk of $\bar{\text{ev}}^\flat(\bar{s})$ at $\bar{w}$ lies in $Q^{\circ,\text{sat}}_\tau \subseteq Q^{\circ,\text{gp}}_\tau$. Further, the stalk necessarily lies in $\chi^{-1}_{\tau'\tau}(0) = F$. The order of vanishing of ψ along the stratum determined by τ is then, by standard toric geometry, equal to the germ of $\bar{\text{ev}}^\flat(\bar{s})$ under the identification of F with $\mathbb{N}$. Dually, since by construction $\bar{s}$ generates $P^{\text{gp}}_\mathfrak{u} \cap \Lambda^\perp_\mathfrak{p}$, this number is the same as

$$d_\tau := |\operatorname{coker}(\Lambda_{\tau_{\text{out}}}/\Lambda_{\tau'_{\text{out}}} \to \Lambda_\mathfrak{u}/\Lambda_\mathfrak{p})|.$$

This is the order of vanishing of ψ along the stratum $\mathfrak{M}_\tau(\mathcal{X}, \tau')$.

Note an elementary diagram chase shows that

$$k_\tau = d_\tau |\operatorname{coker}(\Lambda_{\tau'_{\text{out}}} \to \Lambda_\mathfrak{p})_{\text{tors}}|.$$

Case 3: $h(\tau_{\mathrm{out}})$ intersects the interior of $\mathfrak{u}'$. The same analysis as in Case 2 applies once $\bar{s}$ is replaced by $-\bar{s}$. Hence ψ has a pole of order d_τ along $\mathfrak{M}_\tau(\mathcal{X}, \tau')$.

Putting these three cases together, we see that the divisor of zeros and poles of ψ gives the relation in the Chow group of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau')$:

$$\sum_{\substack{\tau \to \tau' \\ \mathfrak{u} \subseteq h_\tau(\tau_{\mathrm{out}})}} d_\tau [\mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X}, \tau')] = \sum_{\substack{\tau \to \tau' \\ \mathfrak{u}' \subseteq h_\tau(\tau_{\mathrm{out}})}} d_\tau [\mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X}, \tau')].$$

If we then multiply both sides by $|\operatorname{coker}(\Lambda_{\tau'_{\mathrm{out}}} \to \Lambda_{\mathfrak{p}})_{\mathrm{tors}}|$, we get the desired (5.8). □

5.2 Consistency from theta functions

Theorem 5.2 *$\mathscr{S}_{\mathrm{can}}$ is a consistent wall structure.*

Proof Consistency as defined in [39, Def. 3.9] involves checking consistency in codimensions zero, one and two. We check each case in turn.

Consistency in codimension zero Let $\mathfrak{j}$ be a codimension zero joint, $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ the walls containing $\mathfrak{j}$, taken in cyclic order, with $\mathfrak{p}_i$ contained in chambers $\mathfrak{u}_i$ and $\mathfrak{u}_{i+1}$, with indices taken modulo r. With $\theta_{\mathfrak{p}_i} : R_{\mathfrak{u}_i} \to R_{\mathfrak{u}_{i+1}}$ as defined in (5.1), consistency at the joint means [39, Def. 2.13] that

$$\theta := \theta_{\mathfrak{p}_r} \circ \cdots \circ \theta_{\mathfrak{p}_1} = \mathrm{id} \tag{5.9}$$

as an automorphism of R_σ, where $\sigma \in \mathscr{P}$ is the cell containing $\mathfrak{j}$. Note that for $p \in \sigma_{\mathbb{Z}} \setminus \{0\}$ and $x \in \operatorname{Int}(\sigma) \setminus |\mathscr{S}_{\mathrm{can}}|$, $\vartheta_p(x) = z^p \mod \mathfrak{m}_x$, where $\mathfrak{m}_x$ is as in (3.6). Indeed, the trivial broken line with no bends contributes the term z^p, and any other broken line β must have $a_\beta \in \mathfrak{m}_x$. Since the ideal $\mathfrak{m}_x$ is nilpotent in R_σ and z^p is invertible in R_σ, it follows that $\vartheta_p(x)$ is invertible in R_σ. Working inductively modulo powers of $\mathfrak{m}_x$, one then sees that any element of R_σ may be written as a $\Bbbk[Q]/I$-linear combination of ratios $\vartheta_p(x)/\vartheta_{p'}(x)$ for $p, p' \in \sigma_{\mathbb{Z}}$. By Theorem 5.1, (2), applied successively for the walls $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, we obtain $\theta\big(\vartheta_p(x)\big) = \vartheta_p(x)$, and hence θ is the identity on R_σ.

Consistency in codimension zero follows, as this just means each codimension zero joint is consistent.

Consistency in codimension one Let $\mathfrak{j}$ be a codimension one joint, contained in slabs $\mathfrak{b}_1, \mathfrak{b}_2 \subseteq \rho \in \mathscr{P}^{[n-1]}_{\mathrm{int}}$, maximal cones σ, σ', and codimension zero walls $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \subseteq \sigma$ and $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_s \subseteq \sigma'$. We order these so that

$$\mathfrak{b}_1, \mathfrak{p}_1, \ldots, \mathfrak{p}_r, \mathfrak{b}_2, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_s$$

are cyclically ordered about $\mathfrak{j}$. Then consistency at the joint $\mathfrak{j}$ is expressed in [39, Def. 2.14] as follows. We set

$$\begin{aligned}\theta &:= \theta_{\mathfrak{p}_r} \circ \theta_{\mathfrak{p}_{r-1}} \circ \ldots \circ \theta_{\mathfrak{p}_1} : R_\sigma \to R_\sigma \\ \theta' &:= \theta_{\mathfrak{p}'_1} \circ \theta_{\mathfrak{p}'_2} \circ \ldots \circ \theta_{\mathfrak{p}'_s} : R_{\sigma'} \to R_{\sigma'}.\end{aligned}$$

We use the notation $\chi_{\mathfrak{b}_i,\sigma}, \chi_{\mathfrak{b}_i,\sigma'}$ for the localization maps of (5.2). Then consistency is the statement that

$$(\theta \times \theta')\big((\chi_{\mathfrak{b}_1,\sigma}, \chi_{\mathfrak{b}_1,\sigma'})(R_{\mathfrak{b}_1})\big) = (\chi_{\mathfrak{b}_2,\sigma}, \chi_{\mathfrak{b}_2,\sigma'})(R_{\mathfrak{b}_2}).$$

To show this, let $x_i \in \sigma$ be a point in the chamber adjacent to $\mathfrak{b}_i$, and similarly $x'_i \in \sigma'$. If $p \in \rho_{\mathbb{Z}}\backslash\{0\}$, then $\vartheta_p(x_i) = z^p \mod \mathfrak{m}_{x_i}$, and similarly for x'_i, as in the codimension zero case. Thus $\vartheta_p(\mathfrak{b}_i)$, which exists by Theorem 5.1, (3), satisfies $\chi_{\mathfrak{b}_i,\sigma}(\vartheta_p(\mathfrak{b}_i) - z^p)$ nilpotent, and the same for $\chi_{\mathfrak{b}_i,\sigma'}$. By the discussion preceding [39, Def. 2.14], the map $(\chi_{\mathfrak{b}_i,\sigma}, \chi_{\mathfrak{b}_i,\sigma'}) : R_{\mathfrak{b}_i} \to R_\sigma \times R_{\sigma'}$ is injective. Thus we conclude that $\vartheta_p(\mathfrak{b}_i) - z^p$ is nilpotent in $R_{\mathfrak{b}_i}$, and hence $\vartheta_p(\mathfrak{b}_i)$ is invertible, as z^p is invertible. Further, if $p \in \sigma_{\mathbb{Z}}\backslash\{0\}$, we may write $p = p' + a\xi$ with $a \geq 0$ $p' \in \Lambda_\rho$, and then $\vartheta_p(\mathfrak{b}_1) = z^{p'}Z_+^a \mod \mathfrak{m}$, and similarly if $p \in \sigma'_{\mathbb{Z}}\backslash\{0\}$, $p = p' - a\xi$, we have $\vartheta_p(\mathfrak{b}_1) = z^{p'}Z_-^a \mod \mathfrak{m}$. From this we see that any element of $R_{\mathfrak{b}_1}$ may be written as a $\Bbbk[Q]/I$-linear combination of Laurent monomials in theta functions of the form $(\prod \vartheta_{p_i}(\mathfrak{b}_1)^{a_i})/\vartheta_p(\mathfrak{b}_1)$ with $p \in \rho_{\mathbb{Z}}$, $p_i \in \sigma_{\mathbb{Z}} \cup \sigma'_{\mathbb{Z}}$. Note here we only have control over the expressions $\vartheta_p(\mathfrak{b}_1)$ for $p \in \sigma_{\mathbb{Z}} \cup \sigma'_{\mathbb{Z}}$, so we need such Laurent monomials in order to get all elements of $R_{\mathfrak{b}_1}$.

Now it follows from Theorem 5.1, (2) that for $p \in B(\mathbb{Z})\backslash\{0\}$

$$\begin{aligned}&(\theta \times \theta')\big((\chi_{\mathfrak{b}_1,\sigma}, \chi_{\mathfrak{b}_1,\sigma'})(\vartheta_p(\mathfrak{b}_1))\big) = (\theta \times \theta')\big(\vartheta_p(x_1), \vartheta_p(x'_1)\big) \\ &\quad = \big(\vartheta_p(x_2), \vartheta_p(x'_2)\big) = (\chi_{\mathfrak{b}_2,\sigma} \times \chi_{\mathfrak{b}_2,\sigma'})(\vartheta_p(\mathfrak{b}_2)).\end{aligned}$$

Combined with the generation statement of the previous paragraph, we obtain the desired equality.

This shows consistency at codimension one joints, noting there is no condition for codimension one joints contained in ∂B. Thus we obtain consistency in codimension one.

Consistency in codimension two Consistency at codimension two joints is defined in [39, Sect. 3.2]. If $\mathfrak{j}$ is a boundary joint, i.e., contained in ∂B, then it is easy to check that $\mathscr{S}_{\mathrm{can}}$ is convex at $\mathfrak{j}$ in the sense of [39, Def. 3.12]. Indeed, if $\partial B \neq \emptyset$, it follows from Propositions 1.15 and 3.7 that if τ is a wall type, then u_τ is tangent to ∂B. Thus by [39, Thm. 3.13], the joint is consistent.

In case $\mathfrak{j}$ is an interior joint, we refer the reader to [39, Sect. 3.2] for the definition of a corresponding polyhedral affine manifold $(B_{\mathfrak{j}}, \mathscr{P}_{\mathfrak{j}})$, with $\mathscr{P}_{\mathfrak{j}}$

consisting of tangent cones to elements of $\mathscr{P}$ containing $\mathfrak{j}$ along $\mathfrak{j}$. Further, $\mathscr{S}_{\mathrm{can}}$ induces a wall structure on $B_{\mathfrak{j}}$ we denote as $\mathscr{S}_{\mathfrak{j}}$. For $p \in B_{\mathfrak{j}}(\mathbb{Z})\setminus\{0\}$, broken lines on $B_{\mathfrak{j}}$ then define for general $x \in B_{\mathfrak{j}}$ theta functions $\vartheta_p^{\mathfrak{j}}(x)$ as before. Consistency is then the statements (1)–(3) of Theorem 5.1 for $B_{\mathfrak{j}}$ and $\mathscr{S}_{\mathfrak{j}}$ rather than B and $\mathscr{S}_{\mathrm{can}}$.

We do not prove consistency directly from Theorem 5.1 as it is not clear what the relationship between theta functions on B and on $B_{\mathfrak{j}}$ is. Instead, we proceed as follows. Write Star($\mathfrak{j}$) for the open star of $\mathfrak{j}$ with respect to the polyhedral cone decomposition $\mathscr{P}_{\mathscr{B}}$. Then there is a natural embedding of Star($\mathfrak{j}$) in $B_{\mathfrak{j}}$, identifying a cone of $\mathscr{P}_{\mathscr{B}}$ containing $\mathfrak{j}$ and contained in $\sigma \in \mathscr{P}$ with a cone in the tangent wedge of σ along $\mathfrak{j}$, with vertex at the origin of the tangent wedge. Note the walls of $\mathscr{S}_{\mathfrak{j}}$ are in one-to-one correspondence with the walls of $\mathscr{S}_{\mathrm{can}}$ intersecting Star($\mathfrak{j}$).

The tangent space $\Lambda_{\mathfrak{j},\mathbb{R}}$ to $\mathfrak{j}$ acts by translation on $B_{\mathfrak{j}}$ and $\Lambda_{\mathfrak{j}}$ acts on $B_{\mathfrak{j}}(\mathbb{Z})$. Further, all walls of $\mathscr{S}_{\mathfrak{j}}$ are invariant under this translation, and translation takes broken lines to broken lines. In particular, if $p \in B_{\mathfrak{j}}(\mathbb{Z})$, $v \in \Lambda_{\mathfrak{j},\mathbb{R}}$, it is immediate that

$$\vartheta_p^{\mathfrak{j}}(x) = \vartheta_p^{\mathfrak{j}}(x+v)$$

for general x. Further, since the monomials z^v for $v \in \Lambda_{\mathfrak{j}}$ are invariant under crossing all walls in $\mathscr{S}_{\mathfrak{j}}$, we in fact have

$$\vartheta_{p+v}^{\mathfrak{j}}(x) = z^v \vartheta_p^{\mathfrak{j}}(x).$$

Note that the broken lines defining the two theta functions are not precisely the same; rather, if $a_i z^{m_i}$ is a monomial attached to a segment of a broken line contributing to $\vartheta_p^{\mathfrak{j}}(x)$, then the corresponding broken line contributing to $\vartheta_{p+v}^{\mathfrak{j}}(x)$ has attached monomial $a_i z^{m_i+v}$, and thus the actual maps β are different as they have different derivatives. However, there is clearly a one-to-one correspondence between such broken lines. In particular, if items (1)–(3) of Theorem 5.1 hold for a given choice of p, for the functions $\vartheta_p^{\mathfrak{j}}(x)$ rather than $\vartheta_p(x)$, they will also hold for $p+v$ for any $v \in \Lambda_{\mathfrak{j}}$.

We will now show items (1)–(3) of Theorem 5.1 for the functions $\vartheta_p^{\mathfrak{j}}(x)$ for arbitrary non-zero $p \in B_{\mathfrak{j}}(\mathbb{Z})$, which is fixed throughout the following discussion. Because we may replace p with $p+v$ for any $v \in \Lambda_{\mathfrak{j}}$, we may assume, as we do from now on in this proof, that $p \in$ Star($\mathfrak{j}$). Indeed, for any point $x \in B_{\mathfrak{j}}(\mathbb{Z})$, there is some $v \in \mathfrak{j} \cap B(\mathbb{Z})$ such that $x + v \in$ Star($\mathfrak{j}$) under the canonical embedding of Star($\mathfrak{j}$) in $B_{\mathfrak{j}}$.

More generally, it follows easily from this observation that if $Z \subseteq B_{\mathfrak{j}}$ is any compact subset, then there is a $v \in \mathfrak{j} \cap B(\mathbb{Z})$ such that $Z + v \subseteq$ Star($\mathfrak{j}$).

Now suppose given an open subset $U \subseteq B_{\mathrm{j}}$ whose closure Z is compact. For each $x \in Z$ not contained in $|\mathscr{S}_{\mathrm{j}}|$, we have a finite set of broken lines contributing to $\vartheta^{\mathrm{j}}_p(x)$, and these all have asymptotic direction p contained in $\mathrm{Star}(\mathrm{j})$. Thus if β is a broken line contributing to this theta function, $\beta([t_1, 0])$ is a compact subset of B_{j} and we may translate x by some $v \in \mathrm{j}$ so that $\beta([t_1, 0]) \subseteq \mathrm{Star}(\mathrm{j})$. As the asymptotic direction p necessarily lies in a cone of $\mathscr{P}_{\mathscr{B}}$ containing $\beta(t_1)$, we see also that $\beta((-\infty, t_1]) \subseteq \mathrm{Star}(\mathrm{j})$. By summing over the v's for the finite number of broken lines contributing to $\vartheta^{\mathrm{j}}_p(x)$ we may thus translate x by some $v \in \mathrm{j}$ to guarantee that all broken lines contributing to $\vartheta^{\mathrm{j}}_p(x)$ have image contained in $\mathrm{Star}(\mathrm{j})$. Noting that broken lines can be deformed continuously inside some polyhedron without changing their type (see [39, Prop. 3.5]), we may further in fact assume that all broken lines contributing to $\vartheta^{\mathrm{j}}_p(x)$ for $x \in Z \setminus |\mathscr{S}_{\mathrm{j}}|$ have image in $\mathrm{Star}(\mathrm{j})$.

From the construction of $\mathscr{S}_{\mathrm{j}}$, it is now clear that provided x has been translated sufficiently, $\vartheta^{\mathrm{j}}_p(x)$ is the sum of contributions of those broken lines contributing to $\vartheta_p(x)$ with image wholly contained in $\mathrm{Star}(\mathrm{j})$.

One now checks that the proof of Theorem 5.1 applies equally well when $p \in \mathrm{Star}(\mathrm{j})$ if one restricts attention to endpoints $x \in \mathrm{Star}(\mathrm{j})$ translated "sufficiently far" by an element $v \in \mathrm{j}$. Indeed, there are two key points of the argument in the proof of Theorem 5.1 where one needs to control the behaviour of the broken lines being considered.

First, in Step 2, the proof of (5.3) and (5.4) works for any wall structure (not necessarily a consistent one) and hence works for $\mathscr{S}_{\mathrm{j}}$.

Second, in Steps 3 and following, we focus on a single degenerate broken line type τ' and the result follows from the relation (5.8). However, if we fix a point $x \in h_{\tau'}(\tau'_{\mathrm{out}})$ and a neighbourhood U of x with compact closure Z, we then may translate Z by some $v \in \mathrm{j}$ so that all broken lines with endpoints in $Z \setminus |\mathscr{S}_{\mathrm{j}}|$ lie in $\mathrm{Star}(\mathrm{j})$, and thus in (5.8), we only need to consider those types τ with contraction morphisms $\tau \to \tau'$ whose corresponding broken lines with endpoint in Z lie entirely in $\mathrm{Star}(\mathrm{j})$. $\square$

We end this section with a brief observation in the relative case $g : X \to S$, following on from Proposition 1.16 and Remarks 1.17,1.18. Assume that the hypotheses of Proposition 1.16 hold, so that $(B', \mathscr{P}')$ is a polyhedral affine pseudomanifold. In this case, the canonical wall structure $\mathscr{S}_{\mathrm{can}}$ induces a wall structure $\mathscr{S}'_{\mathrm{can}}$ on $(B', \mathscr{P}')$. Indeed, first note that $\Lambda_{B'}$, the local system of integral tangent vectors on B', is a subsheaf of $\Lambda_B|_{B'}$ consisting of those tangent vectors v with $(g_{\mathrm{trop}})_*(v) = 0$. Using φ the MPL function on B given by Construction 1.14, we obtain also an MPL function $\varphi|_{B'}$ as in Remark 1.17. This induces a local system $\mathcal{P}_{B'}$, and $\mathcal{P}_{B'} \subseteq \mathcal{P}_B|_{B'}$ is the subsheaf consisting of those sections m of $\mathcal{P}_B|_{B'}$ such that $(g_{\mathrm{trop}})_*(\bar{m}) = 0$.

Next, given a wall $(\mathfrak{p}, f_\mathfrak{p}) \in \mathscr{S}_{\mathrm{can}}$, we define the *index* of $\mathfrak{p}$ to be

$$\mathrm{ind}(\mathfrak{p}) := |\operatorname{coker}(g_{\mathrm{trop},*} : \Lambda_\mathfrak{p} \to \mathbb{Z})|.$$

We may now define

$$\mathscr{S}'_{\mathrm{can}} := \{(\mathfrak{p} \cap B', f_\mathfrak{p}^{\mathrm{ind}(\mathfrak{p})}) \,|\, (\mathfrak{p}, f_\mathfrak{p}) \in \mathscr{S}_{\mathrm{can}}\},$$

noting that for any $x \in B' \setminus \Delta$, $f_\mathfrak{p} \in \Bbbk[\mathcal{P}^+_{B',x}] \subseteq \Bbbk[\mathcal{P}^+_{B,x}]$ by Proposition 3.7 and the construction of the canonical wall structure.

Proposition 5.3 *Assuming $g : X \to S$ satisfies the hypotheses of Proposition 3.7, then $\mathscr{S}'_{\mathrm{can}}$ is a consistent wall structure.*

Proof This follows from the definition of consistency and the consistency of $\mathscr{S}_{\mathrm{can}}$. Indeed, checking consistency of $\mathscr{S}'_{\mathrm{can}}$ in codimensions zero, one and two involves checking certain properties in rings which are subrings of the corresponding rings for B. For example, for consistency in codimension zero, to verify (5.9) in the ring $R_{\sigma\cap B'}$ defined using the data $(B', \mathscr{P}', \varphi|_{B'})$, it is enough to note that $R_{\sigma\cap B'}$ is a subring of R_σ and that the ring automorphism $\vartheta_\mathfrak{p}$ of R_σ restricts to an automorphism of $R_{\sigma\cap B'}$ given by the wall $(\mathfrak{p}\cap B', f_\mathfrak{p}^{\mathrm{ind}(\mathfrak{p})})$. This is straightforward: see the discussion of [39, Sect. 4.2] as to why the power of $f_\mathfrak{p}$ is necessary. Indeed, [39, Sect. 4.2] goes the opposite way, from a wall structure for B' to a wall structure for $\mathbf{C}B' = B$, see Remark 1.18. Thus there one must pass to a $\mathrm{ind}(\mathfrak{p})^{th}$ root rather than power. We leave the remaining details to the reader. □

6 Comparison with intrinsic mirror symmetry

We continue with $g : X \to S$ in the absolute or relative cases satisfying Assumptions 1.1 or 1.2. Having now constructed a consistent wall structure $\mathscr{S}_{\mathrm{can}}$ on B, we observe that the data $(B, \mathscr{P})$ and $\mathscr{S}_{\mathrm{can}}$ are *conical* in the sense of [39, Def. 3.20]. Thus this data first specifies a scheme[9] $\check{\mathfrak{X}}^\circ$ flat over $\Bbbk[Q]/I$, as constructed in [39, Prop. 2.4] by gluing together copies of spectra of rings $R_\mathfrak{u}$ and $R_\mathfrak{b}$, as well as some additional rings corresponding to $n-1$ dimensional cells of $\mathscr{P}_\mathscr{S}$ contained in ∂B. We refer the reader to the cited result for details. Most importantly for us, [39, Prop. 3.21] tells us that $R := \Gamma(\check{\mathfrak{X}}^\circ, \mathcal{O}_{\check{\mathfrak{X}}^\circ})$ is freely generated as a $\Bbbk[Q]/I$-module by a basis $\{\vartheta_p \,|\, p \in B(\mathbb{Z})\}$. Further, $\check{\mathfrak{X}} := \operatorname{Spec} R$ contains $\check{\mathfrak{X}}^\circ$ as a dense open subscheme and is also flat over

[9] This scheme is written as $\mathfrak{X}^\circ$ in [39]; we include the check here to indicate it is the mirror to (X, D).

$\operatorname{Spec} \Bbbk[Q]/I$. The theta functions ϑ_p restrict on standard pieces $\operatorname{Spec} R_{\mathfrak{u}}$ to the expressions $\vartheta_p(x)$ for $x \in \operatorname{Int}(\mathfrak{u})$ defined in Definition 4.4.

In the relative case, R is naturally a graded ring via $\deg(\vartheta_p) = g_{\mathrm{trop}}(p) \in \mathbb{N}$. Thus to obtain the mirror in the relative case, one considers instead the flat family $\operatorname{Proj} R \to \operatorname{Spec} \Bbbk[Q]/I$; see [39, Sect. 4] and [48, Sect. 1] for more details on this point of view.

[39, Thm. 3.24] gives a description of the structure constants for R. Write, for $p_1, p_2 \in B(\mathbb{Z})\backslash\{0\}$,

$$\vartheta_{p_1} \cdot \vartheta_{p_2} = \sum_{r \in B(\mathbb{Z})} \alpha^{\mathrm{trop}}_{p_1 p_2 r} \cdot \vartheta_r$$

with $\alpha^{\mathrm{trop}}_{p_1 p_2 r} \in \Bbbk[Q]/I$. These structure constants, written as $\alpha_r(p_1, p_2)$ in [39], are defined tropically as follows. Let $\mathfrak{u}$ be a chamber of $\mathscr{S}_{\mathrm{can}}$ such that $r \in \mathfrak{u}$, and let $x \in \mathfrak{u}$ be general. Then

$$\alpha^{\mathrm{trop}}_{p_1 p_2 r} = \sum_{(\beta_1, \beta_2)} a_{\beta_1} a_{\beta_2} \tag{6.1}$$

where the sum is over all pairs (β_1, β_2) of broken lines with asymptotic monomials p_1, p_2 respectively, with endpoint x, and such that $m_{\beta_1} + m_{\beta_2} = r$, viewed as an equation in $\Lambda_{\mathfrak{u}}$. We note that the restriction that $p_1, p_2 \neq 0$ is not important as ϑ_0 was defined as the unit in the ring R.

On the other hand, [48] defines structure constants on the free $\Bbbk[Q]/I$-module with basis $\{\vartheta_p \mid p \in B(\mathbb{Z})\}$ directly in terms of punctured Gromov–Witten invariants. Following the notation of [48], we have a potentially different product rule

$$\vartheta_{p_1} \cdot \vartheta_{p_2} = \sum_{r \in B(\mathbb{Z})} \alpha^{\log}_{p_1 p_2 r} \cdot \vartheta_r.$$

These structure constants are written simply as $\alpha_{p_1 p_2 r}$ in [48]. They are given by a formula

$$\alpha^{\log}_{p_1 p_2 r} = \sum_A N^A_{p_1 p_2 r} t^A \in \Bbbk[Q]/I, \tag{6.2}$$

where the numbers $N^A_{p_1 p_2 r}$ are defined in [48, Def. 3.21] as certain punctured invariants with contact orders p_1, p_2 and $-r$.

In this section, we show that in fact we obtain the same algebra, i.e.,

Theorem 6.1 *For all $p_1, p_2 \in B(\mathbb{Z})\backslash\{0\}$, $r \in B(\mathbb{Z})$,*

$$\alpha^{\mathrm{trop}}_{p_1 p_2 r} = \alpha^{\log}_{p_1 p_2 r}.$$

Before proving this theorem, let us introduce some additional terminology.

Definition 6.2 A *product type* with inputs $p_1, p_2 \in B(\mathbb{Z})$ and output $r \in B(\mathbb{Z})$ is a type $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$ of tropical map to $\Sigma(X)$ such that:

(1) G is a genus zero graph with $L(G) = \{L_1, L_2, L_{\mathrm{out}}\}$ with $p_i \in \boldsymbol{\sigma}(L_i)$, $\mathbf{u}(L_i) = p_i$, $\boldsymbol{\sigma}(L_{\mathrm{out}}) \in \mathscr{P}$, $r \in \boldsymbol{\sigma}(L_{\mathrm{out}})$ and $\mathbf{u}(L_{\mathrm{out}}) = -r$.
(2) τ is realizable and balanced.
(3) Let $h : \Gamma(G, \ell) \to \Sigma(X)$ be the corresponding universal family of tropical maps, and let $\tau_{v_{\mathrm{out}}} \in \Gamma(G, \ell)$ be the cone corresponding to the vertex v_{out} of G adjacent to L_{out}. Then $\dim \tau = \dim h(\tau_{v_{\mathrm{out}}}) = n$.

A *decorated product type* is a decorated type $\boldsymbol{\tau} = (\tau, \mathbf{A})$ with τ a product type.

Identically to Lemmas 3.9 and 3.20, we have

Lemma 6.3 *Let $\boldsymbol{\tau}$ be a decorated product type. Then $\mathscr{M}(X, \boldsymbol{\tau})$ is proper over* Spec $\Bbbk$ *and carries a virtual fundamental class of dimension zero.*

We now define

$$N_{\boldsymbol{\tau}} := \frac{\deg[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau})|}$$

as usual. We also have a map $h_* : \Lambda_{\tau_{v_{\mathrm{out}}}} \to \Lambda_{\boldsymbol{\sigma}(v_{\mathrm{out}})}$, necessarily of finite index, and define

$$k_\tau := |\operatorname{coker} h_*| = |\Lambda_{\boldsymbol{\sigma}(v_{\mathrm{out}})}/h_*(\Lambda_{\tau_{v_{\mathrm{out}}}})|.$$

We split the proof of Theorem 6.1 into several steps.

Theorem 6.4 *Fix $p_1, p_2, r \in B(\mathbb{Z})$, and let $\mathfrak{u}$ be a chamber of $\mathscr{P}_{\mathscr{B}}$ containing r. Then there exists a top-dimensional subcone $\mathfrak{u}' \subseteq \mathfrak{u}$ containing r such that*

$$\alpha^{\log}_{p_1 p_2 r} = \sum_{\boldsymbol{\tau}=(\tau, \mathbf{A})} k_\tau N_{\boldsymbol{\tau}} t^A \in \Bbbk[Q]/I,$$

where the sum is over all isomorphism classes of decorated product types with inputs p_1, p_2, output r, and $\mathfrak{u}' \subseteq h(\tau_{v_{\mathrm{out}}})$. Here A is the total class of $\mathbf{A}$.

Proof Throughout the proof, we assume familiarity with the notation of [48], and do not review it, giving only references where appropriate.

Step 1: Alternative description of $N^A_{p_1 p_2 r}$. Fix a curve class $A \in Q \setminus I$, and let β be the type of punctured map of curve class A, with three punctured points $x_1, x_2, x_{\mathrm{out}}$ with contact orders given by p_1, p_2 and $-r$ respectively. Thus we obtain moduli spaces of punctured maps $\mathscr{M}(X, \beta)$ and $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \beta)$, where evaluation is at x_{out} only.

Let $\sigma_{\mathfrak{u}} \in \mathscr{P}$ be the maximal cell containing $\mathfrak{u}$. Let $P_{\mathfrak{u}}$ be the stalk of $\overline{\mathcal{M}}_X$ at the zero-dimensional stratum $X_{\sigma_{\mathfrak{u}}}$. Let $R := \mathbb{N}^n$ be generated by $e_1, \ldots, e_n$, with dual monoid generated by $e_1^*, \ldots, e_n^*$. Choose a monoid homomorphism $\psi : P_{\mathfrak{u}} \to R$ in such a way that the transpose $\psi^t : R^\vee \to P_{\mathfrak{u}}^\vee = \sigma_{\mathfrak{u}} \cap \Lambda_{\mathfrak{u}}$ is (1) injective; (2) has image contained in $\mathfrak{u}$. Further, if $r \neq 0$, we also require (3) $\psi^t(e_1^*) = r$.

If $r \neq 0$, define W as the log stack quotient of $\mathrm{Spec}(R \to \Bbbk)$ by the $\mathbb{G}_m$ action which acts on the torsor associated to $m \in R$ with weight $\langle e_1^*, m\rangle$. Put another way, $\underline{W} = B\mathbb{G}_m$, $\overline{\mathcal{M}}_W = R$, and the torsor associated to $m \in R$ is $\mathcal{U}^{\otimes\langle e_1^*, m\rangle}$ where $\mathcal{U}$ is the universal torsor on $B\mathbb{G}_m$. If, on the other hand, $r = 0$, we set

$$W := \mathrm{Spec}(R \to \Bbbk) \times B\mathbb{G}_m^\dagger,$$

where $B\mathbb{G}_m^\dagger$ denotes the log structure on $B\mathbb{G}_m$ induced by the inclusion as a divisor $B\mathbb{G}_m \subseteq [\mathbb{A}^1/\mathbb{G}_m]$.

We may then define a morphism $g : W \to \mathscr{P}(X, r)$, where the latter stack is as defined in [48, Sect. 3.1]. Using the notation of [48], write Z_r for the stratum X_σ where $\sigma \in \mathscr{P}$ is the minimal cone containing r. Then $\mathscr{P}(X, r)$ may be described as $[Z_r/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts trivially on $\underline{Z}_r$ and acts on the log structure on Z_r as described in [48, Rem. 3.1]. Note that since $r \in \sigma_{\mathfrak{u}}$, $X_{\sigma_{\mathfrak{u}}} \subseteq Z_r$. There then exists a morphism $g' : \mathrm{Spec}(R \to \Bbbk) \to X_{\sigma_{\mathfrak{u}}}$ which is an isomorphism on underlying schemes and such that $\overline{(g')}^\flat = \psi$. We also denote by g' the composition with the inclusion into Z_r. If $r \neq 0$, then the action of $\mathbb{G}_m$ on $\mathrm{Spec}(R \to \Bbbk)$ used to define W is compatible with the action of $\mathbb{G}_m$ on Z_r using the fact that $\psi^t(e_1^*) = r$ and the description of [48, Rem. 3.1], and hence g' descends to the desired map $g : W \to \mathscr{P}(X, r)$. If, on the other hand, $r = 0$, $\mathscr{P}(X, r) = X \times B\mathbb{G}_m$, and hence we obtain the desired morphism $g : W \to \mathscr{P}(X, r)$ as the product of g' with the morphism $B\mathbb{G}_m^\dagger \to B\mathbb{G}_m$ which is an isomorphism on underlying stacks.

We have canonical morphisms $\mathrm{ev}_{\mathcal{X}} : \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \beta) \to \mathscr{P}(X, r)$ and $\mathrm{ev}_X : \mathscr{M}(X, \beta) \to \mathscr{P}(X, r)$, see [48, Def. 3.5]. We may then define

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \beta)_W = W \times^{\mathrm{fs}}_{\mathscr{P}(X,r)} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \beta),$$
$$\mathscr{M}(X, \beta)_W = W \times^{\mathrm{fs}}_{\mathscr{P}(X,r)} \mathscr{M}(X, \beta).$$

By choosing the image of ψ^t to be sufficiently small, we may assume that g is transverse to $\mathrm{ev}_{\mathcal{X}}$ in the sense of [48, Def. 2.6], see [48, Thm. 2.9] for the necessary tropical criterion. By [48, Def. 7.9], we then obtain a number

$$N^{A,W}_{p_1 p_2 r} = \deg[\mathscr{M}(X, \beta)_W]^{\mathrm{virt}}$$

with the virtual fundamental class defined via an obstruction theory for

$$\varepsilon_W : \mathscr{M}(X,\beta)_W \to \mathfrak{M}^{\mathrm{ev}}(X,\beta)_W$$

pulled back from that for $\mathscr{M}(X,\beta) \to \mathfrak{M}^{\mathrm{ev}}(X,\beta)$. By [48, Lem. 7.8], $\mathfrak{M}^{\mathrm{ev}}(X,\beta)_W$ is pure-dimensional of dimension zero, so that the virtual pull-back of its fundamental class gives the virtual fundamental class of $\mathscr{M}(X,\beta)_W$.

The hypotheses of [48, Thm. 7.10] now hold, and we obtain $N^{A,W}_{p_1p_2r} = N^{A}_{p_1p_2r}$. Thus by (6.2), to prove the theorem it is sufficient to take $\mathfrak{u}' := \psi^t(R^\vee_{\mathbb{R}})$ and show that

$$N^{A,W}_{p_1p_2r} = \sum_{\boldsymbol{\tau}} k_\tau N_{\boldsymbol{\tau}}$$

where the sum is over all isomorphism classes of decorated product types with total class A as in the statement of the theorem.

We now make a slight change in the definition of W, $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$ and $\mathscr{M}(X,\beta)_W$ in the case that $r=0$, to make the remainder of the proof more uniform. Note that in this case,

$$\begin{aligned}\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W &= (\mathrm{Spec}(R\to \Bbbk)\times B\mathbb{G}_m^\dagger)\times^{\mathrm{fs}}_{X\times B\mathbb{G}_m}\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)\\ &= B\mathbb{G}_m^\dagger\times_{B\mathbb{G}_m}((\mathrm{Spec}(R\to\Bbbk)\times B\mathbb{G}_m)\times^{\mathrm{fs}}_{\mathscr{P}(X,r)}\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)).\end{aligned}$$

The stack $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$ then has the same underlying stack as

$$(\mathrm{Spec}(R\to\Bbbk)\times B\mathbb{G}_m)\times^{\mathrm{fs}}_{\mathscr{P}(X,r)}\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta),$$

but with the ghost sheaf on the former being a sum of the ghost sheaf of the latter and the constant sheaf $\underline{\mathbb{N}}$. As we are only concerned about virtual fundamental classes, which are defined using the underlying morphism of stacks $\underline{\mathscr{M}(X,\beta)_W} \to \underline{\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W}$, we may thus replace W with $\mathrm{Spec}(R\to \Bbbk)\times B\mathbb{G}_m$, which also changes $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$ and $\mathscr{M}(X,\beta)_W$. Note that now the definition of W is uniform across the two cases, and whether or not $r=0$, the ghost sheaf of W is R.

Step 2: The structure of $\mathfrak{M}^{\mathrm{ev}}(X,\beta)_W$. We recall from [48, Lem. 7.8] and its proof that the projection $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W \to W$ is integral, log smooth and of fibre dimension and log fibre dimension one. For the latter notion, see the review of [48, Def. A.5].

Let $\bar\eta$ be a generic point of an irreducible component $\mathfrak{M}_{\bar\eta}$ of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$. We view the former as an integral closed substack of the latter, so that

$$[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W] = \sum_{\bar\eta}\mu_{\bar\eta}[\mathfrak{M}_{\bar\eta}]$$

in the Chow group of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$. Here the sum is over all generic points $\bar\eta$, and $\mu_{\bar\eta}$ is the multiplicity of $\mathfrak{M}_{\bar\eta}$ in $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$.

Let $Q_{\bar\eta}$ be the stalk of the ghost sheaf of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$ at $\bar\eta$. We then have

$$\operatorname{rk} Q_{\bar\eta}^{\mathrm{gp}} = \operatorname{rk} \overline{\mathcal{M}}_W^{\mathrm{gp}} = n; \tag{6.3}$$

the first equality comes from [48, (A.1)] and the above-mentioned equality of log fibre dimension and fibre dimension, while the second equality is from the definition of W, as $R=\mathbb{N}^n$. Let $\tau=(G,\boldsymbol{\sigma},\mathbf{u})$ be the type of the punctured map corresponding to $\bar\eta$.

Note that $\varepsilon^!_W[\mathfrak{M}_{\bar\eta}]$ is a zero-dimensional cycle in the Chow group of $\mathscr{M}(X,\beta)_W$. Assume that $\deg \varepsilon^!_W[\mathfrak{M}_{\bar\eta}]\neq 0$. We claim that in this case, τ is a product type. Definition 6.2, (1) is immediate by the definition of β. For Condition (2) of the definition, note that τ is necessarily realizable as it is the type of a punctured map to $\mathcal{X}$. On the other hand, the assumption of non-zero virtual degree implies that for some decoration $\boldsymbol{\tau}$ of τ with total curve class A, $\mathscr{M}(X,\boldsymbol{\tau})$ is non-empty. In particular, one may find a punctured map $C^\circ\to X$ defined over a geometric point whose type is τ' with a contraction $\tau'\to\tau$. Hence by Lemma 2.1, τ' is a balanced type, and hence τ is also balanced.

For Condition (3) of the definition of product type, let Q_τ be the basic monoid associated to the type τ, so that $Q^\vee_{\tau,\mathbb{R}}$ coincides with the cone τ. Note that in general $Q_{\bar\eta}\neq Q_\tau$, as the fibre product construction of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$ changes the log structure. We wish to show that $\dim\tau=n$.

From [4, Prop. 5.2], we have $Q^\vee_{\bar\eta,\mathbb{R}}=\tau\times_{\sigma_{\mathrm{u}}}R^\vee_{\mathbb{R}}$ where the map $R^\vee_{\mathbb{R}}\to\sigma_{\mathrm{u}}$ is just the inclusion ψ^t and the map $\tau\to\sigma_{\mathrm{u}}$, induced by the morphism $\mathrm{ev}_{\mathcal{X}}$, is given by evaluation at the vertex v_{out} adjacent to L_{out}, i.e., is $h|_{\tau_{v_{\mathrm{out}}}}$. See [48, Lem. 3.23] for this latter statement.

Note the images of the two projections $Q^\vee_{\bar\eta,\mathbb{R}}\to\tau$, $R^\vee_{\mathbb{R}}$ intersect the interiors of τ and $R^\vee_{\mathbb{R}}$ respectively. This is because these morphisms of cones are dual to local morphisms of monoids, being induced by the two projections of log stacks $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W\to\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)$, W. From this we may calculate the dimension of $Q^\vee_{\bar\eta,\mathbb{R}}$ by computing its tangent space at a point in the interior of $Q^\vee_{\bar\eta,\mathbb{R}}$ mapping to the interiors of τ and $R^\vee_{\mathbb{R}}$ via the two projections. This tangent space is $Q^*_{\bar\eta,\mathbb{R}}=\tau^{\mathrm{gp}}\times_{\sigma_{\mathrm{u}}^{\mathrm{gp}}}R^*_{\mathbb{R}}=\tau^{\mathrm{gp}}$ as $R^*_{\mathbb{R}}\to\sigma_{\mathrm{u}}^{\mathrm{gp}}$ is an isomorphism of real vector spaces. Thus $\dim\tau=\dim Q^\vee_{\bar\eta,\mathbb{R}}=n$, the latter equality by (6.3). Hence $\dim\tau=\dim h(\tau_{v_{\mathrm{out}}})=n$, as desired.

Step 3: The multiplicity of $\mathfrak{M}_{\bar\eta}$ *in* $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$. We use the log smoothness of the projection $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\eta)_W\to W$ already mentioned in Step 2. Note that W carries a natural idealized log structure given by the ideal $R\setminus\{0\}\subseteq R$. This ideal pulls back to give an idealized structure on $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$, and an idealized strict morphism[10] is idealized log smooth if and only if it is log smooth, see [70, IV,Variant 3.1.22]. Thus $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \beta)_W \to W$ is idealized log smooth, and hence, smooth locally at $\bar\eta$, is given as $\mathcal{A}_{Q_{\bar\eta},K} \to \mathcal{A}_{R,R\setminus\{0\}}$ by [5, Prop. B.4], where $K \subseteq Q_{\bar\eta}$ is the monoid ideal generated by the image of $R\setminus\{0\}$ in $Q_{\bar\eta}$ under the map $R \to Q_{\bar\eta}$.

The multiplicity $\mu_{\bar\eta}$ of the component $\mathfrak{M}_{\bar\eta}$ may now be calculated as the multiplicity of $\mathcal{A}_{Q_{\bar\eta},K}$, or equivalently, $\dim_{\Bbbk} \Bbbk[Q_{\bar\eta}]/K$. Note that $Q^\vee_{\bar\eta,\mathbb{R}} \to R^\vee_{\mathbb{R}}$ is surjective, as follows from the integrality of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \beta)_W \to W$ and [48, Prop. 2.4]. Further, $Q^\vee_{\bar\eta,\mathbb{R}}$ and $R^\vee_{\mathbb{R}}$ are of the same dimension. It thus follows that $R_{\mathbb{R}} \to Q_{\bar\eta,\mathbb{R}}$ is an isomorphism of real cones. Identifying $Q^{\mathrm{gp}}_{\bar\eta}$ with a super-lattice of $R^{\mathrm{gp}} = \mathbb{Z}^n$, the multiplicity is then the number of points of $Q^{\mathrm{gp}}_{\bar\eta}$ contained in $[0, 1)^n \subseteq R_{\mathbb{R}}$. This may be expressed as

$$\mu_\tau := \mu_{\bar\eta} = |\operatorname{coker}(R^{\mathrm{gp}} \to Q^{\mathrm{gp}}_{\bar\eta})|, \tag{6.4}$$

noting this expression only depends on the type τ.

Step 4: Comparison with the $N_{\boldsymbol\tau}$. Consider the commutative diagram

$$\begin{tikzcd}
\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)_W \arrow[r, "j'_\tau"] \arrow[d, "k"'] & \mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X},\beta)_W \arrow[r, "\iota'_\tau"] \arrow[d] & \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W \arrow[d] \\
\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau) \arrow[r, "j_\tau"'] & \mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X},\beta) \arrow[r, "\iota_\tau"'] & \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)
\end{tikzcd}$$

where

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau)_W = W \times^{\mathrm{fs}}_{\mathscr{P}(X,r)} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau),$$
$$\mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X}, \beta)_W = W \times^{\mathrm{fs}}_{\mathscr{P}(X,r)} \mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X}, \beta).$$

In this diagram, all squares are Cartesian in both the log and fs log categories, as all horizontal morphisms are strict. The morphism ι_τ, hence also ι'_τ, is a closed embedding, while j_τ, hence also j'_τ, is a finite morphism of degree $|\operatorname{Aut}(\tau)|$. Further, as $W \to \mathscr{P}(X, r)$ is a finite morphism, so is k.

For a stack M, denote by M_{red} the reduction. Then

$$\iota'_{\tau,*}[\mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X}, \beta)_{W,\mathrm{red}}] = \sum_{\bar\eta} [\mathfrak{M}_{\bar\eta}] \tag{6.5}$$

[10] We recall a morphism of idealized log schemes $f : (X, \mathcal{M}_X, \mathcal{K}_X) \to (Y, \mathcal{M}_Y, \mathcal{K}_Y)$ is idealized strict if $\mathcal{K}_X$ agrees with the monoid ideal generated by $f^\flat(\mathcal{K}_Y)$.

where the sum is over all generic points $\bar{\eta}$ of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$ whose type is τ. Indeed, any such generic point $\bar{\eta}$ corresponds to a punctured map $C^\circ/\bar{\eta} \to \mathcal{X}$ in $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)$ of type τ, hence lies in $\mathfrak{M}^{\mathrm{ev}}_\tau(\mathcal{X},\beta)$. Combining (6.5) with $\deg j'_\tau = |\operatorname{Aut}(\tau)|$ gives

$$(\iota'_\tau \circ j'_\tau)_*[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)_{W,\mathrm{red}}] = |\operatorname{Aut}(\tau)| \sum_{\bar{\eta}} [\mathfrak{M}_{\bar{\eta}}]. \tag{6.6}$$

Let $k_{\mathrm{red}} : \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)_{W,\mathrm{red}} \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)$ be the morphism induced by k. Bearing in mind that $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)$ is reduced, we may now calculate the degree of the finite morphism k_{red} as follows. We take a (strict) geometric generic point $\operatorname{Spec}\kappa \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)$. The desired degree is the length of the reduction of the fibre

$$\begin{aligned} \operatorname{Spec}\kappa \times_{\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)_W &= \operatorname{Spec}\kappa \times_{\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)} (\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau) \times^{\mathrm{fs}}_{\mathscr{P}(X,r)} W) \\ &= \operatorname{Spec}(Q_\tau \to \kappa) \times^{\mathrm{fs}}_{\mathscr{P}(X,r)} W. \end{aligned} \tag{6.7}$$

We may replace $\mathscr{P}(X,r)$ with the strict closed substack Z with underlying closed substack $\underline{X}_{\sigma_{\mathrm{u}}} \times B\mathbb{G}_m$ as the morphism $\operatorname{Spec}\kappa \to \mathscr{P}(X,r)$ factors through this closed substack. Further, the morphism $\operatorname{Spec}\kappa \to Z$ factors through $X_{\sigma_{\mathrm{u}}}$ as all line bundles on $\operatorname{Spec}\kappa$ are trivial. Thus we may rewrite (6.7) as

$$\begin{aligned} &\operatorname{Spec}(Q_\tau \to \kappa) \times^{\mathrm{fs}}_{X_{\sigma_{\mathrm{u}}}} \operatorname{Spec}(R \to \Bbbk) \\ &= (A_{Q_\tau, Q_\tau\setminus\{0\}} \times^{\mathrm{fs}}_{A_{P_{\mathrm{u}}, P_{\mathrm{u}}\setminus\{0\}}} A_{R, R\setminus\{0\}}) \times_{\operatorname{Spec}\Bbbk} \operatorname{Spec}\kappa, \end{aligned} \tag{6.8}$$

again following the notation of [5, Sect. B]. By [48, Prop. A.4, (4)], (6.8) may then be expressed as

$$A_{Q_\tau \oplus^{\mathrm{fs}}_{P_{\mathrm{u}}} R, J} \times_{\operatorname{Spec}\Bbbk} \operatorname{Spec}\kappa$$

where J is the ideal generated by the image of $Q_\tau\setminus\{0\}$ and $R\setminus\{0\}$. We pass to the reduction by replacing J with its radical, which, by the explicit description of the fs push-out of [48, Prop. A.4, (3)], is the complement of the group of invertible elements of $Q_\tau \oplus^{\mathrm{fs}}_{P_{\mathrm{u}}} R$. Further, this group of invertible elements is the torsion part of the cokernel of the map $(\varphi_1, -\varphi_2) : P_{\mathrm{u}}^{\mathrm{gp}} \to Q_\tau^{\mathrm{gp}} \oplus R^{\mathrm{gp}}$, where $\varphi_1 : P_{\mathrm{u}}^{\mathrm{gp}} \to Q_\tau^{\mathrm{gp}}$ and $\varphi_2 : P_{\mathrm{u}}^{\mathrm{gp}} \to R^{\mathrm{gp}}$ are induced by $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau) \to \mathscr{P}(X,r)$ and $W \to \mathscr{P}(X,r)$ respectively. Thus we see that the reduced fibre is

$$\operatorname{Spec}\kappa[\operatorname{coker}(P_{\mathrm{u}}^{\mathrm{gp}} \to Q_\tau^{\mathrm{gp}} \oplus R^{\mathrm{gp}})_{\mathrm{tors}}],$$

whose length of course is given by the order of this group of invertible elements.

In conclusion, we see that

$$k_{\mathrm{red},*}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)_{W,\mathrm{red}}] = |\operatorname{coker}(P_{\mathfrak{u}}^{\mathrm{gp}} \to Q_\tau^{\mathrm{gp}} \oplus R^{\mathrm{gp}})_{\mathrm{tors}}|[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)]. \tag{6.9}$$

Bearing in mind that $Q_{\bar\eta} = (Q_\tau \oplus^{\mathrm{fs}}_{P_{\mathfrak{u}}} R)/\mathrm{tors}$, it again follows from the explict description of the fs pushout in [48, Prop. A.4, (3)] that

$$Q_{\bar\eta}^{\mathrm{gp}} = \operatorname{coker}(P_{\mathfrak{u}}^{\mathrm{gp}} \to Q_\tau^{\mathrm{gp}} \oplus R^{\mathrm{gp}})/\mathrm{tors},$$

and then a simple diagram chase gives

$$\begin{aligned} &|\operatorname{coker}(P_{\mathfrak{u}}^{\mathrm{gp}} \to Q_\tau^{\mathrm{gp}} \oplus R^{\mathrm{gp}})_{\mathrm{tors}}||\operatorname{coker}(R^{\mathrm{gp}} \to Q_{\bar\eta}^{\mathrm{gp}})| \\ &\quad = |\operatorname{coker}(P_{\mathfrak{u}}^{\mathrm{gp}} \to Q_\tau^{\mathrm{gp}})| = |\operatorname{coker}(Q_\tau^* \to P_{\mathfrak{u}}^*)| = k_\tau. \end{aligned} \tag{6.10}$$

We next consider the diagram

$$\begin{tikzcd} \coprod_{\boldsymbol{\tau}=(\tau,\mathbf{A})} \mathscr{M}(X,\boldsymbol{\tau}) \arrow[d, "\varepsilon_\tau"'] & \coprod_{\boldsymbol{\tau}=(\tau,\mathbf{A})} \mathscr{M}(X,\boldsymbol{\tau})_W \arrow[l, "k'"'] \arrow[r] \arrow[d, "\varepsilon_{\tau,W}"'] & \mathscr{M}(X,\beta)_W \arrow[d, "\varepsilon_W"'] \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau) & \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)_W \arrow[l, "k"] \arrow[r, "\iota'_\tau \circ j'_\tau"'] & \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W \end{tikzcd}$$

where the disjoint unions are over all decorations of τ with total degree A. Now observe that

$$\begin{aligned} N_{p_1p_2r}^{A,W} = \deg[\mathscr{M}(X,\beta)_W]^{\mathrm{virt}} &= \deg \varepsilon_W^![\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W] \\ &= \deg \sum_{\bar\eta} \mu_{\bar\eta}\varepsilon_W^![\mathfrak{M}_{\bar\eta}]. \end{aligned}$$

Here we are summing over all generic points $\bar\eta$ of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)_W$, but of course we may restrict the sum to those $\bar\eta$ such that $\deg \varepsilon_W^![\mathfrak{M}_{\bar\eta}] \neq 0$. Thus by (6.4), (6.6) and Step 2, we may rewrite this equation as

$$N_{p_1p_2r}^{A,W} = \deg \sum_\tau \frac{\mu_\tau}{|\operatorname{Aut}(\tau)|}\varepsilon_W^!(\iota'_\tau \circ j'_\tau)_*[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)_{W,\mathrm{red}}]$$

where, by Step 2, the sum is now over all product types τ with inputs p_1, p_2, output r, and $\mathfrak{u}' \subseteq h_\tau(\tau_{v_{\mathrm{out}}})$. Now using the push-pull formula for virtual pull-back of [65, Thm. 4.1] for the first and third equality, as well as (6.4), (6.9) and (6.10) for the third, we obtain

$$
\begin{aligned}
N^{A,W}_{p_1p_2r} &= \deg \sum_{\tau} \frac{\mu_\tau}{|\operatorname{Aut}(\tau)|} \varepsilon^!_{\tau,W}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)_{W,\mathrm{red}}] \\
&= \deg \sum_{\tau} \frac{\mu_\tau}{|\operatorname{Aut}(\tau)|} k'_* \varepsilon^!_{\tau,W}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)_{W,\mathrm{red}}] \\
&= \deg \sum_{\tau} \frac{k_\tau}{|\operatorname{Aut}(\tau)|} \varepsilon^!_{\tau}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)] \\
&= \sum_{\boldsymbol{\tau}=(\tau,\mathbf{A})} k_\tau N_{\boldsymbol{\tau}},
\end{aligned}
$$

as desired. □

Lemma 6.5 *Let $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$ be a product type with $N_{\boldsymbol{\tau}} \neq 0$ for some decoration $\boldsymbol{\tau}$ of τ, and let G' be the spine of G in the sense of Definition 2.4. Then v_{out} is the unique trivalent vertex of G'. Further, v_{out} is also a trivalent vertex of G, and if v_{out} and L_{out} are removed from G, then τ splits into two broken line types τ_1, τ_2.*

Proof Since G' has three legs and no univalent vertices, G' has a unique trivalent vertex. Suppose that v_{out} is not this trivalent vertex. Necessarily $\boldsymbol{\sigma}(v_{\mathrm{out}})$ is a maximal cell $\sigma \in \mathscr{P}^{\max}$ since $\dim h(\tau_{\mathrm{out}}) = n$, and thus for any punctured map $f : C^\circ \to X$ in the non-empty moduli space $\mathscr{M}(X, \boldsymbol{\tau})$, the union of irreducible components of C° corresponding to v_{out} are mapped to the zero-dimensional stratum X_σ. Hence by stability, v_{out} must be at least trivalent in G, and thus there is an edge E adjacent to v_{out} which is not contained in G'. Let $\bar\tau = (\bar G, \bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}})$ be the type obtained by cutting G at the edge E and taking the connected component of the resulting graph not containing L_1, L_2 and L_{out}. Thus E becomes the unique leg of $\bar G$. Let $h_{\bar\tau}$ be the universal tropical map of type $\bar\tau$. If $\bar\tau_E \in \Gamma(\bar G, \bar\ell)$ is the cone corresponding to E, then by Lemma 2.5, (1), $\dim h_{\bar\tau}(\bar\tau_E) \leq n-1$. But since $\dim h(\tau_{v_{\mathrm{out}}}) = n$, we obtain a contradiction. Thus v_{out} is a trivalent vertex of G'.

A similar argument applies to show that v_{out} is also trivalent in G: if the valency is higher than three, with an adjacent edge E not an edge of G', we may again cut at E to obtain a contradiction as above.

Thus by removing $v_{\mathrm{out}}, L_{\mathrm{out}}$ from G, we obtain, for $i = 1, 2$, types τ_i with legs $L_i, L_{i,\mathrm{out}}$, where $L_{1,\mathrm{out}}, L_{2,\mathrm{out}}$ arise from the edges (or legs) E_1, E_2 adjacent to v_{out} other than L_{out}. We only need to show that τ_i is a broken line type.

First consider Condition (1) of Definition 3.19. The only thing to check here is that $\boldsymbol{\sigma}(L_{i,\mathrm{out}}) \in \mathscr{P}$ and that $\mathbf{u}(L_{i,\mathrm{out}}) \neq 0$. For the first statement, note that as $\boldsymbol{\sigma}(v_{\mathrm{out}}) = \sigma$ and $\boldsymbol{\sigma}(v_{\mathrm{out}}) \subseteq \boldsymbol{\sigma}(E_i) = \boldsymbol{\sigma}(L_{i,\mathrm{out}})$, necessarily $\boldsymbol{\sigma}(L_{i,\mathrm{out}}) = \sigma$

also. For the second statement, note that if $\mathbf{u}(L_{i,\mathrm{out}}) = 0$, then we may vary $s \in \tau$ by changing only the affine length of the edge E_i of G, so that $h_s(v_{\mathrm{out}})$ remains unchanged. Since $\dim h(\tau_{v_{\mathrm{out}}}) = n$, this would imply that $\dim \tau > n$, a contradiction.

Condition (2) of Definition 3.19 is immediate. Finally, for Condition (3), it follows from $\dim h(\tau_{v_{\mathrm{out}}}) = n$ that $\dim h_{\tau_i}(\tau_{i,\mathrm{out}}) = n$, and in particular $\dim \tau_i \geq n-1$. However, if $\dim \tau_i > n-1$ for $i = 1$ or 2, it then immediately follows by gluing together tropical maps in the families τ_1, τ_2 that $\dim \tau > n$. Thus we get $\dim \tau_i = n-1$, as desired. □

Lemma 6.6 *Let $\boldsymbol{\tau}$ be a decorated product type with $N_{\boldsymbol{\tau}} \neq 0$, and let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ be the decorated broken line types obtained from splitting $\boldsymbol{\tau}$ at v_{out}. Then*

$$k_\tau N_{\boldsymbol{\tau}} = k_{\tau_1} k_{\tau_2} N_{\boldsymbol{\tau}_1} N_{\boldsymbol{\tau}_2}.$$

Proof The proof is by a straightforward application of Yixian Wu's gluing formula Theorem A.4. We first assume that neither τ_1 nor τ_2 is a trivial broken line, that is, has no vertex. Then v_{out} has two adjacent edges E_1,E_2. Splitting $\boldsymbol{\tau}$ at these two edges leads to, by Lemma 6.5, the decorated broken line types $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and a third decorated type $\boldsymbol{\tau}_0 = (\tau_0, \mathbf{A}_0)$, $\tau_0 = (G_0, \boldsymbol{\sigma}_0, \mathbf{u}_0)$ with only one vertex v_{out} and three adjacent legs (E_1, v_{out}), (E_2, v_{out}) and L_{out}, the outgoing leg of τ. Moreover, τ_0 is a type entirely contained in the maximal cell σ containing $h(\tau_{v_{\mathrm{out}}})$, that is, $\boldsymbol{\sigma}_0(v_{\mathrm{out}}) = \sigma$. In particular, the curve class $\mathbf{A}_0(v_{\mathrm{out}})$ is trivial. Thus any punctured map of type $\boldsymbol{\tau}_0$ is constant on underlying schemes and has domain a $\mathbb{P}^1$ with three punctured points. It then follows as in [48, Claim 3.22] that there is a unique basic punctured map $f : C^\circ/W \to X$ of type $\boldsymbol{\tau}_0$ with $\underline{W} = \operatorname{Spec} \Bbbk$. Since further $\mathscr{M}(X, \boldsymbol{\tau}_0)$ is unobstructed over $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}_0)$, $\mathscr{M}(X, \boldsymbol{\tau}_0)$ is a reduced point.

It is now clear by stability of the tropical situation under small perturbations, and can be checked also explicitly, that the tropical gluing map $\varepsilon^{\mathrm{gp}}_\omega$ in (A.7) is surjective already for $\omega = \tau$. Thus the trivial displacement vector $\nu = 0$ in Definition A.2 is general for τ. Moreover, the dimension formula (A.8) for $\omega = \tau$ holds:

$$\begin{aligned}\dim \widetilde{\tau}_0 + \dim \widetilde{\tau}_1 + \dim \widetilde{\tau}_2 &= (n+2) + n + n = (n+2) + 2n \\ &= \dim \widetilde{\tau} + \sum_E \operatorname{rk} \Lambda_E.\end{aligned}$$

Thus the set of transverse types $\Delta(\nu)$ consist only of the one element τ.

To compute the splitting multiplicity $m(\boldsymbol{\tau})$ (Definition A.2, 3) consider the following commutative diagram with exact rows and columns.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widetilde{\tau}^{\mathrm{gp}}_{\mathbb{Z}} \arrow[r] \arrow[d] & (\widetilde{\tau}^{\mathrm{gp}}_0)_{\mathbb{Z}} \times (\widetilde{\tau}^{\mathrm{gp}}_1)_{\mathbb{Z}} \times (\widetilde{\tau}^{\mathrm{gp}}_2)_{\mathbb{Z}} \arrow[r, "(\varepsilon^{\mathrm{gp}}_{\tau})_{\mathbb{Z}}"] \arrow[d] & \mathrm{im}(\varepsilon^{\mathrm{gp}}_{\tau})_{\mathbb{Z}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda \times \mathbb{Z}^2 \arrow[r] \arrow[d] & (\Lambda \times \mathbb{Z}^2) \times \Lambda \times \Lambda \arrow[r] \arrow[d] & \Lambda \times \Lambda \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}/k_\tau \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/k_{\tau_1}\mathbb{Z} \times \mathbb{Z}/k_{\tau_2}\mathbb{Z} \arrow[r] \arrow[d] & \mathrm{coker}(\varepsilon^{\mathrm{gp}}_{\tau})_{\mathbb{Z}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Here $\Lambda = \sigma^{\mathrm{gp}}_{\mathbb{Z}}$, the first column is the exact sequence defining k_τ, along with a trivial $\mathbb{Z}^2$ factor in the first two entries, the middle column the product of the exact sequences defining k_{τ_1} and k_{τ_2}, and the isomorphism $(\widetilde{\tau}^{\mathrm{gp}}_0)_{\mathbb{Z}} \to \Lambda \times \mathbb{Z}^2$. The third row now shows

$$
m(\boldsymbol{\tau}) = |\,\mathrm{coker}((\varepsilon^{\mathrm{gp}}_{\tau})_{\mathbb{Z}})| = \frac{k_{\tau_1} k_{\tau_2}}{k_\tau}.
$$

Applying Theorem A.4 we conclude

$$
\deg \left[\mathscr{M}(X, \boldsymbol{\tau})\right]^{\mathrm{virt}} = \frac{k_{\tau_1} k_{\tau_2}}{k_\tau} \cdot \deg \left[\mathscr{M}(X, \boldsymbol{\tau}_1)\right]^{\mathrm{virt}} \cdot \deg \left[\mathscr{M}(X, \boldsymbol{\tau}_2)\right]^{\mathrm{virt}}.
$$

The claimed equality $k_\tau N_{\boldsymbol{\tau}} = k_{\tau_1} k_{\tau_2} N_{\boldsymbol{\tau}_1} N_{\boldsymbol{\tau}_2}$ now follows by the definition of $N_{\boldsymbol{\tau}}, N_{\boldsymbol{\tau}_1}, N_{\boldsymbol{\tau}_2}$ and noting that $\mathrm{Aut}(\boldsymbol{\tau}) = \mathrm{Aut}(\boldsymbol{\tau}_1) \times \mathrm{Aut}(\boldsymbol{\tau}_2)$.

If one of τ_1 or τ_2 is a trivial broken line, the same analysis omitting the trivial broken line factor applies to prove the result in this case. If both τ_1, τ_2 are trivial broken lines the statement is trivially true. □

Proof of Theorem 6.1 Take $x \in \mathrm{Int}(\mathfrak{u})$ in the definition of $\alpha^{\mathrm{trop}}_{p_1 p_2 r}$; by consistency of $\mathscr{S}_{\mathrm{can}}$, (6.1) is independent of this choice of x. Further, in (6.1), we may sum over decorated broken lines, rather than broken lines, and get the same value. Thus by Theorem 4.14, we may instead write

$$
\alpha^{\mathrm{trop}}_{p_1 p_2 r} = \sum_{\boldsymbol{\tau}_1 = (\tau_1, \mathbf{A}_1), \boldsymbol{\tau}_2 = (\tau_2, \mathbf{A}_2)} k_{\tau_1} k_{\tau_2} N_{\boldsymbol{\tau}_1} N_{\boldsymbol{\tau}_2} t^{A_1 + A_2}
$$

where now the sum is over admissible decorated broken line types $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ with $\mathfrak{u} \subseteq h_{\boldsymbol{\tau}_i}(\tau_{i,\mathrm{out}})$, $\mathbf{u}_{\boldsymbol{\tau}_i}(L_{i,\mathrm{in}}) = p_i$, and $u_{\tau_1} + u_{\tau_2} = r$. (Note that by definition of $\mathscr{S}_{\mathscr{B}}$, if $x \in h_{\boldsymbol{\tau}_i}(\tau_{i,\mathrm{out}})$ then $\mathfrak{u} \subseteq h_{\boldsymbol{\tau}_i}(\tau_{i,\mathrm{out}})$.)

For such a pair $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$, we obtain a decorated product type by taking a vertex v_{out} and gluing G_1 and G_2 to v_{out} via the legs $L_{1,\mathrm{out}}, L_{2,\mathrm{out}}$, and add an additional leg L_{out} adjacent to v_{out} with $\mathbf{u}(L_{\mathrm{out}}) = -r$. The data $\boldsymbol{\sigma}, \mathbf{u}, \mathbf{A}$ are then determined by $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$, with $\mathbf{A}(v_{\mathrm{out}}) = 0$.

We see $\boldsymbol{\tau}$ is a decorated product type. Indeed, we only need to check conditions (2) and (3) of Definition 6.2. Balancing just needs to be checked at v_{out}, which follows from $r = \mathbf{u}_1(L_{1,\mathrm{out}}) + \mathbf{u}_2(L_{2,\mathrm{out}}) = u_{\tau_1} + u_{\tau_2}$. For realizability and the dimension conditions (3), note that by the corresponding dimension conditions for broken line types, for each point $y \in \mathrm{Int}(\mathfrak{u})$, there exists a unique $s_i \in \tau_i$ such that $y \in h_{\tau_i,s_i}(L_{i,\mathrm{out}})$. Thus by gluing together h_{τ_1,s_1} and h_{τ_2,s_2}, we obtain a tropical map realizing τ which takes v_{out} to y. Furthermore, this map is necessarily the unique such tropical map of type τ, giving both realizability and the dimension statements of (3).

Conversely, by Lemma 6.5, all product types $\boldsymbol{\tau}$ arise in this way. Thus we obtain using Lemma 6.6 that

$$\alpha^{\mathrm{trop}}_{p_1p_2r} = \sum_{\boldsymbol{\tau}=(\tau,\mathbf{A})} k_\tau N_{\boldsymbol{\tau}} t^A,$$

which coincides with $\alpha^{\log}_{p_1p_2r}$ by Theorem 6.4. $\square$

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Appendix A: The gluing formula with toric gluing strata

Let $B = \mathrm{Spec}(Q_B \to \Bbbk)$ be a logarithmic point, X a Zariski log scheme and $X \to B$ a logarithmically smooth, integral, projective morphism. We assume $\Sigma(X)$ monodromy-free for simplicity. As before we write $\mathcal{X}$ for the Artin fan of X. Let $\boldsymbol{\tau} = (\tau, \mathbf{A})$ with $\tau = (G, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a realizable decorated global type of a punctured map of genus 0.[11] Recall from [5, Def. 3.8] the stacks $\mathscr{M}(X, \boldsymbol{\tau})$ and $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau})$ of punctured stable maps to X and to $\mathcal{X}$ marked by $\boldsymbol{\tau}$.

For each vertex $v \in V(G)$ denote by $\boldsymbol{\tau}_v = (\tau_v, \mathbf{A}_v)$, $\tau_v = (G_v, \boldsymbol{\sigma}_v, \bar{\mathbf{u}}_v)$ the decorated global type with $V(G_v) = \{v\}$ and $E(G_v) = \emptyset$ obtained by splitting G at all edges.[12] Note that each τ_v is realizable as well. Then $\mathfrak{M}(\mathcal{X}, \tau)$ is locally pure-dimensional, and hence defines a virtual fundamental class for $\mathscr{M}(X, \boldsymbol{\tau})$ by means of the obstruction theory for $\mathscr{M}(X, \boldsymbol{\tau})$ over $\mathfrak{M}(\mathcal{X}, \tau)$.

According to [5, Cor. 5.13, Prop. 5.15], the splitting morphism

$$\delta : \underline{\mathscr{M}}(X, \boldsymbol{\tau}) \longrightarrow \prod_{v \in V(G)} \underline{\mathscr{M}}(X, \boldsymbol{\tau}_v) \tag{A.1}$$

is finite and representable.[13] The gluing formula expresses $\delta_*[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}}$ in terms of the virtual fundamental classes of strata of $\mathscr{M}(X, \boldsymbol{\tau}_v)$. A stratum is given by a $\boldsymbol{\tau}_v$-marked decorated global type $\boldsymbol{\omega}_v$ as the image of the finite map

$$j_{\boldsymbol{\omega}_v} : \mathscr{M}(X, \boldsymbol{\omega}_v) \to \mathscr{M}(X, \boldsymbol{\tau}_v)$$

defined by changing the marking [5, Rem. 3.28].

We now explain some more notation needed to state the gluing formula. For each edge $E \in E(G)$ we obtain a gluing stratum $\sigma_E = \boldsymbol{\sigma}(E) \in \Sigma(X)$ defined by τ. Note that if $v, v' \in V(G)$ are the vertices adjacent to E and $L \in L(G_v)$, $L' \in L(G_{v'})$ are the legs in G_v, $G_{v'}$ obtained by splitting E, then by the definition of τ_v,

$$\boldsymbol{\sigma}_v(L) = \boldsymbol{\sigma}_{v'}(L') = \sigma_E.$$

We write $L = (E, v)$, $L' = (E, v')$ in the following, where the notation indicates that $v \in V(G_v) \subseteq V(G)$ and $v' \in V(G_{v'}) \subseteq V(G)$ are the vertices adjacent to E.

[11] The restriction to genus 0 is for convenience, since this is the case relevant to us and the general statement is slightly more complicated.

[12] Splitting at only a subset of edges works the same way, with a little more bookkeeping notation necessary. For simplicity of presentation we only discuss splitting at all edges.

[13] We suppress here the log structure which is irrelevant for the formulation of the statement, although it is central for its proof.

Now the restriction of a τ_v-marked basic tropical punctured map, defined say over $\omega_v \in$ **Cones**, to the leg $(E, v) \in L(G_v)$ defines a map of cones

$$\omega_{E,v} = \big\{(h, \lambda) \in \omega_v \times \mathbb{R}_{\geq 0} \,\big|\, \lambda \leq \ell(E, v)(h)\big\} \longrightarrow \sigma, \tag{A.2}$$

for some $\sigma \in \Sigma(X)$ containing $\sigma_E = \boldsymbol{\sigma}_v(E, v)$ as a face. Recall also that ω_v is the basic cone, or real cone with integral points the dual of the basic monoid [5, Def. 2.36], associated to ω_v. We conflate notations for types and associated basic cones and hence write ω_v also for this type unless there is a danger of confusion. Similarly, τ_v denotes both a global type and the basic cone $(Q_{\tau_v})^\vee_{\mathbb{R}}$ associated to the type. Note also that the marking by τ_v induces a face embedding

$$\tau_v = (Q_{\tau_v})^\vee_{\mathbb{R}} \longrightarrow \omega_v. \tag{A.3}$$

of basic cones.

Letting (E, v) in (A.2) run over all the gluing legs, we arrive at the following enlargement of the cone ω_v that records a point on each puncturing leg:

$$\widetilde{\omega}_v = \big\{(h, \lambda_{E,v}) \in \omega_v \times \mathbb{R}^{L(G_v)}_{\geq 0} \,\big|\, \lambda_{E,v} \leq \ell(E, v)(h)\big\}. \tag{A.4}$$

For each leg $(E, v) \in L(G_v)$, the projection forgetting all λ-components but $\lambda_{E,v}$ defines a map of cones

$$\widetilde{\omega}_v \longrightarrow \omega_{E,v}. \tag{A.5}$$

Similarly, for a τ-marked tropical punctured map defined over $\omega \in$ **Cones**, recording a point on each edge $E \in E(G)$ leads to

$$\widetilde{\omega} = \big\{(h, \lambda_E) \in \omega \times \mathbb{R}^{E(G)}_{\geq 0} \,\big|\, \lambda_E \leq \ell(E)(h)\big\}. \tag{A.6}$$

We now add the following assumption:

Assumption A.1 Assume all gluing strata $\underline{Z}_E$, $E \in E(G)$, are complete toric varieties, with $\overline{\mathcal{M}}_X|_{\underline{Z}_E}$ invariant under the torus action.

Under this assumption there is an embedding of the star of $\sigma_E \in \Sigma(X)$ as a face-fitting complex of cones in the real vector space associated to a lattice that we denote Λ_E. Moreover, this embedding identifies the integral affine structures on each cone, and the composition with the quotient by the image of the real subspace with lattice $(\sigma_E)^{\mathrm{gp}}_{\mathbb{Z}} \cap \Lambda_E$ maps the face-fitting complex of cones to a complete fan. If Z_E is strictly embedded as a toric stratum of a toric variety, Λ_E is the lattice underlying the describing fan. We now view each of the cones $\sigma \in \Sigma(X)$ containing σ_E as embedded in $(\Lambda_E)_{\mathbb{R}}$.

Let $\omega = (G_\omega, \boldsymbol{\sigma}_\omega, \mathbf{u}_\omega)$ be a not necessarily realizable global type of punctured map marked by τ. Splitting ω at the edges not contracted by the marking leads to a collection $(\omega_v)_{v\in V(G)}$ of types of tropical punctured maps, with ω_v marked by τ_v and with a leg $(E, v) \in L(G_{\omega_v})$ for each edge $E \in E(G)$ adjacent to v. For each $E \in E(G)$ with v, v' the adjacent vertices, the composition of the embeddings $\boldsymbol{\sigma}_\omega(E, v), \boldsymbol{\sigma}_\omega(E, v') \to (\Lambda_E)_\mathbb{R}$ with the quotient by the diagonal embedding $(\Lambda_E)_\mathbb{R} \to (\Lambda_E)_\mathbb{R} \times (\Lambda_E)_\mathbb{R}$ defines a map

$$\varepsilon_E : \boldsymbol{\sigma}_\omega(E, v) \times \boldsymbol{\sigma}_\omega(E, v') \to (\Lambda_E)_\mathbb{R} \times (\Lambda_E)_\mathbb{R} \to (\Lambda_E)_\mathbb{R}.$$

Explicitly, the arrow on the right can be taken as the difference map $(a, b) \mapsto a - b$.

Taking the product over all $E \in E(G)$ of the composition of ε_E with the maps (A.5) and (A.2) yields the map

$$\varepsilon_\omega : \prod_{v\in V(G)} \widetilde{\omega}_v \longrightarrow \prod_{E,v} \omega_{E,v} \longrightarrow \prod_E (\Lambda_E)_\mathbb{R} \times (\Lambda_E)_\mathbb{R} \longrightarrow \prod_E (\Lambda_E)_\mathbb{R}. \tag{A.7}$$

Thus ε_ω measures the failure of a collection of tropical punctured maps of types ω_v, together with points on the legs arising from splitting, to patch to a tropical punctured map of type ω together with a choice of point on each of the splitted edges. We refer to an element of $\prod_E (\Lambda_E)_\mathbb{R}$ as a *displacement vector*.

Definition A.2 (1) We call a displacement vector $\nu = (\nu_E)_{E\in E(G)} \in \prod_E (\Lambda_E)_\mathbb{R}$ *general* for τ if for each global type ω marked by τ, either $\varepsilon_\omega^{\mathrm{gp}}$ from (A.7) is surjective or $\nu \notin \mathrm{im}(\varepsilon_\omega^{\mathrm{gp}})$.
(2) Let $\nu = (\nu_E)_E \in \prod_E \Lambda_E$ be general for τ. Define the *set* $\Delta(\nu)$ *of transverse types for* ν as the set of isomorphism classes of global types ω marked by τ with $\nu \in \mathrm{im}(\varepsilon_\omega)$, and such that

$$\sum_v \dim \widetilde{\omega}_v = \dim \widetilde{\tau} + \sum_E \mathrm{rk}\, \Lambda_E. \tag{A.8}$$

The set of decorated global types $\boldsymbol{\omega} = (\omega, \mathbf{A})$ marked by $\boldsymbol{\tau}$ and with $\omega \in \Delta(\nu)$ is denoted by $\hat{\Delta}(\nu)$. We confuse a transverse type ω with its splitting $(\omega_v)_{v\in V(G)}$, and similarly in the decorated case.
(3) For a general displacement vector ν and $\boldsymbol{\omega} = (\omega, \mathbf{A}) \in \hat{\Delta}(\nu)$ define the *splitting multiplicity* by

$$m(\boldsymbol{\omega}) = m(\omega) := \Big[\prod_E \Lambda_E : \mathrm{im}(\varepsilon_\omega^{\mathrm{gp}})_\mathbb{Z} \Big].$$

Remark A.3 The set $\Delta(\nu)$ has an interpretation in terms of types of "broken" tropical punctured maps, in the sense that the matching condition along an edge E is replaced by matching translated by ν_E. Specifically, let ω be a global type marked by τ and (ω_v) the collection of global types marked by τ_v obtained by splitting. Then $\omega \in \Delta(\nu)$ iff ω satisfies (A.8) and there exist $h_v \in \omega_v$ and

$\lambda_v, \lambda_{v'} \in \mathbb{R}_{>0}$ with

$$V(h_v) + \lambda_v \cdot u_E = V'(h_{v'}) - \lambda_{v'} \cdot u_E + \nu_E, \tag{A.9}$$

as an equation in $(\Lambda_E)_{\mathbb{R}}$. For the signs we assume E oriented from v to v'. Here $V : \omega_v \to (\Lambda_E)_{\mathbb{R}}$ is the restriction of the map in (A.2) to the face $\lambda = 0$, that is, the image of the vertex v under the tropical punctured map h_v, and analogously for V' and v'.

From this description it is also obvious that $\Delta(\nu)$ does not change by rescaling ν by a positive constant. Moreover, if $\widetilde{\tau}$ is defined analogously to $\widetilde{\omega}$, with λ_E recording the position of a point on the edge E, then $\Delta(\nu)$ is also unchanged by adding to ν an element of the image of the map $\widetilde{\tau} \to \prod_E \Lambda_E$ evaluating at all points on the splitting edges.

The dimension formula (A.8) is equivalent to requiring $\omega = (\omega_v)_v$ to be a minimal type of punctured map with $\varepsilon_\omega^{\mathrm{gp}}$ surjective.

We are now in position to state the gluing formula.

Theorem A.4 [82] *Let $\boldsymbol{\tau} = (\tau, \mathbf{A})$, $\tau = (G, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a realizable decorated global type of punctured maps of genus 0 to X, and $\boldsymbol{\tau}_v$ the associated decorated global type defined at $v \in V(G)$ obtained from splitting all edges. Assume that all gluing strata are toric in the sense of Assumption A.1. Let $\nu = (\nu_E)_{E \in E(G)}$ be a general displacement vector for τ, and $\hat{\Delta}(\nu)$ the set of decorated transverse types for ν (Definition A.2). Denote by δ the splitting morphism from (A.1).*

Then the following equality in the Chow group of $\prod_{v \in V(G)} \underline{\mathscr{M}}(X, \boldsymbol{\tau}_v)$ holds:

$$\delta_*[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}} = \sum_{\boldsymbol{\omega} = (\boldsymbol{\omega}_v)_v \in \hat{\Delta}(\nu)} \frac{m(\boldsymbol{\omega})}{|\operatorname{Aut}(\boldsymbol{\omega}/\boldsymbol{\tau})|} \cdot (j_{\boldsymbol{\omega}_v})_* \Big[\prod_v \mathscr{M}(X, \boldsymbol{\omega}_v)\Big]^{\mathrm{virt}}.$$

References

1. Abouzaid, M.: The family Floer functor is faithful. J. Eur. Math. Soc. (JEMS) **19**, 2139–2217 (2017)
2. Abouzaid, M.: Homological mirror symmetry without corrections. arXiv:1703.07898 [math.SG]
3. Abramovich, D., Chen, Q.: Stable logarithmic maps to Deligne–Faltings pairs II. Asian J. Math. **18**, 465–488 (2014)
4. Abramovich, D., Chen, Q., Gross, M., Siebert, B.: Decomposition of degenerate Gromov–Witten invariants. Compos. Math. **156**, 2020–2075 (2020)
5. Abramovich, D., Chen, Q., Gross, M., Siebert, B.: Punctured Gromov–Witten invariants (2020). arXiv:2009.07720v2 [math.AG]
6. Abramovich, D., Chen, Q., Marcus, S., Wise, J.: Boundedness of the space of stable logarithmic maps. J. Eur. Math. Soc. (JEMS) **19**, 2783–2809 (2017)

7. Abramovich, D., Marcus, S., Wise, J.: Comparison theorems for Gromov–Witten invariants of smooth pairs and of degenerations. Ann. Inst. Fourier (Grenoble) **64**, 1611–1667 (2014)
8. Abramovich, D., Wise, J.: Birational invariance in logarithmic Gromov–Witten theory. Compos. Math. **154**, 595–620 (2018)
9. Argüz, H., Gross, M.: The higher dimensional tropical vertex. arXiv:2007.08347 [math.AG]
10. Auroux, D.: Mirror symmetry and T-duality in the complement of an anticanonical divisor. J. Gökova Geom. Topol. GGT **1**, 51–91 (2007)
11. Battistella, L., Nabijou, N., Ranganathan, D.: Gromov–Witten theory via roots and logarithms. arXiv:2203.17224 [math.AG]
12. Batyrev, V.: Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties. J. Algebr. Geom. **3**, 493–535 (1994)
13. Batyrev, V., Borisov, L.: On Calabi-Yau complete intersections in toric varieties. In: Higher-dimensional complex varieties (Trento, 1994), 39–65, de Gruyter, Berlin (1996)
14. Birkar, C.: Existence of log canonical flips and a special LMMP. Publ. Math. Inst. Hautes Étud. Sci. **115**, 325–368 (2012)
15. Birkar, C.: On connectedness of non-klt loci of singularities of pairs. arXiv:2010.08226 [math.AG]
16. Bossinger, L., Frías-Medina, B., Magee, T., Nájera Chávez, A.: Toric degenerations of cluster varieties and cluster duality. Compos. Math. **156**, 2149–2206 (2020)
17. Bousseau, P.: Tropical refined curve counting from higher genera and lambda classes. Invent. Math. **215**, 1–79 (2019)
18. Bousseau, P.: The quantum tropical vertex. Geom. Topol. **24**(3), 1297–1379 (2020)
19. Bousseau, P.: Quantum mirrors of log Calabi–Yau surfaces and higher-genus curve counting. Compos. Math. **156**(2), 360–411 (2020)
20. Bousseau, P.: Scattering diagrams, stability conditions, and coherent sheaves on $\mathbb{P}^2$. arXiv:1909.02985 [math.AG]
21. Chen, Q.: Stable logarithmic maps to Deligne–Faltings pairs I. Ann. Math. **180**, 455–521 (2014)
22. Cheung, M.-W.: Theta functions and quiver Grassmannians. arXiv:1906.12299 [Math.AG]
23. Cheung, M,-W., Magee, T., Nájera Chávez, A.: Compactifications of cluster varieties and convexity. arXiv:1912.13052 [Math.AG]
24. Cheung, M.-W., Mandel, T.: Donaldson–Thomas invariants from tropical disks. Sel. Math. New Ser. **26**, 57 (2020)
25. Candelas, P., Lynker, M., Schimmrigk, R.: Calabi–Yau manifolds in weighted $\mathbf{P}_4$. Nucl. Phys. B **341**, 383–402 (1990)
26. Candelas, P., de la Ossa, X., Green, P., Parkes, L.: A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory. Nucl. Phys. B **359**, 21–74 (1991)
27. Carl, M., Pumperla, M., Siebert, B.: A tropical view of Landau-Ginzburg models. http://www.math.uni-hamburg.de/home/siebert/preprints/LGtrop.pdf
28. Fukaya, K., Oh, Y.-G., Ohta, H., Ono, K.: Lagrangian intersection Floer theory: anomaly and obstruction. Part I/II, AMS/IP Studies in Advanced Mathematics, 46.1/46.2. AMS (2009)
29. Fan, H., Wu, L., You, F.: Structures in genus-zero relative Gromov–Witten theory. J. Topol. **13**, 269–307 (2020)
30. Fukaya, K.: Multivalued Morse theory, asymptotic analysis and mirror symmetry. In: Graphs and Patterns in Mathematics and Theoretical Physics. Proceedings of Symposia in Pure Mathematics, vol. 73, pp. 205–278. American Mathematical Society, Providence (2005)
31. Ganatra, S., Pomerleano, D.: Symplectic cohomology rings of affine varieties in the topological limit. Geom. Funct. Anal. **30**, 334–456 (2020)
32. Gräfnitz, T.: Tropical correspondence for smooth del Pezzo log Calabi–Yau pairs. arXiv:2005.14018 [math.AG]

33. Greene, B., Plesser, M.: Duality in Calabi–Yau moduli space. Nucl. Phys. B **338**, 15–37 (1990)
34. Gross, M.: Topological mirror symmetry. Invent. Math. **144**, 75–137 (2001)
35. Gross, M.: Toric degenerations and Batyrev–Borisov duality. Math. Ann. **333**, 645–688 (2005)
36. Gross, M.: Mirror symmetry for $\mathbb{P}^2$ and tropical geometry. Adv. Math. **224**, 169–245 (2010)
37. Gross, M., Hacking, P., Keel, S.: Mirror symmetry for log Calabi–Yau surfaces I. Publ. Math. Inst. Hautes Études Sci. **122**, 65–168 (2015)
38. Gross, M., Hacking, P., Keel, S., Kontsevich, M.: Canonical bases for cluster algebras. J. Am. Math. Soc. **31**, 497–608 (2018)
39. Gross, M., Hacking, P., Siebert, B.: Theta functions on varieties with effective anti-canonical class. Mem. Amer. Math. Soc. **278**(1376), 1–103 (2022)
40. Gross, M., Pandharipande, R., Siebert, B.: The tropical vertex. Duke Math. J. **153**, 297–362 (2010)
41. Gross, M., Pomerleano, D., Siebert, B.: Relative quantum cohomology. Work in progress
42. Gross, M., Siebert, B.: Mirror symmetry via logarithmic degeneration data I. J. Differ. Geom. **72**, 169–338 (2006)
43. Gross, M., Siebert, B.: Mirror symmetry via logarithmic degeneration data II. J. Algebraic Geom. **19**, 679–780 (2010)
44. Gross, M., Siebert, B.: From affine geometry to complex geometry. Ann. Math. **174**, 1301–1428 (2011)
45. Gross, M., Siebert, B.: Logarithmic Gromov–Witten invariants. J. Am. Math. Soc. **26**, 451–510 (2013)
46. Gross, M., Siebert, B.: Local mirror symmetry in the tropics. In: Proceedings of the International Congress of Mathematicians—Seoul 2014. vol. II, pp. 723–744. Kyung Moon Sa, Seoul (2014)
47. Gross, M., Siebert, B.: Intrinsic mirror symmetry and punctured Gromov–Witten invariants. In: Algebraic Geometry: Salt Lake City 2015, 199–230. Proceedings of Symposia in Pure Mathematics, vol. 97, no. 2. AMS, Providence (2018)
48. Gross, M., Siebert, B.: Intrinsic mirror symmetry. arXiv:1909.07649v2 [math.AG]
49. Hacking, P., Keating, A.: Homological mirror symmetry for log Calabi–Yau surfaces. arXiv:2005.05010 [math.SG]
50. Hacon, C., Xu, C.: Existence of log canonical closures. Invent. Math. **192**, 161–195 (2013)
51. Hori, K., Vafa, C.: Mirror symmetry. arXiv:hep-th/0002222
52. Keel, S., Yu, T.Y.: The Frobenius structure theorem for affine log Calabi–Yau varieties containing a torus. arXiv:1908.09861 [math.AG]
53. Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, N.: Toroidal Embeddings I. Springer, LNM 339 (1973)
54. Kollár, J.: Singularities of the minimal model program. Cambridge Tracts in Mathematics, vol. 200. CUP (2013)
55. Kollár, J.: Sources of log canonical centers. In: Minimal Models and Extremal Rays (Kyoto, 2011). Advanced Studies in Pure Mathematics, vol. 70, pp. 29–48. Mathematical Society of Japan, Tokyo (2016)
56. Kollár, J., Xu, C.: The dual complex of Calabi–Yau pairs. Invent. Math. **205**, 527–557 (2016)
57. Kollár, J., Nicaise, J., Xu, C.: Semi-stable extensions over 1-dimensional bases. Acta Math. Sin. **34**, 103–113 (2018)
58. Kontsevich, M.: Homological algebra of mirror symmetry. In: Proceedings of the International Congress of Mathematicians, vol. 1, 2 (Zürich, 1994), pp. 120–139. Birkhäuser, Basel (1995)

59. Kontsevich, M., Soibelman, Y.: Homological mirror symmetry and torus fibrations. In: Symplectic Geometry and Mirror Symmetry (Seoul, 2000), pp. 203–263. World Scientific Publishing, River Edge (2001)
60. Kontsevich, M., Soibelman, Y.: Affine structures and non-Archimedean analytic spaces. In: The Unity of Mathematics (P. Etingof, V. Retakh, I.M. Singer, eds.). Progr. Math. vol. 244, pp. 321–385. Birkhäuser (2006)
61. Lau, S.-C.: Gross–Siebert's slab functions and open GW invariants for toric Calabi–Yau manifolds. Math. Res. Lett. **22**, 881–898 (2015)
62. Magee, T.: Littlewood–Richardson coefficients via mirror symmetry for cluster varieties. Proc. Lond. Math. Soc. (3) **121**, 463–512 (2020)
63. Mandel, T.: Classification of rank 2 cluster varieties. SIGMA Symm. Integr. Geom. Methods Appl. **15**, 042 (2019)
64. Mandel, T.: Theta bases and log Gromov–Witten invariants of cluster varieties. arXiv:1903.03042 [math.AG]
65. Manolache, C.: Virtual pull-backs. J. Algebr. Geom. **21**, 201–245 (2012)
66. Morrison, D.: Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians. J. Am. Math. Soc. **6**, 223–247 (1993)
67. Nicaise, J., Xu, C.: The essential skeleton of a degeneration of algebraic varieties. Am. J. Math. **138**, 1645–1667 (2016)
68. Nicaise, J., Xu, C., Yu, T.: The non-archimedean SYZ fibration. Compos. Math. **155**, 953–972 (2019)
69. Oda, T.: Convex Bodies and Algebraic Geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3, p. 15. Springer, Berlin (1988)
70. Ogus, A.: Lectures on Logarithmic Algebraic Geometry. Cambridge University Press, Cambridge (2018)
71. Pascaleff, J.: On the symplectic cohomology of log Calabi–Yau surfaces. Geom. Topol. **23**, 2701–2792 (2019)
72. Pomerleano, D.: On the homological algebra of relative symplectic geometry, Primitive Forms and Related Subjects—Kavli IPMU 2014. Adv. Stud. Pure Math. **83**, 327–355 (2019)
73. Pomerleano, D.: Intrinsic mirror symmetry and categorical crepant resolutions. arXiv:2103.01200 [math.SG]
74. Prince, T.: The tropical superpotential for $\mathbb{P}^2$. Algebr. Geom. **7**, 30–58 (2020)
75. Rietsch, K., Williams, L.: Newton–Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians. arXiv:1712.00447 [math.AG]
76. Perutz, T., Siebert, B.: Homological mirror symmetry for intrinsic mirror pairs via Hamiltonian Floer theory. Work in progress
77. Ruddat, H., Siebert, B.: Period integrals from wall structures via tropical cycles, canonical coordinates in mirror symmetry and analyticity of toric degenerations. Publ. Math. Inst. Hautes Étud. Sci. **132**, 1–82 (2020)
78. Strominger, A., Yau, S.-T., Zaslow, E.: Mirror symmetry is T-duality. Nucl. Phys. B **479**, 243–259 (1996)
79. Tehrani, M., McLean, M., Zinger, A.: Singularities and semistable degenerations for symplectic topology. C. R. Math. Acad. Sci. Paris **356**, 420–432 (2018)
80. Tseng, H.-H., You, F.: A Gromov-Witten theory for simple normal-crossing pairs without log geometry. arXiv:2008.04844 [math.AG]
81. Tu, J.: On the reconstruction problem in mirror symmetry. Adv. Math. **256**, 449–478 (2014)
82. Wu, Y.: Splitting of Gromov–Witten invariants with toric gluing strata. arXiv:2103.14780 [math.AG]