ANALYSIS & PDE

Volume 15 No. 8 2022

WENCAI LIU

QUANTITATIVE INDUCTIVE ESTIMATES FOR GREEN'S FUNCTIONS OF NON-SELF-ADJOINT MATRICES

msp

ANALYSIS AND PDE
Vol. 15, No. 8, 2022
https://doi.org/10.2140/apde.2022.15.2061

QUANTITATIVE INDUCTIVE ESTIMATES FOR GREEN'S FUNCTIONS OF NON-SELF-ADJOINT MATRICES

WENCAI LIU

We provide quantitative inductive estimates for Green's functions of matrices with (sub-)exponentially decaying off-diagonal entries in arbitrary dimensions. Together with Cartan's estimates and discrepancy estimates, we establish explicit bounds for the large-deviation theorem for non-self-adjoint Toeplitz operators. As applications, we obtain the modulus of continuity of the integrated density of states with explicit bounds and the pure point spectrum property for analytic quasiperiodic operators. Moreover, our inductions are self-improved and work for perturbations with low-complexity interactions.

1. Introduction

The dynamics and spectral theory of quasiperiodic operators have seen significant progress in the last 40 years: first, through earlier perturbative methods [Dinaburg and Sinaĭ 1975; Eliasson 1992; Fröhlich et al. 1990; Sinaĭ 1987; Chulaevsky and Sinaĭ 1989; Moser and Pöschel 1984], and then through nonperturbative methods by controlling Green's functions/transfer matrices [Jitomirskaya 1994; 1999; Bourgain and Jitomirskaya 2002; Bourgain 2005a; Bourgain and Goldstein 2000; Bourgain et al. 2001] or by reducibility [Hou and You 2012; Avila et al. 2011]. The case of one-dimensional lattice and one-frequency potentials has been well understood for both small and large coupling constants, with the recent discovery of global theory [Avila 2015] and universal structure [Jitomirskaya and Liu 2018a; 2018b]. In particular, remarkable developments have been achieved for several models motivated by physics: the almost Mathieu operator (the Harper's model), the extended Harper's model and the Maryland model [Avila et al. 2017; Jitomirskaya and Liu 2017; 2018a; 2018b; Simon 1985; Han and Jitomirskaya 2017; Jitomirskaya and Marx 2012; Jitomirskaya 1999; 2021, Liu and Yuan 2015a; 2015b; Avila and Jitomirskaya 2009; Liu 2020; Jitomirskaya et al. 2020a; Jitomirskaya and Krasovsky 2019; Avila and Damanik 2008; Avila 2008; Avila and Krikorian 2006; Liu and Shi 2019; Jitomirskaya and Kachkovskiy 2016; Jitomirskaya and Zhang 2022]. We refer readers to [Marx and Jitomirskaya 2017; You 2018] for more details.

Problems are known to be much more complicated if one increases the underlying dimension b of the torus or the dimension d of the lattice. The high-dimensional picture is still far from clear. For the one-dimensional lattice $d = 1$ and multifrequencies $b \geq 1$, some special cases have been studied by transfer matrices or Schrödinger cocycles [Goldstein et al. 2019; Goldstein and Schlag 2001; Bourgain 2005a; Damanik et al. 2018; Hadj Amor 2009; Cai et al. 2019; Eliasson 1992]. The first multidimensional

MSC2020: primary 81Q10; secondary 37C55, 82B44.
Keywords: multiscale analysis, large-deviation theorem, discrepancy, semialgebraic sets, Cartan's techniques, Anderson localization.

localization result was obtained by perturbative (KAM) methods in [Chulaevsky and Dinaburg 1993] for operators on lattices $\mathbb{Z}^d$ and a torus $\mathbb{T}$ for arbitrary d. Bourgain, Goldstein and Schlag developed a celebrated method in the spirit of nonperturbative approaches from [Bourgain and Goldstein 2000] to handle the two-dimensional and two-frequency case [Bourgain et al. 2002] ($b = d = 2$) and established the Anderson localization for large coupling constants. This is the first high-dimensional lattice and multifrequency result. Moreover, the large-deviation theorem in [Bourgain et al. 2002], which is a key ingredient to proving the Anderson localization, is purely arithmetic in the sense that removed sets of frequencies are independent of the potential. Roughly speaking, by imposing some purely arithmetic condition on $(\omega_1, \omega_2) \in \mathbb{R}^2$, for any algebraic curve $\Gamma \subset [0, 1]^2$ with degree at most N^C, the number of lattice points

$$\{(n_1, n_2) \in \mathbb{Z}^2 : |n_1| \leq N,\ |n_2| \leq N,\ (n_1\omega_1, n_2\omega_2) \bmod \mathbb{Z}^2 \in \Gamma_\tau\} \tag{1}$$

is bounded by $N^{1-\delta}$ for some $\delta > 0$, where Γ_τ is the e^{-N^τ} neighborhood of Γ. The quantity $N^{1-\delta}$ is referred to as the sublinear bound. It is still open whether the analogous statement for $d \geq 3$ is true or not.

Bourgain [2007] developed a new scheme to prove the large-deviation theorem for arbitrary $b = d$ by a delicate study of the semialgebraic sets. Jitomirskaya, Liu and Shi [Jitomirskaya et al. 2020b] extended Bourgain's result to the case of arbitrary b and d (also see [Shi 2022] for some extensions in particular cases). It is worthwhile to mention that the removed set of frequencies in [Bourgain 2007; Jitomirskaya et al. 2020b] depends on the potential.

Bourgain, Goldstein and Schlag [Bourgain et al. 2002] mentioned that the sublinear bound (1) is the only obstruction to establishing an arithmetic version of the large-deviation theorem in high dimensions. However, there is no detailed proof available yet. Our first goal of this paper is to provide such a proof. Moreover, we are going to establish the quantitative version of the main results in [Bourgain et al. 2002] with generalizations, in particular, that it can be applied to quasiperiodic operators on arbitrary lattices $\mathbb{Z}^d$ driven by any dynamics on tori $\mathbb{Z}^b$ under the assumption on sublinear bounds.

Instead of Laplacians or long-range operators, we will study Toeplitz matrices with (sub-)exponentially decaying off-diagonal entries. Among all the motivations of our generalizations, we want to highlight one. Anderson localization receives a lot of attention from both mathematics and physics. The approach to establishing Anderson localization for quasiperiodic operators with analytic potentials turns out to be a breakthrough component to constructing quasiperiodic solutions for nonlinear Schrödinger equations and nonlinear wave equations [Bourgain 2005a; Wang 2016a; 2019b; 2020]. It is known that the quasiperiodic solutions in PDEs are only subexponentially, not exponentially, decaying [Bourgain 1998; 2005a; Wang 2016a]. Therefore, the (sub-)exponentially decaying matrices are more natural settings in PDEs.

In our arguments, the matrices are not necessarily self-adjoint and every entry of the matrices is allowed to be a function. For $d \geq 2$, this is the first study of operators that goes beyond long-range cases. For $d = 1$, our assumptions are weaker than those of [Bourgain 2002]. See Remark 3.2 for details. Moreover, our arguments hold under perturbations with low complexity.

Our proof is definitely inspired by [Bourgain et al. 2002]. However, there are a lot of important ingredients being added into the arguments to make it quantitative in our more general settings. Moreover, we significantly simplify the arguments even for the case appearing in [Bourgain et al. 2002]. The analysis of that work required dealing with many different types of elementary regions, say rectangles and L-shapes in $\mathbb{Z}^2$. We largely reduced the elementary regions to be square related. See Figure 1. Two novelties are added here. Firstly, we introduce the concept "width" of subsets of lattices. In our arguments, we always keep the involved regions Λ having large width so that every lattice point in Λ can be covered by a square-related elementary region with preset size contained in Λ. For example, a region like Figure 2 was not allowed because the width determined by the distance between B and C is too small. Secondly, we reconstruct the exhaustion of x in every elementary region. In our new construction, the annuli with small width are absorbed into bigger ones. See Figure 3.

There are several other technical improvements in this paper, which we believe to be of independent interest. For example, we establish the Cartan's estimates for non-self-adjoint matrices.

We will prove a quantitatively inductive theorem about the Green's functions in high dimensions, as stated in Theorem 2.3. This is a deterministic statement, which can be applied to study operators even without dynamics. Based on a matrix-valued Cartan-type theorem (estimates on subharmonic functions) in [Bourgain et al. 2002], with further developments in [Bourgain 2005a; Goldstein and Schlag 2008; Jitomirskaya et al. 2020b], we will establish the measure estimates in Theorem 2.6. Imposing proper dynamics on tori, the quantitative inductive estimate for Green's functions is obtained (Theorem 2.7). Moreover, the relation among all constants and parameters is displayed clearly so that the whole picture becomes extremely transparent. We will see how arithmetic conditions on frequencies affect the discrepancy, how structures of semialgebraic sets affect the number of bad Green's functions, and how the dimensions of lattices and frequencies contribute to bounds.

Finally, we want to talk about the applications. As far as we know, there is no explicit bound for the large-deviation theorem except for the case $d = 1$ and $b = 1, 2$. Our approaches (Theorems 2.3, 2.6 and 2.7) are the first to establish the explicit bounds in high dimensions and multifrequencies. We show that in the arithmetic sense, for $d = 1$ and any b, the bound is arbitrarily close to $1/b^3$ for shift dynamics and $1/(4^{b-1}b^3)$ for skew-shift dynamics. For $b = 1$ and arbitrary d, we show that the bound is arbitrarily close to 1.

Another application we want to mention is the regularity of the integrated density of states (IDS) of quasiperiodic operators. The log-Hölder continuity of the integrated density of states is quite general [Craig and Simon 1983; Bourgain and Klein 2013]. The Hölder continuity in one-dimensional settings is well established [Bourgain 2000; 2005a; Goldstein and Schlag 2001; 2008; Avila and Jitomirskaya 2010; 2011; Hadj Amor 2009; Liu and Yuan 2015c; Cai et al. 2019; Zhao 2020; Han and Zhang 2020] for both large and small coupling constants. What we will investigate in this paper is the modulus of continuity $f(x) = e^{-\kappa|\log x|^\tau}$. Unfortunately, like the large-deviation theorem, except for the case $d = 1$ and $b = 1, 2$, there are no explicit bounds of τ in the region of large coupling constants. Based on the ingredients from [Bourgain 2000; Schlag 2001] and the large-deviation theorem, the modulus of continuity of the integrated density of states with explicit estimates will be obtained in Theorem 2.10.

Finally, we mention that the quantitative estimates for Green's functions developed in this paper have been used to establish the explicit power law logarithmic bounds of moments for long-range operators [Liu 2022; Shamis and Sodin 2021].

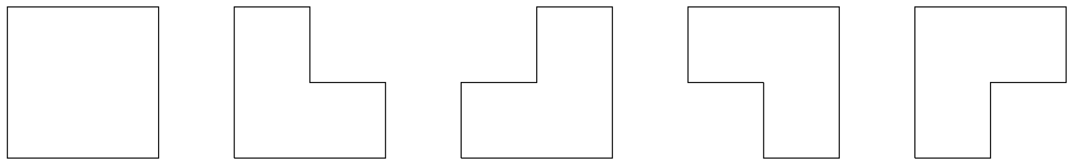

Figure 1. Elementary regions in $\mathbb{Z}^2$.

2. Main results

Let A be a (operator) matrix on $\ell^2(\mathbb{Z}^d)$ satisfying, for any $n \neq n' \in \mathbb{Z}^d$,

$$|A(n,n')| \leq K e^{-c_1|n-n'|^{\tilde{\sigma}}}, \quad K>0,\ c_1>0,\ 0<\tilde{\sigma}\leq 1, \tag{2}$$

where $|n| := \max_{1\leq i\leq d} |n_i|$ for $n=(n_1,n_2,\ldots,n_d)\in\mathbb{Z}^d$. We say that the off-diagonal entries of A are subexponentially decaying if A satisfies (2). Sometimes, we just say A is subexponentially decaying for simplicity.

For $d=1$, the elementary region of size N centered at 0 is given by

$$Q_N = [-N, N].$$

For $d \geq 2$, denote by Q_N an elementary region of size N centered at 0, which is one of the regions

$$Q_N = [-N,N]^d \quad \text{or} \quad Q_N = [-N,N]^d \setminus \{n\in\mathbb{Z}^d : n_i \,\square_i\, 0,\ 1\leq i\leq d\},$$

where $\square_i \in \{<, >, \varnothing\}$ for $i=1,2,\ldots,d$ and at least two $\square_i$ are not $\varnothing$.

Denote by $\mathcal{E}_N^0$ the set of all elementary regions of size N centered at 0. Let $\mathcal{E}_N$ be the set of all translates of elementary regions with center at 0, namely,

$$\mathcal{E}_N := \{n + Q_N : n\in\mathbb{Z}^d,\ Q_N\in\mathcal{E}_N^0\}.$$

We call elements in $\mathcal{E}_N$ elementary regions.

Example 2.1. For $d=2$, there are five types of elementary regions, shown in Figure 1.

The width of a subset $\Lambda\subset\mathbb{Z}^d$ is defined by the maximum $M\in\mathbb{N}$ such that for any $n\in\Lambda$ there exists $\widehat{M}\in\mathcal{E}_M$ such that

$$n\in\widehat{M}\subset\Lambda$$

and

$$\operatorname{dist}(n, \Lambda\backslash\widehat{M}) \geq \tfrac{1}{2}M.$$

Example 2.2. In Figure 2, the width of Λ is determined by the distance between B and C.

A generalized elementary region is defined to be a subset $\Lambda\subset\mathbb{Z}^d$ of the form

$$\Lambda := R\backslash(R+z),$$

where $z\in\mathbb{Z}^d$ is arbitrary and R is a rectangle,

$$R = \{n=(n_1,n_2,\ldots,n_d)\in\mathbb{Z}^d : |n_1-n_1'|\leq M_1,\ldots,|n_d-n_d'|\leq M_d\}.$$

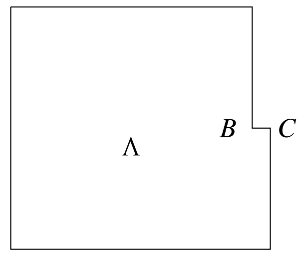

Figure 2. A region with small width.

For $\Lambda \subset \mathbb{Z}^d$, we introduce its diameter,

$$\operatorname{diam}(\Lambda) = \sup_{n,n' \in \Lambda} |n - n'|.$$

Denote by $\mathcal{R}_N$ all generalized elementary regions with diameter less than or equal to N. Denote by $\mathcal{R}_N^M$ all generalized elementary regions in $\mathcal{R}_N$ with width larger than or equal to M. For $\Lambda \subset \mathbb{Z}^d$, let R_Λ be the restriction operator; i.e., $(R_\Lambda u)(n) = u(n)$ for $n \in \Lambda$, and $(R_\Lambda u)(n) = 0$ for $n \notin \Lambda$.

We say an elementary region $\Lambda \in \mathcal{E}_{N'}$ is in class G (good) if

$$|(R_\Lambda A R_\Lambda)^{-1}(n,n')| \le e^{-c_2|n-n'|^{\tilde\sigma}} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N', \tag{3}$$

where $0 < c_2 \le ((5^{\tilde\sigma}-1)/5^{\tilde\sigma})c_1$ and $0 < \tilde\sigma \le 1$. We remark that the upper bound $((5^{\tilde\sigma}-1)/5^{\tilde\sigma})c_1$ arises from $\frac{1}{10}N'$ in (3). See (72) for the explanation. If we change $|n-n'| \ge \frac{1}{10}N'$ to $|n-n'| \ge \sqrt{N'}$ in (3), we can take any c_2 with $0 < c_2 \le (1 - 2^{\tilde\sigma}/N'^{\tilde\sigma/2})c_1$.

Denote by $\lfloor x \rfloor$ the largest integer smaller than or equal to x.

Theorem 2.3. *Assume A satisfies* (2). *Let $\varsigma, \sigma, \xi \in (0,1)$ and $\sigma < \tilde\sigma \le 1$. Let $\widetilde\Lambda_0 \in \mathcal{E}_N$ be an elementary region with the property that for all $\Lambda \subset \widetilde\Lambda_0$, $\Lambda \in \mathcal{R}_L^{N^\xi}$, with $N^\xi \le L \le 2N$, the Green's function $(R_\Lambda A R_\Lambda)^{-1}$ satisfies*

$$\|(R_\Lambda A R_\Lambda)^{-1}\| \le e^{L^\sigma}. \tag{4}$$

Assume that, for any family $\mathcal{F}$ of pairwise disjoint elementary regions in $\widetilde\Lambda_0$ with size $M = \lfloor N^\xi \rfloor$,

$$\#\{\Lambda \in \mathcal{F} : \Lambda \text{ is not in class } G\} \le \frac{N^\varsigma}{N^\xi}. \tag{5}$$

Then, for large N (depending on $K, \varsigma, \sigma, \tilde\sigma, \xi, c_1$ and the lower bound of c_2),

$$|(R_{\widetilde\Lambda_0} A R_{\widetilde\Lambda_0})^{-1}(n,n')| \le e^{-(c_2 - N^{-\vartheta})|n-n'|^{\tilde\sigma}} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N, \tag{6}$$

where $\vartheta = \vartheta(\sigma, \tilde\sigma, \xi, \varsigma) > 0$.

Here are several comments about Theorem 2.3.

Remark 2.4. (1) For $d = 1$ and $\tilde\sigma = 1$, a similar statement was proved in [Bourgain 2002]. For $d = 2$ and $\tilde\sigma = 1$, a similar statement was proved in [Bourgain et al. 2002] for the particular case where A is given by the discrete Laplacian.

(2) The statement in Theorem 2.3 is a robust approach to deal with the spectral theory for quasiperiodic operators and also the construction of quasiperiodic solutions for nonlinear Schrödinger/wave equations. See [Bourgain and Wang 2004; 2008; Bourgain 2002; Bourgain et al. 2001; 2002] for applications. Some particular cases of Theorem 2.3 have been used as ingredients to construct quasiperiodic solutions for PDEs and have been stated in [Bourgain and Wang 2004; 2008; Wang 2016a; 2016b; 2019a] without detailed proof. There are no explicit bound estimates in their arguments either.

(3) In applications, ς is chosen to be arbitrarily close to 1, namely $\varsigma = 1-\varepsilon$ with arbitrarily small $\varepsilon > 0$. Then the upper bound in (5) equals $N^{1-\xi-\varepsilon}$. Theorem 2.3 says that the "goodness" of Green's functions at small size N^{ξ} will ensure the "goodness" of Green's functions at larger size N under the following two conditions:

- The number of bad Green's functions of size N^{ξ} in $[-N,N]^d$ is less than $N^{1-\xi-\varepsilon}$ (referred to as the sublinear bound).
- The Green's functions cannot be "super bad" in the sense that they are controlled by (4). The upper bound $e^{L^{\sigma}}$ with $\sigma < 1$ is referred to as the subexponential bound.

Let $b = \sum_{i=1}^{k} b_i$, where $b_i \in \mathbb{N}$. Let $x = (x_1, x_2, \dots, x_k)$, where $x_i \in \mathbb{T}^{b_i} = (\mathbb{R}/\mathbb{Z})^{b_i}$, $i = 1, 2, \dots, k$. For any $x \in \mathbb{T}^b$ and $1 \le i \le k$, let

$$x_i^{\neg} = (x_1, \dots, x_{i-1}, x_{i+1} \dots, x_k) \in \mathbb{T}^{b-b_j}.$$

For any $y \in \mathbb{T}^{d_1}$ and $X \subset \mathbb{T}^{d_1+d_2}$, denote the y-section of X by

$$X(y) := \{z \in \mathbb{T}^{d_2} : (y,z) \in X\}.$$

Write $\mathrm{Leb}(S)$ for the Lebesgue measure.

Assume each element of the operator A is a function on $\mathbb{T}^b$. Sometimes, we indicate the dependence and denote by the element $A(x;n,n')$. Assume every element $A(z;n,n')$ is analytic in the strip $\{z \in \mathbb{C}^b : |\Im z| \le \rho\}$, $\rho > 0$, and satisfies, for any $n, n' \in \mathbb{Z}^d$ and $x \in \mathbb{T}^b$,

$$|A(x;n,n')| \le K e^{-c_1 |n-n'|^{\tilde{\sigma}}}, \quad K > 0,\ c_1 > 0,\ 0 < \tilde{\sigma} \le 1. \tag{7}$$

Assume there exists $K_1 > 1$ such that for any $x \in \mathbb{T}^b$ and $z \in \{z \in \mathbb{C}^b : |\Im z| \le \rho\}$, with $\|x - z\| \le e^{-(\log(|n|+|n'|+2))^{K_1}}$,

$$|A(x;n,n') - A(z;n,n')| \le K \|x-z\|^{\gamma}, \tag{8}$$

where $\|z\| = \mathrm{dist}(z, \mathbb{Z}^b)$.

Example 2.5. If A satisfies (7) and, for any $n, n' \in \mathbb{Z}^d$, $A(x;n,n')$ is a trigonometric polynomial of degree at most $e^{(\log(|n|+|n'|+2))^{K_1}}$, then (8) holds.

We say an elementary region $\Lambda \in \mathcal{E}_N$ is in class SG_N (strongly good with size N) if

$$\|(R_\Lambda A R_\Lambda)^{-1}\| \le e^{N^{\sigma}}, \tag{9}$$

$$|(R_\Lambda A R_\Lambda)^{-1}(n,n')| \le e^{-c_2 |n-n'|^{\tilde{\sigma}}} \quad \text{for } |n-n'| \ge \tfrac{1}{10} N, \tag{10}$$

where $0 < c_2 \le ((5^{\tilde\sigma}-1)/5^{\tilde\sigma})c_1$ and $0 < \sigma < \tilde\sigma \le 1$. When there is no confusion, we drop the dependence of N from the notation SG_N.

Theorem 2.6. *Assume A satisfies* (7) *and* (8). *Fix $\sigma, \delta, \tilde\sigma, \zeta \in (0,1)$ and $\mu \in (1-\delta, 1)$, $\sigma < \tilde\sigma$. Suppose $\mathcal{R} \subset [-N_3, N_3]^d$ has width at least N_2. For $x \in \mathbb{T}^b$, define $\mathcal{B}_{\mathcal{R}}(x)$ as*

$$\mathcal{B}_{\mathcal{R}}(x) = \{n \in \mathcal{R} : \textit{ there exists } Q_{N_1} \in \mathcal{E}^{\mathbf{0}}_{N_1} \textit{ such that } n + Q_{N_1} \notin \mathrm{SG}_{N_1}\}.$$

Assume that, for any $x \in \mathbb{T}^b$,

$$\#\mathcal{B}_{\mathcal{R}}(x) \le L^{1-\delta}. \tag{11}$$

Assume that there exists a subset $X_{N_2} \subset \mathbb{T}^b$ such that

$$\sup_{1\le i\le k, x_i^{\neg}\in\mathbb{T}^{b-b_i}} \mathrm{Leb}(X_{N_2}(x_i^{\neg})) \le e^{-N_2^{\zeta}}, \tag{12}$$

and, for any $Q_{N_2} \in \mathcal{E}^{\mathbf{0}}_{N_2}$, $x \notin X_{N_2}$ and $n \in \mathcal{R}$, the region $n + Q_{N_2}$ is in class SG_{N_2}. Let

$$\widetilde{X}_{\mathcal{R}}(x) = \{x \in \mathbb{T}^b : \|(R_{\mathcal{R}} A(x) R_{\mathcal{R}})^{-1}\| \ge e^{L^{\mu}}\}.$$

Suppose $N_3 \le e^{N_1^{1/(2K_1)}}$, $N_2 \ge N_1^{2/\zeta}$ and $L \ge N_2^{(2d+b+2)/(\mu-1+\delta)}$. Then there exists

$$N_0 = N_0(K_1, K, c_1, c_2, \tilde\sigma, \sigma, \delta, \gamma, \rho, \mu)$$

(*depending on the lower bound of c_2*) *such that, for any $N_1 \ge N_0$ and $i = 1, 2, \ldots, k$,*

$$\sup_{x_i^{\neg}\in\mathbb{T}^{b-b_i}} \mathrm{Leb}(\widetilde{X}_{\mathcal{R}}(x_i^{\neg})) \le e^{-(L^{\mu-1+\delta}/N_2^{2d+b+2})^{1/b_i}}. \tag{13}$$

Let f be a function from $\mathbb{Z}^d \times \mathbb{T}^b$ to $\mathbb{T}^b$. Assume, for any $m_1, m_2, \ldots, m_d \in \mathbb{Z}^d$ and $n_1, n_2, \ldots, n_d \in \mathbb{Z}^d$,

$$f(m_1+n_1, m_2+n_2, \ldots, m_d+n_d, x) = f(m_1, m_2, \ldots, m_d, f(n_1, n_2, \ldots, n_d, x)).$$

Sometimes, we write down $f^n(x)$ for $f(n,x)$ for convenience, where $n \in \mathbb{Z}^d$ and $x \in \mathbb{T}^b$. We say A is a Toeplitz (operator) matrix on $\ell^2(\mathbb{Z}^d)$ with respect to f if

$$A(x; n+k, n'+k) = A(f^k(x); n, n') \tag{14}$$

for any $n \in \mathbb{Z}^d$, $n' \in \mathbb{Z}^d$ and $k \in \mathbb{Z}^d$. We note that A is not necessarily self-adjoint.

We say the Green's function of an operator $A(x)$ satisfies Property P with parameters (μ, ζ, c_2) at size N if the following statement is true: there exists a subset $X_N \subset \mathbb{T}^b$ such that

$$\sup_{1\le i\le k, x_i^{\neg}\in\mathbb{T}^{b-b_i}} \mathrm{Leb}(X_N(x_i^{\neg})) \le e^{-N^{\zeta}},$$

and, for any $x \notin X_N \mod \mathbb{Z}^b$ and $Q_N \in \mathcal{E}^{\mathbf{0}}_N$,

$$\|(R_{Q_N} A(x) R_{Q_N})^{-1}\| \le e^{N^{\mu}},$$
$$|(R_{Q_N} A R_{Q_N})^{-1}(x; n, n')| \le e^{-c_2|n-n'|^{\tilde\sigma}} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N.$$

Theorem 2.7. *Assume $A(x)$ satisfies* (7), (8) *and* (14), *and*

$$0 < c_2 < (1-5^{-\tilde{\sigma}})c_1, \quad 1-\delta < \sigma < \tilde{\sigma} \le 1, \quad \delta > \iota > 0, \quad \text{and} \quad 0 < \mu < \tilde{\sigma}.$$

Let $c = \frac{1}{2}\min\{1/K_1, \tilde{\sigma}\}$. Fix any sufficiently small $\varepsilon > 0$. There exists a large constant C depending on all parameters such that the following statements are true. Let N_1 be sufficiently large, $N_2 \in [N_1^C, e^{N_1^{c/2}}]$ and $N_3 \in [N_2^C, e^{N_1^c}]$. Assume that the Green's function satisfies Property P with parameters (μ, ζ, c_2) at sizes N_1 and N_2. Assume, for any $L \in [N_3^{\delta-\iota}, N_3]$ and any $x \in \mathbb{T}^b$,

$$\#\{n \in \mathbb{Z}^d : |n| \le L,\, f(n,x) \in X_{N_1} \mod \mathbb{Z}^b\} \le L^{1-\delta}. \tag{15}$$

Then there exists $X_{N_3} \subset \mathbb{T}^b$ such that

$$\sup_{1 \le i \le k, x_i^{\neg} \in \mathbb{T}^{b-b_i}} \operatorname{Leb}(X_{N_3}(x_i^{\neg})) \le e^{-N_3{}^{(\sigma-1)/b_i\,\delta+\delta^2/b_i-\varepsilon}}, \tag{16}$$

and, for any $x \notin X_{N_3}$ and $Q_{N_3} \in \mathcal{E}^0_{N_3}$,

$$\|(R_{Q_{N_3}} A(x) R_{Q_{N_3}})^{-1}\| \le e^{N_3^{\sigma}}, \tag{17}$$

and, for $|n-n'| \ge \frac{1}{10}N_3$,

$$|(R_{Q_{N_3}} A R_{Q_{N_3}})^{-1}(x;n,n')| \le e^{-(c_2 - 2N_1^{-\vartheta_1} - N_3^{-\vartheta_2})|n-n'|^{\tilde{\sigma}}}, \tag{18}$$

where $\vartheta_1 = \vartheta_1(\tilde{\sigma}, \mu, c)$ and $\vartheta_2 = \vartheta_2(\tilde{\sigma}, \sigma, \delta, \varepsilon)$.

Our theorems work for Toeplitz matrices with low-complexity interactions. Let U be an operator on $\ell^2(\mathbb{Z}^d)$ satisfying

$$|U(n,n')| \le K e^{-c_1|n-n'|^{\tilde{\sigma}}}.$$

Given $m \in \mathbb{Z}^d$, define the operator U^m by

$$U^m(n,n') = U(m+n, m+n'), \quad n \in \mathbb{Z}^d,\ n' \in \mathbb{Z}^d.$$

We say U has low complexity if there exists $0 < a < 1$ such that, for any $N > 1$,

$$\#\{R_{Q_N} U^m R_{Q_N} : m \in \mathbb{Z}^d, Q_N \in \mathcal{E}^0_N\} \le K e^{N^a}. \tag{19}$$

For any $m \in \mathbb{Z}^d$, let

$$\tilde{A}^m(x;n,n') = A(x;n,n') + U^m(n,n'). \tag{20}$$

We say that the Green's function of an operator $A(x)$ satisfies Property $\widetilde{\mathrm{P}}$ with parameters (μ, ζ, c_2) at size N if the following statement is true: there exists a set $X_N \subset \mathbb{T}^b$ such that

$$\sup_{1 \le i \le k, x_i^{\neg} \in \mathbb{T}^{b-b_i}} \operatorname{Leb}(X_N(x_i^{\neg})) \le e^{-N^{\zeta}},$$

and, for any $x \notin X_N \mod \mathbb{Z}^b$, $m \in \mathbb{Z}^d$, and $Q_N \in \mathcal{E}^0_N$,

$$\|(R_{Q_N} \tilde{A}^m(x) R_{Q_N})^{-1}\| \le e^{N^{\mu}},$$

$$|(R_{Q_N} \tilde{A}^m R_{Q_N})^{-1}(x;n,n')| \le e^{-c_2|n-n'|^{\tilde{\sigma}}} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N.$$

Theorem 2.8. *Assume $A(x)$ satisfies* (7), (8) *and* (14), *U has low complexity,*

$$0 < c_2 < (1-5^{-\tilde{\sigma}})c_1, \quad 1-\delta < \sigma < \tilde{\sigma} \le 1, \quad \delta > \iota > 0, \quad 0 < \mu < \tilde{\sigma},$$

and

$$a \le \frac{1}{2}\min_i \left\{ \frac{\sigma-1}{b_i}\delta + \frac{\delta^2}{b_i} \right\}. \tag{21}$$

Let $\tilde{A}^m$ be given by (20) *and $c = \frac{1}{2}\min\{1/K_1, \tilde{\sigma}\}$. Fix any sufficiently small $\varepsilon > 0$. Then there exists a large constant C depending on all parameters such that the following statements are true. Let N_1 be sufficiently large, $N_2 \in [N_1^C, e^{N_1^{c/2}}]$ and $N_3 \in [N_2^C, e^{N_1^c}]$. Assume the Green's function satisfies Property $\widetilde{P}$ with parameters (μ, ζ, c_2) at sizes N_1 and N_2. Assume, for any $L \in [N_3^{\delta-\iota}, N_3]$ and any $x \in \mathbb{T}^b$,*

$$\#\{n \in \mathbb{Z}^d : |n| \le L, f(n,x) \in X_{N_1} \mod \mathbb{Z}^b\} \le L^{1-\delta}.$$

Then there exists a subset $X_{N_3} \subset \mathbb{T}^b$ such that

$$\sup_{1\le i\le k, x_i^{\neg} \in \mathbb{T}^{b-b_i}} \operatorname{Leb}(X_{N_3}(x_i^{\neg})) \le e^{-N_3^{(\sigma-1)/b_i\delta+\delta^2/b_i-\varepsilon}},$$

and, for any $x \notin X_{N_3}$, $m \in \mathbb{Z}^d$ and $Q_{N_3} \in \mathcal{E}_{N_3}^0$,

$$\|(R_{Q_{N_3}}\tilde{A}^m(x)R_{Q_{N_3}})^{-1}\| \le e^{N_3^{\sigma}},$$

and, for $|n-n'| \ge \frac{1}{10}N_3$,

$$|(R_{Q_{N_3}}\tilde{A}^m R_{Q_{N_3}})^{-1}(x;n,n')| \le e^{-(c_2 - N_1^{-\vartheta_1} - N_3^{-\vartheta_2})|n-n'|^{\tilde{\sigma}}},$$

where $\vartheta_1 = \vartheta_1(\tilde{\sigma}, \mu, c)$ and $\vartheta_2 = \vartheta_2(\tilde{\sigma}, \sigma, \delta, \varepsilon)$.

Remark 2.9. (1) Theorem 2.7 improves the parameters from (μ, ζ, c_2) to

$$\left(\sigma, \frac{\sigma-1}{b_i}\delta + \frac{\delta^2}{b_i} - \varepsilon, c_2 - N_1^{-\vartheta_1} - N_3^{-\vartheta_2}\right).$$

Theorem 2.7 gives us opportunities to combine perturbative approaches with nonperturbative approaches. After establishing Property P for initial scales by nonperturbative methods, we can adapt the parameters to establish Property P with explicit bounds for larger scales. See Theorems 3.4, 3.5 and 3.6, and Corollaries 3.8, 3.9 and 3.10 for examples.

(2) Roughly speaking Theorem 2.7 says that under the assumption on the sublinear bound, the large-deviation theorem at sizes $N = N_1$ and $N = N_2$ will ensure the large-deviation theorem at size $N = N_3$.

We are going to discuss the modulus of continuity of the integrated density of states (IDS). In order to make it as general as possible, we do not require the existence of the integrated density of states first. Let $E_1 < E_2$ and define

$$k(x, E_1, E_2) = \limsup_{N\to\infty} \frac{1}{(2N+1)^d} \#\{\text{eigenvalues of } R_{[-N,N]^d}A(x)R_{[-N,N]^d} \text{ in } [E_1, E_2]\}. \tag{22}$$

Fix $x \in \mathbb{T}^b$. Assume for any measurable set $\mathcal{S} \subset \mathbb{T}^b$ we have

$$\limsup_{N\to\infty} \frac{1}{(2N+1)^d} \#\{n \in \mathbb{Z}^d : |n| \le N,\, f(n_1, n_2, \dots, n_d, x) \in \mathcal{S}\} \le \mathrm{Leb}(\mathcal{S}). \tag{23}$$

For an operator $A(x)$ on $\ell^2(\mathbb{Z}^d)$, denote by the energy-dependent Green's functions

$$G_\Lambda(E, x) = (R_\Lambda(A(x) - E)R_\Lambda)^{-1}. \tag{24}$$

Instead of $G_\Lambda(E, x)$, we will write G_Λ, $G_\Lambda(E)$, or $G_\Lambda(x)$ when there is no ambiguity. We will write $G_\Lambda(n, n')$, $G_\Lambda(E; n, n')$, $G_\Lambda(x; n, n')$, or $G_\Lambda(E, x; n, n')$ for the element of matrices.

Theorem 2.10. *Assume $A(x)$ is a Toeplitz (operator) matrix on $\ell^2(\mathbb{Z}^d)$ with respect to f in the sense of* (14). *Let $\zeta \in (0, 1)$ and $0 < \sigma < \tilde{\sigma} \le 1$. Assume, for any $E \in \mathbb{R}$, there exists a set $X_N \subset \mathbb{T}^b$ such that*

$$\mathrm{Leb}(X_N) \le e^{-N^\zeta}$$

and, for any $x \notin X_N$ and any $Q_N \in \mathcal{E}_N^0$,

$$\|G_{Q_N}(E, x)\| \le e^{N^\sigma},$$
$$|G_{Q_N}(E, x; n, n')| \le e^{-c|n-n'|^{\tilde{\sigma}}} \quad \text{for } |n - n'| \ge \tfrac{1}{10}N,$$

where $c > 0$. Assume (23) *holds for some $x_0 \in \mathbb{T}^b$. Then, for any $\varepsilon > 0$, we have*

$$|k(x_0, E_1, E_2)| \le e^{-|\log|E_1 - E_2||^{\zeta/\sigma - \varepsilon}},$$

provided that $|E_1 - E_2|$ is sufficiently small.

The rest of this paper is organized as follows. Except for some statements in applications (Section 3), this paper is entirely self-contained. We will introduce many applications to quasiperiodic operators in Section 3. Sections 4, 5, 6, 7 are devoted to prove Theorems 2.3, 2.6, 2.7, 2.8 and 2.10. We will introduce the discrepancy for semialgebraic sets in Section 8. In Section 9, we will give the proof for all the results in Section 3.

3. Applications

Let S be a Toeplitz (operator) matrix on $\ell^2(\mathbb{Z}^d)$ with respect to f, namely,

$$S(x; n + k, n' + k) = S(f^k(x); n, n') \tag{25}$$

for any $n \in \mathbb{Z}^d$, $n' \in \mathbb{Z}^d$ and $k \in \mathbb{Z}^d$. Assume every element $S(z; n, n')$, $n, n' \in \mathbb{Z}^d$, is analytic in a strip $\{z : |\Im z| \le \rho\}$ with $\rho > 0$ and satisfies, for any $x \in \mathbb{R}$ and $n, n' \in \mathbb{Z}^d$,

$$|S(x; n, n')| \le K e^{-c_1|n-n'|}, \quad K > 0,\ c_1 > 0. \tag{26}$$

Assume that there exists $K_1 > 1$ such that, for any $x \in \mathbb{T}^b$ and $z \in \{z \in \mathbb{C}^b : |\Im z| \le \rho\}$ with $\|x - z\| \le e^{-(\log(|n|+|n'|+2))^{K_1}}$,

$$|S(x; n, n') - S(z; n, n')| \le K\|x - z\|^\gamma. \tag{27}$$

Assume, for any $N > 1$, $n, n' \in \mathbb{Z}^d$ with $|n| \le N$ and $|n'| \le N$, there exists a trigonometric polynomial $\widetilde{S}(x; n, n')$ of degree less than $e^{(\log N)^{K_1}}$ such that

$$\sup_{x \in \mathbb{T}^b} |S(x; n, n') - \widetilde{S}(x; n, n')| \le K e^{-N^2}. \tag{28}$$

Define a family of operators $H(x)$ on $\ell^2(\mathbb{Z}^d)$:

$$H(x) = \lambda^{-1} S + v(f(n, x)) \delta_{nn'}, \tag{29}$$

where v is an analytic function on $\mathbb{T}^b$.

In this section, we always assume

- v is nonconstant,
- f is a frequency shift or skew-shift, which is defined explicitly in each subsection,
- except for in Section 3F, S is a Toeplitz (operator) matrix on $\ell^2(\mathbb{Z}^d)$ with respect to f and satisfies (25)–(28).

Example 3.1. • If S is a long-range operator, namely, S does not depend on x and

$$S(n, n') \le K e^{-c_1 |n - n'|}, \quad n, n' \in \mathbb{Z}^d,$$

then (26), (27) and (28) hold.

• Let $\phi_k(x)$, $k \in \mathbb{Z}$, be a trigonometric polynomial on $\mathbb{T}^b$ of degree less than $e^{(\log(1+|k|))^{K_1}}$ satisfying

$$\sup_{x \in \mathbb{T}^b} |\phi_k(x)| \le K e^{-c_1 |k|}.$$

Let

$$S(x; n, n') = \phi_{n-n'}(f(n, x)) + \overline{\phi_{n'-n}(f(n', x))}.$$

Then (26), (27) and (28) hold.

Remark 3.2. For $d \ge 2$, our setting (25)–(28) is the first to allow every entry of S to depend on x, which goes beyond the long-range operators. For $d = 1$, Bourgain [2002] studied the case in Example 3.1 under the assumption that $\phi_k(x)$ is a trigonometric polynomial of degree at most N^C.

Remark 3.3. In the applications, all potentials are assumed to be analytic. It is possible to investigate potentials in the Gevrey class by combining approaches in the present paper with [Klein 2005; 2014].

We will apply Theorems 2.3, 2.6, 2.7 and 2.10 to operators

$$A(x) = H(x) = \lambda^{-1} S + v(f(n, x)) \delta_{nn'}.$$

In this section, the Green's functions always depend on energy E. See (24).

The IDS appearing in applications always exists, namely, the limit

$$k(x, E) = \lim_{N \to \infty} \frac{1}{(2N+1)^d} \#\{\text{eigenvalues of } R_{[-N,N]^d} A(x) R_{[-N,N]^d} \text{ smaller than } E\},$$

converges to $k(E)$ for almost every x. We write $k(E)$ for the IDS when it exists.

For the large-deviation theorem, S is not necessarily self-adjoint and v is not necessarily real-valued. However, to establish the pure point spectrum property or regularity of the IDS, self-adjointness of H is necessary because of the energy elimination. In the following, we always assume that S is self-adjoint and v is a real analytic function if we study the pure point spectrum property or regularity of the IDS.

3A. ***Shifts*: *d* = 1, *arbitrary b*.** Denote by Δ the discrete Laplacian on $\ell^2(\mathbb{Z})$, that is, for $\{u(n)\}\in\ell^2(\mathbb{Z})$,

$$(\Delta u)(n)=\sum_{|n-n'|=1}u(n').$$

We say that $\omega=(\omega_1,\omega_2,\dots,\omega_b)$ satisfies Diophantine condition $\mathrm{DC}(\kappa,\tau)$ if

$$\|k\omega\|\geq\frac{\tau}{|k|^{\kappa}},k\in\mathbb{Z}^b\backslash\{(0,0,\dots,0)\}. \tag{30}$$

By the Dirichlet principle, one has $\kappa\geq b$. When $\kappa>b$, $\bigcup_{\tau>0}\mathrm{DC}(\kappa,\tau)$ has full Lebesgue measure.

We say that $\omega\in\mathbb{R}$ satisfies strong Diophantine conditions if there exist $\kappa>1$ and $\tau>0$ such that

$$\|k\omega\|\geq\frac{\tau}{k(1+\log k)^{\kappa}}\quad\text{for all } k\in\mathbb{N}. \tag{31}$$

It is easy to see that almost every ω satisfies strong Diophantine conditions.

Let

$$f^n(x)=x+n\omega=(x_1+n\omega_1,x_2+n\omega_2,\dots,x_b+n\omega_b) \bmod \mathbb{Z}^b,$$

where $x=(x_1,x_2,\dots,x_b)\in\mathbb{T}^b$, $n\in\mathbb{Z}$ and $\omega=(\omega_1,\omega_2,\dots,\omega_b)\in\mathbb{R}^b$.

Let $H(x)$ on $\ell^2(\mathbb{Z})$ be given by

$$H(x)=\Delta+v(f^n(x))=\Delta+v(x_1+n\omega_1,x_2+n\omega_2,\dots,x_b+n\omega_b)\delta_{nn'}, \tag{32}$$

where $n,n'\in\mathbb{Z}$.

Let

$$A_k^E(x)=\prod_{j=k-1}^{0}A^E(x+j\omega)=A^E(x+(k-1)\omega)A^E(x+(k-2)\omega)\cdots A^E(x) \tag{33}$$

and

$$A_{-k}^E(x)=(A_k^E(x-k\omega))^{-1} \tag{34}$$

for $k\geq 1$, where

$$A^E(x)=\begin{pmatrix}E-v(x) & -1\\ 1 & 0\end{pmatrix}.$$

A_k^E is called the (k-step) transfer matrix. The Lyapunov exponent is given by

$$L(E)=\lim_{k\to\infty}\frac{1}{k}\int_{\mathbb{T}^b}\ln\|A_k^E(x)\|\,dx. \tag{35}$$

Theorem 3.4. *Let $\omega\in\mathrm{DC}(\kappa,\tau)$ and $1-1/(b\kappa)<\sigma<1$. Let $H(x)$ be given by* (32)*. Assume the Lyapunov exponent $L(E)$ is positive. Then, for any $\varepsilon>0$ and large N, there exists a subset $X_N\subset\mathbb{T}^b$ such that*

$$\mathrm{Leb}(X_N)\leq e^{-N^{(\sigma-1)/(b^2\kappa)+1/(b^3\kappa^2)-\varepsilon}},$$

and, for any $x \notin X_N$, we have

$$\|G_{[-N,N]}(E,x)\| \le e^{N^\sigma},$$
$$|G_{[N,-N]}(E,x;n,n')| \le e^{-(L(E)-\varepsilon)|n-n'|} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N.$$

Theorem 3.5. *Let $\omega \in \mathrm{DC}(\kappa,\tau)$ and $H(x)$ be given by* (32). *Suppose the Lyapunov exponent $L(E)$ is positive for every E in an interval I. Then, for any $\varepsilon > 0$,*

$$|k(E_1)-k(E_2)| \le e^{-\left(\log \frac{1}{|E_1-E_2|}\right)^{1/(b^3\kappa^2)-\varepsilon}},$$

provided that $|E_1 - E_2|$ is sufficiently small and $E_1, E_2 \in I$.

Theorem 3.6. *Let $H(x)$ be given by* (32). *Then the following statement is true for almost every ω. Assume the Lyapunov exponent $L(E)$ is positive for every E in an interval I. Then, for any $\varepsilon > 0$,*

$$|k(E_1)-k(E_2)| \le e^{-\left(\log \frac{1}{|E_1-E_2|}\right)^{1/b^3-\varepsilon}},$$

provided that $|E_1 - E_2|$ is sufficiently small and $E_1, E_2 \in I$.

Remark 3.7. Under the same assumptions, the large-deviation theorem and the modulus of continuity of the IDS were shown in [Goldstein and Schlag 2001] (also see [Bourgain 2005a]). When $b = 2$, a better bound $b = \frac{1}{3}$ was obtained in [Goldstein and Schlag 2001]. However, there are no explicit bounds in [Goldstein and Schlag 2001; Bourgain 2005a] when $b \ge 3$.

Putting a coupling constant λ^{-1} in front of the Laplacian Δ, the operator given by (32) becomes

$$H(x) = \lambda^{-1}\Delta + v(x+n\omega)\delta_{nn'}. \tag{36}$$

For large λ only depending on the potential v, the Lyapunov exponent $L(E)$ is positive for every E [Bourgain 2005b]. Therefore, we have the following three corollaries.

Corollary 3.8. *Assume $\omega \in \mathrm{DC}(\kappa,\tau)$ and $1 - 1/(b\kappa) < \sigma < 1$. Let $H(x)$ be given by* (36). *Then there exists $\lambda_0 = \lambda_0(v)$ such that, for any $\varepsilon > 0$, $\lambda > \lambda_0$ and large N, there exists $X_N \subset \mathbb{T}^b$ such that*

$$\mathrm{Leb}(X_N) \le e^{-N^{\sigma-1/(b^2\kappa)+1/(b^3\kappa^2)-\varepsilon}}, \tag{37}$$

and, for any $x \notin X_N$, we have

$$\|G_{[-N,N]}(E,x)\| \le e^{N^\sigma},$$
$$|G_{[N,-N]}(E,x;n,n')| \le e^{-(L(E)-\varepsilon)|n-n'|} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N.$$

Corollary 3.9. *Let $\omega \in \mathrm{DC}(\kappa,\tau)$ and $H(x)$ be given by* (36). *Then there exists $\lambda_0 = \lambda_0(v)$ such that, for any $\varepsilon > 0$ and $\lambda > \lambda_0$,*

$$|k(E_1)-k(E_2)| \le e^{-\left(\log \frac{1}{|E_1-E_2|}\right)^{1/(b^3\kappa^2)-\varepsilon}},$$

provided that $|E_1 - E_2|$ is sufficiently small.

Corollary 3.10. *Let $H(x)$ be given by* (36)*. Then there exists $\lambda_0=\lambda_0(v)$ such that the following statement is true for almost every ω. For any $\varepsilon>0$ and $\lambda>\lambda_0$,*

$$|k(E_1)-k(E_2)|\leq e^{-\left(\log\frac{1}{|E_1-E_2|}\right)^{1/b^3-\varepsilon}},$$

provided that $|E_1-E_2|$ is sufficiently small.

Let $H(x)$ on $\ell^2(\mathbb{Z})$ be given by

$$H(x)=\lambda^{-1}S+v(f^n(x))=\lambda^{-1}S+v(x+n\omega)\delta_{nn'}, \tag{38}$$

where $x,\omega\in\mathbb{R}^b$.

Theorem 3.11. *Let $H(x)$ be given by* (38)*. Assume $\omega\in\mathrm{DC}(\kappa,\tau)$ and $1-1/(b\kappa)<\sigma<1$. Then, for any $\varepsilon>0$, there exists*

$$\lambda_0=\lambda_0(\varepsilon,\kappa,\tau,\rho,\sigma,\gamma,K,K_1,c_1,v)$$

such that, for any $\lambda>\lambda_0$ and any N, there exists $X_N\subset\mathbb{T}^b$ such that

$$\mathrm{Leb}(X_N)\leq e^{-N^{\sigma-1/(b^2\kappa)+1/(b^3\kappa^2)-\varepsilon}},$$

and, for any $x\notin X_N$, we have

$$\|G_{[-N,N]}(E,x)\|\leq e^{N^\sigma},$$
$$|G_{[N,-N]}(E,x;n,n')|\leq e^{-\frac{1}{2}c_1|n-n'|}\quad \textit{for } |n-n'|\geq \tfrac{1}{10}N.$$

Theorem 3.12. *Assume S is self-adjoint and $\omega\in\mathrm{DC}(\kappa,\tau)$. Let $H(x)$ be given by* (38)*. Then, for any $\varepsilon>0$, there exists*

$$\lambda_0=\lambda_0(\varepsilon,\kappa,\tau,\rho,\gamma,K,K_1,c_1,v)$$

such that, for any $\lambda>\lambda_0$,

$$|k(E_1)-k(E_2)|\leq e^{-\left(\log\frac{1}{|E_1-E_2|}\right)^{1/(b^3\kappa^2)-\varepsilon}},$$

provided that $|E_1-E_2|$ is sufficiently small.

Theorem 3.13. *Assume S is self-adjoint. Let $H(x)$ be given by* (38)*. Then for almost every $\omega\in\mathbb{R}^b$ the following is true. For any $\varepsilon>0$, there exists*

$$\lambda_0=\lambda_0(\varepsilon,\omega,\rho,\gamma,K,K_1,c_1,v)$$

such that, for any $\lambda>\lambda_0$,

$$|k(E_1)-k(E_2)|\leq e^{-\left(\log\frac{1}{|E_1-E_2|}\right)^{1/b^3-\varepsilon}},$$

provided that $|E_1-E_2|$ is sufficiently small.

Theorem 3.14. *Let $H(x)$ be given by* (38)*. Then, for any $\varrho>0$, there is $\lambda_0=\lambda_0(\varrho,\rho,\gamma,K,K_1,c_1,v)>0$ such that the following statement holds. For any $\lambda>\lambda_0$ and any $x\in\mathbb{T}$, there exists $\Omega=\Omega(x,\lambda,S,v,\varrho)\subset\mathbb{T}^b$ with $\mathrm{Leb}(\mathbb{T}^b\setminus\Omega)\leq\varrho$ such that, for any $\omega\in\Omega$, $H(x)$ satisfies Anderson localization.*

3B. *Shifts*: $b = 1$, *arbitrary* d. Let v be analytic on $\mathbb{T}$. Let

$$f^n(x) = x + n\omega = x + n_1\omega_1 + n_2\omega_2 + \cdots + n_d\omega_d \mod \mathbb{Z},$$

where $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$ and $x \in \mathbb{T}$. Let $H(x)$ on $\ell^2(\mathbb{Z}^d)$ be given by

$$H(x) = \lambda^{-1}S + v(f^n(x))\delta_{nn'} = \lambda^{-1}S + v(x + n_1\omega_1 + n_2\omega_2 + \cdots + n_d\omega_d)\delta_{nn'}. \tag{39}$$

Theorem 3.15. *Let $\omega \in \mathrm{DC}(\kappa, \tau)$ and $H(x)$ be given by* (39). *Then, for any $\varepsilon > 0$, there exists $\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \rho, \sigma, \gamma, K, K_1, c_1, v)$ such that, for any $\lambda > \lambda_0$ and any N, there exists $X_N \subset \mathbb{T}$ such that*

$$\mathrm{Leb}(X_N) \le e^{-N^{\sigma-\varepsilon}}, \tag{40}$$

and, for any $x \notin X_N$ and any $Q_N \in \mathcal{E}_N^0$, we have

$$\begin{aligned} \|G_{Q_N}(E,x)\| &\le e^{N^\sigma}, \\ |G_{Q_N}(E,x;n,n')| &\le e^{-\frac{1}{2}c_1|n-n'|} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N. \end{aligned} \tag{41}$$

Theorem 3.16. *Assume S is self-adjoint and $\omega \in \mathrm{DC}(\kappa, \tau)$. Let $H(x)$ be given by* (39). *Then, for any $\varepsilon > 0$, there exists*

$$\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \rho, \gamma, K, K_1, c_1, v)$$

such that, for any $\lambda > \lambda_0$,

$$|k(E_1) - k(E_2)| \le e^{-\left(\log \frac{1}{|E_1 - E_2|}\right)^{1-\varepsilon}},$$

provided that $|E_1 - E_2|$ is sufficiently small.

Theorem 3.17. *Assume S is self-adjoint. Let $H(x)$ be given by* (39). *Then, for any $\varrho > 0$, there is $\lambda_0 = \lambda_0(\varrho, \rho, \gamma, K, K_1, c_1, v) > 0$ such that the following statement holds. For any $\lambda > \lambda_0$ and any $x \in \mathbb{T}$, there exists $\Omega = \Omega(x, \lambda, S, v, \varrho) \subset \mathbb{T}^d$ with $\mathrm{Leb}(\mathbb{T}^d \setminus \Omega) \le \varrho$ such that, for any $\omega \in \Omega$, $H(x)$ satisfies Anderson localization.*

Remark 3.18. Theorem 3.17 is a generalization of [Bourgain 2005a, Theorem 2 p. 138] and the main result in [Chulaevsky and Dinaburg 1993].

3C. *Skew-shifts*: $d = 1$, *arbitrary* b. Let $f\colon \mathbb{T}^b \to \mathbb{T}^b$ be the skew-shift defined as

$$f(x_1, x_2, \ldots, x_b) = (x_1 + \omega, x_2 + x_1, \ldots, x_b + x_{b-1}). \tag{42}$$

Let $H(x)$ on $\ell^2(\mathbb{Z})$ be given by

$$H(x) = \lambda^{-1}S(x) + v(f^n(x))\delta_{nn'}, \tag{43}$$

where v is analytic on $\mathbb{T}^b$.

Theorem 3.19. *Let $H(x)$ be given by* (43). *Assume $\omega \in \mathrm{DC}(\kappa, \tau)$ and $1 - 1/(2^{b-1}b\kappa) < \sigma < 1$. Then, for any $\varepsilon > 0$, there exists $\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \rho, \sigma, \gamma, K, K_1, c_1, v)$ such that, for any $\lambda > \lambda_0$ and any N, there exists $X_N \subset \mathbb{T}^b$ such that*

$$\mathrm{Leb}(X_N) \le e^{-N^{\frac{\sigma-1}{2^{b-1}b^2\kappa} + \frac{1}{4^{b-1}b^3\kappa^2} - \varepsilon}},$$

and for any $x \notin X_N$, we have

$$\|G_{[-N,N]}(E,x)\| \le e^{N^\sigma},$$
$$|G_{[N,-N]}(E,x;n,n')| \le e^{-\frac{1}{2}c_1|n-n'|} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N.$$

Remark 3.20. Under the stronger assumption that $\omega \in \mathrm{DC}(2,\tau)$, v and each element of S are nonconstant trigonometric polynomials, the large-deviation theorem appearing in Theorem 3.19 without explicit bounds was proved for $d = 2$ [Bourgain 2005a] and arbitrary d [Shi and Yuan 2020].

Theorem 3.21. *Assume S is self-adjoint and $\omega \in \mathrm{DC}(\kappa,\tau)$. Let $H(x)$ be given by* (43). *Then, for any $\varepsilon > 0$, there exists*

$$\lambda_0 = \lambda_0(\varepsilon,\kappa,\tau,\rho,\gamma,K,K_1,c_1,v)$$

such that, for any $\lambda > \lambda_0$, we have

$$|k(E_1)-k(E_2)| \le e^{-\left(\log \frac{1}{|E_1-E_2|}\right)^{1/(4^{b-1}b^3\kappa^2)-\varepsilon}},$$

provided that $|E_1 - E_2|$ is sufficiently small.

Corollary 3.22. *Assume S is self-adjoint. Let $H(x)$ be given by* (43). *Then, for almost every $\omega \in \mathbb{R}$ the following is true. For any $\varepsilon > 0$, there exists $\lambda_0 = \lambda_0(\varepsilon,\omega,\rho,\gamma,K,K_1,c_1,v)$ such that, for any $\lambda > \lambda_0$,*

$$|k(E_1)-k(E_2)| \le e^{-\left(\log \frac{1}{|E_1-E_2|}\right)^{1/(4^{b-1}b^3)-\varepsilon}},$$

provided that $|E_1 - E_2|$ is sufficiently small.

Assume S is taken to be the particular case $S = \Delta$. Let $b = 2$. In this case, by Corollary 3.22, $1/(4^{b-1}b^3) = \frac{1}{32}$. A bound $\frac{1}{24}$ was shown by Bourgain, Goldstein and Schlag [Bourgain et al. 2001]. By combining the arguments in that work with the proof of Corollary 3.22, we are able to improve the bound.

Corollary 3.23. *Assume S is self-adjoint. Let $b = 2$ and $H(x)$ be given by* (43). *Then, for almost every $\omega \in \mathbb{R}$ the following is true. For any $\varepsilon > 0$, there exists $\lambda_0 = \lambda_0(\varepsilon,\omega,\rho,\gamma,K,K_1,c_1,v)$ such that, for any $\lambda > \lambda_0$,*

$$|k(E_1)-k(E_2)| \le e^{-\left(\log \frac{1}{|E_1-E_2|}\right)^{1/18-\varepsilon}},$$

provided that $|E_1 - E_2|$ is sufficiently small.

3D. *Skew-shifts*: $d = b = 1$. Let P_b be the projection on the b-th coordinate of $\mathbb{T}^b$, namely,

$$P_b(x_1,x_2,\dots,x_b) = x_b,$$

where $(x_1,x_2,\dots,x_b) \in \mathbb{R}^b$. Define $H(x)$ on $\ell^2(\mathbb{Z})$,

$$H(x) = \lambda^{-1}\Delta + v(P_b(f^n(x)))\delta_{nn'}, \tag{44}$$

where v is analytic on $\mathbb{T}$ and f is the skew-shift on $\mathbb{T}^b$.

Theorem 3.24. *Let $H(x)$ be given by* (44). *Assume ω is strong Diophantine and $1-1/(2^{b-1}b)<\sigma<1$. Then there exists $\lambda_0=\lambda_0(v)$ such that for any $\varepsilon>0$, $\lambda>\lambda_0$ and large N, there exists $X_N\subset\mathbb{T}^b$ such that*

$$\mathrm{Leb}(X_N)\le e^{-N^{(\sigma-1)/2^{b-1}+1/4^{b-1}-\varepsilon}},$$

and for any $x\notin X_N$, we have

$$\|G_{[-N,N]}(E,x)\|\le e^{N^\sigma},$$

$$|G_{[N,-N]}(E,x;n,n')|\le e^{-\frac{1}{2}c_1|n-n'|}\quad \text{for } |n-n'|\ge \tfrac{1}{10}N.$$

Theorem 3.25. *Let ω be strong Diophantine and $H(x)$ be given by* (44). *Then there exists $\lambda_0=\lambda_0(v)$ such that, for any $\varepsilon>0$ and $\lambda>\lambda_0$,*

$$|k(E_1)-k(E_2)|\le e^{-\left(\log\frac{1}{|E_1-E_2|}\right)^{1/4^{b-1}-\varepsilon}}, \tag{45}$$

provided that $|E_1-E_2|$ is sufficiently small.

Remark 3.26. • Comparing to Theorems 3.19 and 3.21, there is no dimension (b^3) loss in the bounds of Theorems 3.24 and 3.25. This is because the potential v is defined on $\mathbb{T}$.

• The large-deviation theorem and the modulus of continuity of Lyapunov exponents (the IDS) without explicit bounds were obtained in [Tao 2019a].

• Let $b=2$. The constant in (45) becomes $1/(4^{b-1})=\frac{1}{4}$. It is possible to improve the bound from $\frac{1}{4}$ to $\frac{1}{3}$ by incorporating arguments in [Bourgain et al. 2001]. A weaker result was proved in [Tao 2019b], where a constant $\frac{1}{30}$ was obtained.

3E. *Shifts*: $d=b=2$. Assume v is analytic on $\mathbb{T}^2=(\mathbb{R}/\mathbb{Z})^2$. Let

$$f^n(x)=(x_1+n_1\omega_1,x_2+n_2\omega_2)\bmod\mathbb{Z}^2,$$

where $n=(n_1,n_2)\in\mathbb{Z}^2$, $\omega=(\omega_1,\omega_2)\in\mathbb{R}^2$ and $x=(x_1,x_2)\in\mathbb{T}^2$. Let $H(x)$ on $\ell^2(\mathbb{Z}^2)$ be given by

$$H(x)=\lambda^{-1}S(x)+v(f^n(x))\delta_{nn'}=\lambda^{-1}S(x_1,x_2)+v(x_1+n_1\omega_1,x_2+n_2\omega_2)\delta_{nn'}. \tag{46}$$

Theorem 3.27. *Let $H(x)$ be given by* (46). *Suppose v is nonconstant on any line segment contained in $[0,1)^2$, $\omega_1\in\mathrm{DC}(\kappa,\tau)$ and $\omega_2\in\mathrm{DC}(\kappa,\tau)$ with $1\le\kappa<\frac{13}{12}$. Assume*

$$3\kappa-\tfrac{9}{4}<\sigma<1.$$

Then there exists $\lambda_0=\lambda_0(\varepsilon,\kappa,\tau,\rho,\sigma,\gamma,K,K_1,c_1,v)$ such that for any $\lambda>\lambda_0$ and any N, there exists $X_N\subset\mathbb{T}^2$ such that, for any line segment $L\subset[0,1)^2$,

$$\mathrm{Leb}(X_N\cap L)\le e^{-N^{(\sigma-1)(13/4-3\kappa)+(13/4-3\kappa)^2-\varepsilon}}, \tag{47}$$

and, for any $x\notin X_N$ and $Q_N\in\mathcal{E}_N^0$, we have

$$\|G_{Q_N}(E,x)\|\le e^{N^\sigma},$$

$$|G_{Q_N}(E,x;n,n')|\le e^{-\frac{1}{2}c_1|n-n'|}\quad \text{for } |n-n'|\ge\tfrac{1}{10}N.$$

Theorem 3.28. *Assume S is self-adjoint, v is nonconstant on any line segment contained in* $[0,1)^2$, $\omega_1 \in \mathrm{DC}(\kappa,\tau)$ *and* $\omega_2 \in \mathrm{DC}(\kappa,\tau)$ *with* $1 \le \kappa < \frac{13}{12}$. *Let $H(x)$ be given by* (46). *Then, for any ε, there exists $\lambda_0 = \lambda_0(\varepsilon,\kappa,\tau,\rho,\gamma,K,K_1,c_1,v)$ such that, for any $\lambda > \lambda_0$,*

$$|k(E_1)-k(E_2)| \le e^{-\left(\log \frac{1}{|E_1-E_2|}\right)^{(13/4-3\kappa)^2-\varepsilon}},$$

provided that $|E_1-E_2|$ is sufficiently small.

Corollary 3.29. *Assume S is self-adjoint and v is nonconstant on any line segments contained in* $[0,1)^2$. *Let $H(x)$ be given by* (46). *Then, for almost every $\omega \in \mathbb{R}^2$, the following is true. For any $\varepsilon > 0$, there exists $\lambda_0 = \lambda_0(\varepsilon,\omega,\rho,\gamma,K,K_1,c_1,v)$ such that, for any $\lambda > \lambda_0$,*

$$|k(E_1)-k(E_2)| \le e^{-\left(\log \frac{1}{|E_1-E_2|}\right)^{1/16-\varepsilon}},$$

provided that $|E_1-E_2|$ is sufficiently small.

Remark 3.30. Theorems 3.27 and 3.28 follow from the arguments in [Bourgain and Kachkovskiy 2019]. Our quantitative approaches developed in the paper allow us to obtain the explicit bounds.

3F. ***Subexponentially decaying matrices with interactions.*** Our applications can be wider. Here are several examples. Instead of (26), assume

$$|S(x;n,n')| \le K e^{-c_1|n-n'|^{\tilde\sigma}}, \quad 0<\tilde\sigma\le 1,\ c_1>0, \tag{48}$$

for any $n,n' \in \mathbb{Z}^d$.

Assume for any $N>1$, $n,n' \in \mathbb{Z}^d$, with $|n|\le N$ and $|n'|\le N$, there exists a trigonometric polynomial $\widetilde{S}(x;n,n')$ of degree less than e^{N^a} such that

$$\sup_{x\in\mathbb{T}^b} |S(x;n,n')-\widetilde{S}(x;n,n')| \le K e^{-N^2}. \tag{49}$$

In this subsection, assume S satisfies (27), (48) and (49).

Let $\widetilde{U}$ be a diagonal matrix on $\ell^2(\mathbb{Z}^d)$ satisfying

$$\|U\| \le K.$$

Given $m \in \mathbb{Z}^d$, define the diagonal matrix $\widetilde{U}^m$ on $\ell^2(\mathbb{Z}^d)$ by

$$\widetilde{U}^m(n) = \widetilde{U}(m+n), \quad n \in \mathbb{Z}^d.$$

We say $\widetilde{U}$ has low complexity if there exists $0<a<1$ such that, for any $N>1$,

$$\#\{R_{Q_N} U^m(n)\delta_{nn'} R_{Q_N} : m \in \mathbb{Z}^d,\ Q_N \in \mathcal{E}_N^0\} \le K e^{N^a}. \tag{50}$$

Let

$$\widetilde{H}(x) = H(x) + \lambda^{-1}U + \widetilde{U} = \lambda^{-1}(S+U) + (\widetilde{U}(n) + v(f(n,x)))\delta_{nn'}. \tag{51}$$

For any $m \in \mathbb{Z}^d$, let

$$\widetilde{H}^m(x) = H(x) + \lambda^{-1}U^m + \widetilde{U}^m = \lambda^{-1}(S + U^m) + (\widetilde{U}^m + v(f(n,x)))\delta_{nn'}. \tag{52}$$

Denote by $\widetilde{G}^m$ the Green's function of $\widetilde{H}^m$.

Theorem 3.31. *Assume α is strong Diophantine, and U and $\widetilde{U}$ have low complexity in the sense of* (19) *and* (50) *respectively. Assume*

$$1 - \frac{1}{b} < \sigma < \tilde{\sigma} \quad \text{and} \quad a \le \frac{1}{4}\left\{\frac{1}{K_1}, \frac{\sigma - 1}{b^2} + \frac{1}{b^3}\right\}.$$

Let $H(x)$ and $\widetilde{H}^m(x)$ be given by (38) *and* (52) *respectively. Then, for any $\varepsilon > 0$, there exists*

$$\lambda_0 = \lambda_0(\varepsilon, \alpha, \rho, c_1, \sigma, \tilde{\sigma}, \gamma, K, K_1, c_1, v)$$

such that, for any $\lambda > \lambda_0$ and any N, there exists $X_N \subset \mathbb{T}^b$ such that

$$\mathrm{Leb}(X_N) \le e^{-N^{(\sigma-1)/b^2+1/b^3-\varepsilon}},$$

and, for any $x \notin X_N$ and $m \in \mathbb{Z}$, we have

$$\|\widetilde{G}^m_{[-N,N]}(E,x)\| \le e^{N^\sigma},$$
$$|\widetilde{G}^m_{[-N,N]}(E,x;n,n')| \le e^{-\frac{c}{2}|n-n'|} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N,$$

where $c = (5^{\tilde{\sigma}} - 1)/5^{\tilde{\sigma}}$.

Theorem 3.32. *Assume $\omega \in \mathrm{DC}(\kappa, \tau)$, and U and $\widetilde{U}$ have low complexity. Assume $0 < \sigma < \tilde{\sigma} \le 1$ and $a \le \frac{1}{4}\min\{1/K_1, \sigma\}$. Let $H(x)$ and $\widetilde{H}^m(x)$ be given by* (39) *and* (52) *respectively. Then, for any $\varepsilon > 0$, there exists*

$$\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \sigma, \tilde{\sigma}, \rho, \gamma, K, K_1, c_1, v)$$

such that, for any $\lambda > \lambda_0$ and any N, there exists $X_N \subset \mathbb{T}$ such that

$$\mathrm{Leb}(X_N) \le e^{-N^{\sigma-\varepsilon}}, \tag{53}$$

and, for any $x \notin X_N$, any $m \in \mathbb{Z}^d$ and any $Q_N \in \mathcal{E}^0_N$, we have

$$\|\widetilde{G}^m_{Q_N}(E,x)\| \le e^{N^\sigma},$$
$$|\widetilde{G}^m_{Q_N}(E,x;n,n')| \le e^{-\frac{c}{2}|n-n'|^{\tilde{\sigma}}} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N, \tag{54}$$

where $c = (5^{\tilde{\sigma}} - 1)/5^{\tilde{\sigma}}$.

Theorem 3.33. *Assume S is self-adjoint, $\omega \in \mathrm{DC}(\kappa, \tau)$ and $a \le \frac{1}{4}\min\{1/K_1, \tilde{\sigma}\}$. Let $H(x)$ be given by* (39)*. Then, for any $\varepsilon > 0$, there exists*

$$\lambda_0 = \lambda_0(\varepsilon, \kappa, \tau, \tilde{\sigma}, \rho, \gamma, K, K_1, c_1, v)$$

such that, for any $\lambda > \lambda_0$,

$$|k(E_1) - k(E_2)| \le e^{-\left(\log \frac{1}{|E_1 - E_2|}\right)^{1-\varepsilon}},$$

provided that $|E_1 - E_2|$ *is sufficiently small.*

Using Theorem 2.8 instead of Theorem 2.7, the proofs of Theorems 3.31, 3.32 and 3.33 follow from that of Theorems 3.11, 3.15 and 3.16 respectively. In order to avoid repetitions, we skip the details.

4. Multiscale analysis

4A. ***Exhaustion construction for an elementary region.*** For $m \in \mathbb{Z}^d$ and $\Lambda \subset \mathbb{Z}^d$, define the distance by

$$\mathrm{dist}(m, \Lambda) = \inf_{n \in \Lambda} |m - n|.$$

Fix an elementary region $\Lambda \in \mathcal{E}_N$. Let $x \in \Lambda$. Given $M \le \frac{1}{10} N$, we will construct the exhaustion at x with width M. Set

$$\widetilde{S}_0(x) = (x + [-2M, 2M]^d) \cap \Lambda$$
$$\widetilde{S}_j(x) = \bigcup_{y \in S_{j-1}(x)} (y + [-4M, 4M]^d) \cap \Lambda, \quad 1 \le j \le \tilde{l},$$

where $\tilde{l}$ is the minimum such that $\widetilde{S}_{\tilde{l}}(x) = \Lambda$. We set $S_{-1}(x) = \varnothing$ for convenience.

When $\widetilde{S}_{j-1}(x)$ is very close to the boundary of Λ, $\tilde{A}_j(x) = \widetilde{S}_j(x) \backslash \widetilde{S}_{j-1}(x)$ and $\widetilde{S}_j$ may have width less than M. However, there are at most finitely many j with $0 \le j \le \tilde{l}$, say $C(d)$, such that $\tilde{A}_j(x) = \widetilde{S}_j(x) \backslash \widetilde{S}_{j-1}(x)$ has width less than M, where $C(d)$ is a constant depending on d.

We will delete j if $\tilde{A}_j(x) = \widetilde{S}_j(x) \backslash \widetilde{S}_{j-1}(x)$ has small width and then rearrange the exhaustion. Here are the details. Let $j_0 \in \{0, 1, \dots, \tilde{l} - 1\}$ be the smallest possible number such that both $\widetilde{S}_{j_0}(x)$ and $\widetilde{S}_{\tilde{l}}(x) \backslash \widetilde{S}_{j_0}(x)$ have width at least M. Otherwise, set $j_0 = \tilde{l}$. Let $S_0(x) = \widetilde{S}_{j_0}(x)$. Let $j_1 \in \{j_0, j_0 + 1, \dots, \tilde{l} - 1\}$ be the smallest possible number such that both $\widetilde{S}_{j_1}(x) \backslash \widetilde{S}_{j_0}(x)$ and $\widetilde{S}_{\tilde{l}}(x) \backslash \widetilde{S}_{j_1}(x)$ have width at least M. Otherwise, set $j_1 = \tilde{l}$. Let $S_1(x) = \widetilde{S}_{j_1}(x)$. Suppose we have defined $j_0, j_1 \cdots, j_k$ and corresponding $S_1(x), S_2(x), \dots S_k(x)$. Let $j_{k+1} \in \{j_k, j_k + 1, \dots, \tilde{l} - 1\}$ be the smallest possible number such that $\widetilde{S}_{j_{k+1}}(x) \backslash \widetilde{S}_{j_k}(x)$ and $\widetilde{S}_{\tilde{l}}(x) \backslash \widetilde{S}_{j_{k+1}}(x)$ have width at least M. Otherwise, set $j_{k+1} = \tilde{l}$. Let $S_{k+1}(x) = \widetilde{S}_{j_{k+1}}(x)$. Let l be such that $S_l(x) = \Lambda$. By our constructions, $\tilde{l} - C(d) \le l \le \tilde{l}$.

For example, assume x is located exactly at the uppermost left corner. In Figure 3, $\tilde{A}_k(x) = \widetilde{S}_k(x) \backslash \widetilde{S}_{k-1}(x)$ and $\widetilde{S}_{\tilde{l}}(x) = \widetilde{S}_{\tilde{l}}(x) \backslash \widetilde{S}_{\tilde{l}-1}(x)$ are the only two annuli which have width less than M. Therefore,

- $l = \tilde{l} - 2$,
- $S_j(x) = \widetilde{S}_j(x)$ for $j = 0, 1, 2, \dots, k-2$,
- $S_j(x) = \widetilde{S}_{j+1}(x)$ for $j = k-1, k-2, \dots, l-3$,
- $S_{l-2}(x) = \widetilde{S}_{\tilde{l}}(x)$.

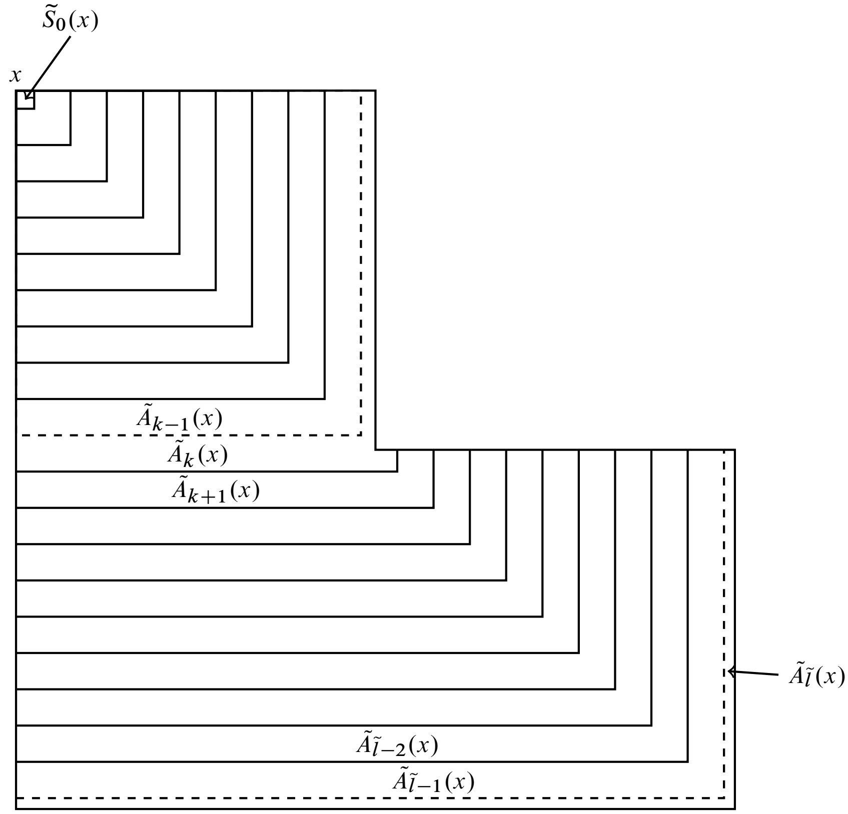

Figure 3. Exhaustion construction.

For any elementary region Λ, $x \in \Lambda$ and M, we call $\{S_j(x)\}_{j=0}^{l}$ the exhaustion of Λ at x with width M. We call $A_j(x) = S_j(x)\backslash S_{j-1}(x)$ the j-th annulus. For any $y \in S_j(x)\backslash S_{j-1}(x)$, $j = 1, 2, \ldots, l$, one has

$$4(j-1)M \le |y-x| \le 4jM + C(d)M. \tag{55}$$

By our constructions, any $\{A_j(x)\}$ has width at least M. Namely, for any $n \in A_j(x)$, there exists $W(n) \in \mathcal{E}_M$ such that

$$n \in W(n) \subset A_j(x)$$

and

$$\operatorname{dist}(n, A_j(x)\backslash W(n)) \ge \tfrac{1}{2}M.$$

4B. ***Resolvent identities.*** For simplicity, assume $K = 1$, namely, for any $n \ne n'$,

$$|A(n,n')| \le e^{-c_1|n-n'|^{\tilde{\sigma}}}, \quad 0 < \tilde{\sigma} \le 1,\ c_1 > 0, \tag{56}$$

for any $n, n' \in \mathbb{Z}^d$. For any $\Lambda \subset \mathbb{Z}^d$, let $A_\Lambda = R_\Lambda A R_\Lambda$, where R_Λ is the restriction on Λ, and the Green's function is given by

$$G_\Lambda = (R_\Lambda A R_\Lambda)^{-1},$$

provided $R_\Lambda A R_\Lambda$ is invertible. Denote by $G_\Lambda(n,n')$ its elements, $n, n' \in \Lambda \subset \mathbb{Z}^d$.

Assume Λ_1 and Λ_2 are two disjoint subsets of $\mathbb{Z}^d$. Namely, $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ and $\Lambda_1 \cap \Lambda_2 = \varnothing$. Let $\Lambda = \Lambda_1 \cup \Lambda_2$. Suppose that $R_\Lambda A R_\Lambda$ and $R_{\Lambda_i} A R_{\Lambda_i}$, $i = 1, 2$, are invertible. Then

$$G_\Lambda = G_{\Lambda_1} + G_{\Lambda_2} - (G_{\Lambda_1} + G_{\Lambda_2})(A_\Lambda - A_{\Lambda_1} - A_{\Lambda_2})G_\Lambda.$$

If $m \in \Lambda_1$ and $n \in \Lambda$, we have

$$|G_\Lambda(m,n)| \le |G_{\Lambda_1}(m,n)|\chi_{\Lambda_1}(n) + \sum_{n' \in \Lambda_1, n'' \in \Lambda_2} e^{-c_1|n'-n''|^{\tilde{\sigma}}} |G_{\Lambda_1}(m,n')||G_\Lambda(n'',n)|. \tag{57}$$

If $n \in \Lambda_2$ and $m \in \Lambda$, we have

$$|G_\Lambda(m,n)| \le |G_{\Lambda_2}(m,n)|\chi_{\Lambda_2}(n) + \sum_{n' \in \Lambda_1, n'' \in \Lambda_2} e^{-c_1|n'-n''|^{\tilde{\sigma}}} |G_\Lambda(m,n')||G_{\Lambda_2}(n'',n)|. \tag{58}$$

Lemma 4.1 (Schur test). *Suppose $A = A_{ij}$ is a matrix. Then*

$$\|A\| \le \sqrt{\Big(\sup_i \sum_j |A_{ij}|\Big)\Big(\sup_j \sum_i |A_{ij}|\Big)}.$$

The following lemma is a generalization of Lemma 3.2 in [Jitomirskaya et al. 2020b].

Lemma 4.2. *Let $c_2 \in [\tilde{c}_1, c_1]$, $\sigma < \tilde{\sigma}$ and $M_0 \le M_1 \le N$. Assume Λ is a subset of $\mathbb{Z}^d$ with* $\operatorname{diam}(\Lambda) \le 2N + 1$. *Suppose that, for any $n \in \Lambda$, there exists some $W = W(n) \in \mathcal{E}_M$ with $M_0 \le M \le M_1$ such that $n \in W \subset \Lambda$,* $\operatorname{dist}(n, \Lambda \backslash W) \ge \frac{1}{2}M$ *and*

$$\|G_{W(n)}\| \le 2e^{M^\sigma}, \tag{59}$$

$$|G_{W(n)}(n,n')| \le 2e^{-c_2|n-n'|^{\tilde{\sigma}}} \quad \text{for } |n-n'| \ge \tfrac{1}{10}M. \tag{60}$$

We assume further that M_0 is large enough so that

$$\sup_{M_0 \le M \le M_1} \sup_{c_2 \in [\tilde{c}_1, c_1]} 2e^{M^\sigma}(2M+1)^d e^{(c_2/10^{\tilde{\sigma}})M^{\tilde{\sigma}}} \sum_{j=0}^{\infty} (M+2j+1)^d e^{-c_2(j+M/2)^{\tilde{\sigma}}} \le \tfrac{1}{2}. \tag{61}$$

Then

$$\|G_\Lambda\| \le 4(2M_1+1)^d e^{M_1^\sigma}.$$

Proof. Under the assumption of (61), it is easy to check that for any M with $M_0 \le M \le M_1$ and any $n \in \Lambda$,

$$2(2M+1)^d e^{M^\sigma + (c_2/10^{\tilde{\sigma}})M^{\tilde{\sigma}}} \sum_{\substack{n_2 \in \Lambda \\ |n_2 - n| \ge M/2}} e^{-c_2|n-n_2|^{\tilde{\sigma}}} \le \tfrac{1}{2}. \tag{62}$$

By (59) and (60), one has

$$|G_{W(n)}(n,n')| \le 2e^{M^\sigma + (c_2/10^{\tilde{\sigma}})M^{\tilde{\sigma}}} e^{-c_2|n-n'|^{\tilde{\sigma}}}. \tag{63}$$

For each $n \in \Lambda$, applying (57) with $\Lambda_1 = W(n)$, one has

$$|G_\Lambda(n,n')| \le |G_{W(n)}(n,n')|\chi_{W(n)}(n') + \sum_{\substack{n_1 \in W(n) \\ n_2 \in \Lambda \backslash W(n)}} e^{-c_1|n_1-n_2|^{\tilde{\sigma}}} |G_{W(n)}(n,n_1)||G_\Lambda(n_2,n')|.$$

It is easy to see, for $0 < \tilde\sigma \le 1$,

$$|x+y|^{\tilde\sigma} \le |x|^{\tilde\sigma} + |y|^{\tilde\sigma}. \tag{64}$$

By (63) and the fact that $|W(n)| \le (2M+1)^d$, one has

$$\begin{aligned}
|G_\Lambda(n,n')| &\le |G_{W(n)}(n,n')|\chi_{W(n)}(n') \\
&\qquad + 2\sum_{\substack{n_1\in W(n)\\ n_2\in\Lambda\backslash W(n)}} e^{M^\sigma + (c_2/10^{\tilde\sigma})M^{\tilde\sigma}} e^{-c_2|n-n_1|^{\tilde\sigma}} e^{-c_1|n_1-n_2|^{\tilde\sigma}} |G_\Lambda(n_2,n')| \\
&\le |G_{W(n)}(n,n')|\chi_{W(n)}(n') \\
&\qquad + 2(2M+1)^d e^{M^\sigma + (c_2/10^{\tilde\sigma})M^{\tilde\sigma}} \sum_{n_2\in\Lambda\backslash W(n)} e^{-c_2|n-n_2|^{\tilde\sigma}} |G_\Lambda(n_2,n')| \\
&\le |G_{W(n)}(n,n')|\chi_{W(n)}(n') \\
&\qquad + 2(2M+1)^d e^{M^\sigma + (c_2/10^{\tilde\sigma})M^{\tilde\sigma}} \sum_{\substack{n_2\in\Lambda\\ |n_2-n|\ge M/2}} e^{-c_2|n-n_2|^{\tilde\sigma}} |G_\Lambda(n_2,n')|.
\end{aligned} \tag{65}$$

where the second inequality holds by (64) and the last holds by the assumption $\operatorname{dist}(n,\Lambda\backslash W(n)) \ge \frac{1}{2}M$.

Summing over $n'\in\Lambda$ in (65) and noticing (62) yields

$$\sup_{n\in\Lambda}\sum_{n'\in\Lambda} |G_\Lambda(n,n')| \le 2(2M_1+1)^d e^{M_1^\sigma} + \tfrac{1}{2}\sup_{n_2\in\Lambda}\sum_{n'\in\Lambda}|G_\Lambda(n_2,n')|. \tag{66}$$

Similarly, using (58) instead of (57), one has

$$\sup_{n\in\Lambda}\sum_{n'\in\Lambda} |G_\Lambda(n',n)| \le 2(2M_1+1)^d e^{M_1^\sigma} + \tfrac{1}{2}\sup_{n_2\in\Lambda}\sum_{n'\in\Lambda}|G_\Lambda(n',n_2)|. \tag{67}$$

Now the lemma follows from (66), (67) and Lemma 4.1. □

4C. *Proof of Theorem 2.3.* Choose a constant ρ with $\rho^2 \in (1, 1+\tilde\sigma-\sigma)$. Calculation shows $\rho^2\sigma < \tilde\sigma$.

Define inductively $M_{j+1} = \lfloor M_j^\rho \rfloor$, $M_0 = M$. Let $\gamma_0 = c_2$. Fix an elementary region $\widetilde\Lambda_1 \in \mathcal{E}_{M_1}$ and $\widetilde\Lambda_1 \subset \widetilde\Lambda_0$. For any $x\in\widetilde\Lambda_1$, consider the exhaustion $\{S_j(x)\}_{j=0}^{l}$ of $\widetilde\Lambda_1$ at x with width M_0. Denote by $\{A_k(x)\}$ the annuli.

We call the annulus $A_k(x)$ good if, for any $y\in A_k(x)$, there exists $W(y)\in\mathcal{E}_{M_0}$ such that

$$y\in W(y)\subset A_k(x), \quad \operatorname{dist}(y, A_k(x)\backslash W(y)) \ge \tfrac{1}{2}M_0,$$

and, for $|n-n'| \ge \frac{1}{10}M_0$,

$$|(R_{W(y)}AR_{W(y)})^{-1}(n,n')| \le e^{-\gamma_0|n-n'|^{\tilde\sigma}}. \tag{68}$$

Otherwise, we call the annulus A_k bad.

Fix $\kappa > 0$, which will be determined later. An elementary region $\widetilde\Lambda_1 \subset \widetilde\Lambda_0$ is called bad if there exists $x\in\widetilde\Lambda_1$ such that the number of bad annuli $\{A_k(x)\}$ exceeds

$$B_1 := \kappa\frac{M_1}{M_0}.$$

Otherwise, we call $\widetilde{\Lambda}_1$ good. Let $\mathcal{F}_1$ be an arbitrary family of pairwise disjoint bad elementary regions in $\mathcal{E}_{M_1}$ contained in $\widetilde{\Lambda}_0$. Since every annulus in $\{A_k\}$ has width at least M_0 by our construction, one has that every bad annulus contains at least one elementary region in $\mathcal{E}_{M_0}$ without satisfying (68) and hence

$$\#\mathcal{F}_1 \leq \frac{N^{\varsigma}}{\kappa M_1}. \tag{69}$$

Assume that $\widetilde{\Lambda}_1 \subset \widetilde{\Lambda}_0$ is a good elementary region in $\mathcal{E}_{M_1}$. We will first show that $\widetilde{\Lambda}_1$ is in class G with slightly smaller γ_0. Consider the exhaustion $\{S_j(x)\}$ of $\widetilde{\Lambda}_1$ at x with width M_0. By the assumption, there are no more than B_1 bad annuli in this exhaustion. Denote by $\{A_j(x)\}_{j=0}^{l}$ annuli. By putting adjacent good annuli or bad annuli together, we obtain a new exhaustion

$$\varnothing = J_{-1} \subset J_0 \subset J_1 \subset \cdots \subset J_g = \widetilde{\Lambda}_1. \tag{70}$$

More precisely, $\{J_s(x)\},\ s = 0, 1, 2, \ldots, g$, satisfies the following rules:

- $J_s(x)\backslash J_{s-1}(x) = \{A_j(x)\}_{j=t_s}^{j=t'_s}$ for some $t_s < t'_s$.
- $x \in J_0(x)$.
- The annuli $A_j(x),\ j = t_s, t_s + 1, \ldots, t'_s$, are either all good or all bad.
- Take $J_s(x)$ maximal with the above three properties.

Recall that $J_s(x)\backslash J_{s-1}(x)$ has width at least M_0 for any $s = 0, 1, 2, \ldots, g$. By our construction, if all annuli in $J_s(x)\backslash J_{s-1}(x)$ are good (bad), then all annuli in $J_{s+1}(x)\backslash J_s(x)$ are bad (good).

For any $n \in \widetilde{\Lambda}_1$, let $k(n)$ be the number of good annuli between x and n. Namely, for any $n \in A_j(x)$,

$$k(n) = \#\{A_t(x) : A_t(x) \text{ is a good annulus}, 0 \leq t \leq j\}.$$

Before we start the estimates, let us give several facts, which will be used constantly later in the proof. By our constructions, J_s is a generalized elementary region, $s = 0, 1, \ldots, g$. By the assumption (4), one has, for all $s = 0, 1, \ldots, g$,

$$(R_{J_s} A R_{J_s})^{-1} \leq e^{M_1^{\sigma}}. \tag{71}$$

Assume

$$0 < c \leq (1 - 5^{-\tilde{\sigma}})c_1.$$

If $|n - n_2| \geq \frac{1}{2}M$ and $|n - n_1| \leq \frac{1}{10}M$, one has

$$c_1|n_1 - n_2|^{\tilde{\sigma}} \geq c_1(|n - n_2|^{\tilde{\sigma}} - |n - n_1|^{\tilde{\sigma}}) \geq c|n - n_2|^{\tilde{\sigma}}. \tag{72}$$

It is clear that, for any $n_1, n_2 \in \widetilde{\Lambda}_1$,

$$4M_0k(n_1) + |n_1 - n_2| \geq 4M_0k(n_2) - 4M_0. \tag{73}$$

Without loss of generality, assume all the annuli in J_0 are bad (the other case is similar).

For any $n \in \widetilde{\Lambda}_1$, define

$$\Gamma_s(n) = \max\{4M_0k(n) - 10(s + 1)M_0, 0\}.$$

By (73), we have, for any $n_1 \in \widetilde{\Lambda}_1$ and $n_2 \in \widetilde{\Lambda}_1$,

$$\Gamma_s(n_1) + |n_2 - n_1| \geq \max\{\Gamma_s(n_2) - 4M_0, 0\}. \tag{74}$$

We shall inductively obtain estimates of the form

$$|G_{J_s(x)}(x,z)| \leq T_s e^{-\gamma_0 \Gamma_s^{\tilde{\sigma}}(z)}, \tag{75}$$

where $z \in J_s,\ s = 0, 1, \ldots, g$.

First step: $s = 0$. Since all annuli in J_0 are bad, one has $k(z) = 0$ and hence

$$\Gamma_0(z) = 0. \tag{76}$$

By (71) and (76), one has, for $z \in J_0(x)$,

$$|G_{J_s}(x,z)| \leq e^{M_1^{\sigma}} = e^{M_1^{\sigma}} e^{-\gamma_0 \Gamma_0^{\tilde{\sigma}}(z)}.$$

It implies that (75) holds for

$$T_0 = e^{M_1^{\sigma}}. \tag{77}$$

Assume (75) holds at s-th step for a proper T_s.

Case 1: All annuli in $J_{s+1} \backslash J_s$ are bad. Pick any $z \in J_{s+1}$. Let $\tilde{n}_1 \in J_s$ and $\tilde{n}_2 \in J_{s+1} \backslash J_s$ be such that

$$\Gamma_s(\tilde{n}_1) + |\tilde{n}_1 - \tilde{n}_2| = \inf_{\substack{n_1 \in J_s \\ n_2 \in J_{s+1} \backslash J_s}} (\Gamma_s(n_1) + |n_1 - n_2|).$$

Case 1_1: $z \in J_{s+1} \backslash J_s$. In this case, for any $n_2 \in J_{s+1} \backslash J_s$, one has

$$k(z) = k(n_2), \quad \Gamma_s(z) = \Gamma_s(n_2), \tag{78}$$

since all annuli in $J_{s+1} \backslash J_s$ are bad.

Applying (57) ($\Lambda_1 = J_s$ and $\Lambda_2 = J_{s+1} \backslash J_s$), one has

$$\begin{aligned}
|G_{J_{s+1}}(x,z)| &\leq \sum_{\substack{n_1 \in J_s \\ n_2 \in J_{s+1} \backslash J_s}} |G_{J_s}(x,n_1)| e^{-c_1 |n_1 - n_2|^{\tilde{\sigma}}} |G_{J_{s+1}}(n_2, z)| \\
&\leq \sum_{\substack{n_1 \in J_s \\ n_2 \in J_{s+1} \backslash J_s}} T_s e^{-\gamma_0 \Gamma_s^{\tilde{\sigma}}(n_1)} e^{-\gamma_0 |n_1 - n_2|^{\tilde{\sigma}}} |G_{J_{s+1}}(n_2, z)| \\
&\leq e^{M_1^{\sigma}} T_s \sum_{\substack{n_1 \in J_s \\ n_2 \in J_{s+1} \backslash J_s}} e^{-\gamma_0 \Gamma_s^{\tilde{\sigma}}(n_1)} e^{-\gamma_0 |n_1 - n_2|^{\tilde{\sigma}}} \\
&\leq (2M_1 + 1)^{2d} e^{M_1^{\sigma}} T_s \sup_{\substack{n_1 \in J_s \\ n_2 \in J_{s+1} \backslash J_s}} e^{-\gamma_0 \Gamma_s^{\tilde{\sigma}}(n_1)} e^{-\gamma_0 |n_1 - n_2|^{\tilde{\sigma}}} \\
&\leq (2M_1 + 1)^{2d} e^{M_1^{\sigma}} T_s e^{-\gamma_0 (\Gamma_s(\tilde{n}_1) + |\tilde{n}_1 - \tilde{n}_2|)^{\tilde{\sigma}}} \\
&\leq (2M_1 + 1)^{2d} e^{M_1^{\sigma}} T_s e^{-\gamma_0 (\max\{\Gamma_s(\tilde{n}_2) - 4M_0, 0\})^{\tilde{\sigma}}} \\
&\leq (2M_1 + 1)^{2d} e^{M_1^{\sigma}} T_s e^{-\gamma_0 (\max\{\Gamma_s(z) - 4M_0, 0\})^{\tilde{\sigma}}},
\end{aligned} \tag{79}$$

where the second inequality holds by the induction (75) and $\gamma_0 \le c_1$, the third inequality holds by (71), the fifth inequality holds by (64), the sixth inequality holds (74), and the last inequality holds by (78).

Case 1_2: $z \in J_s$. In this case, we have, for any $n_2 \in J_{s+1} \backslash J_s$,

$$k(n_2) \ge k(z). \tag{80}$$

Applying (57) ($\Lambda_1 = J_s$ and $\Lambda_2 = J_{s+1} \backslash J_s$), one has

$$\begin{aligned}
|G_{J_{s+1}}(x,z)| &\le |G_{J_s}(x,z)| + \sum_{\substack{n_1 \in J_s \\ n_2 \in J_{s+1} \backslash J_s}} |G_{J_s}(x,n_1)| e^{-c_1 |n_1 - n_2|^{\tilde{\sigma}}} |G_{J_{s+1}}(n_2,z)| \\
&\le T_s e^{-\gamma_0 \Gamma_s^{\tilde{\sigma}}(z)} + \sum_{\substack{n_1 \in J_s \\ n_2 \in J_{s+1} \backslash J_s}} T_s e^{-\gamma_0 \Gamma_s^{\tilde{\sigma}}(n_1)} e^{-\gamma_0 |n_1 - n_2|^{\tilde{\sigma}}} |G_{J_{s+1}}(n_2,z)| \\
&\le T_s e^{-\gamma_0 \Gamma_s^{\tilde{\sigma}}(z)} + (2M_1+1)^{2d} e^{M_1^{\sigma}} T_s \sup_{\substack{n_1 \in J_s \\ n_2 \in J_{s+1} \backslash J_s}} e^{-\gamma_0 (\Gamma_s(n_1) + |n_1 - n_2|)^{\tilde{\sigma}}} \\
&= T_s e^{-\gamma_0 \Gamma_s^{\tilde{\sigma}}(z)} + (2M_1+1)^{2d} e^{M_1^{\sigma}} T_s e^{-\gamma_0 (\Gamma_s(\tilde{n}_1) + |\tilde{n}_1 - \tilde{n}_2|)^{\tilde{\sigma}}} \\
&\le T_s e^{-\gamma_0 \Gamma_s^{\tilde{\sigma}}(z)} + (2M_1+1)^{2d} e^{M_1^{\sigma}} T_s e^{-\gamma_0 (\max\{\Gamma_s(\tilde{n}_2) - 4M_0, 0\})^{\tilde{\sigma}}} \\
&\le T_s e^{-\gamma_0 \Gamma_s^{\tilde{\sigma}}(z)} + (2M_1+1)^{2d} e^{M_1^{\sigma}} T_s e^{-\gamma_0 (\max\{\Gamma_s(z) - 4M_0, 0\})^{\tilde{\sigma}}} \\
&\le 2(2M_1+1)^{2d} e^{M_1^{\sigma}} T_s e^{-\gamma_0 (\max\{\Gamma_s(z) - 4M_0, 0\})^{\tilde{\sigma}}},
\end{aligned} \tag{81}$$

where the second inequality holds by the induction (75), the third inequality holds by (71) and (64), the forth inequality holds by (74), and the fifth inequality holds by (80).

Case 2: All the annuli in $J_{s+1} \backslash J_s$ are good. By our constructions, $J_{s+1} \backslash J_s$ has width at least M_0. Therefore, for any $k \in J_{s+1} \backslash J_s$, there exists some $W = W(k) \in \mathcal{E}_{M_0}$ such that $k \in W \subset \Lambda$,

$$\operatorname{dist}(k, J_{s+1} \backslash J_s \backslash W) \ge \tfrac{1}{2} M_0, \tag{82}$$

and

$$\|G_{W(k)}\| \le e^{M_0^{\sigma}}, \tag{83}$$

$$|G_{W(k)}(n_1,n_2)| \le e^{-\gamma_0 |n_1 - n_2|^{\tilde{\sigma}}} \quad \text{for } |n_1 - n_2| \ge \tfrac{1}{10} M_0, \tag{84}$$

where (83) holds by the assumption (4).

Since M_0 is large enough, one has (61) is satisfied. Applying Lemma 4.2, we have

$$\|G_{J_{s+1} \backslash J_s}\| \le 4(2M_0+1)^d e^{M_0^{\sigma}}. \tag{85}$$

We remark that we cannot use the assumption (4) to bound $G_{J_{s+1} \backslash J_s}$ since $J_{s+1} \backslash J_s$ is not necessary to be a generalized elementary region. It is worth pointing out that $J_{s+1} \backslash J_s$ may not be connected.

We will first prove that, for any $m, n \in J_{s+1} \backslash J_s$,

$$|G_{J_{s+1} \backslash J_s}(m,n)| \le M_1^{10 d M_1^{\tilde{\sigma}} M_0^{\sigma - \tilde{\sigma}}} e^{-\gamma_0 (\max\{|m-n| - 2M_0, 0\})^{\tilde{\sigma}}}. \tag{86}$$

Assume $|m-n| \le 2M_0$. (86) holds by (85).

Assume $|m-n|>2M_0$. Applying (57) with $\Lambda_1=W(m)$ and using that $|m-n|>2M_0$, one has

$$|G_{J_{s+1}\setminus J_s}(m,n)|\le\sum_{\substack{n_1\in W(m)\\ n_2\in J_{s+1}\setminus J_s\setminus W(m)}}e^{-c_1|n_1-n_2|^{\tilde\sigma}}|G_{W(m)}(m,n_1)||G_\Lambda(n_2,n)|.\tag{87}$$

Applying (82) with $k=m$ and by (72), one has, for any n_1 with $|n_1-m|\le\frac{1}{10}M_0$ and $n_2\in J_{s+1}\setminus J_s\setminus W(m)$,

$$c_1|n_1-n_2|^{\tilde\sigma}\ge c_2|m-n_2|^{\tilde\sigma}.\tag{88}$$

By (83), (84) and (87), we have

$$\begin{aligned}
|G_{J_{s+1}\setminus J_s}(m,n)|&\le\sum_{\substack{n_1\in W(m),|n_1-m|\le M_0/10-1\\ n_2\in J_{s+1}\setminus J_s\setminus W(m)}}e^{-c_1|n_1-n_2|^{\tilde\sigma}}|G_{W(m)}(m,n_1)||G_{J_{s+1}\setminus J_s}(n_2,n)|\\
&\quad+\sum_{\substack{n_1\in W(m),|n_1-m|\ge M_0/10\\ n_2\in J_{s+1}\setminus J_s\setminus W(m)}}e^{-c_1|n_1-n_2|^{\tilde\sigma}}|G_{W(m)}(m,n_1)||G_{J_{s+1}\setminus J_s}(n_2,n)|\\
&\le\sum_{\substack{n_1\in W(m),|n_1-m|\le M_0/10-1\\ n_2\in J_{s+1}\setminus J_s\setminus W(m)}}e^{M_0^\sigma}e^{-c_1|n_1-n_2|^{\tilde\sigma}}|G_{J_{s+1}\setminus J_s}(n_2,n)|\\
&\quad+\sum_{\substack{n_1\in W(m),|n_1-m|\ge M_0/10\\ n_2\in J_{s+1}\setminus J_s\setminus W(m)}}e^{-c_1|n_1-n_2|^{\tilde\sigma}}e^{-\gamma_0|m-n_1|^{\tilde\sigma}}|G_{J_{s+1}\setminus J_s}(n_2,n)|\\
&\le\sum_{\substack{n_1\in W(m),|n_1-m|\le M_0/10-1\\ n_2\in J_{s+1}\setminus J_s\setminus W(n_2)}}e^{M_0^\sigma}e^{-\gamma_0|m-n_2|^{\tilde\sigma}}|G_{J_{s+1}\setminus J_s}(n_2,n)|\\
&\quad+\sum_{\substack{n_1\in W(m),|n_1-m|\ge M_0/10\\ n_2\in J_{s+1}\setminus J_s\setminus W(m)}}e^{-\gamma_0|m-n_2|^{\tilde\sigma}}|G_{J_{s+1}\setminus J_s}(n_2,n)|\\
&\le(2M_1+1)^{2d}e^{M_0^\sigma}\sup_{n_2\in J_{s+1}\setminus J_s\setminus W(m)}e^{-\gamma_0|m-n_2|^{\tilde\sigma}}|G_{J_{s+1}\setminus J_s}(n_2,n)|,
\end{aligned}\tag{89}$$

where the third inequality holds because of (88).

Recall that $|m-n_2|\ge\frac12 M_0$. Iterating (89) until $|n_2-n|\le 2M_0$ or at most $\lfloor 2^{\tilde\sigma}|m-n|^{\tilde\sigma}/M_0^{\tilde\sigma}\rfloor+1$ times, we have

$$\begin{aligned}
|G_{J_{s+1}\setminus J_s}(m,n)|&\le e^{M_0^\sigma(2^{\tilde\sigma}|m-n|^{\tilde\sigma}/M_0^{\tilde\sigma}+1)}(2M_1+1)^{2d(2^{\tilde\sigma}|m-n|^{\tilde\sigma}/M_0^{\tilde\sigma}+1)}e^{-\gamma_0(|m-n|-2M_0)^{\tilde\sigma}}\|G_{J_{s+1}\setminus J_s}\|\\
&\le M_1^{9dM_1^{\tilde\sigma}M_0^{\sigma-\tilde\sigma}}e^{-\gamma_0(|m-n|-2M_0)^{\tilde\sigma}}\|G_{J_{s+1}\setminus J_s}\|\\
&\le M_1^{9dM_1^{\tilde\sigma}M_0^{\sigma-\tilde\sigma}}e^{-\gamma_0(|m-n|-2M_0)^{\tilde\sigma}}4(2M_0+1)^d e^{M_0^\sigma}\\
&\le M_1^{10dM_1^{\tilde\sigma}M_0^{\sigma-\tilde\sigma}}e^{-\gamma_0(|m-n|-2M_0)^{\tilde\sigma}},
\end{aligned}\tag{90}$$

where the first inequality holds by $|m-n|\le 2M_1$ and the third inequality holds by (85).

Case 2_1: $z\in J_s$. For this case, following the proof of Case 1_2 (see (81)), one has

$$|G_{J_{s+1}}(x,z)|\le 2(2M_1+1)^{2d}e^{M_1^\sigma}T_s e^{-\gamma_0(\max\{\Gamma_s(z)-4M_0,0\})^{\tilde\sigma}}.\tag{91}$$

Case 2_2: $z \in J_{s+1} \backslash J_s$. Applying (58) ($\Lambda_1 = J_s$ and $\Lambda_2 = J_{s+1}\backslash J_s$), one has

$$\begin{aligned}
|G_{J_{s+1}}(x,z)| &\leq \sum_{\substack{n_1 \in J_s\\ n_2 \in J_{s+1}\backslash J_s}} |G_{J_{s+1}\backslash J_s}(n_2,z)| e^{-c_1|n_1-n_2|^{\tilde\sigma}} |G_{J_{s+1}}(x,n_1)| \\
&\leq 2(2M_1+1)^{2d} e^{M_1^\sigma} T_s \\
&\qquad\qquad \times \sum_{\substack{n_1 \in J_s\\ n_2 \in J_{s+1}\backslash J_s}} |G_{J_{s+1}\backslash J_s}(n_2,z)| e^{-c_1|n_1-n_2|^{\tilde\sigma}} e^{-\gamma_0(\max\{\Gamma_s(n_1)-4M_0,0\})^{\tilde\sigma}} \\
&\leq 2(2M_1+1)^{4d} e^{M_1^\sigma} M_1^{10dM_1^{\tilde\sigma} M_0^{\sigma-\tilde\sigma}} T_s \\
&\qquad \times \sup_{\substack{n_1 \in J_s\\ n_2 \in J_{s+1}\backslash J_s}} e^{-\gamma_0(\max\{|n_2-z|-2M_0,0\})^{\tilde\sigma}} e^{-c_1|n_1-n_2|^{\tilde\sigma}} e^{-\gamma_0(\max\{\Gamma_s(n_1)-4M_0,0\})^{\tilde\sigma}} \\
&\leq M_1^{11dM_1^{\tilde\sigma} M_0^{\sigma-\tilde\sigma}} T_s \sup_{n_1 \in J_s} e^{-\gamma_0(\max\{|n_1-z|-2M_0,0\})^{\tilde\sigma}} e^{-\gamma_0(\max\{\Gamma_s(n_1)-4M_0,0\})^{\tilde\sigma}} \\
&\leq M_1^{11dM_1^{\tilde\sigma} M_0^{\sigma-\tilde\sigma}} T_s e^{-\gamma_0(\max\{\Gamma_s(z)-10M_0,0\})^{\tilde\sigma}},
\end{aligned} \tag{92}$$

where the second inequality holds by (91), the third inequality holds by (86), the fourth inequality holds by (64) and the fifth inequality holds by (64) and (74).

Putting all cases together and by (79), (81), (91) and (92), one has if (75) holds at the s-th step, then

$$|G_{J_{s+1}}(x,z)| \leq M_1^{12dM_1^{\tilde\sigma} M_0^{\sigma-\tilde\sigma}} T_s e^{-\gamma_0(\max\{\Gamma_s(z)-10M_0,0\})^{\tilde\sigma}} \leq M_1^{12dM_1^{\tilde\sigma} M_0^{\sigma-\tilde\sigma}} T_s e^{-\gamma_0\Gamma_{s+1}^{\tilde\sigma}(z)}. \tag{93}$$

By (77) and (93), we obtain that (75) is true for

$$T_0 = e^{M_1^\sigma}, \tag{94}$$

$$T_{s+1} = M_1^{12dM_1^{\tilde\sigma} M_0^{\sigma-\tilde\sigma}} T_s. \tag{95}$$

By (75), (94) and (95), one has

$$|G_{J_g}(x,z)| \leq M_1^{13gdM_1^{\tilde\sigma} M_0^{\sigma-\tilde\sigma}} e^{-\gamma_0\Gamma_g^{\tilde\sigma}(z)}. \tag{96}$$

By the assumption that $\widetilde\Lambda_1$ is good, one has

$$g \leq 2B_1 = 2\kappa\frac{M_1}{M_0}, \tag{97}$$

and hence (by (55))

$$k(z) \geq \frac{|x-z|}{4M_0} - 2B_1 - C(d) \geq \frac{|x-z|}{4M_0} - 2\kappa\frac{M_1}{M_0} - C(d). \tag{98}$$

By (98) and the definition of Γ_s, we have for $|x-z| \geq \frac{1}{10}M_1$,

$$\begin{aligned}
\Gamma_g^{\tilde\sigma}(z) &\geq (|x-z| - 8\kappa M_1 - 10(g+1)M_0)^{\tilde\sigma} \\
&\geq (|x-z| - 30\kappa M_1)^{\tilde\sigma} \geq |x-z|^{\tilde\sigma}(1-160\kappa)^{\tilde\sigma} \geq |x-z|^{\tilde\sigma}(1-200\tilde\sigma\kappa),
\end{aligned} \tag{99}$$

where κ will be chosen to be sufficiently small.

By (96), (97) and (99), we have, for $|x-z| \geq \frac{1}{10}M_1$,

$$|G_{\widetilde{\Lambda}_1}(x,z)| = |G_{J_g}(x,z)| \leq M_1^{13gdM_1^{\tilde{\sigma}}M_0^{\sigma-\tilde{\sigma}}} e^{-\gamma_0 \Gamma_g^{\tilde{\sigma}}(z)}$$
$$\leq e^{-\gamma_0\left(1-200\kappa\tilde{\sigma}-300d\kappa\rho\gamma_0^{-1}\frac{\log M_0}{M_0^{1-\rho+\tilde{\sigma}-\sigma}}\right)|x-z|^{\tilde{\sigma}}}. \tag{100}$$

Inductions: Define

$$\gamma_m = \prod_{i=0}^{m-1} \gamma_0\left(1 - C(d)\kappa\tilde{\sigma} - C(d)\kappa\rho\gamma_i^{-1}\frac{\log M_i}{M_i^{1-\rho+\tilde{\sigma}-\sigma}}\right). \tag{101}$$

Recall that $1-\rho+\tilde{\sigma}-\sigma > 0$. Fix an elementary region $\tilde{\tilde{\Lambda}}_1 \in \mathcal{E}_{M_m}$ with $\tilde{\tilde{\Lambda}}_1 \subset \widetilde{\Lambda}_0$. For any $x \in \tilde{\tilde{\Lambda}}_1$, consider the exhaustion $\{S_j(x)\}_{j=0}^{l}$ of $\tilde{\tilde{\Lambda}}_1$ at x with width M_{m-1}. We say the annulus $A_j(x)$ is good if, for any $y \in A_j(x)$, there exists $W(y) \in \mathcal{E}_{M_{m-1}}$ such that

$$y \in W(y) \subset A_j(x), \ \text{dist}\,(y, A_j(x)\backslash W(y)) \geq \tfrac{1}{2}M_{m-1},$$

and, for $|n-n'| \geq \frac{1}{10}M_{m-1}$,

$$|(R_{W(y)}AR_{W(y)})^{-1}(n,n')| \leq e^{-\gamma_m|n-n'|^{\tilde{\sigma}}}.$$

Otherwise, we call the annulus bad. An elementary region $\widetilde{\Lambda}_1 \subset \widetilde{\Lambda}_0$ is called bad provided for some $x \in \widetilde{\Lambda}_1$ the number of bad annuli $\{A_j(x)\}$ exceeds

$$B_m := \kappa\frac{M_m}{M_{m-1}}.$$

Otherwise, we call $\tilde{\tilde{\Lambda}}_1$ good. Let $\mathcal{F}_m$ be an arbitrary family of pairwise disjoint bad elementary regions in $\mathcal{E}_{M_m}$ contained in $\widetilde{\Lambda}_0$. By induction, it is easy to see that

$$\#\mathcal{F}_m \leq \frac{1}{\kappa^m}\frac{N^{\varsigma}}{M_m}.$$

Replace M_0, M_1, γ_0, B_1 with M_{m-1}, M_m, γ_{m-1}, B_m. By induction and following the proof of (100), for good elementary regions $\tilde{\tilde{\Lambda}}_1 \subset \widetilde{\Lambda}_0$ and $\tilde{\tilde{\Lambda}}_1 \in \mathcal{E}_{M_m}$, we have, for $|x-z| \geq \frac{1}{10}M_m$,

$$|G_{\widetilde{\Lambda}_1}(x,z)| \leq e^{-\gamma_m|x-z|^{\tilde{\sigma}}}. \tag{102}$$

In order to reach $\widetilde{\Lambda}_0$ after k steps, we expect that

$$M_k = M^{\rho^k} = N,$$

and hence

$$\rho^k = \frac{1}{\xi}. \tag{103}$$

To ensure that k is a positive integer, we need to modify the scale at the last step. More precisely, let $k = \lfloor \log\xi^{-1}/\log\rho \rfloor$. For $j \leq k-1$, let

$$M_j = M^{\rho^j}$$

and for $j = k$ let

$$M_k = N.$$

Choose $\kappa = N^{-\delta}$ and

$$\delta = -\frac{1}{2}(1-\varsigma)\frac{\log \rho}{\log \xi^{-1}}.$$

Direct computations show that

$$\#\mathcal{F}_k \leq \frac{1}{\kappa^k}\frac{N^{\varsigma}}{M_k} < 1, \tag{104}$$

which implies that $M_k = N$ is good. Therefore, (6) holds for

$$c_3 = \gamma_k,$$

where k solves (103). Computations show that

$$c_3 = c_2 - N^{-\vartheta},$$

where $\vartheta = \vartheta(\sigma, \tilde{\sigma}, \xi, \varsigma) > 0$. □

5. Proof of Theorem 2.6

The proof of Theorem 2.6 is based on matrix-valued Cartan-type estimates [Bourgain et al. 2002; Bourgain 2005a; Goldstein and Schlag 2008; Jitomirskaya et al. 2020b]. For our purpose, a new version of Cartan's estimate, which works for non-self-adjoint matrices, is necessary. For convenience, we include a proof in the Appendix.

Lemma 5.1. *Let $T(x)$ be an $N \times N$ matrix function of a parameter $x \in [-\delta, \delta]^J$ ($J \in \mathbb{N}$) satisfying the following conditions*:

(i) *$T(x)$ is real-analytic in $x \in [-\delta, \delta]^J$ and has a holomorphic extension to*

$$\mathcal{D}_{\delta,\delta_1} = \Big\{x = (x_i)_{1\leq i\leq J} \in \mathbb{C}^J : \sup_{1\leq i\leq J} |\Re x_i| \leq \delta, \sup_{1\leq i\leq J} |\Im x_i| \leq \delta_1\Big\}$$

satisfying

$$\sup_{x\in\mathcal{D}_{\delta,\delta_1}} \|T(x)\| \leq B_1, \quad B_1 \geq 1. \tag{105}$$

(ii) *For all $x \in [-\delta, \delta]^J$, there is subset $V \subset [1, N]$ with*

$$|V| \leq M,$$

and

$$\|(R_{[1,N]\setminus V} T(x) R_{[1,N]\setminus V})^{-1}\| \leq B_2, B_2 \geq 1. \tag{106}$$

(iii)
$$\text{mes}\{x \in [-\delta, \delta]^J : \|T^{-1}(x)\| \geq B_3\} \leq 10^{-3J} J^{-J} \delta_1^J (1 + B_1)^{-J} (1 + B_2)^{-J}. \tag{107}$$

Let

$$0 < \epsilon \leq (1 + B_1 + B_2)^{-10M}. \tag{108}$$

Then

$$\mathrm{mes}\big\{x \in \big[-\tfrac{1}{2}\delta, \tfrac{1}{2}\delta\big]^J : \|T^{-1}(x)\| \geq \epsilon^{-1}\big\} \leq C\delta^J e^{-c\left(\frac{\log \epsilon^{-1}}{M\log(B_1+B_2+B_3)}\right)^{1/J}}, \tag{109}$$

where $C = C(J)$, $c = c(J) > 0$.

Proof of Theorem 2.6. Without loss of generality, we assume $i = 1$. Fix $x_1 \in \mathbb{T}^{b_1}$ and $x_1^{\neg} \in \mathbb{T}^{b-b_1}$. Recall that $x = (x_1, x_1^{\neg}) \in \mathbb{T}^b$.

Let $\Lambda = \mathcal{R} \subset [-N_3, N_3]^d$. By making $\mathcal{B}_{\mathcal{R}}(x)$ slightly larger, we have there exists $\bar{\Lambda} \subset \Lambda$ such that, for any $j \in \Lambda\backslash\bar{\Lambda}$, there exists $W(j) \in \mathcal{E}_{N_1}$ such that $W(j) \subset \Lambda\backslash\bar{\Lambda}$,

$$\mathrm{dist}(j, \Lambda\backslash\bar{\Lambda}\backslash W(j)) \geq \tfrac{1}{2}N_1$$

and

$$\|G_{W(j)}\| \leq e^{N_1{}^{\sigma}}, \tag{110}$$

$$|G_{W(j)}(n,n')| \leq e^{-c_2|n-n'|^{\tilde{\sigma}}} \quad \text{for } |n-n'| \geq \tfrac{1}{10}N_1, \tag{111}$$

and

$$|\bar{\Lambda}| \leq C(d)L^{1-\delta}N_1^{2d}. \tag{112}$$

Indeed, $\bar{\Lambda}$ can be chosen so that

$$\bar{\Lambda} \subset \{n \in \mathbb{Z}^d : \mathrm{dist}(n, B_{\mathcal{R}}(x))\} \leq C(d)N_1. \tag{113}$$

Let $\eta = c_1/\gamma$. Let $\mathcal{D}$ be the $e^{-\eta N_1}$ neighborhood of x_1 in the complex plane, i,e.,

$$\mathcal{D} = \{z \in \mathbb{C}^{b_1} : |\Im z| \leq e^{-\eta N_1}, |\Re z - x_1| \leq e^{-\eta N_1}\}.$$

By the assumption that $N_3 \leq e^{N_1^{1/(2K_1)}}$, one has, for any $y \in \mathcal{D}$,

$$\|x - y\| \leq e^{-e^{(\log(2N_3+2))^{K_1}}},$$

and hence (by (8))

$$|A(x;n,n') - A(y;n,n')| \leq K\|x-y\|^{\gamma} \leq Ke^{-c_1N_1} \tag{114}$$

for $n, n' \in [-N_3, N_3]^d$ and large N_1. By (110), (111), (114), and standard perturbation arguments, we have, for any $y \in \mathcal{D}$, and $j \in \Lambda\backslash\bar{\Lambda}$,

$$\|G_{W(j)}(x_1+y, x_1^{\neg})\| \leq 2e^{N_1{}^{\sigma}}, \tag{115}$$

$$|G_{W(j)}(x_1+y, x_1^{\neg}; n,n')| \leq 2e^{-c_2|n-n'|^{\tilde{\sigma}}} \quad \text{for } |n-n'| \geq \tfrac{1}{10}N_1. \tag{116}$$

Substituting Λ with $\Lambda\backslash\bar{\Lambda}$ in Lemma 4.2, one has, for any $y \in \mathcal{D}$,

$$\|G_{\Lambda\backslash\bar{\Lambda}}(x_1+y, x_1^{\neg})\| \leq e^{2N_1{}^{\sigma}}. \tag{117}$$

We want to use Lemma 5.1. For this purpose, let

$$T(y) = R_\Lambda A R_\Lambda, \quad J = b_1, \quad \delta = \delta_1 = e^{-\eta N_1}.$$

Now we are in the position to check the assumptions of Lemma 5.1. By (114) and (7), one has $B_1 = O(1)$.

Let $V=\bar{\Lambda}$. By (112) and (117), one has

$$M=|\bar{\Lambda}|\le C(d)L^{1-\delta}N_1^{2d},\quad B_2=e^{2N_1{}^{\sigma}}. \tag{118}$$

Applying Lemma 4.2 with $M_0=M_1=N_2$ and (12), one has

$$\|T^{-1}(y)\|\le 4(2N_2+1)^d e^{N_2{}^{\sigma}}\le e^{2N_2{}^{\sigma}}=:B_3,$$

except on a set of $y\in\mathbb{T}^{b_1}$ with measure less than $e^{-N_2^{\zeta}}$.

Since $N_2\ge N_1^{2/\zeta}$, direct computation shows that

$$10^{-3b_1}b_1^{-b_1}\delta^{b_1}(1+B_1)^{-b_1}(1+B_2)^{-b_1}\ge e^{-N_2^{\zeta}}.$$

This verifies (iii) in Lemma 5.1.

Let $\epsilon=e^{-L^{\mu}}$. By (118) and the assumption that $L\ge N_2^{(2d+b+2)/(\mu-1+\delta)}$, one has

$$\epsilon<(1+B_1+B_2)^{-10M}.$$

Let

$$Y=\{y\in\mathcal{D}:\|T^{-1}(y)\|\ge e^{L^{\mu}}\}.$$

By (109) of Lemma 5.1,

$$\operatorname{mes}(Y)\le Ce^{-c\left(\frac{L^{\mu-1+\delta}}{N_2^{\sigma}N_1^{2d+\sigma}}\right)^{1/b_1}}. \tag{119}$$

By covering $\mathbb{T}^{b_1}$ with balls with radius $e^{-\eta N_1}$, we have

$$\operatorname{Leb}(\widetilde{X}_{\mathcal{R}}(x_1^{\neg}))\le e^{CN_1}e^{-c\left(\frac{L^{\mu-1+\delta}}{N_2^{\sigma}N_1^{2d+\sigma}}\right)^{1/b_1}}\le e^{-\left(\frac{L^{\mu-1+\delta}}{N_2^{2d+b+2}}\right)^{1/b_1}}, \tag{120}$$

where the second inequality holds by the assumption $L\ge N_2^{(2d+b+2)/(\mu-1+\delta)}$. It implies (13). □

6. Proof of Theorems 2.7 and 2.8

Theorem 6.1. *Let* $\sigma,\tilde{\sigma},\kappa,s\in(0,1)$ *and* $\tilde{\sigma}>\kappa$. *Assume* $\operatorname{diam}(\Lambda)\le 2N+1$. *Let* $M_0=(\log N)^{1/s}$. *Assume*

$$c_2\in(0,(1-5^{-\tilde{\sigma}})c_1]. \tag{121}$$

Suppose that, for any $n\in\Lambda$, *there exists some* $W=W(n)\in\mathcal{E}_M$ *with* $M_0\le M\le N^{\kappa}$ *such that* $n\in W$, $\operatorname{dist}(n,\Lambda\backslash W)\ge\frac{1}{2}M$, $W\subset\Lambda$ *and*

$$\|G_W\|\le 2e^{M^{\sigma}},$$

$$|G_W(n,n')|\le 2e^{-c_2|n-n'|^{\tilde{\sigma}}}\quad \textit{for } |n-n'|\ge\tfrac{1}{10}M.$$

Then

$$\|G_{\Lambda}\|\le 4(1+2N^{\kappa})^d e^{N^{\kappa\sigma}}, \tag{122}$$

$$|G_{\Lambda}(n,n')|\le e^{-\bar{c}|n-n'|^{\tilde{\sigma}}}\quad \textit{for } |n-n'|\ge\tfrac{1}{10}N,$$

where

$$\bar{c}=c_2-\frac{O(1)}{M_0^{\tilde{\sigma}-s}}-\frac{O(1)}{M_0^{\tilde{\sigma}-\sigma}}-\frac{O(1)}{N^{\tilde{\sigma}-\kappa}}. \tag{123}$$

Proof. Inequality (122) follows from Lemma 4.2 immediately.

Assume $|n-n'|\geq \frac{1}{10}N$. Applying (57) with $\Lambda_1=W=W(n)$, one has $n'\notin W(n)$ and

$$|G_\Lambda(n,n')|\leq \sum_{\substack{n_1\in W\\ n_2\in\Lambda\setminus W}} e^{-c_1|n_1-n_2|^{\tilde\sigma}}|G_W(n,n_1)||G_\Lambda(n_2,n')|.$$

This implies

$$\begin{aligned}
|G_\Lambda(n,n')| &\leq \sum_{\substack{n_1\in W,\,|n_1-n|\leq M/10-1\\ n_2\in\Lambda\setminus W}} e^{-c_1|n_1-n_2|^{\tilde\sigma}}|G_W(n,n_1)||G_\Lambda(n_2,n')| \\
&\qquad + \sum_{\substack{n_1\in W,\,|n_1-n|\geq M/10\\ n_2\in\Lambda\setminus W}} e^{-c_1|n_1-n_2|^{\tilde\sigma}}|G_W(n,n_1)||G_\Lambda(n_2,n')| \\
&\leq \sum_{\substack{n_1\in W,\,|n_1-n|\leq M/10-1\\ n_2\in\Lambda\setminus W}} e^{M^\sigma}e^{-c_1|n_1-n_2|^{\tilde\sigma}}|G_\Lambda(n_2,n')| \\
&\qquad + \sum_{\substack{n_1\in W,\,|n_1-n|\geq M/10\\ n_2\in\Lambda\setminus W}} e^{-c_1|n_1-n_2|^{\tilde\sigma}}e^{-c_2|n-n_1|^{\tilde\sigma}}|G_\Lambda(n_2,n')| \\
&\leq \sum_{\substack{n_1\in W,\,|n_1-n|\leq M/10-1\\ n_2\in\Lambda\setminus W}} e^{M^\sigma}e^{-c_1|n_1-n_2|^{\tilde\sigma}}|G_\Lambda(n_2,n')| \\
&\qquad + \sum_{\substack{n_1\in W,\,|n_1-n|\geq M/10\\ n_2\in\Lambda\setminus W}} e^{-c_2|n-n_2|^{\tilde\sigma}}|G_\Lambda(n_2,n')| \\
&\leq e^{M^\sigma}\sum_{\substack{n_1\in W,\,|n_1-n|\leq M/10-1\\ n_2\in\Lambda\setminus W}} e^{-c_2|n-n_2|^{\tilde\sigma}}|G_\Lambda(n_2,n')| \\
&\qquad + \sum_{\substack{n_1\in W,\,|n_1-n|\geq M/10\\ n_2\in\Lambda\setminus W}} e^{-c_2|n-n_2|^{\tilde\sigma}}|G_\Lambda(n_2,n')| \\
&\leq e^{M^\sigma}(2N+1)^{2d}\sup_{n_2\in\Lambda\setminus W} e^{-c_2|n-n_2|^{\tilde\sigma}}|G_\Lambda(n_2,n')| \\
&\leq (2N+1)^{2d}\sup_{n_2\in\Lambda\setminus W} e^{-\left(c_2-\frac{O(1)}{M_0^{\tilde\sigma-\sigma}}\right)|n-n_2|^{\tilde\sigma}}|G_\Lambda(n_2,n')|, \qquad (124)
\end{aligned}$$

where the third inequality holds by (64) and the fourth inequality holds by (72).

Iterating (124) until $|n_2-n'|\leq 4N^\kappa$ (but at most $\lfloor 2^{\tilde\sigma}|n-n'|^{\tilde\sigma}/M_0^{\tilde\sigma}\rfloor$ times) and applying (122), we have for $|n-n'|\geq\frac{1}{10}N$,

$$\begin{aligned}
|G_\Lambda(n,n')| &\leq (2N+1)^{\frac{2^{\tilde\sigma}|n-n'|^{\tilde\sigma}}{M_0^{\tilde\sigma}}} e^{-\left(c_2-\frac{O(1)}{M_0^{\tilde\sigma-\sigma}}\right)(|n-n'|-4N^\kappa)^{\tilde\sigma}} 4(1+2N^\kappa)^d e^{N^{\kappa\sigma}} \\
&\leq e^{\frac{4|n-n'|^{\tilde\sigma}}{M_0^{\tilde\sigma}}\log N} e^{-\left(c_2-\frac{O(1)}{M_0^{\tilde\sigma-\sigma}}\right)(|n-n'|-4N^\kappa)^{\tilde\sigma}} 4(1+2N^\kappa)^d e^{N^{\kappa\sigma}} \\
&\leq e^{\frac{4|n-n'|^{\tilde\sigma}}{M_0^{\tilde\sigma}}M_0^{s}} e^{-\left(c_2-\frac{O(1)}{M_0^{\tilde\sigma-\sigma}}\right)(|n-n'|-4N^\kappa)^{\tilde\sigma}} 4(1+2N^\kappa)^d e^{N^{\kappa\sigma}} \\
&\leq e^{-\bar c|n-n'|}, \qquad (125)
\end{aligned}$$

completing the proof. $\square$

It is easy to see that the number of generalized elementary regions in $[-N,N]^d$ with width greater than or equal to N^{ξ} is bounded by $N^{C(d)}$; more precisely, for any $\xi>0$,

$$\#\{\Lambda\subset[-N,N]^d:\Lambda\subset\mathcal{R}_L^{N^{\xi}}\}\le N^{C(d)}. \tag{126}$$

Proof of Theorem 2.7. Since the Green's function satisfies Property P with parameters (μ,ζ,c_2) at size N_2, there exists $\widetilde{X}_{N_2}\subset\mathbb{T}^b$ with

$$\sup_{1\le i\le k,x_i^{\neg}\in\mathbb{T}^{b-b_i}}\mathrm{Leb}(\widetilde{X}_{N_2}(x_i^{\neg}))\le N_3^{C(d)}e^{-N_2^{\zeta}}, \tag{127}$$

such that

$$\|G_{m+Q_{N_2}}(x)\|\le e^{N_2^{\mu}},$$

and, for $|n-n'|\ge\frac{1}{10}N_2$,

$$|G_{m+Q_{N_2}}(x;n,n')|\le e^{-c_2|n-n'|^{\tilde\sigma}}$$

for any $Q_{N_2}\in\mathcal{E}_{N_2}^0$ and $|m|\le N_3$. Indeed, we only need to set

$$\widetilde{X}_{N_2}=\bigcup_{|m|\le N_3}X_{N_2}(f^m(x)).$$

By the assumption $N_3\ge N_2^C$ and $N_2\ge N_1^C$ with large C depending on ε, one has

$$N_2\le N_3^{\varepsilon},\quad N_1\le N_2^{\varepsilon}. \tag{128}$$

Let $\xi=\delta-5\varepsilon$. Applying (15) to Theorem 2.6, and by (126) and (128), there exists $X_{N_3}\subset[0,1)^b$ such that

$$\sup_{x_i^{\neg}\in\mathbb{T}^{b-b_i}}\mathrm{Leb}(X_{N_3}(x_i^{\neg}))\le N_3^{C(d)}e^{-N_3^{\xi(\sigma-1/b_i+\delta/b_i)-\varepsilon}}\le e^{-N_3^{(\sigma-1/b_i)\delta+\delta^2/b_i-\varepsilon}}, \tag{129}$$

and, for any $x\notin X_{N_3}$, $\mathcal{R}\subset\mathcal{R}_L^{N_3^{\xi}}$ with $N_3^{\xi}\le L\le N_3$,

$$\|G_{\mathcal{R}}(x)\|\le e^{L^{\sigma}}. \tag{130}$$

Let $\widetilde{\mathcal{F}}$ be any pairwise disjoint elementary regions in $[-N_3,N_3]^d$ with size $\lfloor N_3^{\xi}\rfloor$. By (15), it is easy to see that there are at most $N_1^{C(d)}N_3^{1-\delta}=N_3^{1-\delta+\varepsilon}$ in $\widetilde{\mathcal{F}}$ that will intersect elementary regions not in SG_{N_1}. By Theorem 6.1, any elementary region in $[-N_3,N_3]^d$ with size $\lfloor N_3^{\xi}\rfloor$, that do not intersect any non-SG_{N_1} elementary regions, will satisfy (3). It implies (5) is true for $\varsigma=1-\varepsilon$. Applying Theorem 2.3 and (130), we obtain Theorem 2.7. Let us explain where the bound $c_2-N_1^{-\vartheta_1}-N_3^{-\vartheta_2}$ in (18) is from. Since $N_3^{\xi}\le e^{\xi N_1^c}$, one has $s=\frac{11}{10}c$ in Theorem 6.1. Applying $M_0=N_1$, $N=N_3^{\xi}$, $\sigma=\mu$ to Theorem 6.1, we obtain the bound $c_2-O(1)N_1^{-(\tilde\sigma-(11/10)c)}-O(1)N_1^{-(\tilde\sigma-\mu)}-N_3^{-\vartheta_2}$. Theorem 2.3 will only contribute $N_3^{-\vartheta_2}$. □

Proof of Theorem 2.8. Fix any $m\in\mathbb{Z}^d$. Applying Theorem 2.7 with $\tilde{A}^m$, one has there exists a subset $X_{N_3}^m\subset\mathbb{T}^b$ such that

$$\sup_{1\le i\le k,x_i^{\neg}\in\mathbb{T}^{b-b_i}}\mathrm{Leb}(X_{N_3}^m(x_i^{\neg}))\le e^{-N_3^{(\sigma-1)/b_i\delta+(\delta^2/b_i)-\varepsilon}},$$

and, for any $x \notin X_{N_3}^m$ and $Q_{N_3} \in \mathcal{E}_{N_3}^0$,

$$\|(R_{Q_{N_3}} \tilde{A}^m(x) R_{Q_{N_3}})^{-1}\| \le e^{N_3^\sigma},$$

and, for $|n-n'| \ge \frac{1}{10} N_3$,

$$|(R_{Q_{N_3}} \tilde{A}^m R_{Q_{N_3}})^{-1}(x; n, n')| \le e^{-(c_2 - N_1^{-\vartheta_1} - N_3^{-\vartheta_2})|n-n'|^{\tilde{\sigma}}}.$$

Let

$$X_{N_3} = \bigcup_{m \in \mathbb{Z}^d} X_{N_3}^m.$$

By (19) and (21) , we have

$$\sup_{1 \le i \le k, x_i^\neg \in \mathbb{T}^{b-b_i}} \operatorname{Leb}(X_{N_3}(x_i^\neg)) \le e^{N_3^a} e^{-N_3^{(\sigma-1/b_i)\delta+\delta^2/b_i-\varepsilon}} \le e^{-N_3^{(\sigma-1/b_i)\delta+\delta^2/b_i-\varepsilon}}. \qquad \square$$

7. Proof of Theorem 2.10

Proof. Once we have the LDT at hand, the modulus of continuity of the IDS is standard. The proof here follows from the corresponding part in [Bourgain 2000; Schlag 2001]. Let $N = |\log|E_1 - E_2||^{1/\sigma - \varepsilon}$. Without loss of generality, assume $E_1 < E_2$ and let E be the center of $[E_1, E_2]$. Therefore,

$$|E_1 - E_2| \le e^{-N^{\sigma+\varepsilon}}. \tag{131}$$

By the assumption, there exists a set $X_N \subset \mathbb{T}^b$ such that

$$\operatorname{Leb}(X_N) \le e^{-N^\zeta},$$

and, for any $x \notin X_N$ and any $Q_N \in \mathcal{E}_N^0$,

$$\|G_{Q_N}(E, x)\| \le e^{N^\sigma},$$
$$|G_{Q_N}(E, x; n, n')| \le e^{-c|n-n'|^{\tilde{\sigma}}} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N,$$

where $c > 0$. We should mention that X_N depends on E. By the assumption (23), for large N_1, one has

$$\#\{n \in \mathbb{Z}^d : |n| \le N_1, f^n(x) \in X_N\} \le 2(2N_1+1)^d e^{-N^\zeta}.$$

Let $\Lambda = [-N_1, N_1]^d$. By making $\#\{n \in \mathbb{Z}^d : |n| \le N_1, f^n(x) \in X_N\}$ slightly larger, we have there exists $\bar{\Lambda} \subset \Lambda$ such that, for all $j \in \Lambda \backslash \bar{\Lambda}$, there exists $W(j) \in \mathcal{E}_N$ such that $W(j) \subset \Lambda \backslash \bar{\Lambda}$,

$$\operatorname{dist}(j, \Lambda \backslash \bar{\Lambda} \backslash W(j)) \ge \tfrac{1}{2} N$$

and

$$\|G_{W(j)}\| \le e^{N^\sigma}, \tag{132}$$
$$|G_{W(j)}(n, n')| \le e^{-c|n-n'|^{\tilde{\sigma}}} \quad \text{for } |n-n'| \ge \tfrac{1}{10}N. \tag{133}$$

and

$$|\bar{\Lambda}| \le C(d) N^{2d} (2N_1+1)^d e^{-N^\zeta}.$$

Here, $\bar{\Lambda}$ is obtained in a similar way as (113).

Substituting Λ with $\Lambda\backslash\bar{\Lambda}$ in Lemma 4.2, we have

$$\|G_{\Lambda\backslash\bar{\Lambda}}(E,x)\| \le 4(2N+1)^d e^{N^\sigma}.$$

By standard perturbation arguments, we have, for any $\widetilde{E}\in[E_1,E_2]$,

$$\|G_{\Lambda\backslash\bar{\Lambda}}(\widetilde{E},x)\| \le 8(2N+1)^d e^{N^\sigma}. \tag{134}$$

Denote by $\xi_j,\ j=1,2,\dots,M$, the normalized eigenfunctions of H_Λ with eigenvalues falling into the interval $[E_1,E_2]$. Let ξ be one of them with eigenvalue $\widetilde{E}$. By definition,

$$R_{\Lambda\backslash\bar{\Lambda}}(H_\Lambda-E)R_{\Lambda\backslash\bar{\Lambda}}\xi+R_{\Lambda\backslash\bar{\Lambda}}(H_\Lambda-E)R_{\bar{\Lambda}}\xi=(\widetilde{E}-E)R_{\Lambda\backslash\bar{\Lambda}}\xi. \tag{135}$$

Applying $G_{\Lambda\backslash\bar{\Lambda}}(E,x)$ to (135), one has

$$R_{\Lambda\backslash\bar{\Lambda}}\xi+G_{\Lambda\backslash\bar{\Lambda}}(E,x)R_{\Lambda\backslash\bar{\Lambda}}(H_\Lambda-E)R_{\bar{\Lambda}}\xi=(\widetilde{E}-E)G_{\Lambda\backslash\bar{\Lambda}}(E,x)R_{\Lambda\backslash\bar{\Lambda}}\xi. \tag{136}$$

Denote by P the projection onto the range of $G_{\Lambda\backslash\bar{\Lambda}}(E,x)R_{\Lambda\backslash\bar{\Lambda}}(H_\Lambda-E)R_{\bar{\Lambda}}$. Clearly, the dimension of this range does not exceed $\bar{\Lambda}$. Thus $\operatorname{rank}(P)\le\bar{\Lambda}$. By (131) and (134), one has

$$\|(\widetilde{E}-E)G_{\Lambda\backslash\bar{\Lambda}}(E,x)R_{\Lambda\backslash\bar{\Lambda}}\xi\| \le \tfrac{1}{100}\|\xi\|. \tag{137}$$

Applying $I-P$ to (136) and by (137), we have

$$\|R_{\Lambda\backslash\bar{\Lambda}}\xi-PR_{\Lambda\backslash\bar{\Lambda}}\xi\| \le \tfrac{1}{100}\|\xi\|. \tag{138}$$

Applying (138) to each ξ_j, we have

$$\begin{aligned} M=\sum_{j=1}^{N}\|\xi_j\|^2 &\le \tfrac12 M+4\sum_{j=1}^{M}\|PR_{\Lambda\backslash\bar{\Lambda}}\xi_j\|^2+2\sum_{j=1}^{M}\|R_{\bar{\Lambda}}\xi_j\|^2\\ &\le \tfrac12 M+4\operatorname{Trace}(PR_{\Lambda\backslash\bar{\Lambda}})+2\operatorname{Trace}(R_{\bar{\Lambda}})\\ &\le \tfrac12 M+6|\bar{\Lambda}|\\ &\le \tfrac12 M+C(d)N^{2d}(2N_1+1)^d e^{-N^\zeta}. \end{aligned}$$

Therefore,

$$M\le C(d)N^{2d}(2N_1+1)^d e^{-N^\zeta},$$

which implies

$$k(x,E_1,E_2)\le C(d)N^{2d}e^{-N^\zeta}\le e^{-\left(\log\frac{1}{E_2-E_1}\right)^{\zeta/\sigma-\varepsilon}}. \qquad \square$$

8. The discrepancy and semialgebraic sets

8A. *Discrepancy.* Let $\vec{x}_1,\dots,\vec{x}_N\in[0,1)^b$ and $\mathcal{S}\subset[0,1)^b$. Let $A(\mathcal{S};\{\vec{x}_n\}_{n=1}^N)$ be the number of $\vec{x}_n$ $(1\le n\le N)$ such that $\vec{x}_n\in\mathcal{S}$. We define the discrepancy of the sequence $\{\vec{x}_n\}_{n=1}^N$ by

$$D_N(\{\vec{x}_n\}_{n=1}^N)=\sup_{\mathcal{S}\in\mathcal{C}}\left|\frac{A(\mathcal{S};\{\vec{x}_n\}_{n=1}^N)}{N}-\operatorname{Leb}(S)\right|, \tag{139}$$

where $\mathcal{C}$ is the family of all intervals in $[0,1)^b$, namely $\mathcal{S}$ has the form of

$$\mathcal{S} = [\varrho_1, \beta_1] \times [\varrho_2, \beta_2] \times \cdots \times [\varrho_b, \beta_b],$$

with $0 \le \varrho_n < \beta_n < 1,\ n = 1, 2, \ldots, b$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_b) \in [0,1)^b$. The b-dimensional sequence $\vec{x}_n = (n\alpha_1, n\alpha_2, \ldots, n\alpha_b) \bmod \mathbb{Z}^b$ ($n\alpha$ for short), $n = 1, 2, \ldots$, is called the Kronecker sequence. We denote by $D_N(\alpha)$ the discrepancy of $\{n\alpha\}_{n=1}^N$. The following lemmas are well known.

Lemma 8.1 [Drmota and Tichy 1997]. *Assume* $\alpha \in \mathrm{DC}(\kappa, \tau)$. *Then*

$$D_N(\alpha) \le C(b, \kappa, \tau) N^{-\frac{1}{\kappa}} (\log N)^2.$$

Lemma 8.2 [Schmidt 1964]. *For almost every* α, *we have*

$$D_N(\alpha) \le C(\alpha) N^{-1} (\log N)^{b+2}.$$

Let $f \colon \mathbb{T}^b \to \mathbb{T}^b$ be defined as

$$T(y_1, y_2, \ldots, y_b) = (y_1 + \alpha, y_2 + y_1, \ldots, y_b + y_{b-1}).$$

Let T^n be the n-th iteration of T and $\vec{Y}_n = T^n(y_1, \ldots, y_b)$.

Lemma 8.3. *Assume* $\alpha \in \mathrm{DC}(\kappa, \tau)$. *Then, for any* $\varepsilon > 0$,

$$D_N(\{\vec{Y}_n\}_{n=1}^N) \le C(b, \kappa, \tau, \varepsilon) N^{-1/(2^{b-1}\kappa) + \varepsilon}.$$

Remark 8.4. Lemma 8.3 follows from the Erdős–Turán inequality (see [Montgomery 1994, Corollary 1.1, p. 8]) and Weyl's method [Montgomery 1994, Theorem 2, p. 41].

The Erdős–Turán inequality and Weyl's method also imply:

Lemma 8.5. *Assume* $\alpha \in \mathrm{DC}(\kappa, \tau)$. *Let* $Y_n = P_b(T^n(y_1, \ldots, y_b))$, *where* P_b *is the* b*-th coordinate projection. Then, for any* $\varepsilon > 0$,

$$D_N(\{Y_n\}_{n=1}^N) \le C(b, \kappa, \tau, \varepsilon) N^{-1/(2^{b-1}\kappa) + \varepsilon}.$$

8B. *Semialgebraic sets.* A set $\mathcal{S} \subset \mathbb{R}^n$ is called a semialgebraic set if it is a finite union of sets defined by a finite number of polynomial equalities and inequalities. More precisely, let $\{P_1, \ldots, P_s\} \subset \mathbb{R}[x_1, \ldots, x_n]$ be a family of real polynomials whose degrees are bounded by d. A (closed) semialgebraic set $\mathcal{S}$ is given by the expression

$$\mathcal{S} = \bigcup_j \bigcap_{\ell \in \mathcal{L}_j} \{x \in \mathbb{R}^n : P_\ell(x) \varsigma_{j\ell} 0\}, \tag{140}$$

where $\mathcal{L}_j \subset \{1, \ldots, s\}$ and $\varsigma_{j\ell} \in \{\ge, \le, =\}$. Then we say that $\mathcal{S}$ has degree at most sd. In fact, the degree of $\mathcal{S}$, which is denoted by $\deg(\mathcal{S})$, means the smallest sd over all representations as in (140).

The following lemma is a special case appearing [Basu 1999]. It is restated in [Bourgain 2005a].

Lemma 8.6 [Bourgain 2005a, Theorem 9.3; Basu 1999, Theorem 1]. *Let* $\mathcal{S} \subset [0,1]^n$ *be a semialgebraic set of degree* B. *Then the number of connected components of* $\mathcal{S}$ *does not exceed* $(1+B)^{C(n)}$.

The following lemma has been stated in [Bourgain 2005a], where the author mentioned that it follows from the Yomdin–Gromov triangulation theorem [Gromov 1987; Yomdin 1987], However, as far as we know, Lemma 8.7 has only been proved recently. We refer readers to [Binyamini and Novikov 2019] for the history and complete proof of the Yomdin–Gromov triangulation theorem.

Lemma 8.7 [Bourgain 2005a, Corollary 9.6]. *Let $\mathcal{S} \subset [0,1]^n$ be a semialgebraic set of degree B. Let $\epsilon > 0$ be a small number and $\mathrm{Leb}(\mathcal{S}) \leq \epsilon^n$. Then $\mathcal{S}$ can be covered by a family of ϵ-balls with total number less than $(1+B)^{C(n)}/\epsilon^{n-1}$.*

Theorem 8.8. *Assume that the discrepancy of the sequence $\{\vec{x}_j\}_{j=1}^N$ satisfies*

$$D_N(\{\vec{x}_j\}_{j=1}^N) \leq N^{-\varsigma}$$

for some $\varsigma > 0$. Let $S \subset [0,1]^n$ be a semialgebraic set with degree less than B. Suppose

$$\mathrm{Leb}(\mathcal{S}) \leq N^{-\varsigma}.$$

Then

$$A(\mathcal{S}; \{\vec{x}_j\}_{j=1}^N) \leq (1+B)^{C(n)} N^{1-\frac{\varsigma}{n}}.$$

Proof. Let $\epsilon = N^{-\varsigma/n}$. By Lemma 8.7, S can be covered by at most $(1+B)^C/\epsilon^{n-1}$ ϵ-balls. Pick one ϵ-ball, say J. By the fact $D_N(\{\vec{x}_j\}_{j=1}^N) \leq N^{-\varsigma}$, one has

$$A(J; \{\vec{x}_j\}_{j=1}^N) \leq CN\epsilon^n + N^{1-\varsigma} \leq CN^{1-\varsigma},$$

where C depends on the dimension n. Since there are at most $(1+B)^C/\epsilon^{n-1}$ balls, we have

$$A(\mathcal{S}; \{\vec{x}_j\}_{j=1}^N) \leq (1+B)^C \frac{1}{\epsilon^{n-1}} N^{1-\varsigma} = (1+B)^C N^{\frac{n-1}{n}\varsigma} N^{1-\varsigma} = (1+B)^C N^{1-\frac{\varsigma}{n}}. \qquad \square$$

Remark 8.9. • Theorem 8.8 says that there is a factor-n loss (referred to as dimension loss) when passing the discrepancy from intervals to semialgebraic sets. The dimension loss is not surprising. For example, there is also a dimension loss passing the discrepancy to the isotropic discrepancy [Kuipers and Niederreiter 1974, Theorem 1.6].

• The proof of Theorem 8.8 is taken from [Bourgain 2005a], where no explicit bounds are given.

For a set $S \subset [0,1)^2$, denote by $l(S)$ the length of the longest line segment contained in S.

Lemma 8.10 [Bourgain and Kachkovskiy 2019, Theorem 5.1]. *Assume $\alpha_1 \in \mathrm{DC}(\kappa,\tau)$ and $\alpha_2 \in \mathrm{DC}(\kappa,\tau)$. Let $S \subset [0,1)^2$ be a semialgebraic set with degree less than B and*

$$l(S) \leq \tfrac{1}{2} \min_{1 \leq |k| \leq 2N} \|k\alpha\|.$$

Then

$$\#\{k = (k_1,k_2) \in \mathbb{Z}^2 : |k| \leq N, (k_1\alpha_1, k_2\alpha_2) \in S \mod \mathbb{Z}^2\} \leq (1+B)^{C(d)} C(\kappa,\tau) N^{3\kappa - \frac{9}{4}}. \tag{141}$$

9. Proof of all the results in Section 3

Applying Theorem 2.10 with $\sigma = 1-\varepsilon$, Theorem 3.5 follows from Theorem 3.4, Theorem 3.12 follows from Theorem 3.11, Theorem 3.16 follows from Theorem 3.15, Theorem 3.21 follows from Theorem 3.19, Theorem 3.25 follows from Theorem 3.24 and Theorem 3.28 follows from Theorem 3.27.

Applying strong Diophantine frequencies to Theorems 3.21 and 3.28, we obtain Corollaries 3.22 and 3.29.

With large-deviation Theorems 3.11 and 3.15 at hand, the proof of Theorems 3.14 and 3.17 is rather standard. We refer the readers to [Bourgain 2005a, Chapter XV; 2007, Section 3; Bourgain et al. 2002, Section 6] for details. We note that the only difference is that the degree of semialgebraic sets is at most $e^{(\log N)^C}$ in our cases, not N^C.

By the discussion above, in order to prove all the results in Section 3, it suffices to prove Theorems 3.4, 3.6, 3.11, 3.13, 3.15, 3.19, 3.24, 3.27 and Corollary 3.23.

In this section, $C(c)$ is always a large (small) constant. It may change even in the same formula.

Lemma 9.1 [Bourgain 2005a, Proposition 7.19]. *Let $H(x)$ be given by* (32) *and the Lyapunov exponent is given by* (35). *Suppose $L(E)>0$. Then there exist $0<\sigma<1$ and $\zeta>0$ such that, for large N, there exists $X_N \subset \mathbb{T}^b$ such that $\mathrm{Leb}(X_N) \le e^{-N^{\zeta}}$ and, for $x \notin X_N$, one of the intervals*

$$\Lambda = [1,N],\ [1,N-1],\ [2,N],\ [2,N-1]$$

will satisfy

$$|G_{\Lambda}(n_1,n_2)| \le e^{-L(E)|n_1-n_2|+N^{\sigma}}.$$

Proof of Theorem 3.4. By Lemma 9.1, there exist $0<\sigma_1<1$ and $\zeta_1>0$ such that, for any large N_1, there exists $X_{N_1} \subset \mathbb{T}^b$ such that

$$\mathrm{Leb}(X_{N_1}) \le e^{-N_1^{\zeta_1}}$$

and, for $x \notin X_{N_1}$, one of the intervals

$$\Lambda(N_1) = [1,N_1],\ [1,N_1-1],\ [2,N_1],\ [2,N_1-1] \tag{142}$$

will satisfy

$$|G_{\Lambda(N_1)}(n_1,n_2)| \le e^{-L(E)|n_1-n_2|+N_1^{\sigma_1}}. \tag{143}$$

By approximating the analytic function with trigonometric polynomials given by (28) and using Taylor expansions, we can further assume that X_{N_1} is a semialgebraic set with degree less than $e^{(\log N_1)^C}$. This argument is quite standard. We refer to [Bourgain 2005a] for details. By Lemma 8.1 and Theorem 8.8,

$$A(X_{N_1};\{n\omega\}_{n=1}^{N_3}) \le N_3^{1-1/b\kappa+\varepsilon}$$

for any

$$e^{(\log N_1)^C} \le N_3 \le e^{N_1^c}.$$

Let $N_2 = N_3^{1/C}$. Applying (143) to N_2, one has

$$|G_{\Lambda(N_2)}(n_1,n_2)| \le e^{-L(E)|n_1-n_2|+N_2^{\sigma_1}}, \tag{144}$$

except for a set of x with measure less than $e^{-N_2^{\sigma_1}}$. Now Theorem 3.4 follows from Theorem 2.7. We should mention that the elementary region is $[-N_1, N_1]$ in Theorem 2.7, which is slightly different from (142). However, the same statement is true. □

Proof of Theorem 3.24. The proof of Theorem 3.24 is similar to that of Theorem 3.4. The difference is that instead of Lemma 9.1, we need to use the corresponding statements in [Tao 2019a, p. 3575] for initial scales. We also need to use Lemma 8.5 instead of Lemma 8.3. □

Proof of Theorem 3.15. Let $N_2 = e^{N_1^c}$. Assume the Green's function in Theorem 3.15 satisfies Property P with parameters (μ, ζ, c_2) at sizes N_1 and N_2. Let $N_3 = N_2^C$. We can assume that X_{N_1} is a semialgebraic set with degree less than $e^{(\log N_1)^C}$. By Lemma 8.6, X_{N_1} is consisted of at most $e^{(\log N_1)^C}$ intervals with measure less than $e^{-N_1^\zeta}$. Let I be one of the intervals. Since ω satisfies the Diophantine condition, for any $x \in \mathbb{T}$, there is at most one $n \in \mathbb{Z}^d$ with $|n| \leq N_3$ such that $x + n\omega \mod \mathbb{Z} \in I$. Therefore,

$$A(X_{N_1}; \{n\omega\}_{n=1}^{N_3}) \leq e^{(\log N_1)^C} \leq N_3^\varepsilon. \tag{145}$$

By Theorem 2.7, we have the Green's function satisfies Property P with parameters $(\sigma, \sigma - \varepsilon, c_2 - N_3^{-\vartheta})$ at size N_3. Standard Neumann series expansion ensures that, for any large N_0, there exists λ_0 such that, for any $\lambda > \lambda_0$, the Green's functions have Property P with parameters $\left(\sigma, \sigma - \varepsilon, \frac{4}{5}c_1\right)$ at all sizes smaller than N_0 [Jitomirskaya et al. 2020b, Theorem 4.3]. Now Theorem 3.15 follows by standard induction. See [Jitomirskaya et al. 2020b, pp. 15–16] for details. □

Proof of Theorem 3.11. Fix N_1. Let $N_2 = e^{N_1^c}$ and $N_3 = N_2^C$. Assume the Green's function in Theorem 3.11 satisfies Property P with parameters (μ, ζ, c_2) at sizes N_1 and N_2. We can again assume that X_{N_1} is a semialgebraic set with degree less than $e^{(\log N_1)^C}$. By Lemma 8.1 and Theorem 8.8,

$$A(X_{N_1}; \{n\omega\}_{n=1}^{N_3}) \leq N_3^{1-1/b\kappa+\varepsilon}. \tag{146}$$

By Theorem 2.7, we have the Green's function satisfies Property P with parameters

$$\left(\sigma, \frac{\sigma - 1}{b^2\kappa} + \frac{1}{b^3\kappa^2} - \varepsilon, c_2 - N_3^{-\vartheta}\right)$$

at size N_3. As the arguments at the end of proof of Theorem 3.15, large λ will ensure the initial scales and hence Theorem 3.11 follows by induction. □

Proof of Theorems 3.6 and 3.13. The proofs of Theorems 3.6 and 3.13 closely follow that of Theorems 3.5 and 3.12. The difference is that we need to use Lemma 8.2 instead of Lemma 8.1. □

Proof of Theorem 3.19. Replacing Lemma 8.1 with Lemma 8.3, Theorem 3.19 follows Theorem 3.11. □

Proof of Corollary 3.23. By formula (3.53) in [Bourgain et al. 2001], one has, for almost every α,

$$A(X_{N_1}; \{n\omega\}_{n=1}^{N_3}) \leq N_3^{1-\frac{1}{3}+\varepsilon}. \tag{147}$$

Let $\delta = \frac{1}{3} - \varepsilon$. Applying $\tilde{\sigma} = 1$, $\sigma = 1 - \varepsilon$ and $b_i = 2$ in Theorem 2.7 and then Theorem 2.10, we obtain Corollary 3.23. Indeed, $\frac{1}{18}$ comes from $\left(\frac{1}{3}\right)^2/b$. □

Proof of Theorem 3.27. The proof of Theorem 3.27 is similar to that of Theorems 3.15 and 3.11. We only point out the modifications.

- The induction goes in the following way. The semialgebraic set X_N intersecting with any line segments contained in $[0,1)^2$ has Lebesgue measure at most $e^{-N^{\zeta}}$. The assumption that v is not constant on any line segments ensures the initial scales.
- Replace (145) or (146) with (141).
- Since the induction is based on semialgebraic sets only on line segments, Cartan's estimate will not lead to dimension loss. In other words, when (16) is used to do the induction, $b_i = 1$. □

Remark 9.2. (1) The calculation of the bound in Theorem 3.27 goes in the following way. By (141), the sublinear bound is

$$3\kappa - \tfrac{9}{4} = 1 - \delta, \quad \text{where } \delta = \tfrac{13}{4} - 3\kappa.$$

Therefore, the bound in (16) becomes ($b_i = 1$)

$$\frac{\sigma-1}{b_i}\delta + \frac{\delta^2}{b_i} = (\sigma-1)\delta + \delta^2 = (\sigma-1)\big(\tfrac{13}{4} - 3\kappa\big) + \big(\tfrac{13}{4} - 3\kappa\big)^2.$$

(2) The induction of Theorem 3.27 follows the corresponding parts in [Bourgain and Kachkovskiy 2019]. Our quantitative approaches developed in the paper allow us to obtain the explicit bound.

Appendix: Cartan's estimates for non-self-adjoint matrices

In the following, we will prove the several-variable matrix-valued Cartan estimate (Lemma 5.1). The proof is similar to that in [Bourgain 2002; 2005a; 2007; Jitomirskaya et al. 2020b; Bourgain et al. 2002]. The improvement is that we do not assume the matrix is self-adjoint.

Lemma A.1. *Let T be the matrix*

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where T_1 is an invertible $n \times n$ matrix, T_2 is an $n \times k$ matrix, T_3 is a $k \times n$ matrix, and T_4 is a $k \times k$ matrix. Let

$$S = T_4 - T_3 T_1^{-1} T_2.$$

Then T is invertible if and only if S is invertible, and

$$\|S^{-1}\| \le \|T^{-1}\| \le C(1 + \|T_2\|)(1 + \|T_3\|)(1 + \|T_1^{-1}\|)^2(1 + \|S^{-1}\|), \tag{148}$$

where C is an absolute constant.

Proof. It is easy to check that

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ T_3 T_1^{-1} & I \end{pmatrix} \begin{pmatrix} I & T_2 \\ 0 & S \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix}. \tag{149}$$

It implies T is invertible if and only if S is invertible. By (149), one has

$$T^{-1}=\begin{pmatrix}T_1 & 0\\ 0 & I\end{pmatrix}^{-1}\begin{pmatrix}I & T_2\\ 0 & S\end{pmatrix}^{-1}\begin{pmatrix}I & 0\\ T_3T_1^{-1} & I\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}T_1^{-1} & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}I & -T_2S^{-1}\\ 0 & S^{-1}\end{pmatrix}\begin{pmatrix}I & 0\\ -T_3T_1^{-1} & I\end{pmatrix} \tag{150}$$

$$=\begin{pmatrix}\star & \star\\ \star & S^{-1}\end{pmatrix}. \tag{151}$$

Now the second inequality of (148) follows from (150) and the first one follows from (151). □

Denote by $\mathcal{D}(z,r)$ the standard disk on $\mathbb{C}$ of center z and radius $r>0$.

Lemma A.2 [Goldstein and Schlag 2008, Lemma 2.15]. *Let $f(z_1,\dots,z_J)$ be an analytic function defined in a polydisk $\mathcal{P}=\prod_{1\le i\le J}\mathcal{D}\big(z_{i,0},\frac{1}{2}\big)$ and $\phi=\log|f|$. Let $\sup_{\underline{z}\in\mathcal{P}}\phi(\underline{z})\le M, m\le\phi(\underline{z}_0)$, $\underline{z}_0=(z_{1,0},\dots,z_{J,0})$. Given sufficiently large F, there exists a set $\mathcal{B}\subset\mathcal{P}$ such that*

$$\phi(\underline{z})>M-C(J)F(M-m)\quad \textit{for any } \underline{z}\in\prod_{1\le i\le J}\mathcal{D}\big(z_{i,0},\tfrac{1}{4}\big)\setminus\mathcal{B}, \tag{152}$$

and

$$\operatorname{mes}(\mathcal{B}\cap\mathbb{R}^J)\le C(J)e^{-F^{1/J}}. \tag{153}$$

Proof of Lemma 5.1. The proof is similar to that of Lemma 3.4 in [Bourgain 2005a]. In the following proof, $C=C(J)$ and $c=c(J)$. Let

$$\mu=10^{-2}J^{-1}\delta_1(1+B_1)^{-1}(1+B_2)^{-1}.$$

Fix

$$x_0\in\big[-\tfrac{1}{2}\delta,\tfrac{1}{2}\delta\big]^J$$

and consider $T(z)$ with $|z-x_0|=\sup_{1\le i\le J}|z_i-x_{0,i}|<\mu$. Thanks to Cauchy's estimate and (105), one obtains, for $|z-x_0|<\mu$,

$$\|\partial_{z_i}T(z)\|\le\frac{4B_1}{\delta_1},\quad i=1,2,\dots,J,$$

which implies

$$\|T(z)-T(x_0)\|\le\frac{4JB_1\mu}{\delta_1}\le 25^{-1}(1+B_2)^{-1}.$$

From assumption (ii) of Lemma 5.1, we can find $V=V(x_0)$ so that $|V|=\widetilde{M}\le M$ and (106) is satisfied. Define $V^c=[1,N]\setminus V$. Thus using the standard Neumann series argument and (106), one has

$$\|(R_{V^c}T(z)R_{V^c})^{-1}\|\le 2B_2\quad\text{for } |z-x_0|<\mu. \tag{154}$$

We define for $|z-x_0|<\mu$ the analytic function

$$S(z)=R_VT(z)R_V-R_VT(z)R_{V^c}(R_{V^c}T(z)R_{V^c})^{-1}R_{V^c}T(z)R_V. \tag{155}$$

Then, by (154) and (155), we have

$$\|S(z)\| \le 3B_1^2 B_2. \tag{156}$$

Recalling Lemma A.1, if $S(z)$ is invertible, so is $T(z)$ and by (148),

$$\|S^{-1}(z)\| \le C\|T^{-1}(z)\| \le CB_1^2 B_2^2 (1 + \|S^{-1}(z)\|). \tag{157}$$

For $x \in \mathbb{R}^J$, one has

$$\|S(x)\|^{\widetilde{M}} \ge |\det S(x)|. \tag{158}$$

Let $\lambda = \min\{|\tilde{\lambda}| : \tilde{\lambda} \in \sigma(S(x))\}$. We have

$$|\det S(x)| \ge \lambda^{\widetilde{M}} \ge \|S^{-1}(x)\|^{-\widetilde{M}}. \tag{159}$$

By Cramer's rule, one has every entry of $S^{-1}(x)$ is bounded by

$$\frac{\|S(x)\|^{\widetilde{M}-1}}{|\det S(x)|}$$

and hence (by (156))

$$\|S^{-1}(x)\| \le \frac{\widetilde{M}(3B_1^2 B_2)^{\widetilde{M}}}{|\det S(x)|}. \tag{160}$$

Let

$$\phi(z) = \log|\det S(x_0 + \mu z)|, \quad |z| < 1.$$

Then, by (158) and (156),

$$\sup_{|z|<1} \phi(z) \le C\widetilde{M} \log(B_1 + B_2). \tag{161}$$

By (107) and the definition of μ, there is some x_1 with $|x_0 - x_1| < \frac{1}{10}\mu$ such that

$$\|T^{-1}(x_1)\| \le B_3. \tag{162}$$

Hence by (157), $\|S^{-1}(x_1)\| \le CB_3$, and from (159),

$$\phi(a) \ge -C\widetilde{M} \log B_3, \tag{163}$$

where $a = (x_1 - x_0)/\mu$, so $|a| < \frac{1}{10}$. Let

$$\mathcal{P} = \prod_{1\le i\le J} \mathcal{D}\big(a_i, \tfrac{1}{2}\big).$$

Therefore, one has

$$\sup_{z\in\mathcal{P}} \phi(z) \le C\widetilde{M} \log(B_1 + B_2), \phi(a) \ge -C\widetilde{M} \log B_3.$$

Applying Lemma A.2 and recalling (152), (153), for any $F \gg 1$, there is some set $\mathcal{B} \subset \prod_{1\le i\le J} \mathcal{D}\big(a_i, \frac{1}{4}\big)$ with

$$\phi(z) \ge -CF\widetilde{M} \log(B_1 + B_2 + B_3) \quad \text{for } z \in \prod_{1\le i\le J} \mathcal{D}\big(a_i, \tfrac{1}{4}\big) \setminus \mathcal{B}, \tag{164}$$

and

$$\operatorname{mes}(\mathcal{B}\cap\mathbb{R}^J)\le Ce^{-F^{1/J}}. \tag{165}$$

For $0<\epsilon<1$, let

$$F=\frac{-c\log\epsilon}{\widetilde{M}\log(B_1+B_2+B_3)}.$$

Then by (164) and (165),

$$\begin{aligned}\operatorname{mes}\bigl\{x\in\mathbb{R}^J:|x-x_1|<\tfrac14\mu\text{ and }|\det(S(x))|\le\epsilon\bigr\}&=\mu^J\operatorname{mes}\bigl\{x\in\mathbb{R}^J:|x-a|<\tfrac14\text{ and }\phi(x)\le\log\epsilon\bigr\}\\&\le C\mu^J e^{-F^{1/J}}.\end{aligned}\tag{166}$$

Since $|x_0-x_1|<\frac{1}{10}\mu$, we have

$$\operatorname{mes}\bigl\{x\in\mathbb{R}^J:|x-x_0|<\tfrac18\mu\text{ and }|\det(S(x))|\le\epsilon\bigr\}\le C\mu^J e^{-c\left(\frac{\log\epsilon^{-1}}{\widetilde{M}\log(B_1+B_2+B_3)}\right)^{1/J}}. \tag{167}$$

Recalling (157), (160) and (108), one has, for $|x-x_0|<\frac18\mu$ and $|\det S(x)|\ge\epsilon$,

$$\|T^{-1}(x)\|\le CB_1^2B_2^2\epsilon^{-1}\widetilde{M}(3B_1^2B_2)^{\widetilde{M}}\le\epsilon^{-2}. \tag{168}$$

Covering $\bigl[-\frac12\delta,\frac12\delta\bigr]^J$ by cubes of side $\frac14\mu$, and combining (167) and (168), one has

$$\begin{aligned}\operatorname{mes}\bigl\{x\in\bigl[-\tfrac12\delta,\tfrac12\delta\bigr]^J:\|T^{-1}(x)\|\ge\epsilon^{-2}\bigr\}&\le C\delta^J e^{-c\left(\frac{\log\epsilon^{-1}}{\widetilde{M}\log(B_1+B_2+B_3)}\right)^{1/J}}\\&\le C\delta^J e^{-c\left(\frac{\log\epsilon^{-1}}{M\log(B_1+B_2+B_3)}\right)^{1/J}}.\end{aligned}$$

□

Acknowledgments

I would like to thank Alan Haynes for telling me about [Montgomery 1994], which led to Lemmas 8.3 and 8.5. I also wish to thank Svetlana Jitomirskaya for comments on earlier versions of the manuscript. This research was supported by NSF DMS-1700314/2015683, DMS-2000345 and DMS-2052572.

References

[Avila 2008] A. Avila, "The absolutely continuous spectrum of the almost Mathieu operator", preprint, 2008. arXiv 0810.2965

[Avila 2015] A. Avila, "Global theory of one-frequency Schrödinger operators", *Acta Math.* **215**:1 (2015), 1–54. MR Zbl

[Avila and Damanik 2008] A. Avila and D. Damanik, "Absolute continuity of the integrated density of states for the almost Mathieu operator with non-critical coupling", *Invent. Math.* **172**:2 (2008), 439–453. MR Zbl

[Avila and Jitomirskaya 2009] A. Avila and S. Jitomirskaya, "The ten martini problem", *Ann. of Math.* (2) **170**:1 (2009), 303–342. MR Zbl

[Avila and Jitomirskaya 2010] A. Avila and S. Jitomirskaya, "Almost localization and almost reducibility", *J. Eur. Math. Soc.* **12**:1 (2010), 93–131. MR Zbl

[Avila and Jitomirskaya 2011] A. Avila and S. Jitomirskaya, "Hölder continuity of absolutely continuous spectral measures for one-frequency Schrödinger operators", *Comm. Math. Phys.* **301**:2 (2011), 563–581. MR Zbl

[Avila and Krikorian 2006] A. Avila and R. Krikorian, "Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles", *Ann. of Math.* (2) **164**:3 (2006), 911–940. MR Zbl

[Avila et al. 2011] A. Avila, B. Fayad, and R. Krikorian, "A KAM scheme for SL$(2, \mathbb{R})$ cocycles with Liouvillean frequencies", *Geom. Funct. Anal.* **21**:5 (2011), 1001–1019. MR Zbl

[Avila et al. 2017] A. Avila, J. You, and Q. Zhou, "Sharp phase transitions for the almost Mathieu operator", *Duke Math. J.* **166**:14 (2017), 2697–2718. MR Zbl

[Basu 1999] S. Basu, "On bounding the Betti numbers and computing the Euler characteristic of semi-algebraic sets", *Discrete Comput. Geom.* **22**:1 (1999), 1–18. MR Zbl

[Binyamini and Novikov 2019] G. Binyamini and D. Novikov, "Complex cellular structures", *Ann. of Math.* (2) **190**:1 (2019), 145–248. MR Zbl

[Bourgain 1998] J. Bourgain, "Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations", *Ann. of Math.* (2) **148**:2 (1998), 363–439. MR Zbl

[Bourgain 2000] J. Bourgain, "Hölder regularity of integrated density of states for the almost Mathieu operator in a perturbative regime", *Lett. Math. Phys.* **51**:2 (2000), 83–118. MR Zbl

[Bourgain 2002] J. Bourgain, "Estimates on Green's functions, localization and the quantum kicked rotor model", *Ann. of Math.* (2) **156**:1 (2002), 249–294. MR Zbl

[Bourgain 2005a] J. Bourgain, *Green's function estimates for lattice Schrödinger operators and applications*, Ann. of Math. Stud. **158**, Princeton Univ. Press, 2005. MR Zbl

[Bourgain 2005b] J. Bourgain, "Positivity and continuity of the Lyapounov [sic] exponent for shifts on $\mathbb{T}^d$ with arbitrary frequency vector and real analytic potential", *J. Anal. Math.* **96** (2005), 313–355. MR Zbl

[Bourgain 2007] J. Bourgain, "Anderson localization for quasi-periodic lattice Schrödinger operators on $\mathbb{Z}^d$, d arbitrary", *Geom. Funct. Anal.* **17**:3 (2007), 682–706. MR Zbl

[Bourgain and Goldstein 2000] J. Bourgain and M. Goldstein, "On nonperturbative localization with quasi-periodic potential", *Ann. of Math.* (2) **152**:3 (2000), 835–879. MR Zbl

[Bourgain and Jitomirskaya 2002] J. Bourgain and S. Jitomirskaya, "Absolutely continuous spectrum for 1D quasiperiodic operators", *Invent. Math.* **148**:3 (2002), 453–463. MR Zbl

[Bourgain and Kachkovskiy 2019] J. Bourgain and I. Kachkovskiy, "Anderson localization for two interacting quasiperiodic particles", *Geom. Funct. Anal.* **29**:1 (2019), 3–43. MR Zbl

[Bourgain and Klein 2013] J. Bourgain and A. Klein, "Bounds on the density of states for Schrödinger operators", *Invent. Math.* **194**:1 (2013), 41–72. MR Zbl

[Bourgain and Wang 2004] J. Bourgain and W.-M. Wang, "Anderson localization for time quasi-periodic random Schrödinger and wave equations", *Comm. Math. Phys.* **248**:3 (2004), 429–466. MR Zbl

[Bourgain and Wang 2008] J. Bourgain and W.-M. Wang, "Quasi-periodic solutions of nonlinear random Schrödinger equations", *J. Eur. Math. Soc.* **10**:1 (2008), 1–45. MR Zbl

[Bourgain et al. 2001] J. Bourgain, M. Goldstein, and W. Schlag, "Anderson localization for Schrödinger operators on $\mathbb{Z}$ with potentials given by the skew-shift", *Comm. Math. Phys.* **220**:3 (2001), 583–621. MR Zbl

[Bourgain et al. 2002] J. Bourgain, M. Goldstein, and W. Schlag, "Anderson localization for Schrödinger operators on $\mathbb{Z}^2$ with quasi-periodic potential", *Acta Math.* **188**:1 (2002), 41–86. MR Zbl

[Cai et al. 2019] A. Cai, C. Chavaudret, J. You, and Q. Zhou, "Sharp Hölder continuity of the Lyapunov exponent of finitely differentiable quasi-periodic cocycles", *Math. Z.* **291**:3-4 (2019), 931–958. MR Zbl

[Chulaevsky and Dinaburg 1993] V. A. Chulaevsky and E. I. Dinaburg, "Methods of KAM-theory for long-range quasi-periodic operators on $\mathbb{Z}^\nu$: pure point spectrum", *Comm. Math. Phys.* **153**:3 (1993), 559–577. MR Zbl

[Chulaevsky and Sinaĭ 1989] V. A. Chulaevsky and Y. G. Sinaĭ, "Anderson localization for the 1-D discrete Schrödinger operator with two-frequency potential", *Comm. Math. Phys.* **125**:1 (1989), 91–112. MR Zbl

[Craig and Simon 1983] W. Craig and B. Simon, "Log Hölder continuity of the integrated density of states for stochastic Jacobi matrices", *Comm. Math. Phys.* **90**:2 (1983), 207–218. MR Zbl

[Damanik et al. 2018] D. Damanik, M. Goldstein, W. Schlag, and M. Voda, "Homogeneity of the spectrum for quasi-periodic Schrödinger operators", *J. Eur. Math. Soc.* **20**:12 (2018), 3073–3111. MR Zbl

[Dinaburg and Sinaĭ 1975] E. I. Dinaburg and Y. G. Sinaĭ, "The one-dimensional Schrödinger equation with quasiperiodic potential", *Funkt. Anal. i Prilozhen.* **9**:4 (1975), 8–21. In Russian; translated in *Funct. Anal. Appl.* **9**:4 (1975), 279–289. MR

[Drmota and Tichy 1997] M. Drmota and R. F. Tichy, *Sequences, discrepancies and applications*, Lecture Notes in Math. **1651**, Springer, 1997. MR Zbl

[Eliasson 1992] L. H. Eliasson, "Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation", *Comm. Math. Phys.* **146**:3 (1992), 447–482. MR Zbl

[Fröhlich et al. 1990] J. Fröhlich, T. Spencer, and P. Wittwer, "Localization for a class of one-dimensional quasi-periodic Schrödinger operators", *Comm. Math. Phys.* **132**:1 (1990), 5–25. MR Zbl

[Goldstein and Schlag 2001] M. Goldstein and W. Schlag, "Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions", *Ann. of Math.* (2) **154**:1 (2001), 155–203. MR Zbl

[Goldstein and Schlag 2008] M. Goldstein and W. Schlag, "Fine properties of the integrated density of states and a quantitative separation property of the Dirichlet eigenvalues", *Geom. Funct. Anal.* **18**:3 (2008), 755–869. MR Zbl

[Goldstein et al. 2019] M. Goldstein, W. Schlag, and M. Voda, "On the spectrum of multi-frequency quasiperiodic Schrödinger operators with large coupling", *Invent. Math.* **217**:2 (2019), 603–701. MR Zbl

[Gromov 1987] M. Gromov, "Entropy, homology and semialgebraic geometry", exposé 663, pp. 225–240 in *Séminaire Bourbaki*, 1985/86, Astérisque **145-146**, Soc. Math. France, Paris, 1987. MR Zbl

[Hadj Amor 2009] S. Hadj Amor, "Hölder continuity of the rotation number for quasi-periodic co-cycles in $\mathrm{SL}(2,\mathbb{R})$", *Comm. Math. Phys.* **287**:2 (2009), 565–588. MR Zbl

[Han and Jitomirskaya 2017] R. Han and S. Jitomirskaya, "Full measure reducibility and localization for quasiperiodic Jacobi operators: a topological criterion", *Adv. Math.* **319** (2017), 224–250. MR Zbl

[Han and Zhang 2020] R. Han and S. Zhang, "Large deviation estimates and Hölder regularity of the Lyapunov exponents for quasi-periodic Schrödinger cocycles", *Int. Math. Res. Not.* (online publication May 2020), art. id. rnz319.

[Hou and You 2012] X. Hou and J. You, "Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems", *Invent. Math.* **190**:1 (2012), 209–260. MR Zbl

[Jitomirskaya 1994] S. Y. Jitomirskaya, "Anderson localization for the almost Mathieu equation: a nonperturbative proof", *Comm. Math. Phys.* **165**:1 (1994), 49–57. MR Zbl

[Jitomirskaya 1999] S. Y. Jitomirskaya, "Metal-insulator transition for the almost Mathieu operator", *Ann. of Math.* (2) **150**:3 (1999), 1159–1175. MR Zbl

[Jitomirskaya 2021] S. Jitomirskaya, "On point spectrum of critical almost Mathieu operators", *Adv. Math.* **392** (2021), art. id. 107997. MR Zbl

[Jitomirskaya and Kachkovskiy 2016] S. Jitomirskaya and I. Kachkovskiy, "L^2-reducibility and localization for quasiperiodic operators", *Math. Res. Lett.* **23**:2 (2016), 431–444. MR Zbl

[Jitomirskaya and Krasovsky 2019] S. Jitomirskaya and I. Krasovsky, "Critical almost Mathieu operator: hidden singularity, gap continuity, and the Hausdorff dimension of the spectrum", preprint, 2019. arXiv 1909.04429

[Jitomirskaya and Liu 2017] S. Jitomirskaya and W. Liu, "Arithmetic spectral transitions for the Maryland model", *Comm. Pure Appl. Math.* **70**:6 (2017), 1025–1051. MR Zbl

[Jitomirskaya and Liu 2018a] S. Jitomirskaya and W. Liu, "Universal hierarchical structure of quasiperiodic eigenfunctions", *Ann. of Math.* (2) **187**:3 (2018), 721–776. MR Zbl

[Jitomirskaya and Liu 2018b] S. Jitomirskaya and W. Liu, "Universal reflective-hierarchical structure of quasiperiodic eigenfunctions and sharp spectral transition in phase", preprint, 2018. arXiv 1802.00781

[Jitomirskaya and Marx 2012] S. Jitomirskaya and C. A. Marx, "Analytic quasi-perodic [sic] cocycles with singularities and the Lyapunov exponent of extended Harper's model", *Comm. Math. Phys.* **316**:1 (2012), 237–267. MR Zbl

[Jitomirskaya and Zhang 2022] S. Jitomirskaya and S. Zhang, "Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators", *J. Eur. Math. Soc.* (*JEMS*) **24**:5 (2022), 1723–1767. MR Zbl

[Jitomirskaya et al. 2020a] S. Jitomirskaya, H. Krüger, and W. Liu, "Exact dynamical decay rate for the almost Mathieu operator", *Math. Res. Lett.* **27**:3 (2020), 789–808. MR Zbl

[Jitomirskaya et al. 2020b] S. Jitomirskaya, W. Liu, and Y. Shi, "Anderson localization for multi-frequency quasi-periodic operators on $\mathbb{Z}^D$", *Geom. Funct. Anal.* **30**:2 (2020), 457–481. MR Zbl

[Klein 2005] S. Klein, "Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function", *J. Funct. Anal.* **218**:2 (2005), 255–292. MR Zbl

[Klein 2014] S. Klein, "Localization for quasiperiodic Schrödinger operators with multivariable Gevrey potential functions", *J. Spectr. Theory* **4**:3 (2014), 431–484. MR Zbl

[Kuipers and Niederreiter 1974] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley, New York, 1974. MR Zbl

[Liu 2020] W. Liu, "Almost Mathieu operators with completely resonant phases", *Ergodic Theory Dynam. Systems* **40**:7 (2020), 1875–1893. MR Zbl

[Liu 2022] W. Liu, "Power law logarithmic bounds of moments for long range operators in arbitrary dimension", preprint, 2022. arXiv 2212.02411

[Liu and Shi 2019] W. Liu and Y. Shi, "Upper bounds on the spectral gaps of quasi-periodic Schrödinger operators with Liouville frequencies", *J. Spectr. Theory* **9**:4 (2019), 1223–1248. MR Zbl

[Liu and Yuan 2015a] W. Liu and X. Yuan, "Anderson localization for the almost Mathieu operator in the exponential regime", *J. Spectr. Theory* **5**:1 (2015), 89–112. MR Zbl

[Liu and Yuan 2015b] W. Liu and X. Yuan, "Anderson localization for the completely resonant phases", *J. Funct. Anal.* **268**:3 (2015), 732–747. MR Zbl

[Liu and Yuan 2015c] W. Liu and X. Yuan, "Hölder continuity of the spectral measures for one-dimensional Schrödinger operator in exponential regime", *J. Math. Phys.* **56**:1 (2015), art. id. 012701. MR Zbl

[Marx and Jitomirskaya 2017] C. A. Marx and S. Jitomirskaya, "Dynamics and spectral theory of quasi-periodic Schrödinger-type operators", *Ergodic Theory Dynam. Systems* **37**:8 (2017), 2353–2393. MR Zbl

[Montgomery 1994] H. L. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, CBMS Reg. Conf. Ser. Math. **84**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[Moser and Pöschel 1984] J. Moser and J. Pöschel, "An extension of a result by Dinaburg and Sinaĭ on quasiperiodic potentials", *Comment. Math. Helv.* **59**:1 (1984), 39–85. MR Zbl

[Schlag 2001] W. Schlag, "On the integrated density of states for Schrödinger operators on $\mathbb{Z}^2$ with quasi periodic potential", *Comm. Math. Phys.* **223**:1 (2001), 47–65. MR Zbl

[Schmidt 1964] W. M. Schmidt, "Metrical theorems on fractional parts of sequences", *Trans. Amer. Math. Soc.* **110** (1964), 493–518. MR Zbl

[Shamis and Sodin 2021] M. Shamis and S. Sodin, "Upper bounds on quantum dynamics in arbitrary dimension", preprint, 2021. arXiv 2111.10902

[Shi 2022] Y. Shi, "Spectral theory of the multi-frequency quasi-periodic operator with a Gevrey type perturbation", *J. Anal. Math.* **148**:1 (2022), 305–338. MR Zbl

[Shi and Yuan 2020] J. Shi and X. Yuan, "Anderson localization for Jacobi matrices associated with high-dimensional skew shifts", *Chin. Ann. Math. Ser. B* **41**:4 (2020), 495–510. MR Zbl

[Simon 1985] B. Simon, "Almost periodic Schrödinger operators, IV: The Maryland model", *Ann. Phys.* **159**:1 (1985), 157–183. MR Zbl

[Sinaĭ 1987] Y. G. Sinaĭ, "Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential", *J. Stat. Phys.* **46**:5-6 (1987), 861–909. MR Zbl

[Tao 2019a] K. Tao, "Non-perturbative positive Lyapunov exponent of Schrödinger equations and its applications to skew-shift mapping", *J. Differential Equations* **266**:6 (2019), 3559–3579. MR Zbl

[Tao 2019b] K. Tao, "Non-perturbative weak Hölder continuity of Lyapunov exponent of discrete analytic Jacobi operators with skew-shift mapping", *Electron. J. Differential Equations* **2019** (2019), art. id. 81. MR Zbl

[Wang 2016a] W.-M. Wang, "Energy supercritical nonlinear Schrödinger equations: quasiperiodic solutions", *Duke Math. J.* **165**:6 (2016), 1129–1192. MR Zbl

[Wang 2016b] W.-M. Wang, "Quasi-periodic solutions for nonlinear Klein–Gordon equations", preprint, 2016. arXiv 1609.00309

[Wang 2019a] W.-M. Wang, "Infinite energy quasi-periodic solutions to nonlinear Schrödinger equations on $\mathbb{R}$", preprint, 2019. arXiv 1908.11627

[Wang 2019b] W.-M. Wang, "Quasi-periodic solutions to nonlinear PDEs", pp. 127–175 in *Harmonic analysis and wave equations*, edited by J.-M. Coron et al., Ser. Contemp. Appl. Math. **23**, Higher Ed., Beijing, 2019. MR Zbl

[Wang 2020] W.-M. Wang, "Space quasi-periodic standing waves for nonlinear Schrödinger equations", *Comm. Math. Phys.* **378**:2 (2020), 783–806. MR Zbl

[Yomdin 1987] Y. Yomdin, "C^k-resolution of semialgebraic mappings", *Israel J. Math.* **57**:3 (1987), 301–317. MR Zbl

[You 2018] J. You, "Quantitative almost reducibility and its applications", pp. 2113–2135 in *Proceedings of the International Congress of Mathematicians*, *III* (Rio de Janeiro, 2018), edited by B. Sirakov et al., World Sci., Hackensack, NJ, 2018. MR Zbl

[Zhao 2020] X. Zhao, "Hölder continuity of absolutely continuous spectral measure for multi-frequency Schrödinger operators", *J. Funct. Anal.* **278**:12 (2020), art. id. 108508. MR Zbl

Received 24 Sep 2020. Revised 29 Mar 2021. Accepted 19 May 2021.

WENCAI LIU: liuwencai1226@gmail.com, wencail@tamu.edu
Department of Mathematics, Texas A&M University, College Station, TX, United States

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2022 is US $370/year for the electronic version, and $580/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 15 No. 8 2022