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Journal of Functional Analysis

www.elsevier.com/locate/jfa



Full Length Article

Irreducibility of the Bloch variety for finite-range
Schrödinger operators



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ARTICLE INFO

Article history:

Received 13 July 2021

Accepted 24 July 2022

Available online 5 August 2022

Communicated by Benjamin Schlein

Keywords:

Irreducibility

Bloch variety

Periodic operator

ABSTRACT

We study the Bloch variety of discrete Schrödinger operators associated with a complex periodic potential and a general finite-range interaction, showing that the Bloch variety is irreducible for a wide class of lattice geometries in arbitrary dimension. Examples include the triangular lattice and the extended Harper lattice.

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1. Introduction

1.1. Setting and main theorem

We will study periodic finite-range Schrödinger operators of the form

$$H = A + V, \tag{1.1}$$

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¹ J. F. was supported in part by Simons Foundation Collaboration Grant #711663.

² W. L. and R. M. were partially supported by NSF DMS-2000345 and DMS-2052572.

acting in $\ell^2(\mathbb{Z}^d)$, where V is periodic and A is a Toeplitz operator given by

$$[A\psi]_n = \sum_{m \in \mathbb{Z}^d} a_{n-m} \psi_m.$$

Here, $\{a_n\}_{n \in \mathbb{Z}^d}$ is finitely supported and V will as usual denote both the potential $V : \mathbb{Z}^d \rightarrow \mathbb{C}$ and the corresponding multiplication operator $[V\psi]_n = V_n \psi_n$. We say that V is q -periodic for $q = (q_1, \dots, q_d) \in \mathbb{N}^d$ if $V_{n+q_j e_j} = V_n$ for all $n \in \mathbb{Z}^d$ and each $1 \leq j \leq d$, where e_j denotes the standard j th basis vector.

In particular, let us note that the approach discussed herein does not rely on reality of the potential or self-adjointness of A . The case in which

$$a_n = \begin{cases} -1 & n = \pm e_j \text{ for some } 1 \leq j \leq d \\ 0 & \text{otherwise} \end{cases}$$

corresponds to $A = -\Delta$, the discrete Laplacian.

Our main result is irreducibility of the Bloch variety for all operators of the form (1.1) subject to a suitable condition on A . In particular, under mild assumptions on $\{a_n\}_{n \in \mathbb{Z}^d}$, the result holds universally for all periodic V , including complex-valued potentials. We will define the Bloch variety precisely later in the manuscript (see Section 1.2); for now, the reader may think of it as a relation between the energy, λ , and the quasi-momentum, k , which is given by the zero set of a function $\mathcal{P}(z, \lambda)$ that is a polynomial in λ and a Laurent polynomial in $z = e^{2\pi i k}$.

Let us describe some of the main objects and assumptions used in this work. Starting with A we generate the Laurent polynomial

$$p(z) = p_A(z) = \sum_{n \in \mathbb{Z}^d} a_n z^{-n}, \quad (1.2)$$

where we employ the standard multi-index notation $z^n = z_1^{n_1} \cdots z_d^{n_d}$.

Let us state our assumptions here in a moderately informal manner. For further definitions, details, and a more precise account, we refer the reader to Section 2, where we define the component of lowest degree and the fundamental domain W . Let h denote the lowest degree component of p in the sense that $p(z) = h(z) + \text{higher order terms}$. Given $q = (q_1, \dots, q_d) \in \mathbb{N}^d$ and $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$, the vector $\mu_n = (\mu_n^1, \dots, \mu_n^d)$ is defined by

$$\mu_n^j = e^{2\pi i n_j / q_j}, \quad 1 \leq j \leq d.$$

Given $z \in \mathbb{C}^d$, we define $\mu_n \odot z = (\mu_n^1 z_1, \dots, \mu_n^d z_d)$. Our main assumptions are the following:

(A₁) The degree of h is negative.

(A₂) The polynomials $h(\mu_n \odot z)$, $n \in W$, are pairwise distinct (cf. (2.2), (2.3), and (2.4)).

Theorem 1.1. *Let $q = (q_1, q_2, \dots, q_d)$ be given and let V be q -periodic. If p_A satisfies Assumptions (A₁) and (A₂), then the Bloch variety of $H = A + V$ is irreducible modulo periodicity.*

Remark 1.2. Let us make a few comments about Theorem 1.1.

- (1) The precise definitions of the Bloch variety and irreducibility modulo periodicity are given in Section 1.2. To prove Theorem 1.1, we use standard Floquet theory to obtain a Laurent polynomial $\mathcal{P}(z, \lambda)$ with the property that (k, λ) belongs to the Bloch variety if and only if $\mathcal{P}(e^{2\pi i k}, \lambda) = 0$. We show that \mathcal{P} is irreducible as a function of z and λ . Since the Bloch variety is defined as a set of pairs (k, λ) , it is only irreducible after one quotients out the relevant \mathbb{Z}^d action. This is indeed necessary as shown by the free Laplacian, see Equation (1.22) and Figure 4 in [14]. The main idea of the proof is to reduce from the operator $A + V$ to A by focusing on the lowest-degree component of \mathcal{P} (after a suitable change of variables). We show that reducibility combined with Assumptions (A₁) and (A₂) would imply mutually contradictory properties of the lowest-degree component.
- (2) Assumption (A₁) only depends on p_A , whereas Assumption (A₂) depends on p_A and q (via the action of the vectors μ_n). As the reader can see, our proof does not require V to be real-valued.
- (3) The strength of the result comes from the generality of the operators under consideration. For instance, this can handle operators in higher dimensions on rather general graphs; specifically, one can handle \mathbb{Z}^d -periodic graphs for which \mathbb{Z}^d acts transitively on vertices (see Remark 6.1 for further details). We emphasize that our result does not necessarily imply irreducibility of the Fermi varieties associated with $A + V$.

Theorem 1.1 is the main motivation for this work. It will follow from a more general result formulated in Theorem 2.6 below.

The above assumptions are satisfied and straightforward to verify in many cases of interest. To illustrate scope of applications, we enumerate some corollaries.

We first note that Theorem 1.1 provides a direct proof of the irreducibility of the Bloch variety for all discrete Schrödinger operators on \mathbb{Z}^d .

Corollary 1.3. *If $A = -\Delta$ denotes the Laplacian on $\ell^2(\mathbb{Z}^d)$, then for any periodic V , the Bloch variety of $A + V$ is irreducible modulo periodicity.*

The corollary above was already known via results about the Fermi variety – see the discussion in Section 1.2 for additional details and references. Thus, we supply an alternative argument, working directly on the Bloch variety.

More significantly, Theorem 1.1 also enables one to prove irreducibility of the Bloch variety for other lattice geometries in arbitrary dimension. To remain concrete, we present a couple of two dimensional examples but the reader may readily recognize from the proofs that generalizations are possible (compare Remark 6.1).

Corollary 1.4. *If A denotes the Laplacian on the extended Harper lattice, q_1 and q_2 are coprime, and V is q -periodic, then the Bloch variety of $A + V$ is irreducible modulo periodicity.*

Corollary 1.5. *If A denotes the Laplacian on the triangular lattice, then for any periodic V , the Bloch variety of $A + V$ is irreducible modulo periodicity.*

Note that irreducibility of the Bloch variety is potentially sensitive to modifications in the hopping terms (i.e., the matrix elements of A). To the best of our knowledge, even the results of Corollaries 1.4 and 1.5 are new. For further details, including definitions of the triangular and extended Harper lattices, see Section 6. To emphasize the distinction between the above models, we present the corresponding polynomials below, recalling that Equation (1.2) provides the dictionary between A and P_A .

(i) For the discrete Laplacian on \mathbb{Z}^d ,

$$p_{-\Delta}(z) = - \left(z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + \cdots + z_d + \frac{1}{z_d} \right)$$

(ii) For the extended Harper lattice

$$p_{\text{EHM}}(z) = - \left(z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + \frac{z_1}{z_2} + \frac{z_2}{z_1} + z_1 z_2 + \frac{1}{z_1 z_2} \right)$$

(iii) For the triangular lattice,

$$p_{\text{tri}}(z) = - \left(z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + \frac{z_1}{z_2} + \frac{z_2}{z_1} \right).$$

In particular, in dimension $d = 2$, $p_{\text{EHM}}(z)$ adds to $p_{-\Delta}(z)$ next nearest neighbor terms and is symmetric with respect to the map $z_j \mapsto z_j^{-1}$ for $j = 1, 2$. The polynomial $p_{\text{tri}}(z)$ does not possess this symmetry, nonetheless the corresponding variety still falls into the scope of Theorem 1.1. The triangular lattice is depicted in Fig. 1. Applying a simple shear transformation reduces the triangular lattice to the square lattice with additional edges, as shown in Fig. 2, and hence places the Laplacian on the triangular lattice into the context of the paper after a suitable change of coordinates.

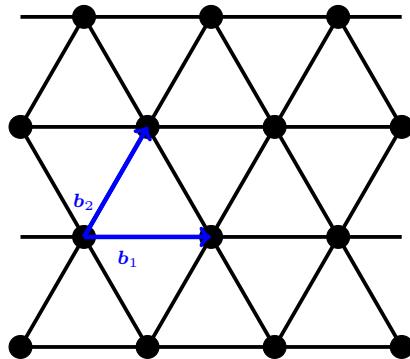


Fig. 1. A portion of the triangular lattice.

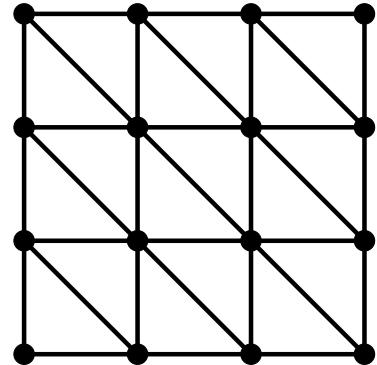


Fig. 2. The triangular lattice after shearing.

1.2. Definitions and context

Let us now give relevant definitions and context. Given $q_i \in \mathbb{N}$, $i = 1, 2, \dots, d$, let $\Gamma = \Gamma_q := q_1\mathbb{Z} \oplus q_2\mathbb{Z} \oplus \dots \oplus q_d\mathbb{Z}$. We say that a function $V : \mathbb{Z}^d \rightarrow \mathbb{C}$ is q -periodic (Γ -periodic, or just periodic) if $V_{n+\gamma} = V_n$ for all $n \in \mathbb{Z}^d$ and all $\gamma \in \Gamma$.

Definition 1.6. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For $z = (z_1, \dots, z_d) \in (\mathbb{C}^*)^d$ and $q = (q_1, \dots, q_d) \in \mathbb{N}^d$, the space $\mathcal{H}(z, q)$ consists of those $\psi : \mathbb{Z}^d \rightarrow \mathbb{C}$ for which

$$\psi_{n+j \odot q} = z^j \psi_n, \forall n, j \in \mathbb{Z}^d, \quad (1.3)$$

where we write $j \odot q = (j_1 q_1, \dots, j_d q_d)$ and use the multi-index notation $z^j = z_1^{j_1} \cdots z_d^{j_d}$. Naturally, $\mathcal{H}(z, q)$ is a Hilbert space of finite dimension $Q := q_1 \cdots q_d$.

If $V : \mathbb{Z}^d \rightarrow \mathbb{C}$ is q -periodic, the corresponding Bloch variety is given by

$$B = B(H) = \{(k, \lambda) \in \mathbb{C}^{d+1} : H\psi = \lambda\psi \text{ enjoys a nonzero solution in } \mathcal{H}(e^{2\pi i k}, q)\}, \quad (1.4)$$

where we write $e^{2\pi i k} = (e^{2\pi i k_1}, \dots, e^{2\pi i k_d}) \in (\mathbb{C}^*)^d$. We employ here a standard abuse of notation in which H represents both the self-adjoint operator in $\ell^2(\mathbb{Z}^d)$ and the difference operator acting in, say, $\ell^\infty(\mathbb{Z}^d)$.

Definition 1.7. We will say the Bloch variety $B(H)$ is *irreducible modulo periodicity* if for every two irreducible components Ω_1 and Ω_2 of $B(H)$, there exists $m \in \mathbb{Z}^d$ such that $\Omega_1 = (m, 0) + \Omega_2$.

Definition 1.8. Given $\lambda \in \mathbb{C}$, the Fermi surface (variety) $F_\lambda(H)$ is defined as the level set of the Bloch variety:

$$F_\lambda(H) = \{k \in \mathbb{C}^d : (k, \lambda) \in B(H)\}.$$

We should mention that reducible Fermi and Bloch varieties are known to occur for periodic graph operators, e.g., [8,24]. One challenging problem in the study of periodic operators is to prove the (ir)reducibility of the Bloch and Fermi varieties [3–5,8–10,13,17, 22,24]. For instance, irreducibility of the Bloch variety implies that in case $B(H) \cap U \neq \emptyset$ for some open set $U \subset \mathbb{C}^{d+1}$, the knowledge of $B(H) \cap U$ allows one to recover $B(H)$. Besides its own importance in algebraic geometry, the (ir)reducibility of these varieties is crucial in the study of spectral properties of periodic elliptic operators. In particular, this has implications for the structure of spectral band edges [7,20], the isospectrality [18,19] and the existence of embedded eigenvalues for operators perturbed by a local defect [1,6, 12,15,16,21,23]. Based on existing evidence, Kuchment conjectures that the Bloch variety of any periodic second-order elliptic operator is irreducible [14, Conjecture 5.17].

There have been many works that address irreducibility of the Bloch and Fermi varieties (see, e.g., [2–5,10,13,20]). In two dimensions, Bättig [2] showed that the Bloch variety $B(-\Delta + V)$ is irreducible (modulo periodicity). In [10], Gieseker, Knörrer and Trubowitz proved that $F_\lambda(-\Delta + V)/\mathbb{Z}^2$ is irreducible except for finitely many values of λ . When $d = 3$, irreducibility of $F_\lambda(-\Delta + V)/\mathbb{Z}^d$ for any $\lambda \in \mathbb{C}$ was proved by Bättig [4]. In a recent paper of the second author [20] it is showed that when $d \geq 3$ $F_\lambda(-\Delta + V)/\mathbb{Z}^d$ is irreducible for any $\lambda \in \mathbb{C}$ and when $d = 2$ $F_\lambda(-\Delta + V)/\mathbb{Z}^2$ is irreducible for all $\lambda \in \mathbb{C} \setminus [V]$, where $[V]$ is the average of V over one periodicity cell. It follows from these works that when $d \geq 2$ the Bloch variety $B(-\Delta + V)$ is irreducible (modulo periodicity). For continuous periodic Schrödinger operators, when $d = 2$ Knörrer and Trubowitz [13] proved that the Bloch variety is irreducible (modulo periodicity) and for $d = 3$, Bättig, Knörrer and Trubowitz proved that the Fermi variety at any level is irreducible (modulo periodicity) for separable periodic potentials [5]. In [17], irreducibility of the Fermi variety for all but finitely many energies is proved for a suitable class of planar periodic graphs.

In [2–5,10,13], algebraic geometry techniques are employed to construct the toroidal and directional compactifications of Fermi and Bloch varieties and understand asymptotics of their defining (Laurent) polynomials. The perspective employed by us in the current manuscript is inspired by [20]. In general terms, the goal is to explicitly calculate asymptotics of the (Laurent) polynomials at $z \in \{z : z_j = 0 \text{ or } z_j = \infty, j = 1, 2, \dots, k\}$ and show that these asymptotics contain enough information about the original variety. Concretely, the proof is based on changing variables, studying of the lowest degree components of a family of (Laurent) polynomials in several variables and degree arguments. With regards to the Bloch variety, we expand the approach of [20] in different directions. As a consequence, for the main result of Theorem 2.6 below, the underlying lattice may be of very general nature and contain somewhat arbitrary finite-range connections (see (A_1) and (A_2) below for a more precise statement of the assumptions). In particular, we obtain irreducibility for the Bloch variety corresponding to periodic Schrödinger operators on the triangular lattice and the extended Harper lattice; see Section 6 for a precise description of these examples. While our approach is inspired by [20], we do not follow

the same path. By working directly with the lowest degree components, we can eschew a discussion of asymptotic statements about the varieties themselves.

The structure of the paper is as follows. We precisely formulate Theorem 2.6, our main result, in Section 2. Section 3 contains preparatory technical results that are then employed in Section 4 to prove Theorem 2.6. We elucidate the connection between this result and periodic operators in Section 5, which also contains some relevant background on periodic long-range Schrödinger operators. We conclude in Section 6 with the proof of Theorem 1.1 and some relevant examples and applications.

Acknowledgments. We are grateful to the anonymous reviewer for carefully reading the manuscript and for helpful suggestions that improved the paper.

2. Main result

To state the main result, we begin by recalling some crucial terminology.

Definition 2.1. Suppose f is a Laurent monomial in m variables, that is, $f(z) = cz^\alpha = cz_1^{\alpha_1}z_2^{\alpha_2}\cdots z_m^{\alpha_m}$ with $\alpha_i \in \mathbb{Z}$ for $i = 1, \dots, m$ and $c \neq 0$. The degree of f is defined as $\deg(f) = \alpha_1 + \alpha_2 + \cdots + \alpha_m$. Abusing notation slightly, we also denote $\deg(\alpha) = \alpha_1 + \alpha_2 + \cdots + \alpha_m$ for the multi-index $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m$.

Definition 2.2. Given a Laurent polynomial

$$p(z) = \sum c_\alpha z^\alpha,$$

let $L_- = \min\{\deg(\alpha) : c_\alpha \neq 0\}$. Then, the *lowest degree component* of p is defined to be the Laurent polynomial

$$h(z) = \sum_{\deg \alpha = L_-} c_\alpha z^\alpha.$$

One of the crucial properties of this notion is the following: denoting the lowest-degree component of p by \underline{p} , one has $\underline{(fg)} = \underline{f} \cdot \underline{g}$, which enables one to relate factorizations of a polynomial to factorizations of its lowest-degree component. Obviously, some care is needed to deduce nontrivial consequences from this observation in the context of our main result.

Let us write $\mathbb{C}[z_1, \dots, z_m] =: \mathbb{C}[z]$ for the set of polynomials in z_1, \dots, z_m . Similarly, we write $\mathbb{C}[z_1, z_1^{-1}, \dots, z_m, z_m^{-1}] =: \mathbb{C}[z, z^{-1}]$ for the set of Laurent polynomials³ in z_1, \dots, z_m .

³ This involves a minor, albeit common abuse of notation, since one has the relation $z_j z_j^{-1} = 1$ in $\mathbb{C}[z, z^{-1}]$.

Definition 2.3. Recall that a polynomial $\mathcal{P} \in \mathbb{C}[z]$ is called *reducible* if there exist non-constant polynomials $f, g \in \mathbb{C}[z]$ such that $\mathcal{P} = fg$ and *irreducible* otherwise. Similarly, we say that a Laurent polynomial $\mathcal{P} \in \mathbb{C}[z, z^{-1}]$ is irreducible if it can not be factorized non-trivially, that is, there are no non-monomial Laurent polynomials f, g such that $\mathcal{P} = fg$.

Notice that nonconstant monomials are units in the algebra of Laurent polynomials, which accounts for a small subtlety. That is, one must be somewhat careful here with zeros at $z = 0$ and $z = \infty$. The polynomial z^2 is reducible in $\mathbb{C}[z]$ but is a unit in $\mathbb{C}[z, z^{-1}]$. In practice, this should cause no confusion, and we will write that \mathcal{P} is irreducible in $\mathbb{C}[z]$ (respectively in $\mathbb{C}[z, z^{-1}]$) if we wish to emphasize the sense in which irreducibility is meant in a specific context.

Remark 2.4. If P is an irreducible Laurent polynomial in m variables, then the corresponding variety $\{z \in (\mathbb{C}^*)^m : P(z) = 0\}$ is irreducible as an analytic set.⁴ Thus, the overall strategy of our work is to show that a suitable Laurent polynomial that describes the Bloch variety is irreducible. Concretely, we may consider the set $\mathcal{B}(H)$ which consists of those $(z, \lambda) \in (\mathbb{C}^*)^d \times \mathbb{C}$ such that $H\psi = \lambda\psi$ enjoys a nontrivial solution $\psi \in \mathcal{H}(z, q)$. By Floquet theory, one may determine a suitable Laurent polynomial $\mathcal{P}(z, \lambda)$ such that $\mathcal{B}(H)$ is precisely the zero set of \mathcal{P} (see Section 5). Thus, since $(k, \lambda) \in \mathcal{B}(H)$ if and only if $(e^{2\pi i k}, \lambda) \in \mathcal{B}(H)$, to show that $\mathcal{B}(H)$ is irreducible modulo periodicity, it suffices to show that the corresponding Laurent polynomial is irreducible.

Let us begin by collecting some notation that we will use throughout the paper. Given $q = (q_1, \dots, q_d) \in \mathbb{N}^d$, we define the lattice Γ by

$$\Gamma = \bigoplus_{j=1}^d q_j \mathbb{Z} = \{n \in \mathbb{Z}^d : q_j | n_j \ \forall 1 \leq j \leq d\} \quad (2.1)$$

and the fundamental cell, W , by

$$W = \{n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d : 0 \leq n_j \leq q_j - 1, j = 1, 2, \dots, d\} = \mathbb{Z}^d \cap \prod_{j=1}^d [0, q_j]. \quad (2.2)$$

Given $n \in W$ and $j \in \{1, \dots, d\}$, let

$$\mu_n^j = e^{2\pi i \frac{n_j}{q_j}}, \quad (2.3)$$

and denote by μ_n the vector $(\mu_n^1, \dots, \mu_n^d)$. We also let

⁴ The converse is clearly false, which may be considered by considering the variety associated with $f(z)^2$, where $f(z)$ is irreducible. This issue can be elegantly resolved using the language of schemes [11].

$$\mu_n \odot (z_1, z_2, \dots, z_d) = (\mu_n^1 z_1, \mu_n^2 z_2, \dots, \mu_n^d z_d). \quad (2.4)$$

Let p be a Laurent polynomial and define

$$p_n(z) = p(\mu_n \odot z), \quad n \in W, \quad z \in (\mathbb{C}^*)^d. \quad (2.5)$$

We shall work with Laurent polynomials in $m = d + 1$ variables z_1, \dots, z_d, λ . Abusing notation somewhat, we write $\mathbb{C}[z, \lambda]$ (respectively $\mathbb{C}[z, \lambda, z^{-1}, \lambda^{-1}]$) for the set of polynomials (respectively the set of Laurent polynomials) in z and λ . The Laurent polynomials of interest in the present work are those of the form

$$\tilde{\mathcal{P}}(z, \lambda) = \prod_{n \in W} (p_n(z) - \lambda) + \sum_{X \in \mathcal{S}} C_X \prod_{n \in X} (p_n(z) - \lambda), \quad (2.6)$$

where the summation runs over X in an arbitrary collection \mathcal{S} of proper subsets of W and $C_X \in \mathbb{C}$. In fact, $\tilde{\mathcal{P}}(z, \lambda)$ is a Laurent polynomial in the variable z and a polynomial in λ . Collecting terms, we see that

$$\tilde{\mathcal{P}}(z, \lambda) = (-1)^Q \lambda^Q + \sum_{k=0}^{Q-1} b_k(z) \lambda^k, \quad (2.7)$$

where $b_k \in \mathbb{C}[z, z^{-1}]$ and $Q = q_1 \cdots q_d$.

Note that we do not exclude the case $\emptyset \in \mathcal{S}$, our convention being that $\prod_{n \in \emptyset} (p_n(z) - \lambda) = 1$. These are exactly the types of polynomials that one produces by expanding the determinant of the Floquet operator associated with a suitable periodic operator, hence their interest in the current work.

For each X , the constant C_X is assumed to be independent of λ and z . Assume further that $\tilde{\mathcal{P}}(z, \lambda)$ is invariant under action of each μ_n , i.e.,

$$\tilde{\mathcal{P}}(z, \lambda) = \tilde{\mathcal{P}}(\mu_n \odot z, \lambda) \text{ for all } n \in W. \quad (2.8)$$

Remark 2.5. The assumptions (2.6) and (2.8) include the central example where

$$\tilde{\mathcal{P}}(z, \lambda) = \det(D + B - \lambda I)$$

and the matrices $D = D(z)$ and B are defined by

$$D(n, n') = p_n(z) \delta_{n, n'} \quad (2.9)$$

$$B(n, n') = \widehat{V} \left(\frac{n_1 - n'_1}{q_1}, \dots, \frac{n_d - n'_d}{q_d} \right), \quad n, n' \in W. \quad (2.10)$$

Here \widehat{V} denotes the discrete Fourier transform of V , defined as in (5.1). For further discussion, see Section 5, especially Proposition 5.3.

Let us note the key properties are that D is a diagonal matrix and the entries of B are independent of z . Consequently, neither self-adjointness of A or real-valuedness of V is a crucial ingredient.

Since $\tilde{\mathcal{P}}(z, \lambda)$ is invariant under the action of each μ_n , it is elementary to check (cf. Lemma 3.1) that there exists $\mathcal{P}(z, \lambda)$ such that

$$\tilde{\mathcal{P}}(z, \lambda) = \mathcal{P}(z_1^{q_1}, z_2^{q_2}, \dots, z_d^{q_d}, \lambda). \quad (2.11)$$

Our goal is to show that $\mathcal{P}(z, \lambda)$ is irreducible as a Laurent polynomial under the assumptions below.

- (A_1) $\deg(h) < 0$, where h denotes the lowest degree component of p , (see Definition 2.2).
- (A_2) Letting $h_n(z) = h(\mu_n \odot z)$, the polynomials $\{h_n(z)\}_{n \in W}$ are pairwise distinct.

The reader may readily check that $p_{n+m}(z) = p_n(\mu_m \odot z)$ (with addition of indices computed mod Γ). Thus, to check Assumption (A_2) in practice, it suffices to show that $h_0 \neq h_n$ for every $n \in W \setminus \{0\}$.

Theorem 2.6. *Let $p \in \mathbb{C}[z, z^{-1}]$, $q \in \mathbb{N}^d$, \mathcal{S} a collection of proper subsets of W , and complex numbers $\{C_X\}_{X \in \mathcal{S}}$ be given. Assume that $\tilde{\mathcal{P}}$ is a polynomial of the form (2.6) obeying (2.8), and let \mathcal{P} be the polynomial given by (2.11). Under Assumptions (A_1) and (A_2), we conclude that \mathcal{P} is irreducible as a Laurent polynomial.*

As mentioned in Remark 2.5, the connection to Schrödinger operators and Theorem 1.1 will be established in Section 5.

Remark 2.7. Let us collect some notation from the previous paragraphs that will be repeatedly used throughout the proofs.

- (1) $\mathbb{C}[z]$ (resp. $\mathbb{C}[z, z^{-1}]$) denotes the set of polynomials (resp. Laurent polynomials) in z_1, \dots, z_d .
- (2) $p \in \mathbb{C}[z, z^{-1}]$.
- (3) $h(z)$ is the lowest degree component of $p(z)$.
- (4) $\Gamma = q_1 \mathbb{Z} \oplus \dots \oplus q_d \mathbb{Z}^d$, $W = \mathbb{Z}^d \cap \prod_{j=1}^d [0, q_j)$, $\mathcal{S} \subset 2^W \setminus \{W\}$ is arbitrary.
- (5) $\mu_n^j = e^{2\pi i n_j / q_j}$, $n \in \mathbb{Z}^d$, $j = 1, \dots, d$.
- (6) For $n \in W$, $\mu_n = (\mu_n^1, \dots, \mu_n^d)$.
- (7) $p_n(z) = p(\mu_n \odot z)$.
- (8) $\tilde{\mathcal{P}}(z, \lambda)$ is given by

$$\tilde{\mathcal{P}}(z, \lambda) = \prod_{n \in W} (p_n(z) - \lambda) + \sum_{X \in \mathcal{S}} C_X \prod_{n \in X} (p_n(z) - \lambda),$$

- (9) $Q = q_1 \cdots q_d$

(10) $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$ for $z \in (\mathbb{C}^*)^d$, $\alpha \in \mathbb{Z}^d$.

(11) $\mathcal{P}(z, \lambda)$ is defined by

$$\tilde{\mathcal{P}}(z, \lambda) = \mathcal{P}(z_1^{q_1}, z_2^{q_2}, \dots, z_d^{q_d}, \lambda).$$

(12) $a \odot b = (a_1 b_1, \dots, a_d b_d)$ for ordered d -tuples $a = (a_1, \dots, a_d)$ and $b = (b_1, \dots, b_d)$.

3. Technical lemmas

Lemma 3.1. *Suppose \tilde{g} is a Laurent polynomial in z and λ . With notation as in Remark 2.7, one has that $\tilde{g}(z, \lambda) \equiv \tilde{g}(\mu_n \odot z, \lambda)$ for every $n \in W$ if and only if there is a Laurent polynomial $g(w, \lambda)$ such that*

$$\tilde{g}(z, \lambda) \equiv g(z_1^{q_1}, \dots, z_d^{q_d}, \lambda). \quad (3.1)$$

Proof. If $g(w, \lambda)$ satisfying (3.1) exists, then it readily follows from the definition of $\mu_n \odot z$ that

$$\tilde{g}(\mu_n \odot z, \lambda) = g((\mu_n^1 z_1)^{q_1}, \dots, (\mu_n^d z_d)^{q_d}, \lambda) = g(z_1^{q_1}, \dots, z_d^{q_d}, \lambda) = \tilde{g}(z, \lambda).$$

To see that the converse implication holds, write

$$\tilde{g}(z, \lambda) = \sum_{\ell \in \mathbb{Z}^d, m \in \mathbb{Z}} \tilde{c}_{\ell, m} z^\ell \lambda^m \text{ and let } g(w, \lambda) = \sum_{\ell \in \mathbb{Z}^d, m \in \mathbb{Z}} c_{\ell, m} w^\ell \lambda^m$$

be another Laurent polynomial. Note that

$$g(z_1^{q_1}, \dots, z_d^{q_d}, \lambda) = \sum_{\ell \in \mathbb{Z}^d, m \in \mathbb{Z}} c_{\ell, m} z^{\ell \odot q} \lambda^m.$$

Thus (3.1) holds if and only if for all $m \in \mathbb{Z}$

$$\tilde{c}_{\ell, m} = \begin{cases} c_{r, m} & \ell = q \odot r \in \Gamma, \\ 0 & \text{otherwise,} \end{cases}$$

and hence g satisfying (3.1) exists if and only if $\tilde{c}_{\ell, m} = 0$ for all $\ell \notin \Gamma$ and $m \in \mathbb{Z}$. Thus, if (3.1) does not hold, we must have $\tilde{c}_{\ell, m} \neq 0$ for some $\ell \notin \Gamma$, say $q_i \not\propto \ell_i$ for some $i \in \{1, \dots, d\}$. Choose $n = e_i \in W$, and note that for this choice of n one has

$$\tilde{g}(z, \lambda) - \tilde{g}(\mu_n \odot z, \lambda) \not\equiv 0,$$

concluding the proof. \square

Definition 3.2. For each $j \in \{1, 2, \dots, d\}$, define $\gamma'_j \geq 0$ as follows. We let $-\gamma'_j$ be the lowest exponent of z_j in $h(z)$ in case this exponent is negative and $\gamma'_j = 0$ otherwise.

Lemma 3.3. Let p be a Laurent polynomial in z_1, \dots, z_d and let h be the lowest degree component of p . Then, the polynomials

$$r_n(z, \tilde{\lambda}) = \tilde{\lambda} z_1^{\gamma'_1} \cdots z_d^{\gamma'_d} h(\mu_n \odot z) - z_1^{\gamma'_1} \cdots z_d^{\gamma'_d}$$

are irreducible in $\mathbb{C}[z, \tilde{\lambda}]$ for each $n \in W$. Moreover, under Assumption (A₂), we conclude that for any $n \neq n' \in W$, r_n and $r_{n'}$ are relatively prime.

Proof. Assume for the sake of contradiction that $r_n(z, \tilde{\lambda})$ is reducible. Since the degree of $\tilde{\lambda}$ in $r_n(z, \tilde{\lambda})$ is one, we must have that

$$r_n(z, \tilde{\lambda}) = f(z, \tilde{\lambda})g(z) \quad (3.2)$$

for non-constant polynomials $f(z, \tilde{\lambda})$ and $g(z)$. Since $\tilde{\lambda}$ does not divide $r_n(z, \tilde{\lambda})$ in $\mathbb{C}[z]$, we see that there exist non-zero polynomials $f_1(z)$ and $f_2(z)$ such that

$$f(z, \tilde{\lambda}) = \tilde{\lambda} f_1(z) - f_2(z).$$

From (3.2) and the definition of $r_n(z, \tilde{\lambda})$ we obtain $f_2(z)g(z) = z_1^{\gamma'_1} \cdots z_d^{\gamma'_d}$. In particular, $g(z) = z_1^{m_1} \cdots z_d^{m_d}$ where m_1, \dots, m_d are integers with $0 \leq m_j \leq \gamma'_j$ for $j \in \{1, \dots, d\}$. Since g is nonconstant, $m_l > 0$ for at least one l . In particular,

$$\gamma'_l \geq m_l > 0.$$

Consequently, (3.2) implies that the polynomial $z_1^{\gamma'_1} \cdots z_d^{\gamma'_d} h(\mu_n \odot z)$ is divisible by z_l for some $l \in \{1, 2, \dots, d\}$. However, the lowest degree of z_l in $h(\mu_n \odot z)$ is, by definition, equal to $-\gamma'_l$. Thus $z_1^{\gamma'_1} \cdots z_d^{\gamma'_d} h(\mu_n \odot z)$ is not divisible by z_l , contradicting (3.2). Consequently, r_n is irreducible.

To prove the second statement of the lemma, assume r_n and $r_{n'}$ share a nontrivial common factor. By irreducibility, they must be constant multiples of one another. However, from the definition, this is only possible if $r_n = r_{n'}$, which contradicts Assumption (A₂). \square

Let us introduce the auxiliary polynomial

$$\tilde{a}(z, \tilde{\lambda}) = \prod_{n \in W} r_n(z, \tilde{\lambda}) \quad (3.3)$$

with $r_n(z, \tilde{\lambda})$ as in Lemma 3.3 for $n \in W$. By a direct calculation, $\tilde{a}(z, \tilde{\lambda})$ is invariant under the replacement⁵ $z \mapsto \mu_n \odot z$, so, as a consequence of Lemma 3.1, there exists $a(z, \tilde{\lambda})$ such that

$$\tilde{a}(z, \tilde{\lambda}) = a(z_1^{q_1}, \dots, z_d^{q_d}, \tilde{\lambda}). \quad (3.4)$$

Lemma 3.4. *Under Assumption (A₂), the polynomial $a(z, \tilde{\lambda})$ given by (3.4) is irreducible in $\mathbb{C}[z, \tilde{\lambda}]$.*

Remark 3.5. It is important that we pass to the lift a here, since \tilde{a} is clearly reducible.

Proof of Lemma 3.4. Suppose for the sake of establishing a contradiction that $a(z, \tilde{\lambda})$ is reducible, and write

$$a(z, \tilde{\lambda}) = f_1(z, \tilde{\lambda})g_1(z, \tilde{\lambda}) \quad (3.5)$$

for non-constant polynomials f_1 and g_1 . Let $\tilde{f}_1(z, \tilde{\lambda}) = f_1(z_1^{q_1}, \dots, z_d^{q_d}, \tilde{\lambda})$ and $\tilde{g}_1(z, \tilde{\lambda}) = g_1(z_1^{q_1}, \dots, z_d^{q_d}, \tilde{\lambda})$. Combining (3.4) and (3.5) yields

$$\tilde{a}(z, \tilde{\lambda}) = \tilde{f}_1(z, \tilde{\lambda})\tilde{g}_1(z, \tilde{\lambda}).$$

Moreover, by definition $\tilde{f}_1(z, \tilde{\lambda})$ and $\tilde{g}_1(z, \tilde{\lambda})$ are both invariant under the action of each μ_n . Recall from Lemma 3.3 that each $r_n(z, \tilde{\lambda})$ is irreducible. Therefore, each $r_n(z, \tilde{\lambda})$ is a factor of either \tilde{f}_1 or \tilde{g}_1 . By invariance of $\tilde{f}_1(z, \tilde{\lambda})$ (respectively $\tilde{g}_1(z, \tilde{\lambda})$) under the action of each μ_n and since, by Lemma 3.3, r_n and $r_{n'}$ are relatively prime for $n \neq n'$, we conclude the following: if $\tilde{f}_1(z, \tilde{\lambda})$ (respectively $\tilde{g}_1(z, \tilde{\lambda})$) has a factor of $r_n(z, \tilde{\lambda})$ then it must have a factor of

$$\prod_{n \in W} r_n(z, \tilde{\lambda}) = \tilde{a}(z, \tilde{\lambda}).$$

However, this, together with (3.5), implies that either $\tilde{f}_1(z, \tilde{\lambda})$ or $\tilde{g}_1(z, \tilde{\lambda})$ must be constant, which is a contradiction. Thus, we conclude that $a(z, \tilde{\lambda})$ is irreducible. \square

Lemma 3.6. *Let $\mathcal{P}(z, \lambda)$ be given by (2.11) and let f be any irreducible factor of \mathcal{P} . Then f must depend on λ .*

Proof. If f is an irreducible factor of \mathcal{P} , then f must depend on λ since otherwise there would be a suitable choice of $z = (z_1, \dots, z_d)$, namely any solution of $f(z) = 0$, for which $\mathcal{P}(z, \lambda) = 0$ for any λ . This, in turn, contradicts the fact that the term of highest degree of λ in $\mathcal{P}(z, \lambda)$ is $(-1)^Q \lambda^Q$ (see (2.7) and (2.11)). \square

⁵ Later on, we will call this the *action* of μ_n on a polynomial.

4. Proof of Theorem 2.6

Before proceeding with the proof of the main result, Theorem 2.6, let us introduce some notation.

Definition 4.1. For each $j \in \{1, 2, \dots, d\}$ denote by $-\gamma_j$ the lowest exponent of z_j in $p(z)$ in case this exponent is negative and $\gamma_j = 0$ otherwise. Clearly, $\gamma_j \geq \gamma'_j$ with γ'_j given in Definition 3.2.

Proof of Theorem 2.6. Let $\tilde{\lambda} = \lambda^{-1}$. Then $\mathcal{P}(z, \lambda) = \mathcal{P}(z, \tilde{\lambda}^{-1})$ is a Laurent polynomial in the variables $(z, \tilde{\lambda})$. Let γ_j , $j = 1, \dots, d$ be as in Definition 4.1. In case $\gamma_j > 0$ for some $j \in \{1, \dots, d\}$, the lowest power of z_j in $\mathcal{P}(z, \tilde{\lambda}^{-1})$ is $-\gamma_j Q/q_j$.

Moreover, the lowest power of $\tilde{\lambda}$ in $\mathcal{P}(z, \tilde{\lambda}^{-1})$ is $\tilde{\lambda}^{-Q}$ (cf. (2.7)), so

$$\mathcal{R}(z, \tilde{\lambda}) = \left(\tilde{\lambda} z_1^{\frac{\gamma_1}{q_1}} \cdots z_d^{\frac{\gamma_d}{q_d}} \right)^Q \mathcal{P}(z, \tilde{\lambda}^{-1}) \quad (4.1)$$

defines a polynomial $\mathcal{R} \in \mathbb{C}[z, \tilde{\lambda}]$.

Claim 4.2. For each $1 \leq j \leq d$, z_j does not divide $\mathcal{R}(z, \tilde{\lambda})$.

Proof of Claim. Indeed, if $\gamma_j > 0$, this is clear from the definitions, since $-\gamma_j$ is the smallest power of z_j in p and hence $-\gamma_j Q/q_j$ is the smallest power of z_j in \mathcal{P} . Otherwise, $\gamma_j = 0$, and the claim can be seen from (2.7). \diamond

Since $\tilde{\lambda}$ also does not divide $\mathcal{R}(z, \tilde{\lambda})$, Claim 4.2 implies that reducibility of the Laurent polynomial $\mathcal{P}(z, \tilde{\lambda}^{-1})$ is equivalent to reducibility of the polynomial $\mathcal{R}(z, \tilde{\lambda})$.

Now, assume for the sake of contradiction that $\mathcal{P}(z, \tilde{\lambda}^{-1})$ is reducible. There exist $m > 1$ and non-constant polynomials $f_l(z, \tilde{\lambda})$, $l = 1, 2, \dots, m$, in $\mathbb{C}[z, \tilde{\lambda}]$ such that

$$\left(\tilde{\lambda} z_1^{\frac{\gamma_1}{q_1}} \cdots z_d^{\frac{\gamma_d}{q_d}} \right)^Q \mathcal{P}(z, \tilde{\lambda}^{-1}) = \prod_{l=1}^m f_l(z, \tilde{\lambda}). \quad (4.2)$$

Let us recall the auxiliary polynomial \tilde{a} given by

$$\tilde{a}(z, \tilde{\lambda}) := \left(\tilde{\lambda} z_1^{\gamma'_1} \cdots z_d^{\gamma'_d} \right)^Q \prod_{n \in W} (h(\mu_n \odot z) - \tilde{\lambda}^{-1}).$$

Let $\tilde{f}_l(z, \tilde{\lambda}) = f_l(z_1^{q_1}, \dots, z_d^{q_d}, \tilde{\lambda})$. Then, by (2.11) and (4.2), we have that

$$\left(\tilde{\lambda} z^{\gamma_1} \cdots z^{\gamma_d} \right)^Q \tilde{\mathcal{P}}(z, \tilde{\lambda}^{-1}) = \prod_{l=1}^m \tilde{f}_l(z, \tilde{\lambda}). \quad (4.3)$$

By definition of $\tilde{\mathcal{P}}$ in (2.6) one sees that replacing $\tilde{\lambda}$ by $\tilde{\lambda}^\gamma$ for $\gamma = -\deg(h) > 0$ allows us to conclude that the lowest degree component of $(\tilde{\lambda}^\gamma z^{\gamma_1} \cdots z^{\gamma_d})^Q \tilde{\mathcal{P}}(z, \tilde{\lambda}^{-\gamma})$ is given by $\tilde{a}_1(z, \tilde{\lambda}^\gamma)$, where

$$\tilde{a}_1(z, \tilde{\lambda}^\gamma) = \left(\tilde{\lambda}^\gamma z_1^{\gamma_1} \cdots z_d^{\gamma_d} \right)^Q \prod_{n \in W} (h(\mu_n \odot z) - \tilde{\lambda}^{-\gamma}) = (z_1^{\gamma_1 - \gamma'_1} \cdots z_d^{\gamma_d - \gamma'_d})^Q \tilde{a}(z, \tilde{\lambda}^\gamma). \quad (4.4)$$

We denote by $\tilde{f}_l^1(z, \tilde{\lambda}^\gamma)$ the lowest degree components of $\tilde{f}_l(z, \tilde{\lambda}^\gamma)$, $l = 1, 2, \dots, m$. From (4.3) it must be that

$$\prod_{l=1}^m \tilde{f}_l^1(z, \tilde{\lambda}^\gamma) = \tilde{a}_1(z, \tilde{\lambda}^\gamma) \quad (4.5)$$

and hence

$$\prod_{l=1}^m \tilde{f}_l^1(z, \tilde{\lambda}) = \tilde{a}_1(z, \tilde{\lambda}). \quad (4.6)$$

Given $l \in \{1, \dots, m\}$, $\tilde{f}_l^1(z, \tilde{\lambda})$ is a polynomial in $z_1^{q_1}, z_2^{q_2}, \dots, z_d^{q_d}$. Thus, there exists $f_l^1(z, \tilde{\lambda})$ such that

$$\tilde{f}_l^1(z, \tilde{\lambda}) = f_l^1(z_1^{q_1}, \dots, z_d^{q_d}, \tilde{\lambda}). \quad (4.7)$$

By (4.4), (4.6) and (4.7), we reach, recalling the definition of $a(z, \tilde{\lambda})$ in (3.4),

$$\prod_{l=1}^m f_l^1(z, \tilde{\lambda}) = \left(z_1^{\frac{\gamma_1 - \gamma'_1}{q_1}} z_2^{\frac{\gamma_2 - \gamma'_2}{q_2}} \cdots z_d^{\frac{\gamma_d - \gamma'_d}{q_d}} \right)^Q a(z, \tilde{\lambda}). \quad (4.8)$$

By Lemma 3.4, $a(z, \tilde{\lambda})$ is irreducible, so there exists $j \in \{1, 2, \dots, m\}$ such that $f_j^1(z, \tilde{\lambda})$ has a factor $a(z, \tilde{\lambda})$. We conclude that the highest power of $\tilde{\lambda}$ in $\tilde{f}_j(z, \tilde{\lambda})$ (hence in $f_j(z, \tilde{\lambda})$) is at least Q . Since $m > 1$ and, by Lemma 3.6 and Claim 4.2, $\tilde{f}_l(z, \tilde{\lambda})$, $l = 1, 2, \dots, m$, must depend on $\tilde{\lambda}$ we reach a contradiction since the highest power of $\tilde{\lambda}$ on the left-hand side of (4.3) is equal to Q . \square

5. Floquet theory for long-range operators

Let us summarize some of the important points about Floquet theory for operators with long-range interactions. This is well-known, especially in the continuum case; see the survey [14] and references therein. We are unaware of a precise reference in the discrete setting for long-range operators, so we included the details for the reader's convenience.

Let us assume that $A : \ell^2(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)$ is bounded. Writing $A_{n,m} = \langle \delta_n, A \delta_m \rangle$ for the matrix elements, we further assume that A is translation-invariant in the sense that

$$A_{n+\ell, m+\ell} = A_{n, m} \quad \forall n, m, \ell \in \mathbb{Z}^d,$$

and that A satisfies the decay estimate

$$|A_{n, m}| \leq C e^{-\nu |n-m|}$$

for constants $C, \nu > 0$. By translation-invariance, A is fully encoded by $\{a_n := A_{n, 0}\}_{n \in \mathbb{Z}^d}$ via

$$[A\psi]_n = \sum_{m \in \mathbb{Z}^d} a_{n-m} \psi_m.$$

We denote the Fourier transform on $\ell^2(\mathbb{Z}^d)$ by $\mathcal{F} : u \mapsto \widehat{u}$, where

$$\widehat{u}(x) = \sum_{n \in \mathbb{Z}^d} e^{-2\pi i \langle n, x \rangle} u_n,$$

for $u \in \ell^1(\mathbb{Z}^d)$ and then extended to ℓ^2 by Plancherel.

By the assumptions on A , the symbol \widehat{a} is analytic, real-analytic whenever $a_n = \overline{a_{-n}}$, and a trigonometric polynomial whenever a is finitely supported. For example, when $A = -\Delta$ denotes the Laplacian on \mathbb{Z}^d ,

$$\widehat{a}(x) = -2 \sum_{j=1}^d \cos(2\pi x_j).$$

Recall that $V : \mathbb{Z}^d \rightarrow \mathbb{C}$ is q -periodic and $\Gamma = \{q \odot m : m \in \mathbb{Z}^d\}$ denotes the period lattice. We define the dual lattice $\Gamma^* = \{(m_1/q_1, \dots, m_d/q_d) : m_j \in \mathbb{Z}\}$ and

$$W^* := \Gamma^* \cap [0, 1]^d = \left\{0, \frac{1}{q_1}, \dots, \frac{q_1 - 1}{q_1}\right\} \times \dots \times \left\{0, \frac{1}{q_d}, \dots, \frac{q_d - 1}{q_d}\right\}.$$

The discrete Fourier transform of a q -periodic function $g : \mathbb{Z}^d \rightarrow \mathbb{C}$ is defined by

$$\widehat{g}_\ell = \frac{1}{\sqrt{Q}} \sum_{n \in W} e^{-2\pi i \langle n, \ell \rangle} g_n, \quad \ell \in W^*. \quad (5.1)$$

Of course, this also makes sense for $\ell \in \Gamma^*$ and satisfies $\widehat{g}_{\ell+n} = \widehat{g}_\ell$ for any $\ell \in W^*$ and any $n \in \mathbb{Z}^d$. One can check the inversion formula

$$\frac{1}{\sqrt{Q}} \sum_{\ell \in W^*} e^{2\pi i \langle \ell, n \rangle} \widehat{g}_\ell = g_n, \quad \forall n \in \mathbb{Z}^d, \quad (5.2)$$

which holds for any q -periodic g .

Let $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ denote the torus.

Proposition 5.1. *For any $f \in L^2(\mathbb{T}^d)$,*

$$[\mathcal{F}A\mathcal{F}^*f](x) = \widehat{a}(x)f(x)$$

and

$$[\mathcal{F}V\mathcal{F}^*f](x) = \frac{1}{\sqrt{Q}} \sum_{\ell \in W^*} \widehat{V}_\ell f(x - \ell).$$

Proof. These follow from direct calculations using the definitions of and assumptions on A and V and the inversion formula (5.2). \square

Let us now define $\mathbb{T}_*^d = \mathbb{R}^d/\Gamma^*$,

$$\mathcal{H}_q = \int_{\mathbb{T}_*^d}^{\oplus} \mathbb{C}^W \frac{dx}{|\mathbb{T}_*^d|} = L^2 \left(\mathbb{T}_*^d, \mathbb{C}^W; \frac{dx}{|\mathbb{T}_*^d|} \right)$$

and $\mathcal{F}_q : \ell^2(\mathbb{Z}^d) \rightarrow \mathcal{H}_q$ by $u \mapsto \widehat{u}$ where

$$\widehat{u}_j(x) = \sum_{n \in \mathbb{Z}^d} e^{-2\pi i \langle n \odot q, x \rangle} u_{j+n \odot q}, \quad x \in \mathbb{T}_*^d, \quad j \in W.$$

As usual, this is initially defined for (say) ℓ^1 vectors, but has a unique extension to a unitary operator on ℓ^2 via Plancherel.

Proposition 5.2. *The operator \mathcal{F}_q is unitary. If V is q -periodic, then*

$$\mathcal{F}_q H \mathcal{F}_q^* = \int_{\mathbb{T}_*^d}^{\oplus} \widetilde{H}(x) \frac{dx}{|\mathbb{T}_*^d|},$$

where $\widetilde{H}(x)$ denotes the restriction of H to W with boundary conditions

$$u_{n+m \odot q} = e^{2\pi i \langle m \odot q, x \rangle} u_n, \quad n, m \in \mathbb{Z}^d. \quad (5.3)$$

Proof. Unitarity of \mathcal{F}_q follows from Parseval's formula. The form of $\mathcal{F}_q H \mathcal{F}_q^*$ follows from a direct calculation. \square

Given $x \in \mathbb{R}^d$, let \mathcal{F}^x be the Floquet-Bloch transform defined on \mathbb{C}^W as follows: for any vector on W , $\{u(n)\}_{n \in W}$, we set

$$[\mathcal{F}^x u]_l = \frac{1}{\sqrt{Q}} \sum_{n \in W} e^{-2\pi i \sum_{j=1}^d \left(\frac{l_j}{q_j} + x_j \right) n_j} u_n, \quad l \in W.$$

Therefore,

$$[(\mathcal{F}^x)^*u]_l = \frac{1}{\sqrt{Q}} \sum_{n \in W} e^{2\pi i \sum_{j=1}^d (\frac{n_j}{q_j} + x_j)l_j} u_n, \quad l \in W.$$

Let $z_j = e^{2\pi i x_j}$, $j = 1, 2, \dots, d$ and define the Laurent series $p(z)$ by

$$p(e^{2\pi i x_1}, e^{2\pi i x_2}, \dots, e^{2\pi i x_d}) = \hat{a}(x_1, x_2, \dots, x_d). \quad (5.4)$$

Using multi-index notation, we may rewrite this as

$$p(z) = \hat{a}(x) = \sum_{n \in \mathbb{Z}^d} e^{-2\pi i \langle n, x \rangle} a_n = \sum_{n \in \mathbb{Z}^d} a_n z_1^{-n_1} z_2^{-n_2} \cdots z_d^{-n_d} = \sum_{n \in \mathbb{Z}^d} a_n z^{-n}. \quad (5.5)$$

Proposition 5.3. *Assume V is q -periodic. Then $\tilde{H}(x)$ given by (5.3) is unitarily equivalent to $D^z + B_V$, where $z_j = e^{2\pi i x_j}$, D^z is a diagonal matrix with entries*

$$D^z(n, n') = p(\mu_n \odot z) \delta_{n, n'}, \quad (5.6)$$

μ_n is the vector from (2.3), and $B = B_V$ has entries related to the discrete Fourier transform of V via

$$B(n, n') = \hat{V} \left(\frac{n_1 - n'_1}{q_1}, \dots, \frac{n_d - n'_d}{q_d} \right).$$

Remark 5.4. In particular, D^z depends on A and is independent of V , while B_V depends only on V with no dependence on A .

Proof of Proposition 5.3. By a direct calculation, we see that \mathcal{F}^x is unitary, so it suffices to prove that $D^z + B_V = (\mathcal{F}^x)\tilde{H}(x)(\mathcal{F}^x)^*$. Let $\tilde{H}_0(x)$ be $\tilde{H}(x)$ with the potential V set to zero. We are going to show $(\mathcal{F}^x)\tilde{H}_0(x)(\mathcal{F}^x)^* = D^z$ and $(\mathcal{F}^x)V(\mathcal{F}^x)^* = B$ separately. To prove that $(\mathcal{F}^x)\tilde{H}_0(x)(\mathcal{F}^x)^* = D^z$, it suffices to show that for any $u = \{u_n\}_{n \in W}$,

$$(\mathcal{F}^x)^* D^z u = \tilde{H}_0(x)(\mathcal{F}^x)^* u.$$

It is worth mentioning that $(\mathcal{F}^x)^*u$ satisfies (5.3) so that $\tilde{H}_0(x)(\mathcal{F}^x)^*u$ is well defined. With the given definitions, for any $m \in W$,

$$\begin{aligned} (\tilde{H}_0(x)(\mathcal{F}^x)^*u)_m &= \sum_{l \in \mathbb{Z}^d} a_{m-l} [(\mathcal{F}^x)^*u]_l \\ &= \frac{1}{\sqrt{Q}} \sum_{l \in \mathbb{Z}^d} a_{m-l} \sum_{n \in W} e^{2\pi i \sum_{j=1}^d (\frac{n_j}{q_j} + x_j)l_j} u_n \\ &= \frac{1}{\sqrt{Q}} \sum_{l \in \mathbb{Z}^d} a_l \sum_{n \in W} e^{2\pi i \sum_{j=1}^d (\frac{n_j}{q_j} + x_j)(m_j - l_j)} u_n \end{aligned}$$

$$= \frac{1}{\sqrt{Q}} \sum_{n \in W} e^{2\pi i \sum_{j=1}^d (\frac{n_j}{q_j} + x_j) m_j} \hat{a} \left(\frac{n_1}{q_1} + x_1, \dots, \frac{n_d}{q_d} + x_d \right) u_n. \quad (5.7)$$

Putting together (2.4) and (5.7),

$$\hat{a} \left(\frac{n_1}{q_1} + x_1, \dots, \frac{n_d}{q_d} + x_d \right) = p(\mu_n \odot z) = D^z(n, n). \quad (5.8)$$

Similarly,

$$((\mathcal{F}^x)^* D^z u)_m = \frac{1}{\sqrt{Q}} \sum_{n \in W} e^{2\pi i \sum_{j=1}^d (\frac{n_j}{q_j} + x_j) m_j} u_n D^z(n, n). \quad (5.9)$$

By (5.7), (5.9) and (5.8), we finish the proof of $(\mathcal{F}^x) \tilde{H}_0(x) (\mathcal{F}^x)^* = D^z$.

The proof of $(\mathcal{F}^x) V (\mathcal{F}^x)^* = B$ is similar. \square

6. Proof of Theorem 1.1 and examples

Proof of Theorem 1.1. The Bloch variety precisely consists of those (k, λ) such that there is a nontrivial solution of $Hu = \lambda u$ satisfying the boundary conditions as in (1.4). Thus, with D and B as in Proposition 5.3, the Bloch variety is the zero set of the Laurent polynomial $\mathcal{P}(z, \lambda)$ defined by (2.11) where

$$\tilde{\mathcal{P}}(z, \lambda) = \det(D^z + B - \lambda I).$$

After using the standard permutation expansion for this determinant, we see that $\tilde{\mathcal{P}}$ is of the form (2.6) (with p given via (5.5)). By a brief calculation, one can check that $\tilde{\mathcal{P}}$ satisfies (2.8). Namely, if S_m denotes the shift $e_n \mapsto e_{n+m}$ with addition computed modulo Γ , one can check that

$$\begin{aligned} \tilde{\mathcal{P}}(\mu_m z, \lambda) &= \det(D^{\mu_m z} + B - \lambda) \\ &= \det(S_m^* D^z S_m + B - \lambda). \end{aligned}$$

Since $S_m^* B S_m = B$, (2.8) follows. Thus, the result follows from Theorem 2.6. \square

Let us conclude by discussing a few examples of how to obtain the generator $p(z)$ for which Theorem 2.6 is applicable. In particular, the examples below show that the framework of the present paper allows one to consider different discrete geometries. We start with the most basic example of the Laplacian on \mathbb{Z}^d , where

$$[A\psi]_n = - \sum_{\|m-n\|_1=1} \psi_m.$$

In this case, it readily follows from (5.5) that

$$p(z) = - \left(z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + \cdots + z_d + \frac{1}{z_d} \right). \quad (6.1)$$

Proof of Corollary 1.3. From (6.1), we see that the minimal degree component of p is precisely

$$h(z) = - \left(\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_d} \right).$$

Here assumptions (A_1) and (A_2) are fulfilled with $\deg(h) = -1$. \square

We then proceed to the description of a couple of two dimensional examples. The triangular lattice is given by specifying the vertex set

$$\mathcal{V} = \{nb_1 + nb_2 : n, m \in \mathbb{Z}\}, \quad b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad b_2 = \frac{1}{2} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

with edges given by $u \sim v \iff \|u - v\|_2 = 1$. Applying the shear transformation $b_1 \mapsto b_1$, $b_2 \mapsto [0, 1]^\top$, one can view this graph as having vertices in \mathbb{Z}^2 and

$$u \sim v \iff u - v \in \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}.$$

In particular, the nearest-neighbor Laplacian on the triangular lattice is equivalent to the operator $A_{\text{tri}} : \ell^2(\mathbb{Z}^2) \rightarrow \ell^2(\mathbb{Z}^2)$ such that

$$[A_{\text{tri}}\psi]_{n_1, n_2} = -\psi_{n_1-1, n_2} - \psi_{n_1+1, n_2} - \psi_{n_1, n_2-1} - \psi_{n_1, n_2+1} - \psi_{n_1-1, n_2+1} - \psi_{n_1+1, n_2-1}.$$

Making use of (5.5) one finds that

$$p_{\text{tri}}(z) = - \left(z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + \frac{z_1}{z_2} + \frac{z_2}{z_1} \right). \quad (6.2)$$

Proof of Corollary 1.5. From (6.2), we see that

$$h_{\text{tri}}(z) = -\frac{1}{z_1} - \frac{1}{z_2},$$

from which it is trivial to check Assumptions (A_1) and (A_2) . \square

Finally, in the Extended Harper Model

$$\begin{aligned} [A_{\text{EHM}}\psi]_{n_1, n_2} = & -\psi_{n_1-1, n_2} - \psi_{n_1+1, n_2} - \psi_{n_1, n_2-1} - \psi_{n_1, n_2+1} \\ & - \psi_{n_1-1, n_2+1} - \psi_{n_1+1, n_2-1} - \psi_{n_1-1, n_2-1} - \psi_{n_1+1, n_2+1}. \end{aligned}$$

Equation (5.5) now implies that

$$p_{\text{EHM}}(z) = - \left(z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + \frac{z_1}{z_2} + \frac{z_2}{z_1} + z_1 z_2 + \frac{1}{z_1 z_2} \right)$$

The lowest degree component is $\frac{1}{z_1 z_2}$. The proof of Corollary 1.4 follows in just the same way as before; notice that we need the periods to be coprime to ensure that Assumption (A_2) is met.

Remark 6.1. To conclude, let us say a few words about when our results can be applied in the general setting of periodic operators on periodic graphs. That is, given a periodic graph and a periodic operator thereupon, when can one apply a suitable change of coordinates as in the case of the triangular lattice to reduce an operator on \mathbb{Z}^d ? In short, this can be done to periodic operators on any \mathbb{Z}^d -periodic graph with transitive vertex action. Let us describe this in a little more detail.

Suppose \mathcal{G} is a locally finite graph with vertices \mathcal{V} on which \mathbb{Z}^d acts freely by graph automorphisms. Denote the action of $n \in \mathbb{Z}^d$ on the vertex $u \in \mathcal{V}$ by $n + u$. This gives a unitary representation of \mathbb{Z}^d viz. $[T^n \psi](u) = \psi(n + u)$, $\psi \in \ell^2(\mathcal{V})$. One may consider operators of the form $H = A + V$ acting in the natural Hilbert space $\ell^2(\mathcal{V})$, where

- (1) A commutes with the \mathbb{Z}^d -action (that is, $AT^n = T^n A$ for all $n \in \mathbb{Z}^d$)
- (2) V is diagonal, that is, $[V\psi](u) = V(u)\psi(u)$ for a suitable function $V : \mathcal{V} \rightarrow \mathbb{C}$.
- (3) V is invariant under the action of a full-rank subgroup of \mathbb{Z}^d , that is, there is a subgroup $F \leq \mathbb{Z}^d$ of rank d such that $VT^m = T^m V$ for all $m \in F$.

In item (3), notice that one can always take F to be of the form $q_1 \mathbb{Z} \oplus \cdots \oplus q_d \mathbb{Z}$ for some $q \in \mathbb{N}^d$, in which case we say V is q -periodic as before. If in addition, \mathbb{Z}^d acts transitively on \mathcal{V} , then choosing arbitrarily some $u_0 \in \mathcal{V}$, one has a one-to-one correspondence $\mathbb{Z}^d \rightarrow \mathcal{V}$ via $n \mapsto n + u_0$. Of course, this induces a unitary operator $Q : \ell^2(\mathbb{Z}^d) \rightarrow \ell^2(\mathcal{V})$ in an obvious manner. In this case, it is clear that $Q^* H Q$ is a periodic operator of the form studied in the present paper.

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