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Research paper

Teaching matters: A longitudinal study of mathematics teachers' knowledge growth

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HIGHLIGHTS

- The development of pedagogical content knowledge (PCK) through teaching is investigated.
- Teachers in their early teaching careers gained PCK by teaching a subject matter.
- The growth of separate components of PCK is related to different factors.
- Subject matter knowledge plays a role in teacher learning of student math thinking.
- The math teaching credential plays a role in teacher learning of math teaching.

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ABSTRACT

Teaching a particular subject provides rich opportunities for teachers to develop subject-specific knowledge and skills, especially for those who are in the early years of their career. Yet supporting evidence is scarce regarding the extent to which knowledge and skills teachers could gain from their teaching experience. This study aims to address this gap by collecting data from a from a national sample of 207 novice mathematics teachers for three years in a row to explore the development of two elements of teachers' pedagogical content knowledge (PCK) of mathematics: (1) knowledge of students' mathematical thinking and (2) knowledge of mathematics teaching. By using linear growth modeling to analyze the data derived from teachers' analyses of videos clips of mathematics classes, we found that teachers increased both elements of PCK, albeit at different rates. Further, the growth of these two important elements of PCK were associated with different teacher-related factors. Having a robust knowledge of mathematics played a key role in teachers' learning of students' mathematical thinking, whereas having a credential in mathematics teaching played a role in the development of teachers' knowledge of mathematics teaching.

Continuous learning is at the heart of the teaching profession. Not only the formal professional support teachers may receive but also informal learning opportunities, such as meetings with colleagues, interactions with students, and working with curriculum materials, can help teachers gain new knowledge. Of these informal opportunities, learning through teaching on one's own is particularly important, given that teachers complete the task of teaching on a daily basis. Thus, understanding the extent to which teachers develop knowledge and skills through teaching has important implications for teacher education and research.

Although the notion of learning through teaching is appealing and has gained attention from scholars (Hiebert et al., 2007; Leikin & Zazkis, 2010), the supporting literature is limited for several reasons. First, teachers can gain a wide range of knowledge and skills from teaching but relatively less attention has been given to which knowledge and skills are developed through teaching (Kyndt et al., 2016). Prior literature has been concentrated on the sense-making of teaching situations and teachers' responses to these situations to highlight potential learning opportunities therein for teachers (Hoekstra et al., 2007; Lloyd, 2008; Lohman & Woolf, 2001; Marcus & Chazan, 2010; Remillard, 2000). Such limited attention to capture what knowledge teachers learn through teaching will hamper efforts to understand the learning opportunities the teaching experience can provide. Second, while teachers are engaged in teaching, they may be learning from teaching on their own or through the external support they receive on the job,

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such as by attending professional development or learning with their colleagues at grade-level meetings. Thus, attributing teachers' learning through teaching to self-learning (rather than external support) requires concurrently paying attention to the role of externally provided sources for learning. Yet the majority of work in this area has either focused on the role of years of teaching experience (e.g., Hill, 2007; Ko & Herbst, 2020) or on professional learning opportunities, such as lesson studies and instructional coaching (Kraft & Blazar, 2017; Leavy & Hourigan, 2016), as indicators of teachers' learning on the job. The extent to which teaching can provide room for teachers to grow professionally on their own remains unknown. Finally, little is known about the extent to which teachers' prior educational preparation, such as how they entered the teaching profession or the credential they hold in teaching, or the indicators of their teaching competency, such as their understanding of the subject matter being taught either, help or hinder their learning. It is possible that teachers from certain educational backgrounds may gain knowledge and skills faster than those from other backgrounds. Although research has shown that teachers' subject matter knowledge (Copur-Gencturk et al., 2019; Kleickmann et al., 2013; Krauss et al., 2008), noticing skills (e.g., Blömeke et al., 2015), credential in mathematics (e.g., Hill, 2007), and training in teacher preparation programs (e.g., Hiebert et al., 2017, 2019; Santagata et al., 2018) are related to their current level of knowledge and the skills needed in teaching a subject matter, such as their pedagogical content knowledge (PCK), research is limited on what role these factors play in the *development* of these knowledge and skill components.

This study contributes to the literature by examining teachers' capability of learning PCK through teaching on their own by taking consideration of all learning activities that participating teachers received on mathematics teaching and learning during the study period. We also explored how teachers' subject matter knowledge, noticing skills, and their educational training were related to teachers' learning progression. Specifically, we focused on the extent to which teachers develop two important elements of the subject-specific knowledge needed in teaching (in this case, PCK of mathematics): (1) knowledge of students' mathematical thinking (KSMT) and (2) knowledge of mathematics teaching (KMT). Our rationale for focusing on these two aspects of PCK of mathematics was twofold. First, these two components of teachers' PCK of mathematics have been widely considered important elements in many conceptualizations of mathematics teachers' PCK in both national and cross-cultural research studies, even though these constructs may have been named or categorized differently (Ball et al., 2008; Copur-Gencturk & Tolar, 2022; Krauss et al., 2008; Schmidt et al., 2007; Tatto et al., 2008). Empirical evidence also supports their importance in the quality of teachers' mathematics instruction and students' mathematics learning (Baumert et al., 2010; Hill & Chin, 2018; Kersting et al., 2010). Second, understanding the development of KSMT and KMT sheds light on teachers' knowledge development in other subject areas. Although KSMT and KMT are specific to mathematics, they are also agreed-on elements of PCK in other subjects, such as science (e.g., knowledge of students' thinking about science and science-specific instructional strategies; see Park & Oliver, 2008; Schneider & Plasman, 2011) and reading and literacy (e.g., teachers' knowledge of decoding and instructional strategies for reading comprehension; see Jordan et al., 2018). Moreover, these two elements of teachers' PCK are instrumental in teachers' instructional quality and students' achievement in other subjects (Keller et al., 2017; Park et al., 2011).

In the following section, we conceptualize KSMT and KMT and review prior literature on the development of these two knowledge components through teaching. We then present the design of this study and the findings. In the discussion section, we elaborate our findings and implications for teacher education and policy in relation to teacher preparation and professional development.

1. Teacher knowledge of students' mathematical thinking and knowledge of mathematics teaching

In this study, we focused on the development of content-specific knowledge, particularly, changes in two theoretically and empirically two important elements of PCK: (1) knowledge of students' math thinking and (2) knowledge of math teaching. Guided by prior literature, we conceptualized knowledge of students' mathematical thinking, as teachers' understanding of students' prior and existing conceptions regarding mathematical issues (Ball et al., 2008; Baumert et al., 2010; Copur-Gencturk & Tolar, 2022; Krauss et al., 2008). Specifically, teachers should be able to anticipate where students will struggle, know the typical strategies students use to solve problems, and be able to evaluate and diagnose student mathematical thinking from their responses (Krauss et al., 2008; Tatto et al., 2008). For example, knowing that students make the mistake of adding the denominators when solving fraction addition problems because they treat the numerator and denominator as separate numbers reflects teachers' knowledge about student math thinking.

Knowledge of mathematics teaching was defined according to prior literature as knowing how to make math content understandable to students (Ball et al., 2008; Baumert et al., 2010; Copur-Gencturk & Tolar, 2022). Specifically, it includes the understanding of affordances and limitations of representations and tools to use to facilitate students' learning of a particular concept (Mohr-Schroeder et al., 2017) and choosing teaching strategies that help students overcome their struggles (e.g., Döhrmann et al., 2014; Tatto et al., 2008). For example, to help students understand fractions, teachers could use fraction rectangles, a visual approach, to help students see the denominator is the number of equal parts in the whole and the numerator shows how many of those parts are included in the fraction.

2. Teaching as a potential teacher self-learning space

Teaching involves a cyclic process of planning a lesson, implementing the planned lesson, and possibly reflecting on the lesson that provides teachers with rich learning opportunities in each phase. During planning, teachers can learn from the curriculum materials (Davis & Krajcik, 2005; Remillard, 2000). During the instruction, teachers can learn about students' mathematical thinking and try out new strategies and tools to facilitate students' learning (e.g., Leikin & Zazkis, 2010; Lloyd, 2008; Remillard & Bryans, 2004). Reflection provides an opportunity for teachers to gain knowledge by identifying and reasoning about the discrepancy between their expectations (i.e., planning) and the reality in teaching (i.e., implementation; Leikin & Zazkis, 2010; Tzur, 2010). Prior work has shown that through well-designed professional development, such as collective lesson studies and instructional coaching, teaching can provide teachers with a rich learning experience to develop knowledge and skills (e.g., Kraft & Blazar, 2017; Lewis & Perry, 2017; Sherin & Van Es, 2009). While providing sustained and structured support is crucial for teachers' continuous growth, the question is how much teachers can learn from teaching on their own. Such question is important, given that teachers in the United States work in a relatively independent teaching environment and high-quality professional development programs that are tailored to teachers' individual needs are not always available (Hoekstra et al., 2009; Kyndt et al., 2016). Thus, understanding how much teachers are able to learn through teaching on their own will help school leaders provide external support as a complement to teachers' professional growth.

A growing body of work has focused on identifying potential teacher learning opportunities in teaching activities such as the usage of curriculum materials in lesson enactment (e.g., Remillard & Bryans, 2004), interactions with students around one task (e.g., Marcus & Chazan, 2010), and trying out new teaching strategies (Cobb & Mcclain, 2001; Lohman & Woolf, 2001). Based on teachers' sensemaking of their own practice in interviews and researchers' classroom observations, it seems that teachers know more about their students through teaching and try to change their instructions accordingly. For example, Marcus and Chazan (2010) found during interactions with students, a novice teacher was able to see students' difficulties in solving equations in two variables and came up with an alternative representation to help students build the connection between solving equations in one variable and two variables. Recognizing students' struggles with certain concepts and changing instructional strategies indicate teachers' awareness of classroom situations, which leads to potential teacher learning. Yet being aware of students' struggles with certain math concepts does not represent that the teacher could correctly interpret students' math thinking behind their solutions (i.e., an indicator of KSMT) and explain the math behind tasks conceptually rather than procedurally using appropriate representations and tools (i.e., an indicator of KMT). Similarly, Remillard and Bryans (2004) found that novice teachers were more likely to adopt suggestions in reform-oriented curriculum materials such as emphasizing student work and talk in class, which provided them with more learning opportunities for understanding student mathematical thinking. However, whether these novice teachers kept increasing the accuracy of interpreting students' mathematical thinking and the capability of unpacking mathematical concepts by using appropriate representations as they obtained more teaching experience is unknown.

Since the main goal of prior studies on teacher learning through teaching were to identify potential learning opportunities embedded in teaching, directly measuring what specific knowledge teachers learned through teaching was not the top priority. Our current understanding of teacher learning through teaching is mostly centered around general aspects of teacher knowledge, such as teachers being aware that students did not follow teachers' planned representations and trying to use an alternative teaching strategy. However, we know little about whether teaching increases teachers' particular pedagogical knowledge of mathematics, such as students' math thinking, and math teaching. For example, does teaching help teachers develop their understanding about specific mathematical concepts students are struggling with and misconceptions that hinder students' understanding of the concept? Does teaching enrich repertories of representations and tools that teachers could draw from in response to different student learning needs? Built upon prior evidence showing that one's own teaching could be a potential space for teacher learning, directly measuring what specific knowledge teachers are able to learn through teaching on their own will further our understanding the affordance and limitations of teaching, as one teacher learning source on the job.

3. Years of teaching experience and teacher knowledge

A few studies explored the relationship between years of teaching experience and teachers' mathematical knowledge for teaching using cross-sectional data (e.g., Herbst & Kosko, 2014; Hill, 2007, 2010; Ko & Herbst, 2020). Nevertheless, to which extent changes in teachers' knowledge can be attributed to teachers' own teaching practice is unknown. Also, we know little about how teaching impacts teachers' development of different components of PCK. For example, in Hill's (2007) study with 591 middle school mathematics teachers, she found each additional year of teaching experience was associated with an increase of only 0.008 point on the overall scale of the mathematical knowledge for teaching. A few studies found a bigger size of this relationship if teachers' years of teaching experience were specific to the same mathematical content (e.g., Herbst & Kosko, 2014; Ko & Herbst, 2020). These findings imply that teachers gained PCK from different learning opportunities on the job but how much of this knowledge was developed on their own is not known. Moreover, teachers' knowledge is multidimensional (Copur-Gencturk & Tolar, 2022), but the development of different components of teacher knowledge has received insufficient attention in prior studies. Using a composite score to indicate teachers' overall PCK likely neglects the possibility that teachers might develop components of PCK differently through teaching. Thus, we argue there is a need to study teacher learning of specific components of teacher PCK through teaching on their own by taking consideration of different learning opportunities teachers might have in the profession simultaneously.

4. Factors potentially contributing to teachers' knowledge development through teaching

The current literature provides limited insight into how teachers' prior training and indicators of teacher competencies might influence teachers' capability of developing knowledge through teaching over time. For instance, although studies found a positive correlation between teachers' subject matter knowledge and their PCK of that subject matter (Copur-Gencturk et al., 2019; Kleickmann et al., 2013; Krauss et al., 2008; Tröbst et al., 2018), little evidence exists regarding how one's subject matter knowledge will impact his/her ability to gain KSMT and KMT, respectively, through teaching over time. Prior evidence shows having a solid understanding of school math makes it easier for teachers to follow the mathematical arguments students make (i.e., KSMT) and choose appropriate representations and instructional strategies to help students understand mathematical concepts (i.e., KMT) (e.g., Charalambous et al., 2011; Ma, 1999). Is it possible that teachers who have a strong understanding of school math will develop KSMT and KMT faster than those who are still struggling with understanding the math they need to teach to their students? This question might be answered by tracking the same groups of teachers' knowledge development for a long period of time.

Likewise, teachers' noticing about students' mathematical thinking and their own instruction may shape their learning opportunities of KSMT and KMT through teaching. Teachers' attention to students' strategies can influence their interpretations of students' thinking behind the strategies and their instructional responses (Jacobs et al., 2010). Empirical work has supported the connection between teachers' noticing and knowledge of students' mathematical thinking, and their mathematics teaching (Blömeke et al., 2015; Copur-Gencturk & Tolar, 2022; Dunekacke et al., 2016) as well as teachers' knowledge of science teaching and learning (Meschede et al., 2017). Limited work, if any, has been conducted solely to investigate how teacher noticing is related to *the development* of teachers' KSMT and KMT. Most work in this area has been conducted with teachers who were participating in professional development programs (e.g., video clubs) or preservice training (e.g., taking a mathematics teaching method course) (Jong et al., 2021; Sherin & Van Es, 2009; Walkoe & Levin, 2018). What remains unclear is whether and to what extent teachers' noticing skills, apart from not the other features of professional development programs, are related to changes in teachers' knowledge and practices.

Lastly, teachers who graduated from teacher education programs may be more equipped to learn from teaching independently, compared with those who entered the profession through alternative programs that devote relatively less time to general and content-specific pedagogy. Scholars have long advocated for preparing prospective teachers in teacher education programs to learn through teaching (Hiebert et al., 2007), and evidence shows that graduates of particular teacher preparation programs that emphasized prospective teachers' ability to analyze teaching continued to learn on the job (Hiebert et al., 2017, 2019; Santagata et al., 2018). However, the extent to which any teacher education programs prepare teachers with skills in learning from teaching on their own has not been tested. In addition, existing research has suggested that teachers with a credential in teaching mathematics seem to have higher PCK compared with those holding a credential in general teaching (Baumert et al., 2010; Hill, 2007; Krauss, Brunner, et al., 2008), but how a teaching credential in math impacts the development of teachers' knowledge, particularly the components of PCK, is unclear.

5. Present study

As summarized briefly above, considerable research has been done on teachers' learning on the job. However, comparatively little has focused on what components of PCK teachers gain on their own through their day-to-day practice. The present study aimed to address this gap in the literature by focusing on the development of teachers' knowledge of students' mathematical thinking (KSMT) and mathematics teaching (KMT) – two components of PCK. To distinguish teachers' independent learning from externally supported learning on the job, we collected data on all learning activities teachers participated in during the study period including both formal professional development and informal learning with colleagues. Additionally, we explored the role of other indicators of teaching proficiency and teachers' pre-service training background, in the development of their KSMT and KMT. Using data collected from more than 207 teachers in 3 consecutive years, we explored the following research questions in this study:

1. To what extent do teachers increase their KSMT and KMT through teaching over time?
2. To what extent is the development of teachers' KSMT and KMT related to their content knowledge and content-specific noticing skills as well as their prior educational preparation (certification pathway and credential type)?

6. Methods

6.1. Sample

The data used in this study were collected from teachers across the United States who were teaching mathematics in elementary and middle school and who had fewer than 3 years of teaching experience at the beginning of data collection. The teachers were invited via email to participate in the study and were then asked to complete a set of surveys annually for 3 school years. The analytic sample was restricted to teachers who provided demographic and prior training information and who completed the content knowledge and noticing measures. Our rationale for this restriction was to ensure that the results were obtained from the same sample across models that included different predictors. As displayed in Table 1, the majority of teachers in our sample were White (70.1%) and female (84.1%). Most teachers (72%) entered the teaching profession through traditional teacher preparation programs. Eighteen percent of the teachers held a credential in mathematics.

Table 1
Background characteristics of teachers in the present study.

Teacher background characteristic	Sample (%)
Gender	
Female	84.1
Male	15.5
Ethnicity	
White	70.1
Black	8.2
Hispanic	9.7
Other (e.g., Asian, multiracial, and other)	12.1
Credential Type	
Credential in mathematics	18.4
Credential in multiple subjects	69.6
Credential in other subjects (e.g., special education)	12.1
Certification Pathways	
Traditional certification programs[a]	72.0
Alternative certification programs[b]	28.0

Note. $N = 207$.
[a] Traditional certification requires one to graduate from a teacher preparation program prior to teaching.
[b] Alternative certification requires one to have a bachelor's degree but not necessarily to be trained as a teacher prior to teaching. Many countries have such alternative programs (e.g., Teach for America, Teach for Australia) that enable educated adults with relatively limited academic coursework in education and supervised student teaching experience to enter the profession (e.g., Blumenreich & Gupta, 2015; Crawford-Garrett, 2017).

6.2. Measures and procedures

6.2.1. Outcome measures

Teachers' KSMT and KMT were measured by teachers' open-ended responses to questions about students' mathematical understanding and mathematics teaching they viewed in authentic video clips (Kersting, 2008). For each construct, teachers watched a set of eight video clips. Each video clip was 2–3 min long and encompassed student–teacher interactions centering around fractions or ratio concepts that are taught in Grades 3–7, which were the grade levels the teachers in our sample were teaching at the time of data collection. Teachers were given a brief description of the context for each video so that they would have the background to understand the instruction presented in the videos.

We operationalized KSMT as teachers' knowledge of students' mathematical understanding based on their responses or strategies. We captured teachers' KSMT by asking them to analyze the mathematical understanding of students in the video clips and coded their responses as indicators of their KSMT. We operationalized KMT as teachers' knowledge of instructional practices and representations to improve students' learning of a particular topic. We captured teachers' KMT by asking teachers to provide suggestions to improve the teaching practices shown in the videos to increase students' mathematical understanding of the concept targeted in the lesson and coded their responses as indicators of their KMT.

Although not the ideal procedure, we used the same survey each year to capture the potential changes in teachers' KSMT and KMT. We found, in another line of work, that teachers did not learn from watching the same videos several times (i.e., their scores did not increase), which Hiebert et al. (2017) also observed in prior research. Thus, if teachers' scores on these items changed, we were confident the observed changes in their scores were not related to multiple administrations of this measure.

Teachers' responses to each question were scored using a 4-point rubric based on prior research to capture their KSMT and KMT (e.g., Hiebert et al., 2017; see Tables S1 and S2 for the rubric

Table 2
Descriptive statistics for the variables used in the reported analyses.

Variable	*n*	*M*[a]	*SD*	Min	Max
Knowledge of Students' Mathematical Thinking (Year 1)	207	13.78	3.48	8	25
Knowledge of Students' Mathematical Thinking (Year 2)	171	14.78	3.49	8	26
Knowledge of Students' Mathematical Thinking (Year 3)	168	16.93	4.27	8	26
Knowledge of Mathematics Teaching (Year 1)	207	10.90	2.32	8	21
Knowledge of Mathematics Teaching (Year 2)	170	11.54	2.37	8	21
Knowledge of Mathematics Teaching (Year 3)	166	12.43	3.32	8	26
Density of Professional Support (Year 1)	207	0.00	0.00	0	0
Density of Professional Support (Year 2)	170	0.93	0.77	0	2
Density of Professional Support (Year 3)	170	0.93	0.77	0	2
Density of Peer Support (Year 1)	207	0.00	0.00	0	0
Density of Peer Support (Year 2)	170	1.03	0.84	0	2
Density of Peer Support (Year 3)	166	1.00	0.87	0	2
Mathematics Content Knowledge	207	0.01	0.47	−2	1
Content-Specific Noticing Skills	207	0.07	0.99	−2	2

Note. Numbers displayed in the Min and Max columns represent the minimum and maximum values observed in the data set and were rounded to the nearest whole number.
[a] The mean score indicates the average score across teachers for each variable.

and sample responses). To prevent scoring bias, each response was coded by two raters who did not know which responses belonged to which year of administration. A Cohen's kappa statistic of 0.92 indicated strong agreement between the two raters. The teachers' total score[1] on these items in each year was used as an indicator of their knowledge for that year (see Table 2 below for descriptive statistics of teachers' KSMT and KMT scores by year).

6.3. Time

To capture whether teachers' KSMT and KMT changed throughout the study, and if so, to what extent they changed, we created a variable to capture the occasion on which the data were collected. The time variable ranged from 0 to 2, with 0 being the first administration of the survey (i.e., the first academic year the study was conducted) and 2 being the third administration of the survey. Thus, the time variable captured the academic years that had elapsed since the study began.

6.3.1. Content knowledge

We also measured teachers' understanding of fractions and ratios, the two content areas in which the KSMT and KMT items were situated. Fourteen constructed-response items identified from prior literature (e.g., Izsák & Beckmann, 2019; Van de Walle et al., 2019) were used to capture teachers' robust understanding of these concepts (for detailed item descriptions and the scoring rubric, see Copur-Gencturk & Olmez, 2021). All responses were coded by two raters to capture the correctness of teachers' final answers as well as the accuracy of the reasoning they provided or the validity of the methods they used to solve these problems. Cronbach's alpha, indicating the internal consistency (reliability) of this scale, was 0.81. We used the average of the standardized item scores in our analyses (see Table 2 for descriptive statistics of the variables used in the study).

6.3.2. Content-specific noticing

We measured teachers' content-specific noticing skills at their entry into the study by asking them to watch four short video clips of mathematics instruction and report the most significant issue they noticed around students' mathematics learning and the teachers' teaching of the targeted mathematics content. The research team coded the teachers' responses by using a 4-point rubric designed to capture the topic the teachers noticed (the teacher's pedagogy, the students' mathematical thinking, the general classroom climate, or other) and how teachers interpreted the events they noticed as these related to the mathematics content (purely descriptive, or analytical and interpretive; see Copur-Gencturk & Rodrigues, 2021). A score of 1 was given to responses that did not mention an issue related to mathematics learning or teaching (i.e., the teacher focused on generic issues). A score of 2 was given to responses that focused on a content-specific issue but gave a descriptive interpretation. A score of 3 was given to interpretations of the content-specific event that were supported by some evidence or a rationale. Finally, a score of 4 was given to responses that focused on a content-specific issue and included an interpretation of the event supported by evidence. Each video was coded independently so that teachers' responses and scores on one video would not affect their scores on other videos. Twelve percent of the data were coded by at least two coders to ensure that the data were coded consistently by the raters. Strong agreement between raters (a Cohen's kappa statistic of 0.81) indicated the data were coded consistently across raters. Teachers' total scores on all the items were used to represent their overall noticing skill. The internal consistency (i.e., Cronbach's alpha) of the scale was 0.66. The standardized scores around the mean were used in the analyses.

6.3.3. Formal learning opportunities

We asked teachers to report the number of hours of formal support they had received from their schools and districts by using items adapted from prior work (Copur-Gencturk, Plowman, et al., 2019; Garet et al., 2016). The formal learning opportunities included time spent with mentor teachers, in induction programs, or in university-level courses or professional development programs. Teachers were asked to report the number of hours of activities they had completed that focused on mathematics teaching and learning between the previous and current administration of the survey.

We then created a variable to capture the intensity of support teachers received in each year. Low formal support indicated teachers received fewer than 14 h of formal support, medium support indicated teachers received more than 14 h but fewer than 48 h of formal support, and substantial support indicated teachers received 49 h or more of formal support (see Table 2 for descriptive statistics). These categories were created based on prior evidence showing that professional development programs lasting, on average, 49 h had a positive impact on student learning, whereas professional development programs lasting fewer than 14 h did not

[1] Teachers' scores on the KSMT and KMT scales can range from 8 to 32 points given that each scale is measured with eight items and calculated by the sum of their scores on the items.

yield a positive impact on student learning (Yoon et al., 2007).

6.3.4. Other informal learning opportunities (learning with peers)

To capture teachers' learning from other informal sources, we asked teachers to report any activities that included structured and unstructured discussions with their peers by using items adapted from prior work (Garet et al., 1999, 2016). These activities included regular meetings with grade-level teachers in their school or group study with other teachers around a common issue or concern.[2] Similar to the formal support variable, we created a variable to capture the intensity of the peer-learning support teachers reported receiving on a 3-point scale. Low peer support indicated teachers spent 10 or fewer hours on peer-support activities, whereas medium peer support indicated teachers spent 11–26 h on activities involving learning with peers, and substantial peer support indicated teachers spent 27 h or more on peer-supported learning activities[3] (see Table 2 for descriptive statistics).

6.4. Teachers' background characteristics

We also included binary variables in our analyses to capture teachers' pathway to entering teaching (i.e., alternative or traditional teacher education program category, with the latter being the reference category) and their teaching credentials (two categories: holding a mathematics teaching credential or not [i.e., holding multiple subject credentials in teaching or holding teaching credentials in other subjects, e.g., special education], with the latter being the reference category). We also created a variable to capture teachers' years of teaching experience at the beginning of the study.

6.5. Analytic plan

Because we wanted to investigate the extent to which teachers improved their KSMT and KMT over the study period, for each knowledge component, we used a linear growth modeling approach to investigate the growth trajectories of teachers' development of such knowledge. Growth modeling handles missing data efficiently by allowing samples in the estimation as long as the outcome variable of the subject has one data point. In this study, we conducted the same set of analyses for each outcome measure (i.e., KSMT and KMT).

In the two-level growth modeling we used, Level 1 parameters characterized each individual teacher's KSMT (or KMT) development trajectory over time (i.e., years elapsed since the study began) and Level 2 parameters characterized how the teacher-level factors might explain the differences in their KSMT (or KMT) development. We allowed the growth patterns to differ across individual teachers by including a random slope for the time variable, and we allowed the initial level of KSMT (or KMT) to differ across teachers by including a random intercept. Specifically, to investigate the change in teachers' KSMT over time, we tested the following model:

$$Level\ 1:\ KSMT_{it} = \beta_{oi} + \beta_{1i}Time_{it} + e_{it}$$

$$Level\ 2:\ \beta_{oi} = \gamma_{00} + r_{0i},$$

$$\beta_{1i} = \gamma_{10} + r_{1i}$$

where $KSMT_{it}$ refers to the KSMT score for teacher i at time t. The intercept, β_{oi}, was the level of KSMT for teacher i at the beginning of the study (i.e., at time 0), whereas β_{1i} indicated the average rate of increase in the KSMT score for teacher i per year.

Although this model allowed us to check whether teachers' KSMT changed over the duration of the study, it did not account for the formal or other informal professional support teachers received throughout the study that might also contributed to their KSMT growth. Because both formal support and peer support vary within and between teachers each year, we included them as Level 1 predictors, as specified below. The equations for the model in which formal and informal support were the predictors are as follows:

$$\begin{aligned} Level\ 1:\ KSMT_{it} = \beta_{oi} + \beta_{1i}Time_{it} + formalsupport_{it} \\ \times [\beta_{2i} + \beta_{3i}Time_{it}] + peersupport_{it} \\ \times [\beta_{4i} + \beta_{5i}Time_{it}] + e_{it} \end{aligned}$$

$$Level\ 2:\ \beta_{oi} = \gamma_{00} + r_{0i}$$

$$\beta_{1i} = \gamma_{10} + r_{1i}$$

To investigate the role of teachers' content knowledge, noticing skills, and prior educational preparation in the growth of their KSMT, we tested models in which these variables were added separately to predict the KSMT score at the beginning of the study (i.e., intercept) and the rate of change in the KSMT score per year (i.e., slope). Specifically, as shown in the equations below, we added these time-invariant predictors (i.e., Level 2 variables, such as teachers' knowledge of school mathematics and their background indicators) to the model separately to predict their role in teachers' initial KSMT scores (β_{oi}) at the beginning of the study and the growth in their KMST per year (β_{1i}) [4]

$$Level\ 1:\ KMST_{it} = \beta_{oi} + \beta_{1i}Time_{it} + e_{it}$$

$$Level\ 2:\ \beta_{oi} = \gamma_{00} + \gamma_{01}Timeinvariantpredictor_{i} + r_{0i}$$

$$\beta_{1i} = \gamma_{10} + \gamma_{11}Timeinvariantpredictor_{i} + r_{1i}$$

We then tested a final model that included all the significant indicators from the previous models to investigate the relative contribution of each to the development of KSMT. All the aforementioned analyses were also conducted for the KMT outcome.

7. Results

7.1. The development of teachers' KSMT through teaching

Table 3 displays estimates of the fixed and random effects of the predictors by each model tested for the KSMT outcome. As shown in the results from Model 1, teachers improved their KSMT throughout the study. Teachers' KSMT scores increased by an average of 1.48 points per year ($p < .001$). To better understand the magnitude of the relationship, we also calculated the effect size [5], which is 0.60 SD in this case. This increase in their KSMT seemed to

[2] We also instructed teachers not to report the same activities several times.

[3] According to recent data in the National Center for Teacher Quality's Teacher Contract Database, only about a third school districts allocated time specifically for teacher collaboration, among which most districts only provided around 45 min per week for teacher collaboration (see https://www.nctq.org/contract-database).

[4] These models were similar to the previous model with support indicators, with the only difference being that these predictors replaced the support indicators in the models.

[5] We calculated the effect size by multiplying the cofficient for the time variable by its standard deviation and then dividing it by the square root of the level-1 error variance. The effect size is .60 SD (=1.48 *.83 /sqrt(4.14))

Table 3
Estimates of the linear growth models for teachers' knowledge of students' mathematical thinking.

	Time	Formal and peer support	Teacher knowledge and skills		Teacher professional background			Final
	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8
Fixed effect								
Intercept	13.64*** (0.23)	13.74*** (0.24)	13.60*** (0.21)	13.57*** (0.23)	14.34*** (0.46)	13.78*** (0.27)	13.36*** (0.25)	13.90*** (0.37)
Time (i.e., rate of change)	1.48*** (0.13)	1.50*** (0.22)	1.46*** (0.13)	1.48*** (0.14)	1.45*** (0.27)	1.44*** (0.16)	1.46*** (0.15)	1.64*** (0.18)
Time-varying formal support		−0.17 (0.45)						
Time-varying formal support × Time		0.17 (0.34)						
Time-varying informal support		−0.63~ (0.37)						−0.39* (0.18)
Time-varying informal support × Time		0.22 (0.29)						
Effect on intercept								
Content knowledge			3.24*** (0.40)					2.87*** (0.42)
Content-specific noticing skills				0.90*** (0.21)				0.55** (0.18)
Entry-level teaching experience					−0.82~ (0.44)			−0.39 (0.33)
Alternative certification						−0.52 (0.54)		
Holding a math teaching credential							1.51* (0.66)	0.30 (0.54)
Effect on slope								
Content knowledge			0.59* (0.24)					0.55* (0.24)
Content-specific noticing skills				−0.06 (0.13)				
Entry-level teaching experience					0.04 (0.27)			
Alternative certification						0.17 (0.29)		
Holding a math teaching credential							0.09 (0.33)	
Random effect								
Intercept variance	7.76*** (1.05)	8.01*** (1.07)	5.38*** (0.90)	6.99*** (1.04)	7.58*** (1.06)	7.70*** (1.05)	7.41*** (1.02)	5.18*** (0.92)
Slope variance	1.09 (0.44)	1.20 (0.43)	0.98 (0.44)	1.09 (0.44)	1.09 (0.44)	1.09 (0.44)	1.09 (0.44)	1.09 (0.43)
Intercept and slope covariance	0.16 (0.53)	−0.03 (0.54)	−0.25 (0.50)	0.21 (0.52)	0.17 (0.53)	0.17 (0.53)	0.13 (0.53)	−0.39 (0.51)
Residual variance	4.14*** (0.45)	3.97*** (0.42)	4.17*** (0.45)	4.13*** (0.45)	4.14*** (0.45)	4.14*** (0.45)	4.14*** (0.45)	4.05*** (0.43)

*$p < .05$; **$p < .01$; ***$p < .001$; ~$p < .10$.

derive from teaching the subject matter itself, given that teachers' formal and informal support did not predict the changes in their KSMT scores (see the results from Model 2). The intensity of formal professional support failed to predict the teachers' KSMT score ($p = .70$), and teachers' gain in KSMT did not differ by the level of formal professional support they received ($p = .61$). The additional peer support that teachers reported receiving was still marginally, yet negatively, associated with their KSMT ($p = .09$), but the rate of change in their KSMT was not different for teachers with different levels of peer support ($p = .45$). Taken together, these findings suggest that the experience of teaching mathematics helped teachers develop their KSMT, rather than the formal support or peer support they received.

Our examination of the role of teaching expertise (i.e., content knowledge and noticing skills) and prior educational training in the development of KSMT underscored the importance of content knowledge in the rate of KSMT change (see Table 3). In particular, having a robust understanding of the mathematics being taught was associated not only with teachers' KSMT level at the beginning of the study (an effect size[6] of 0.66 *SD*; $p < .001$) but also the growth of their KSMT (an effect size[7] of 0.29 *SD*; $p = .01$). This means that teachers with a robust knowledge of the mathematics (i.e., 90th percentile) increased their PCK each year by 1.82 points, whereas teachers with a less robust knowledge of the mathematics (i.e., 10th percentile) increased their PCK each year by 1.11 points (0.72 SD difference in the growth rate, $p = .01$). Teachers' noticing skills were related to their initial KSMT scores (an effect size of 0.33 *SD*; $p < .001$), but not the growth in their KSMT ($p = .65$; see Model 4). Of the indicators of teachers' professional background, teachers holding a credential in teaching mathematics had, on average, a

[6] We calculated the effect size by multiplying the coefficient for the content knowledge variable by the ratio of the standard deviation of the content knowledge variable and the square root of the intercept variance. The effect size is .66 SD (=3.24*.48/sqrt(5.38))

[7] We calculated the effect size by multiplying the coefficient for the content knowledge variable by its standard deviation and dividing by the square root of the slope variance. The effect size is .29 SD (=.59 *.48 /sqrt(.98))

0.55 *SD* higher KSMT score at the beginning of the study ($p = .02$); however, the rate of change in their KSMT was not statistically different from the rate of change for those who did not hold a mathematics teaching credential ($p = .78$). Teachers' initial number of years of experience in mathematics teaching was negatively, albeit marginally significantly, associated with teachers' KSMT score at the beginning of the study (an effect size of 0.15 *SD*, $p = .06$), but it was not linked to the growth rate of their PCK ($p = .89$).

The final model (i.e., Model 8), which included all the significant predictors from the previous models, indicated that teachers' content-specific noticing skills, level of peer support, and knowledge of mathematics were still statistically significantly related to their KSMT at the beginning of the study (an effect size of 0.24 *SD*, $p = .002$ for noticing; an effect size of 0.60 *SD*, $p < .001$ for the subject matter knowledge; an effect size of 0.12 SD, $p = .03$ for peer support). More important, the rate of change in teachers' KMST differed by their subject matter knowledge, even after adjusting for all previously significant indicators (an effect size of 0.25 *SD*, $p = .02$). This result indicates that the growth rate of teachers with a more robust understanding of the school mathematics (i.e., 90th percentile) on average, was 0.68 *SD* higher than those with a less robust understanding of the mathematics (i.e., 10th percentile) (see Fig. 1).

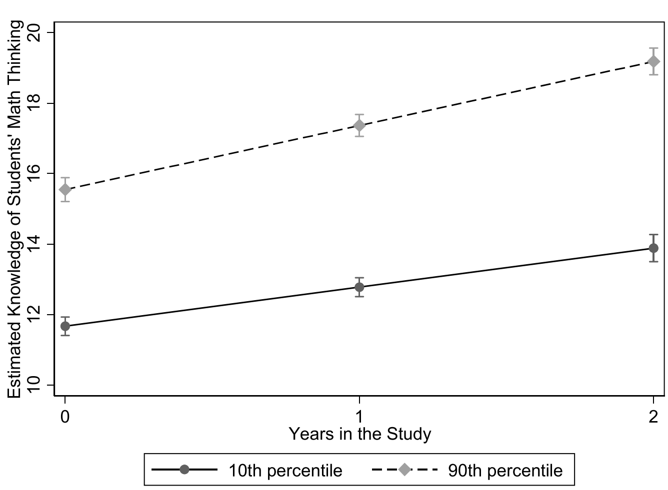

Fig. 1. Growth of knowledge of students' mathematical thinking (KSMT) among teachers with a less or more robust knowledge of the mathematics content at the beginning of the study.
Note. The error bars indicate 66% confidence intervals.

7.2. The development of teachers' KMT through teaching

We conducted a similar analysis to explore the development of teachers' KMT. As displayed in Table 4, teachers also improved their KMT throughout the study but at a slower rate: teachers' KMT scores increased by an average of 0.75 points per year (an effect size of 0.38 SD, $p < .001$). Similar to the pattern observed with the development of their KSMT, the change in teachers' KMT was not predicted by the level of formal or peer support they received throughout the study (see the results for Model 2), suggesting that teaching experience played a key role in the increase in teachers' KMT.

The only predictor of the rate of change in teachers' KMT was their credential type. Teachers holding a credential in teaching mathematics had, on average, a 0.59 *SD* higher KMT score at the beginning of the study ($p = .03$) and gained 0.60 points more KMT per year than did those who did not hold a mathematics teaching credential (an effect size of 0.74 SD, $p = .05$). Teachers' robust understanding of mathematical concepts and their content-specific noticing skills seemed to be associated with their KMT at the beginning of the study (an effect size of 0.98 *SD*, $p < .001$ for teachers' subject matter knowledge; an effect size of 0.42 *SD*, $p < .001$ for noticing), but neither predicted the rate of change in teachers' KMT ($p = .11$ and $p = .35$ for subject matter knowledge and noticing, respectively). Of the indicators of teachers' professional background, teachers' initial number of years of experience in mathematics teaching was negatively associated with their initial KMT scores (an effect size of 0.20 *SD*, $p = .04$), but it was not linked to the growth rate of their PCK ($p = .87$).

When all the significant predictors from previous models were included in the analysis (i.e., Model 8), teachers' subject matter knowledge and content-specific noticing skills were still statistically significantly associated with teachers' KMT scores at their entry into the study (an effect size of 0.34 *SD*, $p = .004$ for noticing; an effect size of 0.98 *SD*, $p < .001$ for mathematics content knowledge). Teachers who held a credential in teaching mathematics did not have a higher entry level of KMT than those who did not have mathematics teaching credential ($p = .90$), after adjusting for teachers' subject matter knowledge and noticing skills. As shown in Fig. 2, teachers with a mathematics teaching credential seemed to gain KMT at a higher rate than did those without a mathematics teaching credential (an effect size of 0.73 *SD*, $p = .06$).

8. Discussion

In this study, we aimed to explore the extent to which teaching can provide learning opportunities for teachers to develop two important components of PCK on their own: knowledge of students' understanding of a subject matter and knowledge of teaching that subject matter. Before we elaborate on the study findings and their implications for theory, research, and practice, we begin with the limitations of the study. First, our sample was a national sample of novice teachers, but not a *nationally representative* sample of teachers. Thus, the extent to which the results are generalizable to novice teachers across the United States remains unclear. The literature will benefit from future replications with a nationally representative sample.

Second, as in any longitudinal study, attrition in this study was inevitable. We tried to minimize the impact of data attrition by using hierarchical linear modeling in our analysis, given that this approach handles missing data. Third, our measures on the learning experience teachers had during our data collection period as well as teachers' pre-service training experience captured only the intensity but not the quality of these experiences. Future research is needed to explore both the intensity and quality of such opportunities in novice teachers' knowledge development. Finally, given that the two components of PCK investigated in the study are specific to mathematics, the extent to which the observed results will hold true for other subjects remains open and requires further investigation. Because of the similarities in the conceptualization of these knowledge components in other subjects (e.g., Jordan et al., 2008; Park & Oliver, 2008; Schneider & Plasman, 2011) and the similar roles these two components of PCK play in teaching and learning other subjects (e.g., Keller et al., 2017; Park et al., 2011), we anticipate that the study findings will be generalizable to other subject matter to some extent. Still, future work is needed in other subject areas.

Our findings provide empirical evidence that teachers continued to increase both their KSMT and KMT on their own through their teaching practice over the 3 years of our study period, even after we considered the time they invested each year in formal learning activities, such as professional development and

Table 4
Estimates of the linear growth models for teachers' knowledge of mathematics teaching.

	Time	Formal and peer support	Teacher knowledge and skills		Teacher professional background			Final
	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8
Fixed effect								
Intercept	10.87*** (0.15)	10.88*** (0.16)	10.85*** (0.13)	10.82*** (0.15)	11.41*** (0.32)	10.90*** (0.17)	10.70*** (0.16)	11.10*** (0.25)
Time (i.e., rate of change)	0.75*** (0.11)	0.63*** (0.17)	0.73*** (0.11)	0.76*** (0.11)	0.72** (0.23)	0.84*** (0.12)	0.63*** (0.11)	0.62*** (0.11)
Time-varying formal support		0.15 (0.37)						
Time-varying formal support × Time		0.06 (0.27)						
Time-varying informal support		−0.27 (0.32)						
Time-varying informal support × Time		0.14 (0.22)						
Effect on intercept								
Content knowledge			2.34*** (0.25)					2.18*** (0.27)
Content-specific noticing skills				0.62*** (0.14)				0.37** (0.13)
Entry-level teaching experience					−0.63* (0.31)			−0.33 (0.23)
Alternative certification						−0.10 (0.38)		
Holding a math teaching credential							0.91* (0.41)	0.05 (0.36)
Effect on slope								
Content knowledge			0.32 (0.20)					
Content-specific noticing skills				−0.11 (0.11)				
Entry-level teaching experience					0.04 (0.22)			
Alternative certification						−0.38 (0.25)		
Holding a math teaching credential							0.60~ (0.31)	0.59~ (0.31)
Random effect								
Intercept variance	2.54*** (0.67)	2.56*** (0.67)	1.30 (0.55)	2.16** (0.65)	2.44** (0.67)	2.54*** (0.68)	2.42*** (0.64)	1.13 (0.54)
Slope variance	0.71 (0.29)	0.71 (0.28)	0.68 (0.29)	0.70 (0.28)	0.71 (0.29)	0.68 (0.28)	0.66 (0.29)	0.65 (0.29)
Intercept and slope covariance	0.46 (0.41)	0.46 (0.40)	0.29 (0.35)	0.52 (0.38)	0.47 (0.40)	0.46 (0.39)	0.37 (0.40)	0.33 (0.35)
Residual variance	2.64*** (0.37)	2.60*** (0.37)	2.64*** (0.37)	2.63*** (0.37)	2.63*** (0.37)	2.63*** (0.37)	2.63*** (0.37)	2.64*** (0.37)

*$p < .05$; **$p < .01$; ***$p < .001$; ~$p < .10$.

mentoring, and informal support from their colleagues specifically in mathematics teaching and learning. Interestingly, by comparing the growth rates of two components of PCK, we found that teachers learned more about students' understanding of the subject matter faster than they learned how to teach the subject matter through teaching over the same period. Overall, the finding that KSMT and KMT both increased through one's teaching practice highlights the situated nature of teacher knowledge and teaching as a situated learning environment (Scheiner et al., 2019). One implication for practice is that school leaders and policy makers could allocate more time for teachers to plan, analyze, and reflect on their teaching practice either independently or collectively with colleagues so that the teachers could gain knowledge that is specific to their own practice. At the same time, the finding that KSMT and KMT grown at different rates sheds light on the multidimensional nature of PCK and how each dimension of PCK develops in and through teaching. Prior work has shown that changes in teachers' instruction seem to be driven by their new perspectives about student thinking (Leikin & Zazkis, 2010; Marcus & Chazan, 2010). This may help explain our finding that teachers developed KSMT faster than KMT during the same period, as KSMT may be formed prior to KMT. Also, since teachers are constantly exposed to students' different approaches to a certain task in class, they may have easy access to the learning of KSMT. In contrast, learning KMT may demand higher-order thinking and effort to solve a teaching puzzle by analyzing student thinking and evaluate their instructional practices (Copur-Gencturk & Rodrigues, 2020). The learning process of KMT may take several rounds of trial and error inside and outside of the class and therefore may take a longer time. Thus, attending to the distinct dimensions of PCK will improve our understand about teachers' learning progression in each knowledge dimension, whereas treating PCK holistically may cover up such differences.

Our findings also provide evidence that teacher-level factors may function differently in the development of different dimensions of teacher knowledge. On one hand, we found that

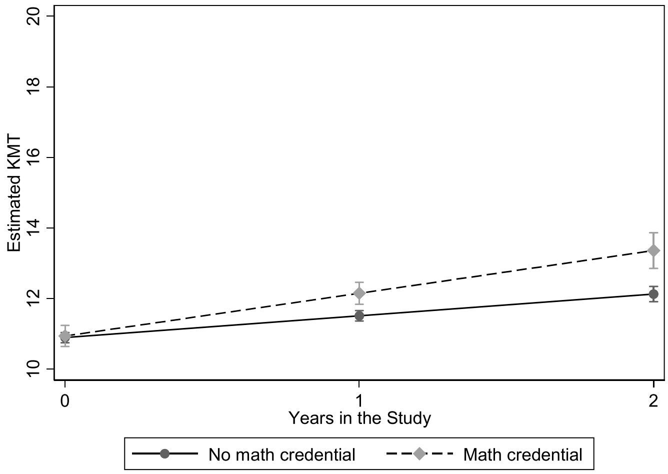

Fig. 2. Growth of Knowledge of Mathematics Teaching (KMT) Among Teachers Holding or not Holding a Mathematics Teaching Credential.
Note. The error bars indicate 66% confidence intervals.

teachers with a more robust understanding of the content developed KSMT faster than their peers with a less robust understanding. This finding suggests that helping prospective and in-service teachers build a solid foundation in the content they need to teach may help them better follow and understand their students' ideas in class and accelerate their attainment of knowledge about student learning of the subject. On the other hand, holding a teaching credential in mathematics was the only significant predictor of the growth rate of KMT, which was still marginally significant after we controlled for both teachers' mathematics knowledge and their noticing skills. This finding is consistent with a common recognition that teachers' subject matter knowledge is not naturally transformed to the knowledge of teaching a subject. Teachers need specialized training in teaching a subject (e.g., by taking mathematics teaching methods courses). Thus, aside from general pedagogy, teacher education programs should devote more time to helping teachers learn the affordances and limitations of subject-specific instructional tools and strategies. For teachers who entered the profession without a subject-specific credential, schools may provide additional support to help them thrive in instruction.

Our results also showed that content-specific noticing skills are positively associated with teachers' initial levels of KSMT and KMT, which is consistent with findings in prior cross-sectional studies (e.g., Copur-Gencturk & Tolar, 2022; Yang et al., 2021). However, noticing did not seem to help teachers gain KSMT and KMT through teaching faster than their peers with weaker noticing skills over time. This result may suggest that teachers with stronger PCK may have greater noticing skills; however, noticing skills may not help teachers learn on their own from teaching.

Overall, our study suggests a necessity for research to take the multidimensional nature of teacher knowledge into consideration when trying to understand teacher learning. We also encourage researchers to identify the impact of different teacher learning experiences on teachers' knowledge development simultaneously. By paying attention to what constitutes teacher knowledge and what learning opportunities can better support the development of each knowledge dimension, we may provide teachers with an optimal combination of support on the job.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The data are still being used by authors for an ongoing project.

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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.tate.2022.103949.

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