

CALDERÓN–ZYGMUND OPERATORS AND COMMUTATORS IN SPACES OF HOMOGENEOUS TYPE: WEIGHTED INEQUALITIES

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Abstract. The recent proof of the sharp weighted bound for Calderón–Zygmund operators has led to much investigation in sharp mixed bounds for operators and commutators, that is, a sharp weighted bound that is a product of at least two different A_p weight constants. The reason why these are sought after is that the product will be strictly smaller than the original one-constant bound. We prove a variety of these bounds in spaces of homogeneous type, using the new techniques of Lerner, for both operators and commutators.

1. Introduction

The theory of sparse domination, that is, pointwise (or bilinearly) bounding an operator by a finite sum of simple, so-called sparse operators, has been an active area of investigation. Since the original preprint of this article appeared, there has been a plethora of theory and applications of sparse bounds to a wide variety of operators and settings. References are too numerous to list, but one could start by looking at [31], [5], [35] and the references therein.

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This article, circulated in preprint form since 2014, uses the older technology developed by Lerner; it is likely that many of our results could be even more updated with the recent technological surge. Our results have proven relevant in several other projects; a preprint form of this article has been cited in [29], [14], [9], [28], [17], for example. With this firmly in mind, we proceed with the introduction.

In the last decades, harmonic analysts have paid much attention to the area of weighted inequalities for singular integrals. Since Muckenhoupt introduced in [38] the A_p classes in the 1970's to answer the necessary and sufficient conditions for the boundedness of the maximal function on weighted L^p spaces, many have sought a deeper understanding of the constants present in such weighted bounds.

The first result in this area was due to Buckley, who asserted the precise dependence of the A_p constant in [6] was

$$(1.1) \quad \|M\|_{L^p(w)} \leq c_p [w]_{A_p}^{\frac{1}{p-1}}.$$

Since then, much work was put into solving the so-called A_2 conjecture, which stated that the constant in the corresponding weighted norm inequality for Calderón–Zygmund singular integrals depended linearly on the A_2 constant. This was solved by Hytönen in [20]. See also [21] for a survey about the history of the conjecture and [30] for a simpler proof.

After the solution of the A_2 conjecture, an improvement of this result was obtained in [24]. This new result can be better understood if we first consider the case of Buckley's estimate (1.1) for the maximal function. For the case $p = 2$, the maximal estimate is given by

$$\|M\|_{L^2(w)} \leq c_n [w]_{A_2}.$$

The idea is to replace a portion of the A_2 constant by another smaller constant defined in terms of the A_∞ constant given by the functional

$$(1.2) \quad [w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q).$$

To be more precise, the improvement of Buckley's theorem is the following " A_2 – A_∞ " estimate:

$$\|M\|_{L^2(w)} \leq c_n [w]_{A_2}^{1/2} [w^{-1}]_{A_\infty}^{1/2},$$

which can be found in [24] along with its L^p counterpart. The A_∞ constant as given by (1.2) was originally introduced by Fujii in [15] and rediscovered

ered later by Wilson in [42]. This definition is more suitable than the more classical condition due to Hrusčev [19], which is defined by the expression

$$[w]_{A_\infty}^H = \sup_Q \left(\frac{1}{|Q|} \int_Q w(t) dt \right) \exp \left(\frac{1}{|Q|} \int_Q \log w(t)^{-1} dt \right),$$

since

$$[w]_{A_\infty} \leq c_n [w]_{A_\infty}^H,$$

as it was observed in [24]. In fact it is shown in the same paper with explicit examples that $[w]_{A_\infty}$ is much smaller (actually exponentially smaller) than $[w]_A^H$.

Considering the case of singular integrals, the mixed sharp A_2 – A_∞ result below obtained in [24] improves the A_2 theorem:

$$\|T\|_{L^2(w)} \leq c_n [w]_{A_2}^{1/2} ([w^{-1}]_{A_\infty}^{1/2} + [w]_{A_\infty}^{1/2}).$$

This is the right estimate when compared with M since this reflects the property that T is (essentially) self-adjoint. There is also a further improvement involving $[w]_{A_p}$ in [23].

In this paper we follow this idea of replacing a portion of the A_p constant by the A_∞ constant for the problems considered in [32] and improved in [33], within the context of spaces of homogeneous type. To do this we will prove first the following theorem which follows essentially from [33], but in this stated form can be found in [39].

THEOREM 1.1. *Let T be a Calderón–Zygmund operator and let $1 < p < \infty$. Then for any weight w and $r > 1$,*

$$(1.3) \quad \|Tf\|_{L^p(w)} \leq C p p' (r')^{\frac{1}{p'}} \|f\|_{L^p(M_r w)},$$

where the weight on the right-hand side is now the “ L^r Hardy–Littlewood maximal function”: $M_r w = \sup_{Q \ni x} (f_Q w^r)^{1/r}$, and the symbol ‘ indicates the dual exponent.

This mixed theorem leads to sharp mixed A_1 – A_∞ weighted bounds for Calderón–Zygmund operators using simple properties of A_p weights:

$$(1.4) \quad \|Tf\|_{L^p(w)} \leq C p p' [w]_{A_\infty}^{1/p'} \|f\|_{L^p(M w)},$$

and if $w \in A_1$,

$$(1.5) \quad \|Tf\|_{L^p(w)} \leq C p p' [w]_{A_\infty}^{1/p'} [w]_{A_1}^{1/p} \|f\|_{L^p(w)},$$

where C is a dimensional constant that also depends on T .

Recent results relating to the simple proof of the A_2 conjecture allow us to simplify and streamline the proof of this theorem, and also extend it to spaces of homogeneous type (SHT). To do this we use the theory of sparse domination in spaces of homogeneous type, which has rapidly developed. However, for our purposes the older technology of [4] suffices (see the next section for details).

A similar approach can also be done with sharp weak bounds. We have the following sharp bound of Lerner, Ombrosi and Pérez in [33],

$$\|T\|_{L^p(w)} \leq Cpp'[w]_{A_1},$$

which led to several endpoint estimates in [24].

Again, we can update this proof by applying recent estimates in sharp weighted theory as well as extend to spaces of homogeneous type. We will employ a new sharp reverse Hölder inequality for spaces of homogeneous type (see Lemma 4.6).

Finally, we show sharp weighted bounds for commutators of Calderón–Zygmund operators with functions in BMO and their iterates. These questions have been considered before in [8] and lately improved in [24] but our bounds are new in the context of spaces of homogeneous type (see also [34], which appeared after this preprint).

The organization of this paper will be as follows. In Section 2 we give some background and definitions which will help us to prove our main results, listed in Section 3. Finally, Section 4 contains all the proofs as well as some remarks.

2. Preliminaries

2.1. Spaces of homogeneous type. We will be working on spaces of homogeneous type, which generalize the Euclidean situation of \mathbb{R}^n with Lebesgue measure. Other examples of spaces of homogeneous type include C^∞ compact Riemannian manifolds, graphs of Lipschitz functions and Cantor sets with Hausdorff measure. These and more examples are described in [7]; some applications of these spaces can be found in [37,43].

DEFINITION 2.1. A space of homogeneous type is an ordered triple (X, ρ, μ) where X is a set, ρ is a quasimetric, that is:

- (1) $\rho(x, y) = 0$ if and only if $x = y$.
- (2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.
- (3) $\rho(x, z) \leq \kappa(\rho(x, y) + \rho(y, z))$, for all $x, y, z \in X$.

for some constant $\kappa > 1$ (quasimetric constant), and the positive measure μ is doubling, that is

$$0 < \mu(B(x_0, 2r)) \leq D_\mu \mu(B(x_0, r)) < \infty,$$

for some constant D_μ (doubling constant).

We will say that a constant is absolute if it only depends on the space (X, ρ, μ) . Particularly, κ and D_μ appearing in the above definition are absolute constants.

Thankfully, many basic constructions and tools for classical harmonic analysis still exist in some form in spaces of homogeneous type, such as certain covering lemmas, however, the Lebesgue Differentiation Theorem does not (see [1] for a thorough description). Therefore, we will always assume that our measure is in addition Borel regular, to avoid technicalities.

We also rely on a dyadic grid decomposition, and can use the construction of Christ [7]; see also the newer construction of [22].

THEOREM 2.2. *There exists a family of sets $D = \bigcup_{k \in \mathbb{Z}} D_k$, called a dyadic decomposition of X , constants $0 < C, \varepsilon < \infty$, and a corresponding family of points $\{x_c(Q)\}_{Q \in D}$ such that:*

- (1) $X = \bigcup_{Q \in D_k} Q$, for all $k \in \mathbb{Z}$.
- (2) If $Q_1 \cap Q_2 \neq \emptyset$, then $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$.
- (3) For every $Q \in D_k$ there exists at least one child cube $Q_c \in D_{k-1}$ such that $Q_c \subseteq Q$.
- (4) For every $Q \in D_k$ there exists exactly one parent cube $\hat{Q} \in D_{k+1}$ such that $Q \subseteq \hat{Q}$.
- (5) If Q_2 is a child of Q_1 then $\mu(Q_2) \geq \varepsilon \mu(Q_1)$.
- (6) $B(x_c(Q), \delta^k) \subset Q \subset B(x_c(Q), C\delta^k)$.

We will refer to the last property as the *sandwich property*.

Later, we will use a multiple L of a dyadic cube Q , defined as $LQ = B(x_c(Q), CL\delta^k)$, a concept that has appeared in other papers including [4] and [22].

A weight w is a nonnegative locally integrable function on (X, μ) that takes values in $(0, \infty)$ almost everywhere. For any $1 < p < \infty$ we define the A_p constant of the weight w on the space of homogeneous type X as follows:

$$(2.1) \quad [w]_{A_p} := \sup_Q \left(\frac{1}{\mu(Q)} \int_Q w(x) d\mu \right) \left(\frac{1}{\mu(Q)} \int_Q w(x)^{1-p'} d\mu \right)^{p-1}.$$

Here we can take Q to be a ball, since the concept of a non-dyadic cube is not defined in spaces of homogeneous type. However, often we will work with an analogue of this constant where Q is a dyadic cube (as defined above). If $w \in A_p$ with respect to balls, then it is dyadic A_p (with respect to cubes) for any dyadic grid D , with a constant independent of D . Conversely, if w is in dyadic A_p with respect to all dyadic grids in an *adjacent dyadic system* (as shown in [22]), then $w \in A_p$ with respect to balls. It will be clear from context if we are working with respect to balls or dyadic cubes.

We define the Fujii–Wilson A_∞ constant in a space of homogeneous type as follows:

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) d\mu,$$

where $w(Q) = \int_Q w d\mu$. While this A_∞ constant is comparable using dyadic cubes or balls, the constant of comparison depends on the measure w (which is doubling since $w \in A_\infty$). Hence to achieve sharp bounds in spaces of homogeneous type we cannot simply switch between these constants defined with respect to cubes or balls since we introduce a w -dependent factor. This is reflected in the difference between the sharp reverse Hölder inequalities involving this A_∞ constant: if defined with respect to cubes we get a sharp reverse Hölder (see Lemma 4.6), but if defined with respect to balls we only get a sharp weak version (see [25]).

We also note that we define the Hardy–Littlewood maximal function with respect to balls, but this is well known to be bounded above and below by a finite sum of dyadic maximal functions (from an adjacent dyadic system) (see [22] or [27], for example).

2.2. Calderón–Zygmund operators and commutators. Next we recall some definitions related to Calderón–Zygmund operators and their commutators in the homogeneous setting.

DEFINITION 2.3. We say that $K: X \times X \setminus \{x = y\} \rightarrow \mathbb{R}$ is a Calderón–Zygmund kernel if there exist $\eta > 0$ and $C < \infty$ such that for all $x_0 \neq y \in X$ and $x \in X$ it satisfies the decay condition

$$(2.2) \quad |K(x_0, y)| \leq \frac{C}{\mu(B(x_0, \rho(x_0, y)))}$$

and the smoothness conditions for $\rho(x_0, x) \leq \eta \rho(x_0, y)$:

$$(2.3) \quad |K(x, y) - K(x_0, y)| \leq \left(\frac{\rho(x, x_0)}{\rho(x_0, y)} \right)^\eta \frac{C}{\mu(B(x_0, \rho(x_0, y)))},$$

and

$$(2.4) \quad |K(y, x) - K(y, x_0)| \leq \left(\frac{\rho(x, x_0)}{\rho(x_0, y)} \right)^\eta \frac{C}{\mu(B(x_0, \rho(x_0, y)))}.$$

DEFINITION 2.4. Let T be a singular integral operator associated to Calderón–Zygmund kernel K . If in addition T is bounded on L^2 , we say that T is a Calderón–Zygmund operator.

THEOREM 2.5 [10]. *Let T be a Calderón–Zygmund operator on a space of homogeneous type. Then T is bounded from L^1 to $L^{1,\infty}$.*

We will be invoking sparse operators to bound our Calderón–Zygmund operators using the formula originally due to Lerner [30]. Before we discuss this, we need to define a sparse family on a dyadic grid $D = \bigcup_k D_k$ as in Theorem 2.2.

DEFINITION 2.6. A sparse family $S = \bigcup_k S_k$, $S_k \in D_k$ on D is a collection of dyadic cubes such that for $Q', Q \in S$,

$$\mu\left(\bigcup_{Q' \subsetneq Q} Q'\right) \leq \frac{\mu(Q)}{2}.$$

DEFINITION 2.7. Given a sparse family S , we define a sparse operator as follows

$$T^S(f) = \sum_{Q \in S} \left(\int_Q f \right) \cdot \chi_Q.$$

We also introduce the decomposition of Lerner, proved in the homogeneous setting in [3].

THEOREM 2.8. *For any Calderón–Zygmund operator T on a space of homogeneous type X , we have*

$$\|Tf\|_Y \leq C \sup_{D,S} \|T^S f\|_Y,$$

where the supremum is taken over all dyadic grids D in an adjacent dyadic system, and over all sparse families with respect to those grids, C only depends on the operator and the space X , and Y is any Banach function space.

Finally, we introduce the definitions of a BMO function and the iterated commutators of Calderón–Zygmund operators with functions in BMO in spaces of homogeneous type.

DEFINITION 2.9. For a locally integrable function $b: X \rightarrow \mathbb{R}$ we define

$$\|b\|_{\text{BMO}} = \sup_Q \frac{1}{\mu(Q)} \int_Q |b(y) - b_Q| d\mu(y) < \infty,$$

where the supremum is taken over all dyadic cubes in X , and

$$b_Q = \frac{1}{\mu(Q)} \int_Q b(y) d\mu(y).$$

DEFINITION 2.10. Given a Calderón–Zygmund operator T with kernel K and a function b in BMO, we define the k -th order commutator with b , for an integer $k \geq 0$, as follows

$$T_b^k(f)(x) = \int_X (b(x) - b(y))^k K(x, y) f(y) d\mu(y).$$

In the particular case when $k = 1$, T_b^1 is the classic commutator and we will denote it by T_b .

Throughout this paper, X will denote a space of homogeneous type equipped with a quasimetric ρ with quasimetric constant κ and a positive doubling measure μ with doubling constant D_μ . We will denote by C a positive constant independent of the weight constant which may change from a line to other. We remind the reader that we call constants absolute if they only depend on parameters of the space, such as μ , but are independent of the weight.

3. Main results

Our goal is to prove the following results.

THEOREM 3.1. *Let T be a Calderón–Zygmund operator and let $1 < p < \infty$. Then for any weight w and $r > 1$,*

$$(3.1) \quad \|Tf\|_{L^p(w)} \leq Cpp' (r')^{\frac{1}{p'}} \|f\|_{L^p(M_rw)},$$

where C is an absolute constant that also depends on T .

From the previous theorem we obtain the following estimates as immediate corollaries.

COROLLARY 3.2. *Let T be a Calderón–Zygmund operator and let $1 < p < \infty$. Then if $w \in A_\infty$ we obtain*

$$(3.2) \quad \|Tf\|_{L^p(w)} \leq Cpp' [w]_{A_\infty}^{1/p'} \|f\|_{L^p(Mw)},$$

and if $w \in A_1$,

$$(3.3) \quad \|Tf\|_{L^p(w)} \leq Cpp' [w]_{A_\infty}^{1/p'} [w]_{A_1}^{1/p} \|f\|_{L^p(w)},$$

where C is an absolute constant that also depends on T .

Note the passage of right-hand side above from $L^p(M_rw)$ to $L^p(Mw)$ to $L^p(w)$. The first passage is due to an application of a reverse Hölder inequality and the second passage gains $[w]_{A_1}^{1/p}$.

As an application of (3.1) we obtain the following endpoint estimate.

THEOREM 3.3. *Let T be a Calderón–Zygmund operator. Then for any weight w and $r > 1$,*

$$(3.4) \quad \|Tf\|_{L^{1,\infty}(w)} \leq C \log(e + r') \|f\|_{L^1(M_rw)},$$

where C is an absolute constant that also depends on T .

Additionally we get the following estimates as corollaries of the above result choosing r as the sharp exponent in the reverse Hölder inequality for weights in the A_∞ class in the setting of spaces of homogeneous type (see Lemma 4.6) and taking into account that $r' \approx [w]_{A_\infty}$.

COROLLARY 3.4. *Let T be a Calderón–Zygmund operator. Then*

(1) *if $w \in A_\infty$,*

$$\|Tf\|_{L^{1,\infty}(w)} \leq C \log(e + [w]_{A_\infty}) \|f\|_{L^1(M, w)};$$

(2) *if $w \in A_1$,*

$$\|Tf\|_{L^{1,\infty}(w)} \leq C [w]_{A_1} \log(e + [w]_{A_\infty}) \|f\|_{L^1(w)}.$$

In both cases C is an absolute constant that also depends on T .

We also prove the following bound for a Calderón–Zygmund operator that is useful to get sharp A_2 – A_∞ bounds for the commutators in spaces of homogeneous type. Throughout define the dual weight $\sigma = w^{\frac{-1}{(p-1)}}$.

THEOREM 3.5. *Let T be a Calderón–Zygmund operator and $w \in A_2$. Then the following sharp weighted bound in an space of homogeneous type holds:*

$$\|T\|_{L^2(w)} \leq C [w]_{A_2}^{1/2} ([w]_{A_\infty} + [\sigma]_{A_\infty})^{1/2}.$$

We remark that one should be able to use the extension of [23] directly to obtain the following extension.

COROLLARY 3.6. *Let T be a Calderón–Zygmund operator and $w \in A_p$. Then the following sharp weighted bound in an space of homogeneous type holds:*

$$\|T\|_{L^p(w)} \leq C [w]_{A_p}^{1/p} ([w]_{A_\infty}^{1/p'} + [\sigma]_{A_\infty}^{1/p}).$$

Finally, as a corollary of the previous result and using a precise version of the John–Nirenberg inequality proved in Section 4.3, we prove the following generalized sharp weighted bound for the k -th iterate commutator of a Calderón–Zygmund operator.

COROLLARY 3.7. *Let T be a Calderón–Zygmund operator defined on a space of homogeneous type and $b \in \text{BMO}$. Then*

$$(3.5) \quad \|T_b^k(f)\|_{L^2(w)} \leq C [w]_{A_2}^{1/2} ([w]_{A_\infty} + [\sigma]_{A_\infty})^{k+1/2} \|b\|_{\text{BMO}} \|f\|_{L^2(w)},$$

where C is an absolute constant. In particular, for the classical commutator we get the estimate

$$(3.6) \quad \|T_b(f)\|_{L^2(w)} \leq C [w]_{A_2}^{1/2} ([w]_{A_\infty} + [\sigma]_{A_\infty})^{3/2} \|b\|_{\text{BMO}} \|f\|_{L^2(w)}.$$

REMARK 3.8. We remark that we can also extend this result to an analogous statement involving $[w]_{A_p}$ by using Corollary 3.6. Finally, the optimality of the exponents in these results follow from the corresponding results in \mathbb{R}^n which were obtained by building specific examples of weights for each operator. However, a new approach to derive the optimality of the exponents without building explicit examples can be found in [36].

4. Proofs

4.1. Proofs of Theorem 3.1 and Corollary 3.2. First, we will prove the next inequality of Coifman–Fefferman type.

PROPOSITION 4.1. *Let T be a Calderón–Zygmund operator and let $1 < p < \infty$. If $w \in A_p$ then*

$$(4.1) \quad \int_X |Tf(x)|w(x) d\mu(x) \leq C[w]_{A_p} \int_X Mf(x)w(x) d\mu(x),$$

where C is an absolute constant that depends also on T .

Before proving Proposition 4.1 we need to recall the following lemma that will allow us to obtain the precise constant in (4.1) and that can be found in [18, Example 9.2.5] as well as in [16, p. 388] in the context of two weights. It is likely that the following result is also true with $[w]_{A_p}$ replaced by $[w]_{A_\infty}$, as this has been remarked to the authors to hold in \mathbb{R}^n . This would improve Proposition 4.1 analogously. However, since we only need the version stated and could not find a reference, we leave this as a remark.

LEMMA 4.2. *Let μ be a positive doubling measure and $1 < p < \infty$. If $w \in A_p$ then*

$$(4.2) \quad \left(\frac{\mu(A)}{\mu(Q)} \right)^p \leq [w]_{A_p} \frac{w(A)}{w(Q)},$$

where $A \subset Q$ is a μ -measurable set and Q is a cube.

PROOF OF PROPOSITION 4.1. It is possible that this argument appeared in exactly the same fashion in \mathbb{R}^n prior to the appearance of our preprint. However, we are unaware of this and therefore choose to include the short proof. We have that

$$\int_X |Tf(x)|w(x) d\mu(x) \leq C_{X,T} \sup_{S,D} \int_X \left| \sum_{Q \in S} \left(\int_Q f(x) \right) \chi_Q(x) \right| w(x) d\mu(x)$$

by the formula of Lerner proved in the homogeneous setting in [3]. Using (4.2), we obtain that

$$\begin{aligned} \int_X |Tf(x)|w(x) d\mu(x) &\leq C_{X,T} \sup_{S,D} \sum_{Q \in S} \left(\int_Q |f(x)| \right) w(Q) \\ &\leq C_{X,T} \sup_{S,D} [w]_{A_p} \sum_{Q \in S} \left(\int_Q |f(x)| \right) w(E(Q)) \\ &\leq C_{X,T} [w]_{A_p} \sup_{D,S} \sum_{Q \in S} \int_{E(Q)} Mf(x)w(x) d\mu(x), \end{aligned}$$

where we have applied Lemma 4.2, noting that the family is $1/2$ -sparse, to arrive at the second line above. Finally, since the family $E(Q)$ is disjoint, we can bound the above by

$$\begin{aligned} \int_X |Tf(x)|w(x) d\mu(x) &\leq C [w]_{A_p} \sup_{D,S} \int_X Mf(x)w(x) d\mu(x) \\ &\leq C [w]_{A_p} \int_X Mf(x)w(x) d\mu(x), \end{aligned}$$

where C is an absolute constant that depends also on T , proving (4.1) as desired. \square

Now we prove the following lemma.

LEMMA 4.3. *Let w be any weight and let $1 \leq p, r < \infty$. Then there is a constant $C = C_{X,T}$ such that*

$$\|Tf\|_{L^p((M_r w)^{1-p})} \leq C p \|Mf\|_{L^p((M_r w)^{1-p})}.$$

The proof of this lemma is based in a variation of the Rubio de Francia algorithm that could be found in [39]. The only main new ingredient is the Coifman–Rochberg theorem in spaces of homogeneous type [13, Proposition 5.32].

By using Lemma 4.3 applied to T^* we can prove Theorem 3.1, whose proof is omitted since again it is analogous for SHT, by replacing the dimensional constants with geometric ones.

PROOF OF COROLLARY 3.2. The proofs of (3.2) and (3.3) are immediate. In the first case, the estimate is derived by applying the sharp reverse Hölder inequality for weights in the A_∞ class proved in Lemma 4.6 to (3.1) and using the fact that $r' \approx [w]_{A_\infty}$. The latter is a direct consequence of (3.2) since $w \in A_1$. \square

4.2. Proof of Theorem 3.3. First we establish a lemma which follows similar ideas of [16, Ch. 4, Lemma 3.3], that we will need for the proof of Theorem 3.3.

LEMMA 4.4. *Let T be a CZO. Let w be a weight and $a \in L^1(w)$ be supported in a cube Q with $\int_Q a(y) d\mu(y) = 0$ and η be the smoothness constant from the definition of a CZO kernel in SHT. Then, if we set $\tilde{Q} = LQ$ for a large $L \geq 1/\eta > 0$, the inequality*

$$(4.3) \quad \int_{X \setminus \tilde{Q}} |T(a)(x)|w(x) d\mu(x) \leq C \int_X |a(x)|Mw(x) d\mu(x),$$

holds with an absolute constant C depending on the kernel K .

Note that all we need is that $L > 1$ above. This is automatically satisfied if $0 < \eta < 1$, otherwise, choose any $L > 1$.

PROOF. Fix $y_0 \in X$ and assume for simplicity that $Q = B(y_0, R)$, with $R > 0$. Now making use of the cancellation property of a , we obtain

$$\begin{aligned} \int_{X \setminus \tilde{Q}} |T(a)(x)|w(x) d\mu(x) &= \int_{X \setminus \tilde{Q}} \left| \int_Q K(x, y)a(y) d\mu(y) \right| w(x) d\mu(x) \\ &\leq \int_Q \int_{X \setminus \tilde{Q}} |K(x, y) - K(x, y_0)|w(x) d\mu(x) |a(y)| d\mu(y) = \int_Q I(y) |a(y)| d\mu(y). \end{aligned}$$

We only need to prove that I is bounded by $CMw(y)$ where $C = C_{X,K}$ is an absolute constant also depending on the kernel K . For every $y \in Q$, using the smoothness property of K in the second variable (since $\rho(y, y_0) \leq \eta\rho(x, y_0)$), we obtain

$$\begin{aligned} I(y) &= \int_{X \setminus \tilde{Q}} |K(x, y) - K(x, y_0)|w(x) d\mu(x) \\ &\leq C_{X,K} \int_{X \setminus \tilde{Q}} \left(\frac{\rho(y, y_0)}{\rho(x, y_0)} \right)^\eta \frac{1}{\mu(B(y_0, \rho(x, y_0)))} w(x) d\mu(x) \\ &= C_{X,K} \sum_{l=1}^{\infty} \int_{2^l Q \setminus 2^{l-1} Q} \left(\frac{\rho(y, y_0)}{\rho(x, y_0)} \right)^\eta \frac{1}{\mu(B(y_0, \rho(x, y_0)))} w(x) d\mu(x) \\ &\leq C_{X,K} \sum_{l=1}^{\infty} \int_{2^l Q} \frac{2^\eta}{2^{l\eta}} \frac{\mu(B(y_0, 2^l R))}{\mu(B(y_0, 2^{l-1} R))} \frac{1}{\mu(2^l Q)} w(x) d\mu(x) \\ &\leq C_{X,K} \sum_{l=1}^{\infty} \frac{1}{2^{l\eta}} \frac{1}{\mu(2^l Q)} \int_{2^l Q} w(x) d\mu(x) \leq C_{X,K} Mw(y). \end{aligned}$$

Above we have used the fact that $\rho(y, y_0) < R$ and since $\rho(x, y_0) > LR$, there exists an $l > 1$ so that $2^{l-1}R < \rho(x, y_0) \leq 2^lR$. Thus we have shown that (4.3) holds. \square

PROOF OF THEOREM 3.3. The proof of Theorem 3.3 follows the proof of [40, Theorem 1.6], with the following two changes. Let Q_j be a cube, $\tilde{Q}_j = 2Q_j$, $\Omega = \bigcup_j Q_j$, $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$ and $x \in Q_j$. Firstly we use the fact that for $r > 1$ and a non-negative function w with $M_r w(x) < \infty$ a.e., we have

$$(4.4) \quad M_r(w\chi_{\tilde{\Omega}^c})(x) \leq C_X \inf_{y \in Q_j} M_r(w\chi_{\tilde{\Omega}^c})(y)$$

where the constant depends only on the doubling constant D_μ . This was proved in [3]. This replaces the use of the classical case with the Hardy–Littlewood maximal operator M , see for instance [16, p. 159]. Finally, we use Lemma 4.3 to replace the analogous lemma used in [40]. \square

4.3. Proofs of Theorem 3.5 and Corollary 3.7. To prove the main result in this section we first need the following lemmas. The first is a sharp reverse Hölder inequality for A_∞ weights in spaces of homogeneous type adapted from an argument due to Hytönen, Pérez, and Rela whose proof can be found in [25]. Before we state and prove this, we note that in this same paper there is a weak version of this inequality stated below. They call this result a weak inequality since on the right-hand side we have the dilation $2\kappa B$ of the ball B .

LEMMA 4.5 [25]. *Let $w \in A_\infty$ and define*

$$r = r_w = 1 + \frac{1}{\tau[w]_{A_\infty}} = 1 + \frac{1}{6(32\kappa^2(4\kappa^2 + \kappa)^2)^{D_\mu} [w]_{A_\infty}}$$

where τ depends on κ , the quasimetric constant of X . Then

$$\left(\int_B w^r d\mu \right)^{1/r} \leq \left(2(4\kappa)^{D_\mu} \int_{2\kappa B} w d\mu \right),$$

for any ball $B \in X$.

However, this lemma is not sufficient for our purposes. The difficulty lies in the fact that the Fujii–Wilson A_∞ constant is comparable when it is defined with respect to cubes or balls, but the constant of comparison depends on the doubling constant of the measure induced by w (which affects the sharp constants that we are trying to achieve). Please refer to [2] for more discussion on the reverse Hölder inequality with respect to balls. Since using balls provides a difficulty in converting between the constants, we needed a sharp reverse Hölder inequality with cubes. Here is the lemma that we use with respect to cubes.

LEMMA 4.6. *Let $w \in A_\infty$ and let*

$$0 < r \leq \frac{1}{\tau[w]_{A_\infty} - 1} = \frac{1}{2D[w]_{A_\infty} - 1},$$

with $D = 1/\varepsilon$, where ε is the absolute constant appearing in the dyadic decomposition of X . Then

$$\int_Q w^{1+r} d\mu \leq 2 \left(\int_Q w d\mu \right)^{1+r},$$

for any cube $Q \subset X$.

The proof will use the following sublemma.

LEMMA 4.7. *Let $w \in A_\infty$ and Q_0 a cube. Then for all subcubes $Q \subseteq Q_0$ and*

$$0 < r \leq \frac{1}{2D[w]_{A_\infty} - 1}$$

we have

$$\int_Q (M(w\chi_{Q_0}))^{1+r} d\mu \leq 2[w]_\infty \left(\int_Q w d\mu \right)^{1+r}$$

where M is the Hardy–Littlewood maximal operator with respect to a fixed dyadic grid D .

PROOF OF LEMMA 4.7. Assume without loss of generality that $w = w\chi_{Q_0}$. Let $\Omega_\lambda = Q_0 \cap \{Mw > \lambda\}$. Then using the layer cake formula

$$\int_{Q_0} |f(x)|^r |f(x)| d\mu = \int_0^\infty r |f(x)|^{r-1} \int_{\Omega_\lambda} |f(x)| d\mu d\lambda$$

with $f = Mw$ we get

$$\begin{aligned} \int_{Q_0} (Mw)^{1+r} d\mu &= \int_0^\infty r \lambda^{r-1} Mw(\Omega_\lambda) d\lambda \\ &= \int_0^{w_{Q_0}} r \lambda^{r-1} d\lambda \int_{Q_0} Mw d\mu + \int_{w_{Q_0}}^\infty r \lambda^{r-1} Mw(\Omega_\lambda) d\lambda. \end{aligned}$$

Note that if $\lambda \leq w_{Q_0}$ then $\Omega_\lambda = Q_0$. In the regime where $w_{Q_0} \leq \lambda$, select a dyadic cube Q_j if it is maximal with respect to the condition $\lambda < w_{Q_j}$. (Since $x \in \Omega_\lambda$, then there must exist some $Q \subset Q_0$ such that this condition holds.) Then $\Omega_\lambda = \bigcup_j Q_j$ where $\lambda < f_{Q_j}$, $w \leq \frac{1}{\varepsilon} \lambda$ and ε is the absolute constant from Theorem 2.2. Hence we have

$$\int_{Q_0} (Mw)^{1+r} d\mu \leq \left(\int_{Q_0} w \right)^r [w]_\infty w(Q_0) + \int_{w_{Q_0}}^\infty r \lambda^{r-1} \sum_j \int_{Q_j} Mw d\mu d\lambda.$$

Now we can localize

$$Mw(x) = M(w\chi_{Q_j})(x)$$

by the maximality of the Q'_j 's for any $x \in Q_j$. Then,

$$\begin{aligned} \int_{Q_j} Mw \, d\mu &= \int_{Q_j} M(w\chi_{Q_j}) \, d\mu \leq [w]_\infty w(Q_j) \leq [w]_\infty w(\hat{Q}_j) \\ &= [w]_\infty \left(\int_{\hat{Q}_j} w \right) \mu(\hat{Q}_j) \leq [w]_\infty \lambda \frac{1}{\varepsilon} \mu(Q_j), \end{aligned}$$

where \hat{Q}_j is the parent of the cube Q_j and we have used the definition of A_∞ and the maximality and containment properties of the cubes. Call $\frac{1}{\varepsilon} = D$. Hence

$$\sum_j \int_{Q_j} Mw \, d\mu \leq \sum_j [w]_\infty \lambda D \mu(Q_j) \leq [w]_\infty \lambda \, d\mu(\Omega_\lambda),$$

so

$$\int_{Q_0} (Mw)^{1+r} \, d\mu \leq \left(\int_{Q_0} w \right)^r [w]_\infty w(Q_0) + r[w]_\infty D \int_{w_{Q_0}}^\infty \lambda^r \mu(\Omega_\lambda) \, d\lambda.$$

Dividing by $w(Q_0)$, we obtain

$$\int_{Q_0} (Mw)^{1+r} \, d\mu \leq \left(\int_{Q_0} w \right)^{1+r} [w]_\infty + \frac{rD[w]_\infty}{1+r} \int_{Q_0} (Mw)^{1+r} \, d\mu,$$

so by subtracting the last term on the left hand side from both sides of the equation, so to get the desired constant of 2 we must have that

$$1 - \frac{rD[w]_\infty}{1+r} \geq \frac{1}{2},$$

which after some calculation results in choosing $0 < r \leq \frac{1}{2D[w]_\infty - 1}$ as stated. Note that we implicitly assumed that $\int_{Q_0} (Mw)^{1+r} \, d\mu < \infty$ (see, for example [11]). \square

PROOF OF LEMMA 4.6. Without loss of generality let $w = w\chi_{Q_0}$. Then

$$\int_{Q_0} w^{1+r} \, d\mu \leq \int_{Q_0} (Mw)^r w \, d\mu = \int_0^\infty r \lambda^{r-1} w(\Omega_\lambda) \, d\lambda,$$

where $\Omega_\lambda = Q_0 \cap \{Mw > \lambda\}$. Note that as in the previous lemma we can decompose $\Omega_\lambda = \bigcup_j Q_j$ where the Q_j are the Calderón–Zygmund cubes. Then splitting up the integral we get

$$\begin{aligned} & \int_0^{w_{Q_0}} r\lambda^{r-1} w(\Omega_\lambda) d\lambda + \int_{w_{Q_0}}^\infty r\lambda^{r-1} w(\Omega_\lambda) d\lambda \\ & \leq w_{Q_0}^r w(Q_0) + \int_{w_{Q_0}}^\infty r\lambda^{r-1} \sum_j w(Q_j) d\lambda. \end{aligned}$$

Now by the decomposition, we have that $w_{Q_j} \leq D\lambda\mu(Q_j)$, where $D = \frac{1}{\varepsilon}$ since the decomposition is with respect to dyadic cubes, so we get

$$\begin{aligned} & \int_{Q_0} (Mw)^r w d\mu \leq w_{Q_0}^r w(Q_0) + rD \int_{w_{Q_0}}^\infty \lambda^r \sum_j \mu(Q_j) d\lambda \\ & \leq w_{Q_0}^r w(Q_0) + rD \int_{w_{Q_0}}^\infty \lambda^r \mu(\Omega_\lambda) d\lambda \leq w_{Q_0}^r w(Q_0) + \frac{rD}{1+r} \int_{Q_0} (Mw)^{1+r} d\mu. \end{aligned}$$

Hence, dividing by $w(Q_0)$ and using Lemma 4.7, we arrive at

$$\int_{Q_0} w^{1+r} d\mu \leq w_{Q_0}^{1+r} + \frac{rD2[w]_\infty}{1+r} \left(\int_{Q_0} w \right)^{1+r} \leq \frac{rD2[w]_\infty + 1 + r}{1+r} \left(\int_{Q_0} w \right)^{1+r}.$$

Therefore, choosing r in the mentioned range, we can make the constant on the right-hand side less than or equal to 2. \square

The next lemma is a precise version of John–Nirenberg inequality in spaces of homogeneous type that will be very useful in the following results.

LEMMA 4.8 (John–Nirenberg inequality). *There are absolute constants $0 \leq \alpha_X < 1 < \beta_X$ such that*

$$(4.5) \quad \sup_Q \frac{1}{\mu(Q)} \int_Q \exp \left(\frac{\alpha_X}{\|b\|_{\text{BMO}}} |b(y) - b_Q| \right) d\mu(y) \leq \beta_X.$$

In fact, we can take $\alpha_X = \ln \sqrt[3]{2a}$, where $0 < a < 1$ is an absolute constant.

Note that we are defining BMO with respect to dyadic cubes; however, by taking a finite family of dyadic grids (an adjacent dyadic system, for example), it is known that BMO with respect to balls is equivalent to BMO defined over each dyadic grid [22]. The proof of Lemma 4.8 follows the scheme of proof showed in [26]; due to the analogous nature; we omit the details.

Now we will prove two lemmas related to the A_2 and A_∞ constants of a particular weight that we will need in the following, extended from those in [25].

LEMMA 4.9. *Let $w \in A_2$. Then $we^{2\operatorname{Re} z b} \in A_2$; moreover, there are absolute constants γ and c such that*

$$[we^{2\operatorname{Re} z b}]_{A_2} \leq c[w]_{A_2}$$

for all

$$|z| \leq \frac{\gamma}{\|b\|_{\operatorname{BMO}}([w]_{A_\infty} + [\sigma]_{A_\infty})},$$

where $\gamma = \alpha_X \max\{\frac{1}{C_1}, \frac{1}{C_2}\}$ with $C_1, C_2 > 0$ are absolute constants.

PROOF OF LEMMA 4.9. We will use the sharp reverse Hölder inequality twice, first for $r = 1 + \frac{1}{\tau[w]_{A_\infty}}$ and then for $r = 1 + \frac{1}{\tau[\sigma]_{A_\infty}}$. With the sharp reverse Hölder inequality for the first choice of r , Hölder's inequality and the sharp John–Nirenberg inequality (4.5), we have

$$\begin{aligned} \int_Q we^{2\operatorname{Re} z b} d\mu &\leq \left(\int_Q w^r d\mu \right)^{1/r} \left(\int_Q e^{r' 2\operatorname{Re} z (b - b_Q)} d\mu \right)^{1/r'} e^{2\operatorname{Re} z b_Q} \\ &\leq \left(2 \int_Q w \right) \cdot \beta_X \cdot e^{2\operatorname{Re} z b_Q}, \end{aligned}$$

for $|z| \leq \frac{\alpha_X}{2C_1\|b\|_{\operatorname{BMO}}[w]_{A_\infty}}$. Note that the constant α_X comes from (4.5) and C_1 is an absolute constant from the sharp reverse Hölder inequality since by our choice of r , $r' = C_1[w]_{A_\infty}$ (we can even calculate that $\tau[w]_{A_\infty} < r' \leq (\tau + 1)[w]_{A_\infty}$). We can also get a similar bound as above for the second choice of $r = 1 + \frac{1}{\tau[\sigma]_{A_\infty}}$, giving us

$$\int_Q w^{-1} e^{-2\operatorname{Re} z b} d\mu \leq \left(2 \int_Q w^{-1} d\mu \right) \cdot \beta_X \cdot e^{-2\operatorname{Re} z b_Q}$$

for $|z| \leq \frac{\alpha_X}{2C_2\|b\|_{\operatorname{BMO}}[\sigma]_{A_\infty}}$. Multiplying these two estimates and taking supremum, we finish the proof by showing that for all z as in the assumption

$$\left(\int_Q we^{2\operatorname{Re} z b} d\mu \right) \left(\int_Q w^{-1} e^{-2\operatorname{Re} z b} d\mu \right) \leq 4\beta_X^2 [w]_{A_2}. \quad \square$$

We also have a similar lemma for the A_∞ weight constant.

LEMMA 4.10. *There are absolute constants γ and c such that*

$$[we^{2\operatorname{Re} z b}]_{A_\infty} \leq c[w]_{A_\infty} \quad \text{for all } |z| \leq \frac{\gamma}{\|b\|_{\operatorname{BMO}}([w]_{A_\infty})},$$

where we can take $\gamma' = \frac{\alpha_X}{4D}$ with D being the absolute constant from Lemma 4.6.

PROOF OF LEMMA 4.10. The proof follows in a similar way as in [8], substituting the appropriate constants from the sharp John–Nirenberg inequality in Lemma 4.5 and the sharp reverse Hölder inequality in Lemma 4.6. \square

Next we will prove Theorem 3.5 where a mixed A_2 – A_∞ bound for Calderón–Zygmund operators in spaces of homogeneous type is obtained. Due to Lerner’s decomposition in spaces of homogeneous type from [3], we can fairly easily prove the mixed result. The proof essentially follows from [21]. Only a brief sketch is given below.

PROOF OF THEOREM 3.5. As stated in [21, Section 2D], Theorem 3.5 follows from verifying the following testing conditions:

- (1) $\|S_Q(\sigma \cdot \chi_Q)\|_{L^2(w)} \leq C_1 \|\chi_Q\|_{L^2(\sigma)},$
- (2) $\|S_Q(w \cdot \chi_Q)\|_{L^2(\sigma)} \leq C_2 \|\chi_Q\|_{L^2(w)}$

where

$$S_Q f = \sum_{L \in S, L \subseteq Q} \left(\int_L f \right) \chi_L,$$

and S is a sparse family (this is a sparse operator). It suffices to still just check these in SHT, see [12], [41].

To verify the testing conditions, one simply follows the argument outlined in [21, Section 5A]. \square

Finally, we can prove the results concerning to the commutator and its iterates.

PROOF OF COROLLARY 3.7. The proof follows as in [8], so we only mention the new ingredients: the A_2 theorem for spaces of homogeneous type [4], and the bound

$$[we^{2\operatorname{Re} z b}]_{A_\infty} \leq c[w]_{A_\infty}$$

from Lemma 4.10 for all $|z| \leq \frac{\delta}{\|b\|_{\operatorname{BMO}}([w]_{A_\infty} + [\sigma]_{A_\infty})}$ where the δ is the minimum of the absolute constants from the corresponding lemmas. \square

REMARK 4.11. Corollary 3.7 can be proved under the weaker assumption that T is a linear operator that satisfies the sharp weak mixed A_2 – A_∞ in spaces of homogeneous type.

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