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K-theoretic balancing conditions and the Grothendieck group of a toric variety



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ABSTRACT

We introduce a ring of $\mathbb{Z}$ -valued functions on a complete fan Δ called Grothendieck weights to describe the ordinary operational K -theory of the associated toric variety X . These functions satisfy a K -theoretic analogue of the balancing condition for Minkowski weights, which is induced by a presentation of the Grothendieck group of X . We explicitly give a combinatorial presentation in low dimensions, and relate Grothendieck weights to other fan-based invariants such as piecewise exponential functions and Minkowski weights. As an application, we give an example of a projective toric surface X such that the forgetful map $K_T^\circ(X) \rightarrow K^\circ(X)$ is not surjective.

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1. Introduction

It is a natural problem to describe geometric and topological invariants of a toric variety X using polyhedral combinatorics of the associated fan Δ . The focus of this document is the operational K -theory ring $\text{op}K^\circ(X)$, introduced by Anderson and Payne in [2]. Like other operational theories, $\text{op}K^\circ(X)$ can be obtained from a bivariant theory

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in the sense of [10], and consequently has properties that make it tractable to describe on singular varieties. Other operational theories such as the ordinary and torus-equivariant Chow rings $A^*(X)$ and $A_T^*(X)$, the torus-equivariant K -theory $\mathrm{op}K_T^\circ(X)$, and torus-equivariant cobordism have already been described on singular toric varieties [2,13,16], but so far, there is no such description for $\mathrm{op}K^\circ(X)$. Our aim is to address this when X is complete.

Our approach to describing $\mathrm{op}K^\circ(X)$ is inspired by work of Fulton and Sturmfels on $A^*(X)$ for complete toric varieties. In [12], they define *Minkowski weights* $\mathrm{MW}^*(\Delta)$ as a group of $\mathbb{Z}$ -valued functions on Δ (viewed as a set of cones) whose values on different cones must satisfy some identities known as a balancing condition. Then, they show that (1) there is a natural isomorphism of groups between $\mathrm{MW}^*(\Delta)$ and $A^*(X)$, and (2) there is a description of the product on $A^*(X)$ in terms of Minkowski weights.

In contrast, we define a group $\mathrm{GW}(\Delta)$ of $\mathbb{Z}$ -valued functions on Δ which is tautologically isomorphic to $\mathrm{op}K^\circ(X)$. We call the elements of $\mathrm{GW}(\Delta)$ *Grothendieck weights* because they can be naturally identified with linear forms on the Grothendieck group of coherent sheaves $K_\circ(X)$. Some of main results in this paper describe the balancing condition that elements of $\mathrm{GW}(\Delta)$ satisfy. The following theorem, which follows from results in Section 3, characterizes Grothendieck weights in dimensions ≤ 3 .

First, some notation: we denote the cones in Δ of dimension k by $\Delta(k)$. Since Δ is a poset of polyhedral cones, we denote containment with $\prec$, and for cones $\alpha \prec \beta$ such that $\dim(\alpha) + 1 = \dim(\beta)$, we let $v_{\beta,\alpha}$ denote the primitive lattice point of $\overline{\beta}$ in the quotient space by the span of α , and write v_ρ for $v_{\rho,\{0\}}$ (see Subsection 1.3).

Theorem 1.1. *Let X be a complete toric variety of dimension $n \leq 3$. A function $g : \Delta \rightarrow \mathbb{Z}$ on the fan of X is a Grothendieck weight if and only if it is*

1. *constant on maximal cones, and*
2. *for σ any maximal cone and $\rho \in \Delta(n-2)$,*

$$\sum_{\substack{\beta \in \Delta(n-1), \\ \beta \succ \rho}} (g(\beta) - g(\sigma))v_{\beta,\rho} = 0,$$

when $\dim(X) \geq 2$, and

3. *for σ any maximal cone,*

$$\sum_{\rho \in \Delta(1)} \left(g(\rho) - \sum_{\substack{\alpha \in \Delta(2), \\ \alpha \succ \rho}} \frac{g(\alpha)}{2} \right) v_\rho = g(\sigma) \left(\sum_{\rho \in \Delta(1)} \left(1 - \sum_{\substack{\alpha \in \Delta(2), \\ \alpha \succ \rho}} \frac{1}{2} \right) v_\rho \right),$$

when $\dim(X) = 3$.

Finding a simple combinatorial balancing condition for Grothendieck weights in higher dimensions is still an open problem, but we provide partial results and a possible way to approach the problem in Section 4. See for example Theorem 4.14, which gives a method to characterize Grothendieck weights on complete simplicial toric varieties by calculating Todd classes.

Another notable result is the following theorem derived from Example 6.8. It appears in the text as Corollary 6.9.

Theorem 1.2. *There is a projective toric surface X such that the forgetful maps $\mathrm{op}K_T^\circ(X) \rightarrow \mathrm{op}K^\circ(X)$ and $K_T^\circ(X) \rightarrow K^\circ(X)$ are not surjective.*

In [1, Theorem 1.7], the authors demonstrate that there is a nonsimplicial toric 3-fold X such that the image of $\mathrm{op}K_T^\circ(X)_\mathbb{Q}$ is a proper linear subspace of $\mathrm{op}K^\circ(X)_\mathbb{Q}$, and from there establish the nonsurjectivity of $\mathrm{op}K_T^\circ(X) \rightarrow \mathrm{op}K^\circ(X)$ and $K_T^\circ(X) \rightarrow K^\circ(X)$. However, the authors deduce this as a corollary of the operational Riemann-Roch theorem that they establish. This approach cannot work to show non-surjectivity if the image of $\mathrm{op}K_T^\circ(X)$ has finite index inside $\mathrm{op}K^\circ(X)$, like when X is simplicial (e.g. our Example 6.8).

Another reason to study $\mathrm{op}K^\circ(X)$ is related to tropical geometry. Minkowski weights, which inspired Grothendieck weights, are well-known in the context of tropical geometry, where they appear as a special case of weighted balanced polyhedral complexes. If one desires a K -theoretic analogue to the methods of tropical geometry, it is our belief that a first step would be to determine the K -theoretic analogue of Minkowski weights and the corresponding balancing condition (in the sense of our Theorem 1.1). In another direction, presentations of $\mathrm{op}K^\circ(X)$ have a direct interpretation in terms of Ehrhart theory, see Proposition 2.12.

Now, we address in more detail the other results in this document. After proving the characterization of Grothendieck weights in low dimensions in Theorem 1.1, we move to higher dimensions in Section 4. There, the problem becomes more difficult, because our approach to characterizing Grothendieck weights relies on finding expressions for Todd classes of simplicial toric varieties in terms of T -invariant subvarieties. Though there is an extensive literature on finding such expressions [3,4,15,17,18], the coefficients of such an expression depend on various choices made in rewriting self-intersections of toric divisors D_i in terms of square-free monomials.

We continue our study of Grothendieck weights by addressing how to compute the product on $\mathrm{GW}(\Delta)$ induced by $\mathrm{op}K^\circ(X)$. By [2, Proposition 6.4], any element of $K_\circ(X \times X)$ has an expression of the form $\sum_{\alpha, \beta \in \Delta} m_{\alpha, \beta} [\mathcal{O}_{V(\alpha) \times V(\beta)}]$, where $V(\alpha)$ and $V(\beta)$ are as defined in Subsection 1.3. Then, Theorem 5.3 reduces the problem of calculating products of Grothendieck weights to calculating coefficients $m_{\alpha, \beta}$ for classes of the form $\delta_*[\mathcal{O}_{V(\gamma)}]$, δ being the diagonal. In the context of Chow groups and Minkowski weights, an elegant method to calculate such coefficients was provided in [12, Theorem 4.2], via

the displacement rule. In K -theory we explain one approach to doing this in subsection 5.6.

After our results on the internal structure of $\mathrm{GW}(\Delta)$, we describe maps between Grothendieck weights and other fan-based invariants in Section 6. There is a map from Minkowski weights to $\mathrm{GW}(\Delta)$ that corresponds to the operational Riemann-Roch transformation of [1]. We relate this map to appropriate expressions for Todd classes. There is also a forgetful map that $\mathrm{op}K^\circ(X)$ receives from $\mathrm{op}K_T^\circ(X)$. Since $\mathrm{op}K_T^\circ(X)$ is isomorphic to the ring of piecewise exponential functions on Δ , there is a map from piecewise exponential functions on Δ to $\mathrm{GW}(\Delta)$, which we give a formula for in Theorem 6.7. Our approach here follows the work of Katz and Payne in [14].

When X is smooth, $\mathrm{op}K^\circ(X)$ agrees with several other Grothendieck rings such as the ring of vector bundles $K^\circ(X)$ (see [2, Section 4]). Then, other descriptions of $\mathrm{op}K^\circ(X)$ are applicable. For instance, Vezzosi and Vistoli showed in [20] that the ring of T -equivariant vector bundles $K_T^\circ(X)$ and also higher K -groups are isomorphic to certain Stanley-Reisner rings. The non-equivariant ring $K^\circ(X)$ is then isomorphic to a quotient of a Stanley-Reisner ring. To calculate the isomorphism from this quotient description of $K^\circ(X)$ to $\mathrm{GW}(\Delta)$, one can choose a representative in $K_T^\circ(X)$ of a class in $K^\circ(X)$, and compute its localization to obtain a piecewise exponential function. Then, the map to $\mathrm{GW}(\Delta)$ can be computed via Theorem 6.7.

1.3. Notation and conventions

We establish notation and conventions. We work in the category of separated schemes of finite type over a fixed base field $\mathbb{k}$, unless stated otherwise. A variety is reduced and irreducible. For toric varieties, as much as possible we follow the conventions of [6] and [8]. Let T be a split algebraic torus over $\mathbb{k}$. A toric variety refers to a normal $\mathbb{k}$ -variety with a T -action and a dense orbit. By modding out by a generic stabilizer, we can assume that T embeds as the dense orbit.

Let $M = \mathrm{Hom}_{\mathrm{alg. gp.}}(T, \mathbb{k}^*)$ be the character lattice of T , and $N = \mathrm{Hom}_{\mathrm{alg. gp.}}(\mathbb{k}^*, T)$ the cocharacter lattice. There is the natural perfect pairing $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathrm{Hom}_{\mathrm{alg. gp.}}(\mathbb{k}^*, \mathbb{k}^*) \cong \mathbb{Z}$ given by composition of maps. We let $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ denote $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$ respectively.

If X is a toric variety, there is a corresponding polyhedral fan Δ , which is a finite set of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ (see e.g. [8, Chapter 1]) satisfying the two additional conditions that (1) if β is a face of α and $\alpha \in \Delta$, then $\beta \in \Delta$, and (2) if α and β are in Δ , then the intersection $\alpha \cap \beta$ is a face of both α and β . If α is a face of β , we write $\alpha \prec \beta$. Let $\Delta(k)$ denote the set of k -dimensional cones in Δ . We use ρ exclusively for 1-dimensional cones (rays) and σ exclusively for n -dimensional cones.

For a cone α , let $\alpha^\vee$ be the cone of linear forms in $M_{\mathbb{R}}$ which are non-negative on α . If $\alpha \in \Delta$, then there is a T -invariant affine open set U_α in X which is naturally isomorphic to $\mathrm{Spec} \mathbb{k}[\alpha^\vee \cap M]$. Inside U_α , there is a unique closed orbit O_α . Let $V(\alpha)$ denote the

closure of O_α in X . There is an order-reversing bijection between the orbit closures $V(\alpha)$ in X and the cones α in Δ .

The subvariety $V(\alpha)$ is also a toric variety: Let T_{O_α} be the stabilizer of O_α . Then, the quotient torus $T_\alpha = T/T_{O_\alpha}$ includes into $V(\alpha)$ as the dense orbit. Let $\alpha^\perp$ be the subspace of $M_\mathbb{R}$ that vanishes on α . The character lattice for T_α is $M_\alpha = M \cap \alpha^\perp$. Let N^α be the $\mathbb{Z}$ -span of lattice points in α , and let $N_\alpha = N/N^\alpha$ be the quotient. Then the perfect pairing $\langle \cdot, \cdot \rangle$ between M and N descends to one between M_α and N_α , which we also refer to by $\langle \cdot, \cdot \rangle$. When $\alpha \prec \beta$, the image of β in $(N_\alpha)_\mathbb{R}$ is a cone which we denote by $\overline{\beta}$. If $\overline{\beta}$ is a ray, we denote the generator of the semigroup $N_\alpha \cap \overline{\beta}$ by $v_{\beta,\alpha}$, or v_β if $\alpha = \{0\}$.

2. K-theory and $\text{GW}(\Delta)$

We start this section by introducing the basic objects $K_\circ(X)$, $K^\circ(X)$, $\text{op}K^\circ(X)$, and $\text{GW}(\Delta)$, and later relate $\text{GW}(\Delta)$ to Ehrhart theory in Proposition 2.12.

2.1. The group $K_\circ(X)$ and the ring $K^\circ(X)$

Let X be a $\mathbb{k}$ -scheme. The group $K_\circ(X)$ is generated by isomorphism classes of coherent sheaves on X , modulo relations $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$ for exact sequences $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$. When X is a smooth algebraic variety, there is a product given by $[\mathcal{E}][\mathcal{F}] = \sum_i (-1)^i [\mathcal{T}or_{\mathcal{O}_X}^i(\mathcal{E}, \mathcal{F})]$, but when X is singular the sum is no longer necessarily finite. For a proper morphism $f : Y \rightarrow X$ and $\mathcal{G}$ a coherent sheaf on Y , there is a pushforward $f_*[\mathcal{G}] = \sum_i (-1)^i [R^i f_* \mathcal{G}]$. A special instance of this when X is a point, in which case $f_*[\mathcal{G}]$ is the Euler characteristic $\chi(X, \mathcal{G}) = \sum_i (-1)^i \dim_{\mathbb{k}} H^i(X, \mathcal{G})$. For a flat morphism $f : Y \rightarrow X$ and $\mathcal{F}$ coherent on X , there is a pullback $f^*[\mathcal{F}] = [f^* \mathcal{F}]$.

We denote by $K^\circ(X)$ the Grothendieck group of vector bundles. Defined analogously to $K_\circ(X)$, $K^\circ(X)$ is a ring with respect to tensor product of vector bundles. For arbitrary morphisms $f : Y \rightarrow X$, there is a pull-back $f^*[E] = [f^*E]$.

If X is smooth, then $K^\circ(X)$ and $K_\circ(X)$ are isomorphic as groups (see, e.g. [21, Chapter II, Theorem 8.2]), but if X is singular this may not be true. For example, it is an easy exercise that if X is the rational nodal curve over $\mathbb{C}$, its Picard group is uncountable. The same is then true for $K^\circ(X)$, which has a surjection onto $\text{Pic}(X)$ via the map given by determinant bundles, e.g. [21, Chapter I]. On the other hand, one can use the localization sequence for $K_\circ$ to show that $K_\circ(X)$ is finitely generated: Let $p \in X$ be the singular point. We have the right-exact sequence

$$K_\circ(p) \rightarrow K_\circ(X) \rightarrow K_\circ(X \setminus \{p\}) \rightarrow 0.$$

Since $X \setminus \{p\} \cong \mathbb{C}^*$, $K_\circ(X)$ is an extension of finitely generated abelian groups, and thus finitely generated itself.

2.2. The operational K -theory ring $\mathrm{op}K^\circ(X)$, and $\mathrm{GW}(\Delta)$

An element c in $\mathrm{op}K^\circ(X)$ is a collection (c_f) of endomorphisms of $K_\circ(Y)$ for each $f : Y \rightarrow X$. The collection (c_f) must be compatible, in the sense that the maps must commute with proper pushforwards, flat pullbacks, and Gysin homomorphisms. Addition and multiplication are defined coordinate-wise, meaning

$$\begin{aligned}(c_f) + (d_f) &= (c_f + d_f), \text{ and} \\ (c_f) \cdot (d_f) &= (c_f \circ d_f).\end{aligned}$$

For further details, we refer the reader to [2, Section 4]. The product is commutative if X is toric, or more generally if it admits a resolution of singularities (via the Kimura sequence [2, Proposition 5.4]). Since we will assume X is complete, we have access to the following special case of [2, Proposition 5.4]:

Theorem 2.3. *Let X be a complete toric variety. The natural map from $\mathrm{op}K^\circ(X)$ to $K_\circ(X)^\vee$ sending (c_f) to $\chi(c_{\mathrm{Id}}(-))$ is an isomorphism of abelian groups.*

We make the following definition:

Definition 2.4. Let Δ be a complete fan. Define the group of *Grothendieck weights* on Δ , denoted by $\mathrm{GW}(\Delta)$, to be the image of $K_\circ(X)^\vee$ in $(\mathbb{Z}^\Delta)^\vee$ under the map

$$f \mapsto (e_\alpha \mapsto f([\mathcal{O}_{V(\alpha)}])).$$

For a proper toric morphism $\phi : X(\Delta) \rightarrow X(\Sigma)$, there is a corresponding pullback $\phi^* : \mathrm{GW}(\Sigma) \rightarrow \mathrm{GW}(\Delta)$.

In other words, elements of $\mathrm{GW}(\Delta)$ are obtained from elements of $K_\circ(X)^\vee$ by recording the value of a linear form on the classes $[\mathcal{O}_{V(\alpha)}]$. Since these classes generate $K_\circ(X)$, the next proposition follows from the previous theorem.

Proposition 2.5. *Let X be a complete toric variety and Δ the associated fan. Then, the natural maps in the sequence*

$$\mathrm{op}K^\circ(X) \rightarrow K_\circ(X)^\vee \rightarrow \mathrm{GW}(\Delta),$$

are isomorphisms of abelian groups.

Now, we would like to focus on the following question:

Question 2.6. How can we characterize which functions on Δ are Grothendieck weights? Is there a balancing condition as in the case of Minkowski weights?

To address this, we first make the following definition:

Definition 2.7. Let $\text{Rel}_{K_\circ(X)}$ be the kernel of the map $\mathbb{Z}^\Delta \rightarrow K_\circ(X)$ that sends e_α to $[\mathcal{O}_{V(\alpha)}]$, and $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$ the same kernel over $\mathbb{Q}$.

By dualizing the exact sequence

$$0 \rightarrow \text{Rel}_{K_\circ(X)} \rightarrow \mathbb{Z}^\Delta \rightarrow K_\circ(X) \rightarrow 0,$$

we obtain

$$0 \rightarrow K_\circ(X)^\vee \rightarrow (\mathbb{Z}^\Delta)^\vee \rightarrow \text{Hom}_\mathbb{Z}(\text{Rel}_{K_\circ(X)}, \mathbb{Z}) \rightarrow 0. \quad (*)$$

So, Grothendieck weights are the $\mathbb{Z}$ -valued functions on Δ that satisfy identities coming from $\text{Rel}_{K_\circ(X)}$.

Remark 2.8. Our focus in Sections 3 and 4 will be to produce a list of conditions that one can check to verify whether a function $g : \Delta \rightarrow \mathbb{Z}$ is a Grothendieck weight. By the above exact sequence (*), one can produce such a list by picking a generating set for $\text{Rel}_{K_\circ(X)}$. Each element will have the form $\sum_\alpha a_\alpha e_\alpha$. Each of the corresponding sums $\sum_\alpha a_\alpha g(\alpha)$ vanish if and only if g is a Grothendieck weight.

Conveniently, $\text{GW}(\Delta)$ naturally includes as a subring into rational Grothendieck weights, denoted by $\text{GW}(\Delta)_\mathbb{Q}$, since it is torsion-free. Rational Grothendieck weights are characterized by vanishing on $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$, so we can characterize $g \in \text{GW}(\Delta)$ as a function $g : \Delta \rightarrow \mathbb{Z}$ whose rational extension vanishes on $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$. Generating sets for $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$ are easier to calculate due to the Riemann-Roch theorem.

Before we finish this section, we note two basic facts about Grothendieck weights. First, we thank an anonymous reviewer for pointing out the following:

Proposition 2.9. *The ring of Grothendieck weights on Δ is independent of the field of the associated toric variety.*

Proof. When Δ is a smooth fan this follows from any of the myriad descriptions of $\text{op}K^\circ(X) \cong K^\circ(X)$ which demonstrate its field independence (e.g. $K^\circ(X)$ is a non-equivariant quotient of either piecewise exponential functions or a Stanley-Reisner ring over the integers).

For Δ a general fan, we use induction on dimension. For Δ a complete fan of dimension ≤ 1 , Δ is smooth. If Δ is higher dimensional, let Δ' be a resolution, and let $\pi : X' \rightarrow X$ be the associated birational toric morphism. Let $p_i : E_i \hookrightarrow X'$ be the components of the exceptional locus, and $\pi_i : E_i \rightarrow S_i$ the corresponding maps to their images. The E_i and S_i are toric varieties whose fans depend only on Δ' and Δ .

The Kimura sequence [2, Proposition 5.4] states that the sequence

$$0 \rightarrow \operatorname{op}K^\circ(X) \rightarrow \operatorname{op}K^\circ(X') \bigoplus \bigoplus_i \operatorname{op}K^\circ(S_i) \rightarrow \bigoplus_i \operatorname{op}K^\circ(E_i),$$

is exact. This implies that $\operatorname{GW}(\Delta)$ includes into $\operatorname{GW}(\Delta')$ as a subring, and $g \in \operatorname{GW}(\Delta)$ is equivalent to a Grothendieck weight g' on Δ' and weights g_i on the fans of the S_i , such that $\pi_i^* g_i = p_i^*(g')$. Since X' is smooth and E_i and S_i have lower dimension, Grothendieck weights on these fans are independent of the field, so those on Δ are as well. $\square$

Here is a fact which we will not use later, but which allows one to think about Grothendieck weights on general fans in terms of Grothendieck weights on smooth fans.

Proposition 2.10. *Let Δ be a complete fan and Δ' a refinement. Then a function $g : \Delta \rightarrow \mathbb{Z}$ is a Grothendieck weight if and only if the function g' on Δ' , determined by*

$$g'(\alpha') = g(\alpha)$$

for α the smallest cone on Δ containing α' , is a Grothendieck weight on Δ' .

Proof. Let X' and X be the toric varieties corresponding to Δ' and Δ , and $\pi : X' \rightarrow X$ the toric birational morphism that corresponds to the refinement. If α is the smallest cone in Δ that contains α' , then $\pi(V(\alpha')) = V(\alpha)$, and standard results on vanishing of higher cohomology imply that $\pi_*([\mathcal{O}_{V(\alpha')}] = [\mathcal{O}_{V(\alpha)}]$.

Grothendieck weights correspond to elements of $K_\circ(X)^\vee$ via the map $g \mapsto \phi_g$ determined by $\phi_g([\mathcal{O}_{V(\alpha)}]) = g(\alpha)$. So, if g is a Grothendieck weight on Δ , g' will be the Grothendieck weight on Δ' that corresponds to the linear form $\phi_g \circ \pi_* : K_\circ(X') \rightarrow \mathbb{Z}$.

Let g be a function such that g' is a Grothendieck weight. As in the previous proposition, we induct on dimension. Once again, let $p_i : E_i \hookrightarrow X'$ be the components of the exceptional locus, and let $\pi_i : E_i \rightarrow S_i$ be the maps to their images in X . By the induction hypothesis, the weight $p_i^* g'$ on the fan of E_i descends to a Grothendieck weight g_i on the fan of S_i . Since we have a Grothendieck weight g' on Δ' , and weights g_i on the fans of the S_i such that $\pi_i^* g_i = p_i^*(g')$, the Kimura sequence [2, Proposition 5.4] implies that g' is the image of a Grothendieck weight under π^* . The only possible preimage of g' is g , so g is a Grothendieck weight. $\square$

2.11. Characterizing $\operatorname{Rel}_{K_\circ(X)_\mathbb{Q}}$ via Ehrhart polynomials

The goal of this subsection is to characterize $\operatorname{Rel}_{K_\circ(X)_\mathbb{Q}}$ in terms of polytopes when X is a projective simplicial toric variety. As before, let Δ be the fan of X . Let P be a lattice polytope in $M_\mathbb{R}$ whose inward normal fan is Δ . For each $\alpha \in \Delta(k)$, there is a corresponding face F_α of P , which has codimension k and is also a lattice polytope.

The Ehrhart polynomial of P , denoted by $\operatorname{Ehr}_P(t)$, is the polynomial determined by $\operatorname{Ehr}_P(t) = |tP \cap M|$ for t a non-negative integer. Let

$$\phi_P : \mathbb{Q}^\Delta \rightarrow \mathbb{Q}[t]$$

send the tuple (a_α) indexed by Δ to $\sum_{\alpha \in \Delta} a_\alpha \text{Ehr}_{F_\alpha}(t)$. The following proposition shows that $\text{Rel}_{K_o(X)_\mathbb{Q}}$ is equal to a submodule of $\mathbb{Q}^\Delta$ defined independently of K -theory.

Proposition 2.12. *Let X be a projective simplicial toric variety, and Δ the corresponding fan. Then*

$$\text{Rel}_{K_o(X)_\mathbb{Q}} = \bigcap_{\substack{P \text{ with} \\ \text{normal fan } \Delta}} \ker(\phi_P).$$

Proof. For the “ $\subset$ ” direction we use some results on vanishing of higher cohomology for nef line bundles on toric varieties (e.g. [6, Theorem 9.2.3]). Suppose $\sum_{\alpha \in \Delta} a_\alpha [\mathcal{O}_{V(\alpha)}] = 0$ in $K_o(X)_\mathbb{Q}$. For P any lattice polytope with normal fan Δ , let D_P denote the associated ample divisor. For each $t_0 \geq 0$, we also have the nef divisor $D_{t_0 P}$. Then, recalling that χ is the Euler characteristic we have

$$0 = \chi([\mathcal{O}(D_{t_0 P})]) \cdot \left(\sum_{\alpha \in \Delta} a_\alpha [\mathcal{O}_{V(\alpha)}] \right) = \sum_{\alpha \in \Delta} a_\alpha \chi([\mathcal{O}(D_{t_0 P})|_{V(\alpha)}]). \quad (\dagger)$$

We would like to show that $\chi([\mathcal{O}(D_{t_0 P})|_{V(\alpha)}]) = \text{Ehr}_{F_\alpha}(t_0)$ so that we can further rewrite the sum on the right. By definition, $\text{Ehr}_{F_\alpha}(t_0) = |t_0 F_\alpha \cap M|$, so we need to verify that $\chi([\mathcal{O}(D_{t_0 P})|_{V(\alpha)}]) = |t_0 F_\alpha \cap M|$.

Since $\mathcal{O}(D_{t_0 P})$ is nef, its restriction to $V(\alpha)$ is as well. By the vanishing of higher sheaf cohomology groups, $\chi([\mathcal{O}(D_{t_0 P})|_{V(\alpha)}]) = \dim_{\mathbb{K}} H^i(V(\alpha), \mathcal{O}(D_{t_0 P})|_{V(\alpha)})$.

To calculate $\dim_{\mathbb{K}} H^i(V(\alpha), \mathcal{O}(D_{t_0 P})|_{V(\alpha)})$, we first translate $t_0 P$ so that $t_0 F_\alpha$ is contained inside $M_\alpha = \alpha^\perp$. This corresponds to replacing $D_{t_0 P}$ by a linearly equivalent divisor that doesn't vanish or have poles containing $V(\alpha)$, so the dimension of H^0 will not change. Now, the invertible sheaf $\mathcal{O}(D_{t_0 P})$ on X is generated on the affine open U_β by the rational function corresponding to any $m \in t_0 F_\beta$, and its restriction to $V(\alpha)$ is generated on the affine opens $U_\beta \cap V(\alpha)$ by the rational function corresponding to any $m \in t_0(F_\beta \cap F_\alpha)$. The global sections $H^0(V(\alpha), \mathcal{O}(D_{t_0 P})|_{V(\alpha)})$ will then have a basis given by the rational functions corresponding to $m \in t_0 F_\alpha$, so $|t_0 F_\alpha \cap M| = \dim_{\mathbb{K}} H^i(V(\alpha), \mathcal{O}(D_{t_0 P})|_{V(\alpha)})$.

So $\chi([\mathcal{O}(D_{t_0 P})|_{V(\alpha)}]) = |t_0 F_\alpha \cap M| = \text{Ehr}_{F_\alpha}(t_0)$. Substituting this into $(\dagger)$ we obtain that

$$0 = \sum_{\alpha \in \Delta} a_\alpha \text{Ehr}_{F_\alpha}(t_0),$$

for $t_0 \geq 0$. Since the polynomial $\sum_{\alpha \in \Delta} a_\alpha \text{Ehr}_{F_\alpha}(t)$ has infinitely many zeros, it follows that $0 = \sum_{\alpha \in \Delta} a_\alpha \text{Ehr}_{F_\alpha}(t)$. This establishes the “ $\subset$ ” direction.

Now, we show the other direction. Suppose that a tuple (a_α) satisfies the identity $\sum_{\alpha \in \Delta} a_\alpha \text{Ehr}_{F_\alpha}(t) = 0$, for any P with normal fan Δ . This implies that

$$0 = \sum_{\alpha \in \Delta} a_{\alpha} \chi(\mathcal{O}(D_P)|_{V(\alpha)}) = \chi \left(\left(\sum_{\alpha \in \Delta} a_{\alpha} [\mathcal{O}_{V(\alpha)}] \right) \cdot [\mathcal{O}(D_P)] \right).$$

Then, the result follows directly from the next lemma if we replace the term “ x ” with “ $\sum_{\alpha \in \Delta} a_{\alpha} [\mathcal{O}_{V(\alpha)}]$ ”. $\square$

Lemma 2.13. *Suppose X is a projective simplicial toric variety and Δ the corresponding fan. If for $x \in K_{\circ}(X)$, we have $\chi(x \cdot [\mathcal{O}(D_P)]) = 0$ for all P with normal fan Δ , then $x = 0$ in $K_{\circ}(X)_{\mathbb{Q}}$.*

Proof. Let $\tau_X : K_{\circ}(X) \rightarrow A_*(X)_{\mathbb{Q}}$ be the Riemann-Roch transformation, which is an isomorphism over $\mathbb{Q}$. Also, let “deg” be the projection map from $A_*(X)$ to $A_0(X)$. By the Riemann-Roch theorem [9, Theorem 18.3],

$$\chi(x \cdot [\mathcal{O}(D_P)]) = \deg(ch(\mathcal{O}(D_P)) \cap \tau_X(x)).$$

So, if we assume the hypothesis of the lemma, we have that for each P with normal fan Δ ,

$$0 = \deg(ch(\mathcal{O}(D_P)) \cap \tau_X(x)). \quad (\dagger)$$

We would like to show that this implies that $\tau_X(x) = 0$.

On a complete toric variety $A^*(X) \cong A_*(X)^{\vee}$ via the map $c \rightarrow \deg(c \cap -)$, by [11, Theorem 3]. By [12, Theorem 4.1], $A^*(X)_{\mathbb{Q}}$ is generated as a vector space by 1 and classes of the form $ch(\mathcal{O}(D_P))$, so we obtain that $\tau_X(x) = 0$. $\square$

We also have the following corollary which follows from the same vanishing theorem as used in the “ $\subset$ ” direction of Proposition 2.12, replacing the ample line bundle $\mathcal{O}(D_P)$ on a projective toric variety with the trivial line bundle on an arbitrary complete toric variety.

Corollary 2.14. *Let X be a complete toric variety, Δ the associated fan, and $\sum_{\alpha} a_{\alpha} e_{\alpha}$ an element of $\text{Rel}_{K_{\circ}(X)}$. Then $0 = \chi(\sum_{\alpha} a_{\alpha} [\mathcal{O}_{V(\alpha)}]) = \sum_{\alpha} a_{\alpha}$.*

3. Grothendieck weights in low codimensions

The goal of this section is to prove some identities (with rational coefficients) that a $\mathbb{Z}$ -valued function on a complete fan will satisfy, if it is a Grothendieck weight. These properties will be enough to characterize Grothendieck weights on fans up to dimension 3. Our method will be to show that certain elements are in $\text{Rel}_{K_{\circ}(X)_{\mathbb{Q}}}$ (see also Remark 2.8).

The end of this section is a proof of Theorem 1.1, which characterizes low-dimensional Grothendieck weights in a more elementary manner than Theorem 4.14 in the next section. We do not require that the fan is simplicial in this section.

3.1. Preparatory results

Let us fix that X is a complete toric variety of dimension n , and Δ is the corresponding complete fan. Fundamental to our approach to finding elements of $\text{Rel}_{K_o(X)_\mathbb{Q}}$ is the Riemann-Roch transformation τ_X . Recall that it is a map $\tau_X : K_o(X) \rightarrow A_*(X)_\mathbb{Q}$, which becomes an isomorphism after tensoring with $\mathbb{Q}$. For a vector bundle E , there is an equality $\tau_X([E]) = \text{ch}(E) \cdot td(X)$, where ch is the Chern character homomorphism $\text{ch} : K^\circ(X) \rightarrow A^*(X)_\mathbb{Q}$ and $td(X) = \tau_X(\mathcal{O}_X) \in K_o(X)_\mathbb{Q}$ is the Todd class. For details regarding the Todd class $td(X)$ for singular varieties and this version of the Riemann-Roch transformation, see [9, Chapter 18].

We also define $\text{Rel}_{A_*(X)_\mathbb{Q}}$:

Definition 3.2. Let $\text{Rel}_{A_*(X)_\mathbb{Q}}$ be the kernel of the map $\mathbb{Q}^\Delta \rightarrow A_*(X)_\mathbb{Q}$ that sends e_α to $[V(\alpha)]$.

Here is the basic strategy. If we choose a suitable lift $\tau_\dagger$ of τ_X to an endomorphism of $\mathbb{Q}^\Delta$, we obtain an isomorphism of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Rel}_{K_o(X)_\mathbb{Q}} & \longrightarrow & \mathbb{Q}^\Delta & \longrightarrow & K_o(X)_\mathbb{Q} \longrightarrow 0 \\ & & \downarrow & & \downarrow \tau_\dagger & & \downarrow \tau_X \\ 0 & \longrightarrow & \text{Rel}_{A_*(X)_\mathbb{Q}} & \longrightarrow & \mathbb{Q}^\Delta & \longrightarrow & A_*(X)_\mathbb{Q} \longrightarrow 0. \end{array}$$

If we can explicitly describe the lift $\tau_\dagger$ and a set of generators $\{r_i\}_{i \in I} \subset \text{Rel}_{A_*(X)_\mathbb{Q}}$, then $\{\tau_\dagger^{-1}(r_i)\}_{i \in I}$ will generate $\text{Rel}_{K_o(X)_\mathbb{Q}}$.

In the next proposition, we reproduce [12, Proposition 2.1(b)], which provides an explicit list of generators for $\text{Rel}_{A_*(X)_\mathbb{Q}}$ (and by extension, $\text{Rel}_{A_i(X)_\mathbb{Q}}$). Recall from subsection 1.3 that if β contains α as a face of codimension 1, then $v_{\beta,\alpha}$ is the generator of the semigroup $\bar{\beta} \cap N_\alpha$.

Proposition 3.3. The space $\text{Rel}_{A_*(X)_\mathbb{Q}}$ splits as a direct sum $\oplus_i \text{Rel}_{A_i(X)_\mathbb{Q}}$, where $\text{Rel}_{A_i(X)_\mathbb{Q}}$ is generated by terms $r_{\alpha,u}$ that have the form

$$r_{\alpha,u} = \sum_{\substack{\beta \in \Delta(n-i), \\ \beta \succ \alpha}} \langle u, v_{\beta,\alpha} \rangle e_\beta,$$

as α varies among cones of dimension $n - i - 1$ and u varies in M_α .

Then, for instance, $\text{Rel}_{A_n(X)_\mathbb{Q}}$ is trivial. To execute the strategy outlined above, we require the following lemmas.

Lemma 3.4. Let X be a complete toric variety, Δ the associated fan, and α an element of $\Delta(k)$. Let τ_X be the Riemann-Roch transformation. Then in $A_*(X)_\mathbb{Q}$, we have

$$\tau_X([\mathcal{O}_{V(\alpha)}]) = [V(\alpha)] + \left(\sum_{\substack{\beta \in \Delta(k+1), \\ \beta \succ \alpha}} \frac{1}{2} [V(\beta)] \right) + c,$$

where $c \in (A_0(X) \oplus \dots \oplus A_{n-k-2}(X))_{\mathbb{Q}}$.

Proof. If f is a proper morphism, we use f_* for the pushforward in both A_* and K_* . Let $i : V(\alpha) \hookrightarrow X$ be the inclusion and $\pi : Y \rightarrow V(\alpha)$ be a toric resolution. By [9, Theorem 18.3 (1)], $f_* \circ \tau_X = \tau_Y \circ f_*$ when $f : X \rightarrow Y$ is proper, so taking $f = i$ yields

$$\tau_X([\mathcal{O}_{V(\alpha)}]) = i_*(\tau_{V(\alpha)}([\mathcal{O}_{V(\alpha)}])).$$

Since $[\mathcal{O}_{V(\alpha)}] = \pi_*([\mathcal{O}_Y])$, we obtain that

$$i_*(\tau_{V(\alpha)}([\mathcal{O}_{V(\alpha)}])) = i_*\pi_*(\tau_Y([\mathcal{O}_Y])).$$

The toric variety Y is smooth and complete, so we can use the following well-known formula for the Todd class (e.g. [6, Theorem 13.1.6]). Let Σ be the fan of Y . Then

$$\tau_Y([\mathcal{O}_Y]) = \prod_{\rho \in \Sigma(1)} \left(\frac{[V(\rho)]}{1 - e^{-[V(\rho)]}} \right).$$

Expanding the product, we obtain

$$\tau_Y([\mathcal{O}_Y]) = 1 + \left(\sum_{\rho \in \Sigma(1)} \frac{1}{2} [V(\rho)] \right) + c,$$

where c represents classes of lower dimension.

If $\rho \in \Sigma(1)$ is in the fan of $V(\alpha)$, π maps $V(\rho)$ birationally onto its image. If not, $\pi(V(\rho))$ has lower dimension than $V(\rho)$. Thus, applying π_* we get

$$\pi_*\tau_Y([\mathcal{O}_Y]) = 1 + \left(\sum_{\substack{\beta \in \Delta(k+1), \\ \beta \succ \alpha}} \frac{1}{2} [V(\beta)] \right) + c'.$$

Applying i_* to both sides, we obtain the lemma. $\square$

3.5. Properties of Grothendieck weights

We will now carry out the strategy outlined in the previous subsection (see also Remark 2.8). Namely, we prove that for $g \in \text{GW}(\Delta)$, the integers $g(\alpha)$ must satisfy certain

identities with coefficients in $\mathbb{Q}$, by showing that certain elements are in $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$ or $\text{Rel}_{K_\circ(X)}$. See also Remark 2.8.

Let X be a complete toric variety of dimension n , and let Δ be the corresponding complete fan. As before, τ_X refers to the Riemann-Roch transformation. Let us fix

$$\tau_{\dagger} : \mathbb{Q}^\Delta \rightarrow \mathbb{Q}^\Delta \quad (\S)$$

to be any lift of τ_X which maps e_α to $e_\alpha + \left(\sum_{\substack{\beta \in \Delta(\dim(\alpha)+1), \\ \beta \succ \alpha}} \frac{1}{2} e_\beta \right) + c$, where c is in the span of e_γ for γ containing α as a face of codimension ≥ 2 . Such a lift exists by Lemma 3.4. Then, $\tau_{\dagger}$ is an isomorphism mapping $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$ to $\text{Rel}_{A_\star(X)_\mathbb{Q}}$.

Proposition 3.6. *If $g : \Delta \rightarrow \mathbb{Z}$ is a Grothendieck weight, it is constant on maximal cones.*

Proof. Let $\sigma, \sigma' \in \Delta$ be maximal cones. Since for any two points p, p' on $\mathbb{P}^1$, $[\mathcal{O}_p] = [\mathcal{O}_{p'}]$, and any two T -fixed points in X are connected by a chain of $\mathbb{P}^1$ - for instance, by T -invariant curves, we have that $[\mathcal{O}_{V(\sigma)}] = [\mathcal{O}_{V(\sigma')}]$. So $e_\sigma - e_{\sigma'}$ is in $\text{Rel}_{K_\circ(X)}$. $\square$

The second proposition is about the values of a Grothendieck weight on cones of codimension ≤ 1 .

Proposition 3.7. *If $g : \Delta \rightarrow \mathbb{Z}$ is a Grothendieck weight, $\sigma \in \Delta(n)$ any maximal cone, and $\alpha \in \Delta(n-2)$, then*

$$\sum_{\substack{\beta \in \Delta(n-1), \\ \beta \succ \alpha}} (g(\beta) - g(\sigma)) v_{\beta, \alpha} = 0.$$

Proof. Recall from (3.2) and (3.3) the definition of $\text{Rel}_{A_i(X)_\mathbb{Q}}$. By Proposition 3.3, the space $\text{Rel}_{A_1(X)_\mathbb{Q}}$ contains

$$\sum_{\substack{\beta \in \Delta(n-1), \\ \beta \succ \alpha}} \langle u, v_{\beta, \alpha} \rangle e_\beta,$$

for each u in M_α .

Applying the $\tau_{\dagger}^{-1}$ chosen in (§), we see that $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$ contains

$$\sum_{\substack{\beta \in \Delta(n-1), \\ \beta \succ \alpha}} \langle u, v_{\beta, \alpha} \rangle (e_\beta - \frac{1}{2} e_{\sigma_1} - \frac{1}{2} e_{\sigma_2}),$$

for σ_1, σ_2 the two maximal cones that contain β .

Since $e_\sigma - e_{\sigma_i}$ is in $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$ by the proof of the previous proposition, it follows that $\sum_{\substack{\beta \in \Delta(n-1), \\ \beta \succ \alpha}} \langle u, v_{\beta, \alpha} \rangle (e_\beta - e_\sigma)$ is also contained in $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$. Thus

$$\sum_{\substack{\beta \in \Delta(n-1), \\ \beta \succ \alpha}} \langle u, v_{\beta, \alpha} \rangle (g(\beta) - g(\sigma)) = 0,$$

for each $u \in M_\alpha$. $\square$

The third proposition, in the same pattern, is about the values of a Grothendieck weight on cones of codimension ≤ 2 .

Proposition 3.8. *If $g : \Delta \rightarrow \mathbb{Z}$ is a Grothendieck weight, $\sigma \in \Delta(n)$ any maximal cone, and $\alpha \in \Delta(n-3)$, then*

$$\sum_{\substack{\beta \in \Delta(n-2), \\ \beta \succ \alpha}} \left(g(\beta) - \sum_{\substack{\gamma \in \Delta(n-1), \\ \gamma \succ \beta}} \frac{g(\gamma)}{2} \right) v_{\beta, \alpha} = g(\sigma) \left(\sum_{\substack{\beta \in \Delta(n-2), \\ \beta \succ \alpha}} \left(1 - \sum_{\substack{\gamma \in \Delta(n-1), \\ \gamma \succ \beta}} \frac{1}{2} \right) v_{\beta, \alpha} \right).$$

Proof. Citing Proposition 3.3 again, we know that for each $u \in M_\alpha$ the element

$$\sum_{\substack{\beta \in \Delta(n-2), \\ \beta \succ \alpha}} \langle u, v_{\beta, \alpha} \rangle e_\beta,$$

is in $\text{Rel}_{A_2(X)_\mathbb{Q}}$. Its inverse image in $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$, with respect to $\tau_\dagger$ as chosen in (§), has the form

$$\sum_{\substack{\beta \in \Delta(n-2), \\ \beta \succ \alpha}} \langle u, v_{\beta, \alpha} \rangle \left(e_\beta - \left(\sum_{\substack{\gamma \in \Delta(n-1), \\ \gamma \succ \beta}} \frac{1}{2} e_\gamma \right) \right) + \sum_{\sigma' \text{ maximal}} a_{\sigma'} e_{\sigma'},$$

for some coefficients $a_{\sigma'}$.

By adding multiples of $e_\sigma - e_{\sigma'}$, the following element is also in $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$:

$$\sum_{\substack{\beta \in \Delta(n-2), \\ \beta \succ \alpha}} \langle u, v_{\beta, \alpha} \rangle \left(e_\beta - \left(\sum_{\substack{\gamma \in \Delta(n-1), \\ \gamma \succ \beta}} \frac{1}{2} e_\gamma \right) \right) + \left(\sum_{\sigma' \text{ maximal}} a_{\sigma'} \right) e_\sigma.$$

Then, Lemma 2.14 implies that

$$\sum_{\substack{\beta \in \Delta(n-2), \\ \beta \succ \alpha}} \langle u, v_{\beta, \alpha} \rangle \left(1 - \left(\sum_{\substack{\gamma \in \Delta(n-1), \\ \gamma \succ \beta}} \frac{1}{2} \right) \right) = - \left(\sum_{\sigma' \text{ maximal}} a_{\sigma'} \right).$$

Thus, a Grothendieck weight g must satisfy

$$\begin{aligned}
& \sum_{\substack{\beta \in \Delta(n-2), \\ \beta \succ \alpha}} \langle u, v_{\beta, \alpha} \rangle \left(g(\beta) - \left(\sum_{\substack{\gamma \in \Delta(n-1), \\ \gamma \succ \beta}} \frac{1}{2} g(\gamma) \right) \right) \\
&= g(\sigma) \sum_{\substack{\beta \in \Delta(n-2), \\ \beta \succ \alpha}} \langle u, v_{\beta, \alpha} \rangle \left(1 - \left(\sum_{\substack{\gamma \in \Delta(n-1), \\ \gamma \succ \beta}} \frac{1}{2} \right) \right),
\end{aligned}$$

for each $u \in M_\alpha$.

This final equality says that two different elements of N_α have the same value when paired with every $u \in M_\alpha$. Since $M_\alpha = N_\alpha^\vee$, the two elements of N_α must be equal. This proves the proposition. $\square$

3.9. Proof of Theorem 1.1

Proof. For X of dimension ≤ 3 , $\text{Rel}_{A_*(X)_\mathbb{Q}} \cong \text{Rel}_{A_0(X)_\mathbb{Q}} \oplus \text{Rel}_{A_1(X)_\mathbb{Q}} \oplus \text{Rel}_{A_2(X)_\mathbb{Q}}$, where the last two factors may be trivial. Thus, the inverse images of the generators of these factors with respect to $\tau_\dagger$ generate $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$. By Remark 2.8, Grothendieck weights are the functions on Δ which send such expressions to 0. $\square$

4. Grothendieck weights on simplicial fans

Theorem 1.1 characterized Grothendieck weights in low dimensions by explicitly producing generators for $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$. In this section, we would like to do the same for higher dimensional toric varieties. Ideally, we would like to address the following question:

Question 4.1. Given X a complete toric variety of arbitrary dimension, how can we calculate explicit generating sets for $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$?

For the sake of explicitness, we restrict to simplicial fans. Our basic strategy is the same as before: if $\tau_\dagger$ is a lift of the Riemann-Roch transformation τ_X , then we obtain an isomorphism of exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Rel}_{K_\circ(X)_\mathbb{Q}} & \longrightarrow & \mathbb{Q}^\Delta & \longrightarrow & K_\circ(X)_\mathbb{Q} \longrightarrow 0 \\
& & \downarrow & & \downarrow \tau_\dagger & & \downarrow \tau_X \\
0 & \longrightarrow & \text{Rel}_{A_*(X)_\mathbb{Q}} & \longrightarrow & \mathbb{Q}^\Delta & \longrightarrow & A_*(X)_\mathbb{Q} \longrightarrow 0,
\end{array}$$

and we can obtain a set of generators of $\text{Rel}_{K_\circ(X)_\mathbb{Q}}$ as the inverse images (with respect to $\tau_\dagger$) of generators of $\text{Rel}_{A_*(X)_\mathbb{Q}}$. Recall that we already know explicit generators for $\text{Rel}_{A_*(X)_\mathbb{Q}}$ by [12, Proposition 2.1(b)], which was reproduced earlier as Proposition 3.3.

4.2. Choosing $\tau_{\dagger}$

In the previous section, we did not need to fully calculate a lift $\tau_{\dagger}$, but because we are now considering X of arbitrary dimension, we change our approach and focus on the following question.

Question 4.3. What is an optimal $\tau_{\dagger}$ that lifts τ_X ? How can we calculate it?

To calculate a suitable $\tau_{\dagger}$, we must choose an expression for $\tau_X(V(\alpha))$ for each $\alpha \in \Delta$ in terms of $[V(\beta)]$, $\beta \succ \alpha$. Finding such expressions is known as Danilov's problem, see [7, Section 11], and [3,15,17,18]. To find such expressions, we follow a strategy from [18], which is to rewrite an expression for the Todd class as a polynomial in T -invariant divisors using [18, Theorem 3].

Because we use it heavily in this section, we start by introducing [18, Theorem 3]. First, we define the multiplicity of a simplicial cone: If α is a k -dimensional cone in N with k extremal rays generated by $v_1, \dots, v_k$, and $N^\alpha = N \cap \mathbb{Q} \cdot \alpha$, then the multiplicity of α is $\text{mult}(\alpha) = [N^\alpha : \mathbb{Z}v_1 + \dots + \mathbb{Z}v_k]$. If $\alpha \prec \beta$, let the relative multiplicity $\text{mult}_\alpha(\beta)$ denote the multiplicity of the image $\bar{\beta}$ in the quotient N_α . Then e.g. $\text{mult}(\alpha) = \text{mult}_{\{0\}}(\alpha)$. Geometrically, $\text{mult}_\alpha(\beta)$ is the Hilbert-Samuel multiplicity of $U_\beta \cap V(\alpha)$ along $V(\beta)$.

In [18], Pommersheim and Thomas define rational numbers t_ρ^α for each cone α and ray ρ such that $\rho \subset \alpha$, which depend on the choice of a generic complete flag $F_\bullet$ in $N_\mathbb{Q}$. These rational numbers are used to rewrite non-squarefree monomials in toric divisors as a linear combination of the classes of T -invariant subvarieties.

Definition 4.4. Let $F_\bullet$ be a complete flag in $N_\mathbb{Q}$, so F_i is an i -dimensional subspace of $N_\mathbb{Q}$. Given $\alpha \in \Delta(k)$, and i from 1 to k , let v_{ρ_i} be the primitive element of the ray ρ_i in α . We impose that $F_\bullet$ is generic, in the sense that $F_{n-k+1} \cap (\mathbb{Q} \cdot \alpha)$ is always 1-dimensional, so it determines a vector (unique up to a common scaling):

$$0 \neq \sum_{i=1}^k t_{\rho_i}^\alpha v_{\rho_i} \in F_{n-k+1} \cap (\mathbb{Q} \cdot \alpha).$$

We impose further that for a generic $F_\bullet$ all such t_ρ^α are non-zero.

The scalar for the t_ρ^α will not matter because they will only appear in expressions which are ratios of homogeneous polynomials of the same degree.

Now, let S be a subset of $\Delta(1)$. For $\rho \in S$, let a_ρ be a positive integer, and let l denote the sum $\sum_{\rho \in S} a_\rho$. Then, a restatement of [18, Theorem 3] is:

Theorem 4.5. Let X be a simplicial toric variety with fan Δ . Then

$$\prod_{\rho \in S} [V(\rho)]^{a_\rho} = \sum_{\substack{\alpha \in \Delta(l), \\ \alpha \text{ contains all } \rho \in S}} \frac{[V(\alpha)]}{\text{mult}(\alpha) \prod_{\rho \subset \alpha} t_\rho^\alpha} \prod_{\rho \in S} (t_\rho^\alpha)^{a_\rho},$$

in $A_{n-l}(X)_{\mathbb{Q}}$.

This allows us to give an explicit formula for certain types of products as linear combinations of subvarieties:

Lemma 4.6. *Let X be a simplicial toric variety with fan Δ , and for each $\rho \in \Delta(1)$, let $f_{\rho}(t) \in \mathbb{Q}[[t]]$ be a series. Then in the $A_*(X)_{\mathbb{Q}}$ we have an identity*

$$\prod_{\rho \in \Delta(1)} f_{\rho}([V(\rho)]) = \sum_{\alpha \in \Delta} \left(\prod_{\rho \subset \alpha} \frac{f_{\rho}(t_{\rho}^{\alpha})}{t_{\rho}^{\alpha}} \right)_{[0]} \left(\prod_{\rho \not\subset \alpha} f_{\rho}(0) \right) \frac{[V(\alpha)]}{\text{mult}(\alpha)}.$$

Proof. Let us write

$$f_{\rho}(t) = \sum_{i \geq 0} a_{i_{\rho}}^{\rho} t^i,$$

so by expanding the product and applying Theorem 4.5, we get

$$\begin{aligned} \prod_{\rho \in \Delta(1)} f_{\rho}([V(\rho)]) &= \sum_{(i_{\rho}) \in \mathbb{Z}_{\geq 0}^{\Delta(1)}} \prod_{\rho \in \Delta(1)} a_{i_{\rho}}^{\rho} [V(\rho)]^{i_{\rho}}, \\ &= \sum_{(i_{\rho}) \in \mathbb{Z}_{\geq 0}^{\Delta(1)}} \sum_{\substack{\alpha \in \Delta(\sum_{\rho} i_{\rho}), \\ \alpha \subset \rho \text{ for all } \rho \\ \text{satisfying } i_{\rho} > 0}} \frac{[V(\alpha)]}{\text{mult}(\alpha) \prod_{\rho \subset \alpha} t_{\rho}^{\alpha}} \prod_{\rho \in \Delta(1)} a_{i_{\rho}}^{\rho} \prod_{\rho \subset \alpha} (t_{\rho}^{\alpha})^{i_{\rho}}, \\ &= \sum_{(i_{\rho}) \in \mathbb{Z}_{\geq 0}^{\Delta(1)}} \sum_{\substack{\alpha \in \Delta(\sum_{\rho} i_{\rho}), \\ \alpha \subset \rho \text{ for all } \rho \\ \text{satisfying } i_{\rho} > 0}} \frac{[V(\alpha)]}{\text{mult}(\alpha)} \prod_{\rho \in \Delta(1)} a_{i_{\rho}}^{\rho} \prod_{\rho \subset \alpha} (t_{\rho}^{\alpha})^{i_{\rho}-1}. \end{aligned}$$

Inside both summations, the index i_{ρ} is zero if $\rho \not\subset \alpha$, so we can split the product $\prod_{\rho \in \Delta(1)} a_{i_{\rho}}^{\rho}$ into $\prod_{\rho \subset \alpha} a_{i_{\rho}}^{\rho} \prod_{\rho \not\subset \alpha} f_{\rho}(0)$. Switching the summations and rearranging, we get that the above can be further rewritten as

$$\sum_{\alpha \in \Delta} \frac{[V(\alpha)]}{\text{mult}(\alpha)} \prod_{\rho \not\subset \alpha} f_{\rho}(0) \sum_{\substack{(i_{\rho}) \in \mathbb{Z}_{\geq 0}^{\Delta(1)}, \\ \sum i_{\rho} = \dim(\alpha), \\ \alpha \subset \rho \text{ for all } \rho \\ \text{satisfying } i_{\rho} > 0}} \prod_{\rho \subset \alpha} a_{i_{\rho}}^{\rho} (t_{\rho}^{\alpha})^{i_{\rho}-1}.$$

The inner sum is equal to $\left(\prod_{\rho \subset \alpha} \frac{f_{\rho}(t_{\rho}^{\alpha})}{t_{\rho}^{\alpha}} \right)_{[0]}$, which proves the corollary. $\square$

The next proposition relates products of divisors in X with products of divisors in $V(\alpha)$. For $\alpha \in \Delta$ and a generic flag $F_{\bullet}$ in N , the images $\overline{F_1} \subset \dots \subset \overline{F_{n-\dim(\alpha)}}$ form a

generic flag in N_α . Thus, for β a cone containing α and ρ a ray in β not contained in α , this choice of generic flag produces numbers t_ρ^β which we can use to rewrite products of divisors in $V(\alpha)$. Like the numbers t_ρ^β , the numbers $t_{\bar{\rho}}^\beta$ are defined up to a scalar which will not matter because they will only appear in expressions which are ratios of homogeneous polynomials of the same degree.

Proposition 4.7. *Let $\alpha \prec \beta$ be simplicial cones in a fan Δ . If ρ is a ray in β that is not contained in α , then $t_\rho^\beta = \frac{\text{mult}(\alpha+\rho)}{\text{mult}(\alpha)} t_\rho^\alpha$.*

Proof. Recall that v_ρ is the generator of the semigroup of lattice points in the ray ρ . Let $\dim(\alpha) = k$. Up to a scalar, the unique vector in $\overline{F_{n-k+1}} \cap (\mathbb{Q} \cdot \bar{\beta})$ is the image of the unique vector in $F_{n-k+1} \cap (\mathbb{Q} \cdot \beta)$, which has the formula

$$\sum_{\rho \subset \beta} t_\rho^\beta v_\rho = \sum_{\substack{\rho \subset \beta, \\ \rho \not\subset \alpha}} t_\rho^\beta \bar{v}_\rho.$$

The image $\bar{v}_\rho$ of v_ρ does not necessarily generate the semigroup of lattice points in $\bar{\rho}$. In other words, $\bar{v}_\rho = b_\rho v_{\bar{\rho}}$ for b_ρ a positive integer. In fact, b_ρ is the index $[\mathbb{Z} v_{\bar{\rho}} : \mathbb{Z} \bar{v}_\rho]$.

Let $\pi_\alpha : N \rightarrow N_\alpha$ be the quotient map, and let $\alpha + \rho$ be the cone generated by α and ρ . Recall that N^α is the kernel of π_α (from subsection 1.3). Then $\pi_\alpha^{-1}(\mathbb{Z} v_{\bar{\rho}}) = N^{\alpha+\rho}$, and $\pi_\alpha^{-1}(\mathbb{Z} \bar{v}_\rho) = N^\alpha + \mathbb{Z} v_\rho$. Thus $b_\rho = [\mathbb{Z} v_{\bar{\rho}} : \mathbb{Z} \bar{v}_\rho] = [N^{\alpha+\rho} : N^\alpha + \mathbb{Z} v_\rho]$.

To compute $[N^{\alpha+\rho} : N^\alpha + \mathbb{Z} v_\rho]$, let us first decompose $\text{mult}(\alpha + \rho)$ as a product. Let $\rho_1, \dots, \rho_k$ be the rays of α .

$$\begin{aligned} \text{mult}(\alpha + \rho) &= [N^{\alpha+\rho} : \mathbb{Z} v_{\rho_1} + \dots + \mathbb{Z} v_{\rho_k} + \mathbb{Z} v_\rho] \\ &= [N^{\alpha+\rho} : N^\alpha + \mathbb{Z} v_\rho] [N^\alpha + \mathbb{Z} v_\rho : \mathbb{Z} v_{\rho_1} + \dots + \mathbb{Z} v_{\rho_k} + \mathbb{Z} v_\rho]. \end{aligned}$$

But $[N^\alpha + \mathbb{Z} v_\rho : \mathbb{Z} v_{\rho_1} + \dots + \mathbb{Z} v_{\rho_k} + \mathbb{Z} v_\rho] = [N^\alpha : \mathbb{Z} v_{\rho_1} + \dots + \mathbb{Z} v_{\rho_k}] = \text{mult}(\alpha)$. So $b_\rho = [N^{\alpha+\rho} : N^\alpha + \mathbb{Z} v_\rho] = \frac{\text{mult}(\alpha+\rho)}{\text{mult}(\alpha)}$.

Thus, we have

$$\sum_{\substack{\rho \subset \beta, \\ \rho \not\subset \alpha}} t_\rho^\beta \bar{v}_\rho = \sum_{\substack{\rho \subset \beta, \\ \rho \not\subset \alpha}} t_\rho^\beta \frac{\text{mult}(\alpha + \rho)}{\text{mult}(\alpha)} v_{\bar{\rho}}$$

is the unique vector in $\overline{F_{n-k+1}} \cap (\mathbb{Q} \cdot \bar{\beta})$. By Definition 4.4, $t_{\bar{\rho}}^\beta$ is the coefficient of $v_{\bar{\rho}}$, which is $t_\rho^\beta \frac{\text{mult}(\alpha+\rho)}{\text{mult}(\alpha)}$. $\square$

Now, we restrict to working over $\mathbb{C}$ temporarily. We use the two previous propositions and a Theorem from [4] to choose a map $\tau_\dagger$ that lifts τ_X .

For each cone $\alpha \in \Delta$, Brion and Vergne defined the finite subgroup $G_\alpha \subset (\mathbb{C}^*)^{\dim(\alpha)}$ to be the kernel of the map $(\mathbb{C}^*)^{\dim(\alpha)} \rightarrow T$ given by

$$(c_\rho)_\rho \mapsto \prod_{\rho \subset \alpha} v_\rho(c_\rho).$$

For nested cones $\alpha \prec \beta$, let G_β^α be the analogous subgroup defined with respect to $\bar{\beta}$ in the fan of $V(\alpha)$. Explicitly:

Definition 4.8. Let G_β^α be the kernel of the map $(\mathbb{C}^*)^{\dim(\beta) - \dim(\alpha)} \rightarrow T_\alpha$ given by

$$(c_\rho)_{\substack{\rho \subset \beta, \\ \rho \not\subset \alpha}} \mapsto \prod_{\substack{\rho \subset \beta, \\ \rho \not\subset \alpha}} v_{\bar{\rho}}(c_\rho).$$

Let k be the number of rays in the quotient fan Δ_α , and let G_{Δ_α} to be the union inside $(\mathbb{C}^*)^k$ over all β containing α . For a ray ρ in β not contained in α , we denote by a_ρ^α the character $G_\beta^\alpha \rightarrow \mathbb{C}^*$ given by projection.

Treating t_i as a variable with degree 1, we will refer to the degree 0 coefficient of a formal Laurent series $\psi(t_1, \dots, t_k)$ by $\psi(t_1, \dots, t_k)_{[0]}$. For example, if

$$\psi = \left(\frac{1}{1 - e^{-t_1}} \right) \left(\frac{1}{1 - e^{-t_2}} \right) = \left(\frac{1}{t_1} + \frac{1}{2} + \frac{t_1}{12} + \mathcal{O}(t_1^2) \right) \left(\frac{1}{t_2} + \frac{1}{2} + \frac{t_2}{12} + \mathcal{O}(t_2^2) \right),$$

then

$$\psi_{[0]} = \frac{1}{4} + \frac{1}{12} \left(\frac{t_1}{t_2} + \frac{t_2}{t_1} \right).$$

Proposition 4.9. Let X be a complete simplicial toric variety over $\mathbb{C}$. Then, we have the following formula for the Riemann-Roch transformation:

$$\tau_X([\mathcal{O}_{V(\alpha)}]) = \sum_{\beta \succ \alpha} \sum_{g \in G_\beta^\alpha} \left(\prod_{\substack{\rho \in \Delta(1), \\ \rho \subset \beta, \\ \rho \not\subset \alpha}} \frac{1}{1 - a_\rho^\alpha(g) e^{-\text{mult}(\alpha + \rho) t_\rho^\beta}} \right)_{[0]} \frac{\text{mult}(\alpha)[V(\beta)]}{\text{mult}(\beta) \prod_{\substack{\rho \subset \beta, \\ \rho \not\subset \alpha}} \frac{\text{mult}(\alpha)}{\text{mult}(\alpha + \rho)}}$$

Proof. Let Δ_α be the fan of $V(\alpha)$ in $(N_\alpha)_\mathbb{R}$. The main theorem in [4, Section 4.2] gives the following formula for the Todd class:

$$\tau_{V(\alpha)}([\mathcal{O}_{V(\alpha)}]) = \sum_{g \in G_{\Delta_\alpha}} \prod_{\rho \in \Delta_\alpha(1)} \frac{[V(\rho)]}{1 - a_\rho^\alpha(g) e^{-[V(\rho)]}}.$$

Let $i : V(\alpha) \hookrightarrow X$ be the inclusion map. Then in $A_*(X)_\mathbb{Q}$, we have

$$\tau_X([\mathcal{O}_{V(\alpha)}]) = \sum_{g \in G_{\Delta_\alpha}} i_* \prod_{\rho \in \Delta_\alpha(1)} \frac{[V(\rho)]}{1 - a_\rho^\alpha(g) e^{-[V(\rho)]}}.$$

Applying Lemma 4.6, and rewriting, we have:

$$\begin{aligned}
 & \sum_{g \in G_{\Delta_\alpha}} i_* \prod_{\rho \in \Delta_\alpha(1)} \frac{[V(\rho)]}{1 - a_\rho^\alpha(g) e^{-[V(\rho)]}} \\
 &= \sum_{g \in G_{\Delta_\alpha}} i_* \sum_{\beta \succ \alpha} \left(\prod_{\substack{\rho \in \Delta_\alpha(1), \\ \rho \subset \bar{\beta}}} \frac{1}{1 - a_\rho^\alpha(g) e^{-t_{\bar{\beta}}}} \right)_{[0]} \left(\prod_{\substack{\rho \in \Delta_\alpha(1), \\ \rho \not\subset \bar{\beta}}} \lim_{t \rightarrow 0} \frac{t}{1 - a_\rho^\alpha(g) e^{-t}} \right) \frac{[V(\bar{\beta})]}{\text{mult}_\alpha(\beta)}, \\
 &= \sum_{g \in G_{\Delta_\alpha}} \sum_{\beta \succ \alpha} \left(\prod_{\substack{\rho \in \Delta(1), \\ \rho \subset \bar{\beta}, \\ \rho \not\subset \alpha}} \frac{1}{1 - a_\rho^\alpha(g) e^{-\frac{\text{mult}(\alpha+\rho)}{\text{mult}(\alpha)} t_{\bar{\beta}}}} \right)_{[0]} \\
 &\quad \times \left(\prod_{\substack{\rho \in \Delta(1), \\ \rho \not\subset \bar{\beta}, \\ \alpha+\rho \in \Delta}} \lim_{t \rightarrow 0} \frac{t}{1 - a_\rho^\alpha(g) e^{-t}} \right) \frac{\text{mult}(\alpha)[V(\beta)]}{\text{mult}(\beta) \prod_{\substack{\rho \subset \beta, \\ \rho \not\subset \alpha}} \frac{\text{mult}(\alpha)}{\text{mult}(\alpha+\rho)}},
 \end{aligned}$$

where in the last line we applied the formula for $t_{\bar{\beta}}$ from Proposition 4.7 and the formula for $\text{mult}_\alpha(\beta)$ from the appendix.

However, the term $\left(\prod_{\substack{\rho \in \Delta(1), \\ \rho \not\subset \bar{\beta}, \\ \alpha+\rho \in \Delta}} \lim_{t \rightarrow 0} \frac{t}{1 - a_\rho^\alpha(g) e^{-t}} \right)$ is 1 if $a_\rho^\alpha(g) = 1$ for all ρ indexing the product, or 0 otherwise. The condition that $a_\rho^\alpha(g) = 1$ for all ρ is equivalent to the condition that $g \in G_\beta^\alpha$. Thus, we obtain that

$$\begin{aligned}
 & \sum_{g \in G_{\Delta_\alpha}} \sum_{\beta \succ \alpha} \left(\prod_{\substack{\rho \in \Delta(1), \\ \rho \subset \bar{\beta}, \\ \rho \not\subset \alpha}} \frac{1}{1 - a_\rho^\alpha(g) e^{-\frac{\text{mult}(\alpha+\rho)}{\text{mult}(\alpha)} t_{\bar{\beta}}}} \right)_{[0]} \\
 &\quad \times \left(\prod_{\substack{\rho \in \Delta(1), \\ \rho \not\subset \bar{\beta}, \\ \alpha+\rho \in \Delta}} \lim_{t \rightarrow 0} \frac{t}{1 - a_\rho^\alpha(g) e^{-t}} \right) \frac{\text{mult}(\alpha)[V(\beta)]}{\text{mult}(\beta) \prod_{\substack{\rho \subset \beta, \\ \rho \not\subset \alpha}} \frac{\text{mult}(\alpha)}{\text{mult}(\alpha+\rho)}}
 \end{aligned}$$

$$= \sum_{\beta \succ \alpha} \sum_{g \in G_{\beta}^{\alpha}} \left(\prod_{\substack{\rho \in \Delta(1), \\ \rho \subset \beta, \\ \rho \not\subset \alpha}} \frac{1}{1 - a_{\rho}^{\alpha}(g) e^{-\frac{\text{mult}(\alpha+\rho)}{\text{mult}(\alpha)} t_{\rho}^{\beta}}} \right) \frac{\text{mult}(\alpha)[V(\beta)]}{\text{mult}(\beta) \prod_{\substack{\rho \subset \beta, \\ \rho \not\subset \alpha}} \frac{\text{mult}(\alpha)}{\text{mult}(\alpha+\rho)}}. \quad [0]$$

Due to the “degree 0” imposition, the $\text{mult}(\alpha)$ factors in the exponents $e^{-\frac{\text{mult}(\alpha+\rho)}{\text{mult}(\alpha)} t_{\rho}^{\beta}}$ cancel, which proves the proposition. $\square$

Here is the relevant consequence.

Corollary 4.10. *Let X be a complete simplicial toric variety over $\mathbb{C}$. Let $\tau_{\dagger} : \mathbb{Q}^{\Delta} \rightarrow \mathbb{Q}^{\Delta}$ send e_{α} to $\sum_{\beta \succ \alpha} \sum_{g \in G_{\beta}^{\alpha}} \left(\prod_{\substack{\rho \subset \beta, \\ \rho \not\subset \alpha}} \frac{\text{mult}(\alpha+\rho)/\text{mult}(\alpha)}{1 - a_{\rho}^{\alpha}(g) e^{-\frac{\text{mult}(\alpha+\rho)}{\text{mult}(\alpha)} t_{\rho}^{\beta}}} \right) \frac{\text{mult}(\alpha)}{\text{mult}(\beta)} e_{\beta}$. Then $\text{Rel}_{K_{\circ}(X)_{\mathbb{Q}}} = \tau_{\dagger}^{-1}(\text{Rel}_{A_{*}(X)_{\mathbb{Q}}})$.*

Example 4.11. We use this proposition to calculate the $\tau_{\dagger}$ given by a particular choice of flag. Let X be the twisted projective space $\mathbb{P}(1, 1, 2, 3)$. Recall that the fan of X has rays $\rho_1 = (1, 0, 0)$, $\rho_2 = (0, 1, 0)$, $\rho_3 = (0, 0, 1)$, and $\rho_4 = (-1, -2, -3)$. The maximal cones are those generated by 3-element subsets of $\{\rho_1, \rho_2, \rho_3, \rho_4\}$. If we choose the flag in $\mathbb{Q}^3$ given by

$$\{0\} \subsetneq \text{span}\{(a, b, c)\} \subsetneq \text{span}\{(a, b, c), (d, e, f)\} \subsetneq \mathbb{Q}^3,$$

where a, b, c, d, e, f are some numbers so that (d, e, f) is not a multiple of (a, b, c) , then the t_{ρ}^{α} are those written in Table 1.

Then, one can write the Todd class of each subvariety in a uniform way with rational functions in t_{ρ}^{σ} as coefficients. For the flag specified by $(a, b, c) = (2, 3, 5)$, $(d, e, f) = (3, 5, 7)$, we get the lift $\tau_{\dagger}$ seen in Table 2.

Definition 4.12. Let $\mu_{\alpha}(\beta)$ be the (β, α) entry of the matrix corresponding to $\tau_{\dagger}$ as defined in Corollary 4.10, with respect to the basis $e_{\alpha}, \alpha \in \Delta$.

The map $\tau_{\dagger} : \mathbb{Q}^{\Delta} \rightarrow \mathbb{Q}^{\Delta}$ is lower triangular with ones on the diagonal, so it is invertible. Let $\nu_{\alpha}(\beta)$ be the (β, α) entry of the matrix corresponding to $\tau_{\dagger}^{-1}$, with respect to the basis $e_{\alpha}, \alpha \in \Delta$. Then, we can write

$$\tau_X^{-1}([V(\alpha)]) = \sum_{\alpha \prec \beta} \nu_{\alpha}(\beta) [\mathcal{O}_{V(\beta)}].$$

Example 4.13. For $\alpha \in \Delta$, we have $\nu_{\alpha}(\alpha) = 1$, and for Δ smooth and t_{ρ}^{α} defined as in 4.4, we have

Table 1
The t_ρ^α of Example 4.11.

Cone (α)	Ray (ρ)	t_ρ^α
σ_{123}	ρ_1	a
	ρ_2	b
	ρ_3	c
σ_{124}	ρ_1	$a - c/3$
	ρ_2	$b - 2c/3$
	ρ_4	$-c/3$
σ_{134}	ρ_1	$a - b/2$
	ρ_3	$c - 3b/2$
	ρ_4	$-b/2$
σ_{234}	ρ_2	$b - 2a$
	ρ_3	$c - 3a$
	ρ_4	$-a$
α_{12}	ρ_1	$af - cd$
	ρ_2	$bf - ce$
α_{13}	ρ_1	$ae - bd$
	ρ_3	$ce - bf$
α_{14}	ρ_1	$3(ae - bd) + 2(cd - af) + (bf - ce)$
	ρ_4	$bf - ce$
α_{23}	ρ_2	$ae - bd$
	ρ_3	$af - cd$
α_{24}	ρ_2	$-(3(ae - bd) + 2(cd - af) + (bf - ce))$
	ρ_4	$af - cd$
α_{34}	ρ_3	$3(ae - bd) + 2(cd - af) + (bf - ce)$
	ρ_4	$ae - bd$
ρ	ρ	1

Table 2
The $\tau_\dagger$ matrix of Example 4.11.

$\{V(-)\} \setminus [\mathcal{O}_{V(-)}]$	X	ρ_1	ρ_2	ρ_3	ρ_4	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	σ_{123}	σ_{124}	σ_{134}	σ_{234}
X	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ρ_1	1/2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
ρ_2	1/2	0	1	0	0	0	0	0	0	0	0	0	0	0	0
ρ_3	1/2	0	0	1	0	0	0	0	0	0	0	0	0	0	0
ρ_4	1/2	0	0	0	1	0	0	0	0	0	0	0	0	0	0
α_{12}	29/48	1/2	1/2	0	0	1	0	0	0	0	0	0	0	0	0
α_{13}	29/48	1/2	0	1/2	0	0	1	0	0	0	0	0	0	0	0
α_{14}	-5/48	1/2	0	0	1/2	0	0	1	0	0	0	0	0	0	0
α_{23}	1/12	0	1/2	1/2	0	0	0	0	1	0	0	0	0	0	0
α_{24}	5/12	0	1/2	0	1/2	0	0	0	0	1	0	0	0	0	0
α_{34}	5/12	0	0	1/2	1/2	0	0	0	0	0	1	0	0	0	0
σ_{123}	31/72	79/180	59/120	31/72	0	1/2	1/2	0	1/2	0	0	1	0	0	0
σ_{124}	1/8	9/20	1/20	0	1/4	1/2	0	1/2	0	1/2	0	0	1	0	0
σ_{134}	1/36	1/9	0	1/9	1/3	0	1/2	1/2	0	0	1/2	0	0	1	0
σ_{234}	5/12	0	11/24	11/24	5/12	0	0	0	1/2	1/2	1/2	0	0	0	1

$$\nu_\alpha(\beta) = \sum_{\substack{\alpha_0 \preceq \alpha_1 \preceq \dots \preceq \alpha_k, \\ \alpha_0 = \alpha, \alpha_k = \beta}} (-1)^k \left(\prod_{l=1}^k \left(\prod_{\substack{\rho \subset \alpha_l, \\ \rho \not\subset \alpha_{l-1}}} \frac{1}{1 - e^{-t_\rho^{\alpha_l}}} \right)_{[0]} \right).$$

Now, let $\tau_\dagger$ be as chosen in Corollary 4.10, and the $\nu_\alpha(\beta)$ as defined above. Then, we have the following theorem. Recall that $v_{\beta,\alpha}$ was defined in subsection 1.3.

Theorem 4.14. *Let Δ be a complete simplicial fan. Then a function $g : \Delta \rightarrow \mathbb{Z}$ is a Grothendieck weight if and only if it satisfies*

$$\sum_{\substack{\beta \succ \alpha, \\ \dim(\beta) = \dim(\alpha) + 1}} \langle u, v_{\beta, \alpha} \rangle \sum_{\gamma \succ \beta} \nu_{\beta}(\gamma) g(\gamma) = 0,$$

for each $\alpha \in \Delta$ and $u \in M_{\alpha}$.

Before the proof, we recall that by Proposition 2.9, $\text{GW}(\Delta)$ is independent of the group field, so even though we have defined $\tau_{\dagger}$ using formulas for the Todd class over $\mathbb{C}$, this theorem describes $\text{op}K^{\circ}$ of the associated toric variety over any field.

Proof. By Proposition 3.3, the elements

$$r_{\alpha, u} = \sum_{\substack{\beta \succ \alpha, \\ \dim(\beta) = \dim(\alpha) + 1}} \langle u, v_{\beta, \alpha} \rangle e_{\beta},$$

generate $\text{Rel}_{A_{*}(X)}$, so the elements $\tau_{\dagger}^{-1}(r_{\alpha, u})$ generate $\text{Rel}_{K_{\circ}(X)_{\mathbb{Q}}}$. By Definition 4.12,

$$\tau_{\dagger}^{-1} \left(\sum_{\substack{\beta \succ \alpha, \\ \dim(\beta) = \dim(\alpha) + 1}} \langle u, v_{\beta, \alpha} \rangle e_{\beta} \right) = \sum_{\substack{\beta \succ \alpha, \\ \dim(\beta) = \dim(\alpha) + 1}} \langle u, v_{\beta, \alpha} \rangle \sum_{\gamma \succ \beta} \nu_{\beta}(\gamma) e_{\gamma}.$$

Since these elements generate $\text{Rel}_{K_{\circ}(X)_{\mathbb{Q}}}$, the group $K_{\circ}(X)^{\vee}$ can be identified with linear forms on $\mathbb{Z}^{\Delta}$ which send such expressions to 0. Grothendieck weights are then (recall Remark 2.8) characterized as the functions $g : \Delta \rightarrow \mathbb{Z}$ satisfying

$$\sum_{\substack{\beta \succ \alpha, \\ \dim(\beta) = \dim(\alpha) + 1}} \langle u, v_{\beta, \alpha} \rangle \sum_{\gamma \succ \beta} \nu_{\beta}(\gamma) g(\gamma) = 0. \quad \square$$

Remark 4.15. Theorem 4.14 in some sense overlaps with Theorem 1.1 in the case that X is simplicial and has dimension ≤ 3 , since one can use the $\tau_{\dagger}$ from Corollary 4.10 in the course of proving Theorem 1.1, but we used the more elementary Lemma 3.4 in Section 3 for simplicity.

We considered using different sets of generators for the Grothendieck group in our definition of Grothendieck weights, e.g. ideal sheaves or canonical sheaves of invariant subvarieties. However, the problem of combinatorially describing the relations between these classes seems equally difficult.

5. Products

If X is a complete toric variety and Δ the associated fan, the natural isomorphism of groups between $\mathrm{op}K^\circ(X)$ and $\mathrm{GW}(\Delta)$ induces a product on Grothendieck weights, which is compatible with the product on $K_\circ(X)^\vee$ induced by the diagonal map by Theorem 5.3. We prove some propositions in this section about how to compute it. Since $\mathrm{op}K^\circ(X)$ and $\mathrm{GW}(\Delta)$ are manifestly torsion-free, we can compute products after extending coefficients to $\mathbb{Q}$, and the calculations will be valid in the original rings.

First, we will need the following special case of [2, Proposition 6.4].

Proposition 5.1. *Let X be a toric variety, and Y arbitrary. Then, the natural map $K_\circ(X) \otimes K_\circ(Y) \rightarrow K_\circ(X \times Y)$ is an isomorphism.*

We will need the following lemma. Recall that an operational class $c = (c_f) \in \mathrm{op}K^\circ(X)$ is a tuple of endomorphisms of $K_\circ(Y)$ for each $f : Y \rightarrow X$. To avoid notational overload, if $f = \mathrm{Id}_X$ is the identity map on X , we write c_{Id} instead of c_{Id_X} for the corresponding endomorphism of $K_\circ(X)$.

Lemma 5.2. *Let X be a toric variety, $\phi : Y \rightarrow X$ be arbitrary, $\gamma_\phi : Y \rightarrow X \times Y$ the graph of ϕ , $z \in K_\circ(Y)$, and $c \in \mathrm{op}K^\circ(X)$.*

Let us identify $(\gamma_\phi)_(z) \in K_\circ(X \times Y)$ with the corresponding element $\sum_i u_i \otimes v_i \in K_\circ(X) \otimes K_\circ(Y)$. Then, we have an equality in $K_\circ(Y)$:*

$$(\phi^*c)_{\mathrm{Id}}(z) = \sum \chi(c_{\mathrm{Id}}(u_i))v_i.$$

Proof. To prove the equality, let π_1 and π_2 be the projections from $X \times Y$ to X and Y respectively. Then, $\pi_2 \circ \gamma_\phi = \mathrm{id}_Y$ and $\pi_1 \circ \gamma_\phi = \phi$. Operational classes satisfy a projection formula, so we have

$$(\phi^*c)_{\mathrm{Id}}(z) = (\mathrm{id}_Y)_*((\phi^*c)_{\mathrm{Id}}(z)) = (\pi_{2*}\gamma_{\phi*})((\gamma_\phi^*\pi_1^*c)_{\mathrm{Id}}(z)) = (\pi_2)_*((\pi_1^*c)_{\mathrm{Id}}(\gamma_\phi)_*(z)).$$

Now, replacing $(\gamma_\phi)_*(z)$ with $\sum_i u_i \otimes v_i \in K_\circ(X) \otimes K_\circ(Y)$ gives us

$$(\pi_2)_*((\pi_1^*c)_{\mathrm{Id}}(\gamma_\phi)_*(z)) = \sum_i (\pi_2)_*((\pi_1^*c)_{\mathrm{Id}}(u_i \otimes v_i)).$$

So, the claim follows if we can show

$$(\pi_2)_*((\pi_1^*c)_{\mathrm{Id}}(u \otimes v)) = \chi(c_{\mathrm{Id}}(u))v,$$

for $u \in K_\circ(X)$, $v \in K_\circ(Y)$. It is enough to consider to the case that $v = [i_*\mathcal{O}_{Y'}]$ is the pushforward of the structure sheaf of a closed subvariety $i : Y' \hookrightarrow Y$, since such classes generate $K_\circ(Y)$.

Recall that $\pi_1 : X \times Y \rightarrow X$ is the projection. Let π'_1 be the projection $X \times Y' \rightarrow X$. Now consider the following commuting square:

$$\begin{array}{ccc} X \times Y' & \xrightarrow{Id \times i} & X \times Y \\ \downarrow \pi'_1 & & \downarrow \pi_1 \\ X & \xrightarrow{=} & X \end{array}$$

Then,

$$u \otimes v = u \otimes [i_* \mathcal{O}_{Y'}] = (Id \times i)_*(\pi'_1)^*(u),$$

so

$$(\pi_1^* c)_{Id}(u \otimes v) = (\pi_1^* c)_{Id}((Id \times i)_*(\pi'_1)^*(u)).$$

By definition of the pullback of operational classes, [2, Section 4.1], $(\pi_1^* c)_{Id} = c_{\pi_1}$, so

$$(\pi_1^* c)_{Id}((Id \times i)_*(\pi'_1)^*(u)) = c_{\pi_1}((Id \times i)_*(\pi'_1)^*(u)).$$

Since $Id \times i$ is proper, $c_{\pi_1} \circ (Id \times i)_* = (Id \times i)_* \circ c_{\pi_1 \circ (Id \times i)}$. Also $\pi_1 \circ (Id \times i) = \pi'_1$, so

$$c_{\pi_1}((Id \times i)_*(\pi'_1)^*(u)) = (Id \times i)_*(c_{\pi'_1}((\pi'_1)^*(u))).$$

Since π'_1 is flat, $c_{\pi'_1}((\pi'_1)^*(u)) = (\pi'_1)^*(c_{Id}(u))$, the latter being equal to $c_{Id}(u) \otimes [\mathcal{O}_{Y'}]$. Thus

$$(Id \times i)_*(c_{\pi'_1}((\pi'_1)^*(u))) = (Id \times i)_*(c_{Id}(u) \otimes [\mathcal{O}_{Y'}]) = c_{Id}(u) \otimes [i_* \mathcal{O}_{Y'}] = c_{Id}(u) \otimes v.$$

Applying $(\pi_2)_*$ finally shows that $(\pi_2)_*((\pi_1^* c)_{Id}(u \otimes v)) = (\pi_2)_*(c_{Id}(u) \otimes v) = \chi(c_{Id}(u))v$, which proves the claim. $\square$

This allows us to show the following fact about products of classes in operational K -theory. It is a K -theoretic analogue of [11, Theorem 4].

Theorem 5.3. *Let X be a complete toric variety. Let $\delta : X \rightarrow X \times X$ be the diagonal map, and let us have classes $z \in K_o(X)$ and $c, d \in \text{op}K^o(X)$. Given an expression $\delta_*(z) = \sum_i m_i u_i \otimes v_i$, with $m_i \in \mathbb{Q}$, the product $c \cdot d$ evaluated on z satisfies*

$$\chi((c \cdot d)_{Id}(z)) = \sum_i m_i \chi(c_{Id}(u_i)) \chi(d_{Id}(v_i)).$$

Proof. Recall (from subsection 2.2) that the product in operational K -theory is defined by point-wise composition and is commutative, so $(c \cdot d)_{Id} = (d \cdot c) = d_{Id} \circ c_{Id}$. Then,

we use the previous lemma, with $Id : X \rightarrow X$ in place of $\phi : Y \rightarrow X$, so the graph γ_{Id} is the diagonal map $\delta : X \rightarrow X \times X$.

The previous lemma implies that

$$c_{Id}(z) = \sum_i m_i \chi(c_{Id}(u_i)) v_i,$$

so

$$(c \cdot d)_{Id}(z) = d_{Id}(c_{Id}(z)) = \sum_i m_i \chi(c_{Id}(u_i)) \chi(d_{Id}(v_i)). \quad \square$$

Remark 5.4. In fact, the proposition and proof are valid for any variety which is linear in the sense of [19].

The following corollary is a direct consequence of Theorem 5.3. We will use the corollary in the next subsection to give a formula for the product of Grothendieck weights induced by $\text{op}K^\circ(X)$.

Corollary 5.5. *Let X be a complete toric variety with fan Δ , and let f and g be Grothendieck weights on Δ . Let $\delta : X \rightarrow X \times X$ be the diagonal map. Given an expression $\delta_*([\mathcal{O}_{V(\alpha)}]) = \sum_{\beta, \gamma} c_{\beta, \gamma} [\mathcal{O}_{V(\beta)}] \otimes [\mathcal{O}_{V(\gamma)}]$ with $c_{\beta, \gamma} \in \mathbb{Q}$, we have*

$$(f \cdot g)(\alpha) = \sum_{\beta, \gamma} c_{\beta, \gamma} f(\beta) g(\gamma).$$

5.6. Decomposing diagonals and product formulas

By using Corollary 5.5, we can compute products of Grothendieck weights if we are able to provide suitable expressions for $\delta_*([\mathcal{O}_{V(\alpha)}])$. Outside of the smooth case where one can use Poincaré duality, we do not know an easy way to do this. Since we have already addressed how to explicitly describe the Riemann-Roch transformation for a complete simplicial complex toric variety in Section 4, we apply it to finding an expression for $\delta_*([\mathcal{O}_{V(\alpha)}])$ in terms of $[\mathcal{O}_{V(\beta)}] \otimes [\mathcal{O}_{V(\gamma)}]$. We recall that by Proposition 2.9, the ring $\text{GW}(\Delta)$ is independent of the underlying field, so we can use an expression for $\delta_*([\mathcal{O}_{V(\alpha)}])$ over the complex numbers to calculate products of operational classes of complete toric varieties over arbitrary fields.

Let Δ be a complete simplicial fan, and suppose that f and g in $\text{GW}(\Delta)$ are given. Their product may be calculated explicitly via the formula in the next theorem. To undertake the calculation we choose a generic vector $v \in N$. Then for three cones α , β , and γ satisfying $\alpha \subset \beta \cap \gamma$, we define $m_{\beta, \gamma}^\alpha$ in the same manner as [12], by

$$m_{\beta, \gamma}^\alpha = \begin{cases} [N : \mathbb{Z} \cdot \beta + \mathbb{Z} \cdot \gamma] & \text{if } \beta \cap (\gamma + v) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Though it is suppressed in the notation, we emphasize that $m_{\beta,\gamma}^\alpha$ depends on the choice of v .

Theorem 5.7. *Let $f, g : \Delta \rightarrow \mathbb{Z}$ be Grothendieck weights on a complete simplicial fan Δ . Let $\tau_{\dagger}^*$ be a lift of the Riemann-Roch map on the associated toric variety over $\mathbb{C}$, which induces $\mu_\alpha(\beta), \nu_\alpha(\beta)$ as in Definition 4.12.*

Then, the product $h = f \cdot g$ is equal to

$$h(\alpha) = \sum_{\beta \succ \alpha} \mu_\alpha(\beta) \sum_{\substack{\gamma, \epsilon \succ \beta, \\ \text{codim}(\gamma) + \text{codim}(\epsilon) \\ = \text{codim}(\beta)}} m_{\gamma, \epsilon}^\beta \sum_{\substack{\zeta \succ \gamma, \\ \eta \succ \epsilon}} \nu_\gamma(\zeta) \nu_\epsilon(\eta) f(\zeta) g(\eta).$$

We point out that $h(\alpha)$ is indeed an integer. Grothendieck weights over the integers include into Grothendieck weights over the rationals as a subring, so the product $f \cdot g$ is unambiguously a weight over the integers, even if we use rational numbers to compute it.

Proof. Recall that $\delta : X \rightarrow X \times X$ is the diagonal map. Then, we have

$$\begin{aligned} \delta_*([\mathcal{O}_{V(\alpha)}]) &= \delta_*(\tau_X^{-1}(\tau_X([\mathcal{O}_{V(\alpha)}]))) \\ &= \tau_{X \times X}^{-1} \left(\sum_{\beta \succ \alpha} \mu_\alpha(\beta) (\delta_*([V(\beta)])) \right). \end{aligned}$$

By [12, Theorem 4.2], we may use the $m_{\gamma, \epsilon}^\beta$ determined by the generic vector v to decompose each $\delta_*([V(\beta)])$, obtaining

$$\begin{aligned} &\tau_{X \times X}^{-1} \left(\sum_{\beta \succ \alpha} \mu_\alpha(\beta) \delta_*([V(\beta)]) \right) \\ &= \tau_{X \times X}^{-1} \left(\sum_{\beta \succ \alpha} \mu_\alpha(\beta) \sum_{\substack{\gamma, \epsilon \succ \beta, \\ \text{codim}(\gamma) + \text{codim}(\epsilon) \\ = \text{codim}(\beta)}} m_{\gamma, \epsilon}^\beta [V(\gamma) \times V(\epsilon)] \right) \\ &= \sum_{\beta \succ \alpha} \mu_\alpha(\beta) \sum_{\substack{\gamma, \epsilon \succ \beta, \\ \text{codim}(\gamma) + \text{codim}(\epsilon) \\ = \text{codim}(\beta)}} m_{\gamma, \epsilon}^\beta \tau_{X \times X}^{-1}([V(\gamma) \times V(\epsilon)]). \end{aligned}$$

But then,

$$\tau_{X \times X}^{-1}([V(\gamma) \times V(\epsilon)]) = \tau_X^{-1}([V(\gamma)]) \otimes \tau_X^{-1}([V(\epsilon)])$$

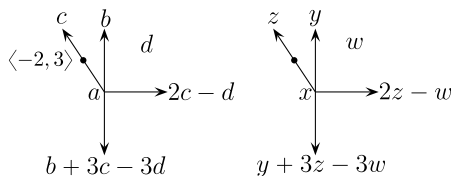


Fig. 1. Two Grothendieck weights on the fan Δ of X .

$$= \sum_{\substack{\zeta \succ \gamma, \\ \eta \succ \epsilon}} \nu_\gamma(\zeta) \nu_\epsilon(\eta) [\mathcal{O}_{V(\zeta)}] \otimes [\mathcal{O}_{V(\eta)}].$$

So, by Corollary 5.5 the theorem follows. $\square$

We can use Theorem 5.7 to show the following basic observations about the structure of $\text{GW}(\Delta)$.

Proposition 5.8. *Let Δ be a complete fan, and Σ a sub-fan. The set of Grothendieck weights on Δ that vanish on the complement of Σ forms an ideal in $\text{GW}(\Delta)$.*

Proof. Let $f, g : \Delta \rightarrow \mathbb{Z}$ be weights such that f vanishes on the cones of Δ . We first assume that Δ is simplicial. There are some coefficients C such that

$$(f \cdot g)(\alpha) = \sum_{\beta \succ \alpha} \sum_{\substack{\gamma, \epsilon \succ \beta, \\ \text{codim}(\gamma) + \text{codim}(\epsilon) = \text{codim}(\beta)}} \sum_{\substack{\zeta \succ \gamma, \\ \eta \succ \epsilon}} C_{\alpha, \beta, \gamma, \epsilon} f(\zeta) g(\eta).$$

If α is not in Σ , then since $\alpha \prec \zeta$ and Σ is a fan, ζ is not in Σ . Thus $f(\zeta)$ is 0 for each term in the sum.

Now, let Δ be an arbitrary complete fan. Let Δ' be a simplicial refinement and Σ' the compatible refinement of Σ . Then the pull-back of $f \cdot g$ on Δ' vanishes on the cones in Σ' , so $f \cdot g$ vanishes on Σ . $\square$

Corollary 5.9. *For a complete fan Δ , the ring $\text{GW}(\Delta)$ is filtered by ideals I_k consisting of weights that vanish on cones of codimension less than k .*

Example 5.10. We calculate the product of Grothendieck weights on a singular toric surface. Let X be the complete toric surface with rays $\rho_1 = \mathbb{R}_{\geq 0} \cdot \langle 0, 1 \rangle$, $\rho_2 = \mathbb{R}_{\geq 0} \cdot \langle 1, 0 \rangle$, $\rho_3 = \mathbb{R}_{\geq 0} \cdot \langle 0, -1 \rangle$, and $\rho_4 = \mathbb{R}_{\geq 0} \cdot \langle -2, 3 \rangle$. The first weight has value d on all maximal cones, and the second has value w . To calculate the product, we need a $\tau_{\dagger}$ that lifts the Riemann-Roch transformation τ_{X_1} . A $\tau_{\dagger}$ as described in Section 4 depends on choosing a complete flag. In two dimensions, this is merely the data of a vector, so we pick the vector $(1, 1)$. The resulting Riemann-Roch lift is shown in Fig. 2.

We must also select a displacement vector v which specifies the values of $m_{\beta, \gamma}^\alpha$. If we choose e.g. $v = (5, 1)$, then $m_{\rho_1, \rho_4}^0 = 2$, $m_{\rho_2, \rho_3}^0 = m_{0, \sigma_{34}}^0 = m_{\sigma_{12}, 0}^0 = m_{\rho_1, \sigma_{14}}^{\rho_1} = m_{\sigma_{12}, \rho_1}^{\rho_1} =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{5}{12} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{12} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{12} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\ \frac{7}{15} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{30} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Fig. 2. The matrix for a lift of the Riemann-Roch transformation on the Hirzebruch surface.

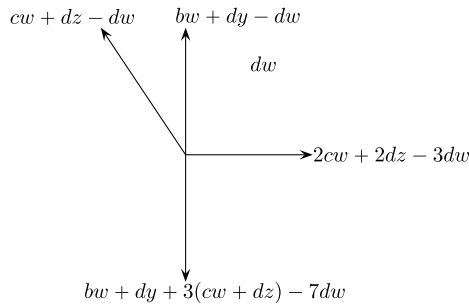


Fig. 3. The product of the two weights in Fig. 1.

$m_{\rho_2, \sigma_{23}}^{\rho_2} = m_{\sigma_{12}, \rho_2}^{\rho_2} = m_{\rho_3, \sigma_{34}}^{\rho_3} = m_{\sigma_{23}, \rho_3}^{\rho_3} = m_{\rho_4, \sigma_{34}}^{\rho_4} = m_{\sigma_{14}, \rho_4}^{\rho_4} = 1$, and the other $m_{\beta, \gamma}^{\alpha}$ are zero. The resulting weight is in Fig. 3, with a value of $aw - 2bw - 8cw + 9dw + dx + 2cy - 2dy + 2bz + 6cz - 8dz$ on the origin.

6. Maps to Grothendieck weights

For a complete fan Δ , there are a few other rings which have geometric origins. The two which we will discuss are the Minkowski weights $\text{MW}^*(\Delta)$ introduced in [12], and the ring of piecewise exponential functions $\text{PExp}(\Delta)$ introduced in [4]. Both of these rings have maps to $\text{GW}(\Delta)$ which correspond to maps of operational theories. We give formulas for these maps and explore some consequences. In the first subsection, we restrict to complete simplicial toric varieties over $\mathbb{C}$, but return to arbitrary complete toric varieties afterwards, in Subsection 6.6.

6.1. Minkowski weights and $\text{GW}(\Delta)$

Let Δ be a complete fan of dimension n . Recall that for $\alpha \in \Delta(k)$ and $\beta \in \Delta(k+1)$, $\beta \succ \alpha$, the generator of the semigroup $\overline{\beta} \cap N(\alpha)$ is denoted by $v_{\beta, \alpha}$. The group of

codimension- i Minkowski weights $\text{MW}^i(\Delta)$ is the set of functions f on $\Delta(n-i)$ that satisfy

$$\sum_{\substack{\beta \in \Delta(n-i), \\ \beta \succ \alpha}} \langle u, v_{\beta, \alpha} \rangle f(\beta) = 0,$$

for each $\alpha \in \Delta(n-i-1)$ and $u \in M(\alpha)$. Fulton and Sturmfels introduced these weights in [12] because $\text{MW}^i(\Delta)$ is naturally isomorphic to $A_i(X)^\vee$ and the Chow cohomology group $A^i(X)$ (compare with Proposition 3.3).

Proposition 6.2. *Let X be a complete simplicial toric variety over $\mathbb{C}$. Let $T : \text{MW}^*(\Delta) \rightarrow \text{GW}(\Delta)_{\mathbb{Q}}$ be the map induced from the isomorphism $\tau_X : K_{\circ}(X)_{\mathbb{Q}} \rightarrow A_*(X)_{\mathbb{Q}}$. Then $T(f) = g \in \text{GW}(\Delta)_{\mathbb{Q}}$ has the formula*

$$g(\alpha) = \sum_{\alpha \prec \beta} \mu_{\alpha}(\beta) f(\beta),$$

with $\mu_{\alpha}(\beta)$ as given in Definition 4.12.

This directly follows from the formula the Riemann-Roch transformation $\tau_X : K_{\circ}(X) \rightarrow A_*(X)_{\mathbb{Q}}$ given in Corollary 4.10. The map T can also be identified with the inverse of the operational Riemann-Roch map of [1].

Remark 6.3. It is possible to use the previous proposition to algorithmically calculate the inverse image under T of a Grothendieck weight. Let $g \in \text{GW}(\Delta)$ be a Grothendieck weight, and suppose that $g \in I_k$. The balancing conditions of Theorem 4.14 combined with Proposition 2.10 imply that $g|_{\Delta(n-k)}$ is a Minkowski weight. Then, $g - T(g|_{\Delta(n-k)})$ is an element of $\text{GW}(\Delta)_{\mathbb{Q}}$, and is in $(I_{k+1})_{\mathbb{Q}}$. One may repeat this process to obtain that $g - T(g|_{\Delta(n-k)}) - T((g - T(g|_{\Delta(n-k)}))|_{\Delta(n-k-1)}) \in (I_{k+1})_{\mathbb{Q}}$, and so on. After $n-k$ iterates, we obtain an identity of the form $g - T(g|_{\Delta(n-k)}) - T((g - T(g|_{\Delta(n-k)}))|_{\Delta(n-k-1)}) - \dots = 0$. Applying T^{-1} produces a formula for $T^{-1}(g)$.

Definition 6.4. For $f \in \text{MW}^k(\Delta)$ we say that an element $g \in \text{GW}(\Delta)$ *lifts* f if $g \in I_k$ and $g|_{\Delta(n-k)} = f$.

Suppose that $f \in \text{MW}^k(\Delta)$. Then for example, $T(f)$ will be contained in $(I_k)_{\mathbb{Q}}$ and will satisfy $T(f)|_{\Delta(n-k)} = f$, but will not generally be an element of $\text{GW}(\Delta)$.

Our next proposition is a sufficient condition for existence of lifts. We no longer need to work over $\mathbb{C}$ and let the base field $\mathbb{k}$ be arbitrary. Let F_i be the i -th piece of the dimension filtration on the Grothendieck group, meaning that it is generated by coherent sheaves with support of dimension at most i .

Proposition 6.5. *Let X be a complete toric variety. Suppose F_k is saturated as a subgroup of $K_{\circ}(X)$. Then every $f \in \text{MW}^k(\Delta)$ has a lift in $\text{GW}(\Delta)$.*

Proof. We have the exact sequence

$$0 \rightarrow F_k/F_{k-1} \rightarrow K_{\circ}(X)/F_{k-1} \rightarrow K_{\circ}(X)/F_k \rightarrow 0.$$

The long exact sequence obtained after applying $(-)^{\vee} = \text{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ is:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (K_{\circ}(X)/F_k)^{\vee} & \longrightarrow & (K_{\circ}(X)/F_{k-1})^{\vee} & \longrightarrow & (F_k/F_{k-1})^{\vee} \longrightarrow \\ & & & & & & \searrow \\ & & & & & & \text{Ext}_{\mathbb{Z}}^1(K_{\circ}(X)/F_k, \mathbb{Z}) \longrightarrow \text{Ext}_{\mathbb{Z}}^1(K_{\circ}(X)/F_{k-1}, \mathbb{Z}) \longrightarrow \text{Ext}_{\mathbb{Z}}^1(F_k/F_{k-1}, \mathbb{Z}). \end{array}$$

Let us consider the first few terms. $(K_{\circ}(X)/F_{k-1})^{\vee}$ may be naturally identified with the ideal I_k of Grothendieck weights which vanish on cones of codimension less than k , as defined in Corollary 5.9. On the other hand, F_k/F_{k-1} is the k -th piece of graded K -theory. The map $A_k(X) \rightarrow F_k/F_{k-1}$ sending $[V]$ to $[\mathcal{O}_V]$ is surjective by dévissage, and has torsion kernel since it is an isomorphism after tensoring with $\mathbb{Q}$ (see [9, Chapter 18]). Thus, $(F_k/F_{k-1})^{\vee} \cong \text{MW}^k(\Delta)$, and the first few terms in the exact sequence become:

$$0 \rightarrow I_{k+1} \rightarrow I_k \rightarrow \text{MW}^k(\Delta) \rightarrow \dots$$

Suppose that a Minkowski weight $f \in \text{MW}^k(\Delta)$ does not have a lift. Then, I_k cannot surject onto $\text{MW}^k(\Delta)$. Thus, the group $\text{Ext}_{\mathbb{Z}}^1(K_{\circ}(X)/F_k, \mathbb{Z})$ cannot be trivial. But this group is isomorphic to the torsion subgroup of $K_{\circ}(X)/F_k$, which is trivial if and only if F_k is saturated as a subgroup of $K_{\circ}(X)$. $\square$

6.6. Piecewise exponential functions and $\text{GW}(\Delta)$

Let the base field $\mathbb{k}$ be arbitrary. Let $R(T)$ be the group algebra $\mathbb{Z}[e^m | m \in M]$. It is a basic fact that $K_{\circ}^T(pt) = R(T)$. The ring of piecewise exponential functions $\text{PExp}(\Delta)$ is the ring of continuous functions on Δ that are given on each cone $\alpha \in \Delta$ by an element of $R(T)$. Anderson and Payne showed that for any fan Δ and associated toric variety X over $\mathbb{k}$, this ring is naturally isomorphic to $\text{op}K_T^{\circ}(X)$. If Δ is complete, $\text{PExp}(\Delta)$ has a map to $\text{GW}(\Delta)$ induced by the forgetful map $\text{op}K_T^{\circ}(X) \rightarrow \text{op}K^{\circ}(X)$, which we now describe.

We require K -theoretic equivariant multiplicities $\epsilon_p^K(V(\alpha))$, for $p \in X^T$. These have been recently introduced in [1]. They satisfy the identity

$$\sum_{p \in X^T} \epsilon_p(V(\alpha)) [i_{p*}(\mathcal{O}_p)] = [\mathcal{O}_{V(\alpha)}],$$

for $i_p : p \hookrightarrow X$ the inclusion.

Theorem 6.7. *Let X be a complete toric variety. There is a commuting square*

$$\begin{array}{ccc} \mathrm{op}K_T^\circ(X(\Delta)) & \xrightarrow{\cong} & \mathrm{PExp}(\Delta) \\ \downarrow \text{forgetful} & & \downarrow \text{forgetful} \\ \mathrm{op}K^\circ(X(\Delta)) & \xrightarrow{\cong} & \mathrm{GW}(\Delta) \end{array}$$

in which the forgetful map from $\mathrm{PExp}(\Delta)$ to $\mathrm{GW}(\Delta)$ sends a piecewise-exponential function ϕ to the limit of the function

$$\alpha \rightarrow \sum_{\sigma \in \Delta(n)} \epsilon_{V(\sigma)}^K(V(\alpha))\phi|_\sigma,$$

as the argument of ϕ approaches $0 \in N$.

Proof. If ϕ is a piecewise exponential function, then via the isomorphism in [2, Theorem 6.1], ϕ corresponds to a $R(T)$ -linear function $\phi_{lin} : K_o^T(X) \rightarrow R(T)$. The function ϕ_{lin} can be written explicitly via the projection formula:

$$\begin{aligned} \phi_{lin}([\mathcal{O}_{V(\alpha)}]) &= \phi_{lin}\left(\sum_{p \in X^T} \epsilon_p^K(V(\alpha))[i_{p*}(\mathcal{O}_p)]\right), \\ &= i_{X^T}^* \phi_{lin}\left(\sum_{p \in X^T} \epsilon_p^K(V(\alpha))[\mathcal{O}_p]\right), \\ &= \sum_{p \in X^T} \epsilon_p^K(V(\alpha))i_p^* \phi_{lin}([\mathcal{O}_p]) = \sum_{p \in X^T} \epsilon_p^K(V(\alpha))\phi|_{\sigma_p}, \end{aligned}$$

where σ_p is the maximal cone corresponding to p .

The forgetful map from $\mathrm{op}K_T^\circ(X)$ to $\mathrm{op}K^\circ(X)$ is induced by the projection $X \times T \rightarrow X$, so it is the pullback from $\mathrm{op}K_T^\circ(X)$ to $\mathrm{op}K_T^\circ(X \times T) \cong \mathrm{op}K^\circ(X)$. Via the identification of $\mathrm{op}K_T^\circ(X)$ with $R(T)$ -linear maps from $K_o^T(X)$ to $R(T)$, and $\mathrm{op}K^\circ(X)$ with $K_o(X)^\vee$, the forgetful map sends $\phi_{lin} : K_o^T(X) \rightarrow R(T)$ to the linear function on $K_o(X)$ sending $[\mathcal{O}_{V(\alpha)}]$ to the equivalence class in $\mathbb{Z}$ of $\phi([\mathcal{O}_{V(\alpha)}])$ (see the appendix of [1] for more details). This is the same as taking the limit as the argument of ϕ approaches $0 \in N$. $\square$

Example 6.8. This example and the following corollary are analogous to [14, Example 4.1 and Theorem 1.5]. Let X be the toric variety whose fan Δ is the complete fan in $N = \mathbb{Z}^2$ with rays $(\pm 1, \pm 1)$. In this case, a generating set for $\mathrm{PExp}(\Delta)$ over $R(T)$ is given by the functions in Fig. 4.

Let ϕ be the piecewise exponential function on the top right of Fig. 4, and let g be its image in $\mathrm{GW}(\Delta)$ under the forgetful map. Since the equivariant multiplicity of a point is just 1, the value of g on any maximal cone is just the value of ϕ at 0, which is 0.

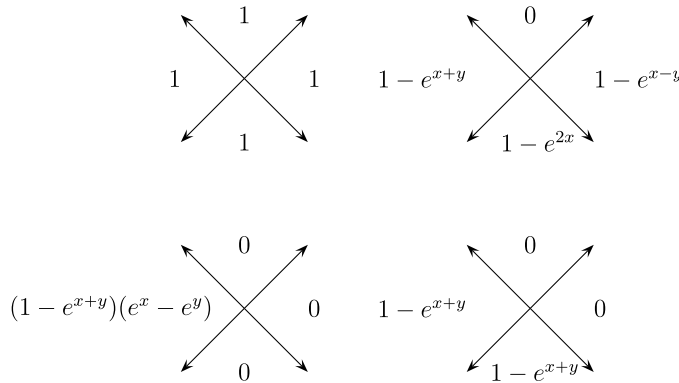


Fig. 4. Generators for the ring of piecewise exponential functions on a toric surface.

Let ρ be the ray generated by $(1, 1)$. Then $V(\rho)$ is a $\mathbb{P}^1$, and at the fixed point corresponding to the maximal cone σ generated by $(1, 1)$ and $(1, -1)$ the character on the tangent space is $y - x$, so the equivariant multiplicity $\epsilon_{v(\sigma)}(V(\rho))$ is $\frac{1}{1 - e^{x-y}}$, by [1, Proposition 6.3]. Let σ' be the maximal cone generated by $(1, 1)$ and $(-1, 1)$. At $V(\sigma')$ in $V(\rho)$, the character is $x - y$, and so $\epsilon_{V(\sigma')}(V(\rho)) = \frac{1}{1 - e^{y-x}}$. The value of g on ρ is then the limit of $\frac{0}{1 - e^{y-x}} + \frac{1 - e^{x-y}}{1 - e^{x-y}}$ as x and y approach 0, which is 1. Similarly, one gets that the value of g on the ray generated by $(1, -1)$ is -1 . The balancing conditions for Grothendieck weights determine the values on the other rays.

Now, we calculate $g(\{0\})$. Since X is singular at each fixed point, we can compute the equivariant multiplicity at the fixed point $V(\sigma)$ by resolving, e.g. by adding the ray $(1, 0)$, and then summing over the new fixed points which map to $V(\sigma)$. One gets

$$\epsilon_{V(\sigma)}(X) = \frac{1}{(1 - ey)(1 - e^{-x-y})} + \frac{1}{(1 - e^{-y})(1 - e^{x+y})} = \frac{1 + e^x}{(1 - e^{x+y})(1 - e^{x-y})}.$$

Let the cone generated by $(-1, -1)$, $(1, -1)$ be σ'' , and let the cone generated by $(-1, 1)$ and $(-1, -1)$ be σ''' . Then

$$\begin{aligned}\epsilon_{V(\sigma'')}(X) &= \frac{1 + e^{-y}}{(1 - e^{x-y})(1 - e^{-x-y})}, \\ \epsilon_{V(\sigma''')}(X) &= \frac{1 + e^{-x}}{(1 - e^{-x-y})(1 - e^{y-x})}.\end{aligned}$$

So, $g(\alpha)$ is the limit of

$$\frac{(1 - e^{x-y})(1 + e^x)}{(1 - e^{x+y})(1 - e^{x-y})} + \frac{(1 - e^{2x})(1 + e^{-y})}{(1 - e^{x-y})(1 - e^{-x-y})} + \frac{(1 - e^{x+y})(1 + e^{-x})}{(1 - e^{-x-y})(1 - e^{y-x})},$$

as the parameters x and y approach 0, which is 2.

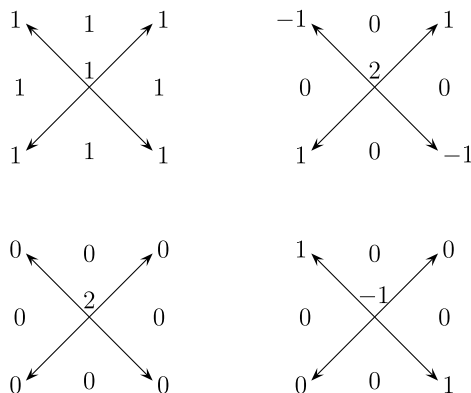


Fig. 5. Images of the piecewise exponential functions in Fig. 4.

This example shows the following (compare with [14, Theorem 1.5] and [1, Theorem 1.7]):

Corollary 6.9. *There is a projective toric surface with a vector bundle which cannot be resolved by a finite sequence of vector bundles that admit a T -equivariant structure.*

Proof. In Example 6.8, X is a complete toric surface, so it is projective. The $\mathbb{Z}$ -linear span of the Grothendieck weights in Fig. 5 does not include the Grothendieck weight with 1 at the origin and 0 elsewhere, so $\text{PExp}(\Delta)$ does not surject onto $\text{GW}(\Delta)$. Thus, the forgetful map from $\text{op}K_T^\circ(X)$ to $\text{op}K^\circ(X)$ is not surjective. Since vector bundles induce linear forms on coherent sheaves by tensor product followed by pushforward to a point, there is a commutative square:

$$\begin{array}{ccc} K_T^\circ(X) & \longrightarrow & \text{op}K_T^\circ(X) \\ \downarrow & & \downarrow \\ K^\circ(X) & \longrightarrow & \text{op}K^\circ(X) \end{array}.$$

Combining [2, Proposition 7.4] and [5, Proposition 5.6] shows that the bottom map is surjective. Comparing the two ways of traversing the diagram, one sees that the map $K_T^\circ(X) \rightarrow K^\circ(X)$ cannot be surjective. This proves the corollary. $\square$

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Appendix A. Multiplicities of cones

Given simplicial cones $\alpha \prec \beta$, let $\text{mult}_\alpha(\beta)$ denote the multiplicity of $\bar{\beta}$ in N_α . If $\alpha = \{0\}$, then $\text{mult}_\alpha(\beta) = \text{mult}(\beta)$ is the usual multiplicity of β . The following lemma describes relative multiplicities of simplicial cones in terms of usual multiplicities. Let α have rays $\rho_1, \dots, \rho_k$, and β have rays $\rho_1, \dots, \rho_k, \rho_{k+1}, \dots, \rho_l$:

Lemma A.1. $\text{mult}_\alpha(\beta) = \frac{\text{mult}(\beta)}{\text{mult}(\alpha)} \prod_{i=k+1}^l \frac{\text{mult}(\alpha)}{\text{mult}(\alpha + \rho_i)}$.

Proof. To simplify notation, we assume that β is a maximal cone. For $v_i \in N$, let $\langle v_1, \dots, v_l \rangle$ denote the sublattice of N generated by $v_1, \dots, v_l$. We have the following diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \langle v_{\rho_1}, \dots, v_{\rho_l} \rangle & \longrightarrow & N & \longrightarrow & N / \langle v_{\rho_1}, \dots, v_{\rho_l} \rangle \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \langle v_{\overline{\rho_{k+1}}}, \dots, v_{\overline{\rho_l}} \rangle & \longrightarrow & N_\alpha & \longrightarrow & N_\alpha / \langle v_{\overline{\rho_{k+1}}}, \dots, v_{\overline{\rho_l}} \rangle \longrightarrow 0, \end{array}$$

where the top and bottom quotient groups on the right have cardinality $\text{mult}(\beta)$ and $\text{mult}_\alpha(\beta)$ respectively. We add the kernels and cokernels to the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \langle v_{\rho_1}, \dots, v_{\rho_k} \rangle & \longrightarrow & N^\alpha & \longrightarrow & A \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \langle v_{\rho_1}, \dots, v_{\rho_l} \rangle & \longrightarrow & N & \longrightarrow & N / \langle v_{\rho_1}, \dots, v_{\rho_l} \rangle \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \langle v_{\overline{\rho_{k+1}}}, \dots, v_{\overline{\rho_l}} \rangle & \longrightarrow & N_\alpha & \longrightarrow & N_\alpha / \langle v_{\overline{\rho_{k+1}}}, \dots, v_{\overline{\rho_l}} \rangle \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & B & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow 0. \end{array}$$

By the snake lemma, the sequence of kernels leading to cokernels is exact. The cokernel of the map $\langle v_{\rho_1}, \dots, v_{\rho_k} \rangle \rightarrow N^\alpha$ has cardinality $\text{mult}(\alpha)$, so $|A| = \text{mult}(\alpha)|B|$. Thus $\text{mult}(\beta) = \text{mult}_\alpha(\beta)|A| = \text{mult}_\alpha(\beta) \text{mult}(\alpha)|B|$. The cardinality of B on the other hand is also easy to determine: the image of $\langle v_{\rho_1}, \dots, v_{\rho_l} \rangle$ in $\langle v_{\overline{\rho_{k+1}}}, \dots, v_{\overline{\rho_l}} \rangle$ is just $\langle \overline{v_{\rho_{k+1}}}, \dots, \overline{v_{\rho_l}} \rangle$. If $\overline{v_\rho} = b_\rho v_{\overline{\rho}}$, the cardinality of the cokernel (i.e. B) is $\prod_{i=k+1}^l b_{\rho_i}$. But in the proof of Proposition 4.7, we saw $b_\rho = \frac{\text{mult}(\alpha + \rho)}{\text{mult}(\alpha)}$, which proves the claim. $\square$

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