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Flat Tori with Large Laplacian Eigenvalues in Dimensions up to Eight*

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Abstract. We consider the optimization problem of maximizing the kth Laplacian eigenvalue, λ_k, over flat d-dimensional tori of fixed volume. For $k=1$, this problem is equivalent to the densest lattice sphere packing problem. For larger k, this is equivalent to the NP-hard problem of finding the d-dimensional (dual) lattice with the longest kth shortest lattice vector. As a result of extensive computations, for $d \leq 8$, we obtain a sequence of flat tori, $T_{k,d}$, each of volume one, such that the kth Laplacian eigenvalue of $T_{k,d}$ is very large; for each (finite) k the kth eigenvalue exceeds the value in (the $k\to\infty$ asymptotic) Weyl's law by a factor between 1.54 and 2.01, depending on the dimension. Stationarity conditions are derived and numerically verified for $T_{k,d}$, and we describe the degeneration of the tori as $k\to\infty$.

Key words. Laplace operator, flat tori, eigenvalue optimization, densest lattice sphere packing problem

MSC codes. 35P15, 49K35, 58J50, 52C17

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1. Introduction.

Lattice models play an essential role in a variety of applications, including describing periodic phenomena or arrangements in physical systems and in computational settings where they can be used as models to transmit or store data. In these settings, it is often of fundamental and practical interest to identify lattices with "extremal" properties. Specific examples of *extremal lattice problems* include the following:

1. In analog-to-digital data conversion, the *best quantizer problem* is to minimize the error when quantizing a source message, and it can be formulated as finding a lattice that minimizes the mean squared error on its Voronoi cell [9, p. 57]. Quantizers are used in, e.g., a typical medium- or long-distance landline telephone system. In designing channel code schemes, the *channel-coding problem* seeks to minimize the error in decoding a signal with added white noise inherently present due to physical signal transmission. This problem can be formulated as finding a lattice of determinant 1 that minimizes the error probability [9, p. 69]. In physical layer security, the objective is to properly take advantage of white noise to minimize the probability

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an eavesdropper can correctly decode a signal. This can be formulated as finding a lattice that minimizes a channel model-dependent value such as *Eve's correct decoding probability* [15], *secrecy gain* [38, 30, 5], etc..

2. Sphere packing problems can sometimes be formulated as extremal lattice problems [9]. For example, the lattice sphere packing density problem, the highest kissing number problem, and the thinnest covering problem are equivalent to finding lattices with the longest shortest vector, the highest count of short vectors, and Voronoi cells with minimal circumradius, respectively. These fundamental problems have been solved in certain low dimensions but are still outstanding in higher dimensions.
3. In a variety of applications, it is of fundamental and applied interest to bound eigenvalues of the Laplacian operator; applications include controlling the frequencies of mechanical vibrations [40, 19], maximizing heat transfer in conducting materials [10, 28, 20], bandgap design in photonic crystals [39, 12, 11, 23, 43], and optimal arrangements of resources in population dynamics [3, 31, 22]. Eigenvalues of the Laplacian operator are generally impossible to explicitly compute, except on domains with simple geometry such as spheres and tori. The explicit formula of eigenvalues on these simple domains are important in understanding the properties and structure of the spectrum. It is well known that there is a one-to-one correspondence between the Laplacian eigenvalues of a flat torus and the lengths of vectors in an associated lattice. Hence, the study of extremal eigenvalues on flat tori yields a simple but interesting model problem that can give insight into the extremal behavior of eigenvalues in these applications, especially in higher dimensions.

Motivated by these applications, there has been a variety of research in developing methods that can compute extremal lattices. In this paper, we consider a particular problem in this area related to eigenvalues of flat tori.

Eigenvalues of flat tori. Consider the d-dimensional lattice $\Gamma_B := B\mathbb{Z}^d$ generated by the basis matrix $B \in GL(d,\mathbb{R})$ and the d-dimensional flat torus $T_B := \mathbb{R}^d/\Gamma_B$. The volume of T_B is given by $\mathrm{vol}(T_B) = |\det B|$. Each eigenpair, (λ, ψ), of the Laplacian, $-\Delta$, on T_B corresponds to an element of the dual lattice, $\Gamma_B^* = B^{-t}\mathbb{Z}^d = \Gamma_{B^{-t}}$:

$$\lambda = 4\pi^2 \|w\|^2, \qquad \psi(x) = e^{2\pi i \langle x, w\rangle} \qquad \text{for all } x \in T_B,\ w \in \Gamma_B^*.$$

The multiplicity of each nonzero eigenvalue is even since $w \in \Gamma_B^*$ and $-w$ correspond to the same eigenvalue. It follows that the eigenvalues of $-\Delta$ on T_B, enumerated in increasing order including multiplicity,

$$0 = \lambda_0 < \lambda_1 = \lambda_2 \le \lambda_3 = \lambda_4 \le \cdots,$$

are characterized by the Courant–Fischer formula,

$$\lambda_k(T_B) = \min_{E \in \mathbb{Z}^d_{k+1}} \max_{v \in E} 4\pi^2 \|B^{-t} v\|^2, \tag{1}$$

where $\mathbb{Z}^d_k := \{E \subset \mathbb{Z}^d \colon |E| = k\}$. Since the multiplicity is even, throughout this manuscript, it will be convenient to use the notation $\kappa := 2\lceil \frac{k}{2} \rceil$, where $\lceil \cdot \rceil$ is the ceiling function.[1] For $k \in \mathbb{N}$, define the *volume-normalized Laplacian eigenvalue*, $\bar{\lambda}_{k,d} \colon GL(d,\mathbb{R}) \to \mathbb{R}$, by

[1] One may remove the systematic double multiplicities from the enumeration of the spectrum and obtain the so-called *desymmetrized spectrum*; however, we prefer to retain the repeated values.

$$\bar{\lambda}_{k,d}(B) = \lambda_k(T_B) \cdot \text{vol}(T_B)^{\frac{2}{d}}. \tag{2}$$

The volume-normalized eigenvalues are scale invariant in the sense that $\bar{\lambda}_{k,d}(\alpha B) = \bar{\lambda}_{k,d}(B)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$. Weyl's law states that, for any $B \in GL(d,\mathbb{R})$,

$$\bar{\lambda}_{k,d}(B) \sim g_d \; \pi^2 \; k^{\frac{2}{d}} \qquad \text{as } k \to \infty, \tag{3}$$

where $g_d = 4(\omega_d)^{-\frac{2}{d}}$ and $\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ is the volume of the unit ball in $\mathbb{R}^d$.

In this work, for fixed $k, d \in \mathbb{N}$, we consider the eigenvalue optimization problem

$$\Lambda_{k,d} = \max_{B \in GL(d,\mathbb{R})} \; \bar{\lambda}_{k,d}(B). \tag{4}$$

The existence of a matrix $B^\star$ attaining the maximum in (4) was proven in [26, Thm. 1.1]. The tori T_A and T_B are isometric if and only if A and B are equivalent in

$$O(d,\mathbb{R}) \setminus GL(d,\mathbb{R}) / GL(d,\mathbb{Z}).$$

Here, $O(d,\mathbb{R})$ is the group of orthogonal matrices, and $GL(d,\mathbb{Z})$ is the group of unimodular matrices. Since the Laplacian spectrum is preserved by isometry, it follows that the solution to the optimization problem in (4) is not unique. Uniqueness up to isometry has been proved only for dimensions $d = 1-8$ and 24 but remains an open problem for other dimensions [42, 44, 8]. Minkowski's first fundamental theorem implies that $\Lambda_{1,d} \leq 4\pi^2 d$; see, e.g., Theorem 22.1 and Corollary 22.1 in [16]. Together with the Courant–Fischer formula (1), this result implies that $\Lambda_{k,d} \leq d\pi^2\kappa^2$.

For general d and k, the maximizer in (4) is unknown. In dimension $d = 1$, it is easy to see that $\Lambda_{k,1} = \pi^2\kappa^2$. In dimension $d = 2$, it was shown by Berger that $\Lambda_{1,2} = \frac{8\pi^2}{\sqrt{3}}$ is attained by the basis $B_{1,2} = \left(\begin{smallmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{smallmatrix}\right)^{-t}$, which generates the equilateral torus [4]. It was shown in [21] that, for $k \geq 1$,

$$B_{k,2} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{\kappa^2-1}}{2} \end{pmatrix}^{-t} \tag{5}$$

is a local maximum with value $\bar{\lambda}_{k,2} = \frac{2\pi^2\kappa^2}{\sqrt{\kappa^2-1}}$. It is shown that this lattice is globally optimal for $k = 1,2,3,4$. For each k, the corresponding eigenvalue has multiplicity six, and as $k \to \infty$, the flat tori generated by these bases degenerate. In Figure 1 (top), we plot the ellipse

$$\left\{ v \in \mathbb{R}^2 \colon \|B_{k,2}^{-t} v\|^2 = \frac{\kappa^2}{4} \right\}$$

for $k = 1,3,5,7$. The kth shortest lattice vectors are indicated with a green dot, which by the Courant–Fischer formula (1) corresponds to λ_k. Note that, for each k, the ellipse intersects 6 points, corresponding to the multiplicity six kth eigenvalue. As k increases, the ellipses elongate in one direction.

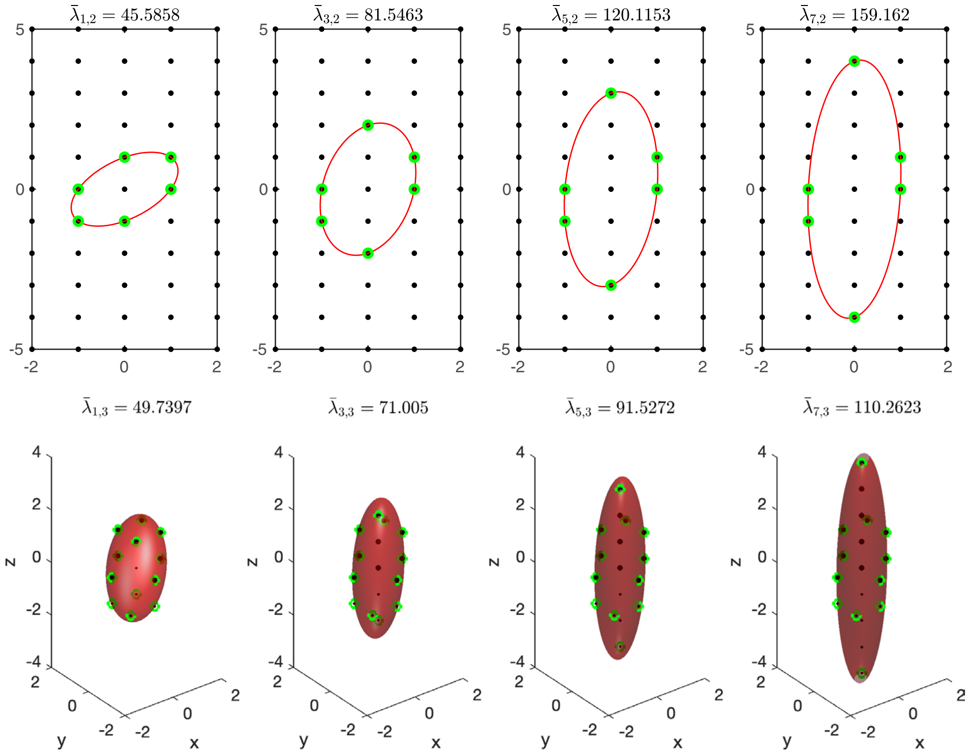

Figure 1. *The ellipses* $(d=2)$ *and ellipsoids* $(d=3)$ *corresponding to the Gram matrix* $G^{\circ}_{k,d}$ *for* $k=1,3,5,7$. *The intersecting lattice points are indicated.*

Principal eigenvalue. We first review the relationship between the principal volume-normalized eigenvalue and the *lattice sphere packing problem*. Recall that, for a given lattice Γ_A, the *density of a sphere packing with centers at* Γ_A is given by

$$\mathcal{P}(A) = \text{proportion of space that is occupied by the spheres.}$$

The *kissing number*, $\tau(A)$, associated with the sphere packing is the number of other spheres that each sphere touches.

Using the Courant–Fischer formula (1) with $k=1$, $\lambda_1(T_B) = \min_{E\in\mathbb{Z}^d\setminus\{0\}} 4\pi^2\|B^{-t}v\|^2$, we see that $\sqrt{\frac{\lambda_1}{4\pi^2}}$ is the length of the shortest vector in the lattice Γ^*_B. The density of a packing of balls with centers on the dual lattice Γ^*_B is

$$\mathcal{P}(B^*) = \frac{\text{volume of ball}}{\text{volume of fundamental region}} = \frac{\omega_d\rho^d}{|\det B^{-t}|} = \omega_d\rho^d|\det B|,$$

where ρ is the radius of the balls. Observing that the shortest vector in the lattice is exactly twice the radius of the ball packing, we have $\sqrt{\frac{\lambda_1}{4\pi^2}} = 2\rho$, giving $\rho^2 = \frac{\lambda_1}{16\pi^2}$. It then follows that $\mathcal{P}^{\frac{2}{d}} = \omega_d^{\frac{2}{d}}\rho^2(\det B)^{\frac{2}{d}} = \omega_d^{\frac{2}{d}}\frac{\lambda_1}{16\pi^2}(\det B)^{\frac{2}{d}} = \omega_d^{\frac{2}{d}}\frac{1}{16\pi^2}\bar{\lambda}_1$. Rearranging gives the following lemma.

Table 1

For dimensions $d = 1, \ldots, 8$, we tabulate the lattice with the largest known density, the corresponding kissing number $\tau(B^*)$, the density $\mathcal{P}(B^*)$, and the volume-normalized eigenvalue of the torus, $\bar{\lambda}_{1,d}(B)$. All values except $\bar{\lambda}_{1,d}(B)$ can be obtained from [9, Table 1.2].

d	Γ_B^*	$\tau(B^*)$	$\mathcal{P}(B^*)$	$\bar{\lambda}_{1,d}(B)$
1	A_1	2	1	$4\pi^2 \approx 39.4784$
2	A_2	6	$\frac{\pi}{2\sqrt{3}} \approx 0.9069$ [27]	$\frac{8\pi^2}{\sqrt{3}} \approx 45.5858$
3	$A_3 = D_3$	12	$\frac{\pi}{3\sqrt{2}} \approx 0.7405$ [14]	$4\pi^2 2^{\frac{1}{3}} \approx 49.7397$
4	D_4	24	$\frac{\pi^2}{16} \approx 0.6169$ [25]	$4\pi^2\sqrt{2} \approx 55.8309$
5	D_5	40	$\frac{4\pi^2}{15} 2^{-\frac{5}{2}} \approx 0.4653$ [25]	$4\pi^2 2^{\frac{3}{5}} \approx 59.8381$
6	E_6	72	$\frac{\pi^3}{48\sqrt{3}} \approx 0.3729$ [6]	$8\pi^2 3^{-\frac{1}{6}} \approx 65.7460$
7	E_7	126	$\frac{\pi^3}{105} \approx 0.2953$ [6]	$4\pi^2 2^{\frac{6}{7}} \approx 71.5131$
8	E_8	240	$\frac{\pi^4}{384} \approx 0.2537$ [6]	$8\pi^2 \approx 78.9568$

Lemma 1.1. *Let $B \in GL(d, \mathbb{R})$, and let $\bar{\lambda}_{1,d}(B) = \lambda_1(T_B) \cdot \mathrm{vol}(T_B)^{\frac{2}{d}}$ be the corresponding principal volume-normalized eigenvalue of the flat torus $T_B := \mathbb{R}^d/\Gamma_B$. Let $\mathcal{P}(B^*)$ be the packing density for the arrangement of balls with centers on the dual lattice, $\Gamma_B^* = B^{-t}\mathbb{Z}^d$. Then*

$$\bar{\lambda}_{1,d}(B) = 16\pi^2 \omega_d^{-\frac{2}{d}} \mathcal{P}(B^*)^{\frac{2}{d}}, \tag{6}$$

where ω_d denotes the volume of a d-dimensional ball. Furthermore, the kissing number, $\tau(B^*)$, of the packing is the multiplicity of $\lambda_1(T_B)$.

A consequence of Lemma 1.1 is that *the eigenvalue optimization problem in* (4) *for $k = 1$ is equivalent to finding the densest lattice packing of balls in d-dimensions.* We can also restate the eigenvalue problem in terms of the *Gram matrix* of $B_{k,d}^{-1}$, denoted

$$G_{k,d} = (B_{k,d})^{-1}(B_{k,d})^{-t} \in \mathbb{R}^{d \times d}.$$

When $k = 1$, (4) can be written as

$$\frac{\Lambda_{1,d}}{4\pi^2} = \max_{G \in \mathcal{S}_{>0}^d} \min_{v \in \mathbb{Z}^d \setminus 0} \frac{v^t G v}{\det G^{\frac{1}{d}}}, \tag{7}$$

where $\mathcal{S}_{>0}^d$ is the space of positive definite quadratic forms. The right-hand-side term of (7) is the so-called *Hermite constant*. It is known that finding Hermite's constant is equivalent to determining the densest lattice sphere packing [42, p. 6].

Much is known about the densest lattice packings for small dimensions, d [9]. In particular, this problem is NP-hard [1, 34], but the densest known lattices for dimension $d = 1, \ldots, 8$ are known via Voronoi's algorithm for the enumeration of perfect positive definite quadratic forms [42]. The corresponding largest volume-normalized Laplacian eigenvalues are tabulated in Table 1. We refer the reader to [9] for details about the lattices and to the website of Nebe [36] for explicit bases and Gram matrices for these lattices. Note that the multiplicity of $\bar{\lambda}_{1,d}$ for these flat tori is very large. We also note that recently, for dimension $d = 8$ and $k = 1$, the E_8 lattice was proven to be the maximizer of (4) using a different technique [44].

In this paper, we focus on dimensions $d \leq 8$, but we briefly remark that this problem for the principal volume-normalized eigenvalue has been studied in higher dimensions. In particular, [9] and [32] give the densest known lattices for higher dimensions, and the Leech lattice was proven to give the densest lattice sphere packing in dimension 24 [8].

Higher eigenvalues. From the above discussion, the optimization problem in (4) can be interpreted as finding the longest kth shortest lattice vector. For higher values of k, this problem has not been as well studied as $k=1$. Recently, Jean Lagacé observed that, using the test lattice basis $\tilde{B}_{k,d}^{-t} = (\frac{\kappa}{2})^{\frac{1}{d}} \mathrm{diag}(1,\ldots,1,\frac{2}{\kappa}) \in \mathbb{R}^{d\times d}$, one can obtain the lower bound on the maximal value,

$$\Lambda_{k,d} \geq \bar{\lambda}_{k,d}(\tilde{B}_{k,d}) = 2^{2-\frac{2}{d}}\ \pi^2\ \kappa^{\frac{2}{d}}, \qquad k,d\in\mathbb{N}. \tag{8}$$

Comparing (8) with Weyl's law (3), he observed that this is a meaningful bound if $\omega_d \leq 2 = \omega_1$, which holds for $2\leq d\leq 10$. He further proved that, for $2\leq d\leq 10$, the optimal tori degenerate as $k\to\infty$ [26].

Summary of main results. As a result of extensive computations, for dimensions $d = 2,\ldots,8$ and all $k\geq 1$, we have identified d-dimensional flat tori $T_{k,d}^{\circ} := \mathbb{R}^d/\Gamma_{B_{k,d}^{\circ}}$, generated by lattices bases, $B_{k,d}^{\circ}$, which have a very large kth volume-normalized eigenvalue, $\bar{\lambda}_{k,d}^{\circ} := \bar{\lambda}_{k,d}(B_{k,d}^{\circ})$. The bases $B_{k,d}^{\circ}$ have the largest objective function for the optimization problem (4) that we were able to identify. Rather than report the basis matrices, we report the corresponding Gram matrices for $(B_{k,d}^{\circ})^{-1}$,

$$G_{k,d}^{\circ} = (B_{k,d}^{\circ})^{-1}(B_{k,d}^{\circ})^{-t} \in \mathbb{R}^{d\times d},$$

which have a nicer form. Define the $\mathbb{Z}^{8\times 8}$ matrix

$$\mathcal{G}_k = \begin{pmatrix} 2\kappa^2 & \kappa^2 & \kappa^2 & 0 & \kappa^2 & 0 & \kappa^2 & -4\\ \kappa^2 & 2\kappa^2 & 0 & 0 & 0 & 0 & \kappa^2 & -4\\ \kappa^2 & 0 & 2\kappa^2 & 0 & \kappa^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 2\kappa^2 & -\kappa^2 & \kappa^2 & -\kappa^2 & 0\\ \kappa^2 & 0 & \kappa^2 & -\kappa^2 & 2\kappa^2 & 0 & \kappa^2 & 0\\ 0 & 0 & 0 & \kappa^2 & 0 & 2\kappa^2 & -\kappa^2 & 0\\ \kappa^2 & \kappa^2 & 0 & -\kappa^2 & \kappa^2 & -\kappa^2 & 2\kappa^2 & -4\\ -4 & -4 & 0 & 0 & 0 & 0 & -4 & 8 \end{pmatrix}.$$

The Gram matrix $G_{k,d}^{\circ}$ is defined to be the $d\times d$ lower-right submatrix of $\mathcal{G}_k$ for each $k\geq 1$. A lattice basis $B_{k,d}^{\circ}$ can be recovered from $G_{k,d}^{\circ}$ via the Cholesky decomposition. The nesting of the Gram matrices is a result of the dual lattices generated by the basis $(B_{k,d}^{\circ})^{-t}$ being *laminated*, i.e., $(B_{k,d}^{\circ})^{-t} = \left(\begin{smallmatrix} b & - & 0 & - \\ & (B_{k,d-1}^{\circ})^{-t} \end{smallmatrix}\right)$ for some *gluing vector* $b\in\mathbb{R}^d$. For example, in dimension $d=2$ we have $G_{k,2}^{\circ} = \left(\begin{smallmatrix} 2\kappa^2 & -4\\ -4 & 8\end{smallmatrix}\right) \propto B_{k,2}^{-1}B_{k,2}^{-t}$, where $B_{k,2}$ is defined in (5).

Table 2
$\frac{\bar{\lambda}^{\circ}_{k,d}}{h_d \pi^2}$, h_d, h_d/g_d, and eigenvalue multiplicities for $k=1$ and $k \geq 2$. See Numerical Observation 1.2 and the following discussion in section 1.

d	1	2	3	4	5	6	7	8
$\frac{\bar{\lambda}^{\circ}_{k,d}}{h_d \pi^2}$	κ^2	$\left(\frac{\kappa^4}{\kappa^2-1}\right)^{\frac{1}{2}}$	$\left(\frac{\kappa^4}{\kappa^2-\frac{4}{3}}\right)^{\frac{1}{3}}$	$\left(\frac{\kappa^4}{\kappa^2-2}\right)^{\frac{1}{4}}$	$\left(\frac{\kappa^4}{\kappa^2-2}\right)^{\frac{1}{5}}$	$\left(\frac{\kappa^4}{\kappa^2-\frac{5}{2}}\right)^{\frac{1}{6}}$	$\left(\frac{\kappa^4}{\kappa^2-\frac{8}{3}}\right)^{\frac{1}{7}}$	$\left(\frac{\kappa^4}{\kappa^2-3}\right)^{\frac{1}{8}}$
h_d	1	2	$4 \cdot 3^{-\frac{1}{3}}$	$2^{\frac{7}{4}}$	4	$2^{\frac{11}{6}}$	$4\left(\frac{16}{3}\right)^{\frac{1}{7}}$	$2^{\frac{5}{2}}$
h_d/g_d	1	$\frac{\pi}{2}$	$\frac{2}{3}2^{\frac{1}{3}}\pi^{\frac{2}{3}}$	$\pi 2^{-\frac{3}{4}}$	$\frac{2\sqrt[5]{2}\pi^{4/5}}{15^{2/5}}$	$\frac{\pi}{\sqrt{2}\sqrt[3]{3}}$	$\frac{2\ 2^{5/7}\pi^{6/7}}{3^{3/7}35^{2/7}}$	$\pi 6^{-\frac{1}{4}}$
$h_d/g_d \approx$	1	1.57	1.80	1.87	1.94	1.54	1.98	2.01
$k=1$ mult.	2	6	12	24	40	72	126	240
$k \geq 2$ mult.	2	6	12	22	38	62	106	182
$\lvert\det G^{\circ}_{k,d}\rvert/8$	1	$2(\kappa^2-1)$	$\kappa^2(3\kappa^2-4)$	$4\kappa^4(\kappa^2-2)$	$4\kappa^6(\kappa^2-2)$	$2\kappa^8(2\kappa^2-5)$	$\kappa^{10}(3\kappa^2-8)$	$2\kappa^{12}(\kappa^2-3)$

The following numerical observation[2] summarizes the results of numerous computations for the flat tori $T^{\circ}_{k,d}$, and their volume-normalized eigenvalues, $\bar{\lambda}^{\circ}_{k,d}$.

Numerical Observation 1.2. *For $k \geq 1$ and $d \leq 8$, the flat tori $T^{\circ}_{k,d}$ have kth volume-normalized eigenvalues $\bar{\lambda}^{\circ}_{k,d} := \bar{\lambda}_{k,d}(B^{\circ}_{k,d})$ as tabulated in the second row of Table 2. The multiplicity of the eigenvalues is given in the sixth and seventh rows of Table 2. The corresponding lattice vectors are of the form $\pm v$, where v is a vector tabulated in Table 3.*

Details on our computations supporting Numerical Observation 1.2 are given in section 2. Magma code with these supporting computations can be found at [24].

In Figure 1, we plot the ellipse/ellipsoid

$$\{v \in \mathbb{R}^d \colon v^t G^{\circ}_{k,d} v = 2\kappa^2\}$$

in dimensions $d=2$ and 3 for $k=1,3,5,7$. The kth shortest lattice vectors are indicated by a green dot. In $d=2$ dimensions, the ellipses intersect six lattice points, and in $d=3$ dimensions, the ellipsoids intersect 12 lattice points. In both dimensions, the ellipse/ellipsoid elongates as k increases in one direction; this is further discussed in subsection 4.3.

We plot k versus $\bar{\lambda}^{\circ}_{k,d}$ for $d \leq 8$ in Figure 2 and tabulate the first few values in Table 4. Using the observation that, for $a > 0$, $\frac{\kappa^4}{\kappa^2-a} \geq k^2$, we have that, for each dimension $d \leq 8$,

$$\bar{\lambda}^{\circ}_{k,d} \geq h_d\ \pi^2\ \kappa^{\frac{2}{d}} \qquad \text{for all } k \geq 1,$$

where $\kappa := 2\lceil \frac{k}{2} \rceil$ and h_d is a constant, which does not depend on k, as tabulated in the third row of Table 2. In particular, for $d \leq 8$, this shows that the optimal value in (4) satisfies

$$\Lambda_{k,d} \geq h_d\ \pi^2\ \kappa^{\frac{2}{d}} \qquad \text{for all } k \geq 1. \tag{9}$$

In the fourth row of Table 2, we compute the value of h_d/g_d, where g_d is the constant appearing in Weyl's law. *Depending on the dimension, the Laplace eigenvalues of $T^{\circ}_{k,d}$ exceed the value*

[2]In this paper, we will use the terminology "numerical observation" to succinctly state results that depend on numerical computations. "Theorem" will be reserved for statements that can be proven without numerical computation.

Table 3

For the $B^{\circ}_{k,d}$-lattice, kth shortest lattice vectors and indexing for $k = 1,2$ and $k \geq 3$. For $d < 8$, as indicated by horizontal lines, we use only vectors that are zero in the first $8 - d$ components. The italicized indices are discussed in subsection 4.2.

$k=1,2$	$k \geq 3$	*lattice vector*							
1	1	0	0	0	0	0	0	0	$k/2$
2	2	0	0	0	0	0	0	1	0
3	3	0	0	0	0	0	0	1	1
4	4	0	0	0	0	0	1	0	0
5	5	0	0	0	0	0	1	1	0
6	6	0	0	0	0	0	1	1	1
7	7	0	0	0	0	1	0	0	0
8	8	0	0	0	0	1	0	-1	0
9	9	0	0	0	0	1	-1	-1	0
10	10	0	0	0	0	1	0	-1	-1
11	11	0	0	0	0	1	-1	-1	-1
12		0	0	0	0	1	-1	-2	-1
13	12	0	0	0	1	0	0	0	0
14	13	0	0	0	1	0	-1	0	0
15	14	0	0	0	1	1	-1	0	0
16	15	0	0	0	1	1	0	0	0
17	16	0	0	0	1	0	0	1	1
18	17	0	0	0	1	0	0	1	0
19	18	0	0	0	1	1	-1	-1	-1
20	19	0	0	0	1	1	-1	-1	0
21	20	0	0	1	0	0	0	0	0
22	21	0	0	1	-1	-1	1	0	0
23	22	0	0	1	0	-1	0	0	0
34	23	0	0	1	-1	-1	0	0	0
25	24	0	0	1	0	-1	0	1	1
26	25	0	0	1	0	-1	0	1	0
27	26	0	0	1	0	-1	1	1	1
28	27	0	0	1	-1	-1	1	1	1
29	28	0	0	1	0	-1	1	1	0
30	29	0	0	1	-1	-1	1	1	0
31	30	0	0	1	-1	-2	1	1	1
32	31	0	0	1	-1	-2	1	1	0
33		0	0	1	-1	-2	2	2	1
34		0	0	1	0	-1	1	2	1
35		0	0	1	0	-2	1	2	1
36		0	0	1	-1	-2	1	2	1
37	32	0	1	0	0	0	0	0	0
38	33	0	1	0	0	0	-1	-1	0
39	34	0	1	0	0	0	0	-1	0
40	35	0	1	0	-1	0	0	-1	0
41	36	0	1	0	0	1	0	-1	0
42	37	0	1	-1	0	1	0	-1	0
43	38	0	1	0	0	1	-1	-1	0
44	39	0	1	-1	0	1	-1	-1	0
45	40	0	1	0	1	1	-1	-1	0
46	41	0	1	-1	1	1	-1	-1	0
47	42	0	1	-1	1	2	-1	-1	0
48	43	0	1	0	0	1	-1	-2	0
49	44	0	1	-1	0	1	-1	-2	0
50	45	0	1	0	0	1	-1	-2	-1
51	46	0	1	-1	0	1	-1	-2	-1
52	47	0	1	-1	1	2	-2	-2	0
53	48	0	1	-1	1	2	-2	-2	-1
54	49	0	1	0	0	0	0	0	1
55	50	0	1	-1	0	2	-1	-2	0
56	51	0	1	-1	1	2	-1	-2	0
57	52	0	1	-1	0	2	-1	-2	-1
58	53	0	1	-1	1	2	-1	-2	-1
59		0	1	-2	1	3	-2	-3	-1
60		0	1	-1	1	2	-2	-3	-1
61		0	1	-1	0	2	-2	-3	-1
62		0	1	-1	0	2	-1	-3	-1
63		0	1	-1	1	3	-2	-3	-1
64	54	1	0	0	0	0	0	0	0
65	55	1	0	0	-1	-1	0	-1	0
66	56	1	0	0	0	0	-1	-1	0
67	57	1	1	-1	0	1	-1	-2	0
68	58	1	0	0	0	0	0	-1	0
69	59	1	0	0	-1	0	0	-1	0
70	60	1	0	-1	0	0	-1	-1	0
71	61	1	-1	0	-1	-1	1	0	0
72	62	1	-1	0	0	-1	0	0	0
73	63	1	-1	0	-1	-1	0	0	0
74	64	1	0	-1	0	0	0	-1	0
75	65	1	0	-1	-1	0	0	-1	0
76	66	1	-1	-1	0	0	0	0	0
77	67	1	-1	0	0	0	0	0	0
78	68	1	0	-1	0	1	0	-1	0
79	69	1	0	-1	0	1	-1	-1	0
80	70	1	0	-1	1	1	-1	-1	0
81	71	1	-1	0	-1	-2	1	1	0
82	72	1	-1	0	0	-1	0	1	0
83	73	1	-1	0	0	-1	0	1	1
84	74	1	0	-1	0	1	-1	-2	-1
85	75	1	0	-1	0	1	-1	-2	0
86	76	1	-1	0	0	-1	1	1	0
87	77	1	-1	0	-1	-1	1	1	0
88	78	1	-1	0	0	-1	1	1	1
89	79	1	-1	0	-1	-1	1	1	1
90	80	1	0	0	-1	-1	1	0	0
91	81	1	-1	0	-1	-2	1	1	1
92	82	1	0	0	0	-1	0	0	0
93	83	1	0	0	-1	-1	0	0	0
94	84	1	0	0	0	0	0	0	1
95	85	1	0	-1	0	0	0	0	0
96	86	1	0	0	-1	-1	1	0	1
97	87	1	0	0	0	-1	0	0	1
98	88	1	0	0	-1	-1	0	0	1
99	89	1	0	-1	0	0	0	0	1
100	90	1	-1	1	-1	-2	1	1	0
101	91	1	-1	1	-1	-2	1	1	1
102		1	-1	0	0	-1	1	2	1
103		1	-1	1	-1	-3	2	2	1
104		1	-1	1	-2	-3	2	2	1
105		1	-1	0	0	-2	1	2	1
106		1	0	1	-1	-2	1	1	1
107		1	-1	0	-1	-2	2	2	1
108		1	0	0	0	-1	0	1	1
109		1	-1	1	0	-2	1	2	1
110		1	0	0	-1	-2	1	1	1
111		1	0	0	-1	-1	1	1	1
112		1	-1	1	-1	-3	2	3	1
113		1	-2	1	-1	-3	2	3	1
114		1	-1	1	-1	-3	2	3	2
115		1	-1	1	-1	-2	2	2	1
116		1	-1	1	-1	-3	1	2	1
117		1	0	0	0	-1	1	1	1
118		1	-1	0	-1	-2	1	2	1
119		1	-1	1	-1	-2	1	2	1
120		2	-1	0	-1	-2	1	1	1

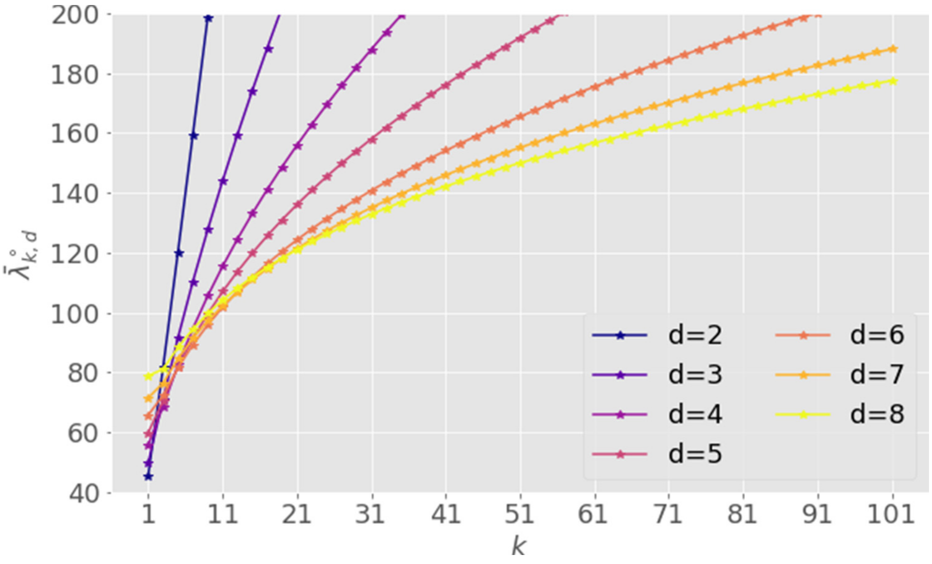

Figure 2. *For indicated dimensions* $d = 1, \ldots, 8$, *a plot of* k *versus* $\bar{\lambda}^{\circ}_{k,d}$.

Table 4
$\bar{\lambda}^{\circ}_{k,d}$ *for the indicated values of the eigenvalue number* k *and dimension* d.

$k \setminus d$	1	2	3	4	5	6	7	8
1,2	39.478	45.586	49.740	55.831	59.838	65.746	71.513	78.957
3,4	157.914	81.546	71.005	68.648	70.596	72.363	76.480	81.033
5,6	355.306	120.115	91.527	82.487	81.768	81.494	84.590	88.336
7,8	631.655	159.162	110.262	94.644	91.275	89.217	91.387	94.461
9,10	986.960	198.387	127.623	105.511	99.567	95.873	97.187	99.662
11,12	1421.223	237.697	143.920	115.401	106.966	101.748	102.262	104.187
13,14	1934.442	277.057	159.365	124.532	113.685	107.029	106.790	108.204
15,16	2526.619	316.446	174.109	133.050	119.864	111.845	110.892	111.826
17,18	3197.752	355.855	188.263	141.062	125.605	116.283	114.651	115.132
19,20	3947.842	395.279	201.909	148.649	130.981	120.410	118.128	118.179

in Weyl's law by a factor between 1.54 *and* 2.01 as indicated in the fifth row of Table 2. Additionally, since $h_d > 2^{2-\frac{2}{d}}$ for each $d = 3, \ldots, 8$, the lower bound on the maximal value in (9) is a sharper result than (8).

The eigenvalue multiplicities listed in Table 2 are very large. This is a consequence of the fact that all lattice vectors $v \in \mathbb{Z}^8$ in Table 3 satisfy $v^t G^{\circ}_{k,d} v = 8 \lceil \frac{k}{2} \rceil^2 = 2\kappa^2$. Note that the first vector in the table is $(0, \ldots, 0, \lceil \frac{k}{2} \rceil) \in \mathbb{Z}^8$. The $k-1$ nontrivial lattice vectors v with a smaller value of $v \mapsto v^t G^{\circ}_{k,d} v$ are of the form $\pm(0, \ldots, 0, \lceil \frac{j}{2} \rceil)$, where $j = 1, \ldots, k-1$. Interestingly, in each dimension $d = 1, \ldots, 8$, the multiplicity of $\lambda^{\circ}_{k,d}$ for $k \geq 2$ is the same (independent of k). Geometrically, this corresponds to the ellipse/ellipsoid corresponding to the Gram matrix, $G^{\circ}_{k,d}$, touching precisely the same number of lattice points (see Figure 1).

In section 3, we give a condition for stationarity; see Theorem 3.4. In subsection 4.2, we show numerically that the bases $B^{\circ}_{k,d}$ satisfy this stationarity condition for $k \geq 1$ and $d \leq 8$. In subsection 4.3, we also show numerically that the tori $T^{\circ}_{k,d}$ degenerate as $k \to \infty$. Supporting

Sage code for these numerical claims is available at [24]. In subsection 4.4, we consider a family of lattices in $d > 8$ dimensions corresponding to gluing the 8-dimensional lattice with gram matrix $\mathcal{G}_k$ to $d-8$ orthogonal lattice vectors that are aligned with the coordinate axes. We show that this lattice yields a meaningful lower bound on $\Lambda_{k,d}$ for $9 \le d \le 15$.

Finally, in Appendix A, we describe the numerical methods that were used to compute the locally maximal solutions to the optimization problem in (4) and compute the bases $B^{\circ}_{k,d}$ described above. Briefly, the optimization problem in (4) was solved by solving a sequence of linearized problems, similar to the method in [32] for the closest packing problem ($k=1$). We have also used these methods to investigate (4) for $d > 8$. Although we have identified locally optimal solutions in higher dimensions for some values of k, we were not able to identify laminated structure in these higher-dimensional lattices.

Other related work. We briefly mention that Milnor used the relationship between flat tori and lattices to find two 16-dimensional compact Riemannian manifolds that have the same Laplace spectrum (isospectral) but are not isometric [35].

In this paper, we count the length of lattice vectors *with multiplicity*. However, we could consider the problem where we enumerate the length of vectors in Γ_B in increasing order *without multiplicity*,

$$0 = \nu_0 < \nu_1 < \nu_2 < \cdots,$$

where ν_k is called the kth length of Γ_B. In this setting, for dimensions 2 to 8, Schaller [41] conjectured that the lattices with best-known sphere packings have maximal lengths; i.e., for all $k > 0$ their kth length is strictly greater than the kth length of any other lattice in the same dimension with the same covolume. This problem is also equivalent to the extremal kth length of closed geodesics among the flat tori of the same dimension and volume. In [45], Willging showed that the conjecture is false in dimension 3 and demonstrated that the 6th shortest vector of the honeycomb lattice is longer than the 6th shortest vector of the face-centered cubic lattice, which is the optimal lattice for sphere packing in dimension 3.

2. Comments on the computations supporting Numerical Observation 1.2.

Here, we discuss the claims in Numerical Observation 1.2 regarding the kth volume-normalized Laplacian eigenvalues of the torus $T^{\circ}_{k,d}$,

$$\lambda^{\circ}_{k,d} := \min_{E \in \mathbb{Z}^d_{k+1}} \max_{v \in E} 4\pi^2 (\det B^{\circ}_{k,d})^{\frac{2}{d}} \|(B^{\circ}_{k,d})^{-t} v\|^2, \tag{10}$$

and the corresponding lattice vectors, E. For $k = 1$, the computation of $\lambda^{\circ}_{k,d}$ is known as the shortest lattice vector problem (SVP) for the dual lattice, $\Gamma^*_{B^{\circ}_{k,d}} = (B^{\circ}_{k,d})^{-t}\mathbb{Z}^d$. The SVP appears in a variety of cryptoanalysis problems and, although it is NP-hard [1, 34], can be solved *for fixed k* in moderately high dimensions [18, 33]. *We are unaware of a method to find the shortest k vectors of the lattice $B^{\circ}_{k,d}$ analytically.*

For fixed (small to moderately large) $k \in \mathbb{N}$, we can compute $\lambda^{\circ}_{k,d}$ in a rigorous way using the `ShortVectors` function[3] in Magma [7]. The enumeration routine underlying this function

[3] http://magma.maths.usyd.edu.au/magma/handbook/text/331.

relies on floating-point approximation but is run in a rigorous way by using the default setting with the parameter `Proof` set to `true` [13], which implements the L^2 algorithm from [37] with tight bounds on the accuracy of computations in order to guarantee Lenstra–Lenstra–Lovász reduction. For each $d=1,\ldots,8$, we checked all values of k from 1 to 500,000 and every value $k\in\{1\times 10^6,\ 2\times 10^6,\ \ldots,\ 9\times 10^6,\ 1\times 10^7,\ 2\times 10^7,\ \ldots,\ 9\times 10^7,\ 1\times 10^8\}$. Magma code with these supporting computations can be found in the `solve-SVP.magma` file at [24]. Indeed, the claims made in Numerical Observation 1.2 hold for these values of k.

3. Eigenvalue perturbation formulae and conditions for stationarity.

Recall that the eigenvalues of $-\Delta$ on a flat torus each have multiplicity of at least two. We will refer to an eigenvalue as a *double eigenvalue* if it has multiplicity of exactly two. We first give the perturbation formula for a double eigenvalue.

Theorem 3.1. *When λ is a double eigenvalue with corresponding lattice vectors $\pm v\in\mathbb{Z}^d$, the variation of the normalized eigenvalue $\bar\lambda$ with respect to the Gram matrix G satisfies*

$$\bar\lambda\,(G_0+\delta G)=\bar\lambda\,(G_0)+\left\langle\frac{\partial\bar\lambda}{\partial G},\delta G\right\rangle_F+o\,(\|\delta G\|)\,,$$

where $\frac{\partial\bar\lambda}{\partial G}=-\frac{\bar\lambda}{d}G^{-1}+4\pi^2\,(\det(G))^{-\frac{1}{d}}\,vv^t$ and $\langle\cdot,\cdot\rangle_F$ denotes the Frobenius inner product.

Proof. For an invertible, symmetric matrix G, Jacobi's formula states that

$$\det(G_0+\delta G)=\det G_0+\det G_0\langle G^{-1},\delta G\rangle+o\,(\|\delta G\|)\,.$$

Since

$$\bar\lambda=4\pi^2\,(\det(G))^{-\frac{1}{d}}\left\langle vv^t,G\right\rangle_F,$$

for fixed lattice vector v, we obtain the desired result using the product rule. ∎

We next give a perturbation formula for eigenvalues of greater multiplicity.

Theorem 3.2. *Suppose the Laplacian eigenvalue λ has even multiplicity $m>2$ with corresponding lattice vectors given by $\pm v_j\in\mathbb{Z}^d$, $j=1,\ldots,\frac{m}{2}$. A perturbation of the Gram matrix of the form $G=G_0+\delta G$ will split the normalized eigenvalue $\Lambda=|\det G|^{-\frac{1}{d}}\lambda$ into up to $\frac{m}{2}$ (unsorted) normalized eigenvalues (each with multiplicity of at least two) given by*

$$\bar\lambda_j\,(G_0+\delta G)=\bar\lambda\,(G_0)+\mu_j+o\,(\|\delta G\|)\,,\qquad j=1,\ldots,\frac{m}{2},$$

where $\mu_j=\langle M_j,\delta G\rangle$ and

$$M_j=-\frac{\bar\lambda}{d}G_0^{-1}+4\pi^2\,(\det G_0)^{-\frac{1}{d}}\,v_jv_j^t. \tag{11}$$

Proof. The volume-normalized eigenvalue, $\bar\lambda$, satisfies

$$4\pi^2\,(\det(G_0))^{-\frac{1}{d}}\,\langle v_j,G_0v_j\rangle=\bar\lambda,\qquad 1\le j\le\frac{m}{2}.$$

Noting that the lattice vectors v_j are fixed, the perturbed volume-normalized eigenvalues satisfy

$$4\pi^2 \left(\det(G_0 + \delta G)\right)^{-\frac{1}{d}} \langle v_j, (G_0 + \delta G)v_j\rangle = \bar{\lambda} + \mu_j + o(\|\delta G\|), \qquad 1 \le j \le \frac{m}{2}.$$

The first order terms give

$$-\frac{4\pi^2}{d}\left(\det(G_0)\right)^{-\frac{1}{d}} \langle G_0^{-1}, \delta G\rangle \langle v_j, G_0 v_j\rangle + 4\pi^2 \left(\det(G_0)\right)^{-\frac{1}{d}} \langle v_j, \delta G v_j\rangle = \mu_j, \qquad 1 \le j \le \frac{m}{2}.$$

Thus

$$\mu_j = 4\pi^2 \left(\det(G_0)\right)^{-\frac{1}{d}} \left\{ \langle v_j, \delta G v_j\rangle - \frac{1}{d}\langle G_0^{-1}, \delta G\rangle \langle v_j, G_0 v_j\rangle \right\}$$

as desired. ∎

Recall that the volume-normalized eigenvalue is scale invariant, i.e., $\Lambda(\alpha G_0) = \bar{\lambda}(G_0)$ for $\alpha \neq 0$. In Theorem 3.2, if we take $\delta G = \varepsilon G_0$ we obtain $\mu_j = 0$ for all $j = 1, \ldots, \frac{m}{2}$ and $\bar{\lambda}_j(G_0 + \delta G) = \bar{\lambda}(G_0) + o(\|G_0\|)$, as we expect.

We next use Theorem 3.2 to derive two necessary conditions for local optimality in the eigenvalue optimization problem (4). We say that G_0 is a *stationary point* for $\bar{\lambda}$ if for every δG we have that there exists at least one $j \in \{1, \ldots, \frac{m}{2}\}$ such that $\mu_j \le 0$. We say that G_0 is a *strict local maximum* for $\bar{\lambda}$ if for every δG satisfying $\langle \delta G, G_0\rangle = 0$ we have at least one $j \in \{1, \ldots, \frac{m}{2}\}$ such that $\mu_j < 0$.

Theorem 3.3. *Using the notation in Theorem 3.2, a necessary condition for G_0 to be a strict local maximum for $\bar{\lambda}$ is that the collection of outer products $\{v_j v_j^t\}_{j=1}^{\frac{m}{2}}$ spans the space of symmetric $\mathbb{R}^{d\times d}$ matrices.*

Proof. Otherwise, there exists a matrix $\delta G = A$ so that, for all $j = 1, \ldots, \frac{m}{2}$, we have

$$\langle A, v_j v_j^t\rangle = 0.$$

In this case, for every $j = 1, \ldots, \frac{m}{2}$, we have

$$\mu_j = \langle M_j, A\rangle = -\frac{\Lambda}{d}\langle G_0^{-1}, A\rangle.$$

Changing the sign of A if necessary, we may assume that $\langle G_0^{-1}, A\rangle \le 0$, implying $\mu_j \ge 0$ for all $j = 1, \ldots, \frac{m}{2}$. ∎

Theorem 3.4. *Using the notation in Theorem 3.2, the Gram matrix G_0 is a stationary point for $\bar{\lambda}$ if and only if there are nonnegative coefficients $c_j \ge 0$, $j = 1, \ldots, \frac{m}{2}$, that are not all zero such that $\sum_{j=1}^{\frac{m}{2}} c_j M_j = 0$.*

Proof. We consider the linear map $U\colon \mathbb{S}^d_{++} \to \mathbb{R}^{\frac{m}{2}}$ defined by

$$U_j(G) := \mu_j(G) = \langle M_j, G\rangle_F, \qquad j = 1, \ldots, \frac{m}{2}.$$

To find the adjoint map of U, denoted $U^*\colon \mathbb{R}^{\frac{m}{2}} \to \mathbb{S}^d_{++}$, for $c \in \mathbb{R}^{\frac{m}{2}}$, we compute

$$\langle U_j(G), c\rangle_{\mathbb{R}^{\frac{m}{2}}} = \sum_{j=1}^{\frac{m}{2}} c_j \langle M_j, G\rangle_F = \left\langle \left(\sum_{j=1}^{\frac{m}{2}} c_j M_j \right), G \right\rangle_F$$

so that $U^* c = \sum_{j=1}^{\frac{m}{2}} c_j M_j$.

Stationarity of G_0 means that $\delta G \mapsto U(\delta G) \in \mathbb{R}^{\frac{m}{2}}$ has at least one nonpositive component for every δG; i.e., there is no solution to the linear system

$$U(\delta G) > 0. \tag{12}$$

We recall Gordan's alternative theorem (see, e.g., [2, Thm. 10.4]) which states that either (12) has a solution or

$$U^* c = 0, \quad c \geq 0, \ c \neq 0 \tag{13}$$

has a solution. Here $c \geq$ should be interpreted componentwise. Thus it is enough to show that there is a nontrivial, nonnegative $c \in \ker(U^*)$. ■

Remark 3.5. For $k=1$, the eigenvalue optimization problem (4) is equivalent to Hermite's constant (7) and determining the densest lattice sphere packing. It was shown by Voronoi that a lattice gives the densest lattice sphere packing if and only if it is perfect and eutactic [42, Thm. 3.9]. It can be seen that the necessary conditions here imply Theorems 3.3 and 3.4 for $k=1$. Note that the lack of convexity for higher eigenvalues makes a sufficiency condition more difficult to state.

4. Properties of flat tori, $T^\circ_{k,d}$, and degeneracy as $k \to \infty$. In this section, we show that $G^\circ_{k,d}$ for $d \leq 8$ and $k \geq 1$ satisfies the necessary condition for strict local maximum given in Theorem 3.3 (see subsection 4.1) and provide numerical evidence that it satisfies the necessary condition for stationarity in Theorem 3.4 (see subsection 4.2). In subsection 4.3, we describe the degeneracy of flat tori $T^\circ_{k,d}$ as $k \to \infty$.

4.1. Linear independence.

Theorem 4.1. *Let $\{v_j\}_{j=1}^{\frac{m}{2}}$ be the collection of lattice vectors given in Table 3. The collection of outer products $\{v_j v_j^t\}_{j=1}^{\frac{m}{2}}$ spans the space of symmetric $\mathbb{R}^{d\times d}$ matrices. Consequently, $G^\circ_{k,d}$ for $d \leq 8$ and $k \geq 1$ satisfies the necessary conditions for a strict local maximum given in Theorem 3.3.*

Proof. We only need $\binom{d}{2} \leq \frac{m}{2}$ outer products $v_j v_j^t$ to span the space of symmetric matrices, so for dimensions, $d = 4,5,6,7,8$, it is not necessary to use all of the lattice vectors listed in Table 3. The lattice vector indices we use are given by

$$J = \{1 \,|\, 2,3 \,|\, 4:6 \,|\, 7:10 \,|\, 12:14, 16, 17 \,|\, 20:25 \,|\, 32:37, 49 \,|\, 54, 56, 64, 66, 67, 82, 83, 84\}.$$

Here, the vertical lines correspond to the horizontal lines in Table 3 and identify the dimension that the lattice vector first appears. For each dimension d, we reshape the upper triangular part of the matrices $v_jv_j^t$ into vectors and stack the vectors as columns of a matrix $F_d \in \mathbb{R}^{\binom{d}{2}\times\binom{d}{2}}$. The matrices F_d are the lower-left $\binom{d}{2}\times\binom{d}{2}$ block of the following matrix, $F \in \mathbb{R}^{36\times 36}$, where to reduce size we use the shorthand $\blacktriangle = 1$, $\blacktriangledown = -1$, and $\blacksquare = \frac{\kappa^2}{4}$:

$$F = \left[\begin{array}{c|cc|ccc|cccc|ccccc|cccccc|ccccccc|cccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\blacktriangle&\blacktriangle&\blacktriangle&\blacktriangle&\blacktriangle&\blacktriangle&\blacktriangle&\blacktriangle\\
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\end{array}\right].$$

We will show that the matrices F_d have nonzero determinant, and hence the outer products are linearly independent. If the matrix F has nonzero determinant, then F_d for each $d \le 8$ also has nonzero determinant. Observing that the upper-left submatrix blocks of F are zero, we see that the determinant of F is the product of the lower-left to upper-right diagonal

We will show that the matrices F_d have non-zero determinant, and hence the outer products are linearly independent. If the matrix F has non-zero determinant, then F_d for each $d \le 8$ also has non-zero determinant. Observing that the upper left submatrix blocks of F are zero, we see that the determinant of F is the product of the lower-left to upper-right diagonal sub-blocks of the matrix F. Judiciously choosing the minors in the Laplace expansion for the determinant[4], we obtain $|\det F| = |\det F_d| = \frac{\kappa^2}{4} \neq 0$. ■

[4] For example, we can expand column-wise: (36, 1) | (34, 2), (35, 3) | (31, 4), (32, 5), (33, 6) | (27, 7) (29, 8), (28, 9), (30, 10) | (22, 11), (24, 12), (23, 13), (25, 15), (26, 14) | (16, 16), (18, 18), (17, 19), (19, 17), (20, 21), (21, 20) | (9, 22), (14, 24), (13, 23), (11, 25), (12, 26), (10, 27), (15, 28) | (1, 29), (2, 33), (3, 32), (7, 31), (6, 30), (5, 34), (4, 35), (8, 36).

Table 5
The values of $a_{k,d}$ and $b_{k,d}$ used in the definition of the vector c°. See subsection 4.2.

d	2	3	4	5	6	7	8
$a_{k,d}$	$\frac{2\kappa^2-4}{\kappa^2}$	$\frac{3\kappa^2-8}{\kappa^2}$	$4\frac{\kappa^2-4}{\kappa^2}$	$6\frac{\kappa^2-4}{\kappa^2}$	$8\frac{\kappa^2-5}{\kappa^2}$	$4\frac{3\kappa^2-16}{\kappa^2}$	$18\frac{\kappa^2-6}{\kappa^2}$
$b_{k,d}$	$\cdot$	$\frac{2\kappa^2-4}{\kappa^2}$	$2\frac{\kappa^2-4}{\kappa^2}$	$2\frac{\kappa^2-3}{\kappa^2}$	$2\frac{\kappa^2-4}{\kappa^2}$	$2\frac{\kappa^2-4}{\kappa^2}$	$\frac{2\kappa^2-9}{\kappa^2}$

subblocks of the matrix F. Judiciously choosing the minors in the Laplace expansion for the determinant,[4] we obtain $|\det F| = |\det F_d| = \frac{\kappa^2}{4} \neq 0$. ■

4.2. Stationarity of tori. Here, for each $k \geq 1$ and $2 \leq d \leq 8$, we give a vector $c = c(k,d) \in \mathbb{R}^{\frac{m}{2}}$ that satisfies the stationarity condition given in Theorem 3.4. We first observe that such a vector c, if one exists, is not unique for $d = 4,5,6,7,8$, as the following argument shows. Reshaping the symmetric matrices M_j as defined in (11) into vectors of length $\binom{d}{2}$, the condition for stationarity in Theorem 3.4 is that a nonnegative linear combination gives zero. Of course, if the number of vectors, $\frac{m}{2}$, exceeds $\binom{d}{2}$, i.e., $m > d(d+1)$, then the columns are linearly dependent. Looking at Table 2, this is the case for $d = 4$, 5, 6, 7, 8.

For $k = 1,2$, for every $2 \leq d \leq 8$, define $c^\circ = (1,\ldots,1) \in \mathbb{R}^{\frac{m}{2}}$. For $k \geq 3$ and $2 \leq d \leq 8$, define the vector $c^\circ \in \mathbb{R}^{\frac{m}{2}}$ by

$$c_i^\circ = \begin{cases} a_{k,d}, & i = 1, \\ b_{k,d}, & i \in I, \\ 1 & \text{otherwise}, \end{cases}$$

where the constants $a_{k,d}$ and $b_{k,d}$ are specified in Table 5 and the index set I is defined

$$I := \{4 \,|\, 7 \,|\, 12:15 \,|\, 20:23 \,|\, 33:42 \,|\, 55:70\}.$$

The indices in I correspond to the lattice vectors in Table 3, where the indices in I are italicized. The vertical lines here correspond to the horizontal lines in Table 3 and identify the dimension that the lattice vector first appears. There is a dot $\cdot$ for $b_{k,d}$ when $d = 2$ since there are no elements in I corresponding to $d = 2$.

Numerical Observation 4.2. *For every $k \geq 1$, and $2 \leq d \leq 8$, the vector $c^\circ \in \mathbb{R}^{\frac{m}{2}}$ satisfies the stationarity condition given in Theorem 3.4.*

It is straightforward to check that Numerical Observation 4.2 holds. Sage code that symbolically verifies the claim is provided in [24]. Comments on how we first identified $c^\circ \in \mathbb{R}^{\frac{m}{2}}$ are made in Appendix A.2.

[4]For example, we can expand columnwise: $(36,1)$ | $(34,2)$, $(35,3)$ | $(31,4)$, $(32,5)$, $(33,6)$ | $(27,7)$ $(29,8)$, $(28,9)$, $(30,10)$ | $(22,11)$, $(24,12)$, $(23,13)$, $(25,15)$, $(26,14)$ | $(16,16)$, $(18,18)$, $(17,19)$, $(19,17)$, $(20,21)$, $(21,20)$ | $(9,22)$, $(14,24)$, $(13,23)$, $(11,25)$, $(12,26)$, $(10,27)$, $(15,28)$ | $(1,29)$, $(2,33)$, $(3,32)$, $(7,31)$, $(6,30)$, $(5,34)$, $(4,35)$, $(8,36)$.

As an example, we verify Numerical Observation 4.2 in dimension $d=3$. We have

$$G^{\circ}_{k,d} = \begin{pmatrix} 2\kappa^2 & -\kappa^2 & 0 \\ -\kappa^2 & 2\kappa^2 & -4 \\ 0 & -4 & 8\kappa^2 \end{pmatrix}$$

so that

$$(G^{\circ}_{k,d})^{-1} = \frac{1}{8(3\kappa^2-4)} \begin{pmatrix} 16\frac{\kappa^2-1}{\kappa^2} & 8 & 4 \\ 8 & 16 & 8 \\ 4 & 8 & 3\kappa^2 \end{pmatrix},$$

and $\det G^{\circ}_{k,d} = 8\kappa^2(3\kappa^2-4)$. We also have $\bar{\lambda}^{\circ}_{k,3} = 4\pi^2(\frac{\kappa^4}{3\kappa^2-4})^{\frac{1}{3}} = 8\pi^2\kappa^2(\det G^{\circ}_{k,d})^{-1/d}$ (see Table 2), and from Tables 3 and 5,

$$c^{\circ}_1 = \frac{3\kappa^2-8}{\kappa^2}, \qquad v_1v_1^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\kappa^2}{4} \end{pmatrix},$$

$$c^{\circ}_2 = 1, \qquad v_2v_2^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$c^{\circ}_3 = 1, \qquad v_3v_3^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$c^{\circ}_4 = \frac{2\kappa^2-4}{\kappa^2}, \qquad v_4v_4^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$c^{\circ}_5 = 1, \qquad v_5v_5^t = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$c^{\circ}_6 = 1, \qquad v_6v_6^t = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We can then compute

$$\sum_{i=1}^{6} c^{\circ}_i M_i = -\frac{\Lambda^{\circ}_{k,3}}{d}\left(\sum_{i=1}^{6} c^{\circ}_i\right)(G^{\circ}_{k,d})^{-1} + 4\pi^2\left(\det G^{\circ}_{k,d}\right)^{-\frac{1}{d}}\sum_{i=1}^{6} c^{\circ}_i v_i v_i^t.$$

Since $\sum_{i=1}^{6} c^{\circ}_i = 3(\frac{3\kappa^2-4}{\kappa^2})$ and

$$\sum_{i=1}^{6} c^{\circ}_i v_i v_i^t = \begin{pmatrix} 4(\kappa^2-1)/\kappa^2 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3\kappa^2/4 \end{pmatrix},$$

we obtain $\sum_{i=1}^{6} c^{\circ}_i M_i = 0$.

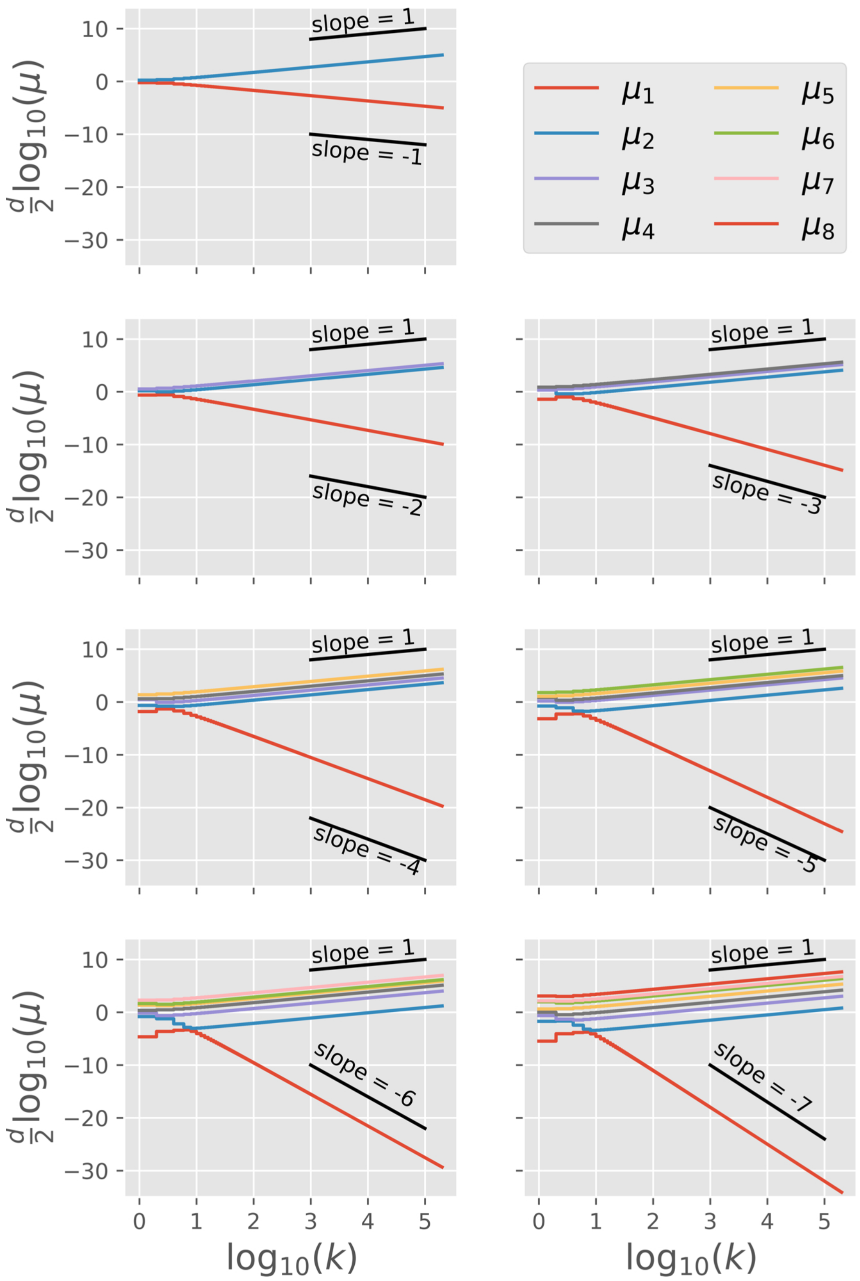

Figure 3. *Log-log plots of k versus the eigenvalues, μ, of the Gram matrix $G^{\circ}_{k,d}$ for dimensions $d = 2,\ldots,8$.*

4.3. Degeneracy of flat tori as $k \to \infty$. For fixed $k \in \mathbb{N}$ and $d \leq 8$, denote the eigenvalues of the normalized Gram matrix $\frac{G^{\circ}_{k,d}}{\det(G^{\circ}_{k,d})^{\frac{1}{d}}}$ by $\mu_1^k \leq \mu_2^k \leq \cdots \leq \mu_d^k$. In Figure 3, we plot k versus $\mu_1^k,\ldots,\mu_d^k$ for $d = 2,\ldots,8$. From Figure 3, we hypothesize that, in each dimension, for large k,

$$\mu_i^k \sim c_i k^{2\cdot p_i/d}$$

for constants c_i, p_i that are independent of k. In particular, $p_1 = -(d-1)$ and $p_2 = \cdots = p_d = 1$, which necessarily satisfy $\sum_{i=1}^d p_i = 0$.

Geometrically, we illustrate the degeneracy in Figure 1 for dimensions $d = 2$ and 3. Here, for $k = 1,3,5,7$, we plot the ellipses/ellipsoids corresponding to the Gram matrix, $\{v \in \mathbb{R}^d \colon v^t G^\circ_{k,d} v = 2\kappa^2\}$, as well as the kth shortest lattice vectors (green points). In both cases, the elongation in one direction corresponds to the first eigenvalue of the Gram matrix scaling as $\mu_1(k) \sim c_1 k^{-2(d-1)/d}$ and the other eigenvalues scaling as $\mu_i(k) \sim c_i k^{2/d}$.

Finally, we conclude with a discussion of the successive minima. Recall that, for $1 \le i \le d$, the *ith successive minimum* of a lattice with basis B is defined by

$$\gamma_i(B) = \min\{\|v_i\| \colon \exists \text{ linearly independent } v_1 \ldots, v_i \in B\mathbb{Z}^d \text{ with } \|v_1\| \le \cdots \le \|v_i\|\}.$$

That is, γ_i is the smallest number γ such that the ellipsoid $\{\|Bx\| \le \gamma\}$ contains i linearly independent vectors (see, e.g., [42]). For each $k \ge 1$, $d \le 8$, we have that $\gamma_1((B^\circ_{k,d})^{-t}) = 2\sqrt{2}$ is attained by the vector $(0,\ldots,1)$ and $\gamma_2((B^\circ_{k,d})^{-t}) = \cdots = \gamma_d((B^\circ_{k,d})^{-t}) = \sqrt{2}\kappa$. In particular, since the injectivity radius of a flat torus, T_B, satisfies $\operatorname{inj}(T_B) = \gamma_1(B) \asymp \gamma_d(B^{-t})^{-1}$, we have that $\operatorname{inj}(T^\circ_{k,d}) \asymp k^{-1}$. If we scale $T^\circ_{k,d}$ by $\alpha = \operatorname{vol}(T^\circ_{k,d})^{-\frac{1}{d}} = |\det G^\circ_{k,d}|^{\frac{1}{2d}} \asymp k^{\frac{d-1}{d}}$ (see Table 2), we obtain that $\operatorname{vol}(\alpha T^\circ_{k,d}) = 1$. We then compute $\operatorname{inj}(\alpha T^\circ_{k,d}) = \gamma_1(\alpha B) = \alpha\gamma_1(B) \asymp \alpha\gamma_d(B^{-t})^{-1} = k^{-\frac{1}{d}}$, which is consistent with [26, Thm. 1.2].

4.4. A lattice model for dimension $d > 8$. We consider the $d > 8$-dimensional lattice with Gram matrix given by

$$\tilde{G}_{k,d} = \begin{pmatrix} 2\kappa^2 I_{d-8} & 0 \\ 0 & \mathcal{G}_k \end{pmatrix}.$$

Here, I_{d-8} is the $(d-8)\times(d-8)$ identity matrix, so this corresponds to gluing the 8-dimensional lattice with gram matrix $\mathcal{G}_k$ to $d-8$ orthogonal lattice vectors that are aligned with the coordinate axes. We compute

$$\det \tilde{G}_{k,d} = 2^{d-4}\kappa^{2d-4}(\kappa^2 - 3).$$

The kth shortest lattice vectors of this lattice still have squared length $v^t \tilde{G}_{k,d} v = 2\kappa^2$. We then compute

$$\bar{\lambda}_{k,d} = 4\pi^2 \cdot 2\kappa^2 \cdot \left(2^{d-4}\kappa^{2d-4}(\kappa^2-3)\right)^{-\frac{1}{d}} = \pi^2\ H_d\ \kappa^{\frac{2}{d}}, \tag{14}$$

where $H_d = 2^{2+\frac{4}{d}}\left(1 - 3\kappa^{-2}\right)^{-\frac{1}{d}} > 2^{2+\frac{4}{d}}$. Comparing this scaling to Weyl's law (3), we have

$$\frac{H_d}{g_d} > (4\omega_d)^{\frac{2}{d}}.$$

We see that $\frac{H_d}{g_d} > 1$ if and only if $\omega_d > \frac{1}{4}$, which holds for $d \leq 15$. To summarize, for $9 \leq d \leq 15$, we obtain a meaningful lower bound on the maximal value,

$$\Lambda_{k,d} \geq \bar{\lambda}_{k,d} = \pi^2 \; H_d \; \kappa^{\frac{2}{d}}, \qquad k \in \mathbb{N}. \tag{15}$$

We do not believe that this lower bound is optimal in these dimensions; however, this constructed sequence of lattices extends the argument in [26] from $d \leq 10$ dimensions to $d \leq 15$ dimensions.

Appendix A. Numerical methods.

In this appendix, we describe a numerical method for approximating solutions to the optimization problem in (4) and for generating a vector $c \in \mathbb{R}^{\frac{m}{2}}$ that satisfies the stationarity condition given in Theorem 3.4.

A.1. Optimization method

In section 2, we explained how we can compute Laplacian eigenvalues of given tori. Here we describe an optimization method for generating those lattices. The optimization problem in (4) can be trivially rewritten as

$$\bar{\lambda}_{k,d} = \max_{\alpha, B} \; \alpha \tag{16a}$$

$$\text{s.t. } \Lambda_j(T_B) \geq \alpha, \qquad j \geq k. \tag{16b}$$

Our strategy for solving (16) is to successively solve its linearization. Writing

$$B^{-t} = (I + \varepsilon) B_0^{-t}$$

for some fixed $B_0 \in GL(d, \mathbb{R})$ and a matrix $\varepsilon \in GL(d, \mathbb{R})$ with small norm, we compute the first order approximations

$$\begin{aligned} \det B^{-t} &= \det B_0^{-t} \det(I + \varepsilon) \\ &= \det B_0^{-t} \left(1 + \langle I, \varepsilon \rangle_F + o(\|\varepsilon\|)\right) \end{aligned}$$

and

$$\begin{aligned} \lambda_k(T_B) &= 4\pi^2 \|B^{-t} v_k\|^2 \\ &= 4\pi^2 \left(v_k^t B_0^{-1} (I + \varepsilon^t)(I + \varepsilon) B_0^{-t} v_k\right) \\ &= 4\pi^2 \|B_0^{-t} v_k\|^2 + 8\pi^2 \langle (B_0^{-t} v_k)(B_0^{-t} v_k)^t, \varepsilon \rangle_F + o(\|\varepsilon\|). \end{aligned}$$

Combining these and assuming $\det B_0^{-t} > 0$, we obtain

$$\begin{aligned} \bar{\lambda}_{k,d}(B) &= \lambda_k(T_B) |\det B^{-t}|^{-\frac{2}{d}} \\ &= \left(4\pi^2 \|B_0^{-t} v_k\|^2 + 8\pi^2 \langle (B_0^{-t} v_k)(B_0^{-t} v_k)^t, \varepsilon \rangle_F + o(\|\varepsilon\|)\right) \\ &\quad \times \det(B_0^{-t})^{-\frac{2}{d}} \left(1 + \langle I, \varepsilon \rangle_F + o(\|\varepsilon\|)\right)^{-\frac{2}{d}} \\ &= \Lambda_k(B_0) + \langle \Sigma_k, \varepsilon \rangle_F + o(\|\varepsilon\|), \end{aligned}$$

where

$$\Sigma_k := 8\pi^2 \det(B_0^{-t})^{-\frac{2}{d}} (B_0^{-t} v_k)(B_0^{-t} v_k)^t - \frac{2}{d} \bar{\lambda}_{k,d}(B_0) I.$$

A linearization of the optimization problem in (16) is then

$$\max_{\alpha,\varepsilon} \alpha \tag{17a}$$

$$\text{s.t. } \Lambda_{j,d}(B_0) + \langle \Sigma_j, \varepsilon \rangle_F \geq \alpha, \qquad j \geq k. \tag{17b}$$

Additionally, for $\beta > 0$, we add the diagonally dominant constraints

$$-\beta \leq \varepsilon_{i,i} \leq \beta, \qquad i \in [d], \tag{17c}$$

$$-\frac{\beta}{d-1} \leq \varepsilon_{i,j} \leq \frac{\beta}{d-1}, \qquad i \neq j, \tag{17d}$$

which ensure $\|\varepsilon\| < 2\beta$. We retain only a finite number of constraints in (17b) by considering only the $j = k, \ldots, k+K+1$ for some integer $K > 1$ shortest lattice vectors. This linearization procedure is similar to that appearing in [32] for the closest packing problem.

The linear optimization problem (17), which depends on the parameter β, is then solved using the Gurobi linear programming library [17] repeatedly until $\|B - B_0\|$ falls below a specified tolerance. The parameter β is treated as a trust-region parameter and adaptively set at each iteration to ensure that the linearization of $\bar{\lambda}_{k,d}(B)$ is faithful.

In these numerical computations, floating-point arithmetic was performed to find the maximal lattice $B_{k,d}$. We then formed the Gram matrix for the dual lattice $B_{k,d}^{-1} B_{k,d}^{-t}$ and observed numerically that all of the elements are multiples of the smallest nonzero element of the Gram matrix, suggesting that the Gram matrix can be rescaled as an integer matrix. We then used the Lenstra–Lenstra–Lovász lattice basis reduction algorithm [29] to simplify the matrix and row/column permutations to obtain the laminated structure of $\mathcal{G}_k$.

A.2. Numerical method for the stationarity condition Here, we explain how, in subsection 4.2, we computed a vector $c^\circ = c^\circ(k,d) \in \mathbb{R}^{\frac{m}{2}}$ that satisfies the stationarity condition given in Theorem 3.4. As explained in subsection 4.2, the vector is not unique, so it is challenging to derive a general formula for c° from an (arbitrarily computed) solution for various k, d. To overcome this obstacle, we specify an addition condition that gives uniqueness. For fixed $k \geq 1$, $2 \leq d \leq 8$, m as the multiplicity of the eigenvalue, and $M_j \in \mathbb{R}^{d \times d}$, $j = 1, \ldots, \frac{m}{2}$ as defined in (11), we consider the quadratic optimization problem

$$\min_{c \in \mathbb{R}^{\frac{m}{2}}} \|c\|_2^2 \tag{18a}$$

$$\text{s.t. } c \geq 0, \tag{18b}$$

$$\sum_{j=1}^{m/2} c_j M_j = 0, \tag{18c}$$

$$c_1 = 1, \tag{18d}$$

which asks for the shortest vector c (in the ℓ^2 sense) that satisfies the desired properties. For each dimension d, we solved this problem for small values of k and were able to deduce the general formula, yielding $c^\circ \in \mathbb{R}^{\frac{m}{2}}$ as given in subsection 4.2.

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