ANALYTIC SPREAD OF FILTRATIONS AND SYMBOLIC ALGEBRAS

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Abstract. In this paper we define and explore the analytic spread $\ell(\mathcal{I})$ of a filtration in a local ring. We show that, especially for divisorial and symbolic filtrations, some basic properties of the analytic spread of an ideal extend to filtrations, even when the filtration is non Noetherian. We also illustrate some significant differences between the analytic spread of a filtration and the analytic spread of an ideal with examples.

In the case of an ideal I, we have the classical bounds $\mathrm{ht}(I) \leq \ell(I) \leq \dim R$. The upper bound $\ell(\mathcal{I}) \leq \dim R$ is true for filtrations $\mathcal{I}$, but the lower bound is not true for all filtrations. We show that for the filtration $\mathcal{I}$ of symbolic powers of a height two prime ideal $\mathfrak{p}$ in a regular local ring of dimension three (a space curve singularity), so that $\mathrm{ht}(\mathcal{I}) = 2$ and $\dim R = 3$, we have that $0 \leq \ell(\mathcal{I}) \leq 2$ and all values of 0,1 and 2 can occur. In the cases of analytic spread 0 and 1 the symbolic algebra is necessarily non-Noetherian. The symbolic algebra is non-Noetherian if and only if $\ell(\mathfrak{p}^{(n)}) = 3$ for all symbolic powers of $\mathfrak{p}$ and if and only if $\ell(\mathcal{I}_a) = 3$ for all truncations $\mathcal{I}_a$ of $\mathcal{I}$.

1. Introduction

The analytic spread of an ideal I in a (Noetherian) local ring R is defined to be

$$\ell(I) = \dim R[I]/m_R R[I] \tag{1}$$

where $R[I] = \bigoplus_{n\geq 0} I^n$ is the Rees algebra of I.

We recall some basic properties of analytic spread from [17] and [25]. We have that upper semicontinuity of fiber dimension holds, that is

$$\ell(I_P) \leq \ell(I_{P'}) \text{ if } P \subset P' \text{ are prime ideals containing } I. \tag{2}$$

This follows for instance by [13, (IV.13.1.5)].

We have inequalities ([17, page 115] and [25, Corollary 8.3.9])

$$\ell(I) \leq \dim R \tag{3}$$

and

$$\mathrm{ht}(I) \leq \ell(I). \tag{4}$$

The lower bound (4) follows from (2) since at a minimal prime Q of I, we have that $\ell(I_Q) = \mathrm{ht}(Q) \geq \mathrm{ht}(I)$ since I_Q is Q_Q-primary.

An ideal I in a local ring R for which the equality $\mathrm{ht}(I) = \ell(I)$ holds is called equimultiple. I is equimultiple if and and only if all fibers of $\pi_0 : \mathrm{Proj}(R[I]/IR[I]) \to \mathrm{Spec}(R/I)$ have the same dimension. This follows since if I is equimultiple and P is a prime ideal of R which contains I, then by (4) and (2),

$$\mathrm{ht}(I) \leq \mathrm{ht}(I_P) \leq \ell(I_P) \leq \ell(I) = \mathrm{ht}(I).$$

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In particular, if I is an equimultiple ideal, then

$$\ell(I_P) = \mathrm{ht}(I_P) \tag{5}$$

for all prime ideals P containing I. For the other direction, we consider a minimal prime P of I such that $\mathrm{ht}(I) = \mathrm{ht}(P)$. If all fibers of π_0 have the same dimension then we have $\mathrm{ht}(I) = \mathrm{ht}(I_P) = \ell(I_P) = \ell(I)$.

We have the following fundamental theorem.

Theorem 1.1. *(*[19], [25, Theorem 5.4.6]*) Let R be a formally equidimensional local ring and I be an ideal in R. Then $m_R \in Ass(R/\overline{I^n})$ for some n if and only if $\ell(I) = \dim R$.*

In this paper we extend the analytic spread of an ideal in a local ring to (not necessarily Noetherian) filtrations, and explore generalizations of the above results to general filtrations, divisorial filtrations and filtrations of symbolic powers.

Let $\mathcal{I} = \{I_n\}$ be a filtration on a local ring R. The Rees ring of the filtration is $R[\mathcal{I}] = \oplus_{n\geq 0} I_n$. Analogously to the case of ideals, we define the analytic spread of the filtration to be

$$\ell(\mathcal{I}) = \dim R[\mathcal{I}]/m_R R[\mathcal{I}]. \tag{6}$$

We show in Lemma 3.6, that the upper bound (3) holds for filtrations $\mathcal{I}$, that is,

$$\ell(\mathcal{I}) \leq \dim R.$$

For an arbitrary filtration, we have that $\sqrt{I_n} = \sqrt{I_1}$ for all n (equation (7)) and we define the height of a filtration $\mathcal{I}$ to be

$$\mathrm{ht}(\mathcal{I}) = \mathrm{ht}(I_1).$$

We may call a filtration $\mathcal{I}$ equimultiple if $\mathrm{ht}(\mathcal{I}) = \ell(\mathcal{I})$.

A simple example of a filtration for which the lower bound (4) is not true is the following.

Example 1.2. *Let R be a local ring of dimension greater than zero. Let $\mathcal{I} = \{I_n\}$ where $I_n = m_R$ for $n \geq 1$. Then $\ell(\mathcal{I}) = 0 < ht(\mathcal{I}) = \dim R$.*

In Example 1.2, all ideals I_n and all truncations (Noetherian approximations) $\mathcal{I}_a$ of $\mathcal{I}$ are equimultiple even though $\mathcal{I}$ is not. This example shows that the "only if" direction of Theorem 1.1 can fail for filtrations.

In the case that $\mathcal{I}$ is a Noetherian filtration, the lower bound $\mathrm{ht}(\mathcal{I}) \leq \ell(\mathcal{I})$ always holds (Proposition 3.7), so that the inequality (4) for ideals continues to hold for Noetherian filtrations.

The condition that a filtration has analytic spread zero has a simple ideal theoretic interpretation (Lemma 3.8). Suppose that $\mathcal{I} = \{I_n\}$ is a filtration in a local ring R. Then the analytic spread $\ell(\mathcal{I}) = 0$ if and only if

$$\text{For all } n > 0 \text{ and } f \in I_n, \text{ there exists } m > 0 \text{ such that } f^m \in m_R I_{mn}.$$

We consider (integral) divisorial filtrations and s-divisorial filtrations in Section 4. Divisorial and s-divisorial filtrations are defined at the beginning of this section. One of the fundamental properties about an m_R-primary ideal I is that $\ell(I) = \dim R$. We saw in Example 1.2 that this property fails for general filtrations. However, it is true for divisorial filtrations of m_R-primary ideals (0-divisorial filtrations). The following theorem shows that the "only if" direction of Theorem 1.1 holds for divisorial filtrations of m_R-primary ideals.

Theorem 1.3. *(Theorem 4.1) Suppose that R is a d-dimensional excellent local domain and $\mathcal{I}$ is a divisorial filtration of m_R-primary ideals on R. Then $\ell(\mathcal{I}) = d$.*

Further, the "if" statement of Theorem 1.1 is true for divisorial filtrations.

Theorem 1.4. *(Theorem 4.7) Suppose that R is a local domain and $\mathcal{I} = \{I_n\}$ is a divisorial filtration on R. Let $I_n = I(\nu_1)_{a_1 n} \cap \cdots \cap I(\nu_r)_{a_r n}$ for $n \geq 1$, some valuations ν_i and some $a_1, \ldots, a_r \in \mathbb{Z}_{>0}$. Suppose that $\ell(\mathcal{I}) = \dim R$. Then for some ν_i, the center $m_{\nu_i} \cap R = \{f \in R \mid \nu_i(f) > 0\}$ is m_R. There exists a positive integer n_0 such that m_R is an associated prime of $I_n = \overline{I_n}$ for all $n \geq n_0$.*

Suppose that $\mathcal{I} = \{I_n\}$ is a filtration in R and $\mathfrak{p}$ is a prime ideal in R. Then the localization of $\mathcal{I}$ at $\mathfrak{p}$ is the filtration $\mathcal{I}_{\mathfrak{p}} = \{(I_n)_{\mathfrak{p}}\}$ in $R_{\mathfrak{p}}$. In a filtration $\mathcal{I} = \{I_n\}$, the ideals I_n have the same minimal primes for all $n \geq 1$.

Corollary 1.5. *(Corollary 4.10) Suppose that R is a local domain and $\mathcal{I} = \{I_n\}$ is an s-divisorial filtration on R (a divisorial filtration consisting of ideals which are equidimensional of dimension s and have no embedded components). Then $\ell(\mathcal{I}_Q) < \dim(R_Q)$ for all prime ideals Q of R which are not minimal primes of I_1.*

The a-the truncation of a filtration $\mathcal{I}$ is the Noetherian approximation of $\mathcal{I}$ generated by the first a terms of $\mathcal{I}$. A formal definition of a truncation is given in Definition 3.2. Knowledge of the analytic spread of the truncations of a filtration can give some information about the analytic spread of the filtration, as is illustrated in the following corollary to Theorem 4.7.

Corollary 1.6. *(Corollary 4.13) Let R be a local domain and $\mathcal{I} = \{I_n\}$ be a divisorial filtration in R where $I_n = \bigcap_{i=1}^r I(\nu_i)_{na_i}$ for all $n \geq 1$. Suppose $I_1 = \cap_{i=1}^r I(\nu_i)_{a_i}$ is a minimal primary decomposition of I_1 and $\ell(\mathcal{I}_a) < \dim R$ for some $a \geq 1$ where $\mathcal{I}_a$ is the a-th truncated filtration of $\mathcal{I}$. Then $\ell(\mathcal{I}) < \dim R$.*

We turn to symbolic algebras in Section 5. Let I be an ideal in a local ring R. For n a positive integer, the n-th symbolic power $I^{(n)}$ of I is

$$I^{(n)} = \cap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} (I^n R_{\mathfrak{p}} \cap R).$$

Symbolic algebras and filtrations have been extensively studied. A survey of some recent progress is given in [10].

We consider in Section 5 the filtration of symbolic powers $\{I^{(n)}\}$ where $I = P_1 \cap \cdots \cap P_r$ is an intersection of prime ideals of a common height in an excellent local ring. If P is a prime ideal in a regular local ring R, then since R_P is a regular local ring, the P_P-adic order on R_P defines a discrete valuation ν of the quotient field of R such that the valuation ideals $I(\nu)_n$ of R are the symbolic powers $I(\nu)_n = P^{(n)}$. Thus the symbolic filtrations in a regular local ring are divisorial filtrations.

There are examples of height two prime ideals P in an excellent regular local ring R of dimension three (space curve singularities) such that the symbolic algebra of P, $\oplus_{n \geq 0} P^{(n)}$, is not a finitely generated R-algebra [24], and even when P is analytically irreducible [12] and [15].

We have a simple characterization of when a symbolic filtration is Noetherian in terms of analytic spread. Suppose that $I \subset J$ are proper ideals in a local ring R. Define $S_J(I) = \oplus_{k \geq 0} I^k : J^\infty$ where $I^k : J^\infty = \cup_{i=1}^\infty I^k :_R J^i$. Let J be the intersection of all asymptotic prime divisors of I which are not minimal primes. Then $I^{(n)} = I^n : J^\infty$ and the symbolic algebra $\oplus_{n \geq 0} I^{(n)} = S_J(I)$. In the case that $I = P$ is the ideal of a space curve singularity, the symbolic algebra is $\oplus_{n \geq 0} P^{(n)} = S_{m_R}(P)$.

Theorem 1.7. *([8, Theorem 2.6], Theorem 5.1) Let $(R, \mathfrak{m})$ be an excellent domain, and let I and J be proper ideals of R. Then the following conditions are equivalent:*

(a) *$S_J(I)$ is finitely generated.*
(b) *There exists an integer $r > 0$ such that $\ell((I^r : J^\infty)_P) < \dim R_P$ for all $P \in V(J)$.*

A related result for ordinary symbolic powers was proven by Katz and Ratliff in Theorem A and Corollary 1 of [14].

We have the following immediate corollary.

Corollary 1.8. *(Corollary 5.2) Suppose that R is an excellent local domain of dimension d and $I = P_1 \cap \cdots \cap P_r$ is an intersection of prime ideals P_i of R of a common height. Then the ring $\bigoplus_{n\geq 0} I^{(n)}$ is a finitely generated R-algebra if and only if there exists $n \in \mathbb{Z}_{>0}$ such that the analytic spread $\ell(I_Q^{(n)}) < ht(Q)$ for all prime ideals Q of R which contain I and are not one of the minimal primes P_i of I.*

With our assumption that R is an excellent local ring of dimension d and $I = P_1 \cap \cdots \cap P_r$ is an intersection of prime ideals P_i of R of a common height in Corollary 1.8, we have that $\ell(I_{P_i}^{(n)}) = \mathrm{ht}(P_i) = \mathrm{ht}(I)$ for the minimal primes P_i of I.

Corollary 1.9. *(Corollary 5.3) Suppose that R is an excellent local domain of dimension d and $I = P_1 \cap \cdots \cap P_r$ is an intersection of prime ideals P_i of R of a common height. If $I^{(n)}$ is equimultiple for some n then the symbolic algebra $\oplus_{n\geq 0} I^{(n)}$ is a finitely generated R-algebra.*

However, there exist ideals I such that the symbolic algebra $\oplus_{n\geq 0} I^{(n)}$ is a finitely generated R-algebra but no symbolic power $I^{(n)}$ is equimultiple (Example 5.4).

In contrast to the conclusions of Corollary 5.2, we have that inequality of analytic spread and height $\ell(\mathcal{I}_Q) < \mathrm{ht}(Q)$ holds at all non minimal primes for symbolic filtrations, irregardless of whether their symbolic algebra is a finitely generated R-algebra. The following proposition follows from Corollary 4.10.

Proposition 1.10. *(Proposition 5.5) Suppose that R is a local domain of dimension d and $I = P_1 \cap \cdots \cap P_r$ is an intersection of prime ideals P_i of R of a common height. Suppose R_{P_i} is a regular local ring for $1 \leq i \leq r$. Let $\mathcal{I} = \{I^{(n)}\}$ be the symbolic filtration of I. Then the analytic spread $\ell(\mathcal{I}_Q) < ht(Q)$ for all prime ideals Q of R which contain I and are not one of the minimal primes P_i of I and $\ell(\mathcal{I}_{P_i}) = ht(P_i) = ht(I)$ for all minimal primes P_i of I.*

The following theorem shows that in the case of the symbolic algebra of a height two prime ideal in a three dimensional local ring, the analytic spread of the symbolic filtration is bounded above by the height, which is 2, and all analytic spreads ≤ 2 occur. Thus the inequality of (4) for ideals is reversed! In contrast, even for non Noetherian divisorial filtrations of m_R-valuations in an excellent local domain, the analytic spread of the filtration must be equal to dimension R by Theorem 1.3.

Theorem 1.11. *(Theorem 5.8) Suppose that R is a regular local ring of dimension 3, and $\mathfrak{p}$ is a height two prime ideal of R. Let $\mathcal{I} = \{\mathfrak{p}^{(n)}\}$ be the symbolic filtration. Then $\ell(\mathcal{I}) \leq 2$ and all values $\ell(\mathcal{I}) = 0, 1, 2$ can occur.*

In Section 6 we construct examples illustrating this theorem with $\ell(\mathcal{I}) = 0$ and 1. A simple example with $\ell(\mathcal{I}) = 2$ is given in the proof of Theorem 1.11.

We have the following ideal theoretic interpretation of analytic spread zero for a symbolic filtration $\mathcal{I}=\{\mathfrak{p}^{(n)}\}$. We have (by Lemma 3.8) that

$\ell(\mathcal{I})=0$ if and only if for all n and $f\in\mathfrak{p}^{(n)}$, there exists $m>0$ such that $f^m\in m_R\mathfrak{p}^{(mn)}$.

In Theorem 5.8, we necessarily have that the symbolic algebra is not finitely generated if $\ell(\mathcal{I})<2$ (by Proposition 3.7). A simple example of a symbolic algebra achieving the maximum analytic spread $\ell(\mathcal{I})=2$ may be constructed by taking $\mathfrak{p}$ to be a regular prime ideal in R ($\mathfrak{p}=(x,y)$ where x,y,z is a regular system of parameters in R). We do not know of an example such that $\ell(\mathcal{I})=2$ but the symbolic algebra is not finitely generated.

If $\ell(\mathcal{I})<2$, then by Corollary 5.2, the analytic spread $\ell(\mathfrak{p}^{(n)})=3$ for all $n>0$ and by Proposition 5.7, we have that $\ell(\mathcal{I}_a)=3$ for all truncations $\mathcal{I}_a$ of $\mathcal{I}$.

We look a little more closely at the most dramatic case of the theorem, when $\ell(\mathcal{I})=0$. The analytic spread $\ell(\mathcal{I})$ being zero has the following interpretation in the geometry of the canonical projection $\varphi:\mathrm{Proj}(R[\mathcal{I}])\to\mathrm{Spec}(R)$. We have that $\varphi^{-1}(\mathfrak{p})=\mathbb{P}^1_{\varkappa(\mathfrak{p})}$, where $\varkappa(\mathfrak{p})=(R/\mathfrak{p})_{\mathfrak{p}}$, since $\mathrm{Proj}(\oplus_{n\ge 0}\mathfrak{p}_{\mathfrak{p}}^n)$ is the blow up of the maximal ideal $\mathfrak{p}_{\mathfrak{p}}$ in the two dimensional regular local ring $R_{\mathfrak{p}}$, so that $\dim\varphi^{-1}(\mathfrak{p})=1$, but $\varphi^{-1}(m_R)=\emptyset$ since $\ell(\mathcal{I})=0$. In particular, the theorem on upper semicontinuity of fiber dimension (2) for ideals fails spectacularly in this non Noetherian situation.

2. Notation

We will denote the nonnegative integers by $\mathbb{N}$ and the positive integers by $\mathbb{Z}_{>0}$, the set of nonnegative rational numbers by $\mathbb{Q}_{\ge 0}$ and the positive rational numbers by $\mathbb{Q}_{>0}$. We will denote the set of nonnegative real numbers by $\mathbb{R}_{\ge 0}$ and the positive real numbers by $\mathbb{R}_{>0}$.

A local ring is assumed to be Noetherian. The maximal ideal of a local ring R will be denoted by m_R. Excellent local rings have many excellent properties which are enumerated in [13, Scholie IV.7.8.3]. We will make use of some of these properties without further reference.

3. The analytic spread of a filtration

A filtration $\mathcal{I}=\{I_n\}_{n\in\mathbb{N}}$ of ideals on a ring R is a descending chain

$$R=I_0\supset I_1\supset I_2\supset\cdots$$

of ideals such that $I_iI_j\subset I_{i+j}$ for all $i,j\in\mathbb{N}$. A filtration $\mathcal{I}=\{I_n\}$ of ideals on a local ring (R,m_R) is a filtration of R by m_R-primary ideals if I_n is m_R-primary for $n\ge 1$. A filtration $\mathcal{I}=\{I_n\}_{n\in\mathbb{N}}$ of ideals on a ring R is called a Noetherian filtration if $\bigoplus_{n\ge 0}I_n$ is a finitely generated R-algebra.

If $I\subset R$ is an ideal, then $V(I)=\{\mathfrak{p}\in\mathrm{Spec}(R)\mid I\subset\mathfrak{p}\}$.

For any filtration $\mathcal{I}=\{I_n\}$ and $\mathfrak{p}\in\mathrm{Spec}\,R$, let $\mathcal{I}_{\mathfrak{p}}$ denote the filtration $\mathcal{I}_{\mathfrak{p}}=\{I_nR_{\mathfrak{p}}\}$.

Let R be a local ring and $\mathcal{I}=\{I_n\}$ be a filtration of R. We define the graded R-algebra $R[\mathcal{I}]=\sum_{m\ge 0}I_mt^m$.

For the rest of this section , suppose that R is a local ring. Let $\mathcal{I}=\{I_n\}$ be a filtration of ideals of R. Then, [9, Lemma 3.1],

$$\text{For all } n\ge 1,\ V(I_1)=V(I_n) \text{ and } \dim R/I_1=\dim R/I_n. \tag{7}$$

Definition 3.1. *Let R be a local ring and $\mathcal{I}=\{I_n\}$ be a filtration of ideals of R. We define the dimension of the filtration $\mathcal{I}$ to be $s(\mathcal{I})=\dim R/I_n$ (for any $n\ge 1$), and define the height $ht(\mathcal{I})$ of $\mathcal{I}$ to be $ht(\mathcal{I})=ht(I_n)$ (for any $n\ge 1$).*

The dimension $s(\mathcal{I})$ and height $\mathrm{ht}(\mathcal{I})$ are well-defined by equation (7). In the case of the trivial filtration $\mathcal{I} = \{I_n\}$, where $I_n = R$ for all n, we have that $s(\mathcal{I}) = -1$.

Suppose that $\mathcal{I} = \{I_n\}$ is a filtration of a local ring R. Then the associated graded rings

$$R[\mathcal{I}] = \sum_{n \geq 0} I_n t^n \text{ and } S[\mathcal{I}] = R[\mathcal{I}][t^{-1}]$$

are subrings of the graded ring $R[t, t^{-1}]$. We have a graded ring

$$T_{\mathcal{I}} := R[\mathcal{I}]/m_R R[\mathcal{I}].$$

Definition 3.2. *Suppose that $\mathcal{I} = \{I_i\}$ is a filtration of ideals on a local ring R. Fix $a \in \mathbb{Z}_{>0}$. The a-th truncated filtration $\mathcal{I}_a = \{I_{a,n}\}$ of $\mathcal{I}$ is defined by*

$$I_{a,n} = \begin{cases} I_n & \text{if } n \leq a \\ \sum_{\substack{i,j>0 \\ i+j=n}} I_{a,i} I_{a,j} & \text{if } n > a. \end{cases}$$

Let $\mathcal{I}_a$ be the a-th truncation of $\mathcal{I}$. Then $R[\mathcal{I}] = \cup_{a \geq 0} R[\mathcal{I}_a]$ and $S[\mathcal{I}] = \cup_{a \geq 0} S[\mathcal{I}_a]$.

The following remark follows from Proposition III.3.2 and Proposition III.3.3 on pages 158 and 159 of [1].

Remark 3.3. *Suppose that $\mathcal{I}$ is a Noetherian filtration. There exists $e > 0$ such that for all $m \geq 1$, $R[\mathcal{I}]$ is a finitely generated $R[I_{me} t^{me}]$-module. In particular, $\dim R[\mathcal{I}] = \dim R[I_{me} t^{me}]$.*

Lemma 3.4. *Let $\mathcal{A}$ be an $\mathbb{N}$ or $\mathbb{Z}$-graded ring. Suppose $\{\mathcal{A}_a\}_{a \geq 1}$ is a collection of Noetherian graded rings with the same grading as $\mathcal{A}$, $\max\{\dim \mathcal{A}_a : a \geq 1\} < \infty$, $\mathcal{A}_{a,n} = \mathcal{A}_n$ for all $n \leq a$ and for each $a \geq 1$ there is a graded ring homomorphism $\varphi_a : \mathcal{A}_a \to \mathcal{A}$ such that $\varphi_a(x) = x$ for all homogeneous elements of $\mathcal{A}_a$ of degree less than or equal to a. Then $\dim \mathcal{A} \leq \max\{\dim \mathcal{A}_a : a \geq 1\}$.*

Proof. Let $P_0 \subset P_1 \subset \cdots \subset P_r$ be a chain of distinct prime ideals in $\mathcal{A}$. There exist $f_i \in P_i \setminus P_{i-1}$ for $1 \leq i \leq r$. Let $a \in \mathbb{Z}_{>0}$ be such that $f_1, \ldots, f_r \in \mathcal{A}_a$. Then the prime ideals in the chain of prime ideals in $\mathcal{A}_a$,

$$\varphi_a^{-1}(P_0) \subset \varphi_a^{-1}(P_1) \subset \cdots \subset \varphi_a^{-1}(P_r)$$

are all distinct. Thus $r \leq \dim \mathcal{A}_a$. ☐

Lemma 3.5. *For any filtration (possibly nonnoetherian) of ideals $\mathcal{I}$ in R, $\dim R[\mathcal{I}] \leq \dim R + 1$, $\dim T_{\mathcal{I}} \leq \dim R$ and $\dim S[\mathcal{I}] \leq \dim R + 1$. In particular, if R is a domain of dimension greater than zero and $I_1 \neq 0$, then $\dim R[\mathcal{I}] = \dim R + 1$, $\dim S[\mathcal{I}] = \dim R + 1$.*

Proof. Let $\mathcal{I}_a$ denote the a-th truncated filtration of $\mathcal{I}$ for all $a \geq 1$. Since $\mathcal{I}_a$ is Noetherian for all $a \geq 1$, there exists $d_a \geq 0$ such that $R[\mathcal{I}_a]$ is a finitely generated $R[I_{a,d_a} t^{d_a}]$-module by Remark 3.3. Thus $\dim R[I_{a,d_a} t^{d_a}] \leq \dim R + 1$ and $\dim T_{\mathcal{I}_a} \leq \dim R$ (formula (1) on page 94 of [25], [25, Proposition 5.1.6]).

Further, by Remark 3.3, there exists $d \geq 0$ such that $S[\mathcal{I}_a]$ is a finitely generated $R[I_{a,d} t^d, t^{-d}]$-module. By formula (2) on page 94 of [25], $\dim R[I_{a,d} t^d, t^{-d}] \leq \dim R + 1$. Thus $\dim S[\mathcal{I}_a] \leq \dim R + 1$.

Since for all $a \geq 1$, we have maps $\varphi_a : R[\mathcal{I}_a] \to R[\mathcal{I}]$ defined by $\varphi_a(xt^n) = xt^n$ for all homogeneous $x \in R[\mathcal{I}_a]$ of degree $n \in \mathbb{N}$, $\psi_a : S[\mathcal{I}_a] \to S[\mathcal{I}]$ defined by $\psi_a(xt^n) = xt^n$ for all homogeneous $x \in S[\mathcal{I}_a]$ of degree $n \in \mathbb{Z}$, and $\chi_a : T_{\mathcal{I}_a} \to T_{\mathcal{I}}$ defined by $\chi_a(x + m_R I_{a,n}) =$

$x+m_RI_n$ for all homogeneous $x\in T_{\mathcal{I}_a}$ of degree $n\in\mathbb{N}$, we get $\dim R[\mathcal{I}]\le \dim R+1$, $\dim T_{\mathcal{I}}\le \dim R$ and $\dim S[\mathcal{I}]\le \dim R+1$ by Lemma 3.4.

Suppose R is domain. Consider the ideal $P=\sum_{n\geqslant 1} I_nt^n\subset R[\mathcal{I}]$. Then height $P\ge 1$. Since $R[\mathcal{I}]/P\cong R$, we have P is a prime ideal in $R[\mathcal{I}]$. Therefore $\dim R[\mathcal{I}]\ge \dim R+1$. Since

$$\dim S[\mathcal{I}]\ge \dim S[\mathcal{I}]_{t^{-1}}=\dim R[t,t^{-1}]=\dim R+1,$$

we have $\dim S[\mathcal{I}]\ge \dim R+1$. □

This allows us to define the analytic spread $\ell(\mathcal{I})$ of a filtration $\mathcal{I}$ by

$$\ell(\mathcal{I})=\dim T_{\mathcal{I}}. \tag{8}$$

This generalizes the classical definition of analytic spread of an ideal I, $\ell(I)=\dim T_I$ where $T_I=R[It]/m_RR[It]$, since if $\mathcal{I}$ is the I-adic filtration $\mathcal{I}=\{I^n\}$, then $T_{\mathcal{I}}=T_I$, so $\ell(\mathcal{I})=\ell(I)$.

From Lemma 3.5 we obtain the following lemma.

Lemma 3.6. *Suppose that $\mathcal{I}$ is an arbitrary filtration of a local ring R. Then*

$$\ell(\mathcal{I})\le \dim R,$$

in agreement with the classical bound for ideals I, $\ell(I)\le\dim R$.

Suppose that I is an ideal in a local ring. Then we have the inequalities

$$\mathrm{ht}(I)\le \ell(I)\le \dim R. \tag{9}$$

(proven for instance in [25, Corollary 8.3.9]). An ideal for which the equality $\mathrm{ht}(I)=\ell(I)$ holds is called equimultiple. The inequalities (9) continue to hold for Noetherian filtrations.

Proposition 3.7. *Suppose that $\mathcal{I}$ is a Noetherian filtration in a local ring R. Then there exists $e>0$ such that $\ell(I_{em})=\ell(\mathcal{I})$ for all $m>0$. In particular, $ht(\mathcal{I})\le\ell(\mathcal{I})$. Further, $ht(\mathcal{I})\le \ell(\mathcal{I})\le\dim R$.*

Proof. Let $e>0$ be such that the conclusions of Remark 3.3 hold. Then

$$m_RR[\mathcal{I}]\cap R[I_{em}t^{em}]=m_RR[\mathcal{I}_{em}t^{em}],$$

so

$$R[I_{em}t^{em}]/m_RR[I_{em}t^{em}]\subset R[\mathcal{I}]/m_RR[\mathcal{I}]$$

is a finite inclusion of Noetherian rings, so

$$\dim T_{I_{em}}=\dim R[I_{em}t^{em}]/m_RR[I_{em}t^{em}]=\dim T_{\mathcal{I}}.$$

□

The condition of analytic spread zero has a simple ideal theoretic interpretation.

Lemma 3.8. *Suppose that $\mathcal{I}=\{I_n\}$ is a filtration in a local ring R. Then the analytic spread $\ell(\mathcal{I})=0$ if and only if*

(10) *For all $n>0$ and $f\in I_n$, there exists $m>0$ such that $f^m\in m_RI_{mn}$.*

Proof. Let $A=R[\mathcal{I}]$. We have that $\ell(\mathcal{I})=0$ if and only if $\dim A/m_RA=0$ which holds if and only if all minimal prime ideals of m_RA are maximal ideals of A. Since m_RA is a homogeneous ideal, all minimal prime ideals of m_RA are homogeneous ([27, Lemma 3, page 153]). The only graded maximal ideal of A is $m_R\oplus I_1\oplus I_2\oplus\cdots$. Thus $\ell(\mathcal{I})=0$ if and only if $\sqrt{m_RA}=m_R\oplus I_1\oplus I_2\oplus\cdots$, which holds if and only if the condition (10) holds. □

4. Divisorial filtrations

Let R be a local domain of dimension d with quotient field K. Let ν be a discrete valuation of K with valuation ring $\mathcal{O}_\nu$ and maximal ideal m_ν. Suppose that $R \subset \mathcal{O}_\nu$. Then for $n \in \mathbb{N}$, define valuation ideals

$$I(\nu)_n = \{f \in R \mid \nu(f) \geq n\} = m_\nu^n \cap R.$$

A divisorial valuation of R ([25, Definition 9.3.1]) is a valuation ν of K such that if $\mathcal{O}_\nu$ is the valuation ring of ν with maximal ideal $\mathfrak{m}_\nu$, then $R \subset \mathcal{O}_\nu$ and if $\mathfrak{p} = \mathfrak{m}_\nu \cap R$ then $\mathrm{trdeg}_{\varkappa(\mathfrak{p})}\varkappa(\nu) = \mathrm{ht}(\mathfrak{p}) - 1$, where $\varkappa(\mathfrak{p})$ is the residue field of $R_\mathfrak{p}$ and $\varkappa(\nu)$ is the residue field of $\mathcal{O}_\nu$. If ν is a divisorial valuation of R such that $m_R = m_\nu \cap R$, then ν is called an m_R-valuation.

By [25, Theorem 9.3.2], the valuation ring of every divisorial valuation ν is Noetherian, hence is a discrete valuation. Suppose that R is an excellent local domain. Then a valuation ν of the quotient field K of R which is nonnegative on R is a divisorial valuation of R if and only if the valuation ring $\mathcal{O}_\nu$ is essentially of finite type over R ([9, Lemma 6.1]).

Suppose that $s \in \mathbb{N}$. An s-valuation of R is a divisorial valuation of R such that $\dim R/p = s$ where $p = \mathfrak{m}_\nu \cap R$.

An integral divisorial filtration of R (which we will refer to as a divisorial filtration in this paper) is a filtration $\mathcal{I} = \{I_m\}$ such that there exist divisorial valuations $\nu_1, \ldots, \nu_r$ and $a_1, \ldots, a_r \in \mathbb{Z}_{\geq 0}$ such that for all $m \in \mathbb{N}$,

$$I_m = I(\nu_1)_{ma_1} \cap \cdots \cap I(\nu_r)_{ma_r}.$$

If $\mathcal{I}$ is a divisorial filtration, then the ideals $I_m = \overline{I_m}$ are integrally closed for all $m \geq 1$. In fact, the Rees algebra $R[\mathcal{I}] = \sum_{n\geq 0} I_n t^n$ is integrally closed in $R[t]$. This is proven in [7, Lemma 5.8]. [7, Lemma 5.8] is stated for divisorial m_R-filtrations but the proof is valid for arbitrary divisorial filtrations.

An integral s-divisorial filtration of R (which we will refer to as an s-divisorial filtration in this paper) is a filtration $\mathcal{I} = \{I_m\}$ such that there exist s-valuations $\nu_1, \ldots, \nu_r$ and $a_1, \ldots, a_r \in \mathbb{Z}_{\geq 0}$ such that for all $m \in \mathbb{N}$,

$$I_m = I(\nu_1)_{ma_1} \cap \cdots \cap I(\nu_r)_{ma_r}. \tag{11}$$

Theorem 4.1. *Suppose that R is a d-dimensional excellent local domain and $\mathcal{I} = \{I_n\}$ is a divisorial filtration of m_R-primary ideals on R. Then $\ell(\mathcal{I}) = d$.*

Proof. There exist m_R-valuations $\nu_1, \ldots, \nu_t$ and $a_1, \ldots, a_t \in \mathbb{Z}_{>0}$ such that $\mathcal{I} = \{I_n\}$ where $I_n = I(\nu_1)_{a_1 n} \cap \cdots \cap I(\nu_t)_{a_t n}$ for $n \geq 0$, with $I(\nu_i)_m = \{f \in R \mid \nu_i(f) \geq m\}$.

Let S be the normalization of R in the quotient field of R. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_u$ be the maximal ideals of S. Let $J(\nu_i)_m = \{f \in S \mid \nu_i(f) \geq m\}$. For each i, there exists $\sigma(i)$ with $1 \leq \sigma(i) \leq u$ such that the ideals $J(\nu_i)_m$ are $\mathfrak{m}_{\sigma(i)}$-primary for all m, and $J(\nu_i)_1 = \mathfrak{m}_{\sigma(i)}$. That is, ν_i is an $\mathfrak{m}_{\sigma(i)}$-valuation. For $n \in \mathbb{N}$, let

$$J_n = J(\nu_1)_{a_1 n} \cap \cdots \cap J(\nu_t)_{a_t n}$$

so that $J_n \cap R = I_n$.

Let $\pi : X \to \mathrm{Spec}(S)$ be the blow up of an ideal K such that $K_{\mathfrak{m}_i}$ is a $(\mathfrak{m}_i)_{\mathfrak{m}_i}$-primary ideal for $1 \leq i \leq u$, X is normal and there exist prime divisors E_i on X such that the valuation rings $\mathcal{O}_{\nu_i} = \mathcal{O}_{X,E_i}$ for $1 \leq i \leq t$. Let A be the effective Cartier divisor on X such that $\mathcal{O}_X(-A) = K\mathcal{O}_X$, so that $-A$ is ample on X. Write $A = \sum_{i=1}^s b_i E_i$ where $s \geq t$, $E_1, \ldots, E_s$ are prime Weil divisors with $\mathcal{O}_{\nu_i} = \mathcal{O}_{X,E_i}$ for $1 \leq i \leq t$ and $b_i \in \mathbb{Z}_{>0}$ for all i.

There exists a unique $\alpha \in \mathbb{Q}_{>0}$ such that $\alpha b_i \geq a_i$ for $1 \leq i \leq t$ and further, there exists an index i_0 such that $\alpha b_{i_0} = a_{i_0}$. Write $\alpha = \frac{c}{d}$ with $c, d \in \mathbb{Z}_{>0}$. Then $mcb_iE_i \geq mda_iE_i$ for $1 \leq i \leq t$ and $mcb_{i_0}E_{i_0} = mda_{i_0}E_{i_0}$ for all $m \geq 0$. Thus

$$\mathcal{O}_X(-mcA) \subset \mathcal{O}_X(-\sum_{i=1}^{t} mda_iE_i)$$

for $m \geq 0$, so that $\Gamma(X, \mathcal{O}_X(-mcA)) \subset J_{md}$ for all $m \in \mathbb{N}$.

Since X is normal and E_{i_0} has codimension 1 in X, there exists a closed point $q \in E_{i_0}$ such that E_{i_0} is the only irreducible component of A which contains q in its support and $\mathcal{O}_{X,q}$ and $\mathcal{O}_{E_{i_0},q}$ are regular local rings. Let $x_1 = 0$ be a local equation of E_{i_0} at q and extend x_1 to a regular system of parameters $x_1, x_2, \ldots, x_d$ in $\mathcal{O}_{X,q}$. Let $P_j = (x_1, x_2, \ldots, x_j)$ for $1 \leq j \leq d$. P_j are regular primes in $\mathcal{O}_{X,q}$ (that is, $\mathcal{O}_{X,q}/P_j$ is a regular local ring for all j). Thus the rule $\omega_j(g) = \mathrm{ord}_{P_j}(g)$ for $g \in \mathcal{O}_{X,q}$ defines a discrete valuation on the quotient field of R, which is a $\mathfrak{m}_{\sigma(i_0)}$-valuation (since E_{i_0} is contracted to $m_{\sigma(i_0)}$). For $1 \leq j \leq d$ let $J(\omega_j)_n = \{f \in S \mid \omega_j(f) \geq n\}$. Then $J(\omega_j)_n = P_j^n \cap S$ for all $n \geq 0$. We have that ω_1 is the valuation ν_{i_0}. Since $x_1, \ldots, x_d$ is a regular system of parameters,

$$P_1^m \cap P_j^{m+1} = P_1^m P_j \text{ for all } m \in \mathbb{N}. \tag{12}$$

Let Z_j be the closed subvariety of X such that its ideal sheaf satisfies $(\mathcal{I}_{Z_j})_q = P_j$ for $1 \leq j \leq d$. Then $Z_1 = E_{i_0}$, $Z_d = q$, $\dim Z_j = d - j$ for all j and $\pi(Z_j) = \mathfrak{m}_{\sigma(i)}$ for all j. For $m \geq 0$ and $1 \leq j \leq d$, we have

$$\begin{aligned} (\mathcal{I}_{Z_j} \otimes \mathcal{O}_X(-mcA))_q &= (\mathcal{I}_{Z_j} \otimes \mathcal{O}_X(-mcb_{i_0}E_{i_0}))_q = (\mathcal{I}_{Z_j} \otimes \mathcal{O}_X(-mda_{i_0}E_{i_0}))_q \\ &= P_1^{mda_{i_0}} P_j = P_1^{mda_{i_0}} \cap P_j^{mda_{i_0}+1}. \end{aligned}$$

Observe that we have inclusions of sheaves

$$\mathcal{I}_{Z_1} \otimes \mathcal{O}_X(-mcA) \subset \mathcal{I}_{Z_2} \otimes \mathcal{O}_X(-mcA) \subset \cdots \subset \mathcal{I}_{Z_d} \otimes \mathcal{O}_X(-mcA) \subset \mathcal{O}_X(-mcA) \subset \mathcal{O}_X.$$

Since $-A$ is ample, for $m \gg 0$, $\mathcal{I}_{Z_j} \otimes \mathcal{O}_X(-mcA)$ is generated by global sections for $1 \leq j \leq d$, so that

$$\begin{aligned} \Gamma(X, \mathcal{I}_{Z_j} \otimes \mathcal{O}_X(-mcA))\mathcal{O}_{X,q} &= P_1^{mda_{i_0}} \cap P_j^{mda_{i_0}+1} \text{ and} \\ \Gamma(X, \mathcal{O}_X(-mcA))\mathcal{O}_{X,q} &= P_1^{mda_{i_0}}. \end{aligned} \tag{13}$$

We have inclusions

$$\begin{aligned} &\Gamma(X, \mathcal{I}_{Z_1} \otimes \mathcal{O}_X(-mcA)) \subset \Gamma(X, \mathcal{I}_{Z_2} \otimes \mathcal{O}_X(-mcA)) \subset \\ &\cdots \subset \Gamma(X, \mathcal{I}_{Z_d} \otimes \mathcal{O}_X(-mcA)) \subset \Gamma(X, \mathcal{O}_X(-mcA)) \subset J_{md}. \end{aligned}$$

By (13), for $1 \leq j \leq d-1$, there exists $f_j \in \Gamma(X, \mathcal{I}_{Z_{j+1}} \otimes \mathcal{O}_X(-mcA)) \subset J_{md}$ such that $f_j \in P_{j+1}^{mda_{i_0}+1}$ but $f_j \notin P_j^{mda_{i_0}+1}$, so that $f_j \in P_{j+1}^{mda_{i_0}+1} \cap J_{md} = J(\omega_{j+1})_{mda_{i_0}+1} \cap J_{md}$, but $f_j \notin P_j^{mda_{i_0}+1} \cap J_{md} = J(\omega_j)_{mda_{i_0}+1} \cap J_{md}$ and there exists $f_d \in J_{dm}$ such that $f_d \notin J(\omega_d)_{mda_{i_0}+1} \cap J_{dm}$.

Let $B = \oplus_{n \geq 0} J_n$, which is a graded ring. Let

$$C_j = \oplus_{m \geq 0} J(\omega_{j+1})_{a_{i_0}m+1} \cap J_m$$

for $0 \leq j \leq d-1$ and

$$C_d = \mathfrak{m}_{\sigma(i_0)} \oplus J_1 \oplus J_2 \oplus \cdots.$$

We will now show that the ideals C_j are prime ideals in B. First observe that none of the C_j are equal to B since $C_j \cap S = J(\omega_{j+1})_1 = \mathfrak{m}_{\sigma(i_0)}$ for $1 \leq j \leq d-1$ and $C_d \cap S = \mathfrak{m}_{\sigma(i_0)}$.

Suppose $1 \leq j \leq d-1$ and $f \in J_m$, $g \in J_n$ are such that $fg \in J(\omega_{j+1})_{a_{i_0}(m+n)+1}$. Then $\omega_{j+1}(fg) \geq a_{i_0}(m+n)+1$. We have that $J_m \subset J(\nu_{i_0})_{a_{i_0}m} \subset J(\omega_{j+1})_{a_{i_0}m}$ so that $\omega_{j+1}(f) \geq a_{i_0}m$. Similarly, $\omega_{j+1}(g) \geq a_{i_0}n$. Thus either $\omega_{j+1}(f) \geq a_{i_0}m+1$ or $\omega_{j+1}(g) \geq a_{i_0}n+1$, so that $f \in J(\omega_{j+1})_{a_{i_0}m+1} \cap J_m$ or $g \in J(\omega_{j+1})_{a_{i_0}n+1} \cap J_n$. Thus the C_j are prime ideals.

We found $f_j \in C_j \setminus C_{j-1}$ for $1 \leq j \leq d$. Thus

$$C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_d$$

is a chain of distinct prime ideals in B.

There is a natural inclusion of graded rings $R[\mathcal{I}] = \oplus_{n\geq 0} I_n \subset B = \oplus_{n \geq 0} J_n$. We will now show that B is integral over $R[\mathcal{I}]$. For $a \in \mathbb{Z}_{>0}$, let $R[\mathcal{I}]_a$ be the a-th truncation of $R[\mathcal{I}]$ and B_a be the a-th truncation of B, so that $R[\mathcal{I}]_a$ is the subalgebra of $R[\mathcal{I}]$ generated by $\oplus_{n\leq a} I_n$ and B_a is the subalgebra of B generated by $\oplus_{n \leq a} J_n$. It suffices to show that homogeneous elements of B are integral over $R[\mathcal{I}]$. Suppose that $f \in J_a$ for some a. Then $f \in B_a$. Let $0 \neq x$ be in the conductor of S over R. Then $xJ_n \subset I_n$ for all n since $I_n = J_n \cap R$. Thus $xB_a \subset R[\mathcal{I}]_a$, so $f^i \in \frac{1}{x} R[\mathcal{I}]_a$ for all $i \in \mathbb{N}$, and so the algebra $R[\mathcal{I}]_a[f] \subset \frac{1}{x} R[\mathcal{I}]_a$. Since $\frac{1}{x}R[\mathcal{I}]_a$ is a finitely generated $R[\mathcal{I}]_a$-module and $R[\mathcal{I}]_a$ is a Noetherian ring, the ring $R[\mathcal{I}]_a[f]$ is a finitely generated $R[\mathcal{I}]_a$-module, so that f is integral over $R[\mathcal{I}]_a$.

We have a chain of prime ideals

$$Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_d$$

in $R[\mathcal{I}]$ where $Q_i := C_i \cap R[\mathcal{I}]$. The Q_i are all distinct since the C_i are all distinct and B is integral over $R[\mathcal{I}]$ (by [2, Theorem A.6 (b)]). It remains to show that $m_R R[\mathcal{I}] \subset Q_0$, so that $\dim R[\mathcal{I}]/m_R R[\mathcal{I}] \geq d$. Since this is the maximum possible dimension of $R[\mathcal{I}]/m_R R[\mathcal{I}]$ by Lemma 3.5, we have that $\ell(\mathcal{I}) = d$.

We now show that $m_R R[\mathcal{I}] \subset Q_0$. First we observe that if $g \in m_R$ then $\nu_{i_0}(g) \geq 1$ since ν_{i_0} is an m_R-valuation. Suppose that $f \in m_R I_n$. Then $f = \sum g_k f_k$ with $g_k \in m_R$ and $f_k \in I_n$. Thus $\nu_{i_0}(g_j f_j) \geq n a_{i_0} + 1$ for all j so that $f \in I(\nu_{i_0})_{na_{i_0}+1}$ and thus $f \in I(\nu_{i_0})_{na_{i_0}+1} \cap I_n$. Since ω_1 is the valuation ν_{i_0}, we have $m_R R[\mathcal{I}] \subset Q_0$.

□

Proposition 4.2. *Let R be a local domain and $\mathcal{I} = \{I_n\}$ be a divisorial filtration in R where $I_n = \bigcap_{i=1}^r I(\nu_i)_{na_i}$ for all $n \geq 1$. Let $m_R \in \operatorname{Ass}(R/I_1)$. Then $m_R \in \operatorname{Ass}(R/\overline{I_{a,n}})$ for all $a, n \geq 1$ where $\mathcal{I}_a = \{I_{a,n}\}$ is the a-th truncated filtration of $\mathcal{I}$. In particular, $m_R \in Ass(R/I_n)$ for all $n \geq 1$.*

Proof. Suppose $\operatorname{Ass}(R/I_1) = \{m_R\}$. Since $\operatorname{Min}\operatorname{Ass}(R/I_1) = \operatorname{Min}\operatorname{Ass}(R/I_{a,n})$ for all $a, n \geq 1$, we have $m_R \in \operatorname{Ass}(R/\overline{I_{a,n}})$ for all $a, n \geq 1$.

Suppose the cardinality of $\operatorname{Ass}(R/I_1)$ is greater than one and $m_R \in \operatorname{Ass}(R/I_1)$. Without loss of generality let us assume that the centers $I(\nu_i)_1$ of ν_i on R are m_R for $1 \leq i \leq c$ and the centers $I(\nu_i)_1$ are not m_R for $i > c$.

Fix a. Since $m_R \in \operatorname{Ass}(R/I_1)$, there exists $y \in R \setminus I_1$ such that $m_R y \in I_1$. Therefore

$$y \in I_1 : m_R^\infty = \bigcap_{j>c}^{r} I(\nu_j)_{a_j}.$$

Thus $y \notin \cap_{i=1}^c I(\nu_i)_{a_i}$. Hence $y^n \in \bigcap_{j>c}^r I(\nu_j)_{na_j} \setminus \cap_{i=1}^c I(\nu_i)_{na_i}$ for all $n \geq 1$. Therefore $y^n \notin I_n$ for all $n \geq 1$. Since $I_{a,n} \subset I_n = \overline{I_n}$, we have $y^n \notin \overline{I_{a,n}}$ for all $n \geq 1$.

Let $v \in m_R^b$ where $b = a_1 + \cdots + a_c$. Then $\nu_i(yv) \geq a_i$ for all $1 \leq i \leq c$. Thus $yv \in \cap_{i=1}^c I(\nu_i)_{a_i}$. Since $y \in \bigcap_{j>c}^r I(\nu_j)_{a_j}$, we have $yv \in I_1$ and hence for all $n \geq 1$,

$$y^n m_R^{nb} \subset I_1^n = I_{a,1}^n \subset I_{a,n} \subset \overline{I_{a,n}}.$$

Let $m \geq 1$ be an integer such that $y^n m_R^m \subseteq \overline{I_{a,n}}$ and $y^n m_R^{m-1} \not\subseteq \overline{I_{a,n}}$. Let $x \in m_R^{m-1} \setminus m_R^m$ such that $y^n x \notin \overline{I_{a,n}}$. Then $m_R = (\overline{I_{a,n}} :_R y^n x)$. Therefore $m_R \in \operatorname{Ass}(R/\overline{I_{a,n}})$ for all $n \geq 1$. □

Lemma 4.3. *Suppose that R is a local domain and $\mathcal{I} = \{I_n\}$ is a divisorial filtration on R. Suppose that P is a prime ideal of R and there exists $t \in \mathbb{Z}_{>0}$ such that $P \in Ass(R/I_t)$. Then there exists $n_0 \in \mathbb{Z}_{>0}$ such that $P \in Ass(R/I_n)$ for all $n \geq n_0$.*

Proof. Let $I_n = I(\nu_1)_{a_1 n} \cap \cdots \cap I(\nu_r)_{a_r n}$ for $n \in \mathbb{N}$. By Lemma 3, page 343 of Zariski Samuel Vol. II, for all $m \in \mathbb{Z}_{>0}$, the ideal $I(\nu_i)_m$ is P_i-primary, where the prime ideal $P_i = I(\nu_i)_1$ is the center of ν_i on R. Let $P_1, \ldots, P_s$ be the distinct centers of the ν_i on R for $1 \leq i \leq r$. For k with $1 \leq k \leq s$ and $n \in \mathbb{Z}_{>0}$, let

$$Q(k)_n = \bigcap_{I(\nu_i)_1 = P_k} I(\nu_i)_{a_i n},$$

which is a P_k-primary ideal. Thus for all $1 \leq k \leq s$, $P_k \in \operatorname{Ass}(R/I_n)$ if and only if $I_n \neq \bigcap_{\substack{1 \leq i \leq s \\ i \neq k}} Q(i)_n$.

Suppose that $P \in \operatorname{Ass}(R/I_t)$. Then $P = P_k$ for some k. After reindexing the ν_i, there exists $c > 0$ such that the centers $I(\nu_i)_1 = P$ if $1 \leq i \leq c$ and $I(\nu_i)_1 \neq P$ if $c < i$.

Thus there exists $f \in \cap_{i>c} I(\nu_i)_{a_i t} \setminus I_t$. Therefore $\nu_i(f) \geq a_i t$ for $i > c$ and there exists j with $1 \leq j \leq c$ such that $\nu_j(f) \leq a_j t - 1$. Let $0 \neq g \in I_1$ be arbitrary. Then $\nu_i(g) \geq a_i$ for all i. Let $\beta = \nu_j(g) \geq a_j$.

Let $n \in \mathbb{N}$. Write $n = mt + s$ with $m \in \mathbb{N}$ and $0 \leq s < t$. $\nu_i(f^m g^s) \geq n a_i$ for $i > c$ and

$$\nu_j(f^m g^s) \leq m(a_j t - 1) + s\beta = (mt + s)a_j + s(\beta - a_j) - m = n a_j + s(\beta - a_j) - m < n a_j$$

for $m > s(\beta - a_j)$. Thus for $m > s(\beta - a_j)$, we have that $f^m g^s \in \cap_{i>c} I(\nu_i)_{a_i n} \setminus I_n$ which implies that $P \in \operatorname{Ass}(R/I_n)$. □

Suppose that R is a local domain and $\mathcal{I} = \{I_n\}$ is a divisorial filtration of R where $I_n = I(\nu_1)_{a_1 n} \cap \cdots \cap I(\nu_r)_{a_r n}$.

Let $S = S[\mathcal{I}]$. Let $I_n = R$ for $n \leq 0$. Then for $r \in \mathbb{Z}_{>0}$.

$$t^{-r} S = \sum_{n \in \mathbb{Z}} I_{n+r} t^n. \tag{14}$$

Lemma 4.4. *Let K be an ideal in R such that $I_1 \subset K$. Suppose that $n \in \mathbb{N}$. Then there exists $r \in \mathbb{Z}_{>0}$ such that $(I_{n+1})^r \subset K I_{rn}$. In particular the ideal $\oplus_{n \geq 0} I_{n+1} t^n \subset \sqrt{KR[\mathcal{I}]}$.*

Proof. For all $r \in \mathbb{Z}_{>0}$, $I_{n+1}^r = I_{n+1} I_{n+1}^{r-1} \subset K I_{n+1}^{r-1}$. Note that $I_{n+1}^{r-1} \subset I_{(n+1)(r-1)}$. Thus if $r \geq n+1$, then $I_{n+1}^{r-1} \subset I_{rn}$. □

Lemma 4.5. *Let R be a local domain and $\mathcal{I} = \{I_n\}$ be a divisorial filtration of ideals in R, where $I_n = I(\nu_1)_{a_1 n} \cap \cdots \cap I(\nu_r)_{a_r n}$. For $1 \leq i \leq r$, let*

$$P_i = \sum_{n \in \mathbb{Z}} I(\nu_i)_{a_i n + 1} \cap I_n t^n.$$

Then P_i is a prime ideal in $S = S[\mathcal{I}]$. Let

$$Q_i = \sum_{n\in\mathbb{Z}} I(\nu_i)_{a_i(n+1)} \cap I_n t^n.$$

Then Q_i is P_i-primary for $1 \le i \le r$ and

$$t^{-1}S = Q_1 \cap \cdots \cap Q_r.$$

Proof. Observe that if $n \le 0$, then the valuation ideal $I(\nu_i)_n = R$, and so $I_n = R$ for $n \le 0$. Since P_i is a graded R-module, to show that it is an ideal in S, it suffices to show that if $f \in I(\nu_i)_{a_i a+1} \cap I_a$ and $g \in I_b$ then $fg \in I(\nu_i)_{a_i(a+b)+1} \cap I_{a+b}$. This follows since $\nu_i(fg) = \nu_i(f) + \nu_i(g) \ge (a_i a + 1) + a_i b \ge a_i(a+b) + 1$ so $fg \in I(\nu_i)_{a_i(a+b)+1}$. $P_i \ne R$ since $R \cap P_i = I(\nu_i)_1$ (which is a prime ideal). Since P_i is graded, to show that P_i is a prime ideal, it suffices to show that if $f \in I_a$ and $g \in I_b$ are such that $fg \in I(\nu_i)_{a_i(a+b)+1} \cap I_{a+b}$, then either $f \in I(\nu_i)_{a_i a+1} \cap I_a$ or $g \in I(\nu)_{a_i b+1} \cap I_b$. This follows since $\nu_i(f) \ge a_i a$, $\nu_i(g) \ge a_i b$ and $\nu_i(fg) = \nu_i(f) + \nu_i(g) \ge a_i(a+b) + 1$ so either $\nu_i(f) \ge a_i a + 1$ or $\nu_i(g) \ge a_i b + 1$.

We now show that Q_i is a primary ideal. It suffices to show that if $f \in I_a$, $g \in I_b$, $fg \in I(\nu_i)_{a_i(a+b+1)} \cap I_{a+b}$ and $f \notin I(\nu_i)_{a_i(a+1)} \cap I_a$, then there exists an $m > 0$ such that $g^m \in I(\nu_i)_{a_i(mb+1)} \cap I_{mb}$. With these assumptions we have that $\nu_i(f) < a_i(a+1)$ and $\nu_i(fg) \ge a_i(a+b+1)$ so that $\nu_i(g) > a_i b$, and thus $\nu_i(g) = a_i b + c$ for some $c > 0$. There exists $m > 0$ such that $mc \ge a_i$. Thus $\nu_i(g^m) = m a_i b + mc \ge a_i(mb+1)$ so that $g^m \in I(\nu_i)_{a_i(mb+1)} \cap I_{mb}$.

We now show that $\sqrt{Q_i} = P_i$. $Q_i \subset P_i$ since $a_i(n+1) \ge a_i n + 1$ for all i and $n \ge 0$. We then have that $\sqrt{Q_i} = P_i$ since $f \in I(\nu_i)_{a_i n+1} \cap I_n$ implies $f^m \in I(\nu_i)_{a_i(mn+1)} \cap I_{mn}$ for $m \ge a_i$.

By (14), $t^{-1}S = \sum_{n\in\mathbb{Z}} I_{n+1} t^n = Q_1 \cap \cdots \cap Q_r$. ☐

Remark 4.6. *With slight modification, Lemma 4.3, Lemma 4.5 and Theorem 4.6 are true for $\mathbb{R}$ divisorial filtrations.*

Theorem 4.7. *Suppose that R is a local domain and $\mathcal{I} = \{I_n\}$ is a divisorial filtration on R. Let $I_n = I(\nu_1)_{a_1 n} \cap \cdots \cap I(\nu_r)_{a_r n}$ for $n \ge 1$, some valuations ν_i and some $a_1, \ldots, a_r \in \mathbb{Z}_{>0}$. Suppose that $\ell(\mathcal{I}) = \dim R$. Then for some ν_i, the center $m_{\nu_i} \cap R = \{f \in R \mid \nu_i(f) > 0\}$ is m_R. There exists a positive integer n_0 such that m_R is an associated prime of $I_n = \overline{I_n}$ for all $n \ge n_0$.*

Proof. Let $S = S[\mathcal{I}]$ and let notation be as in Lemma 4.5. Let J be the graded ideal

$$J = \sum_{n\ge 0} I_{n+1} t^n \subset R[\mathcal{I}]. \tag{15}$$

By assumption, there exists a prime ideal $\overline{U}$ of $T_{\mathcal{I}}$ such that $\dim T_{\mathcal{I}}/\overline{U} = \dim R$. We have isomorphisms of graded R-algebras

$$A := S/(t^{-1}S + m_R S) \cong \sum_{n\ge 0} I_n/(I_{n+1} + m_R I_n) t^n \cong R[\mathcal{I}]/(J + m_R R[\mathcal{I}]).$$

By Lemma 4.4, the nilradical of $R[\mathcal{I}]/(J + m_R R[\mathcal{I}])$ is $\sqrt{m_R R[\mathcal{I}]}/(J + m_R R[\mathcal{I}])$. Thus the quotient of $T_{\mathcal{I}}$ by its nilradical is isomorphic as a graded R-algebra to the quotient of A by its nilradical, and so there exists a prime ideal U' of A such that $\dim A/U' = \dim R$. Let U be the preimage of U' in S. We have that $t^{-1}S + m_R S \subset U$ and $t^{-1} \ne 0$ in the domain S so that $\mathrm{ht}(U) \ge 1$. Since $\dim S/U = \dim R$, we have that

$$1 + \dim R \le \dim S/U + \mathrm{ht}(U) \le \dim S \le \dim R + 1$$

by Lemma 3.5 so $\dim S = \dim R + 1$ and $\mathrm{ht}(U) = 1$. We further have that $U \cap R = m_R$, since $m_R S \subset U$. Now $\sqrt{t^{-1}S} = \cap_{i=1}^r P_i \subset U$ so that $P_i \subset U$ for some i. Thus $P_i = U$ since $\mathrm{ht}(U) = 1$, and so $m_R = P_i \cap R = I(\nu_i)_1 = m_{\nu_i} \cap R$ is the center of ν_i on R.

We will now show that m_R is an associated prime of some I_n. Suppose that m_R is not an associated prime of any I_n. We will derive a contradiction. After reindexing, we may suppose that, for some s, ν_i is an m_R-valuation for $i \le s$ and ν_i is not an m_R-valuation for $i > s$. Since m_R is not an associated prime of I_n for all n, we thus have that $I_n = I(\nu_{s+1})_{a_{s+1}n} \cap \cdots \cap I(\nu_r)_{a_r n}$ for all n. Since none of $\nu_{s+1}, \ldots, \nu_r$ is an m_R-valuation, we have that $\ell(\mathcal{I}) < \dim R$ by the first part of this proof, a contradiction. Thus there is some positive integer n_0 such that m_R is an associated prime of I_{n_0}. Thus m_R is an associated prime of I_n for all $n \gg 0$ by Lemma 4.3.

□

Remark 4.8. *Theorem 4.7 shows that if $\ell(\mathcal{I}) = \dim R$ then one of the prime ideals P_i of Lemma 4.5 is a height one prime ideal in $S[\mathcal{I}]$ such that $P_i \cap R = m_R$.*

Remark 4.9. *The proofs of Lemma 4.5 and 4.7 prove the following more general statement. Let R be a local ring and $\mathcal{J}(i) = \{J(i)_n\}_{n\in\mathbb{N}}$ be filtrations of ideals in R with $J(i)_1 \subsetneq R$ for all $1 \le i \le r$. Suppose $\bigcap_{n\ge 1} J(i)_n = 0$ and $\mathcal{G}_i = \bigoplus_{n\ge 0} J(i)_n/J(i)_{n+1}$ are domains for all $1 \le i \le r$.*

Consider the filtration $\mathcal{J} = \{J_n = J(1)_{a_1 n} \cap \cdots \cap J(r)_{a_r n}\}$ for some fixed $a_1, \ldots, a_r \in \mathbb{Z}_{>0}$. For $1 \le i \le r$, let $P_i = \sum_{n\in\mathbb{Z}} J(i)_{a_i n+1} \cap J_n t^n$. Then P_i is a prime ideal in $S[\mathcal{J}]$. Let $Q_i = \sum_{n\in\mathbb{Z}} J(i)_{a_i(n+1)} \cap J_n t^n$. Then Q_i is P_i-primary for $1 \le i \le r$, and

$$t^{-1}S[\mathcal{J}] = Q_1 \cap \cdots \cap Q_r.$$

Suppose that $\ell(\mathcal{J}) = \dim R$. Then there exists a prime ideal $P_i = \sum_{n\in\mathbb{Z}} J(i)_{a_i n+1} \cap J_n t^n$ in $S[\mathcal{J}]$ for some $i \in \{1, \ldots, r\}$ such that height $P_i = 1$ *and $P_i \cap R = m_R$.*

Corollary 4.10. *Suppose that R is a local domain and $\mathcal{I} = \{I_n\}$ is an s-divisorial filtration on R. Then $\ell(\mathcal{I}_Q) < \dim(R_Q)$ for all prime ideals Q of R which are not minimal primes of I_1.*

Corollary 4.11. *Suppose that R is a local domain and $\mathcal{I}$ is an s-divisorial filtration on R with $s \ge 1$. Then $\ell(\mathcal{I}) < \dim R$.*

Corollary 4.12. *Let R be a local domain and $\mathcal{I} = \{I_n\}$ be a divisorial filtration in R where $I_n = \bigcap_{i=1}^r I(\nu_i)_{na_i}$ for all $n \ge 1$. Suppose $m_R \in \mathrm{Ass}(R/I_1)$. Then $\ell(\mathcal{I}_a) = \dim R$ for all a-th truncated filtration of $\mathcal{I}$ and hence $\ell(\mathcal{I}) \le \ell(\mathcal{I}_a)$ for all $a \ge 1$.*

Proof. By Proposition 4.2, $m_R \in \mathrm{Ass}(R/\overline{I_{a,n}})$ for all $n, a \ge 1$. Since $\mathcal{I}_a$ is a Noetherian filtration, there exists an integer m such that $\ell(\mathcal{I}_a) = \ell(I_{a,m})$. Therefore by [19] ,[25, Theorem 5.4.6], Theorem 1.1, we have $\ell(\mathcal{I}_a) = \dim R$. □

Corollary 4.13. *Let R be a local domain and $\mathcal{I} = \{I_n\}$ be a divisorial filtration in R where $I_n = \bigcap_{i=1}^r I(\nu_i)_{na_i}$ for all $n \ge 1$. Suppose $I_1 = \cap_{i=1}^r I(\nu_i)_{a_i}$ is a minimal primary decomposition of I_1 and $\ell(\mathcal{I}_a) < \dim R$ for some $a \ge 1$ where $\mathcal{I}_a$ is the a-th truncated filtration of $\mathcal{I}$. Then $\ell(\mathcal{I}) < \dim R$.*

Proof. If $\ell(\mathcal{I}) = \dim R$ then by Theorem 4.7, $m_{\nu_i} \cap R = m_R$ for some i. Thus $m_R \in \mathrm{Ass}(R/I_1)$. Therefore by Corollary 4.12, we get $\ell(\mathcal{I}_a) = \dim R$ which is a contradiction. □

Let R be a local ring and I an ideal in R. In [3] Brodmann proved that $\ell(\mathcal{I}) \le \dim R - \mathrm{liminf}_n \,\mathrm{depth}\, R/I^n$ where $\mathcal{I} = \{I^n\}$. If R has infinite residue field then Burch improved the result of Brodmann for the filtration $\mathcal{I} = \{\overline{I^n}\}$ and proved that $\ell(\mathcal{I}) \leq \dim R - \text{liminf}_n \operatorname{depth} R/\overline{I^n}$ [5]. This result was generalized to the filtration $\mathcal{I} = \{I^{(n)}\}$ if the Symbolic Rees algebra of I is finitely generated [4]. We generalize Burch's result for divisorial filtrations under some extra conditions.

Corollary 4.14. (Burch's inequality for divisorial filtration) *Suppose R is a local domain and $\mathcal{I} = \{I_n\}$ is a divisorial filtration in R. Suppose one of the following holds.*

(*i*) *$m_R \in \operatorname{Ass}(R/I_t)$ for some $t \geq 1$.*
(*ii*) *The filtration $\mathcal{I}$ is an 1-divisorial filtration.*

Then $\ell(\mathcal{I}) \leq \dim R - \text{liminf}_n \operatorname{depth} R/I_n$.

Proof. (*i*) By Lemma 4.3, there exists a positive integer n_0 such that $m_R \in \operatorname{Ass}(R/I_n)$ for all $n \geq n_0$. Thus $\text{liminf}_n \operatorname{depth} R/I_n = 0$. Now by Lemma 3.5, $\ell(\mathcal{I}) \leq \dim R$.

(*ii*) If $\dim R \leq 1$ then $I_n = 0$ for all $n \geq 1$ and hence $0 = \ell(\mathcal{I}) \leq \dim R - \text{liminf}_n \operatorname{depth} R/I_n$. Suppose $\dim R \geq 2$. Then by Corollary 4.11, we have $\ell(\mathcal{I}) \leq \dim R - 1$. Since $\mathcal{I}$ is a 1-divisorial filtration, we have $\operatorname{depth} R/I_n \geq 1$ and $\dim R/I_n = 1$ for all $n \geq 1$. Thus $\operatorname{depth} R/I_n = 1$ for all $n \geq 1$. Therefore $\ell(\mathcal{I}) \leq \dim R - \text{liminf}_n \operatorname{depth} R/I_n$. ☐

5. Symbolic Algebras

Suppose that $I \subset J$ are proper ideals in a local ring R. Define $S_J(I) = \oplus_{k\geq 0} I^k : J^\infty$ where $I^k : J^\infty = \cup_{i=1}^{\infty} I^k :_R J^i$.

Theorem 5.1. *(*[8, Theorem 2.6]*) Let $(R, \mathfrak{m})$ be an excellent domain, and let I and J be proper ideals of R. Then the following conditions are equivalent:*

(a) *$S_J(I)$ is finitely generated.*
(b) *There exists an integer $r > 0$ such that $\ell((I^r : J^\infty)_P) < \dim R_P$ for all $P \in V(J)$.*

A related result was proven by Katz and Ratliff in Theorem A and Corollary 1 of [14].

Let I be an ideal in a local ring R. For $n \in \mathbb{Z}_{>0}$, the n-the symbolic power $I^{(n)}$ of I is

$$I^{(n)} = \cap_{\mathfrak{p} \in \operatorname{Ass}(R/I)} (I^n R_{\mathfrak{p}} \cap R).$$

Let J be the intersection of all asymptotic prime divisors of I which are not minimal primes. Then $I^{(n)} = I^n : J^\infty$ and the symbolic algebra $\oplus_{n\geq 0} I^{(n)} = S_J(I)$.

Corollary 5.2. *Suppose that R is an excellent local domain of dimension d and $I = P_1 \cap \cdots \cap P_r$ is an intersection of prime ideals P_i of R of a common height. Then the ring $\bigoplus_{n\geq 0} I^{(n)}$ is a finitely generated R-algebra if and only if there exists $n \in \mathbb{Z}_{>0}$ such that the analytic spread $\ell(I_Q^{(n)}) < ht(Q)$ for all prime ideals Q of R which contain I and are not one of the minimal primes P_i of I.*

With our assumption that R is a local ring of dimension d and $I = P_1 \cap \cdots \cap P_r$ is an intersection of prime ideals P_i of R of a common height in Corollary 5.2, we have that $\ell(I_{P_i}^{(n)}) = \operatorname{ht}(P_i) = \operatorname{ht}(I)$ for the minimal primes P_i of I.

Corollary 5.3. *Suppose that R is an excellent local domain of dimension d and $I = P_1 \cap \cdots \cap P_r$ is an intersection of prime ideals P_i of R of a common height. If $I^{(n)}$ is equimultiple for some n then the symbolic algebra $\oplus_{n\geq 0} I^{(n)}$ is a finitely generated R-algebra.*

Proof. If $I^{(n)}$ is equimultiple, then by (5), $\ell(I_Q^{(n)}) = \mathrm{ht}(I_Q^{(n)})$ for all prime ideals Q containing I, so that if Q is not a minimal prime of I, we have that $\ell(I_Q^{(n)}) = \mathrm{ht}(I_Q) < \dim R_Q$, and so $I^{(n)}$ satisfies the criterion of Corollary 5.2. $\square$

However, there exist ideals I such that the symbolic algebra $\oplus_{n\geq 0} I^{(n)}$ is a finitely generated R-algebra but no symbolic power $I^{(n)}$ is equimultiple, as is shown in the following example.

Example 5.4. *(*[9, Example 8.4]*) There exists a height one prime ideal P in a normal, excellent 3 dimensional local ring R such that no symbolic power of P is equimultiple but the symbolic algebra $\oplus_{n\geq 0} P^{(n)}$ is a finitely generated R-algebra.*

In contrast to the conclusions of Corollary 5.2, we have that inequality of analytic spread and height $\ell(\mathcal{I}_Q) < \mathrm{ht}(Q)$ holds at all non minimal primes for symbolic filtrations, irregardless of whether their symbolic algebra is a finitely generated R-algebra. The following proposition follows from Corollary 4.10.

Proposition 5.5. *Suppose that R is a local domain of dimension d and $I = P_1 \cap \cdots \cap P_r$ is an intersection of prime ideals P_i of R of a common positive height. Suppose R_{P_i} is a regular local ring for $1 \leq i \leq r$. Let $\mathcal{I} = \{I^{(n)}\}$ be the symbolic filtration of I. Then the analytic spread $\ell(\mathcal{I}_Q) < ht(Q)$ for all prime ideals Q of R which contain I and are not one of the minimal primes P_i of I and $\ell(\mathcal{I}_{P_i}) = ht(P_i) = ht(I)$ for all minimal primes P_i of I.*

Proposition 5.6. *Let R be a local domain of positive dimension. Let $\mathfrak{p}$ be a prime ideal in R such that $ht(\mathfrak{p}) = \dim R - 1$ (so that $\dim R/\mathfrak{p} = 1$). Let $d \in \mathbb{Z}_{>0}$. If the $\mathfrak{p}^{(d)}$-adic filtration $\{(\mathfrak{p}^{(d)})^n\}_{n\in\mathbb{N}}$ is a 1-divisorial filtration then $\mathfrak{p}^{(d)}$ is equimultiple.*

Proof. We have that
$$\dim R - 1 = \mathrm{ht}(\mathfrak{p}^{(d)}) \leq \ell(\mathfrak{p}^{(d)}) \leq \dim R$$
and $\ell(\mathfrak{p}^{(d)}) \neq \dim R$ by [19] or [25, Theorem 5.4.6] (or by Theorem 4.7 above). $\square$

Proposition 5.7. *Let R be a normal, excellent local domain of dimension three with an isolated singularity and I be an intersection of (a finite number of) height two prime ideals of R. Let $\mathcal{I} = \{I^{(m)}\}$ be the filtration of symbolic powers of I, so that $\mathcal{I}$ is a 1-divisorial filtration of R. Then $\mathcal{I}$ is not Noetherian if and only if the a-th truncation $\mathcal{I}_a$ of $\mathcal{I}$ (Definition 3.2) satisfies*
$$\ell(\mathcal{I}_a) = 3 \text{ for all } a \in \mathbb{Z}_{>0}.$$

Proof. We have that $\ell(\mathcal{I}) \leq 2$ by Corollary 4.11. If $\mathcal{I}$ is a Noetherian filtration, then $R[\mathcal{I}_a] = R[\mathcal{I}]$ for all a sufficiently large, so that, by Proposition 3.7, $2 = \mathrm{ht}(I) \leq \ell(\mathcal{I}_a) = \ell(\mathcal{I}) \leq 2$.

Suppose that $\mathcal{I}$ is not Noetherian. We will show that $\ell(\mathcal{I}_a) = 3$ for all $a > 0$. We will prove this statement, by assuming that $\ell(\mathcal{I}_a) = 2$ for some a, and deriving a contradiction. Write $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are height two prime ideals in R. Let ν_i be the $\mathfrak{p}_{i\mathfrak{p}_i}$-adic valuation of $R_{\mathfrak{p}_i}$. Then $I(\nu_i)_n = \mathfrak{p}_i^{(n)}$ for $1 \leq i \leq r$ and all $n \in \mathbb{N}$, and $I^{(n)} = I(\nu_1)_n \cap \cdots \cap I(\nu_r)_n$ for all $n \in \mathbb{N}$.

Let $\overline{\mathcal{I}_a} = \{\overline{I_{a,n}}\}$, the filtration of integral closures of the ideals in $\mathcal{I}_a$. Then $R[\overline{\mathcal{I}_a}]$ is finite over $R[\mathcal{I}_a]$. There exists $d > 0$ such that $\overline{I_{a,nd}} = (\overline{I_{a,d}})^n$ for all $n \geq 0$, $\ell(\mathcal{I}_a) = \ell(I_{a,d})$ and $\ell(\overline{\mathcal{I}_a}) = \ell(\overline{I_{a,d}})$ by Remark 3.3 and Proposition 3.7. Thus $\ell(\overline{I_{a,d}}) = 2$.

Let $\pi : X = \text{Proj}(R[\overline{I_{a,d}}]) \to \text{Spec}(R)$ be the blow up of $\overline{I_{a,d}}$. X is normal since the ring $R[\overline{I_{a,d}}] = \sum_{n\ge 0} \overline{I_{a,dn}}t^n$ is integrally closed. Since $\ell(\overline{I_{a,d}}) = 2$, $\dim \pi^{-1}(m_R) = 1$ and so there are no prime divisors on X which contract to m_R. Thus $\overline{I_{a,d}}\mathcal{O}_X = \mathcal{O}_X(-dE)$ where $E = E_1 + \cdots + E_r$ is the sum of prime divisors E_i on X such that the valuation $\nu_{E_i} = \nu_i$. Since X and R are normal, we have that

$$\pi_*\mathcal{O}_X(-nE) = I(\nu_1)_n \cap \cdots \cap I(\nu_r)_n = \mathfrak{p}_1^{(n)} \cap \cdots \cap \mathfrak{p}_r^{(n)} = I^{(n)}$$

for all $n \in \mathbb{N}$.

There exists a graded exact sequence

$$0 \to K \to R[x_0, \ldots, x_m] \to R[\overline{\mathcal{I}_{a,d}}] \to 0,$$

which gives a closed embedding of X into $\mathbb{P}^m_R$, such that $\mathcal{O}_{\mathbb{P}^m_R}(1) \otimes \mathcal{O}_X \cong \mathcal{O}_X(-dE)$.

Sheafify this sequence to get short exact sequences

$$0 \to \mathcal{K}(n) \to \mathcal{O}_{\mathbb{P}^m_R}(n) \to \mathcal{O}_X(-ndE) \to 0$$

and take global sections to get an exact sequence of R-algebras (by [16, Proposition II.5.13])

$$R[x_0, \ldots, x_m] = \oplus_{n\ge 0} H^0(\mathbb{P}^m_R, \mathcal{O}_{\mathbb{P}^m_R}(n)) \to \sum_{n\ge 0} I^{(nd)} \to \oplus_{n\ge 0} H^1(\mathbb{P}^m_R, \mathcal{K}(n)).$$

We have that $\oplus_{n\ge 0}H^1(\mathbb{P}^m_R, \mathcal{K}(n))$ is a finitely generated R-module by [16, Theorem III.5.2(b)]. Thus $A := \oplus_{n\ge 0} I^{(nd)}$ is a finitely generated R-algebra.

Since $\mathcal{O}_X(-dE)$ is an invertible sheaf, for $i, n \in \mathbb{Z}$, the reflexive rank 1 sheaf of the Weil divisor $-(i+nd)E$ is $\mathcal{O}_X(-(i+nd)E) \cong \mathcal{O}_X(-iE) \otimes \mathcal{O}_X(-ndE)$. By [16, Corollary II.5.18], for $i > 0$, there is a short exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \to L \to \sum_{j=1}^{s} \mathcal{O}_X(-n_j dE) \to \mathcal{O}_X(-iE) \to 0. \tag{16}$$

with $n_j \in \mathbb{Z}$. After possibly replacing i with a smaller integer which is equivalent to i modulo d, we may assume that all n_j are positive. Now for all j, $J_j := \oplus_{n\ge 0} \pi_*(\mathcal{O}_X(-(n_j + n)dE))$ is a graded ideal in A, so it is a finitely generated A-module. From (16) we obtain a short exact sequence of A-modules

$$\sum_{j=1}^{s} J_j \to \sum_{n\ge 0} I^{(i+nd)} \to M.$$

where $M = \sum_{n\ge 0} H^1(X, L \otimes \mathcal{O}_X(-ndE))$ is a finitely generated R-module (again by [16, Theorem III.5.2(b)]). Thus $\sum_{n\ge 0} I^{(i+nd)}$ is a finitely generated A-module, and so $\oplus_{n\ge 0} I^{(n)}$ is a finitely generated R-algebra, in contradiction to our assumption. □

We have the following theorem, that uses examples which will be constructed in Section 6.

Theorem 5.8. *Suppose that R is a regular local ring of dimension 3, and $\mathfrak{p}$ is a height two prime ideal of R. Let $\mathcal{I} = \{\mathfrak{p}^{(n)}\}$ be the symbolic filtration of $\mathfrak{p}$. Then $\ell(\mathcal{I}) \le 2$ and all values $\ell(\mathcal{I}) = 0, 1, 2$ can occur.*

Proof. The bound $\ell(\mathcal{I}) \le 2$ follows from Corollary 4.11. Examples 6.1 and 6.6 have analytic spread 0 and 1 respectively. A prime ideal $\mathfrak{p} = (x, y)$ where x, y are part of a regular system of parameters in R gives an example with analytic spread 2. □

We have (by Lemma 3.8) the following ideal theoretic interpretation of analytic spread zero for a symbolic filtration $\mathcal{I}=\{\mathfrak{p}^{(n)}\}$. We have that

$\ell(\mathcal{I})=0$ if and only if for all n and $f\in\mathfrak{p}^{(n)}$, there exists $m>0$ such that $f^m\in m_R\mathfrak{p}^{(mn)}$.

In Theorem 5.8, we necessarily have that the symbolic algebra is not finitely generated if $\ell(\mathcal{I})<2$ (by Proposition 3.7). A simple example of a symbolic algebra achieving the maximum analytic spread $\ell(\mathcal{I})=2$ may be constructed by taking $\mathfrak{p}$ to be a regular prime ideal in R ($\mathfrak{p}=(x,y)$ where x,y,z is a regular system of parameters in R). We do not know of an example such that $\ell(\mathcal{I})=2$ but the symbolic algebra is not finitely generated.

We look a little more closely at the most dramatic case of the theorem, when $\ell(\mathcal{I})=0$. By Proposition 5.7, we have that $\ell(\mathcal{I}_b)=3$ for all truncations $\mathcal{I}_b$ of $\mathcal{I}$. The analytic spread $\ell(\mathcal{I})$ being zero has the following interpretation in the geometry of the canonical projection $\varphi:\mathrm{Proj}(R[\mathcal{I}])\to\mathrm{Spec}(R)$. We have that $\varphi^{-1}(\mathfrak{p})=\mathbb{P}^1_{\varkappa(\mathfrak{p})}$, where $\varkappa(\mathfrak{p})=(R/\mathfrak{p})_{\mathfrak{p}}$, since $\mathrm{Proj}(\oplus_{n\geq 0}\mathfrak{p}_{\mathfrak{p}}^n)$ is the blow up of the maximal ideal $\mathfrak{p}_{\mathfrak{p}}$ in the two dimensional regular local ring $R_{\mathfrak{p}}$, so that $\dim\varphi^{-1}(\mathfrak{p})=1$, but $\varphi^{-1}(m_R)=\emptyset$ since $\ell(\mathcal{I})=0$. In particular, the theorem on upper semicontinuity of fiber dimension (2) for ideals fails in this non Noetherian situation.

Theorem 5.8 shows that the inequality (4) for ideals fails for symbolic filtrations, as we see by taking $\mathfrak{p}$ in Theorem 5.8 such that $\ell(\mathcal{I})<2$, so that $2=\mathrm{ht}(\mathcal{I})>\ell(\mathcal{I})$.

6. Some Examples of Symbolic Algebras

In this section we use famous examples by Nagata and Zariski to compute the analytic spread of some space curve singularities and some related examples.

Example 6.1. *Suppose that $a\geq 0$. Then there exists a prime ideal Q of height $2+a$ in a regular local ring A of dimension $3+a$ such that $\ell(\mathcal{J})=a$, where $\mathcal{J}=\{Q^{(n)}\}$ is the 1-divisorial filtration on A of symbolic powers of Q.*

We make use of a famous example of Nagata. Let s be a positive integer with $s\geq 4$, and $r=s^2$. Let $\alpha_1,\ldots,\alpha_r\in\mathbb{P}^2_{\mathbb{C}}$ be independent generic points of $\mathbb{P}^2$ over $\mathbb{Q}$.

Let $\mathcal{I}_{\alpha_i}$ be the ideal sheaf of α_i in $\mathbb{P}^2$ and let H' be a linear hyperplane section of $\mathbb{P}^2$.

The difficult statement of Theorem 6.2 is proven by Nagata in [21] and in Proposition 1 of Chapter 3, page 18 of [22].

Theorem 6.2. *(Nagata) Let notation be as above.*

1) *Suppose that $d,m_1,\ldots,m_r\in\mathbb{N}$ and $H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(dH')\otimes\mathcal{I}_{\alpha_1}^{m_1}\otimes\cdots\otimes\mathcal{I}_{\alpha_r}^{m_r})\neq 0$. Then*
$$d>\frac{1}{\sqrt{r}}\sum_{i=1}^r m_i.$$
2) *Suppose that r' is a real number such that $r'>\sqrt{r}$. Then there exist $d,m\in\mathbb{Z}_{>0}$ such that $r'>\frac{d}{m}>\sqrt{r}$ and $H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(dH')\otimes\mathcal{I}_{\alpha_1}^m\otimes\cdots\otimes\mathcal{I}_{\alpha_r}^m)\neq 0$.*

Let $\Lambda:X\to\mathbb{P}^2$ be the blow up of the points $\alpha_1,\ldots,\alpha_r$ with exceptional lines $E_1,\ldots,E_r$. Let $H=\Lambda^*(H')$. Since Λ is the blowup of the points $\alpha_1,\ldots,\alpha_r$ on the nonsingular surface $\mathbb{P}^2$, we have that for all $d,m_1,\ldots,m_r\geq 0$,
$$H^0(X,\mathcal{O}_X(dH-m_1E_1-\cdots-m_rE_r))=H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(dH')\otimes\mathcal{I}_{\alpha_1}^{m_1}\otimes\cdots\otimes\mathcal{I}_{\alpha_r}^{m_r}).$$
Let $E=E_1+\cdots+E_r$. The canonical divisor K_X on X is $K_X=-3H+E$.

Lemma 6.3. *Let notation be as above.*

1) Let C be an irreducible reduced curve on X with $C \neq E_i$ for any i. Then $C \sim dH - \sum m_i E_i$ for some $d, m_i \in \mathbb{N}$ with $d > 0$.
2) Let $dH - mE$ be a divisor with $d \geq -2$. Then $H^2(X, \mathcal{O}_X(dH - mE)) = 0$.
3) Let $L = dH - mE$ with $d, m \in \mathbb{Z}_{>0}$. Then L is ample if $\frac{d}{m} > \sqrt{r}$.
4) Let $L = dH - mE$ with $d, m \in \mathbb{Z}_{>0}$. Then $H^1(X, \mathcal{O}_X(dH - mE)) = 0$ if $d > \sqrt{r}m + (\sqrt{r} - 3)$.
5) Suppose that $d, m \geq 0$. Then

$$H^0(X, \mathcal{O}_X(dH - mE)) = H^0(X, \mathcal{O}_X(H))H^0(X, \mathcal{O}_X((d-1)H - mE))$$

if $d \geq \sqrt{r}m + \sqrt{r}$.

Proof. 1). $C \sim dH - \sum m_i E_i$ for some $d, m_i \in \mathbb{Z}$. We have $(H \cdot C) = d \geq 0$ since the complete linear system $|H|$ is base point free. Further, $(E_i \cdot C) = m_i \geq 0$ for $1 \leq i \leq r$. There exists $e > 0$ such that $eH - E$ is ample. If $d = 0$, so that $C \sim \sum -m_i E_i$ with some $m_i > 0$, then $((eH - E) \cdot C) = -\sum m_i < 0$, which is impossible.

2). By Serre duality

$$H^2(X, \mathcal{O}_X(dH - mE)) \cong H^0(X, \mathcal{O}_X(mE - dH + K_X) = H^0(X, \mathcal{O}_X(-(d+3)H + (m+1)E)).$$

The complete linear system $|H|$ is base point free on X and

$$(H \cdot (mE - dH + K_X)) = -(d+3) < 0 \text{ for } d \geq -2,$$

so $H^0(X, \mathcal{O}_X(mE - dH + K_X) = 0$.

3). Suppose that C is an irreducible reduced curve on X. Then by 1) and Theorem 6.2, C is linear equivalent to $eH - \sum n_i E_i$ with $e, n_1, \ldots, n_r \in \mathbb{N}$ and $e > \frac{1}{\sqrt{r}} \sum_{i=1}^r n_i$. Hence $(C \cdot L) = de - m \sum_{i=1}^r n_i > 0$. Further $(L^2) = d^2 - m^2 r > 0$, so L is ample by the Nakai Moishezon criterion ([16, Theorem V.1.10]).

4). The divisor $(dH - mE) - K_X$ is ample if $d > \sqrt{r}m + (\sqrt{r} - 3)$ by 3). Thus, by the Kodaira vanishing theorem ([16, Remark III.7.15]),

$$H^1(X, \mathcal{O}_X(dH - mE)) = H^1(X, \mathcal{O}_X((dH - mE - K_X) + K_X)) = 0$$

if $d > \sqrt{r}m + (\sqrt{r} - 3)$.

5). The statements 2) and 4) imply that $\mathcal{O}_X(-mE)$ is d-regular if $d > \sqrt{r}m + (\sqrt{r} - 2)$; that is, $H^i(X, \mathcal{O}_X(-mE) \otimes \mathcal{O}_X((d-i)H)) = 0$ for $i = 1, 2$. Thus the conclusions of 5) hold by page 99 [20] (also proven in [6, Theorem 17.35]). ☐

Let $S = \mathbb{C}[x_1, x_2, x_3]$ be the homogeneous coordinate ring of $\mathbb{P}^2$ and $\mathfrak{m}$ be the graded maximal ideal of S. Let P_i be the height two prime ideal in S of the point α_i for $1 \leq i \leq r$. Then

$$S = \oplus_{d \geq 0} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(dH')) = \oplus_{d \geq 0} H^0(X, \mathcal{O}_X(dH))$$

and

$$\mathfrak{m} = \oplus_{d > 0} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(dH')) = \oplus_{d > 0} H^0(X, \mathcal{O}_X(dH)).$$

For $d, n_1, \ldots, n_r \in \mathbb{N}$,

$$\begin{aligned} &H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(dH') \otimes \mathcal{I}_{\alpha_1}^{n_1} \otimes \cdots \otimes \mathcal{I}_{\alpha_r}^{n_r}) \\ &= \{F \in S \mid F \text{ is homogeneous of degree } d \text{ and } F \in P_1^{n_1} \cap \cdots \cap P_r^{n_r}\} \end{aligned}$$

and

$$\begin{aligned} P_1^{n_1} \cap \cdots \cap P_r^{n_r} &= \bigoplus_{d > 0} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(dH') \otimes \mathcal{I}_{\alpha_1}^{n_1} \otimes \cdots \otimes \mathcal{I}_{\alpha_r}^{n_r}) \\ &= \bigoplus_{d > 0} H^0(X, \mathcal{O}_X(dH - n_1 E_1 - \cdots - n_r E_r). \end{aligned}$$

In particular,

$$P_1^n \cap \cdots \cap P_r^n = \bigoplus_{d>0} H^0(X, \mathcal{O}_X(dH - nE)).$$

Recall that $s = \sqrt{r} \in \mathbb{Z}_+$ (with $s \geq 4$).

Proposition 6.4. *Given $n \in \mathbb{Z}_{>0}$,*

$$(P_1^n \cap \cdots \cap P_r^n)^s \subset \mathfrak{m}\left(P_1^{sn} \cap \cdots \cap P_r^{sn}\right).$$

Proof. Since S is graded and Noetherian, it suffices to show that if $0 \neq f \in H^0(X, \mathcal{O}_X(dH - nE))$, then $f^s \in H^0(X, \mathcal{O}_X(H))H^0(X, \mathcal{O}_X(sd-1)H - snE)$. We have that $d > sn$ by Theorem 6.2. The statement then follows from 5) of Lemma 6.3, since $f^s \in H^0(X, \mathcal{O}_X(sdH - snE))$ and $sd \geq s(ns+1) = s(sn) + s$. ☐

Let $R = S_{\mathfrak{m}}$, a three dimensional regular local ring, with maximal ideal $m_R = \mathfrak{m}S_{\mathfrak{m}}$. Let $\mathfrak{p}_i = (P_i)_{\mathfrak{m}}$ for $1 \leq i \leq r$. The ideals $\mathfrak{p}_i$ are height two prime ideals in R. Let $\mathcal{I} = \{I_n\}$ where $I_n = \mathfrak{p}_1^n \cap \cdots \cap \mathfrak{p}_r^n$. The filtration $\mathcal{I}$ is the 1-divisorial filtration on R, consisting of the symbolic powers $I_n = I^{(n)}$ of $I = I_1$. Thus $\mathrm{ht}(\mathcal{I}) = \mathrm{ht}(I) = 2$.

By Proposition 6.4, we have that for all $n > 0$,

$$(I^{(n)})^s = (\mathfrak{p}_1^n \cap \cdots \cap \mathfrak{p}_r^n)^s \subset m_R\left(\mathfrak{p}_1^{sn} \cap \cdots \cap \mathfrak{p}_r^{sn}\right) = m_R I^{(sn)}. \tag{17}$$

We will first construct the example when $a = 0$. In [24], a height two prime ideal $\mathfrak{p}$ in $R = \mathbb{C}[x_1, x_2, x_3]_{(x_1,x_2,x_3)}$ and a continuous $\mathbb{C}$-algebra isomorphism $\varphi : \hat{R} \to \hat{R}$ such that $\varphi(\widehat{I^{(n)}}) = \widehat{\mathfrak{p}^{(n)}}$ for all $n \geq 0$ are constructed, where I is the ideal defined before (17). Let $s = \sqrt{r} \in \mathbb{Z}_{>0}$ be the integer defined before Theorem 6.2. By (17), $(I^{(n)})^s \subset m_R I^{(ns)}$ for all $n > 0$. Thus $(\widehat{I^{(n)}})^s \subset m_{\hat{R}} \widehat{I^{(ns)}}$, so applying φ, we obtain

$$\widehat{(\mathfrak{p}^{(n)})^s} = (\widehat{\mathfrak{p}^{(n)}})^s \subset m_{\hat{R}} \widehat{\mathfrak{p}^{(ns)}} = \widehat{m_R \mathfrak{p}^{(ns)}}.$$

Since $R \to \hat{R}$ is faithfully flat, we obtain that for all $n > 0$, we have that

$$\left(\mathfrak{p}^{(n)}\right)^s \subset m_R \mathfrak{p}^{(sn)}. \tag{18}$$

Let ν be the $\mathfrak{p}_{\mathfrak{p}}$-adic valuation of $R_{\mathfrak{p}}$, which is the discrete valuation of $K := \mathbb{C}(x_1, x_2, x_3)$ such that the valuation ideals of ν in R are $I(\nu)_n = \mathfrak{p}^{(n)}$ for all $n \geq 0$.

Let $A = \mathbb{C}[x_1, x_2, x_3, y_1, \ldots, y_a]_{(x_1,x_2,x_3,y_1,\ldots,y_a)}$, the polynomial ring over $\mathbb{C}$ in the variables $x_1, x_2, x_3, y_1, \ldots, y_a$, and $Q = \mathfrak{p}A + (y_1, \ldots, y_a)$, a prime ideal of height $2 + a$ in A. Let ω be the Gauss valuation of $L := \mathbb{C}(x_1, x_2, x_3, y_1, \ldots, y_a)$, defined by

$$\omega(f) = \min\{\nu(b_{i_1,\ldots,i_a}) + i_1 + \cdots + i_a\}$$

if $f = \sum b_{i_1,\ldots,i_a} y_1^{i_1} \cdots y_a^{i_a} \in K[y_1, \ldots, y_a]$ with $b_{i_1,\ldots,i_a} \in K$ for all $i_1, \ldots, i_a$. The valuation ω is a discrete valuation of L which dominates A_Q. Since

$$A_Q = (R_{\mathfrak{p}}[y_1, \ldots, y_a])_{QR_{\mathfrak{p}}[y_1,\ldots,y_a]},$$

we have that ω is the Q-adic valuation of A_Q. Thus the valuation ideals of ω in A are $I(\omega)_n = Q^{(n)}$ for $n \geq 0$. Let $\mathcal{J} = \{Q^{(n)}\}$. The filtration $\mathcal{J}$ is a 1-divisorial filtration on A.

Proposition 6.5. *Let $N = \sqrt{m_A A[\mathcal{J}]}$ be the radical of $m_A A[\mathcal{J}]$ in $A[\mathcal{J}]$. Let $\overline{y}_1, \ldots \overline{y}_a$ be the respective classes of $y_1 t, \ldots, y_a t$ in $A[\mathcal{J}]/N$. Then*

$$A[\mathcal{J}]/N = \mathbb{C}[\overline{y}_1, \ldots, \overline{y}_a]$$

is a standard graded polynomial ring in the variables $\overline{y}_1, \ldots, \overline{y}_a$ over $\mathbb{C}$.

Proof. For $n > 0$,

$$Q^{(n)} = I(\omega)_n = (y_1, \ldots, y_a)^n + (y_1, \ldots, y_a)^{n-1}\mathfrak{p} + (y_1, \ldots, y_a)^{n-2}\mathfrak{p}^{(2)} + \cdots + \mathfrak{p}^{(n)}A.$$

We first show that $A[\mathcal{J}]/N$ is generated by $\overline{y}_1, \ldots, \overline{y}_a$ as a $\mathbb{C}$-algebra. Since N is graded, it suffices to show that if $f \in \mathfrak{p}^{(d)}$ with $d > 0$ and $y_1^{i_1} \cdots y_a^{i_a}$ is such that $i_1, \ldots, i_a \in \mathbb{N}$ and $i_1 + \cdots + i_a = n - d$, then $(y_1^{i_1} \cdots y_a^{i_a})^s f^s \in m_A Q^{(sn)}$. Since $(i_1 + \cdots + i_a)s = (n-d)s$, it suffices to show that $f^s \in m_R \mathfrak{p}^{(sd)}$, which follows from (18).

We now show that the standard graded $\mathbb{C}$-algebra $\mathbb{C}[\overline{y}_1, \ldots, \overline{y}_a]$ is a polynomial ring over $\mathbb{C}$. Suppose otherwise. We will find a contradiction. Then for some $n > 0$, there is a relation

$$\sum_{i_1+\cdots+i_a=n} \lambda_{i_1,\ldots,i_a} \overline{y}_1^{i_1} \cdots \overline{y}_a^{i_1} = 0$$

for some $\lambda_{i_1,\ldots,i_a} \in \mathbb{C}$ not all zero. Let $G = \sum_{i_1+\cdots+i_a=n} \lambda_{i_1,\ldots,i_a} y_1^{i_1} \cdots y_a^{i_a} \in Q^{(n)}$. Since $G \in N$, there exists $m > 0$ such that $G^m \in m_A Q^{(mn)}$. Now G^m has m_A-order mn, and every element of $m_A Q^{(mn)}$ has m_A-order $\geq mn + 1$. Thus $G = 0$, a contradiction to the assumption that some $\lambda_{i_1,\ldots,i_a} \neq 0$. □

Example 6.1 thus has analytic spread $\ell(\mathcal{J}) = a$.

Example 6.6. *There exists a prime ideal $\mathfrak{p}$ of height 2 in a regular local ring R of dimension 3 such that $\ell(\mathcal{J}) = 1$, where $\mathcal{J} = \{\mathfrak{p}^{(n)}\}$ is the 1-divisorial filtration on R of symbolic powers of $\mathfrak{p}$.*

We make use of a famous example of Zariski [26], expositions of which can be found in [18, Section 2.3] and [6, Theorem 20.14]. Let $\alpha_1, \ldots, \alpha_{12} \in \mathbb{P}^2_{\mathbb{C}}$ be independent generic points of an elliptic curve E' of $\mathbb{P}^2$ over $\mathbb{Q}$. The curve E' is defined by the vanishing of an irreducible cubic form $G \in \mathbb{C}[x_1, x_2, x_3]$.

Let $\mathcal{I}_{\alpha_i}$ be the ideal sheaf of α_i in $\mathbb{P}^2$ and let H' be a linear hyperplane section of $\mathbb{P}^2$.

Let $\Lambda : X \to \mathbb{P}^2$ be the blow up of the points $\alpha_1, \ldots, \alpha_{12}$ with exceptional lines $F_1, \ldots, F_{12}$. Let $H = \Lambda^*(H')$. Since Λ is the blowup of the points $\alpha_1, \ldots, \alpha_{12}$ on the nonsingular surface $\mathbb{P}^2$, we have that for all $d, m_1, \ldots, m_r \geq 0$,

$$H^0(X, \mathcal{O}_X(dH - m_1F_1 - \cdots - m_{12}F_{12})) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(dH') \otimes \mathcal{I}_{\alpha_1}^{m_1} \otimes \cdots \otimes \mathcal{I}_{\alpha_r}^{m_r}).$$

Let $F = F_1 + \cdots + F_{12}$. The canonical divisor K_X on X is $K_X = -3H + F$. Let E be the strict transform of E' on X. We have that $\Lambda^*(E') = E + F$, where E is an elliptic curve on X which is isomorphic to E', $(E \cdot E) = -3$ and $\mathcal{O}_X(H + E) \otimes \mathcal{O}_E$ is a degree 0 invertible sheaf on X of infinite order, so that

$$H^0(E, \mathcal{O}_X(m(H + E)) \otimes \mathcal{O}_E) = 0 \text{ for all nonzero integers } m. \tag{19}$$

Further, $(F \cdot F) = -12$.

Lemma 6.7. *Let notation be as above.*

1) *Let C be an irreducible reduced curve on X with $C \neq F_i$ for any i. Then $C \sim dH - \sum m_i F_i$ for some $d, m_i \in \mathbb{N}$ with $d > 0$.*
2) *Let $dH - mF$ be a divisor with $d \geq -2$. Then $H^2(X, \mathcal{O}_X(dH - mF)) = 0$.*
3) *Let $L = dH - mF$ with $m \in \mathbb{Z}_{>0}$ and $d > 4m$. Then L is ample.*
4) *Let $\sigma \in H^0(X, \mathcal{O}_X(3H - F))$ be the section whose divisor is E. Then*

$$\mathcal{O}_X(3H - F) = \sigma \mathcal{O}_X$$

and $H^0(X, \mathcal{O}_X(3H - F)) = \sigma\mathbb{C}$. *Further,*

$$H^0(X, \mathcal{O}_X(3mH - mF)) = H^0(X, \mathcal{O}_X(3H - F))^m$$

for all $m \in \mathbb{Z}_{>0}$.

5) *Suppose that* $d \geq 0$, $m \geq 0$ *and* $d < 3m$. *then* $H^0(X, \mathcal{O}_X(dH - mF)) = 0$.
6) *Let* $L = dH - mF$ *with* $d, m \in \mathbb{Z}_{>0}$. *Then* $H^1(X, \mathcal{O}_X(dH - mF)) = 0$ *if* $d > 4m+1$.
7) *Suppose that* $d, m \geq 0$. *Then*

$$H^0(X, \mathcal{O}_X(dH - mF)) = H^0(X, \mathcal{O}_X(H))H^0(X, \mathcal{O}_X((d-1)H - mF))$$

if $d \geq 4m + 4$.

Proof. The proofs of 1) and 2) are the same as the proofs of 1) and 2) of Lemma 6.3.

3). Suppose C is an integral curve on X other than F_i or E. Then $(C \cdot E) \geq 0$ and $(C \cdot H) = (C \cdot \Lambda^*(H')) = (\Lambda_*(C) \cdot H') > 0$ so $(C \cdot L) > 0$, since $L \sim mE + (d-3m)H$. Further, $(L \cdot E) = 3d - 12m > 0$ and $(L \cdot F_i) = m > 0$ for $1 \leq i \leq 12$, so every irreducible curve of X has positive intersection number with L. Finally, $(L \cdot L) = d^2 - 12m^2 > 4m^2 > 0$, so L is ample by the Nakai Moishezon criterion.

4). We have that $mE \sim 3mH - mF$. Suppose that D is an effective divisor on X which is linearly equivalent to $3mH - mF$. Then $(E \cdot (3mH - mF)) = (E \cdot mE) = -3m < 0$ so E is in the support of D, so $D - E$ is effective. By induction on m, we obtain $D = mE$.

5). Suppose there exists an effective divisor D such that $D \sim dH - mF$. We compute

$$(D \cdot E) = ((dH - mF) \cdot (3H - F)) = 3d - 12m < -3m.$$

Thus E is in the support of D, so $D - E \sim (d-3)H - (m-1)F$ is effective, with $d - 3 < 3(m-1)$. Continuing in this way, we obtain that $(d - 3m)H - 2mF$ is an effective divisor, which is a contradiction to 1).

The proof of 6) is as the proof of 4) of Lemma 6.3, using 3) of this lemma.

The proof of 7) is as the proof of 5) of Lemma 6.3, using 6) and 2) of this lemma.

□

Lemma 6.8. *Suppose that* $0 < 3m < d \leq 4m$. *Then*

1) $H^0(X, \mathcal{O}_X(dH - mF)) > 0$.
2) $H^0(X, \mathcal{O}_X(dH - mF)) = H^0(X, \mathcal{O}_X(3H - F))H^0(X, \mathcal{O}_X((d-3)H - (m-1)F)$ *where* $d - 3 > 3(m-1)$.
3) $H^0(X, \mathcal{O}_X(dH - mF)) = H^0(X, \mathcal{O}_X(3H - F))^r H^0(X, \mathcal{O}_X(d'H - m'F))$ *where* $r = 4m - d + 1$, $d' = d - 3r$, $m' = m - r$ *satisfy* $d' > 4m'$.

Proof. 1). We have that $dH - mF = m(3H - F) + (d - 3m)H \sim mE + (d - 3m)H$, which is a nonzero effective divisor.

2). Recall that σ is a section of $\mathcal{O}_X(3H - F)$ whose divisor is E. Tensor

$$0 \to \mathcal{O}_X(-(3H - F)) \xrightarrow{\sigma} \mathcal{O}_X \to \mathcal{O}_E \to 0$$

with $\mathcal{O}_X(dH - mF)$ to get the exact sequence

$$0 \to H^0(X, \mathcal{O}_X((d-3)H - (m-1)F)) \xrightarrow{\sigma} H^0(X, \mathcal{O}_X(dH - mF)) \to H^0(E, \mathcal{O}_X(dH - mF) \otimes \mathcal{O}_E).$$

The rightmost vector space is zero since $((dH - mF) \cdot E) = 3d - 12m \leq 0$, and by (19).

3). For $t \in \mathbb{Q}$, we have that $(d - 3t)H - (m - t)F$ satisfies $d - 3t \leq 4(m - t)$ if and only if $t \leq 4m - d$. Now 3) follows from 1) and 2) of this lemma.

□

Let $S = \mathbb{C}[x_1, x_2, x_3]$ be the homogeneous coordinate ring of $\mathbb{P}^2$ and $\mathfrak{m}$ be the graded maximal ideal of S. Let P_i be the height two prime ideal in S of the point α_i for $1 \le i \le 12$. Then

$$S = \oplus_{d\ge 0} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(dH')) = \oplus_{d\ge 0} H^0(X, \mathcal{O}_X(dH)),$$
$$\mathfrak{m} = \oplus_{d>0} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(dH')) = \oplus_{d>0} H^0(X, \mathcal{O}_X(dH))$$

and for $n_1, \ldots, n_{12} \ge 0$,

$$\begin{aligned} P_1^{n_1} \cap \cdots \cap P_{12}^{n_{12}} &= \bigoplus_{d>0} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(dH') \otimes \mathcal{I}_{\alpha_1}^{n_1} \otimes \cdots \otimes \mathcal{I}_{\alpha_{12}}^{n_{12}}) \\ &= \bigoplus_{d>0} H^0(X, \mathcal{O}_X(dH - n_1F_1 - \cdots - n_{12}F_{12}). \end{aligned}$$

In particular,

$$P_1^n \cap \cdots \cap P_{12}^n = \bigoplus_{d>0} H^0(X, \mathcal{O}_X(dH - nF)).$$

By 5) of Lemma 6.7,

$$P_1^n \cap \cdots \cap P_{12}^n = \bigoplus_{d\ge 3n} H^0(X, \mathcal{O}_X(dH - nF)). \tag{20}$$

The irreducible cubic form G defining E' is in $H^0(X, \mathcal{O}_X(3H - F))$ and

$$h^0(X, \mathcal{O}_X(3nH - nF)) = 1 \text{ for } n > 0$$

by 4) of Lemma 6.7, so

$$H^0(X, \mathcal{O}_X(n(3H - F)) = G^n\mathbb{C} \text{ for } n > 0. \tag{21}$$

Proposition 6.9. *Given $n \in \mathbb{Z}_{>0}$, and $h \in H^0(X, \mathcal{O}_X(dH - nF))$ with $d > 3n$, there exists $s \in \mathbb{Z}_{>0}$ such that*

$$h^s \in \mathfrak{m}\,(P_1^{sn} \cap \cdots \cap P_{12}^{sn}).$$

Proof. First suppose that $d > 4n$. Then $4d \ge 4(4n) + 4$ implies

$$H^0(X, \mathcal{O}_X(4dH - 4nF)) = H^0(X, \mathcal{O}_X(H))H^0(X, \mathcal{O}_X(4d - 1)H - 4nF))$$

by 7) of Lemma 6.7. Thus $h^4 \in \mathfrak{m}\left(P_1^{4n} \cap \cdots \cap P_{12}^{4n}\right)$.

Now suppose that $3n < d \le 4n$. Then by 3) of Lemma 6.8,

$$H^0(X, \mathcal{O}_X(dH - nF)) = H^0(3H - F))^r H^0(X, \mathcal{O}_X(d'H - m'F))$$

for suitable r, d', m' where $d' > 4m'$ and $n = m' + r$. By the first part of this proof,

$$\begin{aligned} h^4 &\in H^0(X, \mathcal{O}_X(H))H^0(X, \mathcal{O}_X(3H - F))^{4r} H^0(X, \mathcal{O}_X((4d' - 1)H - 4m'F)) \\ &\subset \mathfrak{m}\left(P_1^{4n} \cap \cdots \cap P_{12}^{4n}\right). \end{aligned}$$

□

Let $R = S_{\mathfrak{m}}$, a three dimensional regular local ring, with maximal ideal $m_R = \mathfrak{m}S_{\mathfrak{m}}$. Let $\mathfrak{p}_i = (P_i)_{\mathfrak{m}}$ for $1 \le i \le 12$. The ideals $\mathfrak{p}_i$ are height two prime ideals in R. Let $\mathcal{I} = \{I_n\}$ where $I_n = \mathfrak{p}_1^n \cap \cdots \cap \mathfrak{p}_{12}^n$. The filtration $\mathcal{I}$ is the 1-divisorial filtration on R consisting of the symbolic powers $I_n = I^{(n)}$ of $I = I_1$. Thus $\mathrm{ht}(\mathcal{I}) = \mathrm{ht}(I) = 2$.

Proposition 6.10. *Let $N = \sqrt{m_R R[\mathcal{I}]}$ be the radical of $m_R R[\mathcal{I}]$ in $R[\mathcal{I}]$. Then*

$$R[\mathcal{I}]/N \cong \mathbb{C}[Gt]$$

is a standard graded polynomial ring.

Proof.

$$R[\mathcal{I}]/N \cong \left(\sum_{n\ge 0}\mathfrak{p}_1^{(n)}\cap\cdots\cap\mathfrak{p}_{12}^{(n)}t^n\right)/\sqrt{\mathfrak{m}\left(\sum_{n\ge 0}\mathfrak{p}_1^{(n)}\cap\cdots\cap\mathfrak{p}_{12}^{(n)}t^n\right)}\cong \mathbb{C}[Gt].$$

by (20), (21) and Proposition 6.9. ☐

By the method of [24], we construct a height two prime ideal $\mathfrak{p}$ in $R=\mathbb{C}[x_1,x_2,x_3]_{(x_1,x_2,x_3)}$ and a continuous $\mathbb{C}$-algebra isomorphism $\varphi:\hat{R}\to\hat{R}$ such that $\varphi(\hat{m}_R)=\hat{m}_R$ and $\varphi(\widehat{I^{(n)}})=\widehat{\mathfrak{p}^{(n)}}$ for all $n\ge 0$, where I is the ideal defined before Proposition 6.10. We have that $\hat{\mathfrak{p}}^{(n)}=\widehat{\mathfrak{p}^{(n)}}$ and $\hat{I}^{(n)}=\widehat{I^{(n)}}$ for all n, as explained in the proof of [24, Proposition 1].

Since the Rees algebras of all truncations of $\{\mathfrak{p}^{(n)}\}$ are excellent, we have that

$$\hat{R}\sqrt{m_R R[\mathfrak{p}t,\mathfrak{p}^{(2)}t^2,\ldots,\mathfrak{p}^{(a)}t^a]}=\sqrt{\hat{m}_R\hat{R}[\hat{\mathfrak{p}}t,\hat{\mathfrak{p}}^{(2)}t^2,\ldots,\hat{\mathfrak{p}}^{(a)}t^a]}$$

for all $a\in\mathbb{Z}_{>0}$ and so

$$\hat{R}\sqrt{m_R R[\{\mathfrak{p}^{(n)}\}]}=\sqrt{\hat{m}_R\hat{R}[\{\hat{\mathfrak{p}}^{(n)}\}]}.$$

We thus have that

$$R[\{\mathfrak{p}^{(n)}\}]/\sqrt{m_R R[\{\mathfrak{p}^{(n)}\}]}\cong\left(R[\{\mathfrak{p}^{(n)}\}]/\sqrt{m_R R[\{\mathfrak{p}^{(n)}\}]}\right)\otimes_R\hat{R}\cong\hat{R}[\{\widehat{\mathfrak{p}^{(n)}}\}]/\sqrt{\hat{m}_R\hat{R}[\{\widehat{\mathfrak{p}^{(n)}}\}]}$$
$$\cong\hat{R}[\{\widehat{\mathcal{I}^{(n)}}\}]/\sqrt{\hat{m}_R\hat{R}[\{\widehat{\mathcal{I}^{(n)}}\}]}\cong\left(R[\mathcal{I}]/\sqrt{m_R R[\mathcal{I}]}\right)\otimes_R\hat{R}\cong R[\mathcal{I}]/\sqrt{m_R R[\mathcal{I}]}\cong\mathbb{C}[t]$$

by Proposition 6.10. Thus $\mathcal{J}=\{\mathfrak{p}^{(n)}\}$ fulfills the conditions of the example.

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