ANALYTIC SPREAD OF FILTRATIONS ON TWO DIMENSIONAL NORMAL LOCAL RINGS

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Abstract. In this paper we prove that a classical theorem by McAdam about the analytic spread of an ideal in a Noetherian local ring continues to be true for divisorial filtrations on a two dimensional normal excellent local ring R, and that the Hilbert polynomial of the fiber cone of a divisorial filtration on R has a Hilbert function which is the sum of a linear polynomial and a bounded function. We prove these theorems by first studying asymptotic properties of divisors on a resolution of singularities of the spectrum of R. The filtration of the symbolic powers of an ideal is an example of a divisorial filtration. Divisorial filtrations are often not Noetherian, giving a significant difference in the classical case of filtrations of powers of ideals and divisorial filtrations.

1. Introduction

Divisorial filtrations on two dimensional normal excellent local rings have excellent properties, as we show in this article.

1.1. **Filtrations of powers of ideals and Analytic Spread.** In this subsection we give an outline of how the classical theory of the analytic spread of an ideal admits a simple geometric interpretation in the case of an ideal in a normal excellent local ring. The generalization of analytic spread to divisorial filtrations can then be seen as a natural extension of this theory.

Expositions of the theory of complete ideals, integral closure of ideals and their relation to valuation ideals, Rees valuations, analytic spread and birational morphisms can be found, from different perspectives, in [37], [33], [23] and [25]. The book [33] and the article [25] contain references to original work in this subject. Concepts in this introduction which are not defined in this section or in these references can be found in Section 2 of this paper. A survey of recent work on symbolic algebras is given in [15]. A different notion of analytic spread for families of ideals is given in [16]. A recent paper exploring ideal theory in two dimensional normal local domains using geometric methods is [31].

Let R be a normal excellent local ring with maximal ideal m_R and I be an ideal in R. Let $\pi : X \to \mathrm{Spec}(R)$ be projective and birational (so that π is the blow up of an ideal) and such that X is normal and $I\mathcal{O}_X$ is an invertible sheaf. Let $I\mathcal{O}_X = \mathcal{O}_X(-D)$ where D is an effective and anti-nef divisor (the intersection product $(D \cdot E) \leq 0$ for all exceptional curves E of X). Then $\Gamma(X, \mathcal{O}_X(-nD)) = \overline{I^n}$, the integral closure of I^n, for all $n \in \mathbb{N}$. Write $D = a_1F_1 + \cdots + a_sF_s$ where the F_i are prime divisors. The local rings $\mathcal{O}_{X,F_i}$ are discrete (rank 1) valuation rings. Let ν_{F_i} be the associated valuations. We have that the integral closure of I^n is

$$\overline{I^n} = \Gamma(X, \mathcal{O}_X(-nD)) = I(\nu_{F_1})_{na_1} \cap \cdots \cap I(\nu_{F_s})_{na_s}$$

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where

$$I(\nu_{F_i})_b = \{f \in R \mid \nu_{F_i}(f) \geq b\}$$

are the valuation ideals in R associated to ν_{F_i}. The center of ν_{F_i} on R is the prime ideal $I(\nu_{F_i})_1$. The Rees valuations of I are those ν_{F_j} such that $\overline{I^n} \neq \cap_{i\neq j} I(\nu_{F_i})_{na_i}$. Let Y be the normalization of the blow up $B(I)$ of I and let $I\mathcal{O}_Y = \mathcal{O}_Y(-B)$. Then $Y \to \mathrm{Spec}(R)$ is projective (since R is universally Nagata). The divisor $-B$ is ample on Y and so the Rees valuations of I are exactly the prime components of B. By the universal property of blowing up, π factors through $B(I)$ and since X is normal, π factors through Y. Let $\varphi : X \to Y$ be the induced morphism. Let F be a prime component of D, with associated valuation ν_F. Then ν_F is a Rees valuation of I if and only if φ does not contract F, in which case $\varphi(F) = E$ is a prime component of B and we have that $\mathcal{O}_{X,F} = \mathcal{O}_{Y,E}$.

In the case that $\dim R = 2$, the prime divisor F is contracted by φ if and only if F is exceptional ($\pi(F) = m_R$) and $(D \cdot F) = 0$. Thus the Rees valuations of I are precisely the valuations associated to prime divisors F of X such that either ν_F has center a height one prime of R or F is exceptional for π (the center of ν_F on R is m_R) and $(D \cdot F) < 0$.

Let us return to not having any restrictions on the dimension of R. We have an associated graded ring $R[It] = \sum_{n\geq 0} I^n t^n$ (the Rees algebra of I). The integral closure of $R[It]$ in $R[t]$ is the graded algebra $\overline{R[It]} = \sum_{n\geq 0} \overline{I^n} t^n$, which is a finite extension of $R[It]$ (since R is universally Nagata). The blow up of I is $B(I) = \mathrm{Proj}(R[It])$ and $Y = \mathrm{Proj}(\overline{R[It]})$ is the normalization of the blow up of I, which was introduced earlier. Let $\psi : B(I) \to \mathrm{Spec}(R)$ be the projection.

The blowup $B(I)$ has the important subschemes

$$\psi^{-1}(V(I)) = \mathrm{Proj}(\mathrm{gr}_I(R)) \text{ and } \psi^{-1}(m_R) = \mathrm{Proj}(R[It]/m_R R[It]).$$

The R-algebra $\mathrm{gr}_I(R) = \sum_{n\geq 0} I^n/I^{n+1} t^n$ is the associated graded ring of I and the R-algebra $R[It]/m_R R[It]$ is the fiber cone of I.

Since $\mathrm{Proj}(R[It]) \to \mathrm{Spec}(R)$ and $\mathrm{Proj}(\overline{R[It]}) \to \mathrm{Spec}(R)$ are birational, the dimensions of $\mathrm{Proj}(R[It])$ and $\mathrm{Proj}(\overline{R[It]})$ are the same as the dimension of R. Further, since $\mathrm{Proj}(\mathrm{gr}_I(R))$ is a Cartier divisor on $\mathrm{Proj}(R[It])$, we have that $\dim(\mathrm{Proj}(\mathrm{gr}_I(R)) = \dim R - 1$. Now, since $I \subset m_R$, we have that $\mathrm{Proj}(R[It]/m_R R[It])$ is a subscheme of $\mathrm{Proj}(\mathrm{gr}_I(R))$, so we have $\dim(\mathrm{Proj}(R[It]/m_R R[It])) \leq \dim R - 1$.

Let $\psi_0 : \mathrm{Proj}(\mathrm{gr}_I(R)) \to \mathrm{Spec}(R/I)$ be the projective morphism induced by ψ. Let P be a minimum prime of I. Then $\dim \psi_0^{-1}(P) = \dim R_P - 1$ since I_p is primary for the maximal ideal of R_P. We have that $\dim \psi^{-1}(m_R) = \dim \psi_0^{-1}(m_R) \geq \dim \psi_0^{-1}(P)$ by upper semi-continuity of fiber dimension ([19, Corollary IV.13.1.5]). Thus

$$\mathrm{ht}(I) \leq \dim \psi^{-1}(m_R) + 1.$$

The analytic spread of I is defined to be

$$\ell(I) = \dim R[It]/m_R R[It].$$

Since the dimension of the Proj of a graded ring is one less than the dimension of the ring, we have established in our case of normal excellent local rings the following theorems.

Theorem 1.1. *(*[33, Proposition 5.1.6 and Corollary 8.3.9]*) Let R be a Noetherian local ring and I be an ideal in R. Then*

$$\mathrm{ht}(I) \leq \ell(I) \leq \dim \mathrm{gr}_I(R) = \dim R.$$

Theorem 1.2. *([33, Proposition 5.4.8]) Let R be a Noetherian formally equidimensional local ring and let I be an ideal in R. For every minimal prime ideal P of $\mathrm{gr}_I(R)$, $\dim(gr_I(R)/P) = \dim R$.*

We return to the case that R is a normal excellent local ring of arbitrary dimension. We have that $\ell(I) = \dim R$ if and only if $\dim \psi^{-1}(m_R) = \dim R - 1$. Since

$$Y = \mathrm{Proj}(\overline{R[It]}) \to B(I) = \mathrm{Proj}(R[It])$$

is finite, $\dim \psi^{-1}(m_R) = \dim R - 1$ if and only if there exists a prime divisor E on Y which contracts to m_R; that is, the center of ν_E on R is m_R. Writing

$$I\mathcal{O}_Y = \mathcal{O}_Y(-b_1F_1 - \cdots - b_sF_s)$$

where F_i are prime divisors and $b_i > 0$, we have that

$$\overline{I^n} = I(\nu_{F_1})_{nb_1} \cap \cdots \cap I(\nu_{F_s})_{nb_s}$$

where ν_{F_i} is the discrete rank 1 valuation associated to the valuation ring $\mathcal{O}_{Y,F_i}$. Since $-b_1F_1 - \cdots - b_sF_s$ is ample on Y, we have that $\overline{I^n} \neq \cap_{i\neq j} I(\nu_{F_i})$ for all j and $n \gg 0$ (so that $\nu_{F_1}, \ldots, \nu_{F_s}$ are the Rees valuations of I). Thus $\dim \psi^{-1}(m_R) = \dim R - 1$ holds if and only if $m_R \in \mathrm{Ass}(R/\overline{I^n})$ for some n.

We have established the following theorem in our case of normal excellent local rings.

Theorem 1.3. *([27], [33, Theorem 5.4.6]) Let R be a formally equidimensional local ring and I be an ideal in R. Then $m_R \in Ass(R/\overline{I^n})$ for some n if and only if $\ell(I) = \dim(R)$.*

The assumption of being formally equidimensional is not required for the if direction of Theorem 1.3 (this is Burch's theorem, [6], [33, Proposition 5.4.7]).

Let $k = R/m_R$. Since $R[It]/m_RR[It]$ is a standard graded ring over k (finitely generated in degree 1) it has a Hilbert polynomial P(n) which has degree $d = \ell(I) - 1$; there exists a positive integer n_0 such that

$$\dim_k I^n/m_RI^n = P(n) \text{ for } n \geq n_0. \tag{1}$$

As $\overline{R[It]}/m_R\overline{R[It]}$ is a finitely generated graded ring over k, there exists $e \in \mathbb{Z}_{>0}$ and polynomials $P_0, \ldots, P_{e-1}$ of degree $d = \ell(I) - 1$ such that

$$\dim_k \overline{I^n}/m_R\overline{I^n} = P_i(n) \text{ for } n \geq n_0 \text{ where } i \equiv n \bmod e. \tag{2}$$

1.2. **Filtrations.** Let $\mathcal{I} = \{I_n\}$ be a filtration on a local ring R. The Rees algebra of the filtration is $R[\mathcal{I}] = \oplus_{n\geq 0} I_n$. Analogously to the case of ideals, we define the fiber cone of the filtration $\mathcal{I}$ to be $R[\mathcal{I}]/m_RR[\mathcal{I}]$ and the analytic spread of the filtration of $\mathcal{I}$ to be

$$\ell(\mathcal{I}) = \dim R[\mathcal{I}]/m_RR[\mathcal{I}]. \tag{3}$$

We have that $\mathrm{ht}(I_n) = \mathrm{ht}(I_1)$ for all n ([13, equation (7)]) so it is natural to define $\mathrm{ht}(\mathcal{I}) = \mathrm{ht}(I_1)$.

We always have ([13, Lemma 3.6]) that

$$\ell(\mathcal{I}) \leq \dim R$$

so the second inequality of Theorem 1.1 always holds. However, the first inequality of Theorem 1.1, $\mathrm{ht}(\mathcal{I}) \leq \ell(\mathcal{I})$, fails spectacularly, even attaining the condition that $\ell(\mathcal{I}) = 0$ ([13, Example 1.2, Example 6.1 and Example 6.6]). The last two of these examples are of symbolic algebras of space curves, which are divisorial filtrations. We give a further example where the inequality fails in Example 7.3 of this paper. Example 7.3 is of a symbolic algebra of an intersection of height 1 prime ideals in a two dimensional excellent normal local ring. In the case that $\mathcal{I}$ is a Noetherian filtration ($R[\mathcal{I}]$ is a finitely generated R-algebra), the lower bound $\mathrm{ht}(\mathcal{I}) \le \ell(\mathcal{I})$ always holds ([13, Proposition 3.7]), so that the inequality of Theorem 1.1 for ideals continues to hold for Noetherian filtrations.

The condition that a filtration has analytic spread zero has a simple ideal theoretic interpretation ([13, Lemma 3.8]). Suppose that $\mathcal{I} = \{I_n\}$ is a filtration in a local ring R. Then the analytic spread $\ell(\mathcal{I}) = 0$ if and only if

$$\text{For all } n > 0 \text{ and } f \in I_n, \text{ there exists } m > 0 \text{ such that } f^m \in m_R I_{mn}.$$

1.3. **Divisorial Filtrations.** Let R be a local domain of dimension d with quotient field K. Let ν be a discrete valuation of K with valuation ring V_ν and maximal ideal m_ν. Suppose that $R \subset V_\nu$. Then for $n \in \mathbb{N}$, define valuation ideals

$$I(\nu)_n = \{f \in R \mid \nu(f) \ge n\} = m_\nu^n \cap R.$$

A divisorial valuation of R ([33, Definition 9.3.1]) is a valuation ν of K such that if V_ν is the valuation ring of ν with maximal ideal m_ν, then $R \subset V_\nu$ and if $p = m_\nu \cap R$ then $\mathrm{trdeg}_{\varkappa(p)} \varkappa(\nu) = \mathrm{ht}(p) - 1$, where $\varkappa(p)$ is the residue field of R_p and $\varkappa(\nu)$ is the residue field of V_ν. If ν is divisorial valuation of R such that $m_R = m_\nu \cap R$, then ν is called an m_R-valuation.

By [33, Theorem 9.3.2], the valuation ring of every divisorial valuation ν is Noetherian, hence is a discrete valuation. Suppose that R is an excellent local domain. Then a valuation ν of the quotient field K of R which is nonnegative on R is a divisorial valuation of R if and only if the valuation ring V_ν of ν is essentially of finite type over R ([12, Lemma 5.1]).

In general, the filtration $\mathcal{I}(\nu) = \{I(\nu)_n\}$ is not Noetherian; that is, the graded R-algebra $\sum_{n\ge 0} I(\nu)_n t^n$ is not a finitely generated R-algebra. In a two dimensional normal local ring R, the condition that the filtration of valuation ideals $\mathcal{I}(\nu)$ is Noetherian for all m_R-valuations ν dominating R is the condition (N) of Muhly and Sakuma [29]. It is proven in [9] that a complete normal local ring of dimension two satisfies condition (N) if and only if its divisor class group is a torsion group.

An integral divisorial filtration of R (which we will refer to as a divisorial filtration in this paper) is a filtration $\mathcal{I} = \{I_m\}$ such that there exist divisorial valuations $\nu_1, \ldots, \nu_s$ and $a_1, \ldots, a_s \in \mathbb{Z}_{\ge 0}$ such that for all $m \in \mathbb{N}$,

$$I_m = I(\nu_1)_{ma_1} \cap \cdots \cap I(\nu_s)_{ma_s}.$$

$\mathcal{I}$ is called an $\mathbb{R}$-divisorial filtration if $a_1, \ldots, a_s \in \mathbb{R}_{>0}$ and $\mathcal{I}$ is called a $\mathbb{Q}$-divisorial filtration if $a_1, \ldots, a_s \in \mathbb{Q}$. If $a_i \in \mathbb{R}_{>0}$, then

$$I(\nu_i)_{na_i} := \{f \in R \mid \nu_i(f) \ge na_i\} = I(\nu_i)_{\lceil na_i \rceil},$$

where $\lceil x \rceil$ is the round up of a real number.

Given an ideal I in R, the filtration $\{\overline{I^n}\}$ is an example of a divisorial filtration of R. The filtration $\{\overline{I^n}\}$ is Noetherian if R is universally Nagata.

It is shown in [13, Theorem 4.5] that the "if" statement of Theorem 1.3 is true for divisorial filtrations of a local domain R.

Theorem 1.4. *([13, Theorem 4.5]) Suppose that R is a local domain and $\mathcal{I} = \{I_n\}$ is a divisorial filtration on R such that $\ell(I) = \dim R$. Then $m_R \in Ass(R/\overline{I^n})$ for infinitely many n.*

An interesting question is if the converse of Theorem 1.3 is also true for divisorial filtrations of a local ring R. We prove this for two dimensional excellent normal local rings in this paper (Theorem 7.1, also stated in Theorem 1.5 of this introduction).

1.4. **Divisorial filtrations on normal excellent local rings.** Let R be a normal excellent local ring. Let $\mathcal{I}=\{I_m\}$ where

$$I_m=I(\nu_1)_{ma_1}\cap\cdots\cap I(\nu_s)_{ma_s}.$$

for some divisorial valuations $\nu_1,\ldots,\nu_s$ on R be an $\mathbb{R}$-divisorial filtration on a normal excellent local ring R, with $a_1,\ldots,a_s\in\mathbb{R}_{>0}$. Then there exists a projective birational morphism $\varphi: X\to \mathrm{Spec}(R)$ such that there exist prime divisors $F_1,\ldots,F_s$ on X such that $V_{\nu_i}=\mathcal{O}_{X,F_i}$ for $1\le i\le s$. Let $D=a_1F_1+\cdots+a_sF_s$, an effective $\mathbb{R}$-divisor. Define $\lceil D\rceil=\lceil a_1\rceil F_1+\cdots+\lceil a_s\rceil F_s$, an integral divisor. We have coherent sheaves $\mathcal{O}_X(-\lceil nD\rceil)$ on X such that

$$\Gamma(X,\mathcal{O}_X(-\lceil nD\rceil))=I_n \tag{4}$$

for $n\in\mathbb{N}$. If X is nonsingular then $\mathcal{O}_X(-\lceil nD\rceil)$ is invertible. The formula (4) is independent of choice of X. Further, even on a particular X, there are generally many different choices of effective $\mathbb{R}$-divisors G on X such that $\Gamma(X,\mathcal{O}_X(-\lceil nG\rceil))=I_n$ for all $n\in\mathbb{N}$. Any choice of a divisor G on such an X for which the formula $\Gamma(X,\mathcal{O}_X(-\lceil nG\rceil))=I_n$ for all $n\in\mathbb{N}$ holds will be called a representation of the filtration $\mathcal{I}$.

Given an $\mathbb{R}$-divisor $D=a_1F_1+\cdots+a_sF_s$ on X we have a divisorial filtration $\mathcal{I}(D)=\{I(D)_n\}$ where

$$I(D)_n=\Gamma(X,\mathcal{O}_X(-\lceil nD\rceil))=I(\nu_1)_{\lceil na_1\rceil}\cap\cdots\cap I(\nu_s)_{\lceil na_s\rceil}=I(\nu_1)_{ma_1}\cap\cdots\cap I(\nu_s)_{ma_s}.$$

We write $R[D]=R[\mathcal{I}(D)]$.

1.5. **Summary of principal results in this paper.** Let R be an excellent two dimensional normal excellent local ring with maximal ideal m_R.

All possible analytic spreads $\ell(\mathcal{I}(D))=0,1,2$ can occur for $\mathbb{Q}$-divisors D on R. An example where $\ell(\mathcal{I}(D))=0<\mathrm{ht}(\mathcal{I}(D))=1$ is given in Example 7.3. This example is of a symbolic filtration $\mathcal{I}(D)=\{Q_1^{(n)}\cap Q_2^{(n)}\cap Q_3^{(n)}\}$ where Q_1,Q_2,Q_3 are height one prime ideals in a two dimensional normal excellent local ring R. In contrast, since the filtration $\mathcal{I}(D)$ is not Noetherian, we have (by [13, Corollary 1.9]) that for every $a\in\mathbb{Z}_{>0}$, the analytic spread of the ideal $Q_1^{(a)}\cap Q_2^{(a)}\cap Q_3^{(a)}$ is $\ell(Q_1^{(a)}\cap Q_2^{(a)}\cap Q_3^{(a)})=2$, the largest possible.

We prove that the conclusions of Theorem 1.3 hold for $\mathbb{Q}$-divisorial filtrations on R in Theorem 7.1.

Theorem 1.5. *(Theorem 7.1) Let R be a two dimensional normal excellent local ring. The following are equivalent for a $\mathbb{Q}$-divisorial filtration $\mathcal{I}(D)$ on R.*

1) *The analytic spread $\ell(\mathcal{I}(D))=\dim R[D]/m_RR[D]=2$.*
2) *$m_R\in Ass(R/I(nD))$ for some n.*
3) *There exists $n_0\in\mathbb{Z}_{>0}$ such that $m_R\in Ass(R/I(nD))$ for all $n\ge n_0$.*

We generalize the formula on Hilbert functions of filtrations of powers of ideals in (1) and (2) to $\mathbb{Q}$-divisorial filtrations on R in Theorem 8.1.

Theorem 1.6. *(Theorem 8.1) Suppose that R is a two dimensional normal excellent local ring and $\mathcal{I}(D)$ is a $\mathbb{Q}$-divisorial filtration on R. Then there exist a nonnegative rational number α and a bounded function $\sigma : \mathbb{N} \to \mathbb{Q}$ such that*

$$\ell_R(I(nD)/m_R I(nD)) = \ell_R((R[D]/m_R R[D])_n) = n\alpha + \sigma(n)$$

for $n \in \mathbb{N}$. The constant α is positive if and only if $\dim(R[D]/m_R R[D]) = 2$.

It is unlikely that the function $\sigma(n)$ will always be eventually periodic. It is shown in [14, Theorem 9] that if D has exceptional support then the Hilbert function of $\mathrm{gr}_{\mathcal{I}}(R) = \sum_{n\geq 0} I(nD)/I((n+1)D)t^n$ has an expression

$$\ell_R(I(nD)/I((n+1)D)) = n\beta + \tau(n)$$

where $\beta \in \mathbb{Q}$ and $\tau(n)$ is a bounded function. If R has equicharacteristic zero then it is shown in [14, Theorem 9] that $\tau(n)$ is eventually periodic, and [14, Example 5] gives an example where R has equicharacteristic $p > 0$ and $\tau(n)$ is not eventually periodic.

Suppose that A is an excellent normal local ring of dimension 3. Let $Z \to \mathrm{Spec}(A)$ be a resolution of singularities and D be an effective divisor on Z, all of whose components contract to the maximal ideal m_A. Then the Hilbert polynomial $h(n) = \ell_A(I(nD)/I((n+1)D))$ may be far from being polynomial like. The examples ([14, Example 6] and [10, Theorem 1.4]) have the property that

$$\lim_{n\to\infty} \frac{h(n)}{n^2}$$

is an irrational number. These examples are in three dimensional equicharacteristic rings A of any characteristic. The reason for this irrational behavior in dimension three is because of the lack of existence of Zariski decompositions in dimension three.

We now give an outline of the proof of Theorem 7.1. Let $\pi : X \to \mathrm{Spec}(R)$ be a resolution of singularities such that D is represented on X. Let $E_1, \ldots, E_r$ be the prime exceptional divisors of π. An $\mathbb{R}$-divisor Δ on X is anti-nef if $(E \cdot \Delta) \leq 0$ for all prime exceptional divisors E on X. Since X has dimension two, D has a Zariski decomposition, $\Delta = D + B$ where Δ is an anti-nef divisor and B is an effective divisor with exceptional support such that

$$I(nD) = \Gamma(X, \mathcal{O}_X(-\lceil nD\rceil)) = \Gamma(X, \mathcal{O}_X(-\lceil n\Delta\rceil)) = I(n\Delta)$$

for all $n \in \mathbb{N}$. This decomposition does not exist in higher dimensions, even after blowing up ([8], [30, Section IV.2.10], [21, Section 2.3]).

Proposition 1.7. *(Corollary 6.5) Suppose that Δ is an effective anti-nef $\mathbb{Q}$-divisor on X. Then the following are equivalent.*

1) *There exists n such that $m_R \in Ass(R/I(n\Delta))$.*
2) *There exists n_0 such that $m_R \in Ass(R/I(n\Delta))$ for all $n \geq n_0$.*
3) *There exists j such that E_j is exceptional and $(\Delta \cdot E_j) < 0$.*

Let E_j be an exceptional divisor of π and

$$P_j = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}_X(-\lceil n\Delta\rceil - E_j))$$

for $1 \leq j \leq r$. P_j is a prime ideal in $R[\Delta] = R[D]$. In Proposition 6.7 it is shown that

$$\sqrt{m_R R[\Delta]} = \cap_{i=1}^r P_i.$$

The following proposition computes the dimension of $R[\Delta]/P_j$ in terms of the intersection theory of X.

Proposition 1.8. *(Proposition 6.9) Suppose that Δ is an effective anti-nef $\mathbb{Q}$-divisor on X and E_j is a prime exceptional divisor for $\pi : X \rightarrow Spec(R)$. Then*

1) $\dim R[\Delta]/P_j = 2$ *if* $(\Delta \cdot E_j) < 0$.
2) $\dim R[\Delta]/P_j \leq 1$ *if* $(\Delta \cdot E_j) = 0$.

Since $\sqrt{m_R R[\Delta]} = \cap_{i=1}^r P_i$, we deduce Theorem 7.1 from Propositions 6.5 and 6.9.

The theory of Zariski decomposition was created and developed by Zariski in [35] for projective surfaces over an algebraically closed field. In Section 4, we give the relative version of this theory, over a two dimensional excellent normal local ring, and in Section 5, we extend some results in [35] for numerically effective divisors on a nonsingular projective surface to our situation of a resolution of singularities of a two dimensional normal excellent local ring. We prove the main results of this paper on asymptotic properties of divisors on a resolution of singularities of a two dimensional normal excellent local ring in Section 6. We prove Theorem 7.1 in Section 7 and Theorem 8.1 in Section 8.

1.6. **Notation.** We will denote the nonnegative integers by $\mathbb{N}$ and the positive integers by $\mathbb{Z}_{>0}$, the set of nonnegative rational numbers by $\mathbb{Q}_{\geq 0}$ and the positive rational numbers by $\mathbb{Q}_{>0}$. We will denote the set of nonnegative real numbers by $\mathbb{R}_{\geq 0}$ and the positive real numbers by $\mathbb{R}_{>0}$. If $x \in \mathbb{R}$, then $\lceil x \rceil$ is the smallest integer which is greater than or equal to x.

The maximal ideal of a local ring R will be denoted by m_R. We will denote the length of an R-module M by $\ell_R(M)$. [18, Scholie IV.7.8.3] gives a list of good properties of excellent local rings which we will assume.

2. Divisors on a resolution of singularities of a two dim. local ring

Throughout this paper R will be a two dimensional excellent normal local ring with quotient field K, maximal ideal m_R and residue field $k = R/m_R$.

From this section through Section 6, $\pi : X \rightarrow \text{Spec}(R)$ will be a resolution of singularities such that π is projective and all exceptional prime divisors of π are nonsingular. Such a resolution of singularities exists by [24] or [7]. Let $E_1, \ldots, E_r$ be the exceptional prime divisors for π. A divisor is exceptional if all its prime components map to m_R by π. We will further assume that π is not an isomorphism.

Remark 2.1. *Suppose that $\mathcal{F}$ is a coherent sheaf on X. Then $H^0(X, \mathcal{F})$ is a finitely generated R-module, $H^1(X, \mathcal{F})$ is an R module of finite length and $H^2(X, \mathcal{F}) = 0$.*

Proof. By [20, Theorem III.5.2], $H^0(X, \mathcal{F})$ is a finitely generated R-module. By [20, Theorem III.5.2 and Corollary III.11.2], $H^1(X, \mathcal{F})$ is an R module of finite length and by [20, Corollary III.11.2], $H^2(X, \mathcal{F}) = 0$ since $\dim \pi^{-1}(m_R) = 1$. ☐

An element of the free abelian group $\text{Div}(X)$ on the prime divisors of X is called a divisor. Elements of $\text{Div}(X) \otimes \mathbb{Q}$ are called $\mathbb{Q}$-divisors and elements of $\text{Div}(X) \otimes \mathbb{R}$ are called $\mathbb{R}$-divisors. We will sometimes refer to a divisor as an integral divisor if we want to emphasize this fact. If D_1 and D_2 are $\mathbb{R}$-divisors then write $D_2 \geq D_1$ if $D_2 - D_1$ is an effective divisor. The degree $\deg(\mathcal{L})$ for $\mathcal{L}$ an invertible sheaf on a projective curve is defined in Section 3.

We use the intersection theory on X developed in [23, Sections 12 and 13]. The intersection theory on X is determined by the formula $(D \cdot E) = \deg(\mathcal{O}_X(D) \otimes \mathcal{O}_E)$ if D is a divisor on X and E is a prime exceptional divisor on X.

An $\mathbb{R}$-divisor D is numerically effective (nef) if $(E \cdot D) \geq 0$ for all prime exceptional divisors E of X. An $\mathbb{R}$-divisor D on X is anti-effective or anti-nef if $-D$ is respectively effective or nef. A $\mathbb{Q}$-divisor D is anti-ample if $-D$ is ample and an (integral) divisor D is anti-very ample if $-D$ is very ample.

Let F be a prime divisor on X. Then $\mathcal{O}_{X,F}$ is a (rank 1) discrete valuation ring. Let ν_F be the associated valuation. For $0 \neq f \in K$ the divisor of f on X is $(f) = \sum \nu_F(f)F$ where the sum is over all the prime divisors F of X. Two divisors D_1 and D_2 are linearly equivalent, written $D_1 \sim D_2$ if there exists $f \in K$ such that $(f) = D_2 - D_1$. Two divisors D_1 and D_2 which are linearly equivalent are also numerically equivalent; that is, $(E \cdot D_2) = (E \cdot D_1)$ for all prime exceptional divisors E of π.

Let $D = \sum b_i F_i$ be an integral divisor on X. There is an associated invertible sheaf $\mathcal{O}_X(D)$ on X which is determined by the property that if U is an affine open subset of X and $h \in K$ is such that $h = 0$ is a local equation of D in U, then $\mathcal{O}_X(D) \mid U = \frac{1}{h}\mathcal{O}_U$. Thus

$$\Gamma(X, \mathcal{O}_X(D)) = \{f \in R \mid (f) + D \geq 0\}.$$

Since R is a subset of $\Gamma(X, \mathcal{O}_X)$ in K and R is normal we have that $\Gamma(X, \mathcal{O}_X) = R$ by Remark 2.1, and so if D is an effective divisor then $\Gamma(X, \mathcal{O}_X(-D))$ is an ideal in R.

Let $\lceil a \rceil$ denote the smallest integer that is greater than or equal to a real number a. If $D = \sum_{i=1}^{s} a_i F_i$ with $a_i \in \mathbb{R}$ is an $\mathbb{R}$-divisor, let $\lceil F \rceil = \sum \lceil a_i \rceil F_i$.

Let F be a prime divisor on X. For $\alpha \in \mathbb{R}_{\geq 0}$ define valuation ideals in R by

$$I(\nu_F)_\alpha = \{f \in R \mid \nu_F(f) \geq \alpha\}.$$

We necessarily have that $I(\nu_F)_\alpha = I(\nu_F)_{\lceil \alpha \rceil}$.

For an effective $\mathbb{R}$-divisor $D = a_1 F_1 + \cdots + a_s F_s$, where $F_1, \ldots, F_s$ are prime divisors on X and $a_i \in \mathbb{R}_{\geq 0}$, we have an associated ideal in R

$$I(D) := I(\nu_{F_1})_{a_1} \cap \cdots \cap I(\nu_{F_s})_{a_s} = I(\nu_{F_1})_{\lceil a_1 \rceil} \cap \cdots \cap I(\nu_{F_s})_{\lceil a_s \rceil} = \Gamma(X, \mathcal{O}_X(-\lceil D \rceil)).$$

Let D be a divisor on X. Then $\Gamma(X, \mathcal{O}_X(D)) \neq 0$. The fixed component of D is the largest effective divisor F on X such that

$$\Gamma(X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{O}_X(D - F)).$$

For $n \in \mathbb{N}$, let B_n be the fixed component of nD and let

$$M_i = \{n \in \mathbb{N} \mid E_i \text{ is not a component of } B_n\}.$$

M_i is a numerical semigroup, so if M_i is nonzero, there exists $h_i \in \mathbb{Z}_{>0}$ such that for $n \gg 0$, $n \in M_i$ if and only if h_i divides n.

The global sections $\Gamma(X, \mathcal{O}_X(D))$ of $\mathcal{O}_X(D)$ generate $\mathcal{O}_X(D)$ at a point $q \in X$ if $\mathcal{O}_X(D)_q = \Gamma(X, \mathcal{O}_X(D))\mathcal{O}_{X,q}$. The points $q \in X$ where $\mathcal{O}_X(D)$ is generated by global sections are necessarily disjoint from the support of the fixed component of D.

Lemma 2.2. *Let D be an effective divisor on X and let F be a prime divisor in the support of the fixed component of $-D$. Then the support of F is exceptional.*

Proof. Write $D = \sum_{i=1}^{r} a_i F_i$ where the F_i are distinct prime divisors on X and $a_i \in \mathbb{N}$. Suppose that F_j is not exceptional for π. Let $q_j = \pi(F_j)$, a height one prime ideal in R.

Since π is an isomorphism over $\mathrm{Spec}(R) \setminus m_R$, we have that $R_{q_j} = \mathcal{O}_{X,F_j}$, so

$$\begin{aligned} \mathcal{O}_X(-D)_{F_j} &= (q_j^{a_j})_{q_j} = (I(\nu_j)_{a_j})_{q_j} = \Gamma(X, \mathcal{O}_X(-D))_{q_j} \\ &= \Gamma(X, \mathcal{O}_X(-D))\mathcal{O}_{X,F_j}. \end{aligned}$$

Thus F_j is not in the support of F. □

The intersection matrix of the exceptional curves of π is the $r \times r$ matrix $((E_i \cdot E_j))$ which is negative definite ([23, Lemma 14.1]).

Proposition 2.3. *Let D be a $\mathbb{Q}$-divisor on X. Then D is ample if and only if $(D \cdot E) > 0$ for all prime exceptional divisors E on X.*

This is proved in [23, Theorem 12.1]. As commented in the proof of [23, Theorem 12.1], the additional assumption there that $H^1(X, \mathcal{O}_X) = 0$ is not necessary for this conclusion.

Lemma 2.4. *The support of a nonzero effective anti-nef $\mathbb{R}$-divisor D on X contains all exceptional prime divisors.*

Proof. Let S be the set of exceptional prime divisors which are in the support of D. Write $D = B + \sum_{i=1}^r a_i E_i$ where B is an effective divisor which contains no exceptional prime divisors in its support and all $a_i \geq 0$. For all E_j, we have that

$$0 \geq (D \cdot E_j) = (B \cdot E_j) + \sum_{i \neq j} a_i (E_i \cdot E_j) + a_j (E_j^2),$$

and so

$$-a_j(E_j^2) \geq (B \cdot E_j) + \sum_{i \neq j} a_i (E_i \cdot E_j) \geq 0. \tag{5}$$

If B is nonzero, then there exists E_j such that $(E_j \cdot B) > 0$ and thus $a_j > 0$ and so $E_j \in S$. If $B = 0$ then there exists E_j such that $(E_j \cdot D) < 0$ since $D \neq 0$ and the intersection matrix $((E_i \cdot E_j))$ is nonsingular. Thus S is nonempty. If $E_{j'} \in S$ and E_j is such that $(E_j \cdot E_{j'}) > 0$ then $E_j \in S$ by (5). The exceptional fiber $\pi^{-1}(m_R)$ is connected as R is normal and π is birational (by [20, Corollary III.11.4]). Thus S is the set of all exceptional prime divisors of X. □

Lemma 2.5. *X is the blowup of an m_R-primary ideal.*

Proof. Since the intersection matrix $((E_i \cdot E_j))$ is negative definite, there exists an effective anti-ample $\mathbb{Q}$-divisor A on X with exceptional support (by Proposition 2.3). Thus $-dA$ is very ample for some $d \in \mathbb{Z}_{>0}$. Let $I = \Gamma(X, \mathcal{O}_X(-dA))$. The ideal I is m_R-primary since the support of A is exceptional. The integral closure of $\sum_{n \geq 0} I^n t^n$ in $R[t]$ is

$$\sum_{n \geq 0} \overline{I^n} t^n = \sum_{n \geq 0} \Gamma(X, \mathcal{O}_X(-ndA)) t^n.$$

Since R is excellent, $\sum_{n \geq 0} \overline{I^n} t^n$ is a finitely generated graded R-algebra. Thus after replacing d with a higher power of d we may assume that $I^n = \overline{I^n} = \Gamma(X, \mathcal{O}_X(-ndA))$ for all $n \in \mathbb{Z}_{>0}$ (as follows from [4, Proposition III.3.2 and Proposition III.3.3 on pages 158 and 159]).

Let $Y = \mathrm{Proj}(\oplus_{n \geq 0} I^n)$, which is normal since $\oplus_{n \geq 0} I^n$ is integrally closed. Since $\mathcal{O}_X(-dA)$ is generated by global sections we have that $I\mathcal{O}_X = \mathcal{O}_X(-dA)$. By the universal property of blowing up ([20, Proposition II.7.14]), there exists a unique R-morphism

$\varphi : X \to Y$ such that $\varphi^*\mathcal{O}_Y(1) \cong \mathcal{O}_X(-dA)$. φ is a birational morphism which is an isomorphism away from the preimage of m_R. φ is of finite type since $X \to \mathrm{Spec}(R)$ is. Since $(-A \cdot E) > 0$ for all exceptional curves of X we have that φ does not contract any curves of X and thus φ is quasi-finite. Let $p \in X$ and $q = \varphi(p)$. Let $A = \mathcal{O}_{Y,q}$ and $B = \mathcal{O}_{X,p}$. The birational extension $A \to B$ satisfies $m_A B$ is m_B-primary since φ is quasi-finite. Since A is normal and excellent it is analytically irreducible by [18, Scholie IV.7.8.3(vii)]. Thus by Zariski's main theorem [1, (10.7) page 240] or [11, Proposition 21.53], we have that $A = B$ and so φ is an isomorphism and X is the blowup of the m_R-primary ideal I. $\square$

Lemma 2.6. *Let A be a universally Nagata domain and I be an ideal in A. Let $Y = \mathrm{Proj}(\bigoplus_{n\geq 0} I^n)$. Then the graded ring $\bigoplus_{n\geq 0}\Gamma(Y, I^n\mathcal{O}_Y)$ is a finite $\bigoplus_{n\geq 0} I^n$-module and there exists $n_0 \in \mathbb{Z}_{>0}$ such that $\Gamma(Y, I^n\mathcal{O}_Y) = I^n$ for $n \geq n_0$.*

Proof. This follows from the proof on the last two lines of page 122 through the first half of page 123 of [20, Theorem II.5.19], along with the fact (observed in [20, Remark 5.19.2]) that the integral closure of a Nagata domain in its quotient field is a finite extension (by [26, Proposition 31.B]). $\square$

3. Riemann-Roch theorems for curves

We summarize the famous Riemann-Roch theorems for curves. The following theorems are standard over algebraically closed fields. A reference where they are proven over an arbitrary field k is [22, Section 7.3]. The results that we need are stated in [22, Remark 7.3.33].

Let E be an integral regular projective curve over a field k. For $\mathcal{F}$ a coherent sheaf on E define $h^i(\mathcal{F}) = \dim_k H^i(E, \mathcal{F})$.

Let $D = \sum a_i p_i$ be a divisor on E, where p_i are prime divisors on E (closed points) and $a_i \in \mathbb{Z}$. We have an associated invertible sheaf $\mathcal{O}_X(D)$. Define

$$\deg(D) = \deg(\mathcal{O}_E(D)) = \sum a_i[\mathcal{O}_{E_i,p_i}/m_{p_i} : k].$$

The Riemann-Roch formula is

$$\chi(\mathcal{O}_E(D)) := h^0(\mathcal{O}_E(D)) - h^1(\mathcal{O}_E(D)) = \deg(D) + 1 - p_a(E) \tag{6}$$

where $p_a(E)$ is the arithmetic genus of E.

We further have Serre duality,

$$H^1(E, \mathcal{O}_E(D)) \cong H^0(E, \mathcal{O}_E(K - D)) \tag{7}$$

where $K = K_E$ is a canonical divisor on E. As a consequence, we have

$$\deg D > 2p_a(E) - 2 = \deg(K) \text{ implies } H^1(E, \mathcal{O}_E(D)) = 0. \tag{8}$$

We have the following well known consequence of these formulas, which we record for future reference.

Lemma 3.1. *Let E be an integral regular projective curve over a field k. Let $\{D_n\}_{n\geq 0}$ be an infinite sequence of divisors on E such that $\deg(D_n)$ is bounded from below and let Z be a divisor on E. Then there exists $s \in \mathbb{Z}_{>0}$ such that*

$$h^1(\mathcal{O}_E(D_n + Z)) \leq s \text{ for all } n \in \mathbb{N}.$$

Proof. There exists an integer c such that $\deg(D_n) \geq c$ for all n. Let U be an effective divisor on E of degree larger than $2p_a(E) - 2 + c$. By Serre duality (7),

$$h^1(\mathcal{O}_E(D_n + Z)) = h^0(\mathcal{O}_E(K - (D_n + Z))$$

where K is a cononical divisor on E. We have

$$\deg(K - (Z + D_n)) \leq \deg(K - Z) - c.$$

If $\deg(K - Z) - c < 0$, then certainly $h^0(\mathcal{O}_E(K - (D_n + Z)) = 0$. If $\deg(K - Z) - c \geq 0$, then $h^1(\mathcal{O}_E(K - (D_n + Z) + U) = 0$ by (8) and so

$$\begin{array}{lll} h^0(\mathcal{O}_E(K - (D_n + Z)) & \leq & h^0(\mathcal{O}_E(K - (D_n + Z) + U) \\ & = & \deg(K - (D_n + Z)) + \deg(U) + 1 - p_a(E) \\ & \leq & \deg(K - Z) - c + \deg(U) + 1 - p_a(E). \end{array}$$

□

If $\mathcal{L}$ is an invertible sheaf on E then $\mathcal{L} \cong \mathcal{O}_E(D)$ for some divisor D on E, and we may define $\deg(\mathcal{L}) = \deg(\mathcal{O}_X(D)) = \deg(D)$.

We will apply the above formulas in the case that E is a prime exceptional divisor for a resolution of singularities $\pi : X \to \mathrm{Spec}(R)$ as in Section 2. We take $k = R/m_R$. We have that E is projective over $k = R/m_R$, and E is a nonsingular (by assumption) integral curve. Let D be a divisor on X. Then $\deg(\mathcal{O}_X(D) \otimes \mathcal{O}_E) = (D \cdot E)$.

4. Zariski decomposition

In this section we present a relative form of the Zariski decomposition defined for projective surfaces over an algebraically closed field in [35]. Lemma 4.1 in the case that D is exceptional follows directly from [35] or [3, Theorem 3.3].

We continue with our ongoing assumptions that R is a two dimensional excellent normal local ring with quotient field K, maximal ideal m_R and residue field $k = R/m_R$ and $\pi : X \to \mathrm{Spec}(R)$ is a resolution of singularities such that the exceptional prime divisors $E_1, \ldots, E_r$ are nonsingular.

The proof of the following lemma is a modification of the proof of [3, Theorem 3.3].

Lemma 4.1. *Let D be an effective $\mathbb{R}$-divisor on X. Then there exist unique effective $\mathbb{R}$-divisors Δ and B on X such that the following 1) and 2) hold.*

1) *$\Delta = D + B$ is anti-nef and B has exceptional support.*
2) *$(\Delta \cdot E) = 0$ if E is a component of B.*

Further,

3) *Δ is the unique minimal effective anti-nef $\mathbb{R}$-divisor such that $\Delta - D$ is effective with exceptional support.*
4) *If D is a $\mathbb{Q}$-divisor then Δ and B are $\mathbb{Q}$-divisors.*

The decomposition $\Delta = D + B$ of the conclusions of Lemma 4.1 is called the Zariski decomposition of D.

Proof. For $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$, consider the inequalities

$$0 \leq x_i \text{ for } 1 \leq i \leq r \tag{9}$$

and

$$\left((D + \sum_{i=1}^{r} x_i E_i) \cdot E_j \right) \leq 0 \text{ for } 1 \leq j \leq r. \tag{10}$$

Since the matrix $((E_i \cdot E_j))$ is negative definite and by Proposition 2.3, there exists an anti-ample, effective divisor $A = \sum_{i=1}^{r} a_i E_i$ on X. Thus $a_i > 0$ for all i (by Lemma 2.4) and after possibly replacing A with a positive multiple of A, $x = a = (a_1, \ldots, a_r)$ satisfies (9) and (10). Let

(11) $\quad S = \{x \in \mathbb{R}^r \mid x_i \leq a_i$ for all i and the $2r$ inequalities (9) and (10) are satisfied$\}$.

The set S is nonempty and compact. Thus there is at least one point in S such that $\sum_{i=1}^r x_i$ is minimized on S. Let $b = (b_1, \ldots, b_r)$ be such a point. Let $B = b_1E_1 + \cdots + b_rE_r$ and $\Delta = D + B$. Then Δ is an effective, anti-nef $\mathbb{R}$-divisor and B is an effective $\mathbb{R}$-divisor with exceptional support. Let E_j be a component of B. Since b minimizes $\sum x_i$, $B - \varepsilon E_j$ is effective and $\Delta - \varepsilon E_j$ is not anti-nef for all $\varepsilon > 0$ sufficiently small. But $((\Delta - \varepsilon E_j) \cdot E_i) \leq 0$ for all $i \neq j$ so we must have that $((\Delta - \varepsilon E_j) \cdot E_j) > 0$ for all positive ε and thus $(\Delta \cdot E_j) = 0$ since Δ is anti-nef. Thus the decomposition $\Delta = D + B$ satisfies 1) and 2).

For $b = (b_1, \ldots, b_r)$, $b' = (b'_1, \ldots, b'_r) \in \mathbb{R}^r$, define

$$\min(b, b') = (\min(b_1, b'_1), \ldots, \min(b_r, b'_r)).$$

If b and b' satisfy (9) and (10) then $\min(b, b')$ also satisfies (9) and (10), as we now show. For a fixed j, we may assume that $\min(b_j, b'_j) = b_j$ (after possibly interchanging b and b'). Then since $(E_i \cdot E_j) \geq 0$ if $i \neq j$, we have that

$$((D + \sum_i \min(b_i, b'_i)E_i) \cdot E_j) \leq ((D + \sum_i b_iE_i) \cdot E_j) \leq 0.$$

Suppose that $B = \sum b_iE_i$ and $B' = \sum b'_iE_i$ are effective $\mathbb{R}$-divisors such that $\Delta = D + B$ and $\Delta' = D = B'$ satisfy both 1) and 2). We will show that $B = B'$ and so $\Delta = \Delta'$. Let $\min(B, B') = \sum_i \min(b_i, b'_i)E_i$. There exist $x_i \geq 0$ such that $\min(B, B') = B - \sum_i x_iE_i$. Since $D + \min(B, B')$ is anti-nef, for each element E_j of the support of B we have

$$0 \geq \big((D + \min(B, B')) \cdot E_j\big) = \left((\Delta - \sum_i x_iE_i) \cdot E_j\right) = -\sum_i x_i(E_i \cdot E_j).$$

Thus $\sum_i x_i(E_i \cdot E_j) \geq 0$ and so

$$\left((\sum_i x_iE_i) \cdot (\sum_j x_jE_j)\right) = \sum_i \sum_j x_ix_j(E_i \cdot E_j) \geq 0.$$

Since the matrix $((E_i \cdot E_j))$ is negative definite, we have that $x_i = 0$ for all i. Thus $B = \min(B, B')$. Similarily, $B' = \min(B, B')$ and so $B = B'$. Thus there is a unique effective $\mathbb{R}$-divisor B with exceptional support such that B and $\Delta = D + B$ satisfy 1) and 2).

We now show that Δ is the unique minimal effective and anti-nef $\mathbb{R}$-divisor on X such that $\Delta - D$ is effective with exceptional support. Let U be an effective anti-nef $\mathbb{R}$-divisor on X such that $U - D$ is effective with exceptional support. Let $U' = D + \min(\Delta - D, U - D)$. As shown earlier in the proof, $U' \geq D$ is effective and anti-nef. Write $U' - D = \sum u_iE_i$ and $B = \Delta - D = \sum b_iE_i$. We have $\sum u_i \leq \sum b_i \leq \sum a_i$ so $U' - D \in S$ (defined in (11)). Since $\sum b_i$ is the minimum of $\sum x_i$ on S, we have that $u_i = b_i$ for all i and so $U' = \Delta$. Thus $\Delta \leq U$.

Now suppose that D is an effective $\mathbb{Q}$-divisor on X. Let $\Delta = D + B$ be the Zariski decomposition of D. After possibly reindexing the $E_1, \ldots, E_r$, we may assume that the support of B is $E_1 \cup \cdots \cup E_s$ for some s with $1 \leq s \leq r$. Expand $D = F + \sum_{i=1}^r c_iE_i$ where F is an effective $\mathbb{Q}$-divisor whose support does not contain any prime exceptional divisor and $c_1, \ldots, c_r \in \mathbb{Q}_{\geq 0}$. Then $\Delta = F + \sum_{i=1}^r d_iE_i$ with $c_i \leq d_i$ for all i and $d_i = c_i$

for $s+1 \leq i \leq r$. Further, for $1 \leq j \leq s$, we have $0 = (\Delta \cdot E_j) = \sum_{i=1}^{s} d_i(E_i \cdot E_j) + g_j$ where $g_j = (F \cdot E_j) + \sum_{i=s+1}^{r} c_i(E_i \cdot E_j) \in \mathbb{Q}$. Since the $s \times s$ matrix $((E_i \cdot E_j))_{1 \leq i,j \leq s}$ is negative definite, and thus is nonsingular, we have that $d_1, \ldots, d_s \in \mathbb{Q}$. Thus Δ and B are $\mathbb{Q}$-divisors. $\square$

Remark 4.2. *From 3) of the conclusions of Lemma 4.1, we deduce that if $D_1 \leq D_2$ are effective $\mathbb{R}$-divisors such that $D_2 - D_1$ has exceptional support and the respective anti-nef parts of their Zariski decompositions are Δ_1 and Δ_2, then $\Delta_1 \leq \Delta_2$.*

Lemma 4.3. *Suppose that D is an effective $\mathbb{R}$-divisor on X and $\Delta = D + B$ is the Zariski decomposition of D. Then for all $n \in \mathbb{N}$,*

$$\Gamma(X, \mathcal{O}_X(-\lceil nD \rceil)) = \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil)).$$

Proof. Suppose that $f \in \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil))$. Then $(f) - \lceil n\Delta \rceil \geq 0$. Writing $n\Delta = \lceil n\Delta \rceil - G$ with $G \geq 0$, we have $-n\Delta = G - \lceil n\Delta \rceil$. From

$$-nD = -n\Delta + nB = -\lceil n\Delta \rceil + (G + nB)$$

and the fact that $G + nB \geq 0$, we have that $(f) - nD \geq 0$ so that $f \in \Gamma(X, \mathcal{O}_X(-\lceil nD \rceil))$.

Let S be the set of prime divisors in the support of B. Suppose that

$$f \in \Gamma(X, \mathcal{O}_X(-\lceil nD \rceil)).$$

Then $(f) - nD \geq 0$. Write $(f) - nD = A + C$ where A and C are effective $\mathbb{R}$-divisors on X, no components of A are in S and all components of C are in S. We have that $(f) - n\Delta = A + (C - nB)$. If $E \in S$ then

$$(E \cdot (A + (C - nB))) = (E \cdot ((f) - n\Delta)) = 0$$

which implies $(E \cdot (C - nB)) = -(E \cdot A) \leq 0$. The intersection matrix of the curves in S is negative definite since it is so for the set of all exceptional curves, so $C - nB \geq 0$ (by [35, Lemma 7.1]). Thus $(f) - n\Delta \geq 0$ which implies $(f) - \lceil n\Delta \rceil \geq 0$ since (f) is an integral divisor. Thus $f \in \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil))$. $\square$

5. Nef divisors

In this section we extend to our relative situation $X \to \operatorname{Spec}(R)$ some theorems proven by Zariski in [35] for projective surfaces over an algebraically closed field. We stay as close as possible to Zariski's original proof, although some parts require modification. In [21], and the references in that book, a theory of nef divisors on nonsingular projective varieties of arbitrary dimension over an algebraically closed field of characteristic zero is derived. Much of this theory can be extended to the relative situation, over $\operatorname{Spec}(A)$, where the local ring A is normal and essentially of finite type over an algebraically closed field of characteristic zero, or even of positive characteristic.

We continue with our ongoing assumptions that R is a two dimensional excellent normal local ring with quotient field K, maximal ideal m_R and residue field k, and that $\pi : X \to \operatorname{Spec}(R)$ is a resolution of singularities such that the exceptional prime divisors $E_1, \ldots, E_r$ of π are all nonsingular.

Proposition 5.1. *Let Δ be an effective anti-nef divisor on X. For $n \geq 0$, let B_n be the fixed component of $-n\Delta$. Suppose that E is a prime divisor which is in the support of the fixed component B_n of $-n\Delta$ for infinitely many n. Then E is exceptional for π and $(\Delta \cdot E) = 0$.*

Proof. By Lemma 2.2, E is exceptional. We will assume that $(\Delta \cdot E) < 0$ and derive a contradiction. Since $\Gamma(X, \mathcal{O}_X(-\Delta)) \neq 0$ there exists an effective divisor D on X such that $D \sim -\Delta$. Write $D = U + F_1 + \cdots + F_s$ where U is an effective divisor with no exceptional divisors in its support and $F_1 = E, F_2, \ldots, F_s$ are prime exceptional divisors. Let $\Delta_i = U + F_1 + \cdots + F_i$ for $0 \leq i \leq s$.

We have short exact sequences

$$0 \to \mathcal{O}_X(nD - \Delta_0) \to \mathcal{O}_X(nD) \to \mathcal{O}_X(nD) \otimes \mathcal{O}_{\Delta_0} \to 0.$$

There exists a very ample effective divisor H on X which contains no exceptional prime divisors in its support and whose support is disjoint from Δ_0 by [20, Theorem III.5.2] since Δ_0 intersects $\pi^{-1}(m_R)$ in only a finite number of closed points and so Δ_0 is a closed subscheme of the affine scheme $X \setminus V(H)$ and thus Δ_0 is an affine scheme. We thus have that $H^1(\Delta_0, \mathcal{O}_X(-nD) \otimes \mathcal{O}_{\Delta_0}) = 0$ for all n and so

$$h^1(\mathcal{O}_X(nD)) \leq h^1(\mathcal{O}_X(nD - \Delta_0)) \tag{12}$$

for all $n \in \mathbb{N}$.

For $i < s$ and $n \in \mathbb{N}$, we have short exact sequences

$$0 \to \mathcal{O}_X(nD - \Delta_i - F_{i+1}) \to \mathcal{O}_X(nD - \Delta_i) \to \mathcal{O}_X(nD - \Delta_i) \otimes \mathcal{O}_{F_{i+1}} \to 0.$$

Thus

$$h^1(\mathcal{O}_X(nD - \Delta_i)) \leq h^1(\mathcal{O}_X(nD - \Delta_{i+1}) + h^1(F_{i+1}, O_X(nD - \Delta_i) \otimes \mathcal{O}_{F_{i+1}}).$$

$(D \cdot F_{i+1}) = (-\Delta \cdot F_{i+1}) \geq 0$ implies that there exists $\sigma_i > 0$ such that $h^1(F_{i+1}, O_X(nD - \Delta_i) \otimes \mathcal{O}_{F_{i+1}}) \leq \sigma_i$ for all $n \in \mathbb{N}$ by Lemma 3.1, so

$$h^1(\mathcal{O}_X(nD - \Delta_i)) \leq h^1(\mathcal{O}_X(nD - \Delta_{i+1})) + \sigma_i \tag{13}$$

for all $i \geq 0$ and $n \in \mathbb{N}$.

Now consider the exact sequences

$$0 \to \mathcal{O}_X(nD - \Delta_0 - F_1) \to \mathcal{O}_X(nD - \Delta_0) \to \mathcal{O}_X(nD - \Delta_0) \otimes \mathcal{O}_{F_1} \to 0$$

for $n \in \mathbb{N}$. Since $(F_1 \cdot D) = (F_1 \cdot -\Delta) > 0$ we have that $H^1(F_1, \mathcal{O}_X(nD - \Delta_i) \otimes \mathcal{O}_{F_1}) = 0$ for $n \gg 0$ by (8). From the natural inclusion $\mathcal{O}_X(nD - \Delta_0) \to \mathcal{O}_X(nD)$ we deduce that F_1 is in the support of the fixed locus of $nD - \Delta_0$ if F_1 is in the support of the fixed locus of $-n\Delta$. Thus for n such that F_1 is a component of the base locus B_n of $-n\Delta$, the image of $H^0(X, \mathcal{O}_X(nD - \Delta_0))$ in $H^0(F_1, \mathcal{O}_X(nD - \Delta_i) \otimes \mathcal{O}_{F_1})$ is zero. Thus

$$h^1(\mathcal{O}_X(nD - \Delta_0)) = h^1(\mathcal{O}_X(nD - \Delta_0 - F_1)) - \chi(\mathcal{O}_{F_1}(nD - \Delta_0) \otimes \mathcal{O}_{F_1})$$

so that by the Riemann Roch theorem (6),

$$h^1(\mathcal{O}_X(nD - \Delta_0)) = h^1(\mathcal{O}_X(nD - \Delta_0 - F_1)) + n(\Delta \cdot F_1) + (\Delta_0 \cdot F_1) + p_a(F_1) - 1. \tag{14}$$

As explained before the statement of Lemma 2.2, there exists a positive integer h such that for $n \gg 0$, F_1 is a component of B_n if $h \not| n$.

By (12) and (13), there exists a constant $c > 0$ such that

$$h^1(\mathcal{O}_X(nD)) \leq h^1(\mathcal{O}_X((n-1)D)) + c$$

for all $n \in \mathbb{Z}_{>0}$ and for all $n \gg 0$ such that $h \not| n$ we have by (12), (13) and (14) that

$$h^1(\mathcal{O}_X(nD)) \leq h^1(\mathcal{O}_X((n-1)D) + n(\Delta \cdot F_1) + c.$$

Thus we have $h^1(\mathcal{O}_X(nD)) < 0$ for $n \gg 0$ since we have assumed that $(\Delta \cdot F_1) < 0$. But this is impossible, giving a contradiction and so $(\Delta \cdot F_1) = 0$. $\square$

Proposition 5.2. *Let Γ be an effective divisor on X such that $-\Gamma$ has no fixed component. Then*

1) $\mathcal{O}_X(-n\Gamma)$ *is generated by global sections for all $n \gg 0$.*
2) *There exists $s \in \mathbb{Z}_{>0}$ such that $h^1(X, \mathcal{O}_X(-n\Gamma)) < s$ for all $n \in \mathbb{N}$.*

Proof. The set of base points

$$\Omega = \{p \in X \mid \mathcal{O}_X(-\Gamma)_p \text{ is not generated by global sections}\}$$

of $\Gamma(X, \mathcal{O}_X(-\Gamma))$ is a finite set of closed points, which are necessarily contained in the exceptional fiber of π. Let $C \geq 0$ be an effective divisor on X such that $-C$ is very ample for π. There exists an integer $m > 0$ such that there exists an effective divisor $H \sim -mC$ with no exceptional components in its support and such that Ω is disjoint from its support (by [20, Theorem III.5.2]). After replacing C with this multiple mC we may assume that $H \sim -C$. Let $f \in K$, the quotient field of R, be such that $(f) - C = H$. We may regard the effective divisor H as a closed subscheme of X.

We have a short exact sequence

$$0 \to \mathcal{O}_X(C) \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_H \to 0$$

and tensoring with $\mathcal{O}_X(-i\Gamma - jC)$ we have short exact sequences

$$0 \to \mathcal{O}_X(-i\Gamma - (j-1)C) \xrightarrow{f} \mathcal{O}_X(-i\Gamma - jC) \to \mathcal{O}_X(-i\Gamma - jC) \otimes \mathcal{O}_H \to 0. \tag{15}$$

For $i, j \geq 0$, let $A_{i,j}$ be the natural image of $\Gamma(X, \mathcal{O}_X(-i\Gamma - jC))$ in

$$\Gamma(H, \mathcal{O}_X(-i\Gamma - jC) \otimes \mathcal{O}_H),$$

upon taking global sections of (15). Since the base points of $\Gamma(X, \mathcal{O}_X(-i\Gamma - jC))$ are a subset of Ω and so are disjoint from H, we have that, for all $i, j \geq 0$, $A_{i,j}\mathcal{O}_{H,q} = \mathcal{O}_X(-i\Gamma - jC)_q$ for all $q \in H$.

There exists $n \in \mathbb{Z}_{>0}$ such that there exists an effective divisor G on X such that $G \sim -nC$, the support of G contains no exceptional components of π and $\sup(H) \cap \sup(G) \cap \sup(\pi^{-1}(m_R)) = \emptyset$ (by [20, Theorem III.5.2]). We may regard G as a closed subscheme of X. Thus H is a closed subscheme of the affine scheme $X \setminus G$ and so H is affine, say $H = \mathrm{Spec}(S)$. The restriction of π to H is determined by a ring homomorphism $R \to S$. Now $S = \Gamma(H, \mathcal{O}_H)$ is a finitely generated R-module since π is a projective morphism (by [20, Corollary II.5.20]). As explained in [20, Corollary II.5.5], since S is Noetherian, the functor $M \to \tilde{M}$ gives an equivalence of categories between the category of finitely generated S-modules and the category of coherent $\mathcal{O}_{\mathrm{Spec}(S)}$-modules, with inverse $\mathcal{F} \mapsto \Gamma(\mathrm{Spec}(S), \mathcal{F})$.

In particular, letting $B_{i,j} = \Gamma(H, \mathcal{O}_X(-i\Gamma - jC) \otimes \mathcal{O}_H)$ for $i, j \geq 0$, we have that $\mathcal{O}_X(-i\Gamma - jC) \otimes \mathcal{O}_H = \widetilde{B_{i,j}}$. We also have that $B_{i,j}$ is the tensor product over S of i copies of $B_{1,0}$ and j copies of $B_{0,1}$ ([20, Proposition II.5.2]).

We have that the ring $A_{0,0}$ is a quotient of $\Gamma(X, \mathcal{O}_X) = R$ since π is proper birational and R is normal. Let $A_{0,0}[t_1, t_2]$ be a polynomial ring over $A_{0,0}$, which is bigraded by specifying that $\deg(a) = (0,0)$ if $a \in A_{0,0}$, $\deg(t_1) = (1,0)$ and $\deg(t_2) = (0,1)$. Let M be the bigraded $A_{0,0}$-subalgebra $M := \sum_{i,j \geq 0} A_{i,j} t_1^i t_2^j$ of $A_{0,0}[t_1, t_2]$. Similarly, let B be the bigraded S-subalgebra $B := \bigoplus_{i,j \geq 0} B_{i,j} t_1^i t_2^j$ of $S[t_1, t_2]$.

We have a natural inclusion of graded rings $M \to B$.

Since H is disjoint from Ω we have that

$$A_{1,0}^i A_{0,1}^j S_q = A_{ij} S_q = \mathcal{O}_X(-i\Gamma - jA) \otimes \mathcal{O}_{H,q} = (B_{i,j})_q$$

for all $q \in H$ and $i, j \geq 0$. Thus

$$A_{1,0}^i A_{0,1}^j S = B_{i,j} \text{ for all } i, j \geq 0. \tag{16}$$

Let A be the bigraded $A_{0,0}$-subalgebra $A := A_{0,0}[A_{1,0}t_1, A_{0,1}t_2]$ of M. Now we have a natural surjection $A_{1,0}^i A_{0,1}^j \otimes_R S \to B_{i,j}$ for all $i, j \geq 0$ by (16). Thus the natural homomorphism $A \otimes_R S \to B$ is surjective. Since S is a finitely generated R-module, we have that B is a finitely generated bigraded A-module. Since $A \subset M \subset B$ and A is Noetherian, we have that M is also a finitely generated A-module.

By [35, Lemma 4.3], since A is generated in bidegrees $(1,0)$ and $(0,1)$, and M is a finitely generated bigraded R-module, there exists $N \in \mathbb{Z}_{>0}$ such that

$$A_{i,j} = A_{i,j-1}A_{0,1} \text{ whenever } j \geq N \text{ and } i \geq 0 \text{ is arbitrary} \tag{17}$$

and

$$A_{i,j} = A_{i-1,j}A_{1,0} \text{ whenever } i \geq N \text{ and } j \geq 0 \text{ is arbitrary.} \tag{18}$$

Thus taking global sections in the short exact sequences (15), and applying (18), we have that if $i \geq N$ and $j \geq 0$, then

(19)

$$\Gamma(X, \mathcal{O}_X(-i\Gamma - jC)) = \Gamma(X, \mathcal{O}_X(-i\Gamma - (j-1)C))f + \Gamma(X, \mathcal{O}_X(-(i-1)\Gamma - jC))\Gamma(X, \mathcal{O}_X(-\Gamma)).$$

Since $-C$ is ample, for fixed i, $\mathcal{O}_X(-i\Gamma - jC)$ is generated by global sections for all $j \gg 0$ (by [20, Theorem II.5.17]). Let i be a fixed integer $\geq N$ and let $j > 0$ be such that $\mathcal{O}_X(-i\Gamma - jC)$ is generated by global sections.

The only points $q \in X$ where it is possible for $\mathcal{O}_X(-i\Gamma - (j-1)C)_q$ to not be generated by global sections are the points of Ω. Suppose that $q \in \Omega$. Thus q is not in the support of $H = (f) - C$, and so $f = 0$ is a local equation of C at q and $f\mathcal{O}_{X,q} = \mathcal{O}_X(-C)_q$. Further, since $q \in \Omega$, $\Gamma(X, \mathcal{O}_X(-\Gamma))O_{X,q} \subset m_q\mathcal{O}_X(-\Gamma)$ where m_q is the maximal ideal of $\mathcal{O}_{X,q}$, equation (19) and Nakayama's lemma show that

$$\begin{array}{rcl} \mathcal{O}_X(-i\Gamma - jC)_q & = & \Gamma(X, \mathcal{O}_X(-i\Gamma - jC))\mathcal{O}_{X,q} \\ & = & \Gamma(X, \mathcal{O}_X(-i\Gamma - (j-1)C))f\mathcal{O}_{X,q} \\ & & +\Gamma(X, \mathcal{O}_X(-(i-1)\Gamma - jC)\mathcal{O}_X(-\gamma)m_q \\ & = & \Gamma(X, \mathcal{O}_X(-i\Gamma - (j-1)C))\mathcal{O}_X(-C)_q. \end{array}$$

Thus $\Gamma(X, \mathcal{O}_X(-i\Gamma - (j-1)C))\mathcal{O}_{X,q} = \mathcal{O}_X(-i\Gamma - (j-1)C)_q$, and since this is true for all $q \in \Omega$, $\mathcal{O}_X(-i\Gamma - (j-1)C)$ is generated by global sections.

By descending induction on j, we obtain that $\mathcal{O}_X(-i\Gamma)$ is generated by global sections for all $i \geq N$.

We now prove the second statement of the proposition. Let $g_0, \ldots, g_r \in \Gamma(X, \mathcal{O}_X(-N\Gamma))$ generate $\Gamma(X, \mathcal{O}_X(-N\Gamma))$ as an R-module. Then $g_0, \ldots, g_r$ induce a proper R-morphism $\varphi : X \to \mathbb{P}_R^r$ such that $\varphi^*\mathcal{O}_{\mathbb{P}_R^r}(1) \cong \mathcal{O}_X(-N\Gamma)$ (by [20, Theorem II.7.1, Corollary II.4.8]). In fact, φ is projective, by [17, Proposition II.5.5 (v)] or [34, Lemma 29.43.15, Tag 01W7] and [34, Lemma 29.43.16 (1), Tag 01W7]. Let Z be the image of φ in $\mathbb{P}_R^r$ (which is closed since φ is proper) and let $\mathcal{O}_Z(1) = \mathcal{O}_{\mathbb{P}_R^r}(1) \otimes \mathcal{O}_Z$. Let $\overline{\varphi} : X \to Z$ be the induced projective R-morphism. By [20, Corollary III.11.2], for $s \in \mathbb{Z}$, the support of $R^1\overline{\varphi} * \mathcal{O}_X(-s\Gamma)$ is contained in the finite set of closed points of Z which are the images of curves contracted by $\overline{\varphi}$ (the prime exceptional divisors E of π such that $(E \cdot -\Gamma) = 0$). By [20, Theorem II.5.19], $\Gamma(Z, R^1\overline{\varphi} * \mathcal{O}_X(-s\Gamma))$ is a finitely generated R-module. Since it's support is the maximal ideal of R, the length of $\Gamma(Z, R^1\overline{\varphi} * \mathcal{O}_X(-s\Gamma))$ as an R-module is finite.

From the Leray spectral sequence we obtain exact sequences ([32, Theorem 11.2]) for $m \in \mathbb{Z}$,

$$0 \to H^1(Z, \overline{\varphi}_*\mathcal{O}_X(-m\Gamma)) \to H^1(X, \mathcal{O}_X(-m\Gamma)) \to H^0(Z, R^1\overline{\varphi}_*\mathcal{O}_X(-m\Gamma)).$$

For $m \in \mathbb{N}$, write $m = nN + s$ with $0 \leq s < N$. Then $\mathcal{O}_X(-m\Gamma) \cong \overline{\varphi}^*\mathcal{O}_Z(n) \otimes \mathcal{O}_X(-s\Gamma)$. Then by the projection formula ([20, Exercise III.8.3]), we obtain exact sequences for $n, s \in \mathbb{Z}$

$$\begin{aligned} &0 \to H^1(Z, \mathcal{O}_Z(n) \otimes \overline{\varphi}_*\mathcal{O}_X(-s\Gamma)) \to H^1(X, \overline{\varphi}^*\mathcal{O}_Z(n) \otimes \mathcal{O}_X(-s\Gamma)) \\ &\to H^0(Z, (R^1\overline{\varphi}_*\mathcal{O}_X(-s\Gamma)) \otimes \mathcal{O}_Z(n)). \end{aligned} \tag{20}$$

Let

$$s_1 = \max\{\ell_R\Gamma(Z, R^1\overline{\varphi}_*\mathcal{O}_X(-s\Gamma)) \mid 0 \leq s < N\}.$$

We have that $H^1(Z, \mathcal{O}_Z(n) \otimes \overline{\varphi}_*\mathcal{O}_X(-s\Gamma)) = 0$ for all $0 \leq s < N$ and $n \gg 0$ ([20, Theorem III.5.2]). Let

$$s_2 = \max\{\ell_R H^1(Z, \mathcal{O}_Z(n) \otimes \overline{\varphi}_*\mathcal{O}_X(-s\Gamma)) \mid 0 \leq s < N \text{ and } n \in \mathbb{N}\}$$

s_2 is finite by [20, Proposition III.8.5, III.Theorem 8.8, Corollary III.11.2]. By (20), we have that $\ell_R H^1(X, \mathcal{O}_X(-m\Gamma)) \leq s_1 + s_2$ for all $m \in \mathbb{N}$. □

Proposition 5.3. *Let Δ be an effective anti-nef divisor on X. For $n \geq 0$, let B_n be the fixed component of $-n\Delta$. Then there exists an effective exceptional divisor G on X such that $B_n \leq G$ for all $n \in \mathbb{Z}_{>0}$.*

Proof. To prove the proposition, it suffices to prove it for it for some positive multiple d of Δ, since for $n \in \mathbb{N}$, writing $n = md + s$ with $0 \leq s < d$, we have $B_n \leq B_{md} + B_s$.

Write $-\Delta = \sum_{i=1}^t a_iF_i$. Let

$$M_i = \{n \in \mathbb{N} \mid F_i \text{ is not a component of } B_n\}.$$

M_i is a numerical semigroup, so if M_i is nonzero, there exists $h_i \in \mathbb{Z}_{>0}$ such that for $n \gg 0$, $n \in M_i$ if and only if h_i divides n. Let

$$\mathcal{B}(D) = \{F_i \mid F_i \text{ is a component of } B_n \text{ for infinitely many } n\}.$$

By Proposition 5.1, $F_i \in \mathcal{B}(D)$ implies $(F_i \cdot \Delta) = 0$ and F_i is exceptional for π. After possibly reindexing that F_i, we may assume that the support of $\mathcal{B}(D)$ is $\cup_{i=1}^s F_i$, for some $s \leq t$. We have that $M_i = 0$ or $h_i > 1$ for $1 \leq i \leq s$. Thus the support of B_n is $\cup_{i=1}^s F_i$ if $n \gg 0$ and $h_i \not| n$ for all i such that $1 \leq i \leq s$ and M_i is non zero.

If we replace Δ with $n_0\Delta$ for some $n_0 \gg 0$, we have that the support of B_1 is $\mathcal{B}(D)$. By Proposition 5.2, there exists $s_0 \in \mathbb{N}$ such that the effective divisor $\Gamma = \Delta + B_1$ satisfies the condition that $h^1(\mathcal{O}_X(-n\Gamma)) \leq s_0$ for all $n \geq 1$ since $-\Gamma$ has no fixed component.

For a given $n \in \mathbb{Z}_{>0}$, consider the following conditions on a divisor Z_n.

a) $n\Gamma \geq Z_n \geq n\Delta$
b) $-Z_n$ has no fixed component
c) $h^1(\mathcal{O}_X(-Z_n)) \leq s_0$.

Let C_n be a minimal element in the set of divisors satisfying a), b) and c). Let $B'_n = C_n - n\Delta$. Then $nB_1 \geq B'_n \geq B_n$ (since $-n\Delta = -n\Gamma + nB_1 = -C_n + B'_n$ and $C_n \leq n\Gamma$). Thus it suffices to show that the B'_n are bounded from above.

For $1 \leq i \leq s$ we have short exact sequences

$$0 \to O_X(-C_n) \to \mathcal{O}_X(-C_n + F_i) \to \mathcal{O}_X(-C_n + F_i) \otimes \mathcal{O}_{F_i} \to 0,$$

giving exact sequences

$$\begin{aligned}&0 \to H^0(X, \mathcal{O}_X(-C_n)) \to H^0(X, \mathcal{O}_X(-C_n + F_i)) \to H^0(F_i, \mathcal{O}_X(-C_n + F_i) \otimes \mathcal{O}_{F_i})\\ &\to H^1(X, \mathcal{O}_X(-C_n)) \to H^1(X, \mathcal{O}_X(-C_n + F_i)) \to H^1(F_i, \mathcal{O}_X(-C_n + F_i) \otimes \mathcal{O}_{F_i}) \to 0.\end{aligned}$$

We will show that

$$-(C_n \cdot F_i) \leq \max\{s_0 - (F_i^2) - 1 + p_a(F_i), 2p_a(F_i) - 2 - (F_i^2), 0\} \tag{21}$$

for all n and $1 \leq i \leq s$.

First assume that F_i is not a component of B'_n. Then $(B'_n \cdot F_i) \geq 0$. Since $(F_i \cdot \Delta) = 0$ by Proposition 5.1, we have that $(C_n \cdot F_i) \geq 0$ and so (21) holds.

Now assume that F_i is a component of B'_n. We have that either

$$H^0(X, \mathcal{O}_X(-C_n + F_i)) = H^0(X, \mathcal{O}_X(-C_n)) \tag{22}$$

or

$$h^1(\mathcal{O}_X(-C_n + F_i)) > s_0. \tag{23}$$

If (22) holds, then $h^0(\mathcal{O}_X(-C_n + F_i) \otimes O_{F_i}) \leq s_0$. Thus

$$s_0 \geq h^0(\mathcal{O}_X(-C_n + F_i) \otimes \mathcal{O}_{F_i}) \geq ((-C_n + F_i) \cdot F_i) + 1 - p_a(F_i)$$

by the Riemann-Roch formula (6), and so (21) holds.

Suppose that (23) holds. Then $h^1(F_i, \mathcal{O}_X(-C_n + F_i) \otimes \mathcal{O}_{F_i}) > 0$, and so

$$((-C_n + F_i) \cdot F_i) < 2p_a(F_i) - 2$$

by (8). Thus (21) holds.

For i with $1 \leq i \leq s$, let $\sigma_i = \max\{s_0 - (F_i^2) - 1 + p_a(F_i), 2p_a(F_i) - 2 - (F_i^2), 0\}$. Since $(F_i \cdot \Delta) = 0$ for $1 \leq i \leq s$ by Proposition 5.1, and by (21), we have that

$$(B'_n \cdot F_i) = ((C_n - n\Delta) \cdot F_i) = (C_n \cdot F_i) \geq -\sigma_i.$$

In particular, $\sigma_i \geq -(B'_n \cdot F_i)$.

Since the intersection matrix $((F_i \cdot F_j))$ for $1 \leq i, j \leq s$ is negative definite, and thus is nonsingular, there exists a $\mathbb{Q}$-divisor $\mathcal{E} = c_1F_1 + \cdots + c_sF_s$ such that $(\mathcal{E} \cdot F_i) = -\sigma_i$ for $1 \leq i \leq s$. Then

$$((\mathcal{E} - B'_n) \cdot F_i) = -\sigma_i - (B'_n \cdot F_i) \leq 0$$

for all i implies $\mathcal{E} \geq B'_n$ by ([35, Lemma 7.1]), since the intersection matrix is negative definite. Thus the B'_n are bounded from above. $\square$

Corollary 5.4. *Let Δ be an effective anti-nef $\mathbb{Q}$-divisor on X. Let B_n be the fixed component of $-\lceil n\Delta \rceil$; that is, the largest effective divisor on X such that*

$$\Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil)) = \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil - B_n)).$$

Then

1) *The integral divisor B_n has exceptional support for all $n \in \mathbb{N}$ and*
2) *There exists an effective integral divisor G with exceptional support such that $B_n \leq G$ for all $n \in \mathbb{Z}_{>0}$.*

Proof. Statement 1) follows from Lemma 2.2. If Δ is an integral divisor then Statement 2) follows from Proposition 5.3.

Now assume that Δ is a $\mathbb{Q}$-divisor. Write $\Delta = \sum \frac{b_i}{d} F_i$ with $d \in \mathbb{Z}_{>0}$ and $b_i \in \mathbb{N}$, where the F_i are distinct prime divisors on X. Since $d\Delta$ is an integral divisor, there exists an effective integral divisor C with exceptional support such that $B_{nd} \leq C$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, and write $n = md - c$ with $m \in \mathbb{N}$ and $0 \leq c < d$. Then $\mathcal{O}_X(-\lceil n\Delta \rceil) = \mathcal{O}_X(-md\Delta + \lfloor c\Delta \rfloor)$. Thus $B_n \leq B_{md} + \lfloor c\Delta \rceil \leq C + d\Delta$. $\square$

Lemma 5.5. *Let $\{D_n\}$ with $n \geq 0$ be an infinite sequence of divisors on X and Z be an effective divisor on X. If the sequence $h^1(\mathcal{O}_X(D_n))$ is bounded from above and if for each prime exceptional component E of Z $(D_n \cdot E)$ is bounded from below then $h^1(\mathcal{O}_X(D_n + Z))$ is bounded from above.*

Proof. By induction on the number of components of Z, we may assume that $h^1(\mathcal{O}_X(D_n + Z - F))$ is bounded where F is a prime component of Z. We have a short exact sequence

$$0 \to \mathcal{O}_X(-F) \to \mathcal{O}_X \to \mathcal{O}_F \to 0,$$

giving exact sequences

$$H^1(X, \mathcal{O}_X(D_n + Z - F)) \to H^1(X, \mathcal{O}_X(D_n + Z)) \to H^1(F, \mathcal{O}_X(D_n + Z) \otimes \mathcal{O}_F).$$

If F is exceptional, there exists $s \in \mathbb{Z}_{>0}$ such that $h^1(F, \mathcal{O}_X(D_n + Z) \otimes \mathcal{O}_F) \leq s$ for all $n \geq 0$ by Lemma 3.1, so $h^1(\mathcal{O}_X(D_n + Z))$ is bounded from above. If F is not exceptional, then F is affine and so $H^1(F, \mathcal{O}_X(D_m + Z)) \otimes \mathcal{O}_F) = 0$ for all m, so again $h^1(\mathcal{O}_X(D_n + Z))$ is bounded from above. $\square$

Proposition 5.6. *Let Δ be an effective anti-nef divisor on X. Then $h^1(\mathcal{O}_X(-n\Delta))$ is bounded for $n \in \mathbb{N}$.*

Proof. Let C_n be the effective divisors of the proof of Proposition 5.3, so that $B'_n = C_n - n\Delta$ are effective divisors and there exists an effective divisor G with exceptional support such that $B'_n \leq G$ for all $n \in \mathbb{N}$. Since $-\Delta$ is nef, we have that $(-C_n \cdot E)$ is bounded from below for each prime exceptional component E of G. Further, we have (by the proof of Proposition 5.3) that $h^1(\mathcal{O}_X(-C_n)) \leq s_0$ for all $n \in \mathbb{N}$. For each effective divisor $Z \leq G$, Proposition 5.5 gives us an upper bound for $h^1(\mathcal{O}_X(-C_n + Z))$ over $n \in \mathbb{N}$. The maximum of these bounds is an upper bound for $h^1(\mathcal{O}_X(-n\Delta))$ over $n \in \mathbb{N}$. $\square$

Corollary 5.7. *Let Δ be an effective anti-nef divisor on X and $\mathcal{F}$ be a coherent sheaf on X. Then $h^1(\mathcal{O}_X(-n\Delta) \otimes \mathcal{F})$ is bounded for $n \in \mathbb{N}$.*

Proof. There exists an effective anti-ample divisor A on X with exceptional support by Proposition 2.3. There exists $n_0 \in \mathbb{Z}_{>0}$ such that $\mathcal{F} \otimes \mathcal{O}(-n_0 A)$ is generated by global sections, so there is a surjection $\mathcal{O}_X^s \to \mathcal{F} \otimes \mathcal{O}_X(-n_0 A)$ for some s, giving a short exact sequence of coherent sheaves

$$0 \to \mathcal{K} \to \mathcal{O}_X(n_0 A)^s \to \mathcal{F} \to 0$$

and surjections

$$H^1(X, \mathcal{O}_X(-n\Delta + n_0 A))^s \to H^1(X, \mathcal{O}_X(-n\Delta) \otimes \mathcal{F}).$$

Thus $h^1(\mathcal{O}_X(-n\Delta) \otimes \mathcal{F})$ is bounded above for $n \in \mathbb{N}$ since $-\Delta$ is nef, and by Lemma 5.5 and Proposition 5.6. $\square$

6. Asymptotic properties of divisors on a resolution of singularities

We continue with the notation introduced in the introduction and in Section 2. We assume that R is a two dimensional excellent normal local ring with quotient field K, maximal ideal m_R and residue field k, and that $\pi : X \to \mathrm{Spec}(R)$ is a resolution of singularities such that the exceptional prime divisors $E_1, \ldots, E_r$ of π are all nonsingular.

As explained in the introduction, If F is prime divisor on X and $\alpha \in \mathbb{R}_{\geq 0}$, then there is a valuation ideal $I(\nu_F)_\alpha = \{f \in R \mod \nu_F(f) \geq \alpha\}$ of R, where ν_F is the valuation of the discrete (rank 1) valuation ring $\mathcal{O}_{X,F}$.

Proposition 6.1. *Suppose that $\Delta_1 \subset \Delta_2$ are effective anti-nef $\mathbb{Q}$-divisors on X such that $\Delta_1 \neq \Delta_2$. Then there exists $n_0 \in \mathbb{Z}_{>0}$ such that $\Gamma(X, \mathcal{O}_X(-\lceil n\Delta_2 \rceil)) \neq \Gamma(X, \mathcal{O}_X(-\lceil n\Delta_1 \rceil))$ for all $n \geq n_0$.*

Proof. Write $\Delta_1 = \sum a_i F_i$ and $\Delta_2 = \sum b_i F_i$ where the F_i are distinct prime divisors on X. We have $b_i \geq a_i$ for all i and $b_j > a_j$ for some j. If F_j is not exceptional then certainly $\Gamma(X, \mathcal{O}_X(-\lceil n\Delta_2 \rceil)) \neq \Gamma(X, \mathcal{O}_X(-\lceil n\Delta_1 \rceil))$ for n sufficiently large by Lemma 2.2.

Now suppose that F_j is exceptional. By 2) of Lemma 5.4, there exists an effective exceptional divisor $H = \sum c_i F_i$ such that the fixed component B_n of $\Gamma(X, \mathcal{O}_X(-\lceil n\Delta_1 \rceil))$ satisfies $B_n \leq H$ for all $n \in \mathbb{N}$. Observe that $g \in \Gamma(X, \mathcal{O}_X(-\lceil n\Delta_2 \rceil))$ implies $\nu_j(g) \geq \lceil nb_j \rceil$. By definition of B_n, for $n \in \mathbb{Z}_{>0}$, there exists $f_n \in \Gamma(X, \mathcal{O}_X(-\lceil n\Delta_1 \rceil))$ such that $(f_n) - \lceil n\Delta_1 \rceil = A_n + B_n$ where F_j is not a component of the effective divisor A_n. Thus $\nu_j(f_n) = \lceil na_j \rceil + \delta$ with $\delta \leq c_j$. We have that $n > \frac{c_j+1}{b_j-a_j}$ implies $\lceil na_j \rceil + \delta < \lceil nb_j \rceil$. Thus $\nu_j(f_n) < \lceil nb_j \rceil$ so that $f_n \notin \Gamma(X, \mathcal{O}_X(-\lceil n\Delta_2 \rceil))$. ☐

Corollary 6.2. *Suppose that $\Delta_1 \subset \Delta_2$ are effective anti-nef $\mathbb{Q}$-divisors on X. Then the following are equivalent.*

1) $\Gamma(X, \mathcal{O}_X(-\lceil n\Delta_1 \rceil)) = \Gamma(X, \mathcal{O}_X(-\lceil n\Delta_2 \rceil))$ *for infinitely many $n \in \mathbb{Z}_{>0}$.*
2) $\Gamma(X, \mathcal{O}_X(-\lceil n\Delta_1 \rceil)) = \Gamma(X, \mathcal{O}_X(-\lceil n\Delta_2 \rceil))$ *for all $n \gg 0$*
3) $\Delta_1 = \Delta_2$.

Proof. Proposition 6.1 proves the essential implication 1) implies 3). The directions 3) implies 2) and 2) implies 1) are immediate. ☐

Proposition 6.3. *Let $\Delta = \sum_{i=1}^s a_i F_i$ be an effective anti-nef $\mathbb{Q}$-divisor on X and E be a prime exceptional divisor on X. Then $E = F_j$ for some j with $a_j > 0$. The following are equivalent*

1) *There exists $n \in \mathbb{Z}_{>0}$ such that*
$$I(n\Delta) = \cap_{i=1}^s I(\nu_{F_i})_{na_i} \neq \cap_{i \neq j} I(\nu_{F_i})_{na_i}.$$
2) *There exists $n_0 \in \mathbb{Z}_{>0}$ such that*
$$I(n\Delta) = \cap_{i=1}^s I(\nu_{F_i})_{na_i} \neq \cap_{i \neq j} I(\nu_{F_i})_{na_i}.$$
for all $n \geq n_0$.
3) $(\Delta \cdot F_j) < 0$.

Proof. It follows from Lemma 2.4 that $E = F_j$ for some j with $a_j > 0$.

Let $D_1 = \sum_{i \neq j} a_i F_i$, so that $D_1 \leq \Delta$. Let $\Delta_1 = D_1 + B_1$ be the Zariski decomposition of D_1. We have that $\Delta_1 \leq \Delta$ by Remark 4.2, and so $0 \leq \Delta - \Delta_1 = a_j F_j - B_1$ so that $0 \leq B_1 \leq a_j F_j$. Thus $\Delta_1 = \Delta - \lambda F_j$ with $0 \leq \lambda \leq a_j$.

If $\Delta_1 \neq \Delta$ then $\lambda > 0$, and so

$$(F_j \cdot \Delta) = (F_j \cdot \Delta_1) + \lambda(F_j^2) < 0. \tag{24}$$

If $\Delta_1 = \Delta$ then $B_1 = a_j F_j$. Since $a_j > 0$, we have that

$$0 = (\Delta_1 \cdot F_j) = (\Delta \cdot F_j). \tag{25}$$

by 2) of Lemma 4.1.

Suppose that 1) holds. Then $\Delta_1 \neq \Delta$ so that $(F_j \cdot \Delta) < 0$ by (24), so that 1) implies 3) holds. Certainly 2) implies 1) is true, so we are reduced to proving 3) implies 2). Now 3) implies $\Delta_1 \neq \Delta$ by (24) and (25). If 2) doesn't hold then there exist infinitely many $n \in \mathbb{Z}_{>0}$ such that $\Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil)) = \Gamma(X, \mathcal{O}_X(-\lceil n\Delta_1 \rceil))$ so that $\Delta_1 = \Delta_2$ by Corollary 6.2, giving a contradiction. □

Corollary 6.4. *Let $\Delta = \sum_{i=1}^s a_i F_i$ be an effective anti-nef $\mathbb{Q}$-divisor on X and E be a prime exceptional divisor on X so that $E = F_j$ for some j with $a_j > 0$. The following are equivalent*

1) $I(n\Delta) = \cap_{i=1}^r I(\nu_{F_i})_{na_i} = \cap_{i \neq j} I(\nu_{F_i})_{na_i}$ *for all* $n \in \mathbb{Z}_{>0}$.
2) $(\Delta \cdot F_j) = 0$.

Corollary 6.5. *Suppose that Δ is an effective anti-nef $\mathbb{Q}$-divisor on X. Then the following are equivalent.*

1) *There exists n such that $m_R \in Ass(R/I(n\Delta))$.*
2) *There exists n_0 such that $m_R \in Ass(R/I(n\Delta))$ for all $n \geq n_0$.*
3) *There exists a prime exceptional divisor E for π such that $(\Delta \cdot E) < 0$.*

Proof. Write $\Delta = \sum_{i=1}^s a_i F_i$, so that $I(n\Delta) = \cap_{i=1}^s I(\nu_{F_i})_{na_i}$. For a fixed n, we have that $m_R \in \mathrm{Ass}(R/\cap_{i=1}^s I(\nu_{F_i})_{na_i})$ if and only if

$$\cap_{i=1}^s I(\nu_{F_i})_{na_i} \neq \cap_{F_i \text{ which are not exceptional}} I(\nu_{F_i})_{na_i}$$

which occurs if and only if there exists j such that F_j is exceptional and

$$\cap_{i=1}^s I(\nu_{F_i})_{na_i} \neq \cap_{i \neq j} I(\nu_{F_i})_{na_i}$$

Thus by Proposition 6.3, the three conditions of the corollary are equivalent. □

Let $\Delta = \sum_{i=1}^s a_i F_i$ be an effective and anti-nef $\mathbb{Q}$-divisor on X. By Lemma 2.4, all prime exeptional divisors $E_1, \ldots, E_r$ are in the support of Δ. After permuting the F_i, we may assume that $F_i = E_i$ and $a_i > 0$ for $1 \leq i \leq r$. We have that

$$R[\Delta] := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil)) = \bigoplus_{n \geq 0} \cap_{i=1}^s I(\nu_{F_i})_{na_i}.$$

Let $P_j = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil - E_j))$ for $1 \leq j \leq r$. We have that

$$\Gamma(X, \mathcal{O}_X(-E_j)) = \{f \in R \mid \nu_{E_j}(f) > 0\} = m_R \tag{26}$$

for $1 \leq j \leq r$. for all j. Suppose that $f \in \Gamma(X, \mathcal{O}_X(-\lceil m\Delta \rceil))$ and $g \in \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil))$ are such that $fg \in \Gamma(X, \mathcal{O}_X(-\lceil (m+n)\Delta \rceil - E_j)$. Then

$$\nu_{E_j}(f) + \nu_{E_j}(g) = \nu_{E_j}(fg) \geq (m+n)a_j + 1$$

implies $\nu_{E_j}(f) \geq ma_j + 1$ or $\nu_{E_j}(g) \geq na_j + 1$ so that $f \in \Gamma(X, \mathcal{O}_X(-\lceil m\Delta \rceil - E_j)$ or $g \in \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil - E_j)$. Thus P_j is a prime ideal in $R[\Delta]$.

If $f \in m_R$, then $\nu_{E_j}(f) \geq 1$ for $1 \leq j \leq r$ so that

$$m_R R[\Delta] \subset P_j. \tag{27}$$

We have exact sequences

$$0 \to P_j \to R[\Delta] \to \bigoplus_{n \geq 0} \Gamma(E_j, \mathcal{O}_X(-\lceil n\Delta \rceil) \otimes \mathcal{O}_{E_j}).$$

Remark 6.6. *Suppose that Δ is an effective anti-nef $\mathbb{Q}$-divisor on X. Then* $\dim R[\Delta]/P_j = 0$ *if and only if* $R[\Delta]/P_j = R/m_R$.

Proof. Suppose that for some $m > 0$ there exists $f \in \Gamma(X, \mathcal{O}_X(-\lceil m\Delta \rceil))$ such that it's class $\overline{f}$ in $\Gamma(X, \mathcal{O}_X(-\lceil m\Delta \rceil))/\Gamma(X, \mathcal{O}_X(-\lceil m\Delta \rceil - E_j))$ is nonzero. Then

$$\overline{f}t^m \in \sum_{n=0}^{\infty} \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil))/\Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil - E_j))t^n = R[\Delta]/P_j$$

is nonzero. The element $\overline{f}t^m$ is not a unit since it is homogeneous of positive degree and it is not nilpotent since $R[\Delta]/P_j$ is an integral domain. Thus $\dim R[\Delta]/P_j > 0$. Thus by (26), $\dim R[\Delta]/P_j = 0$ implies $R[\Delta]/P_j = R/m_R$. $\square$

Proposition 6.7. *Suppose that Δ is an effective anti-nef $\mathbb{Q}$-divisor on X. Then*

$$\sqrt{m_R R[\Delta]} = \cap_{i=1}^r P_i.$$

Proof. We have that $\sqrt{m_R R[\Delta]} \subset \cap_{i=1}^r P_i$ by (27).

Let $h \in \cap_{i=1}^r P_i$. We will show that $h^n \in m_R R[\Delta]$ for some $n \in \mathbb{Z}_{>0}$, which will establish the proposition. We may assume that h is homogeneous, so that

$$h \in \cap_{i=1}^r \Gamma(X, \mathcal{O}_X(-\lceil a\Delta \rceil - E_i)) = \Gamma(X, \mathcal{O}_X(-\lceil a\Delta \rceil - E_1 - \cdots - E_r)$$

for some $a \in \mathbb{N}$. We must show that $h^n \in m_R \Gamma(X, \mathcal{O}_X(-\lceil an\Delta \rceil))$ for some $n \in \mathbb{Z}_{>0}$.

First suppose that $a = 0$. We have that $\Gamma(X, \mathcal{O}_X(-E_1 - \cdots - E_r) = m_R$ so we already have that $h \in m_R \Gamma(X, \mathcal{O}_X) = m_R$.

Now suppose that $a > 0$. After replacing Δ with a positive multiple of Δ and h with a power of h we may assume that Δ is an integral divisor and $h \in \Gamma(X, \mathcal{O}_X(-\Delta - \sum_{i=1}^r E_i))$. By Lemma 2.5, there exists an m_R-primary ideal I in R such that X is the blowup of I, so that $X = \mathrm{Proj}(\bigoplus_{n \geq 0} I^n)$ and $I\mathcal{O}_X = \mathcal{O}_X(-C)$ is very ample, where C is an effective divisor whose support is the union of all exceptional prime divisors $E_1, \ldots, E_r$. The graded ring $\bigoplus_{n \geq 0} \Gamma(X, I^n \mathcal{O}_X)$ is a finite $\bigoplus_{n \geq 0} I^n$-module and there exists $n_0 \in \mathbb{Z}_{>0}$ such that the R-ideal $\Gamma(X, I^n \mathcal{O}_X) = I^n$ for $n \geq n_0$ by Lemma 2.6. Since R and X are normal, $\Gamma(X, I^n \mathcal{O}_X) = \overline{I^n}$ for all $n \geq 0$.

After possibly replacing I with a positive power of I we may assume that $\Gamma(X, I^n \mathcal{O}_X) = I^n$ for all $n \in \mathbb{N}$ and that there exists an effective divisor $H \sim -C$ on X with no exceptional prime divisors in its support. Let $f \in \Gamma(X, \mathcal{O}_X(-C)) = I$ be such that $(f) - C = H$. We have a short exact sequence

$$0 \to \mathcal{O}_X(C) \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_H \to 0.$$

There exists $\alpha \in \mathbb{Q}_{>0}$ such that $F := \sum_{i=1}^r E_i - \alpha C \geq 0$. There exists $e \in \mathbb{Z}_{>0}$ such that $e\alpha C$ is an integral divisor and so eF is an integral divisor. Thus for $n \in \mathbb{Z}_{>0}$, we have that

$$\begin{array}{rcl} h^{n2e} & \in & \Gamma(X, \mathcal{O}_X(-n2e\Delta - n2e(\sum_{i=1}^r E_i)) = \Gamma(X, \mathcal{O}_X(-n2e\Delta - n2e\alpha C - n2eF) \\ & \subset & \Gamma(X, \mathcal{O}_X(-n2e\Delta - n2e\alpha C)) = \Gamma(X, \mathcal{O}_X(-n2e(\Delta + \frac{\alpha}{2}C) - n2e\frac{\alpha}{2}C)). \end{array}$$

Now the effective integral divisor $2e(\Delta + \frac{\alpha}{2}C)$ is anti-ample by Proposition 2.3, since Δ is anti-nef. Thus there exists $n_0 \in \mathbb{Z}_{>0}$ such that $\mathcal{O}_X(-n2e(\Delta + \frac{\alpha}{2}C))$ is generated by global sections for all $n \geq n_0$. Let $\Gamma = n_0 2e(\Delta + \frac{\alpha}{2}C)$. By the argument of the proof of Proposition 5.2, applying (17), there exists $N > 0$ such that

$$\Gamma(X, \mathcal{O}_X(-i\Gamma - jC)) = \Gamma(X, \mathcal{O}_X(-i\Gamma - (j-1)C)\Gamma(X, \mathcal{O}_X(-C)) + f\Gamma(X, \mathcal{O}_X(-i\Gamma - (j-1)C)$$

whenever $j \geq N$ and $i \geq 0$. Since $f \in \Gamma(X, \mathcal{O}_X(-C))$, we have that

$$\begin{array}{rcl} \Gamma(X, \mathcal{O}_X(-i\Gamma - jC)) & = & \Gamma(X, \mathcal{O}_X(-i\Gamma - (j-1)C))\Gamma(X, \mathcal{O}_X(-C)) \\ & = & I\Gamma(X, \mathcal{O}_X(-i\Gamma - (j-1)C)) \subset I\Gamma(X, \mathcal{O}_X(-in_0 2e\Delta)). \end{array}$$

Thus

$$h^{nn_0 2e} \in \Gamma(X, \mathcal{O}_X(-n\Gamma - \frac{nn_0 2e\alpha}{2} C)) \subset I\Gamma(X, \mathcal{O}_X(-nn_0 2e\Delta)) \subset m_R\Gamma(X, \mathcal{O}_X(-nn_0 2e\Delta))$$

whenever n is so large that $n \geq \frac{N}{n_0 e\alpha}$. ☐

Corollary 6.8. *Suppose that Δ is an effective anti-nef $\mathbb{Q}$-divisor on X. Then*

$$\dim R[\Delta]/m_R R[\Delta] = 0$$

if and only if the image of $\Gamma(X, \mathcal{O}_X(-\lceil n\Delta\rceil))$ in $\Gamma(E_j, \mathcal{O}_X(-\lceil n\Delta\rceil) \otimes \mathcal{O}_{E_j})$ is zero for $1 \leq j \leq r$ and for all $n > 0$.

Proof. By Proposition 6.7, we have that $\dim R[\Delta]/m_R R[\Delta] = 0$ if and only if $\dim R[\Delta]/P_j = 0$ for all j, and this second conditions holds if and only if $R[\Delta]/P_j = R/m_R$ for all j by Remark 6.6. ☐

Proposition 6.9. *Suppose that Δ is an effective anti-nef $\mathbb{Q}$-divisor on X and E_j is a prime exceptional divisor for $\pi : X \to Spec(R)$. Then*

1) $\dim R[\Delta]/P_j = 2$ *if* $(\Delta \cdot E_j) < 0$.
2) $\dim R[\Delta]/P_j \leq 1$ *if* $(\Delta \cdot E_j) = 0$.

Proof. Suppose that $(\Delta \cdot E_j) < 0$. We have short exact sequences

$$0 \to \mathcal{O}_X(-\lceil n\Delta\rceil - E_j) \to \mathcal{O}_X(-\lceil n\Delta\rceil) \to \mathcal{O}_X(-\lceil n\Delta\rceil) \otimes \mathcal{O}_{E_j} \to 0.$$

Taking global sections we have short exact sequences

$$\begin{aligned} &0 \to \Gamma(X, \mathcal{O}_X(-\lceil n\Delta\rceil - E_j)) \to \Gamma(X, \mathcal{O}_X(-\lceil n\Delta\rceil)) \\ &\to \Gamma(E_j, O_X(-\lceil n\Delta\rceil) \otimes \mathcal{O}_{E_j}) \to H^1(X, \mathcal{O}_X(-\lceil n\Delta\rceil - E_j)). \end{aligned}$$

There exists $d \in \mathbb{Z}_{>0}$ such that $d\Delta$ is an integral divisor. By Corollary 5.7, applied to $d\Delta$ and the coherent sheaves $\mathcal{O}_X(-\lceil s\Delta\rceil - E_j)$ for $0 \leq s < d$, we have that

$$h^1(X, \mathcal{O}_X(-\lceil n\Delta\rceil - E_j))$$

is bounded for positive n. Since $(-\Delta \cdot E_j) > 0$, we have (by the Riemann Roch theorem (6)) that there exists $c' > 0$ such that

$$h^0(\mathcal{O}_X(-\lceil n\Delta\rceil) \otimes \mathcal{O}_{E_j}) > c'n$$

for $n \gg 0$. Thus there exists $c > 0$ such that the image $A_n := \mathrm{Im}(\Gamma(X, \mathcal{O}_X(-\lceil n\Delta\rceil))$ in $B_n := \Gamma(E_j, \mathcal{O}_X(-\lceil n\Delta\rceil) \otimes \mathcal{O}_{E_j})$ satisfies

$$\ell_R(\Gamma(X, \mathcal{O}_X(-\lceil n\Delta\rceil))/\Gamma(X, \mathcal{O}_X(-\lceil n\Delta\rceil - E_j)) = \ell_R(A_n) = \dim_k A_n \geq cn \tag{28}$$

for $n \gg 0$.

Let $A = \oplus_{n\geq 0} A_n$. We have that B_0 is a finite field extension of $k = R/m_R = A_0$. Now $\mathcal{O}_X(-d\Delta) \otimes \mathcal{O}_{E_j}$ is ample on the projective curve E_j, so there exists $e \in \mathbb{Z}_{>0}$ which is divisible by d such that $\mathcal{O}_X(-e\Delta) \otimes \mathcal{O}_{E_j}$ is very ample and $\overline{B} = \oplus_{m\geq 0} B_{me}$ is a finitely generated B_0-algebra which is generated by its terms of the lowest positive degree me ([20, Theorem II.5.19 and Exercise II.5.14]). Thus $\overline{B}$ is the coordinate ring of a projective embedding of the curve E_j in a projective space over B_0, determined by a B_0-basis of $\Gamma(E_j, \mathcal{O}_X(-e\Delta) \otimes \mathcal{O}_{E_j})$. Thus $\overline{B}$ has dimension two. Let $\overline{A} = \bigoplus_{m\geq 0} A_{me}$.

By (28), for $n \gg 0$, there exists $F \in A_{ne}$ such that $0 \neq F$. The ring $\overline{B}_{(F)}$ of elements of degree zero in the localization $\overline{B}_F$ is such that $\mathrm{Spec}(\overline{B}_{(F)})$ is the affine variety $E_j \setminus V(F)$, with maximal ideals in $\overline{B}_{(F)}$ corresponding to height one homogeneous prime ideals in $\mathrm{Proj}(\overline{B})$ which do not contain F (by [20, Proposition II.2.5]). Thus there exists a homogeneous height one prime ideal $Q = \oplus_{n>0} Q_{ne}$ in $\overline{B}$ which does not contain F.

Let $P = \overline{A} \cap Q$, where $P = \oplus_{n>0} P_{ne}$ with $P_{ne} = Q_{ne} \cap A_{ne}$. $\dim \overline{B}/Q = 1$ implies that there exists $d \in \mathbb{Z}_{>0}$ such that $\dim_k(B_{ne}/Q_{ne}) < d$ for all n (by [5, Theorem 4.1.3]). Thus by (28) we have that $P \neq 0$. P is not the graded maximal ideal $\oplus_{n\geq 0} A_{ne}$ of $\overline{A}$ since $F \notin P$.

We have constructed a chain of distinct homogeneous prime ideals $0 \subset P \subset \bigoplus_{n>0} A_{ne}$ in $\overline{A}$ and thus $\overline{A}$ has dimension ≥ 2. The extension $\overline{A} \to A$ is integral so $\dim A \geq 2$ by the going up theorem ([2, Theorem 5.11]). We have that $m_R\Gamma(X, \mathcal{O}_X(-\lceil n\Delta\rceil)) \subset \Gamma(X, \mathcal{O}_X(-\lceil n\Delta\rceil - E_j))$ for all $n \geq 0$ by (27). We thus have a surjection $R[\Delta]/m_R R[\Delta] \to A$ and so $\dim A \leq \dim R[\Delta]/m_R R[\Delta]$. But $\dim R[\Delta]/m_R R[\Delta] \leq 2$ by [14, Lemma 3.6], so that $\dim A = 2$.

Now suppose that $(\Delta \cdot E_j) = 0$. Let $B_n = \Gamma(E_j, \mathcal{O}_X(-\lceil n\Delta\rceil) \otimes \mathcal{O}_{E_j})$ and A_n be the natural image of $\Gamma(X, \mathcal{O}_X(-\lceil n\Delta\rceil))$ in B_n. We have that $A_0 \cong R/m_R = k$ and B_0 is a finite field extension of k. Let $A = \sum_{n\geq 0} A_n t^n$ where t is an indeterminate. We have that $A \cong R[\Delta]/P_j$.

By the Riemann-Roch Theorem (6) and Lemma 3.1, there exists $d > 0$ such that $\dim_k(B_n) < d$ for all $n \in \mathbb{N}$.

For $a \in \mathbb{Z}_{>0}$, define ${}_aA = \sum_{n\geq 0} {}_aA_n t^n$ to be the graded subring of A defined by ${}_aA = k[A_1 t, A_2 t^2, \ldots, A_a t^a]$. The ring ${}_aA$ is a finitely generated graded k-algebra. For fixed a, there exists $e \in \mathbb{Z}_{>0}$ such that ${}_aA^{(e)} = \sum_{n\geq 0} {}_aA_{en} t^{en}$ is generated in degree e (as follows from [4, Proposition III.3.2 and Proposition III.3.3 on pages 158 and 159]). Since ${}_aA$ is a finitely generated ${}_aA^{(e)}$-module, we have that $\dim {}_aA = \dim {}_aA^{(e)}$. Since $\dim_k {}_aA_n^{(e)} < d$ for all $n \in \mathbb{N}$, we have that $\dim {}_aA \leq 1$ for all $a \in \mathbb{Z}_{>0}$ by [5, Theorem 4.1.3]. Suppose that $Q_0 \subset Q_1 \subset \cdots \subset Q_s$ is a chain of distinct prime ideals in A. Since $\cup_{a\geq 0}({}_aA) = A$, for all $a \gg 0$, $Q_0 \cap {}_aA \subset Q_1 \cap {}_aA \subset \cdots \subset Q_s \cap {}_aA$ is a chain of distinct prime ideals in A. Thus $\dim A \leq 1$. ☐

Corollary 6.10. *Suppose that Δ is an effective anti-nef $\mathbb{Q}$-divisor on X. Then*

$$\dim R[\Delta]/m_R R[\Delta] = 2$$

if and only if there exists an exceptional prime divisor E of π such that $(\Delta \cdot E) < 0$

Proof. This follows from Propositions 6.7 and 6.9. ☐

7. Analytic spread of divisorial filtrations

Theorem 7.1 is a generalization to (not necessarily Noetherian) divisorial filtrations on a two dimensional normal local ring of a theorem of McAdam, for filtrations of powers of ideals, in [27] and [33, Theorem 5.4.6]. We recall the exact statement of McAdam's theorem in Theorem 1.3 of the introduction. The concept of a divisiorial filtration $\mathcal{I}(D) = \{I(nD)\}$ is defined in the introduction.

Theorem 7.1. *Let R be a two dimensional normal excellent local ring. The following are equivalent for a $\mathbb{Q}$-divisorial filtration $\mathcal{I}(D)$ on R.*

1) *The analytic spread $\ell(\mathcal{I}(D)) = \dim R[D]/m_R R[D] = 2$.*
2) *$m_R \in Ass(R/I(nD))$ for some n.*
3) *There exists $n_0 \in \mathbb{Z}_{>0}$ such that $m_R \in Ass(R/I(nD))$ for all $n \geq n_0$.*

Proof. Let $\pi : X \to \mathrm{Spec}(R)$ be a resolution of singularities such $D = \sum_{i=1}^{s} a_i F_i$ for some prime divisors F_i on X and the exceptional divisors $E_1, \ldots, E_r$ of π are nonsingular. Let $\Delta = D + B$ be the Zariski decomposition of D on X, so that $\mathcal{I}(D) = \mathcal{I}(\Delta)$ and $R[D] = R[\Delta]$ (by Lemma 4.3). Then this theorem follows from Corollary 6.10 and 6.5. ☐

Corollary 7.2. *Let R be a two dimensional normal excellent local ring and $\mathcal{I}(D)$ be a $\mathbb{Q}$-divisorial filtration on R. Then $\dim R[D]/m_R R[D] \le 1$ if and only if there exist height one prime ideals $Q_1, \ldots, Q_s$ in R and $b_1, \ldots, b_r \in \mathbb{Q}_{>0}$ such that $I(nD) = Q_1^{(\lceil nb_1 \rceil)} \cap \cdots \cap Q_s^{(\lceil nb_s \rceil)}$ for all $n \in \mathbb{N}$.*

Proof. We have that $I(nD) = Q_1^{(\lceil nb_1 \rceil)} \cap \cdots \cap Q_s^{(\lceil nb_s \rceil)}$ for all $n \in \mathbb{N}$ if and only if $m_R \notin \mathrm{Ass}(R/I(nD))$ for all n which holds if and only if $\dim R[D]/m_R R[D] \le 1$ by Corollary 7.1. ☐

Example 7.3. *There exists a $\mathbb{Q}$-divisorial filtration $\mathcal{I}(D)$ on a two dimensional normal excellent local ring R such that the analytic spread $\ell(\mathcal{I}(D)) = 0$ and height*

$$\mathrm{ht}(\mathcal{I}(D)) = \mathrm{ht}(I(D)) = 1,$$

giving an example where $\mathrm{ht}(\mathcal{I}(D)) > \ell(\mathcal{I}(D))$. The Rees algebra of the example is a Non Noetherian symbolic algebra $R[D] = \sum_{n \ge 0} Q_1^{(n)} \cap Q_2^{(n)} \cap Q_3^{(n)}$ where Q_1, Q_2, Q_3 are height one prime ideals in R.

Proof. Let k be an algebraically closed field and F be an irreducible cubic form in the polynomial ring $k[x, y, z]$ such that $E = \mathrm{Proj}(k[x, y, z]/(F))$ is an elliptic curve. Let $R = k[[x, y, z]]/(F)$, a complete, normal excellent local ring of dimension two with maximal ideal $m_R = (x, y, z)$. Let $\pi : X \to \mathrm{Spec}(R)$ be the blow up of the maximal ideal m_R of R. X is nonsingular with $\pi^{-1}(m_R) \cong E$, $m_R \mathcal{O}_X = \mathcal{O}_X(-E)$, $\mathcal{O}_X(-E) \otimes \mathcal{O}_E \cong \mathcal{O}_E(1)$ and $(E^2) = -3$. We have that $\mathcal{O}_X(-E) \otimes \mathcal{O}_E \cong \mathcal{O}_E(q_1 + q_2 + q_3)$ for some closed points $q_1, q_2, q_3 \in E$. Let $p_1, p_2, p_3 \in E$ be distinct closed points on E such that the degree 0 invertible sheaf $\mathcal{L} = \mathcal{O}_E(q_1 + q_2 + q_3 - p_1 - p_2 - p_3)$ has infinite order in $\mathrm{Pic}^0(X)$. Then $h^0(\mathcal{L}^n) = 0$ for all $n \in \mathbb{Z}$. In each regular local ring $\mathcal{O}_{X,p_i}$, let u_i, v_i be a regular system of parameters such that $u_i = 0$ is a local equation of E at p_i. Let F_i be the Zariski closure of $v_i = 0$ in X, which is an integral curve. Let $\pi(F_i) = Q_i \in \mathrm{Spec}(R)$. R/Q_i is Henselian since it is complete, so by [28, Theorem 4.2 page 32], we have that E intersects the integral curve F_i only at the point p_i. F_i intersects E transversally at p_i so that $(E \cdot F_i) = 1$. Let $D = F_1 + F_2 + F_3$. The Zariski decomposition of D is $\Delta = D + E$. We have that $\mathcal{O}_X(-n\Delta) \otimes \mathcal{O}_E \cong \mathcal{L}^n$ for all n. Thus $\Gamma(X, \mathcal{O}_X(-n\Delta - E)) = \Gamma(X, \mathcal{O}_X(-n\Delta))$ for all $n \in \mathbb{Z}_{>0}$, and so by Proposition 6.7,

$$R[\Delta]/\sqrt{m_R R[\Delta]} = \bigoplus_{n \ge 0} \Gamma(X, \mathcal{O}_X(-n\Delta))/\Gamma(X, \mathcal{O}_X(-n\Delta - E)) = R/m_R = k.$$

Thus

$$\dim R[\Delta]/m_R R[\Delta] = \dim R[\Delta]/\sqrt{m_R R[\Delta]} = 0.$$

Since $0 = \ell(\mathcal{I}(D)) < 1 = \mathrm{ht}(\mathcal{I}(D))$, we have that $R[D]$ is Non Noetherian (by [13, Proposition 3.7]). ☐

8. The Hilbert function of $R[D]/m_R R[D]$

Theorem 8.1. *Suppose that R is a two dimensional normal excellent local ring and $\mathcal{I}(D)$ is a $\mathbb{Q}$-divisorial filtration on R. Then there exist a nonnegative rational number α and a* *bounded function $\sigma : \mathbb{N} \to \mathbb{Q}$ such that*

$$\ell_R(I(nD)/m_R I(nD)) = \ell_R((R[D]/m_R R[D])_n) = n\alpha + \sigma(n)$$

for $n \in \mathbb{N}$. The constant α is positive if and only if $\dim(R[D]/m_R R[D]) = 2$.

The function σ is bounded from both above and below. The proof gives an explicit calculation of the constant α in terms of the intersection theory of a suitable resolution of singularities in equation (35). The constant α is a nonnegative integer if Δ is an integral divisor in the Zariski decomposition $D = \Delta + B$.

Proof. There exists a resolution of singularities $\pi : X \to \mathrm{Spec}(R)$ such that D is an effective $\mathbb{Q}$-divisor on X, $m_R \mathcal{O}_X$ is invertible and the prime exceptional divisors $E_1, \ldots, E_r$ of X are all nonsingular. Let G be the effective exceptional divisor such that $m_R \mathcal{O}_X = \mathcal{O}_X(-G)$. Let $\Delta = D + B$ be the Zariski decomposition of D on X. There exists $d \in \mathbb{Z}_{>0}$ such that $d\Delta$ is an integral divisor.

Suppose that the ideal m_R is generated by $f_1, \ldots, f_b$. We have an induced short exact sequence of coherent sheaves on X

$$0 \to \mathcal{K} \to \mathcal{O}_X^b \to m_R \mathcal{O}_X \to 0.$$

Tensoring with $\mathcal{O}_X(-\lceil n\Delta \rceil)$ and taking global sections, we have short exact sequences

$$0 \to m_R \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil) \to \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil - G)) \to H^1(X, \mathcal{K} \otimes \mathcal{O}_X(-\lceil n\Delta \rceil)).$$

Thus there exists $c_1 \in \mathbb{Z}_{>0}$ such that

$$\ell_R(\Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil - G))/m_R \Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil)) \le c_1 \tag{29}$$

for all $n \in \mathbb{N}$ by Corollary 5.7, applied to the effective anti-nef divisor $d\Delta$ and the coherent sheaves $\mathcal{F} = \mathcal{K} \otimes \mathcal{O}_X(-\lceil s\Delta \rceil)$ for $0 \le s < d$. From the short exact sequences

$$0 \to \mathcal{O}_X(-\lceil n\Delta \rceil - G) \to \mathcal{O}_X(-\lceil n\Delta \rceil) \to \mathcal{O}_X(-\lceil n\Delta \rceil) \otimes \mathcal{O}_G \to 0$$

we have inclusions for $n \in \mathbb{N}$

$$\Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil))/\Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil - G)) \to \Gamma(G, \mathcal{O}_X(-\lceil n\Delta \rceil) \otimes \mathcal{O}_G)$$

and by Corollary 5.7, there exists $c_2 \in \mathbb{Z}_{>0}$ such that

$$|\ell_R(\Gamma(G, \mathcal{O}_X(-\lceil n\Delta \rceil) \otimes \mathcal{O}_G)) - \ell_R(\Gamma(X, \mathcal{O})X(-\lceil n\Delta \rceil))/\Gamma(X, \mathcal{O}_X(-\lceil n\Delta \rceil)))| \le c_2. \tag{30}$$

We are reduced to computing $h^0(\mathcal{O}_X(-\lceil n\Delta \rceil) \otimes \mathcal{O}_G)$ for $n \in \mathbb{N}$. Write $G = \sum_{i=1}^r a_i E_i$ with $a_i \in \mathbb{Z}_{>0}$.

Let $e = \sum_{i=1}^r a_i$. There exists a function $\tau : \{1, \ldots, e\} \to \{1, \ldots, r\}$ such that letting $C_1 = E_{\tau(1)}$ and $C_{j+1} = C_j + E_{\tau(j+1)}$ for $1 \le j < e$, we have that $C_e = G$. We have short exact sequences

$$0 \to \mathcal{O}_X(-C_j) \otimes \mathcal{O}_{E_{\tau(j+1)}} \to \mathcal{O}_{C_{j+1}} \to \mathcal{O}_{C_j} \to 0 \tag{31}$$

for $1 \le j < e$. The cohomology groups $h^1(\mathcal{O}_X(-\lceil n\Delta \rceil - mE_j) \otimes \mathcal{O}_{E_{\tau(j+1)}})$ are bounded for $1 \le j < e$ and $n \in \mathbb{N}$ by Lemma 3.1. Let

$$f = \max\{h^1(\mathcal{O}_X(-\lceil n\Delta \rceil - mE_j) \otimes \mathcal{O}_{E_{\tau(j+1)}}) \mid 1 \le j < e \text{ and } n \in \mathbb{N}\}.$$

Tensoring the sequences (31) with $\mathcal{O}_X(-\lceil n\Delta \rceil)$ and taking cohomology, we find that

$$|h^0(\mathcal{O}_X(-\lceil n\Delta \rceil) \otimes \mathcal{O}_{C_{j+1}}) - h^0(\mathcal{O}_X(-\lceil n\Delta \rceil) \otimes \mathcal{O}_{C_j}) - h^0(\mathcal{O}_X(-\lceil n\Delta \rceil - C_j) \otimes \mathcal{O}_{E_{\tau(j+1)}})| \le f \tag{32}$$

for $1 \leq j < e$ and $n \in \mathbb{N}$. Setting $C_0 = 0$, we have that there exists $\lambda \in \mathbb{Z}_{>0}$ such that

$$|h^0(X, \mathcal{O}_X(-\lceil n\Delta \rceil) \otimes \mathcal{O}_G) - \sum_{i=0}^{e-1} h^0(X, \mathcal{O}_X(-\lceil n\Delta \rceil - C_i) \otimes \mathcal{O}_{E_{\tau(i+1)}})| < \lambda \tag{33}$$

for all $n \in \mathbb{N}$. Writing $n = md + s$ with $0 \leq s < d$, we have

$$h^0(\mathcal{O}_X(-\lceil n\Delta \rceil - C_j) \otimes \mathcal{O}_{E_{\tau(j+1)}}) = h^0(\mathcal{O}_X(-md\Delta - \lceil s\Delta \rceil - C_j) \otimes \mathcal{O}_{E_{\tau(j+1)}}).$$

By Lemma 3.1 and the Riemann-Roch theorem (6), there exists $g \in \mathbb{Z}_{>0}$ such that

$$|h^0(\mathcal{O}_X(-md\Delta - \lceil s\Delta \rceil - C_j) \otimes \mathcal{O}_{E_{\tau(j+1)}}) - md(-\Delta \cdot E_{\tau(j+1)})| \leq g \tag{34}$$

for $1 \leq j < e$ and $m \in \mathbb{N}$. Thus the theorem holds with

$$\alpha = (-\Delta \cdot G). \tag{35}$$

□

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