LOCAL UNIFORMIZATION OF ABHYANKAR VALUATIONS

STEVEN DALE CUTKOSKY

ABSTRACT. We prove local uniformization of Abhyankar valuations of an algebraic function field K over a ground field k. Our result generalizes the proof of this result, with the additional assumption that the residue field of the valuation ring is separable over k, by Hagen Knaf and Franz-Viktor Kuhlmann. The proof in this paper uses different methods, being inspired by the approach of Zariski and Abhyankar.

1. INTRODUCTION

In this paper we prove local uniformization of Abhyankar valuations ν of an algebraic function field K over a ground field k. An Abhyankar valuation is a valuation which satisfies equality in Abhyankar's inequality (1). These valuations are particularly well behaved. Abhyankar [1] showed that the value groups of these valuations are finitely generated, and that the residue fields of their valuation rings are finitely generated field extensions of k. In [20, Theorem 1.1], Knaf and Kuhlmann prove that with the additional assumption that the residue field of the valuation is separable over the ground field k, local uniformization holds for Abhyankar valuations of algebraic function fields. A version of this theorem, valid for Abhyankar valuations in complete local rings over an algebraically closed field, is proven by Teissier in [27, Theorem 5.5.1]. In this paper, we prove that local uniformization holds for Abhyankar valuations in algebraic function fields, without any extra assumptions. Our local uniformization theorems are given in Theorems 1.1, 1.2 and 1.3, stated later in this introduction, and proven in this paper.

The proof of Knaf and Kuhlmann [20], which has the assumption that the residue field of the Abhyankar valuation ν is separable over k, shows that there is a regular local ring R of K which is dominated by the valuation ν such that R is smooth over the ground field k. Without the assumption that the residue field of ν is separable over k, this may not be possible to achieve. However, we prove in the general case of an Abhyankar valuation, that there exists a regular local ring R of K which is dominated by the valuation ν.

Our proof is a generalization of the proof of Zariski for maximal rational rank valuations in a characteristic zero algebraic function field, [28]. This method was used by Abhyankar to prove local uniformization of Abhyankar valuations in two dimensional algebraic function fields over an algebraically closed ground field in [2, Section 1]. The proofs in [28] and [2] both make use of the values of derivations of K/k to achieve reduction of multiplicity. We only use the definition of a regular local ring: it has a regular system of parameters. Zariski used Perron transforms in [28] to prove local uniformization for rank 1 valuations in characteristic zero algebraic function fields, and made a reduction argument to use this result to prove local uniformization of arbitrary rank valuations in characteristic zero algebraic function fields. Our approach is influenced by that of Samar El Hitti in [14],

where local uniformization is proven in characteristic zero algebraic function fields for an arbitrary valuation, via a uniform use of higher rank Perron transforms.

A delicate point in the construction of a proof of local uniformization of an Abhyankar valuation in the general case, when the residue field of the valuation is not separable over the ground field k, is that it may not be possible to find a coefficient field of the completion of a given local ring dominated by the valuation which contains k.

Before stating our local uniformization theorems, we give some necessary background about valuations. We refer to [29], [15] and [16] for basic facts about valuations. Let K be an algebraic function field over a field k, and ν be a valuation of K/k; that is, a valuation of K which is trivial on k. Let V_ν be the valuation ring of ν with maximal ideal m_ν and Γ_ν be its valuation group. Let t be the rank of ν, and

$$(0) = P_{t+1}^\nu \subset P_t^\nu \subset \cdots \subset P_2^\nu \subset P_1^\nu = m_\nu$$

be the chain of prime ideals in V_ν. Let

$$0 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_t = \Gamma_\nu$$

be the chain of isolated subgroups. For $1 \leq i \leq t$, let ν_i be the specialization of ν with valuation ring $V_{\nu_i} = V_{P_i^\nu}$. The maximal ideal m_{ν_i} of V_{ν_i} is $m_{\nu_i} = P_i^\nu V_{\nu_i}$; in particular, $\nu_1 = \nu$. The value group of ν_i is $\Gamma_{\nu_i} = \Gamma_\nu/\Gamma_{i-1}$.

Abhyankar's inequality ([1] and [29, Proposition 2, Appendix 2, page 331]) is

$$(1) \qquad\qquad \mathrm{rrank}\ \nu + \mathrm{trdeg}_k V_\nu/m_\nu \leq \mathrm{trdeg}_k K$$

where rrank ν is the rational rank of ν. When equality holds in (1), we have that $\Gamma_\nu \cong \mathbb{Z}^n$ as a group for some n and V_ν/m_ν is a finitely generated field over k. This is proven in [1], and [29, Proposition 3, page 335, Appendix 2]. Valuations that satisfy equality in (1) are called Abhyankar valuations.

The following three theorems, establishing local uniformization along an Abyhankar valuation in an algebraic function field, are proven in Section 5 of this paper, as a consequence of the theory developed in Section 4. Any notation used in the statements of our local uniformization theorems, which is not defined above, can be found in Section 2. If V_ν/m_ν is separable over k, these theorems are a consequence of [20, Theorem 1.1]

Theorem 1.1. *Suppose that K is an algebraic function field over a field k and ν is an Abhyankar valuation of K/k. Then there exists a regular algebraic local ring R of K such that ν dominates R. Further,*

1) *R has a regular system of parameters $x_{1,1}, \ldots, x_{t,r_t}$ such that $\nu_j(x_{j,1}), \ldots, \nu_j(x_{j,r_j})$ is a $\mathbb{Z}$-basis of Γ_j/Γ_{j-1} and $P_j(R) = P_j^\nu \cap R$ is the regular prime ideal $(x_{j,1}, \ldots, x_{t,r_t})$ for $1 \leq j \leq t$.*
2) *We have that*

$$(R/P_i(R))_{P_i(R)} \cong (V/P_i^\nu)_{P_i^\nu} \cong V_{\nu_i}/m_{\nu_i}$$

for $1 \leq i \leq t$.

Regular parameters as in 1) are called very good parameters of R (Definition 4.1).

Primitive transforms are defined in Definition 4.2. They are a particularly simple type of birational transform of a regular local ring.

Theorem 1.2. *Suppose that R satisfies the conclusions of Theorem 1.1.*

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1) *Suppose that $0 \neq f \in R$. Then there exists a sequence of primitive transforms (3) along ν, $R \to R(1)$, such that $R(1)$ with the resulting very good parameters $x_{1,1}(1), \ldots, x_{t,r_t}(1)$, satisfies the conclusions of Theorem 1.1, and*

$$f = x_{1,1}(1)^{a_{1,1}} \cdots x_{t,r_t}(1)^{a_{t,r_t}} u$$

where $a_{1,1}, \ldots, a_{t,r_t} \in \mathbb{N}$ and $u \in R(1)$ is a unit.

2) *Suppose that $I \subset R$ is an ideal. Then there exists a sequence of primitive transforms (3) along ν, $R \to R(1)$, and $a_{1,1}, \ldots, a_{t,r_t} \in \mathbb{N}$ such that*

$$IR(1) = x_{1,1}(1)^{a_{1,1}} \cdots x_{t,r_t}(1)^{a_{t,r_t}} R(1).$$

3) *Suppose that $0 \neq f \in V_\nu$. Then there exits a sequence of primitive transforms (3) along ν, $R \to R(1)$, such that*

$$f = x_{1,1}(1)^{a_{1,1}} \cdots x_{t,r_t}(1)^{a_{t,r_t}} u$$

where $a_{1,1}, \ldots, a_{t,r_t} \in \mathbb{N}$ and $u \in R(1)$ is a unit.

Theorem 1.3. *Suppose that K is an algebraic function field over a field k and ν is an Abhyankar valuation of K. Suppose that S is an algebraic local ring of K which is dominated by ν. Then there exists a birational extension $S \to R$ such that R is a regular algebraic local ring of K which is dominated by ν and satisfies the conclusions 1) and 2) of Theorem 1.1.*

1.1. **Defect of extensions of valuations.** It has become apparent that the possibility of defect in a finite extension of valued fields is the essential obstruction to local uniformization in positive characteristic ([22], [26] and [11]). This is somewhat surprising, since defect does not appear explicitly in the proofs that do exist of local uniformization of arbitrary valuations in a positive characteristic algebraic function field of dimension ≤ 3, including [2], [24], [18], [4], [3], [7], [5]. No general results of local uniformization of arbitrary valuations exist, at the time of this writing, in positive characteristic algebraic function fields of dimension larger than 3.

We now define the classical ramification and inertia indices and the defect of a finite extension of valued fields.

Suppose that K is a field and ν is a valuation of K. Let V_ν be the valuation ring of ν with maximal ideal m_ν and Γ_ν be the value group of ν. Suppose that $K \to L$ is a finite field extension and ω is an extension of ν to L. We have associated ramification and inertia indices of the extension ω over ν,

$$e(\omega/\nu) = [\Gamma_\omega : \Gamma_\nu] \text{ and } f(\omega/\nu) = [V_\omega/m_\omega : V_\nu/m_\nu].$$

The defect of the extension of ω over ν is

$$\delta(\omega/\nu) = \frac{[L^h : K^h]}{e(\omega/\nu)f(\omega/\nu)}$$

where K^h and L^h are henselizations of the valued fields K and L. This is a positive integer (as shown in [16]) which is 1 if V_ν/m_ν has characteristic zero and is a power of p if V_ν/m_ν has positive characteristic p.

The following theorem is a consequence of [21, Theorem 1].

Theorem 1.4. *Let K/k be an algebraic function field and ν be an Abhankar valuation of K/k. Suppose that L is a finite extension field of K and ω is an extension of ν to L. Then the defect $\delta(\omega/\nu) = 1$.*

This theorem plays an essential role in the proof of the local uniformization theorem of [20].

It is shown in [11], that Zariski's local uniformization algorithm, which takes place in a finite extension of arbitrary valued fields, converges if and only if there is no defect. In particular, if a projection to a regular local ring is chosen in which defect occurs, then the resolution algorithm which we use will not converge.

In our proof of local uniformization (Theorems 1.1 - 1.3) the fact that there is no defect in an extension of Abhyankar valuations does not appear explicitly, and we do not use Theorem 1.4. However, since we show explicitly that Zariski's local uniformization algorithm converges in the completion of our local ring, it is implicit in the proof that there is no defect in our finite extension.

1.2. **Essentially finitely generated extensions of valuation rings.** In this subsection, we discuss a very interesting question proposed by Hagen Knaf, and give an application of our local uniformization theorem to improve a positive result on this question from [12].

Let H be an ordered subgroup of an ordered abelian group G. The initial index $\varepsilon(G/H)$ of H in G is defined ([15, page 138]) as

$$\varepsilon(G/H) = |\{g \in G_{\geq 0} \mid g < H_{>0}\}|,$$

where

$$G_{\geq 0} = \{g \in G \mid g \geq 0\} \text{ and } H_{>0} = \{h \in H \mid h > 0\}.$$

We define the initial index $\varepsilon(\omega/\nu)$ of the finite extension $K \to L$ as $\varepsilon(\Gamma_\omega/\Gamma_\nu)$.

We always have that $\varepsilon(\omega/\nu) \leq e(\omega/\nu)$ ([15, (18.3)]).

Let $D(\nu, L)$ be the integral closure of V_ν in L. The localizations of $D(\nu, L)$ at its maximal ideals are the valuation rings V_{ω_i} of the extensions ω_i of ν to L. We have the following remarkable theorem.

Theorem 1.5. ([15, Theorem 18.6]) *The ring $D(\nu, L)$ is a finite V_ν-module if and only if*

$$\delta(\omega_i/\nu) = 1 \text{ and } \varepsilon(\omega_i/\nu) = e(\omega_i/\nu)$$

for all extensions ω_i of ν to L.

Hagen Knaf proposed the following interesting question, asking for a local form of the above theorem. Essential finite generation is defined in Section 2.

Question 1.6. *(Knaf) Suppose that ω is an extension of ν to L. Is V_ω essentially finitely generated over V_ν if and only if*

$$\delta(\omega/\nu) = 1 \text{ and } \varepsilon(\omega/\nu) = e(\omega/\nu)?$$

Knaf proved the "only if" direction of his question; his proof is reproduced in [12, Theorem 4.1].

The "if" direction of the question is true if L/K is normal or ω is the unique extension of ν to L by [12, Corollary 2.2]. In [23, Theorem 1.3], it is shown that if L is an inertial extension of K, then V_ω is essentially finitely generated over V_ν.

The "if" direction of the question is proven when K is the quotient field of an excellent two-dimensional excellent local domain and ν dominates R in [12, Theorem 1.4]. The proof of [12, Theorem 1.4] uses the existence of a resolution of excellent surface singularities ([24] or [4]) and local monomialization of defectless extensions of two dimensional excellent local domains ([8, Theorem 3.7] and [13, Theorem 7.3]).

The "if" direction is proven when K is an algebraic function field over a field k of characteristic zero and ν is arbitrary in [9, Theorem 1.3].

We obtain the following theorem, which is proven in Section 5, as a consequence of Theorems 1.1 and 1.3.

Theorem 1.7. *Let K be an algebraic function field over a field k, and let ν be an Abhyankar valuation on K. Assume that L is a finite extension of K and that ω is an extension of ν to L. If $\varepsilon(\omega/\nu) = e(\omega/\nu)$, then V_ω is essentially finitely generated over V_ν.*

The defect $\delta(\omega/\nu) = 1$ with the assumptions of Theorem 1.7 by Theorem 1.4 as ν is an Abhyankar valuation.

Theorem 1.7 is proven in [12, Theorem 1.5], with the additional assumption that V_ω/m_ω is separable over k. To prove the stronger Theorem 1.7, we must only modify the proof of [12, Theorem 1.5] by observing that the statement of [12, Theorem 7.2] is true without the assumption that V_ν/m_ν is separable over k, using Theorems 1.1 and 1.2 of this paper in place of [20, Theorem 1.1].

A complete positive answer to Question 1.6 has been found by Rankeya Datta at the time that this article was in press, in the article "Essential finite generation of extensions of valuation rings", arXiv:2101.08377.

2. Notation

We will denote the non-negative integers by $\mathbb{N}$ and $\mathbb{Z}_{>0}$ will denote the positive integers.

If R is a local ring we will denote its maximal ideal by m_R. A regular prime ideal in a Noetherian local ring is a prime ideal P such that R/P is a regular local ring. If A is a domain then $\mathrm{QF}(A)$ will denote the quotient field of A. Suppose that A is a subring of a ring B. We will say that B is essentially finitely generated over A (or that B is essentially of finite type over A) if B is a localization of a finitely generated A-algebra. If R and S are local rings such that R is a subring of S and $m_S \cap R = m_R$ then we say that S dominates R.

Suppose that k is a field and K/k is an algebraic function field over k. An algebraic local ring of K is a local domain which is essentially of finite type over k and whose quotient field is K. A birational extension $R \to R_1$ of an algebraic local ring R of K is an algebraic local ring R_1 of K such that R_1 dominates R.

If ν is a valuation of a field K, we will denote the valuation ring of ν by V_ν and its maximal ideal by m_ν. If a valuation ring V_ν dominates A we will also say that ν dominates A. If A is a subring of a valuation ring V_ν, we will write A_ν for the localization of A at $m_\nu \cap A$.

A valuation ν of a function field K/k is a valuation of K which is trivial on k.

A pseudo-valuation μ on a local domain R is a prime ideal P of R and a valuation μ on the quotient field of R/P which dominates R/P. If μ is a pseudo-valuation on a domain R, we will write $\mu(f) = \infty$ if f is in the kernel P of the map from R to V_μ. We will write $P = P(\mu)_R$.

3. Extensions of Pseudo-Valuations

Suppose that T is a normal excellent local ring. Let $P(\omega)_T$ be a prime ideal of T and ω be a rank one valuation of the quotient field of $T/P(\omega)_T$ which dominates $T/P(\omega)_T$. The valuation ω induces a pseudo-valuation of T (as defined in the above Section 2), where we define $\omega(f) = \omega(\overline{f})$ if the class $\overline{f}$ of f in $T/P(\omega)_T$ is nonzero, and define $\omega(f) = \infty$ if

$f \in P(\omega)_T$. Let

(2) $\qquad Q(T) = \left\{ \begin{array}{l} \text{Cauchy sequences } \{f_n\} \text{ in } T \text{ such that for all } l \in \mathbb{Z}_{>0}, \\ \text{there exists } n_l \in \mathbb{Z}_{>0} \text{ such that } \omega(f_n) \geq l\omega(m_T) \text{ if } n \geq n_l \end{array} \right\}.$

We have that $Q(T)$ is a prime ideal in $\hat{T}$ and $Q(T) \cap T = P(\omega)_T$. There is a unique extension of ω to a valuation of the quotient field of $\hat{T}/Q(T)$ which dominates $\hat{T}/Q(T)$. It is an immediate extension (there is no extension of the value group or the residue fields of the valuation rings). Thus there is a unique extension of ω to a pseudo-valuation of $\hat{T}$ such that $P(\omega)_{\hat{T}} = Q(T)$. We define

$$\sigma(T) = \dim \hat{T}/Q(T).$$

The objects $Q(T)$ and $\sigma(T)$ are defined in [6], [10] and [14]. Concepts of this type are studied in [19].

The following lemma is proven in the case that ω is a rank 1 valuation dominating T (and not just a pseudo-valuation) in [6, Lemma 6.3]. The proof is essentially the same here, although we require a little more notation.

Lemma 3.1. *Let notation be as in this section. We further suppose that V_ω/m_ω is an algebraic field extension of T/m_T.*

1) *Let I be a nonzero ideal in T such that $I \not\subset P(\omega)_T$. Let $f \in I$ be such that $\omega(f) = \omega(I)$. Let*

$$J = \cup_{j=1}^\infty \left(P(\omega)_T T[\frac{I}{f}] : I^j T[\frac{I}{f}] \right),$$

which is the strict transform of the ideal $P(\omega)_T$ in $T[\frac{I}{f}]$. Then J is a prime ideal in $T[\frac{I}{f}]$, the map $T/P(\omega)_T \to T[\frac{I}{f}]/J$ is birational ($T[\frac{I}{f}]/J$ is of finite type over $T/P(\omega)_T$ and both rings have the same quotient field) and there exists a maximal ideal n of $T[\frac{I}{f}]$ containing J such that ω dominates $(T[\frac{I}{f}]/J)_n$ and so ω is a pseudo-valuation on $T_1 = T[\frac{I}{f}]_n$ with $P(\omega)_{T_1} = J_n$.

2) *Suppose that T_1 is normal. Then $\sigma(T_1) \leq \sigma(T)$.*

Proof. We first consider Statement 1). Let $\overline{f}$ be the class of f in $T/P(\omega)_T$. Since $I \not\subset P(\omega)_T$ we have that $\overline{f} \neq 0$ and

$$T\left[\frac{I}{f}\right]/J = (T/P(\omega)_T)\left[\frac{I(T/P(\omega)_T)}{\overline{f}}\right]$$

is a birational extension of $T/P(\omega)_T$ and all its elements have nonnegative ω-value. Let n be the prime ideal in $T[\frac{I}{f}]$ of elements of positive ω-value. $T/m_T \subset T[\frac{I}{f}]/n \subset V_\omega/m_\omega$ and V_ω/m_ω is assumed to be algebraic over T/m_T. Thus $T[\frac{I}{f}]/n$ is finite over T/m_T. By [25, Theorem 15.6], n is a maximal ideal of $T[\frac{I}{f}]$.

We now establish statement 2). The completion of T_1 at it's maximal ideal is $\hat{T}_1 = \widehat{\hat{T}[\frac{I\hat{T}}{f}]}_{\tilde{n}}$ where $\tilde{n} = m_{\hat{T}_1} \cap \hat{T}[\frac{I\hat{T}}{f}]$. Let

$$\tilde{Q} = \cup_{j=1}^\infty \left(Q(T)\hat{T}\left[\frac{I\hat{T}}{f}\right]_{\tilde{n}} : I^j \hat{T}\left[\frac{I\hat{T}}{f}\right]_{\tilde{n}} \right),$$

the strict transform of $Q(T)$ in $\hat{T}[\frac{I\hat{T}}{f}]_{\tilde{n}}$. Since $I \not\subset P(\omega)_{\tilde{T}}$ we have that $\omega(f) = \infty$ if $f \in \tilde{Q}$. Thus $\tilde{Q} \subset Q(T_1)$. Now $\hat{T}/Q(\hat{T}) \to \hat{T}[\frac{I\hat{T}}{f}]_{\tilde{n}}/\tilde{Q}$ is birational and the residue field extension is finite, so by the dimension formula [25, Theorem 15.6],

$$\sigma(T) = \dim \hat{T}[\frac{I\hat{T}}{f}]_{\tilde{n}}/\tilde{Q} = \dim \hat{T}_1/\tilde{Q}T_1$$

since completion is flat. Thus $\sigma(T) \geq \dim \hat{T}_1/Q(T_1) = \sigma(T_1)$. $\qquad\square$

4. Abhyankar valuations

Let K be an algebraic function field of a field k and let ν be an Abhyankar valuation of K/k. Since ν is Abhyankar, there exists a transcendence basis

$$x_{0,1}, \ldots, x_{0,r_0}, x_{1,1}, \ldots, x_{1,r_1}, x_{2,1}, \ldots, x_{t,1}, \ldots, x_{t,r_t} \in V_\nu$$

of K over k such that the classes $\overline{x}_{0,1}, \ldots, \overline{x}_{0,r_0}$ of $x_{0,1}, \ldots, x_{0,r_0}$ in V_ν/m_ν are a transcendence basis of V_ν/m_ν over k and $\nu(x_{i,1}), \ldots, \nu(x_{i,r_i})$ is a $\mathbb{Z}$-basis of Γ_i/Γ_{i-1} for $1 \leq i \leq t$. In particular, $\nu(x_{i,1}), \ldots, \nu(x_{t,r_t})$ is a $\mathbb{Z}$-basis of Γ_{ν_i} for all i, and the set of classes of $x_{0,1}, \ldots, x_{i-1,r_{i-1}}$ in V_{ν_i}/m_{ν_i} is a transcendence basis of V_{ν_i}/m_{ν_i} over k.

Definition 4.1. *Suppose that A is a regular algebraic local ring of K which is dominated by ν_i. A regular system of parameters $z_{i,1}, \ldots, z_{t,r_t}$ in A such that $\nu_i(z_{j,1}), \ldots, \nu_i(z_{j,r_j})$ is a basis of Γ_j/Γ_{j-1} for $i \leq j \leq t$ is called a very good regular system of parameters in A.*

Definition 4.2. *Suppose that A is a regular algebraic local ring of K which is dominated by ν_i. Suppose that $z_{i,1}, \ldots, z_{t,r_t}$ is a very good regular system of parameters in A. A primitive transform along ν_i, $A \to A_1$, is defined by*

$$(3) \qquad\qquad\qquad z_{j,k} = z_{j,k}(1)z_{l,m}$$

where $\nu_i(z_{j,k}) > \nu_i(z_{l,m})$. We define $A_1 = A[z_{j,k}(1)]_{\nu_i}$. Then A_1 is a regular algebraic local ring of K which has the good regular system of parameters $z_{i,1}(1), \ldots, z_{t,r_t}(1)$ where $z_{\alpha,\beta}(1) = z_{\alpha,\beta}$ if $(\alpha, \beta) \neq (j, k)$.

The following proposition is [12, Proposition 7.4] or [11, Lemma 4.1], which is a generalization of [28, Theorem 2].

Proposition 4.3. *Suppose that A is a regular algebraic local ring of K which is dominated by ν_i and $z_{i,1}, \ldots, z_{t,r_t}$ is a very good regular system of parameters in A. Suppose that $M_1 = \prod_{\alpha,\beta} z_{\alpha,\beta}^{a_{\alpha,\beta}}$ and $M_2 = \prod_{\alpha,\beta} z_{\alpha,\beta}^{b_{\alpha,\beta}}$ are monomials such that $\nu_i(M_1) < \nu_i(M_2)$. Then there exists a sequence of primitive transforms along ν_i,*

$$A \to A_1 \to \cdots \to A_s,$$

such that M_1 divides M_2 in A_s.

We remark that if $\nu_i(M_1) = \nu_i(M_2)$ in the statement of Proposition 4.3, then we have that $M_1 = M_2$.

Proof. There exists a largest index l such that $\prod_j z_{l,j}^{a_{l,j}} \neq \prod_j z_{l,j}^{b_{l,j}}$. Then $\nu_i(\prod_j z_{l,j}^{a_{l,j}}) < \nu_i(\prod_j z_{l,j}^{b_{l,j}})$. By [28, Theorem 2], there exists a sequence of primitive transforms $A \to A_s$ along ν_i in the variables $z_{l,j}(m)$ from the regular parameters of A_m as j varies, such that

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$\prod_j z_{l,j}^{a_{l,j}}$ divides $\prod_j z_{l,j}^{b_{l,j}}$ in A_s. Writing M_1 and M_2 in the regular parameters $z_{i,j}(s)$ of A_s as

$$M_1 = \prod z_{i,j}(s)^{a_{i,j}(s)} \text{ and } M_2 = \prod z_{i,j}(s)^{b_{i,j}(s)},$$

we have that

$$M_2 = \left(\prod_{i<l,j} z_{i,j}(s)^{b_{i,j}(s)}\right)\left(\prod_j z_{l,j}(s)^{b_{l,j}(s)}\right)\left(\prod_{i>l,j} z_{i,j}(s)^{a_{i,j}(s)}\right)$$

with $b_{l,j}(s) - a_{l,j}(s) \geq 0$ for all j and for some j, $b_{l,j}(s) - a_{l,j}(s) > 0$. Without loss of generality, this occurs for $j = 1$. (If $b_{l,j}(s) = a_{l,j}(s)$ for all j, then $a_{l,j} = b_{l,j}$ for all j in contradiction to our choice of l.)

Now perform a sequence of primitive transforms $A_s \to A_m$ along ν_i defined by $z_{l,1}(t) = z_{l,1}(t+1)z_{\alpha,\beta}(t+1)$ for $\alpha < l$ and β such that $b_{\alpha,\beta}(t) < a_{\alpha,\beta}(t)$ where

$$M_1 = \prod z_{\alpha,\beta}(t)^{a_{\alpha,\beta}(t)} \text{ and } M_2 = \prod z_{\alpha,\beta}(t)^{b_{\alpha,\beta}(t)}$$

to achieve that M_1 divides M_2 in A_m. $\qquad\square$

4.1. Construction of an algebraic local ring which is dominated by ν and has some good properties.

Let $L = k(x_{0,1}, \ldots, x_{t,r_t})$. $L \to K$ is a finite field extension. Let $\omega = \nu|L$ and $\omega_i = \nu_i|L$.

Let

$$R_i = k[x_{0,1}, \ldots, x_{t,r_t}]_{\omega_i} = k(x_{0,1}, \ldots, x_{i-1,r_{i-1}})[x_{i,1}, \ldots, x_{t,r_t}]_{(x_{i,1},\ldots,x_{t,r_t})},$$

a regular local ring.

Lemma 4.4. *Suppose that $f \in V_{\omega_i}$. Then there exists a sequence of primitive transforms along ω_i,*

$$R_i \to R_i^1 = R_i[x_{i,1}(1), \ldots, x_{t,r_t}(1)]_{(x_{i,1}(1),\ldots,x_{r,t_r}(1))}$$

such that $f = uM$ where M is a monomial in $x_{i,1}(1), \ldots, x_{r,t_r}(1)$ and u is a unit in R_i^1.

It follows that $V_{\omega_i}/m_{\omega_i} = k(x_{0,1}, \ldots, x_{i-1,r_{i-1}})$.

Proof. Write $f = \frac{g}{h}$ with $g, h \in R_i$. Expand $g = \sum \alpha_{\lambda_{i,1},\ldots,\lambda_{t,r_t}} x_{i,1}^{\lambda_{i,1}} \cdots x_{t,r_t}^{\lambda_{t,r_t}}$ and $h = \sum \beta_{\lambda_{i,1},\ldots,\lambda_{t,r_t}} x_{i,1}^{\lambda_{i,1}} \cdots x_{t,r_t}^{\lambda_{t,r_t}}$ in $\hat{R}_i = \varkappa[[x_{i,1}, \ldots, x_{t,r_t}]]$ with

$$\alpha_{\lambda_{i,1},\ldots,\lambda_{t,r_t}}, \beta_{\lambda_{i,1},\ldots,\lambda_{t,r_t}} \in \varkappa = k(x_{0,1}, \ldots, x_{0,r_0}).$$

Let $I = (x_{i,1}^{\lambda_{i,1}} \cdots x_{t,r_t}^{\lambda_{t,r_t}} \mid \alpha_{\lambda_{i,1},\ldots,\lambda_{t,r_t}} \neq 0)$ and $J = (x_{i,1}^{\lambda_{i,1}} \cdots x_{t,r_t}^{\lambda_{t,r_t}} \mid \beta_{\lambda_{i,1},\ldots,\lambda_{t,r_t}} \neq 0)$ which are ideals in R_i. By Proposition 4.3, there exists a sequence of primitive transforms along ω_i, $R_i \to R(1) = R_i[x_{i,1}(1), \ldots, x_{t,r_t}(1)]_{(x_{i,1}(1),\ldots,x_{r,t_r}(1))}$ and $d_{i,1}, \ldots, d_{t,r_t}, e_{i,1}, \ldots, e_{r,t_r} \in \mathbb{N}$ such that $IR(1) = x_{i,1}(1)^{d_{i,1}} \cdots x_{t,r_t}(1)^{d_{r,t_r}} R(1)$ and $JR(1) = x_{i,1}(1)^{e_{i,1}} \cdots x_{t,r_t}(1)^{e_{r,t_r}} R(1)$ and $d_{k,l} \geq e_{k,l}$ for all k, l. Thus $g = x_{i,1}(1)^{d_{i,1}} \cdots x_{t,r_t}(1)^{d_{t,r_r}} \gamma$ and

$$h = x_{i,1}(1)^{e_{i,1}} \cdots x_{t,r_t}(1)^{e_{t,r_r}} \delta$$

where $\gamma, \delta \in \widehat{R(1)}$ are units. Now $\gamma, \delta \in K \cap \widehat{R(1)} = R(1)$ are units (by [2, Lemma 2]). Thus $f = x_{i,1}(1)^{d_{i,1}-e_{i,1}} \cdots x_{t,r_t}(1)^{d_{t,r_t}-e_{t,r_r}} \gamma \delta^{-1}$ has the desired form in $R(1)$.

$\qquad\square$

Remark 4.5. *The sequence of primitive transforms $R_i \to R_i^1$ of Lemma 4.4 induces a sequence of primitive transforms*

$$R_1 \to R_1^1 = R_1[x_{i,1}(1), \ldots, x_{t,r_t}(1)]_{(x_{1,1}, \ldots, x_{i-1,r_{i-1}}, x_{i,1}(1), \ldots, x_{t,r_t}(1))}$$

along ω such that $(R_1^1)_{(x_{i,1}(1), \ldots, x_{t,r_t}(1))} = R_i^1$.

Now $V_{\omega_i}/m_{\omega_i} = k(x_{0,1}, \ldots, x_{i-1,r_{i-1}}) \to V_{\nu_i}/m_{\nu_i}$ is a finite algebraic extension since ν_i is Abhyankar. Thus there exist $\overline{\alpha}_{i,1}, \ldots, \overline{\alpha}_{i,s_i} \in V_{\nu_i}/m_{\nu_i}$ such that

$$V_{\nu_i}/m_{\nu_i} = (V_{\omega_i}/m_{\omega_i})(\overline{\alpha}_{i,1}, \ldots, \overline{\alpha}_{i,s_i}).$$

Let T_i be the integral closure of V_{ω_i} in L. There exists a maximal ideal m of T_i such that $(T_i)_m = V_{\nu_i}$ and so $V_{\nu_i}/m_{\nu_i} = T_i/m$. Let $\alpha_{i,1}, \ldots, \alpha_{i,s}$ be lifts of $\overline{\alpha}_{i,1}, \ldots, \overline{\alpha}_{i,s_i}$ to T_i. Let $f_j(x) \in V_{\omega_i}[x]$ be the minimal polynomial of $\alpha_{i,j}$ over L. Let $a_{i,j}^k$ be the coefficients of the $f_j(x)$. By Lemma 4.4, and Remark 4.5, there exists a sequence of primitive transforms along ω, $R_1 \to R_1^1$, such that

$$a_{i,j}^k \in (R_1^1)_{\omega_i} = (R_1^1)_{(x_{i,1}(1), \ldots, x_{t,r_t}(1))}$$

for all j, k.

We may thus construct a sequence of primitive transforms along ω

$$R_1 \to R(1) \to \cdots \to R(t)$$

such that

$$R(t) = k(x_{0,1}, \ldots, x_{0,r_0})[x_{1,1}(t), \ldots, x_{t,r_t}(t)]_{(x_{1,1}(t), \ldots, x_{t,r_t}(t))},$$

and

$$(R(t))_{\omega_i} = k(x_{0,1}, \ldots, x_{0,r_0}, x_{1,1}(1), \ldots, x_{i-1,r_{i-1}}(t))[x_{i,1}(t), \ldots, x_{t,r_t}(t)]_{(x_{i,1}(t), \ldots, x_{t,r_t}(t))}$$

for $1 \le i \le t$ and $a_{i,j}^k \in (R(t))_{\omega_i}$ for all j, k and i.

Replacing R_1 with $R(t)$, we may assume that $a_{i,j}^k \in (R_1)_{\omega_i}$ for all i, j, k.

Let B be the integral closure of $R = R_1$ in K. Let $P_i(R) = P_i^\nu \cap R = P_i^\omega \cap R$. Then the localization $B_{P_i(R)}$ of the R-module B at the prime ideal $P_i(R)$ of R is the integral closure in K of $R_{\omega_i} = R_{P_i(R)}$ for all i. Thus $\alpha_{i,j} \in B_{P_i(R)}$ for all j. Now $B_{P_i(R)}$ is a finite R_{ω_i} module for all i and $B_{P_i(R)} \cap P_i^\nu$ is a maximal ideal in $B_{P_i(R)}$. Thus there exists a maximal ideal m_i in $B_{P_i(R)}$ such that $m_i = B_{P_i(R)} \cap m_{\nu_i}$. Thus $\widehat{B_{\nu_i}}$ is the m_i-adic completion of $B_{P_i(R)}$ with respect to m_i and $B_{\nu_i}/m B_{\nu_i} = B_{P_i(R)}/m_i$. We have

(4)
$$V_{\omega_i}/m_{\omega_i} = R_{\omega_i}/m_{R_{\omega_i}} = k(\overline{x}_{0,1}, \ldots, \overline{x}_{i-1,r_{i-1}}) \subset k(\overline{x}_{0,1}, \ldots, \overline{x}_{i-1,r_{i-1}})(\overline{\alpha}_{i,1}, \ldots, \overline{\alpha}_{i,s_i})$$
$$\subset B_{P_i(R)}/m_i = B_{\nu_i}/m_{B_{\nu_i}} \subset V_{\nu_i}/m_{\nu_i} = (V_{\omega_i}/m_{\omega_i})(\overline{\alpha}_{i,1}, \ldots, \overline{\alpha}_{i,s_i})$$

so $B_{P_i(R)}/m_i = B_{\nu_i}/m_{B_{\nu_i}} = V_{\nu_i}/m_{\nu_i}$ for all i.

Now $B_{P_i(R)}$ is a finite R_{ω_i}-module which implies that

$$P_i(R)(B_{P_i(R)})_{m_i} = (x_{i,1}, \ldots, x_{t,r_t})B_{m_i}$$

is an m_i-primary ideal in B_{m_i}, so that $x_{i,1}, \ldots, x_{t,r_t}$ is a system of parameters in B_{m_i}, since

(5) $$\dim B_{m_i} = \dim R_{\omega_i} = r_i + \cdots + r_t.$$

Thus $x_{i,1}, \ldots, x_{t,r_t}$ is a system of parameters in $\widehat{B_{m_i}} = \widehat{B_{\nu_i}}$.

Let k_i be a coefficient field of $\widehat{B_{\nu_i}}$ (so $k_i \cong V_{\omega_i}/m_{\omega_i}$). $k_i[[x_{i,1}, \ldots, x_{t,r_t}]] \subset \widehat{B_{\nu_i}}$ is a power series ring, by [29, Corollary 2, page 293], and $\widehat{B_{\nu_i}}$ is a finite module over $k_i[[x_{i,1}, \ldots, x_{t,r_t}]]$

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by [29, Remark on page 293]. By our construction, $\nu_i(x_{i,1}), \ldots, \nu_i(x_{t,r_t})$ is a $\mathbb{Z}$-basis of Γ_{ω_i} and $k_i \cong V_{\omega_i}/m_{\omega_i}$ for $1 \le i \le t$. Let $P_{i+1}(B_{\nu_i})$ be the prime ideal

$$P_{i+1}(B_{\nu_i}) = (P_{i+1}^\nu V_{\nu_i}) \cap B_{\nu_i} \subset (P_i^\nu V_{\nu_i}) \cap B_{\nu_i} = m_{B_{\nu_i}}.$$

Equation (5) implies that

$$\dim B_{\nu_i} = r_i + \cdots + r_t = \dim R_{\omega_i}$$

for all i. Thus

$$(6) \quad \dim B_{\nu_i}/P_{i+1}(B_{\nu_i}) = \dim B_{\nu_i} - \dim(B_{\nu_i})_{P_{i+1}(B_{\nu_i})} = \dim B_{\nu_i} - \dim B_{\nu_{i+1}} = r_i.$$

Now $\widehat{B_{\nu_i}}$ is reduced and equidimensional of the same dimension $r_i + \cdots + r_t$ as B_{ν_i} and

$$(7) \qquad\qquad P_{i+1}(B_{\nu_i})\widehat{B_{\nu_i}} = I_1 \cap \cdots \cap I_u$$

where the I_j are prime ideals in $\widehat{B_{\nu_i}}$ such that

$$\dim \widehat{B_{\nu_i}}/I_j = \dim B_{\nu_i}/P_{i+1}(B_{\nu_i}) = r_i$$

for all j by [17, Scholie IV.7.8.3]. Define a prime ideal $Q_{i+1} = Q_{i+1}(\widehat{B}_{\nu_i})$ in $\widehat{B_{\nu_i}}$ of the Cauchy sequences $\{f_j\}$ in B_{ν_i} such that given $\gamma \in \Gamma_i/\Gamma_{i-1}$, there exists j_0 such that $\nu_i(f_j) > \gamma$ for $j \ge j_0$. The ideal $Q_{i+1}(\widehat{B_{\nu_i}})$ is the ideal $Q(\widehat{B_{\nu_i}})$ defined in (2).

We have an injective finite map

$$(8) \quad k_i[[x_{i,1}, \ldots, x_{i,r_i}]] \cong k_i[[x_{i,1}, \ldots, x_{t,r_t}]]/Q_{i+1} \cap k_i[[x_{i,1}, \ldots, x_{t,r_t}]] \to \widehat{B_{\nu_i}}/Q_{i+1}$$

so

$$(9) \qquad\qquad\qquad \dim \widehat{B_{\nu_i}}/Q_{i+1} = r_i.$$

Now $P_{i+1}(B_{\nu_i})\widehat{B_{\nu_i}} \subset Q_{i+1}$ implies one of the I_j in (7) is Q_{i+1}. Thus after possibly reindexing the I_j, we have that $Q_{i+1} = I_1$ and

$$(10) \qquad\qquad P_{i+1}(B_{\nu_i})\widehat{B_{\nu_i}} = Q_{i+1} \cap I_2 \cap \cdots \cap I_u.$$

ν_i is composite with the valuation $\overline{\nu}_i$ with valuation ring $V_{\overline{\nu}_i} = (V_\nu/P_{i+1}^\nu)_{P_i^\nu}$, value group $\Gamma_{\overline{\nu}_i} = \Gamma_i/\Gamma_{i-1}$ and residue field $(V_\nu/P_i^\nu)_{P_i^\nu}$. $\overline{\nu}_i$ naturally induces a valuation on $\widehat{B_{\nu_i}}/Q_{i+1}$.

$$(11) \qquad \begin{array}{l} \text{There exists a regular local ring } D \text{ which is essentially of finite type over } k, \\ \text{such that } B_{\nu_1} \text{ is a quotient of } D. \end{array}$$

Let $P_i(D)$ be the preimage of P_i^ν under the composition $D \to B_{\nu_1} \to V_\nu$ and let $D_i = D_{P_i(D)}$. Then D_i are regular local rings which are essentially of finite type over k with regular parameters $x_{i,1}, \ldots, x_{i,r_i}, x_{i+1,1}, \ldots, x_{t,r_t}, y_1, \ldots, y_m$ in D_i such that B_{ν_i} is a quotient of D_i (we identify the $x_{i,j}$ with their image in B).

We have that ν_i induces a pseudo-valuation on D_i with kernel $P_{t+1}(U_i)$. Let

$$(12) \qquad A_i = \widehat{D_i} = k_i[[x_{i,1}, \ldots, x_{i,r_i}, x_{i+1,1}, \ldots, x_{t,r_t}, y_1, \ldots, y_m]].$$

Let $Q = Q_{i+1}(A_i)$ be the preimage of Q_{i+1} in A_i. Then $\overline{\nu}_i$ induces a pseudo-valuation on $\widehat{B_{\nu_i}}$ with kernel Q_{i+1} and induces a pseudo-valuation on A_i with kernel $Q = Q_{i+1}(A_i)$. $\overline{\nu}_i$ induces a pseudo-valuation on D_i, with kernel

$$(13) \qquad\qquad P_{i+1}(D_i) = Q(A_i) \cap D_i$$

which is the preimage of P_{i+1}^ν in D_i.

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More generally, suppose that

(14)

U_i is a regular local ring which is essentially of finite type over k with quotient field K such that U_i dominates D_i and the pseudo-valuation $\bar{\nu}_i$ dominates U_i.

We then have a natural homomorphism $\pi : U_i \to V_{\nu_i}$. For j such that $i \leq j \leq t+1$, define

(15)
$$P_j(U_i) = \pi^{-1}(P_j^\nu V_{\nu_i}).$$

We have a chain of prime ideals

$$P_{t+1}(U_i) \subset P_t(U_i) \subset \cdots \subset P_i(U_i) = m_{U_i}.$$

Suppose that

(16)
$$x_{i,1}, \ldots, x_{i,r_i}, x_{i+1,1}, \ldots, x_{i+1,r_{i+1}}, \ldots, x_{t,r_t}, y_1, \ldots, y_m$$

is a regular system of parameters in U_i such that $\nu_j(x_{j,1}), \ldots, \nu_j(x_{j,t_j})$ is a $\mathbb{Z}$-basis of Γ_j/Γ_{j-1} for $i \leq j \leq t$. Such a regular system of parameters will be called a good regular system of parameters, or simply "good regular parameters". Sometimes we will abuse notation and allow a good system of parameters to be a permutation of an ordered list (16).

4.2. Transforms. We define four types of transformations $U_i \to U(1)$ along ν_i.

Type $(1,j)$ with $i \leq j$. This is a transform

$$x_{j,k} = \prod_{l=1}^{r_j} x_{j,l}(1)^{a_{k,l}}, \text{ for } 1 \leq k \leq r_j$$

where $a_{k,l} \in \mathbb{N}$, $\mathrm{Det}(a_{k,l}) = \pm 1$ and $\nu_j(x_{j,l}(1)) > 0$ for all l. We define

$$U(1) = U_i[x_{j,1}(1), \ldots, x_{j,r_j}(1)]_{\nu_i}.$$

Type $(2,j)$ with $i \leq j$. Suppose $u \in \{x_{j+1,1}, \ldots, x_{t,r_t}, y_1, \ldots, y_m\}$ with $\nu_i(u) \in \Gamma_{j+1}$. Let $a_1, \ldots, a_{r_j} \in \mathbb{N}$. Define

$$u = x_{j,1}^{a_1} \cdots x_{j,r_j}^{a_{r_j}} u(1)$$

and $U(1) = U_i[u(1)]_{\nu_i}$.

Type $(3,j)$ with $i \leq j$. Suppose $\nu_i(y_k) \in \Gamma_j$ and $\nu_j(y_k) = a_1 \nu_j(x_{j,1}) + \cdots + a_{r_j} \nu_j(x_{j,r_j})$ with $a_1, \ldots, a_{r_j} \in \mathbb{N}$ and

$$\nu_i\left(\frac{y_k}{x_{j,1}^{a_1} \cdots x_{j,r_j}^{a_{r_j}}}\right) \geq 0.$$

Define $y_k = x_{j,1}^{a_1} \cdots x_{j,r_j}^{a_{r_j}} y_k(1)$ and $U(1) = U_i[y_k(1)]_{\nu_i}$.

Type $(4,j)$ with $i \leq j$. Suppose that $u \in \{y_1, \ldots, y_m\}$ with $\nu_i(u) \in \Gamma_j$. Suppose that $a_1, \ldots, a_{r_j} \in \mathbb{N}$ are such that $\nu_i(u) > \nu_i(x_{j,1}^{a_1} \cdots x_{j,r_j}^{a_{r_j}})$. Define $u = x_{j,1}^{a_1} \cdots x_{j,r_j}^{a_{r_j}} u(1)$ and $U(1) = U_i[u(1)]_{\nu_i}$.

In all four cases, $U(1)$ has a natural good system of regular parameters.

4.3. **Formal Transforms.** We now construct sequences of formal transforms along $\overline{\nu}_i$

(17) $$A_i = \widehat{U}_i \to A(1) \to \cdots \to A(l).$$

We suppose that

$$x_{i,1}, \ldots, x_{t,r_t}, y_1, \ldots, y_e$$

is a given good regular system of parameters in U_i. In $A(0) = A_i = \widehat{U}_i$, we suppose that

$$x_{i,1}(0), \ldots, x_{t,r_t}(0), y_1(0), \ldots, y_e(0)$$

is a regular system of parameters such that $x_{1,1}(0) = x_{1,1}, \ldots, x_{t,r_t}(0) = x_{t,r_t}$ and $y_1(0), \ldots, y_e(0)$ may be formal (not in U_i). Suppose that we have inductively defined $A(0) \to \cdots \to A(n)$, with a regular system of parameters

$$x_{i,1}(n), \ldots, x_{t,r_t}(n), y_1(n), \ldots, y_e(n).$$

Formal transforms of type $(1, i)$. We define a formal transform $A(n) \to A(n+1)$ of type $(1, i)$ as follows. Set

$$x_{i,k}(n) = \prod_{l=1}^{r_i} x_{i,l}(n+1)^{a_{i,l}}, \text{ for } 1 \le k \le r_i$$

where $a_{i,l} \in \mathbb{N}$, $\text{Det}(a_{i,l}) = \pm 1$ and $\overline{\nu}_i(x_{i,l}(n+1)) > 0$ for all l. By Lemma 3.1, since the ring $A(n)[x_{i,1}(n+1), \ldots, x_{i,r_i}(n+1)]$ is the blow up of an ideal in $A(n)$ generated by monomials in $x_{i,1}(n), \ldots, x_{i,r_i}(n)$ which is thus not contained in $Q(A(n))$, the pseudo-valuation $\overline{\nu}_i$ extends uniquely to a pseudo-valuation which dominates a local ring

$$(A(n)[x_{i,1}(n+1), \ldots, x_{i,r_i}(n+1)])_{m_n}$$

where m_n is a maximal ideal of $A(n)[x_{i,1}(n+1), \ldots, x_{i,r_i}(n+1)]$, and extends uniquely to a pseudo-valuation which dominates it's completion

$$A(n+1) = (A(n)[x_{i,1}(n+1), \ldots, x_{i,r_i}(n+1)]_{m_n})\widehat{}.$$

We extend $x_{i,1}(n+1), \ldots, x_{i,r_i}(n+1)$ to a regular system of parameters

$$x_{i,1}(n+1), \ldots, x_{i,r_i}(n+1), x_{i+1,1}(n+1), \ldots, x_{t,r_t}(n+1), y_1(n+1), \ldots, y_e(n+1)$$

in $A(n+1)$ where $x_{j,k}(n+1) = x_{j,k}(n)$ for $j > i$ and $y_1(n+1), \ldots, y_e(n+1)$ can be chosen arbitrarily to make a regular system of parameters.

Formal transforms of type $(2, i)$. We define a formal transform $A(n) \to A(n+1)$ of type (2,i) as follows. Suppose that $u \in \{x_{i+1,1}(n), \ldots, x_{t,r_t}(n), y_1(n), \ldots, y_e(n)\}$ and $\overline{\nu}_i(u) = \infty$. Suppose that $a_1, \ldots, a_{r_j} \in \mathbb{N}$. Define $u = x_{i,1}(n)^{a_1} \cdots x_{i,r_i}(n)^{a_{r_j}} u'$.

By Lemma 3.1, since the ring $A(n)[u']$ is the blow up of the ideal $(x_{i,1}(n)^{a_1} \cdots x_{i,r_i}(n)^{a_{r_j}}, u)$ in $A(n)$ which is not contained in $Q(A(n))$, the pseudo-valuation $\overline{\nu}_i$ extends uniquely to a pseudo-valuation which dominates a local ring

$$(A(n)[u'])_{m_n}$$

where m_n is a maximal ideal of $A(n)[u']$, and extends uniquely to a pseudo-valuation which dominates it's completion

$$A(n+1) = A\widehat{(n)[u']}_{m_n}.$$

We define a regular system of parameters

$$x_{i,1}(n+1), \ldots, x_{i,r_i}(n+1), x_{i+1,1}(n+1), \ldots, x_{t,r_t}(n+1), y_1(n+1), \ldots, y_e(n+1)$$

in $A(n+1)$.

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If $u \in \{y_1(n), \ldots, y_e(n)\}$, then $x_{j,k}(n+1) = x_{j,k}(n)$ for $j \geq i$ and $y_1(n+1), \ldots, y_e(n+1)$ can be chosen arbitrarily to make a regular system of parameters. If $u = x_{a,b}(n)$ for some $a > i$ and b, then

$$x_{j,k}(n+1) = \begin{cases} u' & \text{if } (j,k) = (a,b) \\ x_{j,k}(n) & \text{if } (j,k) \neq (a,b) \end{cases}$$

and $y_1(n+1), \ldots, y_e(n+1)$ can be chosen arbitrarily to make a regular system of parameters.

Formal transforms of type $(3, i)$.

We define a formal transform $A(n) \to A(n+1)$ of type $(3, i)$ as follows. Suppose $\bar{\nu}_i(y_1(n)) \neq \infty$, and $\bar{\nu}_i(y_1(n)) = a_1 \bar{\nu}_i(x_{i,1}(n)) + \cdots + a_{r_j} \bar{\nu}_i(x_{i,r_i})$ with $a_1, \ldots, a_{r_j} \in \mathbb{N}$ and

$$\bar{\nu}_i \left(\frac{y_1(n)}{x_{i,1}(n)^{a_1} \cdots x_{i,r_i}(n)^{a_{r_j}}} \right) \geq 0.$$

Define $y_1(n) = x_{i,1}(n)^{a_1} \cdots x_{i,r_i}(n)^{a_{r_i}} y_1'$.

By Lemma 3.1, since the ring $A(n)[y_1']$ is the blow up of the ideal $(x_{i,1}(n)^{a_1} \cdots x_{i,r_i}(n)^{a_{r_i}}, y_1(n))$ in $A(n)$ which is not contained in $Q(A(n))$, the pseudo-valuation $\bar{\nu}_i$ extends uniquely to a pseudo-valuation which dominates a local ring

$$(A(n)[y_1'])_{m_n}$$

where m_n is a maximal ideal of $A(n)[y_1']$, and extends uniquely to a pseudo-valuation which dominates it's completion

$$A(n+1) = A\widehat{(n)[y_1']}_{m_n}.$$

We define a regular system of parameters

$$x_{i,1}(n+1), \ldots, x_{i,r_i}(n+1), x_{i+1,1}(n+1), \ldots, x_{t,r_t}(n+1), y_1(n+1), \ldots, y_e(n+1)$$

in $A(n+1)$ where $x_{j,k}(n+1) = x_{j,k}(n)$ for $j \geq i$ and $y_1(n+1), \ldots, y_e(n+1)$ can be chosen arbitrarily to make a regular system of parameters.

Formal transforms of type $(4, i)$.

We define a formal transform $A(n) \to A(n+1)$ of type $(4,i)$ as follows. Suppose that $\bar{\nu}_i(y_1(n)) \neq \infty$ and $a_1, \ldots, a_{r_j} \in \mathbb{N}$ are such that $\nu_i(y_1(n)) > \nu_i(x_{i,1}(n)^{a_1} \cdots x_{i,r_i}(n)^{a_{r_j}})$. Define $y_1(n) = x_{i,1}(n)^{a_1} \cdots x_{i,r_i}(n)^{a_{r_j}} y_1'$.

By Lemma 3.1, since the ring $A(n)[y_1']$ is the blow up of the ideal $(x_{i,1}(n)^{a_1} \cdots x_{i,r_i}(n)^{a_{r_j}}, y_1(n))$ in $A(n)$ which is not contained in $Q(A(n))$, the pseudo-valuation $\bar{\nu}_i$ extends uniquely to a pseudo-valuation which dominates a local ring

$$(A(n)[y_1'])_{m_n}$$

where m_n is a maximal ideal of $A(n)[y_1']$, and extends uniquely to a pseudo-valuation which dominates it's completion

$$A(n+1) = A\widehat{(n)[y_1']}_{m_n}.$$

We define a regular system of parameters

$$x_{i,1}(n+1), \ldots, x_{i,r_i}(n+1), x_{i+1,1}(n+1), \ldots, x_{t,r_t}(n+1), y_1(n+1), \ldots, y_e(n+1)$$

in $A(n+1)$ where $x_{j,k}(n+1) = x_{j,k}(n)$ for $j \geq i$ and $y_1(n+1), \ldots, y_e(n+1)$ can be chosen arbitrarily to make a regular system of parameters.

Remark 4.6. *In a sequence (17) of formal transforms, we have that* $\sigma(A(n)) = \dim B_{\nu_i}$ *for all* n.

This follows since for all n, we have that

$$\dim B_{\nu_i}/P_{i+1}(B_{\nu_i}) = \sigma(A(0)) \geq \sigma(A(n)) \geq \text{rrank } \bar\nu_i = \dim B_{\nu_i}/P_{i+1}(B_{\nu_i})$$

by (9), (6), Lemma 3.1 and [1, Proposition 2] *or Proposition 2 page 331* [29].

We will call a regular system of parameters

$$(18) \qquad\qquad x_{i,1}(n), \ldots, x_{t,r_t}(n), y_1(n), \ldots, y_e(n)$$

as constructed in the above sequence of formal transforms (17) a good regular system of parameters in $A(n)$. We will find it convenient to permute the variables in a good regular system of parameters (18) and write it as

$$(19) \qquad x_1(n), \ldots, x_r(n), z_1(n), \ldots, z_m(n), x_{i+1,1}(n), \ldots, x_{t,r_t}(n), w_1(n), \ldots, w_l(n)$$

where

$$x_1(n) = x_{i,1}(n), \ldots, x_r(n) = x_{i,r_i}(n),$$

$z_1(n), \ldots, z_m(n), w_1(n), \ldots, w_l(n)$ is a permutation of $y_1(n), \ldots, y_e(n)$ and

$$w_1(n), \ldots, w_l(n) \in Q(A(n)).$$

We remind the reader that $x_{i+1,1}(n), \ldots, x_{t,r_t}(n) \in Q(A(n))$. We will also call (19) a good system of parameters in $A(n)$.

Remark 4.7. *If* $U(n) \to U(n+1)$ *is a tranform or* $A(n) \to A(n+1)$ *is a formal transform then we have*

$$k_i[[x_{i,1}(n), \ldots, x_{i,r_i}(n)]] \subset k_i[[x_{i,1}(n+1), \ldots, x_{i,r_i}(n+1)]].$$

4.4. **Setting up the reduction algorithm.** We now fix i. Let $U_i = D_i$ and $A_i = \hat{D}_i$ (12). Let

$$x_1, \ldots, x_r, z_1, \ldots, z_m, x_{i+1,1}, \ldots, x_{t,r_t}, w_1, \ldots, w_l$$

be a good system of regular parameters in U_i of the form of (19), so that $w_1, \ldots, w_l \in P_{i+1}(U_i)$.

Let $\bar{z}_j$ be the class of z_j in $A_i/Q_{i+1}(A_i) \cong \widehat{B_{\nu_i}}/Q_{i+1}(\widehat{B_{\nu_i}})$ for $1 \leq j \leq m$. The element $\bar{z}_j$ is integral over $C_i = k_i[[x_1, \ldots, x_r]]$ by (8). Thus there exists a relation $\bar{z}_j^{n_j} + a_{j,n_j-1}\bar{z}_j^{n_j-1} + \cdots + a_{j,0} = 0$ with all $a_{j,t} \in C_i$. Thus $z_j^{n_j} + a_{j,n_j-1}z_j^{n_j-1} + \cdots + a_{j,0} \in Q_{i+1}(A_i)$. That is, $\bar\nu_i(z_j^{n_j} + a_{j,n_j-1}z_j^{n_j-1} + \cdots + a_{j,0}) = \infty$. Set

$$(20) \qquad\qquad g^{(j)}(x_1, \ldots, x_r, x) = x^{n_j} + a_{j,n_j-1}x^{n_j-1} + \cdots + a_{j,0}$$

for $1 \leq j \leq m$ which are monic polynomials in $C_i[x]$ such that $\bar\nu_i(g^{(j)}(z_j)) = \infty$. We may assume that each $g^{(j)}(x)$ is an irreducible polynomial in $C_i[x]$.

Lemma 4.8. *There exists a sequence of transforms* $U \to U(1)$ *of types (1,i), (3,i) and (4,i) such that* $U(0)$ *has very good parameters*

$$x_1(0), \ldots, x_r(0), z_1(0), \ldots, z_m(0), x_{i+1,1}(1), \ldots, x_{t,r_t}(1), w_1(0), \ldots, w_l(1)$$

such that, with $A(0) = \widehat{U(0)}$,

1) $x_{i+1,1}(1), \ldots, x_{t,r_t}(1), w_1(0), \ldots, w_l(1) \subset P_{i+1}(U(0))$.

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2) *There exist monic polynomials $f^{(j)}(x) \in k_i[[x_1(0), \ldots, x_r(0)]][x]$ such that*

$$\bar{\nu}_i(f^{(j)}(z_j(0))) = \infty$$

for $1 \le j \le m$.

3) *Suppose that $\alpha \in k_i$ and $q \in \mathbb{Z}_{>0}$. Then there exists $\alpha' \in U(0)$ and $h \in m_{A(0)}^q$ such that*

(21)
$$\alpha = \alpha' + (x_1(0) \cdots x_r(0))^q h$$

with $h \in A(0)$.

Proof. For each j, the monic polynomial $g^{(j)}(x) = x^{n_j} + a_{j,n_j-1} x^{n_j-1} + \cdots + a_{j,0} \in C_i[x]$ is irreducible. Further, we have that $\nu_i(a_{j,0}) > 0$ so x divides the residue of $g^{(j)}(x)$ in $C_i/m_{C_i}[x] \cong k_i[x]$. Thus by Hensel's lemma (Theorem 14 on page 279 [29]), all $a_{j,t} \in m_{C_i}$. Since $\nu_i(z_j) \in \mathbb{Z}\nu(x_1) + \cdots + \mathbb{Z}\nu(x_r)$, by Proposition 4.3, there exists a transform of type (1,i), $U \to U'$, defined by $x_s = \prod x_k(0)^{c_{s,k}}$ such that performing the further transforms of types (i,4) and (i,3), setting $z_j = x_1(0) \cdots x_r(0) z_j(0)$ for $1 \le j \le m$ and $w_j = x_1(0) \cdots x_r(0) w_j(0)$ for $1 \le j \le l$, we have that

$$b_{j,t} = \frac{a_{j,t}}{(x_1(0) \cdots x_r(0))^{n_j-t}} \in k_i[[x_1(0), \ldots, x_r(0)]]$$

for $0 \le t \le n_j - 1$. Thus setting $f^{(j)}(x) = x^n + b_{j,n_j-1} x^{n_j-1} + \cdots + b_{j,0}$ we have that $f^{(j)}(x) \in k_i[[x_1(0), \ldots, x_r(0)]][x]$ and $\bar{\nu}_i(f^{(j)}(z_j(0))) = \infty$ for all j. $\qquad\square$

Replacing U_i with $U(0)$, we may suppose that the good properties of Lemma 4.8 hold in U_i and A_i.

Suppose that $m \ge 1$, and let $z = z_1$. In the algorithms of Subsections 4.5 - 4.7, we will show that we can construct a sequence of transforms of types (i,1) and (i,3) giving a reduction of m. In our reduction algorithm, none of the variables

$$z_2, \ldots, z_m, x_{i+1,1}, \ldots, x_{t,r_t}, w_1, \ldots, w_l$$

will be effected, so we need only keep track of the change in the first variables $x_1, \ldots, x_r, z$.

In Theorem 4.9, we will prove that we can apply this algorithm m times to obtain the condition that Q_{i+1} is a regular prime.

4.5. **The reduction algorithm.** Let $z = z_1$ and let

$$f(x) = f^{(1)}(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_0.$$

Set $\mu = \text{ord } f(0, \ldots, 0, x)$. We have that $1 \le \mu \le n$. If $\mu = 1$ we replace z with $f(x_1, \ldots, x_r, z)$, and obtain a reduction in m. So suppose that $\mu \ge 2$. Set $a_n = 1$ and let $\rho = \min\{\bar{\nu}_i(a_j z^j)\}$. Write

$$a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n = a_{i_1} z^{i_1} + \cdots + a_{i_s} z^{i_s} + a_{i_{s+1}} z^{i_{s+1}} + \cdots + a_{i_{n+1}} z^{i_{n+1}}$$

where $\rho = \bar{\nu}_i(z^{i_1}) = \cdots = \bar{\nu}_i(a_{i_s} z^{i_s})$, with $i_1 < \cdots < i_s$ and $\rho < \bar{\nu}_i(a_{i_j} z^{i_j}) \le \bar{\nu}_i(a_{i_{j+1}} z^{i_{j+1}})$ if $j > s$.

There exist $d_i \in \mathbb{Z}$ such that $\bar{\nu}_i(z) = \sum_{i=1}^{r} d_i \bar{\nu}_i(x_i)$. There exists a formal transform $A_i \to A(1)$ along $\bar{\nu}_i$ of type (1,i), $x_j = \prod_k x_k(1)^{b_{jk}}$ for $1 \le j \le r$, as in Lemma 4.4, such that $A(1)$ has regular parameters $x_1(1), \ldots, x_r(1), z, \ldots$ such that if a_{i_j} is nonzero, then a_{i_j} is a unit $\bar{a}_{i_j}$ in $k_i[[x_1(1), \ldots, x_r(1)]]$ times a monomial in $x_1(1), \ldots, x_r(1)$. Further, by Proposition 4.3, $x_1^{d_1} \cdots x_r^{d_r}$ is a monomial $x_1(1)^{e_1} \cdots x_r(1)^{e_r}$ (all e_i are nonnegative).

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Now perform the formal transform $A(1) \to A(2)$ along $\bar{\nu}_i$ of type (3,i) defined by

$$ (22) \qquad\qquad z = x_1(1)^{e_1} \cdots x_r(1)^{e_r} \tilde{z}_1 $$

so that $\bar{\nu}_i(\tilde{z}_1) = 0$. Now $A(2)/m_{A(2)} \subset V_{\nu_i}/m_{\nu_i} \cong k_i$ so k_i continues to be a coefficient field of $A(2)$ and there exists a unit $\alpha \in k_i$ such that setting

$$ (23) \qquad\qquad z_1 = \tilde{z}_1 - \alpha, $$

$x_1(1), \ldots, x_r(1), z_1, \ldots$ are good regular parameters in $A(2)$.

(24)   If $\bar{\nu}_i(z_1) = \infty$ we terminate the algorithm.
             Since $z_1 \in Q_{i+1}(2) = Q_{i+1}(A(2))$, we have a reduction of m in $A(2)$.

Suppose that $\bar{\nu}_i(z_1) \neq \infty$, We have that

$$
\begin{aligned}
f(x_1, \ldots, x_r, z) &= a_{i_1} z^{i_1} + \cdots + a_{i_s} z^{i_s} + a_{i_{s+1}} z^{i_{s+1}} + \cdots + a_{i_{n+1}} z^{i_{n+1}} \\
&= x_1(1)^{g_1} \cdots x_r(1)^{g_r} \left(\bar{a}_{i_1}(z_1 + \alpha)^{i_1} + \cdots + \cdots + \bar{a}_{i_s}(z_1 + \alpha)^{i_s} \right) \\
&\quad + x_1(1)^{g_{1,i_{s+1}}} \cdots x_r(1)^{g_{r,i_{s+1}}} \bar{a}_{i_{s+1}}(z_1 + \alpha)^{i_{s+1}} + \cdots \\
&\quad + x_1(1)^{g_{1,i_{n+1}}} \cdots x_r(1)^{g_{r,i_{n+1}}} \bar{a}_{i_{n+1}}(z_1 + \alpha)^{i_{n+1}}.
\end{aligned}
$$

Now perform a formal transform along $\bar{\nu}_i$ $A(2) \to A(3)$ of type (1,i) in $x_1(1), \ldots, x_r(1)$ so that $x_1(1)^{g_1} \cdots x_r(1)^{g_r}$ divides $x_1(1)^{g_{1,i_j}} \cdots x_r(1)^{g_{r,i_j}}$ for all j. Setting

$$ f_1(x_1(1), \ldots, x_r(1), z_1) = \frac{f}{x_1(1)^{g_1} \cdots x_r(1)^{g_r}} \in k_i[[x_1(1), \ldots, x_r(1)]][z_1] $$

we have that $\bar{\nu}_i(f_1) = \infty$. We expand

$$ (25) \qquad f_1 = \bar{a}_{i_1}(z_1 + \alpha)^{i_1} + \cdots + \bar{a}_{i_s}(z_1 + \alpha)^{i_s} + x_1(2) \cdots x_r(2)\Omega $$

with $\Omega \in k_i[[x_1(2), \ldots, x_r(2)]][z_1]$. We have that

$$ f_1(0, \ldots, 0, z_1) = \tilde{a}_{i_1}(z_1 + \alpha)^{i_1} + \cdots + \tilde{a}_{i_s}(z_1 + \alpha)^{i_s} \in k_i[z_1]. $$

where $\tilde{\alpha}_{i_j}$ is the residue of $\bar{a}_{i_j}$ in $k_i \cong A(2)/m_{A(2)}$. We have that $1 \leq \mathrm{ord}\, f_1(0, \ldots, 0, z_1) \leq \mu$ (since $\bar{\nu}_i(f_1) = \infty$).

(26)
     If $\mathrm{ord}\, f_1(0, \ldots, 0, z_1) < \mu$ then we have a reduction. Go back to the beginning of
     Subsection 4.5 with z replaced by z_1, A_i replaced with $A(3)$, C_i replaced with
     $k_i[[x_1(2), \ldots, x_r(2)]]]$, f replaced with f_1 and μ replaced with $\mathrm{ord}\, f_1(0, \ldots, 0, z_1)$.

Of course it may now be that z is formal.
     If $\mathrm{ord}\, f_1(0, \ldots, 0, z_1) = \mu$, then $i_s = \mu$, a_{i_s} is a unit in C_i (since $\mathrm{ord}\, f(0, \ldots, 0, x) = \mu$) and $f_1(0, \ldots, 0, z_1) = a_{i_s} z_1^\mu$. Thus

$$ f_1(0, \ldots, 0, z_1 - \alpha) = \tilde{a}_{i_1} z_1^{i_1} + \cdots + \tilde{a}_{i_s} z_1^{i_s} = \tilde{a}_{i_s}(z_1 - \alpha)^{i_s}. $$

So, $i_1 = 0$ and $a_{i_1} \neq 0$. Thus

$$
\begin{aligned}
\bar{\nu}_i(z) &= \tfrac{1}{\mu} \bar{\nu}_i(a_{i_1}) = \tfrac{1}{\mu} \bar{\nu}_i(a_0) \\
&\in \tfrac{1}{\mu} \left(\mathbb{N}\nu_i(x_1) + \cdots + \mathbb{N}\nu_i(x_r) \right) \cap \left(\mathbb{Z}\nu_i(x_1) + \cdots + \mathbb{Z}\nu_i(x_r) \right) \\
&= \mathbb{N}\nu_i(x_1) + \cdots + \mathbb{N}\nu_i(x_r).
\end{aligned}
$$

Thus there exist $l_1, \ldots, l_r \in \mathbb{N}$ such that $\bar{\nu}_i(z) = \nu_i(x_1^{l_1} \cdots x_r^{l_r})$, and so there exists $\lambda \in k_i \cong V_{\nu_i}/m_{\nu_i}$ such that $\bar{\nu}_i(z - \lambda x_1^{l_1} \cdots x_r^{l_r}) > \nu_i(z)$. Set $z_1 = z - \lambda x_1^{l_1} \cdots x_r^{l_r}$ and

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$f_1(x_1, \ldots, x_r, z_1) = f(x_1, \ldots, x_r, z)$ (this f_1 is different from the f_1 of (25)). We have that ord $f_1(0, \ldots, 0, z_1) = \mu$.
(27)

Go back to the beginning of Subsection 4.5 with z replaced by z_1 and f replaced with f_1, and run the reduction algorithm in A_i.

Of course it may now be that z is formal.

4.6. Termination of the reduction algorithm.

We either terminate after a finite number of iterations of the reduction algorithm, or after a finite number s of reductions (26) of μ, we never find a reduction in μ after that, performing the operation of equation (27) infinitely many times.

We thus construct a sequence

$$A_i = G(0) \to G(1) \to \cdots \to G(s)$$

where each $G(j) \to G(j+1)$ is an iteration of the reduction algorithm, culminating in the reduction step (26) and such that the algorithm proceeding from $G(s)$ never produces a further reduction in μ. In each $G(j)$ we start with good parameters $\overline{x}_1(j), \ldots, \overline{x}_r(j), \overline{z}'_j$ and make a change of variables replacing $\overline{z}'_j$ with $\overline{z}_j$. This notation is chosen so that we may differentiate between different iterations of the reduction algorithm. After introducing this notation and explaining the construction of $G(j-1) \to G(j)$, we will explain the three possibilities that the algorithm from $G(s)$ can take.

In $G(0) = A_i$, we start with $\overline{z}'_0 \in U_i$ with $\overline{\nu}_i(\overline{z}'_0) < \infty$, and $\overline{x}_1(0), \ldots, \overline{x}_r(0), \overline{z}'_0$ which are the first part of a good regular system of parameters in U_i. If $s > 0$, we make a change of variables

$$(28) \qquad \overline{z}_0 = \overline{z}'_0 - \sum \lambda_{0,l} \overline{x}_1(0)^{u_{1,l}(0)} \cdots \overline{x}_r(0)^{u_{r,l}(0)}$$

with $\lambda_{0,l} \in k_i$ and the sum is finite. We then apply the reduction algorithm to $\overline{z}_0$, to construct $G(0) \to G(1)$.

Each $G(j-1) \to G(j)$ terminates with a new variable $\overline{z}'_j$, which is derived from $\overline{z}_{j-1}$ in the reduction algorithm (these are the variables named z_1 and z respectively in the reduction step (26)). The reduction algorithm gives $\overline{x}_1(j), \ldots, \overline{x}_r(j), \overline{z}'_j$ which are part of a regular system of parameters in $G(j)$. If $j < s$, we make a change of variables

$$(29) \qquad \overline{z}_j = \overline{z}'_j - \sum \lambda_{j,l} \overline{x}_1(j)^{u_{1,l}(j)} \cdots \overline{x}_r(j)^{u_{r,l}(j)}$$

with $\lambda_{j,l} \in k_i$ and the sum is finite, and perform the reduction algorithm on $\overline{z}_j$ in $G(j)$.

We thus construct $\overline{z}_0, \overline{z}_1, \ldots, \overline{z}'_s$ by performing the reduction algorithm of subsection 4.5, each time obtaining a reduction in μ, giving a formal sequence of transforms

$$A_i = G(0) \to \cdots \to G(s)$$

along $\overline{\nu}_i$.

From $G(s)$ there are three possible paths that the algorithm can take.

1) After a change of variables of the from of (29) in $\overline{z}'_s$ we obtain $\overline{z}_s$ such that $\overline{\nu}_i(\overline{z}_s) = \infty$ or there exists $g(\overline{x}_1(s), \ldots, \overline{x}_r(s), \overline{z}_s) \in k_i[[\overline{x}_1(s), \ldots, \overline{x}_r(s)]][\overline{z}_s]$ such that $\overline{\nu}_i(g) = \infty$ and ord $g(0, \ldots, 0, \overline{z}_s) = 1$. In both cases we terminate the algorithm with a reduction in m.

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2) We make a change of variables of the form of (29) in $\overline{z}'_s$ to obtain $\overline{z}_s$. Then we perform a formal transform of type (i,1) $\overline{x}_i(s) = \prod_k \overline{x}_k(s,1)^{b_{i,k}(s)}$ for $1 \leq i \leq r$ followed by a formal transform of type (i,3) of (22)

$$\overline{z}_s = \overline{x}_1(s,1)^{e_1} \cdots \overline{x}_r(s,1)^{e_r} \tilde{z}_s$$

followed by the change of variables (23),

$$\overline{z}'_{s+1} = \tilde{z}_s - \alpha_{s+1}$$

with $\alpha_{s+1} \in k_i$ such that $\overline{\nu}_i(\overline{z}'_{s+1}) = \infty$, terminating the algorithm in (24) with a reduction in m.

3) We perform the operation of (27) infinitely many times, never terminating the algorithm.

Let us assume that this third case occurs. We set $\overline{z}_s = \overline{z}'_s$. We repeat the algorithm of subsection 4.5 infinitely many times in $G(s)$, each time culminating in step (27), constructing $\alpha_{s+i} \overline{x}_1(s)^{g_1(s+i)} \cdots \overline{x}_r(s)^{g_r(s+i)}$ for $i \geq 0$ with $\alpha_{s+i} \in k_i$ and $g_1(s+i), \ldots, g_r(s+i) \in \mathbb{N}$ such that for all i,

$$\overline{\nu}_i(\overline{x}_1(s)^{g_1(s+i+1)} \cdots \overline{x}_r(s)^{g_r(s+i+1)}) > \overline{\nu}_i(\overline{x}_1(s)^{g_1(s+i)} \cdots \overline{x}_r(s)^{g_r(s+i)})$$

and the sequence

$$\overline{z}_{s+i+1} = \overline{z}_{s+i} - \alpha_{s+i} \overline{x}_1(s)^{g_1(s+i)} \cdots \overline{x}_r(s)^{g_r(s+i)}$$

satisfies

$$\overline{\nu}_i(\overline{z}_{s+i}) = \overline{\nu}_i(\overline{x}_1(s)^{g_1(s+i)} \cdots \overline{x}_r(s)^{g_r(s+i)}), \ \ \overline{\nu}_i(\overline{z}_{s+i+1}) > \overline{\nu}_i(\overline{z}_{s+i}).$$

Since $\nu_i(\overline{x}_1(s)^{g_1(s+i)} \cdots \overline{x}_r(s)^{g_r(s+i)})$ is an increasing sequence in the semigroup $\mathbb{N}\nu_i(\overline{x}_1(s)) + \cdots + \mathbb{N}\nu_i(\overline{x}_r(s))$, we have that $\nu_i(\overline{x}_1(s)^{g_1(s+i)} \cdots \overline{x}_r(s)^{g_r(s+i)}) \mapsto \infty$ as $i \mapsto \infty$. Thus $\overline{\nu}_i(\overline{z}_{s+i}) \mapsto \infty$ as $i \mapsto \infty$. Let z_∞ be the limit in $G(s)$ of the Cauchy sequence $\{\overline{z}_{s+i}\}$. We have that the regular parameter z_∞ satisfies $\overline{\nu}_i(z_\infty) = \infty$, so $z_\infty \in Q(s) = Q_{i+1}(G(s))$ and so we have a reduction of m in $G(s)$. Thus the third case produces a change of variables

(30) $$z_\infty = \overline{z}'_s - \sum \lambda_{s,l} \overline{x}_1(s)^{u_{1,l}(s)} \cdots \overline{x}_r(s)^{u_{r,l}(s)}$$

in $G(s)$ with $\lambda_{s,l} \in k_i$ and where the sum is infinite.

4.7. The algorithm comes from an algebraic sequence of transforms. We use the notation of Section 4.6.

Recall that we constructed a sequence

$$A_i = G(0) \to G(1) \to \cdots \to G(s)$$

such that either we obtained a reduction of m in $G(s)$, or we obtained a reduction after making a final sequence $G(s) \to H$ where H is a composition of a formal transform along ν_i of type (i,1) and a formal transform along ν_i of type (3,i).

We have good regular parameters $\overline{x}_1(j), \ldots, \overline{x}_r(j), \overline{z}'_j, \ldots$ in $G(j)$ and in (29) we make a change of variables obtaining a new system of good regular parameters

$$\overline{x}_1(j), \ldots, \overline{x}_r(j), \overline{z}_j, \ldots.$$

Each $G(j) \to G(j+1)$ has a factorization

$$G(j) = E_j(0) \to E_j(1) \to E_j(2) \to E_j(3) = G(j+1).$$

The parameters $\overline{x}_1(j), \ldots, \overline{x}_r(j), \overline{z}_j, \ldots$ are good regular parameters in in $E_j(0)$. The map $E_j(0) \to E_j(1)$ is the formal transform along $\overline{\nu}_i$ of type (i,1) given by $\overline{x}_i(j) =$

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$\prod_k \overline{x}_k(j,1)^{b_{i,k}(j)}$ for $1 \le i \le r$. Thus $\overline{x}_1(j,1), \ldots, \overline{x}_r(j,1), \overline{z}_j, \ldots$ are good regular parameters in $E_j(1)$. The map $E_j(1) \to E_j(2)$ is a formal transform along $\overline{\nu}_i$ of type (3,i), given by a substitution

(31) $$\overline{z}_j = \overline{x}_1(j,1)^{e_1(j)} \cdots \overline{x}_r(j,1)^{e_r(j)} \tilde{z}_j$$

such that $\overline{\nu}_i(\tilde{z}_j) = 0$. We then set

(32) $$\overline{z}'_{j+1} = \tilde{z}_j - \alpha_j$$

where $\alpha_j \in k_i$ so that

(33) $$\overline{x}_1(j,1), \ldots, \overline{x}_r(j,1), \tilde{z}'_{j+1}, \ldots$$

are good regular parameters in $E_j(2)$. The map $E_j(2) \to E_j(3) = G(j+1)$ is a formal transform of type (1,i) given by substitutions

(34) $$\overline{x}_i(j,1) = \prod_k \overline{x}_k(j+1)^{c_{i,k}(j+1)}$$

for $1 \le i \le r$, resulting in the good regular parameters $\overline{x}_1(j+1), \ldots, \overline{x}_r(j+1), \overline{z}'_{j+1}, \ldots$ in $E_j(3) = G(j+1)$.

If we do not find a reduction of m in $G(s)$, then we are in the case 2) discussed in Section 4.6. In this case we have a sequence of maps $G(s) \to H_1 \to H$, giving a reduction of m in H.

We make a change of variables of the form of (29) in $\overline{z}'_s$ to obtain $\overline{z}_s$, giving good regular parameters $\overline{x}_1(s), \ldots, \overline{x}_r(s), \overline{z}_s, \ldots$ in $G(s)$. The map $G(s) \to H_1$ is a formal transform along $\overline{\nu}_i$ of type (i,1) given by a substitution $\overline{x}_i(s) = \prod_k \overline{x}_k(s,1)^{b_{i,k}(j)}$ for $1 \le i \le r$, giving good regular parameters $\overline{x}_1(s,1), \ldots, \overline{x}_r(s,1), \overline{z}_s, \ldots$ in H_1. The map $H_1 \to H$ is a formal transform of type (i,3) of the form of (22),

(35) $$\overline{z}_s = \overline{x}_1(s,1)^{e_1(s)} \cdots \overline{x}_r(s,1)^{e_r(s)} \tilde{z}_s$$

followed by the change of variables (23),

(36) $$\overline{z}'_{s+1} = \tilde{z}_s - \alpha_{s+1}$$

with $\alpha_{s+1} \in k_i$ such that $\overline{\nu}_i(\overline{z}'_{s+1}) = \infty$, terminating the algorithm in (24) with a reduction in m, and giving good regular parameters $\overline{x}_1(s,1), \ldots, \overline{x}_r(s,1), \overline{z}'_{s+1}, \ldots$ in H.

We will show that there exists a sequence

$$U_i = V(0) \to V(1) \to \cdots \to V(s)$$

such that each sequence $V(j) \to V(j+1)$ is a sequence of transforms along ν_i such that $G(j) = \widehat{V(j)}$ for all j. If we have a final sequence $G(s) \to H$, then we will construct a final sequence of transforms along ν_i, $V(s) \to J$, such that $H = \hat{J}$.

We will construct the $V(j)$ by induction, so that $V(j)$ has good regular parameters

$$\overline{x}_1(j), \ldots, \overline{x}_r(j), z_j^*, \ldots$$

such that

(37) $$z_j^* = \overline{z}_j + (\overline{x}_1(j) \cdots \overline{x}_r(j))^{\sigma(j)} h_j$$

with $h_j \in G(j) = \widehat{V(j)}$, and such that we can take $\sigma(j)$ arbitrarily large. In particular, we have $\nu_i(z_j^*) = \overline{\nu}_i(\overline{z}_j)$.

If $j = 0$, we define z_0^* as in (38) below.

Suppose that we have constructed $V(0) \to V(j)$ and $j < s$. We will construct $V(j) \to V(j+1)$.

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Define a transform $V(j) \to F(1)$ along ν_i of type (1,i) by the substitutions $\overline{x}_i(j) = \prod_k \overline{x}_k(j,1)^{b_{i,k}(j)}$ for $1 \le i \le r$. Thus $\overline{x}_1(j,1), \ldots, \overline{x}_r(j,1), z_j^*, \ldots$ are good regular parameters in $F(1)$ and $\widehat{F(1)} = E_j(1)$.

We have that $\overline{x}_1(j) \cdots \overline{x}_r(j)$ is a monomial in $\overline{x}_1(j,1), \ldots, \overline{x}_r(j,1)$ in which all variables have positive exponents. Recall the substitution (31) used to define the map $E_j(1) \to E_j(2)$. Taking $\sigma(j)$ sufficiently large in (37), we have that $\overline{x}_1(j,1)^{e_1(j)} \cdots \overline{x}_r(j,1)^{e_r(j)}$ divides $(\overline{x}_1(j) \cdots \overline{x}_r(j))^{\sigma(j)}$ in $F(1)$ and

$$\frac{(\overline{x}_1(j) \cdots \overline{x}_r(j))^{\sigma(j)}}{\overline{x}_1(j,1)^{e_1(j)} \cdots \overline{x}_r(j,1)^{e_r(j)}} = \overline{x}_1(j,1)^{v_1} \cdots \overline{x}_r(j,1)^{v_r} \in m_{F(1)}$$

for some $v_1, \ldots, v_r \in \mathbb{Z}_{>0}$. Let $F(1) \to F(2)$ be the transform along ν_i of type (3,i) defined by

$$z_j^* = \overline{x}_1(j,1)^{e_1(j)} \cdots \overline{x}_r(j,1)^{e_r(j)} \tilde{v}_{j+1}.$$

Then

$$\tilde{z}_{j+1} = \tilde{v}_{j+1} - \overline{x}_1(j,1)^{v_1} \cdots \overline{x}_r(j,1)^{v_r} h_j$$

and so $\widehat{F(2)} = E(2)$.

The variable $\overline{z}_{j+1}'$ of $E(2)$ is defined by (32). By (21), there exists $\alpha' \in U_i$ such that $\alpha_{j+1} = \alpha' + (\overline{x}_1(j,1) \cdots \overline{x}_r(j,1))^{\tau(j)} h$ where $h \in E(2)$ and $\tau(j)$ can be arbitrarily large.

Set $v_{j+1}' = \tilde{v}_{j+1} - \alpha'$. We have that $\overline{x}_1(j,1), \ldots, \overline{x}_r(j,1), \overline{z}_{j+1}'$ is part of a regular system of parameters in $E(2)$ and $\overline{x}_1(j,1), \ldots, \overline{x}_r(j,1), v_{j+1}'$ are part of a regular system of parameters in $F(2)$ such that

$$v_{j+1}' = \overline{z}_{j+1}' + (\overline{x}_1(j,1) \cdots \overline{x}_r(j,1))^{\beta(j)} g_j$$

with $g_j \in E(2)$ and $\beta(j)$ can be arbitrarily large.

Finally define a transform along ν_i of type (1,i) $F(2) \to F(3)$ by the substutions

$$\overline{x}_i(j,1) = \prod_k \overline{x}_k(j+1)^{c_{i,k}(j+1)}$$

for $1 \le i \le r$, so that $\widehat{F(3)} = E(3)$. We have thus constructed $V(j+1) = F(3)$.

Now in the change of variables of (29), we have

$$\overline{z}_{j+1} = \overline{z}_{j+1}' - \sum_l \lambda_{j+1,l} \overline{x}_1(j+1)^{u_{1,l}(j+1)} \cdots \overline{x}_r(j+1)^{u_{r,l}(j+1)}$$

for $1 \le l \le r$. We apply (21) to find $\lambda_{j+1,l}' \in U_i$ such that

$$\lambda_{j+1,l} = \lambda_{j+1,l}' + (x_1(j+1) \cdots x_r(j+1))^{\omega(j+1)} h_{j+1,l}$$

where $h_{j+1,l} \in G(j+1)$ and $\omega(j+1)$ can be arbitrarily large, and set

$$(38) \qquad z_{j+1}^* = v_{j+1}' - \sum_l \lambda_{j+1,l}' x_1(j+1)^{u_{1,l}(j+1)} \cdots x_r(j+1)^{u_{r,l}(j+1)}.$$

The construction of the final sequence $V(s) \to J$ is a simplification of the argument for constructing $V(j-1) \to V(j)$.

We may now prove the following theorem.

Theorem 4.9. *Let $U_i = D_i$. Then there exists a sequence of transforms of types (1,i), (2,i), (3,i) and (4,i), $U_i \to U(1)$ along ν_i such that $Q_{i+1}(1) = Q_{i+1}(\widehat{U(1)})$ is a regular prime ideal in $\widehat{U(1)}$; that is, $\widehat{U(1)}/Q_{i+1}(1)$ is a regular local ring.*

Proof. Let
$$x_1, \ldots, x_r, z_1, \ldots, z_m, x_{i+1,1}, \ldots, x_{t,r_t}, w_1, \ldots, w_l$$
be a system of good regular parameters in U_i of the form of (19), so that
$$x_{i+1,1}, \ldots, x_{t,r_t}, w_1, \ldots, w_l \in Q(U_i).$$
Suppose that $m \geq 1$. As shown above, there exists a sequence of transforms along ν_i $U_i \to W(1)$ such that $\widehat{W(1)}$ has good regular parameters
$$x_1(1), \ldots, x_r(1), z_2, \ldots, z_m, x_{i+1,1}, \ldots, x_{t,r_t}, w'_1, w_1, \ldots, w_l$$
where
$$x_{i+1,1}, \ldots, x_{t,r_t}, w'_1, w_1, \ldots, w_l \in Q(W(1)).$$
Since $z_2, \ldots, z_m \in U_i$ are "algebraic", and they have relations in $k_i[[x_1(1), \ldots, x_r(1)]]$ by Lemma 4.8, we may continue the algorithm, finding a sequence of transforms along ν_i $U_i \to W(1) \to \cdots \to W(m)$ so that there are regular parameters
$$x_1(1), \ldots, x_r(1), x_{i+1,1}, \ldots, x_{t,r_t}, w'_1, \ldots, w'_m, w_1, \ldots, w_l$$
in $\widehat{W(m)}$ such that
$$Q(W(m)) = (x_{i+1,1}, \ldots, x_{t,r_t}, w'_1, \ldots, w'_m, w_1, \ldots, w_l).$$
Thus the conclusions of the theorem hold in $W(m)$. $\qquad\square$

4.8. Resolution in the smallest rank. Suppose that U_i is as in (14) and there exist
$$w_1, \ldots, w_l \in P = P_{i+1}(U_i) = Q_{i+1}(A_i) \cap U_i \subset U_i$$
and $\hat{z}_1, \ldots, \hat{z}_m \in Q = Q_{i+1}(A_i) \subset A_i = \widehat{U(i)}$ such that

(39)
$$Q = (\hat{z}_1, \ldots, \hat{z}_m, x_{i+1,1}, \ldots, x_{t,r_t}, w_1, \ldots, w_l)$$
and
$$x_1, \ldots, x_r, \hat{z}_1, \ldots, \hat{z}_m, x_{i+1,1}, \ldots, x_{t,r_t}, w_1, \ldots, w_l$$
is a good regular system of parameters in A_i.

Good regular systems of parameters are defined at the beginning of SubSection 4.3.

We will show that if $m \geq 1$, then there exists a sequence of transforms $U_i \to U(1)$ along ν_i such that there exist an expression (39) in $U(1)$ and $A(1) = \hat{U}(1)$ with a decrease of m (and increase in l).

We will prove this by descending induction on m. By Theorem 4.9, we can assume that there is such an expression with $m = \dim U_i - \dim B_{\nu_i}$ (and $l = 0$).

From equation (10), we have a reduced primary decomposition $PA_i = Q \cap I_2 \cap \cdots \cap I_u$. Thus there exists $f_1, \ldots, f_l \in P$ and $a_1, \ldots, a_l \in (A_i)_Q$ such that $a_1 f_1 + \cdots + a_l f_l = \hat{z}_1$. Write $a_i = \frac{b_i}{c_i}$ where $b_i \in A_i$ and $c_i \in A_i \setminus Q$. Set $d_i = b_i \prod_{j \neq i} c_j$. Then
$$d_1 f_1 + \cdots + d_l f_l = c\hat{z}_1$$
where $c = \prod c_i \notin Q$ (so $\overline{\nu}_i(c) < \infty$).

We will show that there exists $g \in P$ such that g has an expansion

(40)
$$g = \alpha_1 \hat{z}_1 + \alpha_2 \hat{z}_2 + \cdots + \alpha_n w_l \text{ with } \alpha_i \in A_i \text{ and } \alpha_1 \notin Q.$$

If one of the f_i has such an expansion, then we set $g = f_i$. Otherwise,
$$f_i \in (\hat{z}_1^2, \hat{z}_2, \ldots, w_l)$$
for all i, and so, $c\hat{z}_1 = \sum_{i=1}^l d_i f_i \in (\hat{z}_1^2, \hat{z}_2, \ldots, w_l)$. But $c \notin Q$ implies $c\hat{z}_1 \notin (\hat{z}_1^2, \hat{z}_2, \ldots, w_l)$, a contradiction. Thus some f_i has an expansion (40).

Suppose $g \in A_i$ has an expansion (40). Let $\tau = \overline{\nu}_i(\alpha_1) < \infty$. For $1 \le j \le m$, there exists $z_j \in U_i$ such that $\hat{z}_j = z_j + h_j$ where $h_j \in m_{A_i}^{3\tau}$. Substituting into (40), we obtain an expansion

$$g = \delta_0 + \delta_1 z_1 + \delta_2 z_2 + \cdots + \delta_n w_l$$

with $\delta_0 \in (m_{k_i[[x_1,\ldots,x_r]]})^{3\tau}$, $\delta_1, \ldots, \delta_n \in A_i$ and $\overline{\nu}_i(\delta_1) = \tau < \infty$.

Expand

$$\delta_1 = \sum_{1 \le i} \beta_i x_1^{b_{1,i}} \cdots x_r^{b_{r,i}} + \gamma_1 z_1 + \cdots + \gamma_n w_l$$

where $0 \ne \beta_i \in k_i$, $\nu_i(x_1^{b_{1,i}} \cdots x_r^{b_{r,i}}) < \nu_i(x_1^{b_{1,i+1}} \cdots x_r^{b_{r,i+1}})$ for all i, and $\gamma_1, \ldots, \gamma_n \in A_i$, so that $\overline{\nu}_i(\beta_i x_1^{b_{1,1}} \cdots x_r^{b_{r,1}}) = \tau$.

Expand

$$\delta_0 = \sum_{1 \le i} \varepsilon_i x_1^{c_{1,i}} \cdots x_r^{c_{r,i}}$$

where $0 \ne \varepsilon_i \in k_i$ and $\nu_i(x_1^{c_{1,i}} \cdots x_r^{c_{r,i}}) < \nu_i(x_1^{c_{1,i+1}} \cdots x_r^{c_{r,i+1}})$ for all i.

Let $J = (\{x_1^{b_{1,i}} \cdots x_r^{b_{r,i}}\}, \{x_1^{c_{1,j}} \cdots x_r^{c_{r,j}}\})$ be the ideal generated by all of the $x_1^{b_{1,i}} \cdots x_r^{b_{r,i}}$ and $x_1^{c_{1,j}} \cdots x_r^{c_{r,j}}$. There exists a transform $U_i \to U(1)$ along ν_i of type (1,i) in $x_1, \ldots, x_r$ and $x_1(1), \ldots, x_r(1)$ such that $JU(1)$ is generated by $x_1^{b_{1,1}} \cdots x_r^{b_{r,1}} = x_1(1)^{e_1} \cdots x_r(1)^{e_r}$.

Now define a sequence of transforms $U(1) \to U(2)$ along ν_i of types (4,i) and (2,i) by

$$z_1 = x_1(1)^{e_1} \cdots x_r(1)^{e_r} z_1(1),$$

$$z_i = x_1(1)^{2e_1} \cdots x_r(1)^{2e_r} z_i(1) \text{ for } 2 \le i \le m,$$

$$x_{j,k} = x_1(1)^{e_1} \cdots x_r(1)^{e_r} x_{j,k}(1) \text{ for } i+1 \le j \le t \text{ and all } k,$$

$$w_j = x_1(1)^{2e_1} \cdots x_r(1)^{2e_r} w_j(1) \text{ for all } j.$$

Taking the sequence of completions of these rings, $A_i \to A(1) = \widehat{U(1)} \to A(2) = \widehat{U(2)}$, we have an induced sequence of formal transforms along $\overline{\nu}_i$. Then $g = x_1(1)^{e_1} \cdots x_r(1)^{e_r} g_1$ where $g_1 \in m_{A(2)}$ satisfies $g_1 \equiv \beta_1 z_1(1) \bmod m_{A(2)}^2$. Now

$$\frac{g}{x_1(1)^{e_1} \cdots x_r(1)^{e_r}} \in \widehat{U(2)} \cap \mathrm{QF}(U(2)) = U(2)$$

by [2, Lemma 2], and so $g_1 \in U(2) \cap Q_{i+1}(A(2)) = P_{i+1}(U(2))$. We thus have a reduction of m in $A(2)$.

By induction on m, we have thus established the following theorem.

Theorem 4.10. *Suppose that U_i is as in (14) and U_i has a good regular system of parameters satisfying (39). Then there exists a sequence of transforms $U_i \to U(1)$ along ν_i of types (i,1), (i,2), (i,3) and (i,4)such that $P_{i+1}(U(1))$ is a regular prime ideal in $U(1)$ and*

$$Q_{i+1}(\widehat{U(1)}) = P_{i+1}(U(1))\widehat{U(1)}.$$

4.9. **Resolution in arbitrary rank.** Suppose that U_i is as in (14). Regular parameters

$$x_{i,1}, \ldots, x_{i,r_i}, y_1, \ldots, y_m$$

in U_i such that $\nu_i(x_{i,1}), \ldots, \nu_i(x_{i,r_i})$ is a $\mathbb{Z}$-basis of Γ_i/Γ_{i-1} and the prime ideal $P_{i+1}(U_i)$ of (15) is $P_{i+1}(U_i) = (y_1, \ldots, y_m)$ so that $P_{i+1}(U_i)$ is a regular prime ideal are called ν_i-good regular parameters in U_i.

Lemma 4.11. *Suppose that U_i is as in (14) and that U_i has ν_i-good regular parameters*

$$x_{i,1}, \ldots, x_{i,r_i}, y_1, \ldots, y_m.$$

1) *Suppose that $f \in U_i \setminus P_{i+1}(U_i)$. Then there exists a sequence of transforms $U_i \to U(1)$ along ν_i of types (1,i) and (2,i) such that f has an expression*

$$f = x_{i,1}(1)^{d_1} \cdots x_{i,r_i}(1)^{d_{r_i}} \gamma,$$

where $d_1, \ldots, d_{r_i} \in \mathbb{N}$ and $\gamma \in U(1)$ is a unit.

2) *Suppose that $f \in P_{i+1}(U_i)$ and $\rho \in \Gamma_i/\Gamma_{i-1}$ is given. Then there exists a sequence of transforms $U_i \to U(1)$ along ν_i of types (1,i) and (2,i) such that $f = x_{i,1}(1)^{d_1} \cdots x_{i,r_i}(1)^{d_{r_i}} \gamma$, where $d_1, \ldots, d_{r_i} \in \mathbb{N}$, $\gamma \in U(1)$ and*

$$\nu_i(x_{i,1}(1)^{d_1} \cdots x_{i,r_i}(1)^{d_{r_i}}) > \rho.$$

The proof of Lemma 4.11 is similar to the proof of Lemma 4.4.

Proof. First assume that f is in Case 1). Expand

$$f = \sum \alpha_{\lambda_{i,1}, \ldots, \lambda_{i,r_i}} x_{i,1}^{\lambda_{i,1}} \cdots x_{i,r_i}^{\lambda_{i,r_i}} + h_1 y_1 + \cdots + h_m y_m$$

in $A_i = \hat{U}_i$ with $\alpha_{\lambda_{i,1}, \ldots, \lambda_{i,r_i}} \in k_i$ not all zero and $h_1, \ldots h_m \in A_i$. Let

$$I = (x_{i,1}^{\lambda_{i,1}} \cdots x_{i,r_i}^{\lambda_{i,r_i}} \mid \alpha_{\lambda_{i,1}, \ldots, \lambda_{i,r_i}} \neq 0),$$

an ideal in U_i. By Proposition 4.3, there exists a sequence of transforms of type (1,i) along ν_i, $U_i \to U(1)$, where $U(1)$ has ν_i-good parameters $x_{i,1}(1), \ldots, x_{i,r_i}(1), y_1, \ldots, y_m$ and there exist $d_{i,1} \ldots, d_{i,r_i} \in \mathbb{N}$, such that

$$IU(1) = x_{i,1}(1)^{d_{i,1}} \cdots x_{i,r_i}(1)^{d_{i,r_i}} U(1).$$

Now perform a sequence of transforms of type (2,i) along ν_i, $U(1) \to U(2)$ defined by

$$y_j = x_{i,1}(1)^{d_{i,1}} \cdots x_{i,r_i}(1)^{d_{i,r_i}} y_j(1) \text{ for } 1 \le j \le m,$$

to obtain $f = x_{i,1}(1)^{d_{i,1}} \cdots x_{i,r_i}(1)^{d_{i,r_i}} \gamma$ where $\gamma \in \widehat{U(2)}$ is a unit. Now $\gamma \in \widehat{U(2)} \cap K$ implies $\gamma \in U(2)$, achieving the conclusions of 1) in $U(2)$.

Now suppose that f is in Case 2). Then $f = h_1 y_1 + \cdots + h_m y_m$ with $h_1, \ldots, h_m \in U_i$. Then perform a sequence of transforms along ν_i of type (2,i), $U_i \to U(1)$, defined by $y_j = x_{i,1}^{d_1} \cdots x_{i,r_i}^{d_{r_i}} y_j(1)$ for $1 \le i \le m$, with $d_1, \ldots, d_{r_i} \in \mathbb{N}$, such that $\nu_i(x_{i,1}^{d_1} \cdots x_{i,r_i}^{d_{r_i}}) > \rho$, to obtain the conclusions of 2) in $U(1)$.

$\square$

Lemma 4.12. *Suppose that U_i is as in (14) and that U_i has ν_i-good regular parameters*

$$x_{i,1}, \ldots, x_{i,r_i}, y_1, \ldots, y_m.$$

Suppose that $z_1, \ldots, z_m$ are regular parameters in $U_{i+1} = (U_i)_{P_{i+1}(U_i)}$. Then there exists a sequence of transforms along ν_i, $U_i \to U(1)$ of types (1,i) and (2,i) such that $U_{i+1} = U(1)_{P_{i+1}(U(1))}$ and $U(1)$ has ν_i-good regular parameters $x_{1,1}(1), \ldots, x_{1,r_1}(1), y_1(1), \ldots, y_m(1)$ such that $\nu_i(x_{i,1}(1)), \ldots, \nu_i(x_{1,r_1}(1))$ is a $\mathbb{Z}$-basis of Γ_i/Γ_{i-1}, $P_{i+1}(U(1)) = (y_1(1), \ldots, y_m(1))$ is a regular prime and for $1 \le j \le m$,

$$z_j = x_{i,1}(1)^{d_j^1} \cdots x_{i,r_i}(1)^{d_j^{r_i}} y_j(1)$$

with $d_k^j \in \mathbb{N}$ for all j, k.

Proof. We have that

$$(y_1, \ldots, y_m)U_{i+1} = (z_1, \ldots, z_m)U_{i+1} = P_{i+1}(U_i)U_{i+1}.$$

Thus there exist $a_{j,k} \in U_{i+1}$ such that for $1 \leq j \leq m$,

$$z_j = \sum_{k=1}^{m} a_{j,k} y_k.$$

There exist $b_{j,k} \in U_i$ and $c_{j,k} \in U_i \setminus P_{i+1}(U_i)$ such that $a_{j,k} = \frac{b_{j,k}}{c_{j,k}}$. By Lemma 4.11, there exists a sequence of transforms $U_i \to U(1)$ along ν_i of types (1,i) and (2,i) such that $U(1)$ has ν_i-good regular parameters

$$x_{i,1}(1), \ldots, x_{i,r_i}(1), y_1(1), \ldots, y_m(1)$$

such that for all j, k,

$$c_{j,k} = x_{i,1}(1)^{d_{jk}^1} \cdots x_{i,r_i}(1)^{d_{jk}^{r_i}} \gamma_{jk}$$

where γ_{jk} are units in $U(1)$ and $d_{jk}^l \in \mathbb{N}$. Now perform a sequence of transforms $U(1) \to U(2)$ along ν_i of type (2,i)

$$y_k(1) = x_{i,1}(1)^{e_1^k} \cdots x_{i,r_i}(1)^{e_{r_i}^k} y_k(2)$$

for $1 \leq k \leq m$ so that we have expansions

$$z_j = \sum_{k=1}^{m} f_{jk} y_k(2) \text{ for } 1 \leq j \leq m$$

with $f_{jk} \in U(2)$ for all j, k. We continue to have $U(2)_{P_{i+1}(U(2))} = U_{i+1}$. Since $\mathrm{Det}(f_{jk})$ is a unit in U_{i+1}, there exists an f_{jk} such that $f_{jk} \notin P_{i+1}(U(2))$. Without loss of generality, $j = 1$. By Lemma 4.11, there exists a sequence of transforms along ν_i, $U(2) \to U(3)$, of types (1,i) and (2,i) such that $U(3)$ has ν_i-good regular parameters $x_{i,1}(3), \ldots, x_{i,r_i}(3), y_1(3), \ldots, y_m(3)$ such that

$$f_{jk} = x_{i,1}(3)^{\alpha_{j,k}^1} \cdots x_{i,r_i}(3)^{\alpha_{j,k}^{r_i}} g_{j,k}$$

where $g_{j,k} \in U(3)$ is a unit if $f_{jk} \notin P_{i+1}(U(3))$ and $\nu_i(x_{i,1}(3)^{\alpha_{jk}^1} \cdots x_{i,r_i}(3)^{\alpha_{jk}^{r_i}})$ is arbitrarily large if $g_{jk} \in P_{i+1}(U(3))$. After performing a transform $U(3) \to U(4)$ along ν_i of type (i,1), and permuting the $y_k(3)$, we have an expression

$$z_1 = x_{i,1}(4)^{\beta_1} \cdots x_{i,r_i}(4)^{\beta_{r_i}} \left[g_{1,1} y_1(4) + \sum_{k=2}^{m} g_{1,k}' y_k(4) \right]$$

where $g_{jk}' \in U(4)$ and g_{11} is a unit in $U(4)$. We then make a change of variables in $U(4)$, replacing $y_1(4)$ with $g_{1,1} y_1(4) + \sum_{k=2}^{m} g_{jk}' y_k(4)$, giving equations

$$\begin{aligned} z_1 &= x_{i,1}(4)^{\beta_1} \cdots x_{i,r_i}(4)^{\beta_{r_i}} y_1(4) \\ z_j &= \sum_{k=1}^{m} g_{jk}' y_k(4) \text{ for } 2 \leq j \leq m \end{aligned}$$

with $g_{jk}' \in U(4)$. We thus have that

$$\mathrm{Det} \begin{pmatrix} g_{2,2}' & \cdots & g_{2,m}' \\ \vdots & & \vdots \\ g_{m,2}' & \cdots & g_{m,m}' \end{pmatrix} \notin P_{i+1}(U(4)),$$

so that some $g'_{j,k} \notin P_{i+1}(U(4))$, with $2 \leq j, 2 \leq k$. We may thus continue as above to construct $U(4) \to U(5)$ such that $U(5)_{P_{i+1}(U(5))} = U_{i+1}$ and

$$
\begin{aligned}
z_1 &= x_{i,1}(5)^{\beta_{1,1}} \cdots x_{i,r_i}(5)^{\beta_{1,r_i}} y_1(5) \\
z_2 &= x_{i,1}(5)^{\beta_{2,1}} \cdots x_{i,r_i}(5)^{\beta_{2,r_i}} y_2(5) \\
z_j &= \sum_{k=1}^{m} g'_{jk} y_k(5) \text{ for } 3 \leq j \leq m.
\end{aligned}
$$

By induction, we continue, to achieve the conclusions of the lemma. $\qquad\square$

Lemma 4.13. *Suppose that U_i is as in (14) and that U_i has ν_i-good regular parameters*

$$x_{i,1}, \ldots, x_{i,r_i}, y_1, \ldots, y_m.$$

Suppose that $U_{i+1} = (U_i)_{P_{i+1}(U_i)}$ has good regular parameters

$$\overline{x}_{i+1,1}, \ldots, \overline{x}_{i+1}, \ldots, \overline{x}_{t,1}, \ldots, \overline{x}_{t,r_t}, \overline{y}_1, \ldots, \overline{y}_{\overline{m}}.$$

Then there exist a sequence of transforms of types of types (1,i) and (2,i) $U_i \to U(1)$ along ν_i such that $U(1)$ has good regular parameters

$$x_{i,1}(1), \ldots, x_{i,r_i}(1), \ldots, x_{t,1}(1), \ldots, x_{t,r_t}(1), y_1(1), \ldots, y_{\overline{m}}(1)$$

such that

$$\overline{x}_{a,b} = x_{i,1}(1)^{d^1_{a,b}} \cdots x_{i,r_i}(1)^{d^{r_i}_{a,b}} x_{a,b}(1)$$

for $a \geq i+1$ and

$$\overline{y}_l = x_{i,1}(1)^{e^1_l} \cdots x_{i,r_i}(1)^{e^{r_i}_l} y_l(1)$$

for $1 \leq l \leq m$.

    Suppose that $U_{i+1} \to X$ is a transform along ν_{i+1} of one of the types (1,j), (2,j), (3,j) or (4,j) with $i+1 \leq j$. Then there exists a sequence of transforms $U(1) \to V(1)$ along ν_i of types (1,k), (2,k), (3,k) and (4,k) with $k = i$ or $k = j$ such that $V(1)_{P_{i+1}(V(1))} = X$.

Proof. The existence of the map $U_i \to U(1)$ having the properties asserted in the Lemma is a consequence of Lemma 4.12.

    Suppose that $U_{i+1} \to X$ is of type (1,j). Then $X = U_{i+1}[\overline{x}_{j,1}(1), \ldots, \overline{x}_{j,r_j}(1)]_{\nu_{i+1}}$ where $\overline{x}_{j,k} = \prod_{l=1}^{r_j} \overline{x}_{j,l}(1)^{a_{kl}}$ for $1 \leq k \leq r_j$. We have that for $1 \leq l \leq r_j$,

$$\overline{x}_{j,l}(1) = \prod_{k=1}^{r_j} x_{k,l}(1)^{b_{k,l}} \delta_l$$

where $\delta_l = \prod_{k=1}^{r_j} (x_{i,1}(1)^{d^1_{k,l}} \cdots x_{i,r_i}(1)^{d^{r_i}_{k,l}})^{b_{k,l}}$ is a unit in U_{i+1}.

    Now $\overline{x}_{j,l}(1) = \prod_{k=1}^{r_j} \overline{x}_{j,l}^{b_{k,l}}$ where $(b_{k,l}) = (a_{k,l})^{-1}$ is a matrix with integral coefficients. Thus $\nu_j(\prod_{k=1}^{r_k} x_{k,l}(1)^{b_{k,l}}) > 0$ for $1 \leq l \leq r_j$. Defining $U(1) \to U(2)$ by $U(2) = U(1)[x_{j,1}(2), \ldots, x_{j,r_j}(2)]_{\nu_i}$ where $x_{j,k}(1) = \prod_{l=1}^{r_j} x_{j,l}(2)^{a_{k,l}}$ for $1 \leq k \leq r_j$, we thus have that $U(2)_{P_{i+1}(U(2))} = X$.

    Now suppose that $U_{i+1} \to X$ is of type (3,j). Then $X = U_{i+1}[\overline{y}_k(1)]_{\nu_{i+1}}$ where $\overline{y}_k = \overline{x}_{j,1}^{a_1} \cdots \overline{x}_{j,r_j}^{a_{r_j}} \overline{y}_k(1)$ with $\nu_j(\overline{y}_k(1)) = 0$ and $\nu_{i+1}(\overline{y}_k(1)) \geq 0$. Now

$$\nu_{i+1}\left(\frac{y_k(1)}{x_{j,1}(1)^{a_1} \cdots x_{j,r_j}(1)^{a_{r_j}}}\right) = \nu_{i+1}\left(\frac{\overline{y}_k}{\overline{x}_{j,1}^{a_1} \cdots \overline{x}_{j,r_j}^{a_{r_j}}}\right) \geq 0$$

so if

$$\nu_i\left(\frac{y_k(1)}{x_{j,1}(1)^{a_1} \cdots x_{j,r_j}(1)^{a_{r_j}}}\right) < 0, \text{ then } \nu_{i+1}\left(\frac{y_k(1)}{x_{j,1}(1)^{a_1} \cdots x_{j,r_j}(1)^{a_{r_j}}}\right) = 0,$$

and so there exists k such that $a_k \neq 0$ and $n \in \mathbb{N}$ such that

$$na_k\nu_i(x_{i,1}(1)) + \nu_i\left(\frac{y_k(1)}{x_{j,1}(1)^{a_1}\cdots x_{j,r_j}(1)^{a_{r_j}}}\right) \geq 0.$$

Let $U(1) \to U(2)$ be the transform of type (2,i) defined by $x_{j,k}(1) = x_{i,1}(1)^n x_{j,k}(2)$. Then, setting $x_{j,l}(2) = x_{j,k}(1)$ for $l \neq k$, define $U(2) \to U(3)$ to be the transform of type (3,j) defined by $y_k(1) = x_{j,1}(2)^{a_1} x_{j,2}(2)^{a_2} \cdots x_{j,r_j}(1)^{a_{r_j}} y_k(2)$.

The remaining two cases, transforms of types (2,j) and (4,j), have a similar but simpler analysis. $\square$

Theorem 4.14. *Let D be the local ring of (11). There exists a sequence of transforms $D \to D(1)$ along ν such that there exist good regular parameters*

$$x_{1,1}, \ldots, x_{t,r_t}, y_1, \ldots, y_m$$

in $D(1)$ such that $\nu_j(x_{j,1}), \ldots, \nu_j(x_{j,r_j})$ is a $\mathbb{Z}$-basis of Γ_j/Γ_{j-1} for $1 \leq j \leq t$ and

$$\nu(y_1) = \cdots = \nu(y_m) = \infty.$$

In particular, $P_j(D(1))$ are regular primes in $D(1)$ for $1 \leq j \leq t+1$ and thus $D(1)/P_{t+1}(D(1))$ is a regular local ring which is dominated by ν and dominates B_{ν_1}. We further have that

$$(D(1)/P_i(D(1)))_{P_i(D(1))} \cong V_{\nu_i}/m_{\nu_i}$$

for $1 \leq i \leq t$.

Proof. We will prove the theorem by descending induction on i with $1 \leq i \leq t$. By Theorems 4.9 and 4.10, there exists a sequence of transforms $D_t \to E_1$ along ν_t of types (1,t), (2,t), (3,t) and (4,t) such that $P_{t+1}(E_1)$ is a regular prime in E_1.

Suppose, by induction, that we have constructed a sequence of transforms $D_{i+1} \to E_1$ along ν_{i+1} of types (1,j), (2,j), (3,j) and (4,j) with $j \geq i+1$ such that $P_j(E_1)$ are regular primes in E_1 for $j \geq i+1$. By Theorems 4.9 and 4.10 and Lemmas 4.12 and 4.13, there exists a sequence of transform $D_i \to F_1$ along ν_i of types (1,j), (2,j), (3,j) and (4,j) with $j \geq i$ such that $(F_1)_{P_{i+1}(F_1)} = E_1$. By Theorem 4.10, there exists a sequence of transforms along ν_i, $F_1 \to F_2$, such that $(F_2)_{P_{i+1}(F_2)} = E_1$ and $P_{i+1}(F_2)$ is a regular prime in F_2. By Lemmas 4.12 and 4.13, there exists a sequence of transforms along ν_i, $F_2 \to F_3$, such that F_3 has good regular parameters $x_{i,1}, \ldots, x_{t,r_t}, z_1, \ldots, z_m$ such that $P_j(F_3) = (x_{j,1}, \ldots, y_m)$ for all $j \geq i$ and $\nu_i(x_{j,1}), \ldots, \nu_i(x_{j,t_j})$ is a $\mathbb{Z}$-basis of Γ_j/Γ_{j-1} for $i \leq j \leq t$.

The last statement of the theorem follows from (4). $\square$

5. Local Uniformization of Abhyankar valuations

In this section we prove Theorems 1.1, 1.2, 1.3 and 1.7.

Proof of Theorem 1.1: This is immediate from Theorem 4.14, taking R to be

$$D(1)/P_{t+1}(D(1)).$$

We remark that the regular parameters in R of the conclusions of Theorem 1.2 are good regular parameters (Definition 4.1).

Proof of Theorem 1.2: We first prove 1). In $\hat{R} = k_1[[x_{1,1}, \ldots, x_{t,r_t}]]$, where $k_1 \cong V_\nu/m_\nu$ is a coefficient field of $\hat{R}$, we have an expansion

$$(41) \qquad f = \sum \alpha_{b_{1,1},\ldots,b_{t,r_t}} x_{1,1}^{b_{1,1}} \cdots x_{t,r_t}^{b_{t,r_t}}$$

with $\alpha_{b_{1,1},\ldots,b_{t,r_t}} \in k_1$. Let J be the ideal

$$J = (x_{1,1}^{b_{1,1}} \cdots x_{t,r_t}^{b_{t,r_t}} \mid \alpha_{b_{1,1},\ldots,b_{t,r_t}} \neq 0).$$

By Proposition 4.3, there exists a sequence of primitive transforms $R \to R(1)$ along ν such that

$$JR(1) = x_{1,1}(1)^{a_{1,1}} \cdots x_{t,r_t}(1)^{a_{t,r_t}} R(1).$$

Then f has an expression

$$f = x_{1,1}(1)^{a_{1,1}} \cdots x_{t,r_t}(1)^{a_{t,r_t}} u$$

with $u \in \widehat{R(1)}$ a unit. By [2, Lemma 2], $u \in K \cap \widehat{R(1)} = R(1)$, giving the desired expression of f in $R(1)$.

To prove 2), take generators $f_1, \ldots, f_m$ of I. By part 1) of this theorem, there exists a sequence of primitive transforms $R \to R(1)$ along ν such that each f_i is a monomial in $x_{1,1}(1), \ldots, x_{t,r_t}(1)$ times a unit in $R(1)$. By Proposition 4.3, we many now apply another sequence of primitive transforms $R(1) \to R(2)$ along ν to achieve the conclusions of 2).

The proof of 3) is a variation on the proof of 1), as in Lemma 4.4.

Proof of Theorem 1.3: There exist $f_1, \ldots, f_m \in V_\nu$ such that $S = k[f_1, \ldots, f_m]_\nu$. Let R be the regular local ring of the conclusions of Theorem 1.1. By Theorem 1.2, there exists a sequence of primitive transforms $R \to R(1)$ such that $f_1, \ldots, f_m \in R(1)$ and $R(1)$ satisfies the conclusions of Theorem 1.1. Thus $R(1)$ dominates S and so $R(1)$ satisfies the conclusions of Theorem 1.3.

Proof of Theorem 1.7: To prove Theorem 1.7, we need only modify the proof of [12, Theorem 1.5] by observing that the statement of [12, Theorem 7.2] is true without the assumption that V_ν/m_ν is separable over k, using Theorems 1.1 and 1.2 of this paper in place of [20, Theorem 1.1].

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Steven Dale Cutkosky, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

Email address: cutkoskys@missouri.edu