

Journal Pre-proof

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PII: S0022-5096(23)00096-0

DOI: <https://doi.org/10.1016/j.jmps.2023.105292>

Reference: MPS 105292

To appear in: *Journal of the Mechanics and Physics of Solids*

Received date: 21 July 2022

Revised date: 31 January 2023

Accepted date: 27 March 2023

Please cite this article as: O. Mattei, G.W. Milton and M. Putinar, Determining the volume fraction in 2-phase composites and bodies using time varying applied fields. *Journal of the Mechanics and Physics of Solids* (2023), doi: <https://doi.org/10.1016/j.jmps.2023.105292>.

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Highlights

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- We tailor the time-dependent boundary conditions so that one boundary measurement at any moment of time allows one to determine the volume fraction of a lossy two-phase composite (or body).
- The response of a two-phase lossy composite (or body) at a single frequency can be recovered by applying time-dependent boundary conditions not oscillating at such a frequency.

Determining the volume fraction in 2-phase composites and bodies using time varying applied fields

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Abstract

A body Θ containing two phases, which may form a periodic composite with microstructure much smaller than the body, or which may have structure on a length scale comparable to the body, is subjected to slowly time varying boundary conditions that would produce an approximate uniform field in Θ were it filled with homogeneous material. Here slowly time varying means that the wavelengths and attenuation lengths of waves at the frequencies associated with the time variation are much larger than the size of Θ , so that we can make a quasistatic approximation. At least one of the two phase does not have an instantaneous response but rather depends on fields at prior times. The fields may be those associated with electricity, magnetism, fluid flow in porous media, or antiplane elasticity. We find, subject to these approximations, that the time variation of the boundary conditions can be designed so boundary measurements at a specific time $t = t_0$ exactly yield the volume fractions of the phases, independent of the detailed geometric configuration of the phases. Moreover, for specially tailored time variations, the volume fraction can be exactly determined from measurements at any time t , not just at the specific time $t = t_0$. We also show how time varying boundary conditions, not oscillating at the single frequency ω_0 , can be designed to exactly retrieve the response at ω_0 .

Keywords: Composites, volume fraction estimation, bounds on transient response, Markov functions

1. Introduction

Consider a body Θ containing a periodic composite material of two isotropic phases with the cell size being much smaller than Θ . We ask: can one exactly determine the volume fractions of the two phases from the response of the body to time varying applied electric, magnetic, or elastic quasistatic fields? Here quasistatic means that the frequencies associated with the time variation have waves with wavelengths and attenuation lengths much larger than Θ .

Cherkaev [8] realized the answer is yes, that in principle one can recover the volume fraction exactly. One makes measurements of, say, the electrical response, as governed by the effective electrical permittivity $\epsilon_*(\omega)$ at each of a continuum of frequencies ω , such that the ratio $\epsilon_1(\omega)/\epsilon_2(\omega)$ of the permittivities $\epsilon_1(\omega)$, and $\epsilon_2(\omega)$ of the phases traces an arc in the complex plane. In principle this allows one by analytic continuation to determine the function $\epsilon_*(\epsilon_1, \epsilon_2)$ and hence to obtain the measure entering the Stieltjes function representation of ϵ_*/ϵ_2 as a function of ϵ_1/ϵ_2 . Then, the integral of the measure determines the volume fraction. Besides the difficulty of measuring the response at sufficiently many frequencies that approximate a continuum of frequencies, the analytic continuation is ill-posed. Nevertheless, one can get approximations to the measure and thus to the volume fraction. In prior work [28, 12] and subsequent work [35, 10, 9] this was done either by estimation of the distributions of poles and zeros or poles and residues when the measure is discrete or approximated by a discrete one, or by extraction of a continuous measure. Other work based on the analytic properties of $\epsilon_*(\epsilon_1, \epsilon_2)$ include using in an inverse way [28, 29, 11, 13] volume fraction dependent bounds on $\epsilon_*(\epsilon_1(\omega), \epsilon_2(\omega))$ at one frequency ω or correlating the values of $\epsilon_*(\epsilon_1(\omega_i), \epsilon_2(\omega_i))$ at n frequencies $\omega_i, i = 1, 2, \dots, n$. To yield accurate approximations to the volume fraction these approaches usually require measurements at many frequencies.

Here we show how the volume fractions can be exactly obtained from a single time varying quasistatic field, composed of a continuum of frequencies. By carefully tailoring the time varying applied field, the response at a selected time t_0 only depends on the volume fractions and not on the detailed microstructure. To our surprise, in many cases we find that the response at any time t , and not just t_0 , only depends on the volume fractions and not on the detailed microstructure.

Our work is an extension of that in [25, 26, 27]. In [25, 26] bounds were

obtained for antiplane elasticity on the response of the average stress $\langle \tau \rangle(t)$, given a time dependent average strain $\langle \epsilon \rangle(t) = \epsilon_0 H(t)$, where the angular brackets denote a volume average over the cell of periodicity, ϵ_0 is a fixed vector and $H(t)$ is the Heaviside function, 0 for $t < 0$ and 1 for $t > 0$. The bounds were volume fraction dependent, but otherwise independent of the microstructure (and some bounds were also independent of the volume fraction). Remarkably, in some examples the bounds were exceedingly tight at particular times. These bounds can be used in an inverse way to provide tight bounds on the volume fraction given measurements of the response $\langle \tau \rangle(t)$ at these particular times.

For some choices of the frequency dependent shear moduli $\mu_1(\omega)$ and $\mu_2(\omega)$ of phase 1 and phase 2 the bounds in [25, 26] were quite wide at all times. While the tightness or lack of tightness of the bounds was explained, it was unclear how to tailor $\langle \epsilon \rangle(t)$ to get tight bounds at a given time $t = t_0$, and whether this was at all possible when the bounds with $\langle \epsilon \rangle(t) = \epsilon_0 H(t)$ were not tight. The case of applied fields that were a finite sum of n terms with a $e^{-i\omega_j t}$ time dependence, with complex frequencies $\omega_1, \omega_2, \dots, \omega_n$ was studied in [27]. To ensure that the input function did not diverge to infinity in the distant past, we required that $\text{Im } \omega_j \geq 0$ for all j . Designing these input signals to ensure tight bounds on the response at time $t = t_0$ boiled down to one of approximation on the real interval $[-1, 1]$ of a polynomial of degree $m + 1$ by one of degree m , achieved by letting the difference be an appropriately scaled Chebyshev polynomial. Additionally, we found the input function could be designed to approximately reproduce at time t_0 the response at another frequency ω_0 .

Our approach here uses judicious applications of the residue theorem to recover the volume fraction exactly. Moreover we can extract additional information such as the exact first moment of the measure and the exact response at a frequency ω_0 . We emphasize that is important for the wavelengths and attenuation lengths of the frequencies associated with the time variation of the applied field be much bigger than Θ , and not just the microstructure. Otherwise, the frequency components will dephase or attenuate differently as one moves inside Θ .

Exact values of the volume fractions can sometimes be obtained using another approach. It has an entirely different character. While the macroscopic response of composite materials, as governed by their effective moduli, is generally microstructure dependent, there is a plethora of examples where effective moduli satisfy microstructure independent relations, or relations

that only involve the volume fractions of the phases. A classic example is Hill's formula for the effective Lame modulus λ_* of a mixture of two elastic phases sharing the same shear modulus μ , having Lame moduli λ_1 and λ_2 , and occupying volume fractions f_1 and $f_2 = 1 - f_1$:

$$\frac{1}{\lambda_* + 2\mu} = \frac{f_1}{\lambda_1 + 2\mu} + \frac{f_2}{\lambda_2 + 2\mu}. \quad (1.1)$$

So if λ_* and the moduli of the phases are known, we can determine the volume fractions f_1 and f_2 exactly, without having to know anything about the detailed microstructure. Many examples of such exact relations are surveyed in chapters 3,4,5, and 6 in [31]. A general theory of exact relations was developed [15, 17] and was reviewed in chapter 17 of [31] and in [16]. It led to an enormous number of new exact relations [16]. According to this theory, a manifold \mathcal{M} is an exact relation if whenever a material has a tensor field $\mathbf{L}(\mathbf{x}) \in \mathcal{M}$ for all \mathbf{x} then its associated effective tensor \mathbf{L}_* also lies in \mathcal{M} . Under an appropriate fractional linear transformation, \mathcal{M} is transformed to a linear subspace that must satisfy certain algebraic properties. The theory is easily extended to include volume fractions by letting \mathcal{M} consist of pairs (f, \mathbf{L}) , where f represents the volume fraction, or volume fractions, of the phases.

Our results are also applicable to recovering the volume and shape of an inclusion, or inclusions, in the body Θ from exterior boundary measurements. This is an important inverse problem that has received much attention: see [6, 21, 22, 34, 33, 19, 1, 18, 5, 7, 3, 2, 20, 32, 23] and references therein. For the extension of our work to this problem the reader can refer to Sections 8,9, and 10 of [27]. The generalization to recovering the volume of an inclusion, or inclusions, in the body Θ from exterior boundary measurements is straightforward and will be outlined in Section 2.4.

2. Preliminaries

The effective magnetic permeability tensor (matrix) $\boldsymbol{\mu}_*$ of a periodic composite containing two phases with isotropic permeabilities μ_1 and μ_2 is given by solving the equations

$$\nabla \cdot \mathbf{b} = 0, \quad \mathbf{h} = -\nabla\psi, \quad \mathbf{b}(\mathbf{x}) = \mu(\mathbf{x})\mathbf{h}(\mathbf{x}), \quad (2.1)$$

where \mathbf{b} is the magnetic induction, \mathbf{h} the magnetic field, ψ the magnetic scalar potential and the permeability $\mu(\mathbf{x})$ takes the value μ_1 in phase 1 and

μ_2 in phase 2, and $\mu(\mathbf{x})$, $\mathbf{h}(\mathbf{x})$, and $\mathbf{b}(\mathbf{x})$ are all periodic functions of \mathbf{x} . Let $\langle \cdot \rangle$ denote an average over the unit cell Q of periodicity:

$$\langle f \rangle = \frac{1}{|Q|} \int_Q f(\mathbf{x}) d\mathbf{x}, \quad (2.2)$$

where $|Q|$ denotes the volume of Q . As the response field $\langle \mathbf{b} \rangle$ depends linearly on the applied field $\langle \mathbf{h} \rangle$, we may write

$$\langle \mathbf{b} \rangle = \boldsymbol{\mu}_* \langle \mathbf{h} \rangle, \quad (2.3)$$

which thus determines the effective permeability tensor $\boldsymbol{\mu}_*$.

We could equivalently deal with electricity, fluid flow in porous media, and antiplane elasticity as they are governed by the same equations, with the electric permittivity, fluid permeability, and shear modulus playing the role of the permeability. For example, in antiplane elasticity in an isotropic material one assumes that the shear modulus μ , which takes the role of the magnetic permeability, is independent of x_3 , $\mu = \mu(x_1, x_2)$, and one looks for a solution where the only non-zero displacement component is u_3 and that is independent of x_3 . Then the only non-zero components of the strain $\boldsymbol{\epsilon}(\mathbf{x}) = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]/2$ are $\epsilon_{13} = \epsilon_{31} = \frac{1}{2}\partial u_3/\partial x_1$ and $\epsilon_{23} = \epsilon_{32} = \frac{1}{2}\partial u_3/\partial x_2$. Because $\boldsymbol{\epsilon}(\mathbf{x})$ is a pure shear (trace-free), the stress is $\boldsymbol{\tau}(x_1, x_2) = 2\mu(x_1, x_2)\boldsymbol{\epsilon}(x_1, x_2)$ and its only non-zero components are

$$\tau_{13} = \tau_{31} = 2\mu\epsilon_{13} = \mu\partial u_3/\partial x_1, \quad \tau_{23} = \tau_{32} = 2\mu\epsilon_{23} = \mu\partial u_3/\partial x_2. \quad (2.4)$$

Since $\nabla \cdot \boldsymbol{\tau} = 0$, we get $\partial \tau_{13}/\partial x_1 + \partial \tau_{23}/\partial x_2 = 0$. Thus, the two-dimensional version of the equations (2.1) hold with the replacements

$$\mathbf{b} \rightarrow \begin{pmatrix} \tau_{13} \\ \tau_{23} \end{pmatrix}, \quad \mathbf{h} \rightarrow \begin{pmatrix} \partial u_3/\partial x_1 \\ \partial u_3/\partial x_2 \end{pmatrix}, \quad \mu(\mathbf{x}) \rightarrow \mu(x_1, x_2). \quad (2.5)$$

2.1. Analytic properties

The effective permeability as $\boldsymbol{\mu}_*$ is an analytic function of μ_1 and μ_2 , except at negative real values of the ratio μ_1/μ_2 with the properties:

$$\begin{aligned} \text{Im}[\boldsymbol{\mu}_*(\mu_1, \mu_2)] &> 0 \text{ if } \text{Im} \mu_1 > 0 \text{ and } \text{Im} \mu_2 > 0, \\ \boldsymbol{\mu}_*(\alpha\mu_1, \alpha\mu_2) &= \alpha\boldsymbol{\mu}_*(\mu_1, \mu_2), \quad \boldsymbol{\mu}_*(1, 1) = \mathbf{I}, \\ \left. \frac{\partial \boldsymbol{\mu}_*(\mu_1, 1)}{\partial \mu_1} \right|_{\mu_1=1} &= f_1 \mathbf{I}, \quad \text{Tr} \left. \frac{\partial^2 \boldsymbol{\mu}_*(\mu_1, 1)}{\partial \mu_1^2} \right|_{\mu_1=1} = -2f_1 f_2, \end{aligned} \quad (2.6)$$

where Tr denotes the trace of the matrix. Taking $\alpha = i$ yields the corollary that

$$\text{Re}[\boldsymbol{\mu}_*(\mu_1, \mu_2)] > 0 \text{ if } \text{Re } \mu_1 > 0 \text{ and } \text{Re } \mu_2 > 0. \quad (2.7)$$

As a result of these analytic properties $\boldsymbol{\mu}_*(\mu_1, \mu_2)$ has the integral representation [4, 30, 14]

$$\boldsymbol{\mu}_*(\mu_1, \mu_2) = \mu_2 \mathbf{I} + 2f_1 \mu_2 \mathbf{G}_\eta(z), \text{ with } z = \frac{\mu_1 + \mu_2}{\mu_2 - \mu_1} \quad (2.8)$$

where

$$\mathbf{G}_\eta(z) = \int_{-1}^1 \frac{d\boldsymbol{\eta}(\lambda)}{\lambda - z}, \quad (2.9)$$

is a Markov function of z . $\mathbf{G}_\eta(z)$ has positive-semidefinite matrix valued measure $\boldsymbol{\eta}$ dependent on the material microstructure and having the identity as its mass,

$$\int_{-1}^1 d\boldsymbol{\eta}(\lambda) = \mathbf{I}. \quad (2.10)$$

Furthermore, as implied by (2.6), the first moment of the measure satisfies

$$\text{Tr} \int_{-1}^1 \lambda d\boldsymbol{\eta}(\lambda) = 2f_2 - d, \quad (2.11)$$

where $d = \text{Tr} \mathbf{I}$ is the dimensionality, $d = 2$ or 3 . In the case where the composite has square (for $d = 2$) or cubic symmetry (for $d = 3$), or is isotropic, then $d\boldsymbol{\eta}(\lambda)$ is proportional to the identity matrix and (2.11) implies

$$\int_{-1}^1 \lambda d\boldsymbol{\eta}(\lambda) = (2f_2/d - 1)\mathbf{I}. \quad (2.12)$$

Notice from (2.8) that f_1 can be absorbed into the measure, in which case the integral of the measure yields the volume fraction.

2.2. Time dependent fields

If there is some time dependence, then in general the magnetic induction at a given point and time depends on the magnetic field at that point at previous times (memory effect). Thus, the last relation in (2.1), called the constitutive equation, gets replaced by

$$\mathbf{b}(\mathbf{x}, t) = \int_{-\infty}^t K(\mathbf{x}, t - \tilde{t}) \mathbf{h}(\mathbf{x}, \tilde{t}) d\tilde{t}, \quad (2.13)$$

where by causality $K(\mathbf{x}, s) = 0$ if $s < 0$. Physically $\mathbf{h}(\mathbf{x}, t)$ is always real but it is useful to also consider complex fields. In particular, we may consider fields $\mathbf{h}(\mathbf{x}, t)$ with a time dependence $e^{-i\omega t}$ where the complex frequency ω has positive imaginary part, to ensure that the applied field is small in the distant past. Then, (2.13), with \tilde{t} replaced by $t - s$, implies

$$\mathbf{b}(\mathbf{x}, t) = \mu(\mathbf{x}, \omega)\mathbf{h}(\mathbf{x}, t), \quad \mu(\mathbf{x}, \omega) = \int_0^\infty K(\mathbf{x}, s)e^{i\omega s} ds \quad (2.14)$$

Thus, the constitutive relation in (2.1) still holds, but with complex fields and complex permeabilities. Since $K(\mathbf{x}, s)$ is real, the complex permeability $\mu(\mathbf{x}, \omega)$ satisfies

$$\overline{\mu(\mathbf{x}, \omega)} = \mu(\mathbf{x}, -\overline{\omega}). \quad (2.15)$$

The constraints that $\nabla \cdot \mathbf{b} = 0$ and $\nabla \times \mathbf{h} = 0$ are valid in the quasistatic limit, where the wavelength is much larger than the size of the body Θ . We can take applied fields $\langle \mathbf{h} \rangle(t)$ that are a superposition $\mathbf{u}(t)$ of a continuum of complex frequencies

$$\langle \mathbf{h} \rangle(t) = \mathbf{u}(t) = \int_0^1 \boldsymbol{\beta}(s)e^{-i\omega(s)(t-t_0)} ds, \quad (2.16)$$

where we are free to choose the amplitude vector field $\boldsymbol{\beta}(s)$. To ensure that (2.16) approximately holds for all unit cells Q within Θ , we need to impose boundary conditions that would ensure uniformity of the field in Θ were it filled with a homogeneous medium: these are affine Dirichlet boundary conditions on ψ at $\partial\Theta$: see Section 2.4.

The associated field $\langle \mathbf{b} \rangle(t)$ in the case when the composite is entirely phase 2, is

$$\langle \mathbf{b} \rangle_2(t) = \int_0^1 \mu_2(\omega(s))\boldsymbol{\beta}(s)e^{-i\omega(s)(t-t_0)} ds. \quad (2.17)$$

The corresponding output field can be considered to be

$$\mathbf{v}(t) = \langle \mathbf{b} \rangle(t) - \langle \mathbf{b} \rangle_2(t) = \int_0^1 2f_1\mu_2(\omega(s))\mathbf{G}_\eta(z(\omega(s)))\boldsymbol{\beta}(s)e^{-i\omega(s)(t-t_0)} ds, \quad (2.18)$$

with

$$z(\omega) = \frac{\mu_1(\omega) + \mu_2(\omega)}{\mu_2(\omega) - \mu_1(\omega)}, \quad (2.19)$$

The physical applied field will be $\text{Re}\langle \mathbf{h} \rangle(t)$, and the physical output field will be

$$\text{Re} \mathbf{v}(t) = \text{Re}\langle \mathbf{b} \rangle(t) - \text{Re}\langle \mathbf{b} \rangle_2(t). \quad (2.20)$$

2.3. The dual problem

Alternatively, one may choose $\mathbf{u}(t)$ to be the average magnetic induction $\langle \mathbf{h}(t) \rangle$, and the constitutive and effective constitutive law can be rewritten as

$$\mathbf{h}(\mathbf{x}) = \mu(\mathbf{x})^{-1} \mathbf{b}(\mathbf{x}), \quad \langle \mathbf{h} \rangle = \boldsymbol{\mu}_*^{-1} \langle \mathbf{b} \rangle. \quad (2.21)$$

Rather than considering the function $\boldsymbol{\mu}_*(\mu_1, \mu_2)$ we can consider $\boldsymbol{\mu}_*^{-1}$ as a function $\boldsymbol{\mu}_*^{-1}(1/\mu_1, 1/\mu_2)$ of $1/\mu_1$ and $1/\mu_2$. All but the last property in (2.6) hold if we make the replacements

$$\mu_1 \rightarrow 1/\mu_1, \quad \mu_2 \rightarrow 1/\mu_2, \quad \boldsymbol{\mu}_* \rightarrow \boldsymbol{\mu}_*^{-1}. \quad (2.22)$$

Consequently $\boldsymbol{\mu}_*^{-1}(1/\mu_1, 1/\mu_2)$ has the integral representation

$$\boldsymbol{\mu}_*^{-1}(1/\mu_1, 1/\mu_2) = \mathbf{I}/\mu_2 + 2f_1 \mathbf{H}_\tau(z)/\mu_2, \quad \text{with } z = -\frac{1/\mu_1 + 1/\mu_2}{1/\mu_2 - 1/\mu_1} = \frac{\mu_1 + \mu_2}{\mu_2 - \mu_1}, \quad (2.23)$$

where $\mathbf{H}_\tau(z)$ is another Markov function of z ,

$$\mathbf{H}_\tau(z) = \int_{-1}^1 \frac{d\boldsymbol{\tau}(\lambda)}{\lambda - z}, \quad (2.24)$$

with positive-semidefinite matrix valued measure $\boldsymbol{\tau}$ dependent on the material microstructure and having the identity as its mass. With $\mathbf{u}(t)$ as in (2.16) (but with $\langle \mathbf{h} \rangle(t)$ replaced by $\langle \mathbf{b} \rangle(t)$), we can take our output field to be

$$\mathbf{v}(t) = \langle \mathbf{h} \rangle(t) - \langle \mathbf{h} \rangle_2(t) = \int_0^1 \frac{2f_1 \mathbf{H}_\tau(z(\omega(s)))}{\mu_2(\omega(s))} \boldsymbol{\beta}(s) e^{-i\omega(s)(t-t_0)} ds, \quad (2.25)$$

where

$$\langle \mathbf{h} \rangle_2(t) = \int_0^1 \boldsymbol{\beta}(s) e^{-i\omega(s)(t-t_0)} / \mu_2(\omega(s)) ds. \quad (2.26)$$

is the field $\langle \mathbf{h} \rangle(t)$ in the case when the composite is entirely phase 2, and $z(\omega)$ is given by (2.19). The analysis proceeds in a parallel way. Again, to ensure that $\langle \mathbf{b} \rangle(t)$ is almost independent of the unit cell of periodicity Q , we need to impose Neumann boundary conditions on $\mathbf{b}(\mathbf{x})$ at $\partial\Theta$ that would ensure uniformity of the field in Θ were it filled with a homogeneous medium: see Section 2.4.

2.4. Determining the volume occupied by inclusions in a body

Rather than assuming Θ contains a microstructured material, we can treat the case where Θ contains two phases, with structure not necessarily small compared to Θ . For instance Θ may contain inclusions of one phase surrounded by the second phase, where the sizes of the inclusions may be comparable to the size of Θ . The volume fraction f_1 , or $f_2 = 1 - f_1$, is now the volume occupied by phase 1, or phase 2, divided by $|\Theta|$, the volume of Θ which may be presumed to be known.

We redefine our average as

$$\langle f \rangle = \frac{1}{|\Theta|} \int_{\Theta} f(\mathbf{x}) d\mathbf{x}, \quad (2.27)$$

where $|\Theta|$ is the volume of Θ . We choose affine Dirichlet boundary conditions on the potential ψ (which may make more sense in an electrical setting or in the setting of antiplane elasticity) that are dictated by our input signal $\mathbf{u}(t)$:

$$\psi(\mathbf{x}, t) = -\mathbf{u}(t) \cdot \mathbf{x} \text{ on } \partial\Theta. \quad (2.28)$$

Equivalently, as more appropriate to magnetism, one can specify the tangential components of \mathbf{h} at $\partial\Theta$. As $\mathbf{h} = -\nabla\psi$ a straightforward calculation shows that $\langle \mathbf{h} \rangle(t) = \mathbf{u}(t)$. Since $\nabla \cdot \mathbf{b} = 0$ it then follows that

$$\langle \mathbf{b} \rangle = \langle \nabla \cdot (\mathbf{b}\mathbf{x}) \rangle = \frac{1}{|\Theta|} \int_{\partial\Theta} (\mathbf{b} \cdot \mathbf{n}) \mathbf{x} dS, \quad (2.29)$$

where \mathbf{n} is the outwards normal to $\partial\Theta$, and $\mathbf{b}\mathbf{x}$ is the outer product between the vectors \mathbf{b} and \mathbf{x} . So $\langle \mathbf{b} \rangle$ can be determined by measurements at the boundary of Θ . The relation between $\langle \mathbf{b} \rangle$ and $\langle \mathbf{h} \rangle$ defines some sort of effective tensor $\boldsymbol{\mu}_*^D$ (D for Dirichlet):

$$\langle \mathbf{b} \rangle = \boldsymbol{\mu}_*^D \langle \mathbf{h} \rangle, \quad (2.30)$$

and the function $\boldsymbol{\mu}_*^D(\mu_1, \mu_2)$ has the same analytic properties as $\boldsymbol{\mu}_*(\mu_1, \mu_2)$ in (2.6), excepting the last property, and can be represented in terms of an associated non-negative measure $d\boldsymbol{\eta}^D(\lambda)$. Therefore the analysis carries over with $\mathbf{u}(t) = \langle \mathbf{h} \rangle(t)$ given by (2.16).

We can also consider the dual problem where we specify Neumann boundary conditions on \mathbf{b} :

$$\mathbf{b} \cdot \mathbf{n} = \mathbf{u}(t) \cdot \mathbf{n} \text{ on } \partial\Theta, \quad (2.31)$$

which from (2.29) implies $\langle \mathbf{b} \rangle(t) = \langle \mathbf{u}(t) \rangle = \mathbf{u}(t)$. We can measure

$$\langle \mathbf{h} \rangle = \langle -\nabla \psi \rangle = -\frac{1}{|\Theta|} \int_{\partial\Theta} \psi \mathbf{n} \, dS, \quad (2.32)$$

and the relation between $\langle \mathbf{h} \rangle$ and $\langle \mathbf{b} \rangle$ defines some other sort of effective tensor $\boldsymbol{\mu}_*^N$ (N for Neumann):

$$\langle \mathbf{h} \rangle = (\boldsymbol{\mu}_*^N)^{-1} \langle \mathbf{b} \rangle. \quad (2.33)$$

Again $(\boldsymbol{\mu}_*^N)^{-1}$ as a function of $1/\mu_1$ and $1/\mu_2$ has the same analytic properties as $\boldsymbol{\mu}_*(\mu_1, \mu_2)$ in (2.6), excepting the last property, and can be represented in terms of a non-negative associated measure $d\tau^N(\lambda)$.

If the body does contain a periodic composite material of two isotropic phases with the cell size being much smaller than Θ , then we may equate the Dirichlet and Neumann effective tensors,

$$\boldsymbol{\mu}_*^D = \boldsymbol{\mu}_*^N, \quad (2.34)$$

as the boundary conditions (2.31) will produce fields \mathbf{h} and \mathbf{b} that are almost periodic and consequently (2.28) will be satisfied in the homogenized limit.

3. Choosing an input signal that allows the volume fraction to be exactly determined

In the previous sections we introduced a general framework to handle both the problem of determining the volume fractions in a periodic composite and that of determining the volume occupied by inclusions in a body, by applying affine boundary conditions of either the Dirichlet or Neumann type. Within such a framework, the input signal (that is $\langle \mathbf{h} \rangle(t)$, or $\langle \mathbf{b} \rangle(t)$ for the dual problem) is

$$\mathbf{u}(t) = \int_0^1 \boldsymbol{\beta}(s) e^{-i\omega(s)(t-t_0)} \, ds, \quad (3.1)$$

where the $\omega(s)$ parameterizes a curve in the upper half complex plane, and the vector valued function $\boldsymbol{\beta}(s)$ needs to be determined so that the volume fraction is determined at least at the given time t_0 , as showed in the remaining of this section. The output is (see (2.18), and (2.25) for the dual problem)

$$\mathbf{v}(t) = \int_0^1 a_0 \mathbf{F}_\gamma(z(\omega(s))) \boldsymbol{\alpha}(s) e^{-i\omega(s)(t-t_0)} \, ds, \quad (3.2)$$

in which $\mathbf{F}_\gamma(z)$ is given by either $\mathbf{G}_\eta(z)$ (2.9) or $\mathbf{H}_\tau(z)$ (2.24), and

$$\boldsymbol{\alpha}(s) = \boldsymbol{\beta}(s)c(\omega(s)), \quad (3.3)$$

where the functions $z(\omega)$ and $c(\omega)$ depend on ω in some known way, with

$$z(\omega) = \overline{z(-\bar{\omega})}. \quad (3.4)$$

Specifically, $z(\omega)$ takes the same expression for both problems (see eqs. (2.19) and (2.23)), whereas $c(\omega) = \mu_2(\omega)$ for the direct problem and $c(\omega) = 1/\mu_2(\omega)$ for the dual problem. The real constant $a_0 = 2f_1$ and the unknown measure $d\gamma$, corresponding to either $d\boldsymbol{\eta}$ or $d\boldsymbol{\tau}$, depend on the system. Note that, due to equation (3.3), choosing the function $\boldsymbol{\beta}(s)$ is equivalent to choosing the function $\boldsymbol{\alpha}(s)$, since $c(\omega(s))$ is known.

As it is only the real part of $\mathbf{v}(t)$ that has physical significance, we can write

$$\operatorname{Re} \mathbf{v}(t_0) = a_0 \int_{-1}^1 d\gamma(\lambda) \mathbf{g}(\lambda), \quad (3.5)$$

where

$$\begin{aligned} \mathbf{g}(\lambda) &= \frac{1}{2} \int_0^1 \frac{\boldsymbol{\alpha}(s)}{\lambda - z(\omega(s))} ds + \frac{1}{2} \int_0^1 \frac{\overline{\boldsymbol{\alpha}(s)}}{\lambda - \overline{z(\omega(s))}} ds \\ &= \int_{C \cup \bar{C}} \frac{\mathbf{r}(z)}{\lambda - z} dz, \end{aligned} \quad (3.6)$$

in which C is the curve traced out by $z(\omega(s))$ as s increases from 0 to 1, \bar{C} is the curve traced out by $\overline{z(\omega(s))}$ as s decreases from 1 to 0, and for $z \in C$, $\mathbf{r}(z)$ is obtained from

$$\mathbf{r}(z(\omega(s))) = \boldsymbol{\alpha}(s)/(2z'(\omega(s))\omega'(s)), \quad (3.7)$$

while for $z \in \bar{C}$,

$$\mathbf{r}(z) = -\overline{\mathbf{r}(\bar{z})}. \quad (3.8)$$

Let us assume $C \cup \bar{C}$ is a closed curve encircling the interval $[-1, 1]$ once anticlockwise. We choose $\boldsymbol{\alpha}(s)$ so that $\mathbf{r}(z)$ satisfies (3.8) and is analytic, with no poles, in $C \cup \bar{C}$. Then, this allows one to apply the residue theorem in equation (3.6) to obtain

$$\mathbf{g}(\lambda) = 2\pi i \mathbf{r}(\lambda). \quad (3.9)$$

The corresponding output at time t_0 , see equation (3.5), would then be

$$\operatorname{Re} \mathbf{v}(t_0) = 2\pi i a_0 \int_{-1}^1 d\gamma(\lambda) \mathbf{r}(\lambda). \quad (3.10)$$

In order for the output to depend only on a_0 and eventually the first moment of the measure, $\alpha(s)$ and, therefore $\mathbf{r}(z)$, by equation (3.7), have to be suitably chosen. For example, if one takes

$$\mathbf{r}(z) = -i(1 + kz)\mathbf{n}_0/(2\pi), \quad (3.11)$$

where \mathbf{n}_0 is a constant real unit vector and k is a real coefficient, we obtain

$$\operatorname{Re} \mathbf{v}(t_0) = a_0 \int_{-1}^1 (1 + k\lambda) d\gamma(\lambda) \mathbf{n}_0 = a_0 \mathbf{n}_0 + a_0 k \int_{-1}^1 \lambda d\gamma(\lambda) \mathbf{n}_0. \quad (3.12)$$

Thus, with this choice of $\mathbf{r}(z)$ the fact that $\operatorname{Re} \mathbf{v}(t_0)$ only depends on a_0 and the first moment of the measure is simply a consequence of the residue theorem. Note that there is considerable freedom in the choice of $\mathbf{r}(z)$, still leading to (3.12). Indeed, for example, to $\mathbf{r}(z)$ we may add $A(1/z)\mathbf{n}_0$ where A is an entire function, zero at the origin. A deformation of the contour path $C \cup \bar{C}$ to infinity shows that this does not affect the output.

4. Conditions that allow one to generate appropriate trajectories

To apply the results of the previous section we need to choose $\omega(s)$ so that $C \cup \bar{C}$ is a closed curve encircling the interval $[-1, 1]$ once. Recall that $z(\omega)$ takes the same expression for both problems, see (2.19) and (2.23), where $\mu_i(\omega)$, $i = 1, 2$, are the responses of the two phases satisfying

$$\mu_i(\omega) = \overline{\mu_i(-\bar{\omega})}, \quad (4.1)$$

with

$$\operatorname{Im}(\mu_i(\omega)) \geq 0 \text{ in the quadrant } \operatorname{Re} \omega > 0, \operatorname{Im} \omega > 0. \quad (4.2)$$

To begin let us suppose that $\mu_2(\omega) = \mu_2$ a positive real constant independent of ω . As

$$z(\omega) = \frac{2\mu_2}{\mu_2 - \mu_1(\omega)} - 1, \quad (4.3)$$

we see that $\operatorname{Im} z(\omega) > 0$ in the quadrant $\operatorname{Re} \omega > 0, \operatorname{Im} \omega > 0$. The identity (4.1) implies that $z(\omega)$ is real on the positive imaginary axis. $\mu_1(\omega)$ and hence $z(\omega)$ might also be real along intervals of the real ω axis.

Assume $z(\omega_1)$ and $z(\omega_2)$ are real at frequencies ω_1 and ω_2 each either on the positive imaginary axis or positive real axis, with $z(\omega_1) > 1$ and $z(\omega_2) < -1$. This is typically guaranteed if $\mu_1(\omega_*) = \mu_2$ for some ω_* on the positive imaginary or real axis since $z(\omega)$ will then have a simple pole at $\omega = \omega_*$. We may then take ω_1 and ω_2 on opposite sides of ω_* and sufficiently close to ω_* . We take a trajectory Γ linking ω_1 and ω_2 that remains in the quadrant $\text{Re } \omega > 0$, $\text{Im } \omega > 0$ except at the endpoints. Its image is a curve $C = z(\Gamma)$ linking $z(\omega_1)$ and $z(\omega_2)$ that remains in the upper half plane $\text{Im } z > 0$. Thus, $C \cup \bar{C}$ is a closed curve encircling the interval $[-1, 1]$ once, anticlockwise. The curve C may have loops that do not cross the real axis. Alternatively, if $\mu_1(\omega) = \mu_1$ is a positive real constant independent of ω , while $\mu_2(\omega)$ depends on ω , then the curve $C = z(\Gamma)$ linking $z(\omega_1)$ and $z(\omega_2)$ will remain in the lower half plane $\text{Im } z < 0$. Thus $C \cup \bar{C}$ will be a closed curve encircling the interval $[-1, 1]$ once, clockwise. The latter case is treated by a simple modification of the arguments in the previous section.

More generally, if $\mu_1(\omega)$ and $\mu_2(\omega)$ both depend on ω , we may consider those curves in the quadrant $\text{Re } \omega > 0$, $\text{Im } \omega > 0$ where $\mu_1(\omega)/\mu_2(\omega)$, and hence $z(\omega)$ is real, and look for a point ω_1 on one of these curves where $z(\omega_1) > 1$ and another point ω_2 on the same or nearest neighboring curve where $z(\omega_2) < -1$. Again we join ω_1 and ω_2 by a trajectory Γ that avoids these curves. This trajectory has an image curve $C = z(\Gamma)$ that lies in the upper half-plane or in the lower half plane.

In particular, if there is a ω_0 in the quadrant $\text{Re } \omega_0 > 0$, $\text{Im } \omega_0 > 0$ such that $\mu_1(\omega_0) = \mu_2(\omega_0)$ then there will be a curve in the ω plane (not a smooth curve if ω_0 is not a simple zero of $\mu_1(\omega) - \mu_2(\omega)$) along which $\mu_1(\omega)/\mu_2(\omega)$ is real and $z(\omega)$ takes real values $z(\omega_1) \gg 1$ and $z(\omega_2) \ll -1$ at points ω_1 and ω_2 on this curve close to ω_0 . Then, we join ω_1 and ω_2 by a trajectory Γ that avoids the curves where $\mu_1(\omega)/\mu_2(\omega)$ is real.

5. Recovering the response at another frequency

Suppose that we want to determine the response of the material at the frequency ω_0 :

$$\mathbf{v}_0(t) = a_0 \int_{-1}^1 \frac{d\gamma(\lambda) \mathbf{n}_0}{\lambda - z_0} e^{-i\omega_0(t-t_0)}, \quad (5.1)$$

where both the real constant a_0 and $z_0 = z(\omega_0)$ are known, as well as the constant real unit vector \mathbf{n}_0 . Specifically, assume that we are interested in

the output function at time t_0 :

$$\mathbf{v}_0(t_0) = a_0 \int_{-1}^1 \frac{d\gamma(\lambda) \mathbf{n}_0}{\lambda - z_0}. \quad (5.2)$$

Suppose that probing the material with an input function at the same frequency ω_0 is not feasible, but it is possible to apply an input with a continuous spectrum of frequencies, such as the one in equation (3.1). The corresponding output will then be (3.2), and the goal of this section is to determine the function $\beta(s)$ in (3.1) so that the value taken by $\mathbf{v}(t)$ in (3.2) at $t = t_0$ will be equal to $\mathbf{v}_0(t_0)$. We start by assuming, as before, that the curve C traced out by $z(\omega(s))$ is such that $C \cup \bar{C}$ is a closed curve encircling the interval $[-1, 1]$ once, anticlockwise. Then, the following choice of $\mathbf{r}(z)$ given by (3.7) (note that $\alpha(s)$ in (3.7) is related to $\beta(s)$ by (3.3) so that, as before, prescribing $\mathbf{r}(z)$ is equivalent to choosing $\beta(s)$):

$$\mathbf{r}(z) = \frac{i\mathbf{n}_0}{4\pi(z - z_0)} + \frac{i\mathbf{n}_0}{4\pi(z - \bar{z}_0)}, \quad (5.3)$$

will ensure that (3.8) holds and, if z_0 is outside $C \cup \bar{C}$, $\text{Re } \mathbf{v}(t_0) = \text{Re } \mathbf{v}_0(t_0)$. Indeed, if z_0 is outside $C \cup \bar{C}$, then, upon application of the residue theorem, the above choice of $\mathbf{r}(z)$ will lead to the following value of the output function (3.2) at $t = t_0$

$$\begin{aligned} \text{Re } \mathbf{v}(t_0) &= a_0 \int_{-1}^1 d\gamma(\lambda) \left(\int_{C \cup \bar{C}} \frac{\mathbf{r}(z)}{\lambda - z} dz \right) \\ &= \frac{a_0}{2} \int_{-1}^1 d\gamma(\lambda) \left(\frac{\mathbf{n}_0}{\lambda - z_0} + \frac{\mathbf{n}_0}{\lambda - \bar{z}_0} \right) \\ &= \text{Re } \mathbf{v}_0(t_0), \end{aligned} \quad (5.4)$$

and we are finished. If the curve C traced out by $z(\omega(s))$ is such that $C \cup \bar{C}$ encircles $[-1, 1]$ once, clockwise, then we need to reverse the sign of $\mathbf{r}(z)$ in (5.3). On the other hand, if z_0 is inside $C \cup \bar{C}$, a deformation of this closed curve towards infinity and Cauchy's theorem imply the contour integral is zero. Then, $\text{Re } \mathbf{v}(t_0) = 0$ and we cannot proceed to obtain $\text{Re } \mathbf{v}_0(t_0)$.

6. Measure independent results valid for all time

In Sections 3 and 5 we obtained results for $\text{Re } v(t_0)$ that only depended on a_0 and possibly also the first moment of the measure or only on the response

at a fixed frequency. Remarkably, and much to our surprise, numerical simulations showed that similar results hold for all time t when $\omega(0)$ and $\omega(1)$ both lie on the imaginary axis. To explain this, we take real a and b and assume that

$$\omega(0) = -ia, \quad \omega(1) = -ib, \quad (6.1)$$

have images

$$z(\omega(0)) = A > 1, \quad z(\omega(1)) = B < -1, \quad (6.2)$$

where (3.4) implies A and B are real. For $s \in (0, 1)$ we assume $\omega(s)$ traces out a curve Γ in the quadrant $\operatorname{Re} \omega > 0$, $\operatorname{Im} \omega > 0$, while $z(\omega(s))$ takes values in the upper half plane tracing anticlockwise the curve $C = z(\Gamma)$ as s increases from 0 to 1.

Then, assuming without loss of generality that $t_0 = 0$, from (3.2) we have

$$\begin{aligned} \operatorname{Re} \mathbf{v}(t) &= a_0 \operatorname{Re} \int_0^1 \mathbf{F}_\gamma(z(\omega(s))) \boldsymbol{\alpha}(s) e^{-i\omega(s)t} ds \\ &= a_0 \operatorname{Re} \int_0^1 e^{-i\omega(s)t} \left(\int_{-1}^1 \frac{d\gamma(\lambda)}{\lambda - z(\omega(s))} \right) \boldsymbol{\alpha}(s) ds. \end{aligned} \quad (6.3)$$

We choose an applied field with $\boldsymbol{\alpha}(s)$ such that $\mathbf{r}(z)$, given by (3.7), takes the form $\mathbf{r}(z) = -i\mathbf{n}_0/(2\pi)$, that is the expression taken by $\mathbf{r}(z)$ in (3.11) when $k = 0$. Then, switching variables from s to ω , (6.3) implies

$$\operatorname{Re} \mathbf{v}(t) = a_0 \int_{-1}^1 d\gamma(\lambda) \left(\operatorname{Re} \frac{\mathbf{n}_0}{i\pi} \int_\Gamma e^{-i\omega t} \frac{z'(\omega)}{\lambda - z(\omega)} d\omega \right). \quad (6.4)$$

Additionally substituting $\zeta = i\omega$, $h(\zeta) = z(-i\zeta)$ and letting $D = i\Gamma$ gives

$$\operatorname{Re} \mathbf{v}(t) = a_0 \int_{-1}^1 d\gamma(\lambda) \left(\operatorname{Re} \frac{\mathbf{n}_0}{i\pi} \int_D E(\zeta) \frac{h'(\zeta)}{\lambda - h(\zeta)} d\zeta \right), \quad (6.5)$$

where $E(\zeta) = e^{-\zeta t}$. Let us take Ω as the domain inside $D \cup \overline{D}$. Suppose $\lambda = h(\alpha_j)$ with multiplicity $m_j \geq 1$ for $j = 1, 2, \dots, m$ with $\alpha_j \in \Omega$ and $h(\beta_k) = \infty$ with multiplicity $n_k \geq 1$ for $k = 1, 2, \dots, n$ with $\beta_k \in \Omega$. Observe that the logarithmic derivative $h'(\zeta)/(h(\zeta) - \lambda)$ has poles at $\zeta = \beta_k$, $k = 1, 2, \dots, n$, with residues $-n_k$, and at $\zeta = \alpha_j$, $j = 1, 2, \dots, m$, with residues m_j . Then, noting that $h(\zeta)$ and $E(\zeta)$ are real symmetric

$$h(\bar{\zeta}) = \overline{h(\zeta)}, \quad E(\bar{\zeta}) = \overline{E(\zeta)}, \quad (6.6)$$

a simple application of the residue theorem shows that

$$\begin{aligned} \operatorname{Re} \frac{1}{i\pi} \int_D E(\zeta) \frac{h'(\zeta)}{\lambda - h(\zeta)} d\zeta &= \frac{1}{2i\pi} \int_{D \cup \bar{D}} E(\zeta) \frac{h'(\zeta)}{\lambda - h(\zeta)} d\zeta \\ &= \sum_{k=1}^n n_k E(\beta_k) - \sum_{j=1}^{m(\lambda)} m_j(\lambda) E(\alpha_j(\lambda)), \end{aligned} \quad (6.7)$$

where we replaced m , m_j and α_k with $m(\lambda)$, $m_j(\lambda)$ and $\alpha_k(\lambda)$ to emphasize their dependence on λ . Note that (6.6) implies that the points α_j and β_k come in complex conjugate pairs and this ensures that the right hand side of (6.7) is real. This relation (6.7) holds for any analytic function $E(\zeta)$ defined in an open neighborhood of the closure of Ω that is real symmetric, and not just $E(\zeta) = e^{-\zeta t}$. Thus,

$$\operatorname{Re} \mathbf{v}(t) = a_0 \mathbf{n}_0 \sum_{k=1}^n n_k e^{-t\beta_k} - a_0 \int_{-1}^1 d\gamma(\lambda) \mathbf{n}_0 \sum_{j=1}^{m(\lambda)} m_j(\lambda) e^{-t\alpha_j(\lambda)} \quad (6.8)$$

will be measure independent for all time t if and only if $h(\zeta) = z(-i\zeta)$ does not take real values in $[-1, 1]$ throughout Ω . If it is measure independent, then we can obtain the volume fraction $f_1 = a_0/2$ from measurements of $\operatorname{Re} \mathbf{v}(t)$ at any time t .

If $m(\lambda)$ and $m_j(\lambda)$ do not depend on λ , then at $t = t_0 = 0$ (6.7) implies

$$\frac{1}{2\pi} \operatorname{Re} \int_{D \cup \bar{D}} \frac{h'(\zeta)}{\lambda - h(\zeta)} d\zeta = \sum_{k=1}^n n_k - \sum_{j=1}^m m_j, \quad (6.9)$$

and the right hand side can be identified with the winding number of $C \cup \bar{C} = h(D \cup \bar{D})$ about $z = \lambda$. This is ± 1 if C lies in the upper half plane and $C \cup \bar{C}$ encloses the interval $[-1, 1]$ and we get agreement with (3.12). We assumed D is traced anticlockwise around $(a+b)/2$ as s is increased from 0 to 1. This will not be the case if the right hand side of (6.9) is -1 . Rather D will be traced clockwise so that C is traced anticlockwise around the origin. This will reverse the sign of (6.9) giving agreement with (3.12).

Note that if μ_2 (or μ_1) is real, positive, and frequency independent, then as observed in Section 4, (4.3) implies $z(\omega)$ will have strictly positive (or negative, respectively) imaginary part when ω is in the quadrant $\operatorname{Re} \omega >$

$0, \text{Im } \omega > 0$. Thus, the poles β_k of $h(\zeta)$ and zeros α_j of $h(\zeta) - \lambda$ will be simple and alternate along the real axis.

One can also get time independent results that only incorporate the first moment of the measure. Again, we take an applied field with $\alpha(s)$ such that $\mathbf{r}(z) = -i\mathbf{n}_0 z/(2\pi)$ but this time we set $E(\zeta) = h(\zeta)e^{-\zeta t}$. Now $E(\zeta)$ is not analytic in an open neighborhood of the closure of Ω , but has poles at the points $\beta_k, k = 1, 2, \dots, n$. However, we have

$$\frac{h'(\zeta)h(\zeta)}{\lambda - h(\zeta)} = -h'(\zeta) + \frac{\lambda h'(\zeta)}{\lambda - h(\zeta)}. \quad (6.10)$$

If we assume simple poles, i.e. $n_k = 1$ for all k , then near the pole at β_k , $h(\zeta)$ has an expansion

$$h(\zeta) = \frac{b_k}{\zeta - \beta_k} + c_k + d_k(\zeta - \beta_k) + \dots, \quad (6.11)$$

giving

$$\frac{h'(\zeta)h(\zeta)e^{-\zeta t}}{\lambda - h(\zeta)} = \frac{b_k e^{-\beta_k t}}{(\zeta - \beta_k)^2} - \frac{b_k t e^{-\beta_k t}}{\zeta - \beta_k} + \frac{\lambda e^{-\beta_k t}}{\zeta - \beta_k} + \dots \quad (6.12)$$

Consequently, we get

$$\begin{aligned} \text{Re } \mathbf{v}(t) &= a_0 \sum_{k=1}^n e^{-t\beta_k} \int_{-1}^1 \lambda d\gamma(\lambda) \mathbf{n}_0 - a_0 \mathbf{n}_0 \sum_{k=1}^n b_k t e^{-\beta_k t} \\ &\quad - a_0 \int_{-1}^1 \sum_{j=1}^{m(\lambda)} m_j(\lambda) \lambda e^{-t\alpha_j(\lambda)} d\gamma(\lambda) \mathbf{n}_0. \end{aligned} \quad (6.13)$$

This will be measure dependent for all times $t \neq t_0$ if $z(-i\zeta)$ does not take real values in $[-1, 1]$ throughout Ω . Otherwise, the result depends for all time only on the measure through its first moment.

7. Revisiting the problem of recovering the response at any frequency

Now suppose we choose an applied field with $\alpha(s)$ such that $\mathbf{r}(z)$ is given by (5.3), in which $z_0 = z(\omega_0)$ where ω_0 is some, possibly complex, frequency

at which we desire to determine the response. We assume z_0 does not lie on the interval $[-1, 1]$ on the real axis, where λ lies. We let

$$E(\zeta) = \frac{e^{-\zeta t}}{2(h(\zeta) - z_0)} + \frac{e^{-\zeta t}}{2(h(\zeta) - \bar{z}_0)}. \quad (7.1)$$

This will have poles at points $z_0 = h(\kappa_\ell)$ with multiplicity $p_\ell \geq 1$ for $\ell = 1, 2, \dots, p$ with $\kappa_\ell \in \Omega$, and poles at points $\bar{z}_0 = h(\bar{\kappa}_\ell)$ with multiplicity $p_\ell \geq 1$ for $\ell = 1, 2, \dots, p$ with $\bar{\kappa}_\ell \in \Omega$. When z_0 is real, the κ_ℓ include their complex conjugates. Observing that

$$\begin{aligned} E(\zeta) \frac{h'(\zeta)}{\lambda - h(\zeta)} &= \left[\frac{1}{2(\lambda - z_0)} + \frac{1}{2(\lambda - \bar{z}_0)} \right] \left[\frac{h'(\zeta)e^{-\zeta t}}{\lambda - h(\zeta)} \right] \\ &+ \frac{1}{2(\lambda - z_0)} \left[\frac{h'(\zeta)e^{-\zeta t}}{h(\zeta) - z_0} \right] + \frac{1}{2(\lambda - \bar{z}_0)} \left[\frac{h'(\zeta)e^{-\zeta t}}{h(\zeta) - \bar{z}_0} \right], \end{aligned} \quad (7.2)$$

and proceeding as before, we obtain for complex z_0 that

$$\begin{aligned} \operatorname{Re} \frac{1}{i\pi} \int_D E(\zeta) \frac{h'(\zeta)}{\lambda - h(\zeta)} d\zeta \\ = - \left[\sum_{j=1}^{m(\lambda)} m_j(\lambda) e^{-t\alpha_j(\lambda)} \right] \left[\frac{1}{2(\lambda - z_0)} + \frac{1}{2(\lambda - \bar{z}_0)} \right] \\ + \left[\sum_{\ell=1}^p p_\ell e^{-t\kappa_\ell} \right] \frac{1}{2(\lambda - z_0)} + \left[\sum_{\ell=1}^p p_\ell e^{-t\bar{\kappa}_\ell} \right] \frac{1}{2(\lambda - \bar{z}_0)}, \end{aligned} \quad (7.3)$$

where we have assumed the integral over D is anticlockwise. This gives

$$\begin{aligned} \operatorname{Re} \mathbf{v}(t) &= a_0 \int_{-1}^1 d\gamma(\lambda) \mathbf{n}_0 \sum_{j=1}^{m(\lambda)} m_j(\lambda) e^{-t\alpha_j(\lambda)} \left[\frac{1}{2(\lambda - z_0)} + \frac{1}{2(\lambda - \bar{z}_0)} \right] \\ &- a_0 \sum_{\ell=1}^p p_\ell e^{-t\kappa_\ell} \int_{-1}^1 \frac{d\gamma(\lambda) \mathbf{n}_0}{2(\lambda - z_0)} \\ &- a_0 \sum_{\ell=1}^p p_\ell e^{-t\bar{\kappa}_\ell} \int_{-1}^1 \frac{d\gamma(\lambda) \mathbf{n}_0}{2(\lambda - \bar{z}_0)}. \end{aligned} \quad (7.4)$$

If $z(-i\zeta)$ does not take real values in $[-1, 1]$ throughout Ω ($m = 0$) and $p \neq 0$ then we can recover the response at frequency ω_0 from measurements at any

two times t_1 and $t_2 \neq t_1$. To see this, we take $p = 1$, which as we will see shortly is necessarily the case when $m = 0$ and $p \neq 0$, and set

$$\xi = \int_{-1}^1 \frac{d\gamma(\lambda) \mathbf{n}_0}{2(\lambda - z_0)}. \quad (7.5)$$

Then (7.4) implies the linear equations

$$\begin{aligned} \operatorname{Re} \mathbf{v}(t_1) &= -a_0 \xi e^{-t_1 \kappa_1} - a_0 \bar{\xi} e^{-t_1 \bar{\kappa}_1} \\ \operatorname{Re} \mathbf{v}(t_2) &= -a_0 \xi e^{-t_2 \kappa_1} - a_0 \bar{\xi} e^{-t_2 \bar{\kappa}_1}, \end{aligned} \quad (7.6)$$

which can be solved for ξ , and hence its real part, giving $v(t_0)$. On the other hand, if $z(-i\zeta)$ takes real values in $[-1, 1]$ inside Ω , then we cannot recover the response at frequency ω_0 , for all measures γ , unless $t = 0$.

When z_0 is real, (7.4) should be replaced by

$$\begin{aligned} \operatorname{Re} \mathbf{v}(t) &= a_0 \int_{-1}^1 d\gamma(\lambda) \mathbf{n}_0 \sum_{j=1}^{m(\lambda)} m_j(\lambda) e^{-t \alpha_j(\lambda)} \left[\frac{1}{\lambda - z_0} \right] \\ &\quad - a_0 \left[\sum_{\ell=1}^p p_\ell e^{-t \kappa_\ell} \right] \int_{-1}^1 \frac{d\gamma(\lambda) \mathbf{n}_0}{\lambda - z_0}, \end{aligned} \quad (7.7)$$

and for $t \neq 0$ and $p \neq 0$ we can recover the response at frequency ω_0 for all measures γ if and only if $z(-i\zeta)$ does not take real values in $[-1, 1]$ throughout Ω ($m = 0$).

Since the point z_0 lies outside $C \cup \bar{C}$, the winding number of $C \cup \bar{C}$ around z_0 is zero, that is:

$$\sum_{\ell=1}^p p_\ell - \sum_{k=1}^n n_k = 0. \quad (7.8)$$

On the other hand, assuming the function $h : D \rightarrow C$ is orientation preserving, the winding number of $C \cup \bar{C}$ around the point λ is equal to one:

$$\sum_{j=1}^{m(\lambda)} m_j(\lambda) - \sum_{k=1}^n n_k = 1. \quad (7.9)$$

This implies $m \neq 0$, and we cannot recover $\operatorname{Re} \mathbf{v}(t_0)$ from measurements of $\operatorname{Re} \mathbf{v}(t)$ except at $t = 0$. However, if the mapping $h : D \rightarrow C$ reverses

orientation, then

$$\sum_{j=1}^{m(\lambda)} m_j(\lambda) - \sum_{k=1}^n n_k = -1, \quad (7.10)$$

and m can only be zero if $n = p = 1$.

By setting $t = 0$ in (7.4) and (7.7) and using these identities we recover the conclusion (5.4) of section 5. Note that if $h : D \rightarrow C$ reverses orientation, we need to change signs of the right hand sides of (7.4) and (7.7). If C reverses orientation this sign change is a consequence of the sign change of $r(z)$ given by (5.3) needed to ensure (5.4) holds. Alternatively if D reverses orientation, then the right hand side of (7.3) changes sign and there is no change in the sign of $r(z)$.

Throughout all cases discussed above it was implicitly assumed that the curve D is simple, that is, without self-intersections. Therefore, $D \cup \bar{D}$ is a Jordan curve, the boundary of the domain Ω . We can relax this condition, simply assuming that D is a curve contained in the upper half-plane, with boundaries at the real points. Leaving aside the domain Ω , we still can invoke Cauchy's theorem in this more general setting, with the necessary modification of the weights $n_k, m_j(\lambda), p_\ell$. Specifically, these numbers will consist of the respective multiplicities, times the winding number with respect to $D \cup \bar{D}$ of the associated zero or pole. Then it may well happen that some winding numbers are equal to zero, thus erasing the contribution of the respective singularity of the integrand.

8. Numerical results

Let us start by considering a composite made of two phases, one of which, say phase 2, is lossless, so that $z(\omega)$ takes the form (4.3). Furthermore, assume that $\mu_1(\omega) = 1 + i/\omega$, thus mimicking the low frequency dielectric response of a lossy dielectric material, and $\mu_2 = 2$. According to the procedure illustrated in Section 4, we can generate an appropriate trajectory Γ traced by $\omega(s)$, with $s \in [0, 1]$, by taking two frequencies ω_1 and ω_2 on opposite sides of the frequency ω_* for which $\mu_1(\omega_*) = \mu_2$. In this example, $\omega_* = i$, so let us take $\omega_1 = 1.5i$ and $\omega_2 = 0.5i$. We verify that $z(\omega_2) = -5 < -1$ and $z(\omega_1) = 11 > 1$, so, then, we can take a trajectory Γ linking ω_1 and ω_2 that remains in the quadrant $\text{Re } \omega > 0$, $\text{Im } \omega > 0$. Consider, for instance, $\omega(s) = -2(1+i)s^2 + (2+i)s + 1.5i$, for which $\omega(0) = \omega_1$ and $\omega(1) = \omega_2$. Then, $z(\omega(s))$ takes values in the upper half plane tracing anticlockwise the

curve $C = z(\Gamma)$ as s increases from 0 to 1, so that the curve $C \cup \overline{C}$ is a closed curve encircling the interval $[-1, 1]$ once, see Figure 1(a).

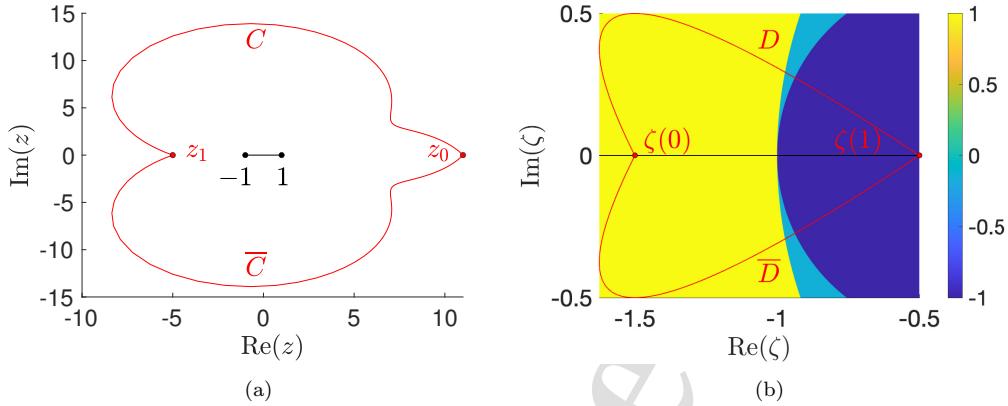


Figure 1: (a) Given $\mu_1 = 1 + i/\omega$ and $\mu_2 = 2$, we choose the trajectory $\omega(s) = -2(1 + i)s^2 + (2 + i)s + 1.5i$ with $s \in [0, 1]$, so that its image through the function $z(\omega)$ given by (4.3) is the curve connecting the points $z_0 = z(\omega(0)) = 11$ and $z_1 = z(\omega(1)) = -5$. Then, the curve $C \cup \overline{C}$ is a closed curve encircling the interval $[-1, 1]$ once. (b) In red is the curve D traced by $\zeta(s) = i\omega(s)$ as s is increased from 0 to 1, and the curve \overline{D} traced by $\overline{\zeta(s)}$. The domain inside $D \cup \overline{D}$ is Ω . In black is represented the line where the function $h(\zeta) = z(-i\zeta)$ takes real values, the yellow region is where the real part of $h(\zeta)$ takes values bigger than 1 and the purple region where it takes values smaller than -1. The color bar indicates that the real part of $h(\zeta)$ never takes values $[-1, 1]$ in the domain Ω , especially along the black line, where the imaginary part of $h(\zeta)$ is zero. Notice that the function has a single pole at $\zeta = -1$: indeed, $z(\omega)$ has a pole at $\omega_* = i$.

Note that the trajectory chosen is rather special because the function $h(\zeta) = z(-i\zeta)$ does not take real values in $[-1, 1]$ throughout Ω (the domain inside $D \cup \overline{D}$), see Figure 1(b). This ensures that the bounds on $\text{Re } v(t)$ are measure-independent not only at time $t = t_0$, but for any time t . To see this, for simplicity, we consider the one-dimensional forward problem in which the input function $\text{Re } u(t)$ (3.1) has only one non-zero component, that we will denote by $\text{Re } u(t)$, and we look at the response (3.2) of the material in the same direction, that we will denote with $\text{Re } v(t)$. First, we assume that the first moment of the measure is not known, and set $k = 0$ in the formula (3.11) that determines the relevant component of the function \mathbf{r} . The corresponding input function is depicted in Figure 2(a), whereas Figure 2(b) shows the bounds on the output function $\text{Re } v(t)$.

For any moment of time, the bounds are found by optimizing over the

measure: the maximum and minimum values are attained when the relevant component of the matrix-valued measure $\gamma(\lambda)$, that we denote with $\gamma(\lambda)$, is an extreme measure, namely the point mass $\gamma(\lambda) = \delta(\lambda - \lambda_0)$, where λ_0 is varied over $[-1, 1]$. The outer bounds, in blue, in Figure 2(b) correspond to the case when the volume fraction is not given (note that the upper bound is always zero), and therefore a_0 , is not known, whereas the inner ones do incorporate the volume fraction ($a_0 = 0.6$). Notice that the latter are indeed coincident and not only the value they take at $t = t_0 = 0$ equals a_0 , as predicted by (3.12), but the value they take at any moment of time t is exactly the one provided by (6.8), where $\beta_1 = -1$ is the only simple pole ($n_1 = 1$) of the function $h(\zeta)$ in Ω , and the second part of the formula is zero as $h(\zeta)$ does not have any zero in the domain Ω , see also Figure 1(b). Therefore, according to (6.8), the two coincident bounds incorporating the volume fraction have the following analytical expression: $\text{Re } v(t) = -a_0 \exp(t)$, which is indeed in agreement with the result shown in Figure 2(b). Note that any other trajectory $\omega(s)$ having the same features of the one chosen in this example, that are, the corresponding closed curve $C \cup \overline{C}$ encircles $[-1, 1]$, and the corresponding closed curve $D \cup \overline{D}$ is such that $h(\zeta)$ does not take real values in $[-1, 1]$ inside it, would have led to the same bounds depicted in Figure 2(b).

Now suppose that we know the first moment of the measure: $M_1 = \int_{-1}^1 \lambda d\gamma(\lambda)$. Then, we apply an input function $\text{Re } u(t)$ so that the corresponding output function will incorporate such a piece of information by choosing, for instance, $k = 1$ in (3.11). Therefore, $\alpha(s)$ (3.7) and $\beta(s)$ (3.3) are known and the input function (3.1) is determined, see Figure 3(a). We compute the bounds on the output function $\text{Re } v(t)$, by considering the extreme measure $\gamma(\lambda) = w_0\delta(\lambda - \lambda_0) + w_1\delta(\lambda - \lambda_1)$, where the weights w_0 and w_1 are chosen so that the values of the zeroth and first moments of the measure are the desired ones, and λ_0 and λ_1 are varied over the interval $[-1, 1]$ in search for the minimum and maximum values of $\text{Re } v(t)$ at any time t , see Figure 3(b). The outer bounds in blue correspond to the case where only the volume fraction is known, whereas the inner bounds in red incorporate also the first moment of the measure. Like in the previous case, the latter are coincident, due to the special choice of the trajectory $\omega(s)$, and their analytical expression is given by the sum of the relations (6.8) (which does not include M_1) and (6.13) (which includes M_1 only). For the case under study, $h(\zeta)$ has only one single pole, $\beta_1 = -1$ with corresponding residue $b_1 = -4$, and no zeros in the domain Ω . Therefore, the analytical expression of the

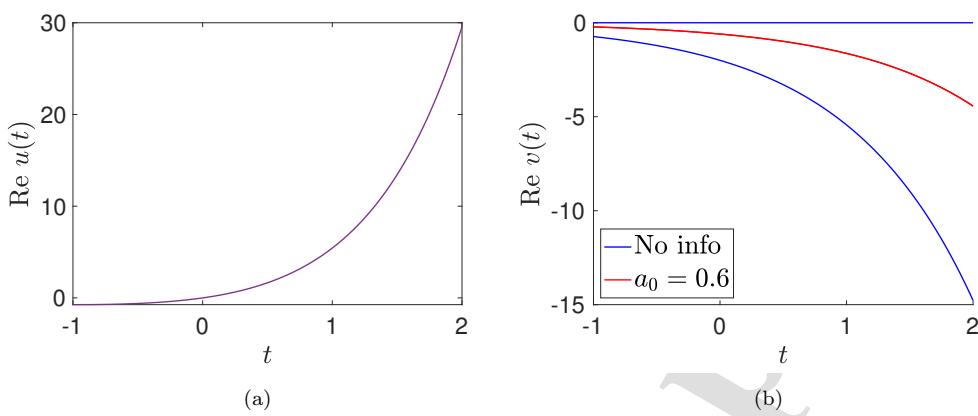


Figure 2: We take $\mu_1 = 1 + i/\omega$, $\mu_2 = 2$, and $\omega(s) = -2(1+i)s^2 + (2+i)s + 1.5i$. (a) The relevant component of the input function (3.1), $\text{Re}[u(t)]$, is found by choosing the relevant component of the function \mathbf{r} according to (3.11), with $k = 0$. Therefore, $\alpha(s)$ is given by (3.7), and then the coefficient $\beta(s)$ appearing in (3.1) is known through (3.3). (b) By optimizing over the measure with one point mass for any moment of time, we obtain bounds on the output function $\text{Re}[v(t)]$ in two different scenarios: in blue are the bounds when the volume fraction, and therefore a_0 , is not given, whereas in red are the bounds incorporating the volume fraction f_1 ($a_0 = 2f_1 = 0.6$). The upper blue bound is clearly always zero, whereas the lower blue bound takes value -2 at $t = 0$ and decreases in time. Interestingly enough, the red bounds are coincident: indeed, the spectrum of frequencies $\omega(s)$ chosen is such that the bounds are measure-independent at any moment of time. Furthermore, their analytical expression is exactly determined by equation (6.8) which, upon substituting the value $\beta_1 = -1$ of the only simple pole, $n_1 = 1$, of the function $h(\zeta)$ in Ω , provides $\text{Re}[v(t)] = -a_0 \exp(t)$ (according to Figure 1(b), $h(\zeta)$ does not have any zero in Ω and the curve D is traced clockwise).

output function, when $M_1 = 0.4$, is $\text{Re } v(t) = -a_0 \exp(t)(1.4 + 4t)$ which is, indeed, in agreement with the numerical results in Figure 3(b).

Finally, suppose that we want to determine the response of the material $\text{Re } v_0(t)$ at a very specific frequency, ω_0 , by applying the spectrum of frequencies $\omega(s)$ chosen earlier. As explained in Section 5, this is only possible if $z(\omega_0)$ is outside $C \cup \overline{C}$. This is true if, for instance, we choose $\omega_0 = 29/27i$, for which $z(\omega_0) = 30$, see Figure 1(a). Then, by choosing the input function given by the choice (5.3), the corresponding output function $\text{Re } v(t)$ is such that at $t = t_0$ it provides the value $\text{Re } v_0(t_0)$, as shown by (5.4). Such a result holds true for any measure γ . In Figure 4(a), we plot the bounds on both output functions, $\text{Re } v_0(t)$ and $\text{Re } v(t)$, by optimizing over the position of the point mass λ_0 in $\gamma = \delta(\lambda - \lambda_0)$. Not only the bounds are coincident

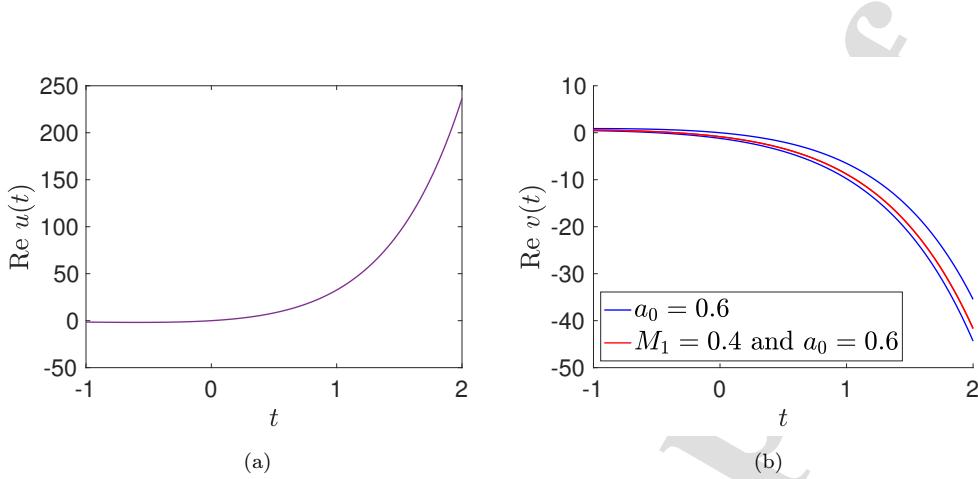


Figure 3: We take $\mu_1 = 1 + i/\omega$, $\mu_2 = 2$, and $\omega(s) = -2(1 + i)s^2 + (2 + i)s + 1.5i$. Furthermore, we assume that the first moment of the measure M_1 is known, say $M_1 = 0.4$. (a) The input function $\text{Re}[u(t)]$ is found by setting $k = 1$ in (3.11). Therefore, $\alpha(s)$ is given by (3.7), and then the coefficient $\beta(s)$ appearing in (3.1) is known through (3.3). (b) In blue are the bounds when only $a_0 = 2f_1 = 0.6$ is given, whereas in red are the bounds incorporating also the first moment of the measure $M_1 = 0.4$. The upper blue bound is clearly always zero, whereas the lower blue bound takes value $-a_0$ at $t = t_0 = 0$ and decreases in time. Again, the red bounds incorporating the first moment of the measure and the volume fraction are coincident: like in the case when the first moment of the measure was not known, the spectrum of frequencies $\omega(s)$ chosen is such that the bounds are independent of the measure, aside from its first moment, at any moment of time.

at $t = t_0 = 0$, in agreement with (5.4), but they are coincident for any moment of time. Notice that, if the measure is known, then the two output functions take exactly the same value at any moment of time t , as shown in Figure 4(b). Since z_0 is real, the analytical expression of the output function is given by (7.7), where the first term equals zero, given that $h(\zeta)$ does not take any real value between -1 and 1 in Ω , see Figure 1(b), and the second term reads $-a_0 \exp(31/27t)/(\lambda_0 - z(\omega_0))$.

Similar results hold when we consider different models for the material properties μ_1 and μ_2 . Suppose, for instance, that both materials are dispersive, meaning that they both are frequency dependent. Assume that $\mu_1(\omega) = 1 - 1/\omega^2$, thus mimicking the response of plasma in the dielectric problem, and $\mu_2(\omega) = 1 + i/\omega$. An appropriate trajectory $\omega(s)$ that generates a curve C so that $C \cup \overline{C}$ is a closed curve that encircles the interval $[-1, 1]$ once is the one chosen in the previous example, that is $\omega(s) = -2(1 + i)s^2 + (2 + i)s + 1.5i$, as shown in Figure 5(a). Further-

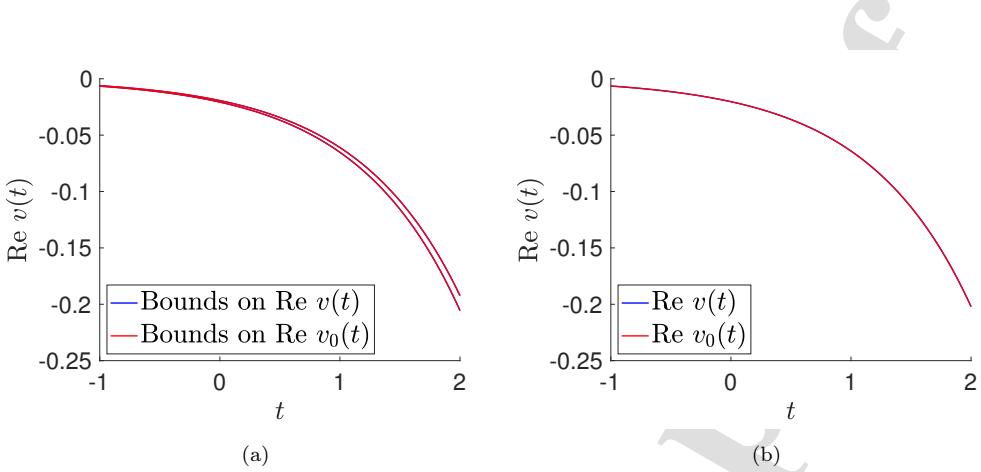


Figure 4: We take $\mu_1 = 1 + i/\omega$, and $\mu_2 = 2$. The value at $t = t_0$ of the output function $\text{Re}[v_0(t)]$, corresponding to an input function at the frequency $\omega_0 = 29/27i$, is determined by applying an input function of the type (3.1) with $\omega(s) = -2(1+i)s^2 + (2+i)s + 1.5i$. We choose $\beta(s)$ in (3.1) by selecting $\alpha(s)$ in (3.3) according to (3.7), with $r(z)$ given by (5.3). (a) Here the measure is unknown and bounds on $\text{Re}[v_0(t)]$, in red, and $\text{Re}[v(t)]$, in blue, are computed by optimizing over all measures. The bounds coincide not only at $t = t_0 = 0$ but any moment of time t . (b) Here the measure is known to be $\gamma(\lambda) = \delta(\lambda - 0.5)$: the value at any t of $\text{Re}[v_0(t)]$, corresponding to an input at the frequency ω_0 , is equal to the value of $\text{Re}[v(t)]$, corresponding to the input (3.1) with $\omega(s)$ chosen as described above. The analytical expression is $-a_0 \exp(31/27t)/(\lambda_0 - z(\omega_0))$.

more, the chosen trajectory is such that the curve D traced clockwise by $\zeta(s) = i\omega(s)$ is such that the function $h(\zeta) = z(-i\zeta)$ does not take real values in $[-1, 1]$ throughout Ω (the domain inside $D \cup \overline{D}$), as shown in Figure 5(b). This ensures that the bounds on $\text{Re}[v(t)]$ when the volume fraction is known are measure-independent not only at time $t = t_0$, but for any time t , as shown in Figure 5.

Finally, we engineer an example in which the trajectory $\omega(s)$ is such that the bounds are measure dependent at any t , except at $t = t_0 = 0$, where they are coincident. To this purpose, consider a material for which $z(\omega) = (i\omega + \alpha_2)/(\omega^2 - i(\alpha_1 + \alpha_3)\omega - \alpha_1\alpha_3)$, with $\alpha_1 < \alpha_2 < \alpha_3$ real coefficients, and the following trajectory $\omega(s) = (\alpha_1 + \epsilon)i + bs - (b + (2\epsilon + \alpha_1 - \alpha_3)i)s^2$, where ϵ is a sufficiently small real number and b is possibly complex. By choosing the parameters wisely, as in Figure 7(a), the closed curve $C \cup \overline{C}$ traced by $z(\omega(s))$ and $\overline{z(\omega(s))}$, as s varies in $[0, 1]$, encircles the interval $[-1, 1]$, while the function $h(\zeta) = z(-i\zeta)$ takes real values between -1 and 1 in the domain Ω , see Figure 7(b). As a result, the bounds on the output

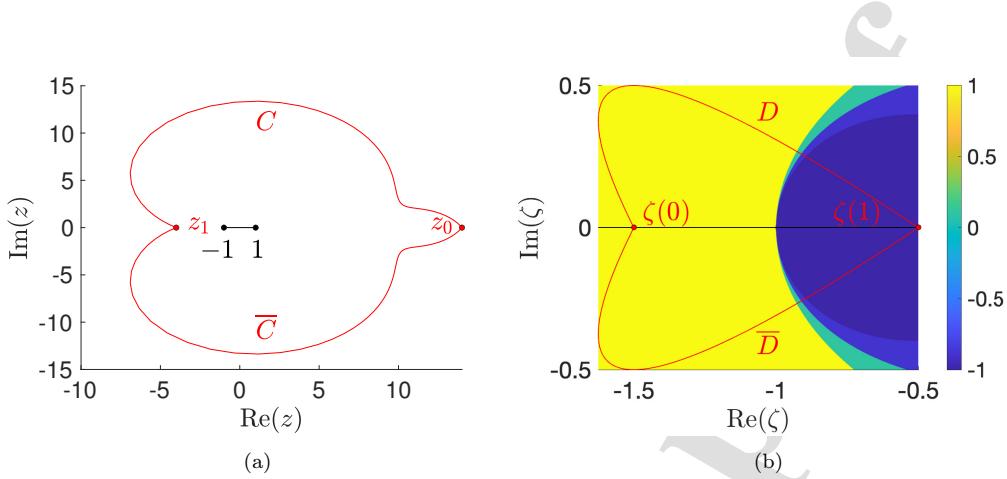


Figure 5: (a) Given $\mu_1 = 1 - 1/\omega^2$ and $\mu_2 = 1 + i/\omega$, we choose the trajectory $\omega(s) = -2(1+i)s^2 + (2+i)s + 1.5i$, with $s \in [0, 1]$, so that its image through the function $z(\omega)$ given by (4.3) is the red curve connecting the points $z(\omega(0)) = 14$ and $z(\omega(1)) = -4$. Then, $C \cup \bar{C}$ is a closed curve encircling the interval $[-1, 1]$ once. (b) In red is the curve D traced by $\zeta(s) = i\omega(s)$ as s is increased from 0 to 1, and the curve \bar{D} traced by $\bar{\zeta(s)}$. The domain inside $D \cup \bar{D}$ is Ω . In black is the straight line where the function $h(\zeta) = z(-i\zeta)$ takes real values, the yellow region is where the real part of $h(\zeta)$ takes values bigger than 1 and the purple region where it takes values smaller than -1. The color bar indicates that the real part of $h(\zeta)$ never takes values $[-1, 1]$ in the domain Ω . Notice that $h(\zeta)$ has a single pole at $\zeta = -1$: indeed, $z(\omega)$ has a pole at $\omega_* = i$.

function are coincident only at $t = t_0 = 0$, as shown in Figure 8.

9. Conclusions

In this paper, we consider the response in time of a two-phase composite in which at least one of the two constituent materials has a non-local response in time. We found that when the applied field is suitably chosen, the measurement of the response of the composite at a specific moment of time exactly yields the volume fraction of the phases, independent of the microstructure of the composite. To achieve such a result, one has to choose affine boundary conditions where the applied field is composed by a continuous spectrum of complex frequencies. Such spectrum has to satisfy certain properties that are related to the response of the constituent materials, which are supposed to be rational functions of the frequency. This is to guarantee that there exists a range of frequencies where the absorption is zero (to this regard, see for instance the experiment described in [24], where the authors

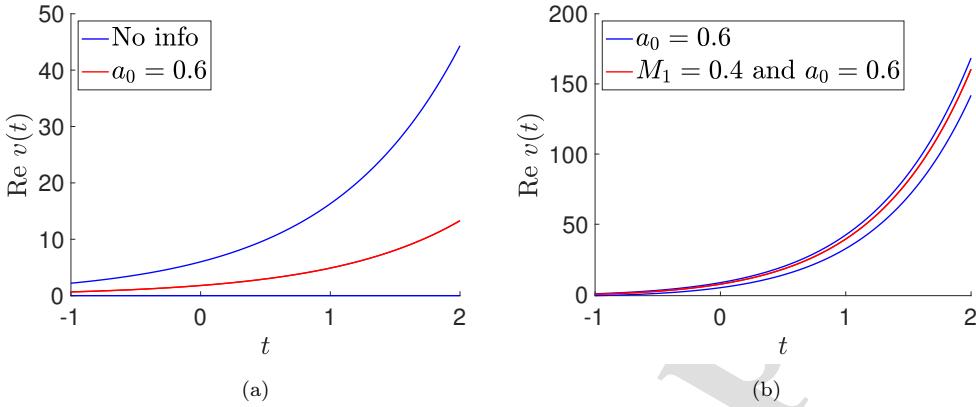


Figure 6: The case where $\mu_1 = 1 - 1/\omega^2$, $\mu_2 = 1 + i/\omega$, and $\omega(s) = -2(1+i)s^2 + (2+i)s + 1.5i$. (a) Bounds on the output function $\text{Re}[v(t)]$: in blue are the bounds when the volume fraction, and therefore a_0 , is not given, whereas in red are the bounds incorporating the volume fraction f_1 ($a_0 = 2f_1 = 0.6$). The latter bounds are coincident: indeed, the spectrum of frequencies $\omega(s)$ chosen is such that the bounds are measure-independent at any moment of time. Their analytical expression is exactly determined by (6.8). (b) Bounds on $\text{Re}[v(t)]$ when only the the volume fraction f_1 ($a_0 = 2f_1 = 0.6$) is given (outer blue bounds), whereas in red are the bounds incorporating also the first moment of the measure $M_1 = 0.4$. Again, the latter bounds are coincident: like in the case when the first moment of the measure was not known, the spectrum of frequencies $\omega(s)$ chosen is such that the bounds are measure-independent at any moment of time. Furthermore, their analytical expression is exactly determined by the sum of equations (6.8) and (6.13).

studied viscous damping in a material over 7 decades of frequency). If that were not the case, then one could expect the trajectory of frequencies to fail to satisfy some of the properties that would ensure the exact determination of the volume fraction (specifically, the curve C would not cross the horizontal axis and, therefore, $C \cup \bar{C}$ would not form a closed loop around the interval $[-1, 1]$). If the continuous spectrum of frequencies satisfies the further property that the initial and final frequencies are purely imaginary, then, remarkably, the volume fraction can be retrieved by measuring the response of the composite at any moment of time. Note that a slight modification of the boundary conditions will also allow one to recover the first moment of the measure. Additionally, one can recover the response of the composite at a certain frequency by suitably applying boundary conditions not oscillating at such a frequency. Finally, following [27], all the analysis developed here carries through to determining exactly the Fourier components of an inclu-

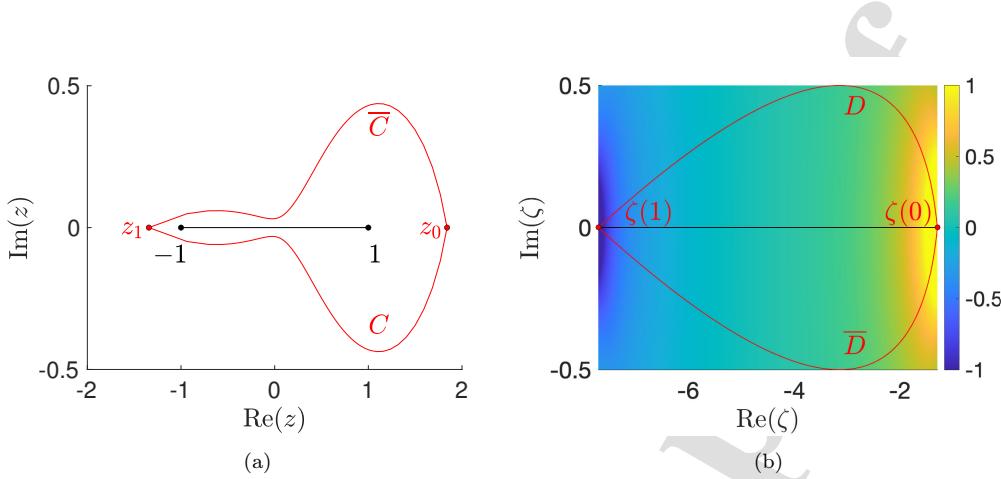


Figure 7: (a) Given $z(\omega) = (i\omega + \alpha_2)/(\omega^2 - i(\alpha_1 + \alpha_3)\omega - \alpha_1\alpha_3)$, with $\alpha_1 = 1$, $\alpha_2 = 5$, and $\alpha_3 = 8$, we choose the trajectory $\omega(s) = (\alpha_1 + \epsilon)i + bs - (b + (2\epsilon + \alpha_1 - \alpha_3)i)s^2$, with $\epsilon = 0.3$ and $b = 2 + i$, with $s \in [0, 1]$, so that its image through the function $z(\omega)$ given by (4.3) is the red curve connecting the points $z(\omega(0)) = 1.8408$ and $z(\omega(1)) = -1.3433$. Then, $C \cup \overline{C}$ is a closed curve encircling the interval $[-1, 1]$ once. (b) In red is the curve D traced by $\zeta(s) = i\omega(s)$ as s is increased from 0 to 1, and the curve \overline{D} traced by $\overline{\zeta(s)}$. The domain inside $D \cup \overline{D}$ is Ω . In black is the straight line where the function $h(\zeta) = z(-i\zeta)$ is real. The choice of parameters, for which $h(\zeta)$ has two single poles at $\zeta = -8$ and at $\zeta = -1$, while it has a zero at $\zeta = -5$, is such that $h(\zeta)$ does take real values between -1 and 1 in the domain Ω , as indicated by the color bar.

sion in a body, from which the shape of the inclusion and not just its size can theoretically be recovered.

Note that our results assume the absence of noise and that the responses of the phases are exactly known. Noise in the measurements of the output at a given time, say $t = t_0$ is easily handled. Different volume fractions would correspond to a different output at time $t = t_0$. Those outputs compatible with the error bars of the measured response at $t = t_0$ give us the range of possible volume fractions. Dealing with uncertainty in the responses of the phases is more involved. The functions $\mu_1(\omega)$ and $\mu_2(\omega)$ need to have the required analytic properties. So, instead of these being exactly known, they may belong to sets of functions Ψ_1 and Ψ_2 satisfying these properties. Then for every pair of functions, one in each set, associated bounds need to be calculated on the volume fraction from the measured response at time $t = t_0$. However, the input function should be that for one specific pair and such that the volume fraction would be uniquely determined by measurements at $t = t_0$ were this pair the actual responses of the two phases. In the end, one

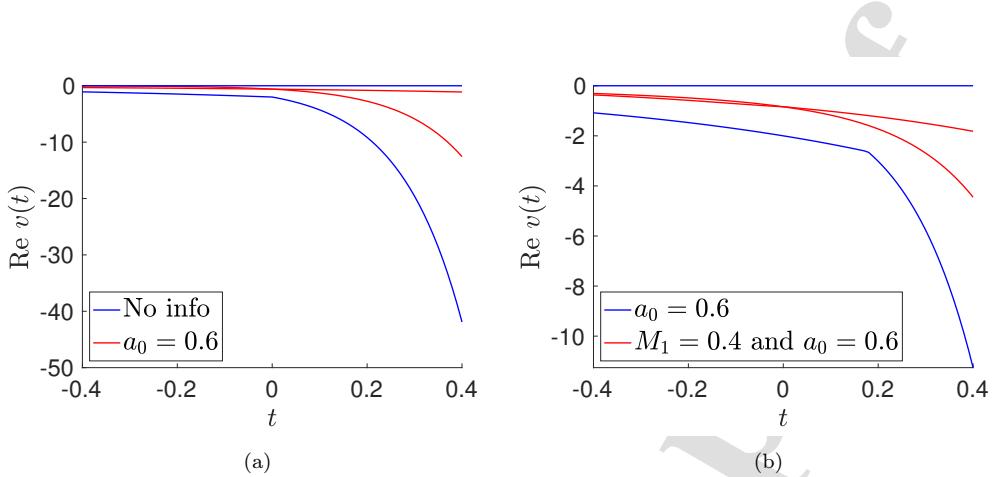


Figure 8: Consider $z(\omega) = (i\omega + \alpha_2)/(\omega^2 - i(\alpha_1 + \alpha_3)\omega - \alpha_1\alpha_3)$, with $\alpha_1 = 1$, $\alpha_2 = 5$, and $\alpha_3 = 8$, and trajectory $\omega(s) = (\alpha_1 + \epsilon)i + bs - (b + (2\epsilon + \alpha_1 - \alpha_3)i)s^2$, with $\epsilon = 0.3$ and $b = 2 + i$, with $s \in [0, 1]$. (a) Bounds on the output function $\text{Re}[v(t)]$: in blue are the bounds when the volume fraction, and therefore a_0 , is not given, whereas in red are the bounds incorporating the volume fraction f_1 ($a_0 = 2f_1 = 0.6$). Contrariwise to the previous cases, the latter bounds are coincident only at $t = t_0 = 0$, where they take value $-a_0$: indeed, the spectrum of frequencies $\omega(s)$ chosen is such that the bounds are measure-dependent at any moment of time (except $t = t_0$). (b) Bounds on $\text{Re}[v(t)]$ when only the the volume fraction f_1 ($a_0 = 2f_1 = 0.6$) is given (outer blue bounds), whereas in red are the bounds incorporating also the first moment of the measure $M_1 = 0.4$. Again, the latter bounds are coincident only at time $t = t_0 = 0$.

should take the union of these bounds on the volume fraction. In addition to the complexity, the effect of incorporating a range of functions $\mu_1(\omega)$ and $\mu_2(\omega)$ would likely be highly dependent on the chosen example. So, faced with a particular problem it would be prudent for a researcher to investigate these effects.

Acknowledgment

The authors are grateful to the National Science Foundation for support through Research Grants DMS-1814854, DMS-2107926, and DMS-2008105. M.P. was partially supported by a Simons Foundation collaboration grant for mathematicians. Davit Harutyunyan is thanked for his help in the initial stages of this work. The authors also wish to thank the reviewers for their insightful comments and suggestions that helped improved the manuscript.

References

- [1] Giovanni Alessandrini and Edi Rosset. The inverse conductivity problem with one measurement: Bounds on the size of the unknown object. *SIAM Journal on Applied Mathematics*, 58(4):1060–1071, August 1998.
- [2] H. Ammari and H. Kang. *Polarization and moment tensors: with applications to inverse problems and effective medium theory*, volume 162. Springer Science & Business Media, New York, 2007.
- [3] Habib Ammari and Hyeonbae Kang. *Reconstruction of Small Inhomogeneities from Boundary Measurements*, volume 1846 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, Germany / Heidelberg, Germany / London, UK / etc., 2004.
- [4] David J. Bergman. The dielectric constant of a composite material — A problem in classical physics. *Physics Reports*, 43(9):377–407, July 1978.
- [5] Martin Brühl, Martin Hanke, and Michael S. Vogelius. A direct impedance tomography algorithm for locating small inhomogeneities. *Numerische Mathematik*, 93(4):635–654, February 2003.
- [6] Alberto-P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and its Applications to Continuum Physics: 24 a 28 de Março 1980*, volume 12 of *Coleção Atas*, pages 65–73. Sociedade Brasileira de Matemática, Rio de Janeiro, Brazil, 1980.
- [7] Yves Capdeboscq and Michael S. Vogelius. Optimal asymptotic estimates for the volume of internal inhomogeneities in terms of multiple boundary measurements. *Mathematical Modelling and Numerical Analysis = Modélisation mathématique et analyse numérique: M²AN*, 37(2):227–240, March/April 2003.
- [8] Elena Cherkaev. Inverse homogenization for evaluation of effective properties of a mixture. *Inverse Problems*, 17(4):1203–1218, August 2001.
- [9] Elena Cherkaev and Carlos Bonifasi-Lista. Characterization of structure and properties of bone by spectral measure method. *Journal of Biomechanics*, 44(2):345–351, January 2011.

- [10] Elena Cherkaev and Miao-Jung Yvonne Ou. Dehomogenization: reconstruction of moments of the spectral measure of the composite. *Inverse Problems*, 24(6):065008, October 2008.
- [11] Elena Cherkaeva and Kenneth M. Golden. Inverse bounds for microstructural parameters of composite media derived from complex permittivity measurements. *Waves in Random Media*, 8(4):437–450, 1998.
- [12] A. R. Day, M. F. Thorpe, A. R. Grant, and A. J. Sievers. The spectral function of a composite from reflectance data. *Physica. B, Condensed Matter*, 279(1–3):17–20, April 2000.
- [13] C. Engström. Bounds on the effective tensor and the structural parameters for anisotropic two-phase composite material. *Journal of Physics D: Applied Physics*, 38(19):3695–3702, September 2005.
- [14] Kenneth M. Golden and George C. Papanicolaou. Bounds for effective parameters of heterogeneous media by analytic continuation. *Communications in Mathematical Physics*, 90(4):473–491, 1983.
- [15] Yury Grabovsky. Exact relations for effective tensors of polycrystals. I: Necessary conditions. *Archive for Rational Mechanics and Analysis*, 143(4):309–329, 1998.
- [16] Yury Grabovsky. *Composite Materials: Mathematical Theory and Exact Relations*. IOP Publishing, Bristol, UK, 2016.
- [17] Yury Grabovsky, Graeme W. Milton, and Daniel S. Sage. Exact relations for effective tensors of composites: Necessary conditions and sufficient conditions. *Communications on Pure and Applied Mathematics (New York)*, 53(3):300–353, March 2000.
- [18] M. Ikehata. Size estimation of inclusion. *Journal of Inverse and Ill-Posed Problems*, 6(2):127–140, 1998.
- [19] Hyeonbae Kang, Jin Keun Seo, and Dongwoo Sheen. The inverse conductivity problem with one measurement: Stability and estimation of size. *SIAM Journal on Mathematical Analysis*, 28(6):1389–1405, November 1997.

- [20] Andreas Kirsch. *An Introduction to the Mathematical Theory of Inverse Problems*, volume 120 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, Germany / Heidelberg, Germany / London, UK / etc., second edition, 2011.
- [21] Robert V. Kohn and Michael S. Vogelius. Determining conductivity by boundary measurements. *Communications on Pure and Applied Mathematics (New York)*, 37(3):289–298, May 1984.
- [22] Robert V. Kohn and Michael S. Vogelius. Relaxation of a variational method for impedance computed tomography. *Communications on Pure and Applied Mathematics (New York)*, 40(6):745–777, November 1987.
- [23] T. Kolokolnikov and A. E. Lindsay. Recovering multiple small inclusions in a three-dimensional domain using a single measurement. *Inverse Problems in Science and Engineering*, 23(3):377–388, 2015.
- [24] R. S. Lakes and J. Quackenbush, Viscoelastic behaviour in indium-tin alloys over a wide range of frequencies and times. *Philosophical Magazine Letters*, 74(4):227–232, 1996.
- [25] Ornella Mattei. *On bounding the response of linear viscoelastic composites in the time domain: The variational approach and the analytic method*. Ph.D. thesis, University of Brescia, Brescia, Italy, 2016.
- [26] Ornella Mattei and G. W. Milton. Bounds for the response of viscoelastic composites under antiplane loadings in the time domain. In G. W. Milton, editor, *Extending the Theory of Composites to Other Areas of Science*, pages 149–178. Milton–Patton Publishers, P.O. Box 581077, Salt Lake City, UT 85148, USA, 2016. See also arXiv:1602.03383 [math-ph].
- [27] Ornella Mattei, Graeme W. Milton, and Mihai Putinar. An extremal problem arising in the dynamics of two-phase materials that directly reveals information about the internal geometry. *Communications on Pure and Applied Mathematics (New York)*, 2022. To appear. Available as arXiv:2007.13964 [math-ph].
- [28] Ross C. McPhedran, D. R. McKenzie, and Graeme W. Milton. Extraction of structural information from measured transport properties

of composites. *Applied Physics A: Materials Science & Processing*, 29(1):19–27, September 1982.

[29] Ross C. McPhedran and Graeme W. Milton. Inverse transport problems for composite media. *Materials Research Society Symposium Proceedings*, 195:257–274, 1990.

[30] Graeme W. Milton. Bounds on the complex permittivity of a two-component composite material. *Journal of Applied Physics*, 52(8):5286–5293, August 1981.

[31] Graeme W. Milton. *The Theory of Composites*, volume 6 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, UK, 2002. Series editors: P. G. Ciarlet, A. Iserles, Robert V. Kohn, and M. H. Wright.

[32] Jennifer L. Mueller and Samuli Siltanen. *Linear and Nonlinear Inverse Problems with Practical Applications*. Computational Science & Engineering. SIAM Press, Philadelphia, 2012.

[33] John Sylvester. Linearizations of anisotropic inverse problems. In Lassi Päivärinta and Erkki Somersalo, editors, *Inverse Problems in Mathematical Physics: Proceedings of The Lapland Conference on Inverse Problems Held at Saariselkä, Finland, 14–20 June 1992*, volume 422 of *Lecture Notes in Physics*, pages 231–241, Berlin, Germany / Heidelberg, Germany / London, UK / etc., 1993. Springer-Verlag.

[34] John Sylvester and Gunther Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Annals of Mathematics*, 125(1):153–169, 1987.

[35] Dali Zhang and Elena Cherkaev. Padé approximations for identification of air bubble volume from temperature- or frequency-dependent permittivity of a two-component mixture. *Inverse Problems in Science and Engineering*, 16(4):425–445, 2008.

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