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A Levine–Tristram invariant for knotted tori

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We define a new topological invariant of an embedded torus in a homology $S^1 \times S^3$, analogous to the Levine–Tristram invariant of a knot. We compare it to an invariant of smooth tori, defined recently by Echeverria using gauge theory for singular connections.

57K41, 57K45

1 Introduction

The Levine–Tristram invariant, $\sigma_\alpha(K)$, of an odd-dimensional knot — see Levine [30], Tristram [53] (also Conway's survey [8]) — is a function from the unit circle to the integers. We will use this invariant in dimension 3, where it is defined for a knot K in an oriented homology 3–sphere. By definition, $\sigma_\alpha(Y, K)$ is the signature of a certain Hermitian form (constructed from the Seifert form) that depends on $\alpha \in S^1$. The Levine–Tristram invariant plays a central role in the calculation of the knot concordance groups in higher dimensions, and has interesting connections to gauge theory in dimension 3.

In this paper we construct, by topological means, a 4–dimensional version of the Levine–Tristram invariant for certain knotted tori, and discuss its relationship to a gauge-theoretic invariant of embedded tori introduced by Echeverria [11]. To state our result, let us define a homology $S^1 \times S^3$ to be an oriented 4–manifold X with the homology of $S^1 \times S^3$. A homology orientation of X (usually suppressed in the notation, but still lurking in the background) is a choice, γ, of generator of the first cohomology $H^1(X) \cong \mathbb{Z}$.

Theorem A *Let T be a locally flat torus in a homology oriented homology $S^1 \times S^3$ such that the inclusion map induces a surjection on the first homology groups. Then, for any $\alpha \in S^1$, there is an invariant $\sigma_\alpha(X, T) \in \mathbb{Z}$ with the following properties:*

(1) *If K is a knot in a homology 3–sphere Y, then $\sigma_\alpha(S^1 \times Y, S^1 \times K) = \sigma_\alpha(Y, K)$.*

(2) *Reversing either the orientation or the homology orientation of X changes the sign of $\sigma_\alpha(X, T)$.*

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The signature invariant $\sigma_\alpha(X,T)$ in Theorem A will be constructed as follows. From a torus, as in Theorem A, we will construct an oriented manifold V with the cohomology of $T^2\times S^2$ and with a distinguished class in $H_3(V)$. If α has prime-power order, then $\sigma_\alpha(X,T)$ will be a standard Atiyah–Singer invariant of a 3–manifold carrying that homology class, together with a U(1) representation determined by α. The function $\sigma_\alpha(X,T)$ will be extended to all α using a standard averaging argument.

1.1 Motivation and background

To motivate Theorem A we briefly discuss the interpretation of $\sigma_\alpha(Y,K)$ in terms of 3–dimensional gauge theory. This comes from independent work of Herald [20] and Heusener and Kroll [22], building on an insightful paper of X-S Lin [32] and a suggestion of the author (see also Heusener [21]). They identified $\frac{1}{2}\sigma_{\alpha^2}(K)$ with a (signed) count of flat SU(2) connections on the knot complement having holonomy conjugate to the matrix

$$\begin{pmatrix}\bar\alpha & 0\\ 0 & \alpha\end{pmatrix} \tag{1}$$

on the meridian of K. The signs are determined using spectral flow, as in Taubes' gauge-theoretic interpretation [52] of the Casson invariant. In Herald's version, the knot can lie in a homology sphere Y, and the Casson invariant — see Akbulut and McCarthy [1], Saveliev [50] — of Y enters into the formula.

A recent preprint by Echeverria [11] constructs an invariant $\lambda_{\mathrm{FO}}(X,T,\alpha)$ of knotted tori, similar to the count used in [20; 22]. (Our notation is not quite the same as in [11], where α denotes an element of $(0,1)$; here, α is the corresponding element of S^1.) The context, extending work of Furuta and Ohta [16], is the following. Consider an oriented 4–manifold X with the homology of $S^1\times S^3$, with a choice of generator of $H^1(X)$. One is given a torus T^2 embedded in X so that the map $H_1(T)\to H_1(X)$ is surjective; let us call that an essential embedding of T. Then Echeverria shows how to count flat SU(2) connections on $X-T$ having specified holonomy (1) on the meridian μ_T of T, by interpreting such flat connections as elements of moduli spaces introduced by Kronheimer and Mrowka [27; 28].

There are some restrictions on the scope of Echeverria's invariant. One is a restriction on the topology of $X-T$, analogous to the condition introduced by Furuta and Ohta [16] in defining a Casson-type invariant λ_{FO} associated to a homology $S^1\times S^3$. Moreover, $\lambda_{\mathrm{FO}}(X,T,\alpha)$ is defined only for α of finite order, or, in other words, of the form

$\alpha = e^{2\pi iq/n}$. The restriction stems from the use of orbifold instanton theory in the basic setup. In the product case,

$$(X, T) = S^1 \times (Y, K),$$

Echeverria [11, Corollary 41] shows that

$$\lambda_{\mathrm{FO}}(X, T, \alpha) = 8\lambda(Y) + \sigma_{\alpha^2}(Y, K), \tag{2}$$

where $\lambda(Y)$ is the Casson invariant [1] of Y.

Echeverria [11] posed the problem of finding a topological invariant $\sigma_\alpha(X, T)$, defined in a fashion similar to the Levine–Tristram invariant, that satisfies a version of (2) with $\lambda(Y)$ replaced by $\lambda_{\mathrm{FO}}(X)$ (when defined) and the knot signature replaced by $\sigma_\alpha(X, T)$. One might even hope to establish such a relation via methods similar to [20; 22]. Such a Levine–Tristram invariant should have the property that, in the product case, $\sigma_\alpha(X, T) = \sigma_\alpha(Y, K)$. In this paper, we construct such a topological invariant $\sigma_\alpha(X, T)$.

After the posting of the first version of this paper, Langte Ma [36] showed that our topological invariant is related to Echeverria's gauge-theoretic invariant $\lambda_{\mathrm{FO}}(X, T, \alpha)$ in the hoped-for fashion.

Theorem (Ma 2021) *When both invariants $\lambda_{\mathrm{FO}}(X, T, \alpha)$ and $\lambda_{\mathrm{FO}}(X)$ are defined,*

$$\lambda_{\mathrm{FO}}(X, T, \alpha) = 8\lambda_{\mathrm{FO}}(X) + \sigma_{\alpha^2}(X, T). \tag{3}$$

To state the main theorem, we make use of a special case of an invariant of Atiyah and Singer [3]. There is a good deal of variation in the terminology and notation for such invariants; we follow the version in Atiyah, Patodi and Singer [2]. Let Y be an oriented 3–manifold and choose a representation $\varphi\colon \pi_1(Y) \to \mathrm{U}(1)$. Assume that there is an oriented 4–manifold W with boundary Y and that the representation φ extends to $\pi_1(W)$. Then define

$$\rho_\varphi(Y) = \mathrm{sign}(W) - \mathrm{sign}_\varphi(W), \tag{4}$$

where $\mathrm{sign}_\varphi(W)$ denotes the signature of the Hermitian intersection form on the twisted homology $H_2(W; \mathbb{C}_\varphi)$. For example, if Y is 0–framed surgery on an oriented knot K with meridian μ_K and $\varphi_\alpha(\mu_K) = \alpha$, then $\rho_{\varphi_\alpha}(Y)$ is — see Kauffman and Taylor [23] and Viro [54] — exactly the Levine–Tristram invariant $\sigma(K, \alpha)$.

Now we can outline the definition of our invariant $\sigma_\alpha(X,T)$, with the proof that it is well defined discussed in the following sections. Let T be an oriented embedded torus, as in Theorem A. One can perform (see Ma [37, Section 2.3]) the analogue of "0–framed surgery" on X along this torus to obtain a manifold V having the homology of $T^2\times S^2$. (This process is not canonical, but in Section 3.1 we will show that the resulting invariant is independent of all choices.) A simple but crucial observation is that in fact V is a cohomology $T^2\times S^2$, or, in other words, has the same cohomology ring as $T^2\times S^2$. A second observation is that the homology orientation $\gamma\in H^1(X)$ gives rise to a primitive cohomology class in $H^1(V)$. We will continue to call this class γ.

Let $g\in H_1(T)$ be a homology class for which $\langle\gamma,g\rangle=1$. Then g, together with the meridian $\mu=\mu_T$, give a basis for $H_1(V)$. For any $\alpha\in S^1$, consider the U(1) representation φ with $\varphi(g)=1$ and $\varphi(\mu)=\alpha$. Choose a connected oriented 3–manifold $M\subset V$ that is Poincaré dual to γ. Then φ restricts to a U(1) representation of $\pi_1(M)$, and we will continue to use the same notation for the restriction. Note that while the choice of g is not canonical, the restriction of φ to $\pi_1(M)$ is still uniquely determined.

The definition then proceeds in two steps. First we show (Proposition 2.5) that if α has prime-power order, then $\rho_\varphi(M)$ is independent of the choice of 3–manifold dual to γ. The second step is to extend this definition to arbitrary characters φ, using the fact that $\rho_\varphi(M)$ is a piecewise-constant function of α, with finitely many jumps.

As it turns out, it is not essential that V come from surgery on a torus, so we start by defining invariants for a pair (V,γ) consisting of an arbitrary cohomology $T^2\times S^2$ and a primitive class $\gamma\in H^1(V)$. We detail the properties needed in the next section.

1.2 Notation and conventions

Throughout the paper, V will denote an oriented cohomology $T^2\times S^2$. We fix a primitive element $\gamma\in H^1(V)$ and choose a second class η so that $\{\gamma,\eta\}$ is a basis for $H^1(V)$. Write $\{g,h\}$ for the dual basis for $H_1(V)$. The condition on the cohomology of V amounts to saying that $\gamma\cup\eta$ generates a summand of $H^2(V)$. A complementary summand is generated by a class $\zeta\in H^2(V)$ such that $\zeta\cup\zeta=0$ and $\gamma\cup\eta\cup\zeta$ gives the orientation of V.

In the course of the proof, we will need to compare the cohomology of V with that of the actual $T^2\times S^2$. To that end, we denote the corresponding cohomology classes in

$T^2 \times S^2$ as follows. The basis for $H^1(T^2 \times S^2)$ is written as $\{\xi, \beta\}$; we can assume that $\langle \xi \cup \beta, [T] \rangle = 1$, where $[T]$ is the homology class of the torus in $H_2(T^2 \times S^2)$. The distinguished element of $H^1(T^2 \times S^2)$ will be ξ. The other generator $\delta \in H^2(T^2 \times S^2)$ is the Poincaré dual of $[T]$.

We will denote by A_x the infinite cyclic covering space of a space A determined by a cohomology class $x \in H^1(A; \mathbb{Z})$. Similarly, if $\beta \in H^1(A; \mathbb{Z}_d)$ then A_x will denote the corresponding finite cyclic covering. If $x \in H^1(A; \mathbb{Z})$ then x_d will denote its reduction mod d, lying in $H^1(A; \mathbb{Z}_d)$. For the rest of the paper, p will denote a prime, and $\mathbb{F}_p$ the field with p elements.

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2 Infinite cyclic coverings

Our approach is based on Milnor's duality theorem [38] which states that the infinite cyclic cover of an $(n+1)$–manifold has, under some circumstances, the homological properties of a closed n–manifold. We will also make use of an extension of this principle [46; 47] that says that, under appropriate hypotheses, the infinite cyclic cover of a $2n$–manifold can have ρ invariants for a U(1) representation φ as if it were a closed $(2n-1)$–manifold. The main point to establish is that the cyclic cover (corresponding to φ) of the infinite cyclic cover also satisfies Poincaré duality. One might say that this extends Milnor's theorem to certain twisted coefficients.

In this section, we prove such an extended duality theorem for V, a cohomology $T^2 \times S^2$. We need a preliminary lemma.

Lemma 2.1 *There is a degree one map from V to $T^2 \times S^2$ inducing isomorphisms on integral homology and cohomology, preserving the distinguished generators of the first cohomology group.*

Proof First choose maps f_1 and f_2 from V to S^1 such that $f_1^*(\xi) = \gamma$ and $f_1^*(\beta) = \eta$. The product of these two gives a map f from V to T^2. We claim that there is a map $h\colon V \to S^2$ such that h^* of the generator of $H^2(S^2)$ is ζ. By standard obstruction theory, there is a map $h_1\colon V \to \mathbb{C}P^2$ such that $h_1^*(\delta') = \zeta$, where δ' is the generator of $H^2(\mathbb{C}P^2)$. But the degree of h_1 must be 0, as the following calculation shows.

$$\begin{aligned}\deg(h_1) &= \langle \delta' \cup \delta', \deg(h_1)[\mathbb{C}P^2]\rangle = \langle \delta' \cup \delta', (h_1)_*([V])\rangle \\ &= \langle h_1^*(\delta' \cup \delta'), [V]\rangle = \langle \zeta \cup \zeta, [V]\rangle = 0.\end{aligned}$$

It follows (possibly [44] is the original reference) that h_1 is homotopic to a map that misses a point of $\mathbb{C}P^2$, and hence is homotopic to a map $h\colon V \to S^2$ with the desired property. Now the map $f_1 \times f_2 \times h$ is a degree one map that induces an isomorphism on homology (and hence on cohomology). □

Now we consider the infinite cyclic covering space $\pi\colon V_\gamma \to V$ induced by the class $\gamma \in H^1(V)$. The homology groups (resp. mod p homology groups) of V_γ are modules over the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$ (resp. $\Lambda_p = \mathbb{F}_p[t, t^{-1}]$). Note that while Λ_p is a PID, Λ is not.

Lemma 2.2 *V_γ satisfies 3–dimensional Poincaré duality with coefficients in any field $\mathbb{F}$. Moreover, if $M \subset V$ is Poincaré dual to γ, then it lifts to V_γ and represents the fundamental class of V_γ.*

Proof According to Milnor's duality theorem [38], the conclusions of the lemma hold whenever $H_*(V_\gamma; \mathbb{F})$ is finitely generated. (The fact that M carries the fundamental class of V_γ is implicit in Milnor's paper, and is proved explicitly in [24].) Papadima and Suciu [41] and Pajitnov [42] (see also [12]) show that this finite generation holds if the *Aomoto complex*

$$\cdots \to H^j(V; \mathbb{F}) \xrightarrow{\cup\gamma} H^{j+1}(V; \mathbb{F}) \xrightarrow{\cup\gamma} H^{j+2}(V; \mathbb{F}) \to \cdots \tag{5}$$

is exact. (Rather strikingly, an analytic version of this fact, which applies when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, is part of Taubes' theory of Fredholm complexes for manifolds with periodic ends [51, Theorem 3.1]; compare [39].) Exactness of (5) for $T^2 \times S^2$ follows directly from the Künneth theorem; so, by hypothesis, it holds for V as well. □

The main result of this section is a version of Lemma 2.2 for some further d–fold covering spaces of V_γ, where $d = p^r$ and p is a prime. We continue (and slightly extend) the notation from above. Consider the covering space $V_{\eta_d} \to V$, where η_d is

the mod d reduction of the generator $\eta \in H^1(V)$. This fits into a diagram of covering spaces, where the vertical arrows are infinite cyclic coverings and the horizontal ones are $\mathbb{Z}_{p^r}$ coverings,

$$\begin{array}{ccc} V_{\gamma,\eta_d} & \xrightarrow{p^r\text{–fold cover}} & V_\gamma \\ \mathbb{Z}\text{–cover}\downarrow & & \downarrow\mathbb{Z}\text{–cover} \\ V_{\eta_d} & \xrightarrow{p^r\text{–fold cover}} & V \end{array}$$

Proposition 2.3 *The homology groups $H_*(V_{\gamma,\eta_d};\mathbb{C})$ are finite-dimensional. In particular, V_{γ,η_d} satisfies 3–dimensional Poincaré duality with complex coefficients.*

Proposition 2.3 will be proved by comparing $H_*(V_{\gamma,\eta_d};\mathbb{C})$ with $H_*(V_{\gamma,\eta_d};\mathbb{F}_p)$, where $\mathbb{F}_p$ denotes the field with p elements. The first step is to investigate the mod p cohomology of V_{η_d}.

Lemma 2.4 *V_{η_d} is an $\mathbb{F}_p$ cohomology $T^2\times S^2$.*

Before embarking on the proof, it is worth considering what happens for the actual $T^2\times S^2$. Recall the convention that γ for a general V corresponds to $\xi \in H^1(T^2\times S^2)$. Then $(T^2\times S^2)_\xi = \mathbb{R}\times S^1\times S^2$ and so $(T^2\times S^2)_{\xi,\beta_d} = \mathbb{R}\times S^1\times S^2$ has finitely generated homology.

Proof The infinite cyclic cover V_η corresponding to η factors through the cyclic p^r–fold cover $V_{\eta_d}\to V$:

$$\begin{array}{ccc} V_\eta & \xrightarrow{\mathbb{Z}\text{–cover}} & V_{\eta_d} \\ & {\scriptstyle\mathbb{Z}\text{–cover}}\searrow \quad \swarrow{\scriptstyle p^r\text{–fold cover}} & \\ & V & \end{array} \tag{6}$$

If t denotes the generator of the covering transformations of $V_\eta\to V$, then t^d generates the covering transformations of $V_\eta\to V_{\eta_d}$.

The mod p cohomology of V_η is a finitely generated module over the PID $\Lambda_p = \mathbb{F}_p[t,t^{-1}]$, and hence splits as a finite sum of free Λ_p modules and cyclic modules of the form $\Lambda_p/\langle q(t)\rangle$. It is easy to see that the fact that V is an $\mathbb{F}_p$ cohomology $T^2\times S^2$ implies that the Aomoto complex (5) is acyclic. According to [41, Proposition 9.4], this means that each homology group $H_k(V;\Lambda_p)$ is finite-dimensional, and moreover any summand of the form $\Lambda_p/((t-1)^j)$ has $j=1$.

To compute the homology of V_{η_d}, we examine the Milnor exact sequences with $\mathbb{F}_p$ coefficients for the two infinite cyclic covering spaces in (6), taking account of the relationship between their covering transformations. (To simplify the notation, the coefficient groups are not indicated.) The first reads

$$H_2(V) \to H_1(V_\eta) \xrightarrow{t-1} H_1(V_\eta) \xrightarrow{\pi_*} H_1(V) \to H_0(V_\eta) \xrightarrow{t-1} H_0(V_\eta). \tag{7}$$

Since $H_0(V_\eta) \cong \mathbb{F}_p$ and t acts by the identity, the first sequence ends

$$\cdots \to H_1(V) = \mathbb{F}_p \oplus \mathbb{F}_p \to \mathbb{F}_p \to 0.$$

In particular, we must have that $\operatorname{coker}[t-1 \colon H_1(V_\eta) \to H_1(V_\eta)] \cong \mathbb{F}_p$.

We remark that if the Laurent polynomial $q(t)$ is relatively prime to $t-1$, then $t-1$ is an isomorphism on $\Lambda_p/(q(t))$. We conclude that $H_1(V_\eta)$ must have exactly one summand of the form $\Lambda_p/(t-1) \cong \mathbb{F}_p$ and none of the form $\Lambda_p/((t-1)^j)$ with $j > 1$.

Now we look at the Milnor sequence relating the homology of V_{η_d} to that of V_η,

$$H_2(V_{\eta_d}) \to H_1(V_\eta) \xrightarrow{t^d-1} H_1(V_\eta) \xrightarrow{\pi_*} H_1(V_{\eta_d}) \to H_0(V_\eta) \xrightarrow{t^d-1} H_0(V_\eta). \tag{8}$$

It is standard that (mod p, of course!) $t^d - 1 = (t-1)^d$, which is an isomorphism on all but one summand in $H_1(V_\eta)$. On the other hand, the cokernel of $(t-1)^d$ on $\Lambda_p/(t-1)$ is again $\mathbb{F}_p$, and so the homology group $H_1(V_{\eta_d})$ is isomorphic to $\mathbb{F}_p \oplus \mathbb{F}_p$.

Since V_{η_d} finitely covers V_η, which has Euler characteristic 0, the Euler characteristic of V_{η_d} must be 0 as well. By Poincaré duality, we see that the mod p homology groups of V_{η_d} are the same as those of $T^2 \times S^2$. So it remains to be seen that the cup product $H^1(V_{\eta_d}) \times H^1(V_{\eta_d}) \to H^2(V_{\eta_d})$ is nonsingular.

To this end, recall the homology equivalence $f \colon V \to T^2 \times S^2$ constructed in Lemma 2.1. It lifts to a degree one map $f_{\eta_d} \colon V_{\eta_d} \to (T^2 \times S^2)_{\beta_d}$. Being a degree one map, f_{η_d} induces a surjection in homology with any coefficients. But with $\mathbb{F}_p$ coefficients, the homology groups of V_{η_d} and $(T^2 \times S^2)_{\beta_d}$ are isomorphic, so this surjection must in fact be an isomorphism. Dually, it is an isomorphism in cohomology, so, by naturality, the cup product on $H^1(V_{\eta_d})$ is nonsingular. □

With these preliminaries in hand, we can now establish the finite-dimensionality of the homology of V_{γ,η_d}, which is a $\mathbb{Z} \times \mathbb{Z}_d$ cover of a cohomology $T^2 \times S^2$.

Proof of Proposition 2.3 By construction, V_{γ,η_d} is an infinite cyclic cover of V_{η_d}, which is itself a $\mathbb{Z}_d$ cover of a cohomology $T^2 \times S^2$. We showed in Lemma 2.4 that

V_{η_d} is an $\mathbb{F}_p$ cohomology torus, from which it follows that the cohomology of V_{γ,η_d} with coefficients in $\mathbb{F}_p$ is finite-dimensional. But by [6, Lemma 6] this implies that the homology of V_{γ,η_d} with complex coefficients is also finite-dimensional. Milnor's duality theorem [38] shows that V_{γ,η_d} satisfies Poincaré duality over $\mathbb{C}$. □

2.1 The ρ invariant of a cohomology $T^2 \times S^2$

From the preceding discussion, we can now define ρ invariants of a cohomology $T^2 \times S^2$ with distinguished generator $\gamma \in H^1(V)$. We continue with the notation established in Section 1.2, and consider the character φ with $\varphi(h) = \alpha$ and $\varphi(g) = 1$. Suppose that M is an oriented 3–manifold embedded in V that is Poincaré dual to γ.

Proposition 2.5 *Suppose that α has prime-power order. Then the quantity*

$$\rho_\alpha(V) = \rho_\alpha(M) \tag{9}$$

is independent of the choice of M.

Proof Let M and N be two such choices, and choose lifts of M and N, say M' and N', to the infinite cyclic cover $V_\gamma \to V$ of V corresponding to γ. Taking the image N' under a high positive power of the covering transformation, we may assume that M' and N' are disjoint, and that N' is to the right of M'. Composing φ with the maps induced by the projection π, we get that $\rho_\varphi(M) = \rho_{\varphi\circ\pi_*}(M')$, and similarly for N.

Now M' and N' cobound a compact submanifold W of V_γ, and $\varphi \circ \pi_*$ extends to $\pi_1(W)$. So by definition (cf (4)) we know

$$\rho_{\varphi\circ\pi_*}(N') - \rho_{\varphi\circ\pi_*}(M') = \operatorname{sign}(W) - \operatorname{sign}_{\varphi\circ\pi_*}(W).$$

We make a final appeal to Milnor's duality theorem: M' carries the fundamental class of V_γ, thought of as a space satisfying 3–dimensional Poincaré duality. In other words, the inclusion map of M' into V_γ is a degree one map, and hence induces a surjection in rational homology. It follows that the intersection form on $H_2(V_\gamma;\mathbb{Q})$ vanishes. The same is true for W, and so $\operatorname{sign}(W) = 0$. Similarly, Proposition 2.3 implies that the p^r–fold cover of V_γ coming from φ has vanishing intersection form. It follows that the twisted signature $\operatorname{sign}_{\varphi\circ\pi_*}(W)$ vanishes as well, so that

$$\rho_\varphi(N) = \rho_{\varphi\circ\pi_*}(N') = \rho_{\varphi\circ\pi_*}(M') = \rho_\varphi(M).$$ □

In applications to knot concordance [34] it is sometimes convenient to replace the Levine–Tristram invariant by its average value around points on the circle where it jumps. We record a similar definition for the torus version of the invariant. Choose a submanifold M as above and note that $\rho_\varphi(M)$ is defined for arbitrary $\alpha \in S^1$. As a function on the circle, it is piecewise constant, with finitely many jumps at those characters φ_{α_0} for which

$$\dim H^1(M;\mathbb{C}_{\varphi_{\alpha_0}}) > \lim_{\alpha\to\alpha_0^-} \dim H^1(M;\mathbb{C}_{\varphi_\alpha}) \quad \text{or} \quad \lim_{\alpha\to\alpha_0^+} \dim H^1(M;\mathbb{C}_{\varphi_\alpha}).$$

Informally, this means that the twisted cohomology jumps at α_0, and there are finitely many $\alpha_0 \in S^1$ where this happens. Let us normalize by defining $\bar\rho_\varphi(M)$ to be the average value of the one-sided limits as $\beta \to \alpha^\pm$. In other words,

$$\bar\rho_\varphi(M) = \tfrac{1}{2}\big(\lim_{\beta\to\alpha^-} \rho_{\varphi_\beta}(M) + \lim_{\beta\to\alpha^+} \rho_{\varphi_\beta}(M)\big).$$

It is again a piecewise-constant function on S^1 with finitely many jumps.

Since $\bar\rho$ is an average of integers, it is possible that it takes values in $\frac{1}{2}\mathbb{Z}$, rather than in $\mathbb{Z}$. The following lemma shows that it is in fact an integer.

Lemma 2.6 *Let Y be an oriented 3–manifold, $\eta \in H^1(Y)$, and φ_α be the $\mathrm{U}(1)$ representation of $\pi_1(Y)$ given by the composition $\eta\colon H_1(Y) \to \mathbb{Z} \to \mathrm{U}(1)$ where $1 \in \mathbb{Z}$ is sent to $\alpha \in \mathrm{U}(1)$. Then the parity of the invariant ρ_{φ_α} is constant off of a finite set. In particular, $\bar\rho_\alpha(Y)$ is an integer for all values of α.*

Chuck Livingston pointed out the idea used in the proof of diagonalizing over the ring of rational functions.

Proof Since the cobordism group $\Omega_3(\mathbb{Z})$ vanishes, we can choose a 4–manifold W over which η extends. After doing surgery on loops in $\ker[\pi_1(W) \to \mathbb{Z}]$ we may assume that in fact $\pi_1(W) \cong \mathbb{Z}$. It follows that all of the representations φ_α extend to W, and so it suffices to show that the parity of the signatures $\mathrm{sign}_{\varphi_\alpha}(W)$ that appear in the definition of ρ_{φ_α} is constant on a dense set of α.

The local coefficient homology group $H_2(W;\Lambda)$, corresponding to the homomorphism $\eta\colon H_1(W) \to \mathbb{Z}$, supports the equivariant intersection form q_Λ. It is Hermitian with respect to the involution on Λ given by $\bar t = t^{-1}$; see for instance [19; 55]. For $\alpha \in \mathrm{U}(1)$,

the form q_Λ determines the intersection form on $H_2(W;\mathbb{C}_\alpha)$ as follows. An application of the Künneth spectral sequence as in [15] shows that

$$H_2(W;\mathbb{C}_\alpha) \cong H_2(W;\Lambda) \otimes_\Lambda \mathbb{C},$$

where t acts on $\mathbb{C}$ by multiplication by α. This isomorphism respects intersections.

To compute signatures, at least for generic values of α (to be specified in a moment), we pass to the quotient field $\mathbb{Q}(t)$ of Λ, and then tensor with $\mathbb{R}$ to get a form $q_{\mathbb{R}(t)}$. The form $q_{\mathbb{R}(t)}$ can then be diagonalized [17; 25] in the following sense. With respect to a basis, represent $q_{\mathbb{R}(t)}$ by a matrix $B(t)$. Then there is an invertible matrix $A(t)$ with $AB(t)A^* = D(t)$, a diagonal matrix. Let the nonzero entries of $D(t)$ be rational functions $\lambda_1(t),\dots,\lambda_k(t)$.

As long as α is not a pole of any $\lambda_j(t)$ or of any entry in $A(t)$, then $D(\alpha)$ is a representative for $q_\Lambda \otimes C_\alpha$. Hence (with finitely many exceptions) $\operatorname{sign}_\alpha(W)$ is the number of positive values of $\lambda_j(\alpha)$ minus the number of negative ones. With the exception of the (finitely many) zeros of the $\lambda_j(t)$, this is given by $k \pmod 2$. □

The arguments in [17; 25] give more information about the location of the jumps in ρ_{φ_α} and their relation to the Alexander polynomial in the special case when Y is 0–surgery on a knot.

The extension of the ρ invariant of V to arbitrary characters is now a formal consequence of what has come before.

Definition 2.7 Let T be a torus embedded in X, a homology $S^1\times S^3$. Let V be the associated cohomology $T^2\times S^2$ as before, and let M be an oriented 3–manifold dual to γ. Then, for any $\alpha\in \mathrm{U}(1)$, define

$$\rho_\alpha(V) = \bar\rho_\varphi(M). \tag{10}$$

The final result of this section is then:

Theorem 2.8 *Let V be a cohomology $T^2\times S^2$, and let $\{g,h\}$ be a basis for $H_1(V)$. Then, for any $\alpha\in S^1$, the quantity $\rho_\alpha(V)$ in (10) is well defined.*

Proof To see that $\rho_\alpha(V,\gamma)$ is well defined, suppose that M and N are submanifolds of V dual to γ. Then $\rho_\varphi(M)$ and $\rho_\varphi(N)$ are both well-defined piecewise-continuous functions on S^1 with finitely many discontinuities. On the other hand, Proposition 2.5 says that they agree on the dense subset of points of prime-power order. Hence the normalized versions $\bar\rho_\varphi(M)$ and $\bar\rho_\varphi(N)$ must agree at all points. □

It is worth remarking that $\rho_\alpha(V,\gamma)$, while independent of the choice of 3–manifold M, may well depend on the choice of γ.

Example 2.9 Consider $V = S^1 \times S^3_0(K)$. Then, for γ pulled back from the circle, we will show below that $\rho_\alpha(V,\gamma)$ is just the Levine–Tristram invariant of K and can well be nonzero. On the other hand, if γ is the pullback of the generator of $H^1(S^3_0(K))$, then the dual to γ is represented by $S^1 \times F$, where F is a Seifert surface for K, capped off in the surgered manifold. But $S^1 \times F = \partial(S^1 \times H)$, where H is a handlebody. Since $H_2(S^1 \times F) \to H_2(S^1 \times H)$ is surjective, and the same is true for the twisted homology, both the ordinary and twisted signatures vanish. Hence $\rho_\alpha(V,\gamma) = 0$ for this choice of γ.

3 The Levine–Tristram invariant of a torus

As described in the introduction, the idea is to construct, from a torus $T \subset X$ where X is an oriented and homology oriented $S^1 \times S^3$, a cohomology $T^2 \times S^2$, say V. Then the Levine–Tristram invariant of (X,T) will be defined as the ρ invariant of V from the preceding section. A small complication is that we need to see that the homology orientation gives rise to a particular class $\gamma \in H^1(V)$. As observed in Example 2.9, the ρ invariant of V may well depend on this choice.

In brief, we choose a diffeomorphism $f : T^2 \times \partial D^2 \to \partial\nu$. Then the manifold V is defined to be

$$V_f = T^2 \times \partial D^2 \cup_f X - \mathrm{int}(\nu). \tag{11}$$

We refer to this operation as a torus surgery. A useful model to keep in mind is $(X,T) = S^1 \times (Y,K)$ with Y an integral homology sphere, and to have the gluing represent S^1 times an ordinary 0–framed surgery on K. This model is a bit oversimplified, as there are many gluing maps that will produce a homology $T^2 \times S^2$. Because many self-diffeomorphisms of a 3–torus extend over $T^2 \times D^2$, many of these manifolds are diffeomorphic. They are presumably not all diffeomorphic, but it will turn out that they all give rise to the same ρ invariants.

3.1 Torus surgery

Surgery along a torus is a standard operation [4; 13] in 4–manifold topology, and is thoroughly discussed in in [43]. We start with a copy of $T^2 \times D^2$, whose boundary is

parametrized as a product of circles $a \times b \times m$, where m is the boundary of D^2. One bit of notation for this section: we will use the same letter for a circle and the homology class that it carries.

The torus $T \subset X$ may be parametrized as $g \times l$, where g represents the preferred generator of $H_1(X)$ and l is null-homologous in X. Note that l is determined by this requirement, but g is only well defined up to replacing g by $g + nl$ for an arbitrary integer n. Pick a framing of $\nu(T)$, which is specified by a section ψ of the normal circle bundle. In particular, $\partial\nu(T)$ is now parametrized as $\psi(g) \times \psi(l) \times \mu_T$, where T is the (oriented) meridian circle. Note that there is an action of $H^1(T)$ on the set of choices for ψ. We can cut down this indeterminacy by choosing $\psi(l)$ to be null-homologous in the complement of T (or in other words to have linking number 0 with T). There does not seem to be a canonical choice for $\psi(g)$.

It is not hard to check, via the Mayer–Vietoris sequence, that V is a cohomology $T^2 \times S^2$ if and only if $f(m) = \psi(l)$, and we choose one such gluing map. We refer to any such choice as a 0–surgery on X along T. The next item to establish is that the cohomology class $\gamma \in H_1(X)$ naturally leads to a cohomology generator, also called γ, in $H^1(V)$. This can also be done by analyzing the Mayer–Vietoris sequence, but we prefer the following more geometric argument.

Choose a connected oriented submanifold Y of X that is Poincaré dual to γ and transverse to T. Transversality implies that $Y \cap K$ is an oriented link, which may be simpliefied, as follows.

Lemma 3.1 *One can choose Y so that $Y \cap T = K$, a knot with homology class l.*

Proof The intersection $Y \cap T$ is a union of oriented circles in T, and Poincaré duality says that the sum of their homology classes is l. Hence $Y \cap T$ consists of a collection of circles that bound disks, together with a collection of $2n + 1$ curves isotopic to l, with $n + 1$ having the same orientation as l, and n oriented the opposite way. By a standard innermost disk/annulus argument, one can do surgery on Y along the disks and an "annulus surgery" along pairs of oppositely oriented copies of l to remove all but one curve. □

A related argument is given in [37, Section 2.4].

Note that the normal bundle of K in Y is the restriction of the normal bundle of T in X to K. In particular, the (canonical) framing of l induces a framing of K. Let M

be the result of surgery on $K \subset Y$ with that framing. Because of our choice of framing, the surgery on Y can be done simultaneously with (and inside of) the torus surgery on X.

It follows that, no matter what gluing f we choose, M becomes a submanifold of V. Now we can define the cohomology class γ to be the Poincaré dual of M. Because M is connected and nonseparating, it follows that γ is a primitive class, and hence generates a summand of $H^1(V)$. Finally, we have our definition.

Definition 3.2 Let X be an oriented homology $S^1 \times S^3$, and let $\gamma \in H^1(X)$ be a homology orientation. Let V be any cohomology $T^2 \times S^2$ resulting from 0–surgery along T, and let $\gamma \in H^1(V)$ be the cohomology generator described above. Then, for $\alpha \in S^1$, define

$$\sigma_\alpha(X, T, \gamma) = \rho_\alpha(V, \gamma),$$

where $\rho_\alpha(V, \gamma)$ is defined in (10).

In using this definition, it is important to recall that for α of prime-power order, $\rho_\alpha(V, \gamma)$ refers to the ρ invariant of the 3–manifold M, whereas, for general α, it refers to the average of the one-sided limits.

With these preliminary results in hand, we can establish the main theorem.

Proof of Theorem A The discussion above shows that $\sigma_\alpha(X, T, \gamma)$ is well defined, and we have already verified in Lemma 2.6 that it is an integer. So we only need to verify the two properties stated in the theorem.

The first property (the computation for a product $S^1 \times (Y, K)$ with Y a homology sphere) is a consequence of an interpretation of the standard Levine–Tristram invariant of (Y, K) as a ρ invariant. This seems to be standard for knots in the 3–sphere — for α of finite order this is [5, Lemma 3.1], while, for arbitrary α, see for instance [14; 31; 35; 45]. The idea is that one can use the complement of a Seifert surface for K, pushed into B^4, to compute the ρ invariant of $Y_0(X)$. The key computation relates the twisted signatures of this complement to the Seifert form of K. For an arbitrary homology sphere Y, much the same proof works, with B^4 replaced by an arbitrary compact 4–manifold with boundary Y. We record the result here.

Proposition 3.3 *For any $\alpha \in S^1$, we have $\sigma_\alpha(Y, K) = \rho_\alpha(Y_0(K))$.*

Now consider a knot K in a homology sphere Y, so that the product $(S^1 \times Y, S^1 \times K)$ is an embedded essential torus. The cohomology class γ pulled back from the standard generator of $H^1(S^1)$ provides a homology orientation. Then we need to show that

$$\sigma_\alpha(S^1 \times Y, S^1 \times K) = \sigma_\alpha(Y, K).$$

Referring back to notation from Section 3.1, the curve $l \subset T$ is just a copy of K, and its framing (restricted to $T \cap Y$) is exactly the usual 0–framing of K in Y. Hence the manifold M in that section is just $Y_0(K)$, and the desired equation is just the definition of $\sigma_\alpha(S^1 \times Y, S^1 \times K)$ coupled with Proposition 3.3.

The second property is straightforward from the definitions. Changing either orientation will reverse the orientation of the manifold M, and this will change the sign of $\rho_\alpha(M)$ for all prime-power α. It follows that the averaged invariants $\bar{\rho}_\alpha(M)$ all change signs as well, which implies that $\sigma_\alpha(X, T)$ changes sign. □

The fact that for M given by 0–surgery on a knot, the ρ invariant used in our definition of $\sigma_\alpha(X, T)$ can be computed via the Seifert form of the knot has a generalization due to Neumann [40]. The key observation is that, for any M and α, the representation φ_α factors through a homomorphism $H_1(M) \to \mathbb{Z}$. Equivalently, φ_α comes from a cohomology class $\eta \in H^1(M; \mathbb{Z})$. The main result of [40] then interprets $\rho_{\varphi_\alpha}(M)$ in terms of an isometric structure (an algebraic construction using the cohomology of a surface dual to η). In the case of 0–surgery, such an isometric structure is equivalent [26; 29] (up to a cobordism relation, respecting signatures) to the Seifert matrix that defines the Levine–Tristram invariant.

4 Examples and comparison with $\lambda_{\mathbf{FO}}(X, T, \alpha)$

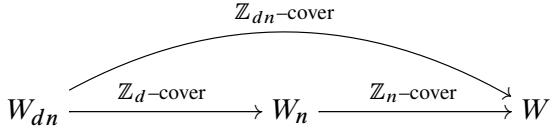
We compute the Levine–Tristram invariant for three constructions of embedded tori. The first two come from the mapping torus of the n–fold branched cover of a knot. In this mapping torus, we can consider the torus swept out by the fixed-point set of the covering transformation, or alternatively the torus swept out by an invariant knot disjoint from the fixed-point set. The third comes from taking a nontrivial circle bundle over a 3–manifold. Along with the product case, the first and last of these constructions were treated by Echeverria [11]. As verified more generally in [36], the answers we get agree with those in [11].

4.1 Branched covers: the branch set

The product case discussed in the proof of Theorem A has a nice generalization. Consider a knot K in a homology sphere Y. Then the n–twist spin [56] of K is a fibered knot in a homology 4–sphere. (The description of twist-spinning in Section 2 of [33] applies to a knot in an arbitrary homology sphere.) Surgery along this knot produces a homology $S^1 \times S^3$, which we will call X. It is fibered with fiber Σ, the n–fold branched cover of (Y, K), and the monodromy is a generator τ of the covering transformations. Let J be the preimage of K in Σ. It is the fixed-point set of τ and hence sweeps out a torus T in X. The twist-spinning construction is not strictly necessary here; one could instead pass directly to the definition of (X, T) as the (pairwise) mapping torus of τ on (Σ, J). In [11] this construction was denoted by (X_τ, T_τ).

It seems natural to expect that $\sigma_\alpha(X, T)$ should be expressible in terms of the Levine–Tristram invariants of (Y, K), and we show that this is indeed the case by establishing a relation between the ρ invariants of surgery on Y along K and those of surgery on Σ along J. The ideas here are well known; compare [2; 5]. Let Y_0 be the result of 0–framed surgery along K, and let Σ_0 be its n–fold cyclic cover. It is standard that $H_1(\Sigma_0)$ splits as $\mathbb{Z} \oplus H_1(\Sigma)$, where the first summand is generated by the meridian of J. Writing $d = p^r$ and $\alpha = e^{2\pi i/d}$, we then want to compute $\sigma_{\alpha^k}(X, T)$. By definition, this is a ρ invariant associated to a U(1) representation of $H_1(\Sigma_0)$ that vanishes on the second summand and takes the meridian of J to α^k.

Since $\Omega_3(\mathbb{Z}) = 0$, there is a 4–manifold W with $\partial W = Y_0$ and such that the inclusion map $H_1(Y_0) \to H_1(W)$ is an isomorphism. We will use W and its covering spaces to relate the various ρ invariants. Consider the tower of covering spaces



and let t denote the generator of the covering transformations of $W_{dn} \to W$. Then t^n generates the covering transformations of $W_{dn} \to W_n$. Write $\omega = e^{2\pi i/dn}$, so that $\alpha = \omega^n$. Denote by $E(t, \omega^j)$ the ω^j–eigenspace of t acting on $H_2(W_{dn}; \mathbb{C})$; it is the same as the twisted cohomology $H_2(W; \mathbb{C}_\varphi)$ where φ sends the generator of $H_1(W)$ to ω^{-j}. Using the symmetries of the Levine–Tristram signature [8, Proposition 2.3],

$$\sigma_{\omega^j}(Y, K) = \sigma_{\omega^{-j}}(Y, K) = \operatorname{sign}(W) - \operatorname{sign}(E(t, \omega^j)),$$

where the last term means the signature of the Hermitian intersection form on $E(t,\omega^j)$. There is a similar decomposition of $H_2(W_{dn};\mathbb{C})$ into eigenspaces $E(t^n,\omega^{nk})$ for the action of t^n.

Theorem 4.1 *For any k, the invariant $\sigma_{\alpha^k}(X,T)$ is a sum of Levine–Tristram invariants of K,*

$$\sigma_{\alpha^k}(X,T)=\sigma_{\alpha^k}(\Sigma,J)=-\sum_{j=1}^{n-1}\sigma_{\omega^{dj}}(Y,K)+\sum_{j=0}^{n-1}\sigma_{\omega^{dj+k}}(Y,K). \tag{12}$$

Proof The main point is the decomposition

$$E(t^n,\omega^{nk})=\bigoplus_{j=0}^{n-1}E(t,\omega^{dj+k}) \tag{13}$$

of $E(t^n,\omega^{nk})$ into eigenspaces of t. Since the different eigenspaces are orthogonal with respect to the intersection form, this translates into an identity on signatures. By definition of the Levine–Tristram signature,

$$\operatorname{sign}(E(t,\omega^{dj+k}))=\operatorname{sign}(W)-\sigma_{\omega^{dj+k}}(Y,K),$$

so the decomposition (13) thus gives

$$\begin{aligned}\sigma_{\alpha^k}(\Sigma,J)&=\operatorname{sign}(W_n)-\operatorname{sign}(E(t^n,\omega^{nk}))\\&=\operatorname{sign}(W_n)-n\operatorname{sign}(W)+\sum_{j=0}^{n-1}\sigma_{\omega^{dj+k}}(Y,K).\end{aligned}$$

Similarly, the decomposition of $H_2(W_n;\mathbb{C})$ into eigenspaces gives that

$$\begin{aligned}\operatorname{sign}(W_n)&=\operatorname{sign}(W)+\sum_{j=1}^{n-1}j^{\text{th}}\text{ eigenspace signature for }W_n\\&=n\operatorname{sign}(W)-\sum_{j=1}^{n-1}\sigma_{\omega^{dj}}(Y,K).\end{aligned}$$

Putting these together (note that the signature of W cancels, as expected) gives the statement of the theorem. □

Comparing this calculation with [11, Theorem 44], and accounting for the somewhat different notation, we see that the expected relation (3) holds.

4.2 Branched covers: periodic knots

To carry out the mapping torus construction above, all we needed was a knot in Σ that was invariant under a suitable transformation τ. Here is a more general version. Suppose that $L = (K, J)$ is an oriented link, and let $\pi : \Sigma \to S^3$ be the n–fold cover of S^3 branched along K, with τ the generator of group of covering transformations. Assuming that n is relatively prime to the linking number of J and K, the preimage $\tilde{J} = \pi^{-1}(J)$ is a knot in Σ that is invariant under τ.

Definition 4.2 The torus $T_{L,n}$ is given by the mapping torus

$$S^1 \times_\tau \tilde{J} \subset S^1 \times_\tau \Sigma = X.$$

By construction, $T_{L,n}$ is essential, and we calculate its Levine–Tristram invariant. To compute $\sigma_\alpha(X, T_{L,n})$ for arbitrary $\alpha \in S^1$, it suffices to compute it for all α of finite (indeed prime-power) order. The answer is similar to the formula in Theorem 4.1 and is expressed in terms of an extension to links of the Levine–Tristram invariant due to Cooper [9; 10]. This invariant associates to a 2–component oriented link and a pair of unit complex numbers $(\omega_1, \omega_2) \in (S^1 - \{1\}) \times (S^1 - \{1\})$ a signature $\sigma_L(\omega_1, \omega_2)$. The scope of σ_L was extended by Cimasoni and Florens [7] to include invariants of "colored links," with one ω for each color. Let us adopt the convention that $\sigma_L(\omega_1, 1)$ is the usual Levine–Tristram signature of the first component, and similarly for $\sigma_L(1, \omega_2)$.

Theorem 4.3 *If $\alpha \in S^1$ has prime-power order, then the invariant $\sigma_{\alpha^k}(X, T)$ is a sum of Levine–Tristram invariants of L,*

$$\sigma_\alpha(X, T_{L,n}) = \sum_{j=0}^{n-1} \sigma_L(e^{2\pi ij/n}, \alpha) - \sum_{j=1}^{n-1} \sigma_L(e^{2\pi ij/n}, 1). \tag{14}$$

Combined with Ma's theorem [36], we get a calculation of Echeverria's invariant $\lambda_{\mathrm{FO}}(X, T, \alpha)$ in this case. Again the answer is in terms of Levine–Tristram invariants of the link.

Corollary 4.4 *For $\alpha \in S^1$ of prime-power order d,*

$$\lambda_{\mathrm{FO}}(X, T_{L,n}, \alpha) = \sum_{j=0}^{n-1} \sigma_L(e^{2\pi ij/n}, \alpha^2).$$

To prove the corollary, recall that Ma showed that $\lambda_{\mathrm{FO}}(X,T,\alpha)=8\lambda_{\mathrm{FO}}(X)+\sigma_{\alpha^2}(X,T)$. On the other hand, the author and Saveliev [48] computed

$$\lambda_{\mathrm{FO}}(X)=\frac{1}{8}\sum_{j=1}^{n-1}\sigma_L(e^{2\pi ij/n},1),$$

which cancels the second term in (14).

Proof of Theorem 4.3 The proof is similar in spirit to that of Theorem 4.1, with some additional signature calculations. The details are basically an adaptation of well-known arguments, notably the "4–dimensional interpretation" of $\sigma_L(1,\omega_2)$ as discussed in [7, Section 6], and so we will be a bit sketchy in our treatment. To simplify notation a little, write T for the torus $T_{L,n}$.

Choose a pair of transversally intersecting surfaces F_K and F_J in B^4 with boundary L. The n–fold cyclic branched covering $\pi\colon\Sigma\to S^3$ over K extends to a branched cover $\pi\colon W_n\to B^4$ branched over F_K. Let $\tilde{J}$ (resp. $\widetilde{F}_J$) denote the preimage of J (resp. F_J). Denote by $\Sigma_0(\tilde{J})$ the 0–framed surgery along $\tilde{J}$ in Σ, where the 0–framing is the one that extends over $\widetilde{F}_J$. Note that $\Sigma_0(\tilde{J})$ embeds in V, the result of 0–surgery on T, and is Poincaré dual to $\gamma\in H^1(V)$, so that, by definition,

$$\sigma_\alpha(X,T)=\rho_{\varphi_\alpha}(\Sigma_0(\tilde{J})).$$

In principle, to compute this one would need a 4–manifold over which the d–fold cyclic covering of $\Sigma_0(\tilde{J})$ corresponding to the character φ_α extends. But a now-standard argument [5, Section 2] shows that it can be computed instead by extending to a branched covering. In addition to the equivariant signatures of the branched cover, there is a term proportional to the self-intersection of the branch set. Let W_n^0 denote the result of adding a 0–framed 2–handle to W_n along $\tilde{J}$. Then the covering corresponding to φ_α extends to a branched covering $W_{nd}^0\to W_n^0$ with branch set equal to the union of $\widetilde{F}_J$ with the core of the 2–handle.

By construction, the self-intersection of the branch set is 0. Hence $\rho_{\varphi_\alpha}(\Sigma_0(\tilde{J}))$ can be computed as the difference between the signature of W_n^0 and an appropriate equivariant signature of W_{nd}^0. It is straightforward to prove (from the choice of framing being 0) that the signature of W_n^0 is the same as the signature of W_n, and that the same is true for the equivariant signatures of W_{nd}^0 and W_{nd}, the d–fold cover of W branched along $\widetilde{F}_J$.

Decomposing the homology of W_n into eigenspaces for the action of the covering transformation from the branched covering $W_n \to B^4$ gives that

$$\operatorname{sign}(W_n) = \sum_{j=1}^{n-1} \sigma_L(e^{2\pi ij/n}, 1).$$

Similarly, W_{nd} is an nd–fold branched cover over B^4 with branch set $F_K \cup F_J$, so its homology has an action of $\mathbb{Z}_n \times \mathbb{Z}_d$. In particular, each eigenspace for the $\mathbb{Z}_d$ action is preserved by the $\mathbb{Z}_n$ action, and therefore has a further eigenspace decomposition. It follows that the equivariant signature of W_{nd} is given by

$$\sum_{j=0}^{n-1} \sigma_L(e^{2\pi ij/n}, \alpha),$$

and the theorem follows. □

4.3 Circle bundles

Some time ago, Baldridge suggested to the author and Saveliev the following construction of a homology $S^1 \times S^3$, which was discussed in [49]. Let N be a homology $S^1 \times S^2$, with a generator $\gamma_N \in H^1(N)$. Let $\pi : X \to N$ be the oriented circle bundle whose Euler class evaluates to 1 on the Poincaré dual of γ_Y. Then X is a homology $S^1 \times S^3$, and $\gamma = \pi^* \gamma_N$ is a homology orientation. In order for $\lambda_{\mathrm{FO}}(X)$ to be defined, N must have the $\mathbb{Z}[\mathbb{Z}]$ homology of $S^1 \times S^2$, and that restriction implies [49, Section 8] that in fact $\lambda_{\mathrm{FO}}(X) = 0$.

In this same situation, Echeverria constructs a torus embedded in X and conjectures (see [11, Conjecture 51]) that, for α for which it is defined, his invariant is 0. Here is a description that is equivalent to his, but easier for our purposes. Consider an oriented knot J in N such that $\langle \gamma_N, J \rangle = 1$. Then let $T = \pi^{-1}(J)$. This of course depends on the precise choice of J, but the calculation does not.

Proposition 4.5 *For any $\alpha \in S^1$, the invariant $\sigma_\alpha(X, T)$ vanishes.*

Proof To compute $\sigma_\alpha(X, T)$, we need a submanifold M dual to γ. Choose an oriented surface $F \subset N$ that is Poincaré dual to γ_N, and let $M = \pi^{-1}(N)$. It is readily verified that M is then Poincaré dual to γ.

By construction, M is the circle bundle over F with Euler class 1, and the Gysin sequence shows that its fibers are null-homologous in N (and of course in X as well). In particular, the fiber of $\pi : T \to J$ is the curve l appearing in the definition of $\sigma_\alpha(X, T)$. When we do a 0–surgery on T, the resulting surgery on M is therefore

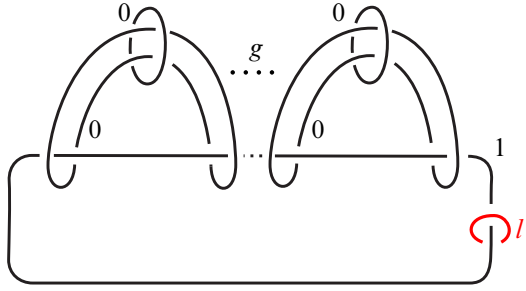


Figure 1: Fiber l in the Euler class 1 circle bundle over F_g.

the 0–framed surgery along J. The result of this surgery is just $S^1 \times F$. This may be verified by carefully thinking about the clutching function for the bundle $\pi : M \to F$, but some readers may prefer the handle picture presented in Figure 1. In either event, the ρ invariants for $S^1 \times F$ all vanish, because any representation φ_α will extend over $S^1 \times H$, where H is a handlebody.

To verify that surgery on M is a product, consider the standard picture [18] of the Euler class 1 circle bundle over F. If F has genus g, then the pattern is repeated g times; the red curve at the right of the picture is the fiber. The 0–framed pushoff of l is the one that has blackboard framing 1 in the picture, so we are doing surgery on l with coefficient 1. Blow down the red curve, changing the framing of the curve so that it links to 0, and the figure becomes the standard picture of $S^1 \times F$. □

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