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Learning to Teach Reasoning and Proof in an Online Setting: The Case of Nancy

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Preparing prospective secondary teachers (PSTs) to teach mathematics with a focus on reasoning and proving is an important goal for teacher education programs. A capstone course, Mathematical Reasoning and Proving for Secondary Teachers, *was designed to address this goal. One component of the course was a school-based experience in which the PSTs designed and taught four proof-oriented lessons in local schools, video recorded these lessons, and reflected on them. In this paper, we focus on one PST – Nancy, who took the course in Fall 2020 during the pandemic, when the school-based experience moved online. We analyzed how Nancy's Mathematical Knowledge for Teaching Proof (MKT-P) evolved through her attempts to teach proof online and through repeated cycles of reflection.*

Keywords: Reasoning and Proof, Prospective Secondary Teachers, Online Teaching, Reflection.

Supporting prospective teachers developing skills needed to teach mathematical reasoning and proof is an important goal of mathematics teacher preparation (Association of Mathematics Teacher Educators, 2017). However, there is limited theoretical or practical knowledge on how to provide prospective teachers with such support (Stylianides, Stylianides, & Weber, 2017). To address this knowledge gap, Buchbinder and McCrone (2018, 2020) developed a capstone course *Mathematical Reasoning and Proof for Secondary Teachers* and studied the development of PSTs' dispositions towards proof and Mathematical Knowledge for Teaching Proof (MKT-P), with a particular focus on the manifestations of MKT-P in classroom practices. Among other things, this involves an ability to plan and enact proof-oriented lessons.

The breakout of the global pandemic and the schools' pivoting to online teaching in 2020 put additional demands on PSTs who had to learn how to teach proof online "on the spot." We report on a case study of one such PST, Nancy (a pseudonym), who successfully surmounted these challenges. Rather than choosing a "representative" case, by focusing on a successful and articulate PST like Nancy and studying her learning processes and strategies, we hoped to gain insights that could inform the mathematics education community (Seawright & Gerring, 2008).

Theoretical Framing

Mathematical Knowledge for Teaching Proof

It has been suggested that in order to support students' learning of reasoning and proof, teachers need a special type of knowledge: Mathematical Knowledge for Teaching Proof (MKT-P). Several researchers (e.g., Lesseig, 2016; Stylianides, 2011; Steele & Rogers, 2012) proposed frameworks delineating the components of MKT-P and the relationships among them. Akin to Steele and Rogers's (2012) approach, Buchbinder and McCrone (2020) conceptualized MKT-P beyond declarative knowledge captured by written tests to include related classroom practices. Their resulting framework comprised three facets: Knowledge of the Logical Aspects of Proof (KLAP), Knowledge of Content and Students (KCS-P), and Knowledge of Content and Teaching (KCT-P). Each knowledge facet has its corresponding classroom practices. In the context of mathematics classrooms, KLAP corresponds to the use of precise mathematical language and notation adjusted to students' grade level, and the remediation of students' reasoning errors.

KCS-P corresponds to facilitating discussions that address students' common proof-related misconceptions and make proof concepts explicit. KCT-P corresponds to designing and enacting proof-oriented tasks. The three knowledge facets are interrelated, e.g., designing proof-related tasks (KCT-P) requires knowledge of students' proof-related conceptions (KCS-P) and proof-specific subject matter knowledge (KLAP). Distinctions between the MKT-P facets aided the design of learning experiences enhancing PSTs' MKT-P and the design of MKT-P assessments both in written form and through lesson planning and enacting (Buchbinder & McCrone, 2021).

Reflective Noticing

Teachers advance their professional expertise by reflecting on teaching (Seidel et al., 2011), which, in turn, entails teachers noticing elements of classroom environments that are most likely to support student learning (Sherin, Jacobs & Philipp, 2011). Although many definitions of noticing exist, we follow Stockero's (2021) definition of noticing as comprised of attending and interpreting. While noticing is often tacit, reflecting requires conscious engagement and processing. In this paper, we use the term *reflective noticing* to capture both of these processes.

There are several types of reflection according to timing: reflection-*in*-action, which occurs during teaching; reflection-*on*-action, which occurs after teaching (McDuffie, 2004; Schön, 1987), and reflection-*for*-action, which connects particular events to future actions (Jay & Johnson, 2002). Since merely descriptive, anecdotal, or non-critical accounts of teaching have little benefits for learning, researchers suggested that *productive* reflection entails attending to multiple aspects of classroom environments; interpreting, analyzing, and integrating them, and connecting them to theoretical principles, future actions, and past experiences (Moore-Russo & Wilsey, 2014). We take the last part of this definition - connecting to the past experiences –as yet another type of reflection: reflection-*back*.

Although PSTs often do not have access to classrooms, whenever possible, PSTs should be encouraged to reflect on their teaching (Jacobs et al., 2010; van Es, 2011). Learning to notice and analyze classroom instruction has been shown to benefit PSTs' professional development (Stockero, 2021). Similarly, Buchbinder et al. (2021) identified a variety of learning opportunities afforded by PSTs reflecting on their own teaching using 360° video technology.

In this study, we examine an overarching question: "How did Nancy's MKT-P evolve as a result of her planning, enacting, and reflecting on four proof-oriented lessons? We operationalize this question by examining:

1. How did Nancy integrate reasoning and proof in her planned and enacted lessons?
2. What did Nancy notice in the video recording of her lessons; how did she reflect on her lessons, and what learning was afforded by this?

Methods

This study is a part of the larger project that designed and studied the capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020). The course includes four modules, each focusing on one proof theme: (1) direct proof and argument evaluation (DP); (2) conditional statements (CS), (3) quantification and the role of examples in proving (RE), and (4) indirect reasoning (IR). Each module includes activities to help PSTs *crystalize* their subject matter knowledge of a particular proof theme, *connect* that knowledge to students' proof-related (mis)conceptions and to secondary curriculum, and *apply* that knowledge through a structured, school-based teaching experience. In this experience, the PSTs *plan* a 50-minute lesson on a particular proof theme, *enact* the lesson with a group of students from local schools, record the lesson using 360° cameras that capture both students and the PST, and *reflect* on the lesson. This process repeats four times during a semester, once for each proof theme. In Fall 2020, due to the pandemic, the PSTs taught their lessons online, via *Zoom*. Otherwise, the structure of the course remained the same.

The Case of Nancy

Nancy was a senior mathematics education major in a high school certification track. Prior to taking the capstone course, Nancy completed a prerequisite course on Mathematical Proof, a proof-writing-intensive Geometry course, and one Mathematics Education course. Nancy was a straight-A student in both mathematical and educational coursework. She regularly tutored undergraduate students taking entry-level calculus courses at the university tutoring center but had no classroom teaching experience. Like other PSTs in the course, Nancy completed the pre- and post- MKT-P questionnaire and Dispositions towards Proof survey (Buchbinder & McCrone, 2021). Her high scores on the pre- instruments indicated strong mathematical knowledge and positive dispositions towards proof. In addition, Nancy was very articulate and active during the class discussions, making her a strong case to study (Seawright & Gerring, 2008).

Nancy was placed in Ms. Meyer's high school geometry classroom. Due to the pandemic, a yearly geometry course was condensed to one semester, and Ms. Mayer relied on Nancy to plan lessons closely aligned with her curriculum. Ms. Meyer taught the whole class via *Zoom* for about 30 minutes and then divided students into two groups: one group remained with her, while another group (6 – 7 students) learned with Nancy. Ms. Meyer determined the geometric topic of the lesson, while the capstone course schedule dictated the proof theme.

Data Sources and Analytic Techniques

Data sources include four of Nancy's lesson plans, four Zoom video recordings of the lessons, and four reflection reports. To analyze planned and enacted lessons, we used analytic techniques developed in our prior research (Buchbinder & McCrone, 2020), outlined here briefly.

We analyzed the *lesson plans* by first noting the percent of time planned for reasoning and proof integration. We also ranked each lesson plan on a three-point scale (high, medium, or low) on three dimensions: (1) the extent to which the plan focused on the intended proof theme, (2) the alignment between the objectives and the tasks, (3) how appropriate was the choice of technology for proof integration.

The videos of the *enacted lessons* were analyzed using the *Lesson Enactment Rubric* (Buchbinder & McCrone, 2020), aligned with the MKT-P framework. The rubric has three dimensions: quality of proof-specific language (KLAP), making the proof theme explicit to students (KCS-P), and actions for promoting student engagement with proof (KCT-P). Each lesson was ranked on a three-point scale (high, medium, low) on each of these three dimensions.

When completing the *reflection reports,* the PSTs watched the video on *Canvas* Learning Management System and used the commenting feature to write reflective comments for every 5-minutes, about 8-9 total comments per lesson. Using open coding (Strauss & Corbin, 1994) in conjunction with the noticing literature (e.g., Stockero, 2021; van Es, 2011), we identified four main categories of noticing in Nancy's comments. These are (1) *instructional decisions*, e.g., 'One teaching move that I liked during this lesson was creating a theme for the lesson."; (2) *student engagement*, e.g., "I was impressed that one of the students was able to see that"; (3) *technology*, e.g., "I think the transition from the *Prezi* presentation to *GeoGebr*a was pretty smooth," and (4) *time*, e.g., "I felt pretty crunched for time." We also coded whether Nancy reflected on-, for-, or back- on her teaching (see examples in the Results section).

Results

Abbreviated Summaries of Nancy's Proof-Oriented Lesson Plans

Lesson 1. *Direct proof and argument evaluation: Supplementary and Vertical angles.* Initial group discussion: "what makes a good two-column proof"? Expected responses: generality, mathematical correctness, but can follow different paths. Teacher-led exploration of Vertical angles theorem using *GeoGebra*; first with specific angle values, followed by a generalization. Students contribute ideas as the teacher writes the proof.

Lesson 2: *Conditional statements: Isosceles and Equilateral Triangles***.** Introduction of a conditional statement and its converse and using mathematical examples (*Prezi*). Student-led exploration in *GeoGebra* of three conditional statements about triangles. For each statement, students first determine if it is true or false (if false, construct a counterexample), then write a converse and determine whether the converse is true or false.

Lesson 3: *Quantification and the role of examples: Triangle Similarity Theorems.* Introduction of universal and existential statements; the role of examples in proving/disproving universal statements. A reminder of the three similarity theorems. Students work on one SSS similarity proof by typing their work on an individual slide in a shared *Google Slides* document. Next, students find counterexamples to three universal statements (e.g., All isosceles triangles are similar) and find confirming examples proving two existential statements (e.g., There exist two right triangles that are similar).

Lesson 4: *Indirect reasoning: Coordinate proofs.* The lesson is structured as a game, "The Quadrilateral Detective," where students use distance and slope formulas to determine what type of quadrilateral is given by a set of four coordinates. Next, students create two statements of the form "This quadrilateral cannot be ___, because otherwise ___," e.g., "The quadrilateral cannot be a kite because otherwise it would have no pairs of parallel sides." After one teacher-led example, students work individually; write their proofs on paper, take a picture and paste it into a shared *Google Slides* document. Indirect reasoning is defined during the lesson summary as "a type of reasoning that shows that something is impossible since it leads to a contradiction."

As the description above shows, Nancy succeeded in integrating each of the four proof themes with the mathematical topics requested by Ms. Meyer in creative and engaging ways. Each lesson had 3-4 objectives, all focused on reasoning and proof, e.g., "Students will come up with counterexamples to disprove mathematical statements," and additional objectives related to student engagement in mathematical discussions. The tasks were closely aligned with the proof theme and with the mathematical content of the lesson. The lesson plans were written in a high level of detail. The percent of time planned for reasoning and proof in each lesson was above 67%, while the rest of the time was devoted to icebreakers in the beginning of the lesson and exit tickets at the end. Nancy's plans used a variety of technological tools: *GeoGebra, Prezi, Google Slides* to facilitate active student engagement.

Nancy's Enactment of the Four Proof-Oriented Lessons

Nancy's *enacted lessons* closely matched the planned ones in terms of content, but not the time. Nancy's lessons were planned for 30 minutes, but since the students had no other class afterward, they stayed between 34 to 56 minutes, allowing Nancy to finish all the planned activities. Table 1 summarizes the analysis of the enacted lessons. Throughout the lessons, Nancy used precise mathematical language and proof-specific vocabulary to discuss proof themes with the students. She used appropriate visual, symbolic, and verbal methods to make the main concepts and key ideas of the proof themes explicit to students. The percent of lesson time devoted to reasoning and proof was very high (above 80%), although the time devoted to the proof themes differed between the four lessons. The two proof themes: Role of Examples and Indirect Reasoning received much less class time and were less of the focal point of the lesson, which is consistent with the lesson plans. Nevertheless, Nancy still explicitly addressed these proof themes in her lessons. The first two enacted lessons - Direct Proof and Conditional Statements - were highly successful on almost all dimensions of the *Lesson Enactment Rubric*.

Table 1. Analysis of Nancy's Four Proof-oriented Enacted Lessons

Dimension on the *Lesson Enactment Rubric*	Lesson			
	1DP	2CS	3RE	4IR
Quality of proof-specific language (KLAP)	High	High	High	Medium
Explicating specific proof-theme (KCS-P)	High	High	Medium	Medium
Actions to promote student engagement (KCT-P)	Medium	High	High	High
Percent of time devoted to proof (KCT-P / KCS-P)	83%	84%	91%	82%
Percent of time devoted to the proof theme	83%	84%	30%	25%

Nancy's Reflective Noticing on the Enacted Lessons

Figure 1 shows the distribution of Nancy's categories on noticing in percent of the total number of comments, which varied by lesson. In the first lesson, Nancy mainly reflected on how her instructional decisions affected student engagement. In lessons two and three, the focus of reflection shifted away from instructional decisions towards technology and time management. This was when Nancy started having students themselves interact with technology (*GeoGebra*, *Google Slides*) providing space for productive struggle and free exploration. By doing so, Nancy had less control over the time spent on each planned activity and thus encountered challenges in coordinating technology, content, and time. The Indirect Reasoning lesson was most challenging for Nancy to enact, as evident in the percent of reflective comments focused on her instruction.

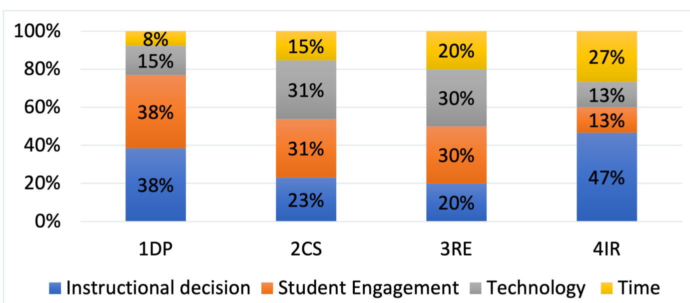

Figure 1. Distribution of Nancy's categories of noticing across four lessons

Table 2 shows the distribution of Nancy's reflective comments along with the four categories of noticing and the three types of reflection: *on, for* and *back*.

Table 2. Distribution of Nancy's categories of noticing and types of reflection

Categories of Noticing	Reflection on	Reflection for	Reflection back	Total
Instructional decisions	11	5	1	17
Student Engagement	6	6	2	14
Technology	6	4	1	11
Time	5	2	2	9
Total	28	17	6	51

Table 2 shows that the total number of Nancy's reflections-*on* (only) was much larger than the number of reflections-*for*, which, in turn, was larger than the total number of reflections-*back*. Most of Nancy's reflections-*on* focused on her instructional decisions. When reflecting-*for,* Nancy focused almost exclusively on instruction, students, and technology; when reflecting-*back* Nancy attended to all four categories.

The following excerpt illustrates the three types of reflection in Nancy's comments. In lesson #1, Nancy attempted to develop a set of criteria for a "good" proof with the students.

Nancy: What do you need [in order] to make our proof a good proof?

S1: Uhm, you need a proof.

Nancy: Yeah, so, we need a proof, right? So we need to be able to show that we can start with our given and then come up with our solution, right? So what are the parts are there in the proof? So we have two parts, right? So what are they to make up the two parts or the two columns?

When reflecting-*on* this exchange while watching the video, Nancy wrote: "one student said that you need a proof, when asked what makes up a "good" proof. This wasn't exactly what I was looking for. It was too general, but I tried to guide her response and make it more specific". Nancy followed by a comment reflecting-*for* the future: "What I should have done was ask the student directly and say something like "I like that idea, S1, what do you mean by that?" Then it would give her a chance to elaborate." In lesson #2, Nancy had a chance to act on her intentions and reflect-*back* on her improvement. She wrote: "I asked them what bisecting means. This is important because I wanted to make sure that they understood what the conditional statement was saying. […] I liked this part is because I asked this question as a follow up to a student's answer and that was something that I had mentioned wanting to work on after last time".

This progression shows how the three types of Nancy's reflections played out to support her gradual improvement of responding to student inputs and leading discussions about proof. Figure 2 shows the distribution of different types of Nancy's reflections in each lesson.

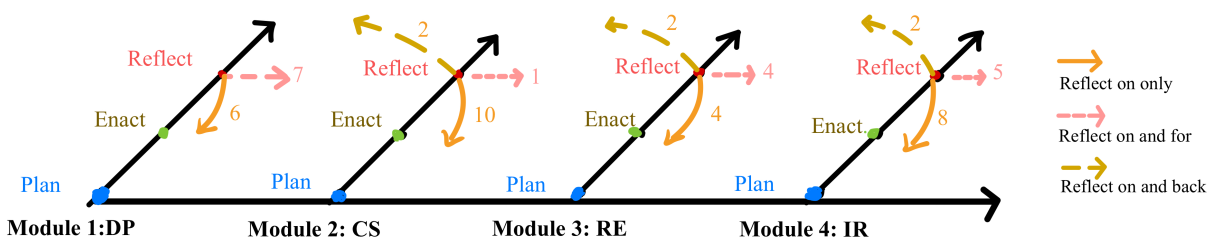

Figure 2. *Distribution of Nancy's types of reflection across four lessons*

Not surprisingly, the majority of Nancy's reflective comments were on the particular lesson she taught. Nancy learned from the teaching experience by making decisions on how to improve her teaching (reflecting-*for*) and kept herself accountable for these changes by reflecting-*back*. These processes supported Nancy's learning through reflecting on her own teaching practices.

Discussion

This paper examined the case of Nancy – a PSTs with a strong mathematical background and positive dispositions towards proof – as she progressed through a capstone course *Mathematical Reasoning and Proof for Secondary Teachers*. We attempted to trace how Nancy's MKT-P classroom practices evolved throughout the teaching experience component of the course. In particular, we focused on Nancy's ability to plan and enact proof-oriented lessons. Our analysis revealed that Nancy integrated the four proof themes with the topics from a regular geometry curriculum using creative and engaging activities, as evidenced in the description of her lesson plans. This is a non-trivial accomplishment, especially since the choices of the mathematical topics and the proof themes were outside of Nancy's control, and due to the shift online.

The analysis above shows that the Role of Examples and Indirect Reasoning proof-themes were most challenging for Nancy to integrate into the lesson plans and to enact (Table 1). This outcome concurs with the results of the previous study (Buchbinder & McCrone, 2020). The fact that the Indirect Reasoning lesson occurs towards the end of the course is probably a contributing factor, especially when the natural end-of-semester fatigue is exacerbated by the pandemic.

Not surprisingly, Nancy's enacted lessons differed from the planned ones (Stein, Remillard & Smith, 2007). When judging the enacted lessons, it is important to note that Nancy's educational coursework did not prepare her to teach online – she had to come up with teaching strategies and technological tools that were new to her and to the students. This willingness to take pedagogical risks makes Nancy's teaching performance even more impressive.

While reflecting on the video of her lessons, Nancy noticed four main aspects: instructional moves, student engagement, timing, and technology. The first three categories of noticing are consistent with those of Sherin and van Es (2005) and Stockero (2021). Nancy's focus on technology is understandable due to the unusual and unfamiliar circumstances of online teaching. As Nancy became more comfortable with the students and her own teaching, she tried new teaching approaches, e.g., having students explore conjectures in *GeoGebra* and naturally encountered new challenges, causing her noticing to shift towards technology and time.

Further examination revealed that Nancy used three types of reflections: *on* a particular lesson, *for* future practice, and *back* to past lessons. Collectively, these three types of reflection are characteristic of productive reflection (Jay & Johnson, 2002; Moore-Russo & Wilsey, 2014). Indeed, Nancy went through repeated cycles of identifying areas for improvement (reflect-*on*), devising a course of action (reflect-*for*), and checking her progress with respect to previous lessons (reflecting-*back*) (Figure 2). Developing purposeful and explicit reflective practices allowed Nancy to leverage her challenges into learning opportunities (Tekkumru-Kisa et al., 2020) and helped her to learn how to learn from teaching (Hiebert, Morris, & Glass, 2003).

Interpreting Nancy's teaching performance in terms of the MKT-P framework suggests that her areas of strength were the quality of proof-oriented language (KLAP) and actions for promoting student engagement (KCT) (Table 1). The areas of improvement, which Nancy reflected –*on*, *for,* and *back* were facilitating discussions (KCS) and using productive teaching moves (KCT), as evidenced in the data excerpt.

Nancy's case serves as a "proof of existence" that it is possible to support PSTs learning teach reasoning and proof through a structured educational experience of the kind provided by the capstone course described above. Nancy's challenges are not unique and sometimes are even more pronounced with less advanced PSTs (Buchbinder and McCrone, 2020). What we found unique and enlightening is how Nancy addressed those challenges and how her reflective practices seem to support her professional growth. Teacher educators can model and promote the use of such reflective practices to support PSTs' professional growth in other teacher preparation programs.

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References

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics.* Available online at amte.net/standards.

Buchbinder, O., Brisard, S., Butler, R., & McCrone, S. (2021). Preservice secondary mathematics teachers' reflective noticing from 360-degree video recordings of their own teaching. *Journal of Technology and Teacher Education, 29*(3), 279-308.

Buchbinder, O., & McCrone, S. (2020). Preservice teachers learning to teach proof through classroom implementation: Successes and challenges. *The Journal of Mathematical Behavior*, 58, 100779.

Buchbinder, O. & McCrone, S. (2021). Characterizing mathematics teachers' proof-specific knowledge, dispositions and classroom practices. Paper presented at the *ICME 14th International Congress on Mathematical Education, Shanghai, 2020.*

Hiebert, J., Morris, A. K., & Glass, B. (2003). Learning to learn to teach: An "experiment" model for teaching and teacher preparation in mathematics. *Journal of Mathematics Teacher Education, 6*(3), 201-222.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202.

Jay, J. K., & Johnson, K. L. (2002). Capturing complexity: A typology of reflective practice for teacher education. *Teaching and teacher education*, *18*(1), 73-85.

Lesseig, K. (2016). Investigating mathematical knowledge for teaching proof in professional development. *International Journal of Research in Education and Science*, *2*(2), 253–270. https://doi.org/10.21890/ijres.13913

McDuffie, A. R. (2004). Mathematics teaching as a deliberate practice: An investigation of elementary pre-service teachers' reflective thinking during student teaching. *Journal of mathematics teacher education*, *7*(1), 33-61.

Moore-Russo, D. A., & Wilsey, J. N. (2014). Delving into the meaning of productive reflection: A study of future teachers' reflections on representations of teaching. *Teaching and Teacher Education*, *37*, 76-90.Schön, D. (1987). *Educating the reflective practitioner.* San Francisco: Jossey-Bass.

Seidel, T., Stürmer, K., Blomberg, G., Kobarg, M., & Schwindt, K. (2011). Teacher learning from analysis of videotaped classroom situations: Does it make a difference whether teachers observe their own teaching or that of others?. *Teaching and Teacher Education*, *27*(2), 259-267. https://doi.org/10.1016/j.tate.2010.08.009

Seawright, J., & Gerring, J. (2008). Case selection techniques in case study research: A menu of qualitative and quantitative options. *Political research quarterly*, *61*(2), 294-308. https://doi.org/10.1177/1065912907313077

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (2011). Situating the study of teacher noticing. *In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), Mathematics teacher noticing: Seeing through teachers eyes* (pp. 3–14). London: Routledge.

Sherin, M., & van Es, E. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education*, *13*(3), 475-491.

Steele, M. D., & Rogers, K. C. (2012). Relationships between mathematical knowledge for teaching and teaching practice: The case of proof. *Journal of Mathematics Teacher Education*, *15*(2), 159–180. https://doi.org/10.1007/s10857-012-9204-5.

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319-370). Charlotte, NC: Information Age Publishing.

Stockero, S. L. (2021). Transferability of teacher noticing. *ZDM–Mathematics Education*, *53*(1), 73-84.
Strauss, A., & Corbin, J. (1994). Grounded theory methodology: An overview. In N. K. Denzin, & Y. S. Lincoln (Eds.). *Handbook of qualitative research* (pp. 273–285). Thousand Oaks: Sage Publications.
Stylianides, A. J. (2011). Towards a comprehensive knowledge package for teaching proof: A focus on the misconception that empirical arguments are proofs. *Pythagoras*, *32*(1), 10.
Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for research in mathematics education.* pp. (237–266). National Council of Teachers of Mathematics.
Tekkumru-Kisa, M., Stein, M. K., & Doyle, W. (2020). Theory and research on tasks revisited: Task as a context for students' thinking in the era of ambitious reforms in mathematics and science. *Educational Researcher*, *49*(8), 606-617.
van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes*. New York: Routledge