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EDITED BY
Oliver Allanson,
University of Exeter, United Kingdom

REVIEWED BY
Anton Artemyev,
University of California, Los Angeles,
United States

*CORRESPONDENCE
Alain J. Brizard,
abrizard@smcvt.edu

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Hamiltonian formulations of quasilinear theory for magnetized plasmas

Alain J. Brizard[1]* and Anthony A. Chan[2]

[1]Department of Physics, Saint Michael's College, Colchester, VT, United States, [2]Department of Physics and Astronomy, Rice University, Houston, TX, United States

Hamiltonian formulations of quasilinear theory are presented for the cases of uniform and nonuniform magnetized plasmas. First, the standard quasilinear theory of Kennel and Engelmann (Kennel, Phys. Fluids, 1966, 9, 2377) is reviewed and reinterpreted in terms of a general Hamiltonian formulation. Within this Hamiltonian representation, we present the transition from two-dimensional quasilinear diffusion in a spatially uniform magnetized background plasma to three-dimensional quasilinear diffusion in a spatially nonuniform magnetized background plasma based on our previous work (Brizard and Chan, Phys. Plasmas, 2001, 8, 4762–4771; Brizard and Chan, Phys. Plasmas, 2004, 11, 4220–4229). The resulting quasilinear theory for nonuniform magnetized plasmas yields a 3 × 3 diffusion tensor that naturally incorporates quasilinear radial diffusion as well as its synergistic connections to diffusion in two-dimensional invariant velocity space (e.g., energy and pitch angle).

KEYWORDS

quasilinear theory, guiding-center approximation, wave-particle resonance, Hamiltonian formulation, action-angle coordinates

1 Introduction

The complex interaction between charged particles and electromagnetic-field wave fluctuations in a magnetized plasma represents a formidable problem with crucial implications toward our understanding of magnetic confinement in laboratory and space plasmas (Kaufman and Cohen, 2019). These wave-particle interactions can be described either linearly, quasi-linearly, or nonlinearly, depending on how the background plasma is affected by the fluctuating wave fields and the level of plasma turbulence associated with them (Davidson, 1972).

In linear plasma wave theory (Stix, 1992), where the field fluctuations are arbitrarily small, the linearized perturbed Vlasov distribution of each charged-particle species describes the charged-particle response to the presence of small-amplitude electromagnetic waves which, when coupled to the linearized Maxwell wave equations, yields a wave spectrum that is supported by the uniform background magnetized plasma (Stix, 1992).

In weak plasma turbulence theory (Sagdeev and Galeev, 1969; Galeev and Sagdeev, 1983), the background plasma is considered weakly unstable so that a (possibly discrete) spectrum of field perturbations grow to finite but small amplitudes. While these small-amplitude fluctuations interact weakly among themselves, they interact strongly with resonant particles, which satisfy a wave-particle resonance condition in particle phase space (described in terms of unperturbed particle orbits). These resonant wave-particle interactions, in turn, lead to a quasilinear modification of the background Vlasov distribution on a long time scale compared to the fluctuation time scale (Kaufman, 1972a; Dewar, 1973).

Lastly, in strong plasma turbulence theory (Dupree, 1966), nonlinear wave-wave and wave-particle-wave interactions cannot be neglected, and wave-particle resonances include perturbed particle orbits (Galeev and Sagdeev, 1983). The reader is referred to a pedagogical review by Krommes (Krommes, 2002) on the theoretical foundations of plasma turbulence as well as a recent study on the validity of quasilinear theory (Crews and Shumlak, 2022). In addition, the mathematical foundations of quasilinear theory for inhomogeneous plasma can be found in the recent work by Dodin (Dodin, 2022).

1.1 Motivation for this work

The primary purpose of the present paper is to present complementary views of two-dimensional quasilinear diffusion in a uniform magnetized plasma. First, we review the quasilinear theory derived by Kennel and Engelmann (Kennel and Engelmann, 1966), which represents the paradigm formulation upon which many subsequent quasilinear formulations are derived (Stix, 1992). (We mainly focus our attention on non-relativistic quasilinear theory in the text and summarize the extension to relativistic quasilinear theory in Supplementary Appendix A) As an alternative formulation of quasilinear theory, we present a Hamiltonian formulation that relies on the use of guiding-center theory for a uniform magnetic field (Cary and Brizard, 2009). In this Hamiltonian formulation, the quasilinear diffusion equation is described in terms of a diffusion tensor whose structure is naturally generalized to three-dimensional quasilinear diffusion in a nonuniform magnetized plasma, as shown in the works of Brizard and Chan (Brizard and Chan, 2001; Brizard and Chan, 2004).

Next, two formulations of three-dimensional quasilinear theory are be presented. First, we present a generic quasilinear formulation based on the action-angle formalism (Kaufman, 1972b; Mahajan and Chen, 1985), which applies to general magnetic-field geometries. This formulation is useful in highlighting the modular features of the quasilinear diffusion tensor. Our second three-dimensional quasilinear formulation is developed for the case of an axisymmetric magnetic field $\mathbf{B}_0 = \nabla\psi \times \nabla\varphi$, for which the drift action $\mathsf{J}_\mathrm{d} = q\psi/c$ is expressed simply in terms of the magnetic flux ψ. The presentation of this case is based on a summary of the non-relativistic limit of our previous work (Brizard and Chan, 2004).

1.2 Notation for quasilinear theory in a uniform magnetized plasma

In a homogeneous magnetic field $\mathbf{B}_0 = B_0\,\hat{\mathsf{z}}$, the unperturbed Vlasov distribution $f_0(\mathbf{v})$ (for a charged-particle species with charge q and mass M) is a function of velocity $\mathbf{v}$ alone and the perturbed Vlasov-Maxwell fields (δf, $\delta\mathbf{E}$, $\delta\mathbf{B}$) can be decomposed in terms of Fourier components: $\delta f = \delta\tilde{f}(\mathbf{v})\,\exp(i\vartheta) + \text{c.c.}$ and $(\delta\mathbf{E},\delta\mathbf{B}) = (\delta\tilde{\mathbf{E}},\delta\tilde{\mathbf{B}})\,\exp(i\vartheta) + \text{c.c.}$, where the wave phase is $\vartheta(\mathbf{x}, t) = \mathbf{k}\cdot\mathbf{x} - \omega\, t$ and the dependence of the eikonal (Fourier) amplitudes $(\delta\tilde{f},\delta\tilde{\mathbf{E}},\delta\tilde{\mathbf{B}})$ on ($\mathbf{k}$, ω), which is denoted by a tilde, is hidden. According to Faraday's law, we find $\delta\tilde{\mathbf{B}} = (\mathbf{k}c/\omega)\times\delta\tilde{\mathbf{E}}$, which implies $\mathbf{k}\cdot\delta\tilde{\mathbf{B}} = 0$. For the time being, however, we will keep the perturbed electric and magnetic fields separate, and assume that the uniform background plasma is perturbed by a monochromatic wave with definite wave vector $\mathbf{k}$ and wave frequency ω.

Following the notation used by Kennel and Engelmann (Kennel and Engelmann, 1966), the velocity $\mathbf{v}$ and wave vector $\mathbf{k}$ are decomposed in terms of cylindrical components

$$\left.\begin{aligned}\mathbf{v} &= v_\parallel\,\hat{\mathsf{z}} + v_\perp\left(\cos\phi\,\hat{\mathsf{x}} + \sin\phi\,\hat{\mathsf{y}}\right)\\ \mathbf{k} &= k_\parallel\,\hat{\mathsf{z}} + k_\perp\left(\cos\psi\,\hat{\mathsf{x}} + \sin\psi\,\hat{\mathsf{y}}\right)\end{aligned}\right\}, \tag{1.1}$$

so that $\mathbf{k}\cdot\mathbf{v} = k_\parallel v_\parallel + k_\perp v_\perp\cos(\phi-\psi)$, where ϕ is the gyroangle phase and ψ is the wave-vector phase. We note that the unperturbed Vlasov equation $\partial f_0/\partial\phi = 0$ implies that $f_0(\mathbf{v})$ is independent of the gyroangle ϕ, i.e., $f_0(v_\parallel, v_\perp)$. In what follows, we will use the definition

$$\begin{aligned}\frac{\mathbf{k}_\perp}{k_\perp} &= \cos\psi\,\hat{\mathsf{x}} + \sin\psi\,\hat{\mathsf{y}} = \frac{1}{2}\,e^{i\psi}\left(\hat{\mathsf{x}} - i\,\hat{\mathsf{y}}\right) + \frac{1}{2}\,e^{-i\psi}\left(\hat{\mathsf{x}} + i\,\hat{\mathsf{y}}\right)\\ &\equiv \frac{1}{\sqrt{2}}\left(\hat{\mathsf{K}} + \hat{\mathsf{K}}^*\right),\end{aligned} \tag{1.2}$$

and the identity

$$\frac{\mathbf{v}_\perp}{v_\perp} \equiv \hat{\perp} = \cos\phi\,\hat{\mathsf{x}} + \sin\phi\,\hat{\mathsf{y}} \equiv e^{i(\phi-\psi)}\hat{\mathsf{K}}\Big/\sqrt{2} + e^{-i(\phi-\psi)}\hat{\mathsf{K}}^*\Big/\sqrt{2}. \tag{1.3}$$

We note that, in the work of Kennel and Engelmann (Kennel and Engelmann, 1966), the right-handed polarized electric field is $\delta\tilde{E}_R \equiv \delta\tilde{\mathbf{E}}\cdot\hat{\mathsf{K}}\,e^{-i\psi}$ and the left-handed polarized electric field is $\delta\tilde{E}_L \equiv \delta\tilde{\mathbf{E}}\cdot\hat{\mathsf{K}}^*\,e^{i\psi}$; we will refrain from using these components in the present work.

2 Kennel-Engelmann quasilinear diffusion equation

In this Section, we review the quasilinear theory presented by Kennel and Engelmann (Kennel and Engelmann, 1966) for the case of a uniform magnetized plasma. Here, we make several changes in notation from Kennel and Engelmann's work in preparation for an alternative formulation presented in Section 3.

2.1 First-order perturbed Vlasov equation

The linearized perturbed Vlasov equation is expressed in terms of the first-order differential equation for the eikonal amplitude $\delta\tilde{f}(\mathbf{v})$:

$$\begin{aligned}-i(\omega-\mathbf{k}\cdot\mathbf{v})\delta\tilde{f}-\Omega\frac{\partial\delta\tilde{f}}{\partial\phi} &\equiv -\Omega\,e^{i\Theta}\frac{\partial}{\partial\phi}\left(e^{-i\Theta}\,\delta\tilde{f}\right) \\ &= -\frac{q}{M}\left(\delta\tilde{\mathbf{E}}+\frac{\mathbf{v}}{c}\times\delta\tilde{\mathbf{B}}\right)\cdot\frac{\partial f_0}{\partial\mathbf{v}}\end{aligned} \tag{2.1}$$

where $\Omega = qB_0/(Mc)$ denotes the (signed) gyrofrequency and the solution of the integrating factor $\partial\Theta/\partial\phi \equiv \Omega^{-1}d\vartheta/dt = (\mathbf{k}\cdot\mathbf{v}-\omega)/\Omega$ yields

$$\begin{aligned}\Theta(\phi) &= \left(\frac{k_\parallel v_\parallel-\omega}{\Omega}\right)\phi+\frac{k_\perp v_\perp}{\Omega}\,\sin(\phi-\psi) \\ &\equiv \varphi(\phi)+\lambda\,\sin(\phi-\psi),\end{aligned} \tag{2.2}$$

where $\lambda = k_\perp v_\perp/\Omega$. The perturbed Vlasov Eq. 2.1 is easily solved as

$$\delta\tilde{f}(\mathbf{v}) = \frac{q\,e^{i\Theta}}{M\Omega}\int^{\phi}e^{-i\Theta'}\left(\delta\tilde{\mathbf{E}}+\frac{\mathbf{v}'}{c}\times\delta\tilde{\mathbf{B}}\right)\cdot\frac{\partial f_0}{\partial\mathbf{v}'}\,d\phi', \tag{2.3}$$

where a prime denotes a dependence on the integration gyroangle ϕ'. Here, we can write the perturbed evolution operator

$$\frac{q}{M\Omega}\left(\delta\tilde{\mathbf{E}}+\frac{\mathbf{v}}{c}\times\delta\tilde{\mathbf{B}}\right)\cdot\frac{\partial}{\partial\mathbf{v}} \equiv \delta\tilde{V}_\parallel\frac{\partial}{\partial v_\parallel}+\delta\tilde{V}_\perp\frac{\partial}{\partial v_\perp}+\delta\tilde{\phi}\,\frac{\partial}{\partial\phi}, \tag{2.4}$$

which is expressed in terms of the velocity-space eikonal amplitudes.

$$\delta\tilde{V}_\parallel = \frac{q}{M\Omega}\left(\delta\tilde{\mathbf{E}}+\frac{\mathbf{v}_\perp}{c}\times\delta\tilde{\mathbf{B}}\right)\cdot\hat{\mathsf{z}}, \tag{2.5}$$

$$\delta\tilde{V}_\perp = \frac{q}{M\Omega}\left(\delta\tilde{\mathbf{E}}+\frac{v_\parallel\hat{\mathsf{z}}}{c}\times\delta\tilde{\mathbf{B}}\right)\cdot\hat{\perp}, \tag{2.6}$$

$$\delta\tilde{\phi} = \frac{q}{M\Omega}\left(\delta\tilde{\mathbf{E}}+\frac{v_\parallel\hat{\mathsf{z}}}{c}\times\delta\tilde{\mathbf{B}}\right)\cdot\frac{\hat{\phi}}{v_\perp}-\frac{\delta\tilde{B}_\parallel}{B_0}, \tag{2.7}$$

where $\hat{\phi} = \partial\hat{\perp}/\partial\phi = \hat{\mathsf{z}}\times\hat{\perp}$. Whenever direct comparison with the work of Kennel and Engelmann (Kennel and Engelmann, 1966) is needed, we will use Faraday's law to express $\delta\tilde{\mathbf{B}} = (\mathbf{k}c/\omega)\times\delta\tilde{\mathbf{E}}$. With this substitution (see Supplementary Appendix A for details), for example, we note that Eqs. 2.4.7.–.Eqs. 2.2.7 agree exactly with Eq. 2.12 of Kennel and Engelmann (Kennel and Engelmann, 1966).

We now remark that, since $\partial f_0(v_\parallel, v_\perp)/\partial\phi$ vanishes, only the first two terms in Eq. 2.4 are non-vanishing when applied to f_0. Hence, Eq. 2.3 contains the integrals.

$$e^{i\Theta}\int^{\phi}e^{-i\Theta'}d\phi', \tag{2.8}$$

$$e^{i\Theta}\int^{\phi}e^{-i\Theta'}\hat{\perp}'d\phi'. \tag{2.9}$$

In order to evaluate these integrals, we use the Bessel-Fourier decomposition $e^{i\Theta} = e^{i\varphi}\sum_{\ell=-\infty}^{\infty}J_\ell(\lambda)\,e^{i\ell(\phi-\psi)}$, so that the scalar integral Eq. 2.8 becomes

$$e^{i\Theta}\int^{\phi}e^{-i\Theta'}d\phi' = \sum_{m,\ell=-\infty}^{\infty}i\,\Delta_\ell\,J_m(\lambda)\,J_\ell(\lambda)\,e^{i(m-\ell)(\phi-\psi)}, \tag{2.10}$$

where the resonant denominator is

$$\Delta_\ell \equiv \frac{\Omega}{k_\parallel v_\parallel+\ell\,\Omega-\omega}, \tag{2.11}$$

while, using the identity Eq. 1.3, the vector integral Eq. 2.9 becomes

$$e^{i\Theta}\int^{\phi}e^{-i\Theta'}\hat{\perp}'\,d\phi' = \sum_{m,\ell=-\infty}^{\infty}i\,\Delta_\ell\,J_m(\lambda)\,\mathbb{J}_{\perp\ell}(\lambda)\,e^{i(m-\ell)(\phi-\psi)}, \tag{2.12}$$

where we introduced the vector-valued Bessel function

$$\mathbb{J}_{\perp\ell}(\lambda) \equiv \frac{\hat{\mathsf{K}}}{\sqrt{2}}\,J_{\ell+1}(\lambda)+\frac{\hat{\mathsf{K}}^*}{\sqrt{2}}\,J_{\ell-1}(\lambda), \tag{2.13}$$

with the identity

$$\mathbf{k}\cdot\mathbb{J}_{\perp\ell} = (J_{\ell+1}+J_{\ell-1})\,k_\perp/2 = (\ell\Omega/v_\perp)\,J_\ell, \tag{2.14}$$

which follows from a standard recurrence relation for Bessel functions. The perturbed Vlasov distribution (Eq. 2.3) is thus expressed as

$$\delta\tilde{f} = \sum_{m,\ell}i\,\Delta_\ell\,J_m(\lambda)\,e^{i(m-\ell)(\phi-\psi)}\left(\delta\tilde{V}_{\parallel\ell}\,\frac{\partial f_0}{\partial v_\parallel}+\delta\tilde{V}_{\perp\ell}\,\frac{\partial f_0}{\partial v_\perp}\right), \tag{2.15}$$

where the Bessel-Fourier components are

$$\delta\tilde{V}_{\parallel\ell} = \frac{q}{M\Omega}\,\delta\tilde{E}_\parallel\,J_\ell(\lambda)-v_\perp\,\hat{\mathsf{z}}\times\frac{\delta\tilde{\mathbf{B}}}{B_0}\cdot\mathbb{J}_{\perp\ell}(\lambda), \tag{2.16}$$

$$\delta\tilde{V}_{\perp\ell} = \frac{q}{M\Omega}\left(\delta\tilde{\mathbf{E}}+\frac{v_\parallel\hat{\mathsf{z}}}{c}\times\delta\tilde{\mathbf{B}}\right)\cdot\mathbb{J}_{\perp\ell}(\lambda). \tag{2.17}$$

Once again, Eqs. 2.15.17.–.Eqs. 2.2.17 agree exactly with Eq. 2.19 of Kennel and Engelmann (Kennel and Engelmann, 1966) when Faraday's law is inserted in Eqs. 2.16, 2.17; see Supplementary Appendix A for details. The relativistic version of Eqs. 2.15.17.–.Eqs. 2.2.17, which was first derived by Lerche (Lerche, 1968), is also shown in Supplementary Appendix A.

2.2 Quasilinear diffusion in velocity space

We are now ready to calculate the expression for the quasilinear diffusion equation for the slow evolution ($\tau = \epsilon^2 t$) of the background Vlasov distribution

$$\frac{1}{\Omega}\frac{\partial f_0}{\partial \tau} = -\mathrm{Re}\left[\left\langle \frac{q}{M\Omega}\left(\delta\tilde{\mathbf{E}}^* + \frac{\mathbf{v}}{c}\times\delta\tilde{\mathbf{B}}^*\right)\cdot\frac{\partial\delta\tilde{f}}{\partial\mathbf{v}}\right\rangle\right]$$
$$= -\mathrm{Re}\left[\left\langle\left(\delta\tilde{V}_\parallel^*\frac{\partial}{\partial v_\parallel} + \delta\tilde{V}_\perp^*\frac{\partial}{\partial v_\perp} + \delta\tilde{\phi}^*\frac{\partial}{\partial\phi}\right)\delta\tilde{f}\right\rangle\right], \tag{2.18}$$

where ϵ denotes the amplitude of the perturbation fields, $\langle\ \rangle$ denotes a gyroangle average, and $(\delta\tilde{V}_\parallel^*, \delta\tilde{V}_\perp^*, \delta\tilde{\phi}^*)$ are the complex conjugates of Eqs. 2.5–2.7. In addition, the real part appears on the right side of Eq. 2.18 as a result of averaging with respect to the wave phase ϑ. We note that Kennel and Engelmann (Kennel and Engelmann, 1966) ignore the term $\partial f_2/\partial t$ on the left side of Eq. 2.18, which is associated with the second-order perturbed Vlasov distribution f_2 generated by non-resonant particles (Kaufman, 1972a; Dewar, 1973). While this term was shown by Kaufman (Kaufman, 1972a) to be essential in demonstrating the energy-momentum conservation laws of quasilinear theory, it is also omitted here and the right side of Eq. 2.18 only contains resonant-particle contributions.

First, since Eqs. 2.5, 2.6 are independent of $v_\parallel$ and $v_\perp$, respectively, we find

$$\langle\delta\tilde{V}_\parallel^*\frac{\partial\delta\tilde{f}}{\partial v_\parallel} + \delta\tilde{V}_\perp^*\frac{\partial\delta\tilde{f}}{\partial v_\perp}\rangle = \frac{\partial}{\partial v_\parallel}\langle\delta\tilde{V}_\parallel^*\delta\tilde{f}\rangle + \frac{\partial}{\partial v_\perp}\langle\delta\tilde{V}_\perp^*\delta\tilde{f}\rangle$$
$$= \frac{\partial}{\partial v_\parallel}\langle\delta\tilde{V}_\parallel^*\delta\tilde{f}\rangle + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}$$
$$\times\left(v_\perp\,\langle\delta\tilde{V}_\perp^*\delta\tilde{f}\rangle\right) - \langle\frac{\delta\tilde{V}_\perp^*}{v_\perp}\,\delta\tilde{f}\rangle, \tag{2.19}$$

where we took into account the proper Jacobian $(v_\perp)$ in cylindrical velocity space $(v_\parallel, v_\perp, \phi)$. On the other hand, the third term in Eq. 2.18 can be written as

$$\left\langle\delta\tilde{\phi}^*\frac{\partial\delta\tilde{f}}{\partial\phi}\right\rangle = -\left\langle\left(\frac{\partial\delta\tilde{\phi}^*}{\partial\phi}\right)\delta\tilde{f}\right\rangle$$
$$= \frac{q}{M\Omega}\left(\delta\tilde{\mathbf{E}}^* + \frac{v_\parallel\hat{\mathsf{z}}}{c}\times\delta\tilde{\mathbf{B}}^*\right)\cdot\left\langle\frac{\hat{\perp}}{v_\perp}\,\delta\tilde{f}\right\rangle$$
$$\equiv \left\langle\frac{\delta\tilde{V}_\perp^*}{v_\perp}\,\delta\tilde{f}\right\rangle,$$

where the last term in Eq. 2.7 is independent of the gyroangle ϕ. Since this term cancels the last term in Eq. 2.19, the quasilinear diffusion Eq. 2.18 becomes

$$\frac{1}{\Omega}\frac{\partial f_0}{\partial\tau} = -\frac{\partial}{\partial v_\parallel}\left(\mathrm{Re}\langle\delta\tilde{V}_\parallel^*\delta\tilde{f}\rangle\right) - \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left(v_\perp\,\mathrm{Re}\langle\delta\tilde{V}_\perp^*\delta\tilde{f}\rangle\right). \tag{2.20}$$

Next, using the identity Eq. 1.3, we find

$$\sum_m J_m(\lambda)\langle(1,\hat{\perp})\,e^{i(m-\ell)(\phi-\psi)}\rangle = (J_\ell(\lambda), \mathbb{J}_{\perp\ell}(\lambda)),$$

so that, from Eq. 2.15, we find

$$\sum_m J_m(\lambda)\langle\delta\tilde{V}_\parallel^* e^{i(m-\ell)(\phi-\psi)}\rangle = \frac{q}{M\Omega}\,\delta\tilde{E}_\parallel^*\,J_\ell(\lambda) - v_\perp\hat{\mathsf{z}}$$
$$\times\frac{\delta\tilde{\mathbf{B}}^*}{B_0}\cdot\mathbb{J}_{\perp\ell}(\lambda) \tag{2.21}$$
$$\equiv \delta\tilde{V}_{\parallel\ell}^*,$$

$$\sum_m J_m(\lambda)\langle\delta\tilde{V}_\perp^* e^{i(m-\ell)(\phi-\psi)}\rangle = \frac{q}{M\Omega}\left(\delta\tilde{\mathbf{E}}^* + \frac{v_\parallel\hat{\mathsf{z}}}{c}\times\delta\tilde{\mathbf{B}}^*\right)$$
$$\cdot\,\mathbb{J}_{\perp\ell}(\lambda) \equiv \delta\tilde{V}_{\perp\ell}^*. \tag{2.22}$$

Hence, the quasilinear diffusion Eq. 2.20 can be written as

$$\frac{1}{\Omega}\frac{\partial f_0}{\partial\tau} = -\frac{\partial}{\partial v_\parallel}\left\{\mathrm{Re}\left[\sum_{\ell=-\infty}^{\infty} i\,\Delta_\ell\,\delta\tilde{V}_{\parallel\ell}^*\left(\delta\tilde{V}_{\parallel\ell}\frac{\partial f_0}{\partial v_\parallel} + \delta\tilde{V}_{\perp\ell}\frac{\partial f_0}{\partial v_\perp}\right)\right]\right\}$$
$$-\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left\{v_\perp\,\mathrm{Re}\left[\sum_{\ell=-\infty}^{\infty} i\,\Delta_\ell\,\delta\tilde{V}_{\perp\ell}^*\left(\delta\tilde{V}_{\parallel\ell}\frac{\partial f_0}{\partial v_\parallel}\right.\right.\right.$$
$$\left.\left.\left.+\delta\tilde{V}_{\perp\ell}\frac{\partial f_0}{\partial v_\perp}\right)\right]\right\} \equiv \frac{\partial}{\partial\mathbf{v}}\cdot\left(\mathsf{D}\cdot\frac{\partial f_0}{\partial\mathbf{v}}\right), \tag{2.23}$$

where the diagonal diffusion coefficients are

$$\mathsf{D}^{\parallel\parallel} \equiv \hat{\mathsf{z}}\cdot\mathsf{D}\cdot\hat{\mathsf{z}} = \sum_{\ell=-\infty}^{\infty}\mathrm{Re}(-i\,\Delta_\ell)\,|\delta\tilde{V}_{\parallel\ell}|^2, \tag{2.24}$$

$$\mathsf{D}^{\perp\perp} \equiv \hat{\perp}\cdot\mathsf{D}\cdot\hat{\perp} = \sum_{\ell=-\infty}^{\infty}\mathrm{Re}(-i\,\Delta_\ell)\,|\delta\tilde{V}_{\perp\ell}|^2, \tag{2.25}$$

while the off-diagonal diffusion coefficients are

$$\mathsf{D}^{\parallel\perp} \equiv \hat{\mathsf{z}}\cdot\mathsf{D}\cdot\hat{\perp} = \sum_{\ell=-\infty}^{\infty}\mathrm{Re}(-i\,\Delta_\ell)\,\mathrm{Re}\left(\delta\tilde{V}_{\parallel\ell}^*\,\delta\tilde{V}_{\perp\ell}\right), \tag{2.26}$$

$$\mathsf{D}^{\perp\parallel} \equiv \hat{\perp}\cdot\mathsf{D}\cdot\hat{\mathsf{z}} = \sum_{\ell=-\infty}^{\infty}\mathrm{Re}(-i\,\Delta_\ell)\,\mathrm{Re}\left(\delta\tilde{V}_{\perp\ell}^*\,\delta\tilde{V}_{\parallel\ell}\right), \tag{2.27}$$

which are defined to be explicitly symmetric (i.e., $\mathsf{D}^{\parallel\perp} = \mathsf{D}^{\perp\parallel}$). Here, using the Plemelj formula (Stix, 1992), we find

$$\mathrm{Re}(-i\,\Delta_\ell) = \mathrm{Re}\left[\frac{i\,\Omega}{(\omega - k_\parallel v_\parallel - \ell\Omega)}\right] = \pi\,\Omega\,\delta(\omega_r - k_\parallel v_\parallel - \ell\Omega), \tag{2.28}$$

where we assumed $\omega = \omega_r + i\,\gamma$ and took the weakly unstable limit $\gamma\to 0^+$. Hence, the quasilinear diffusion coefficients (2.24)–(2.27) are driven by resonant particles, which satisfy the resonance condition $k_\parallel v_{\parallel\mathrm{res}} \equiv \omega - \ell\Omega$. The reader is referred to the early references by Kaufman (Kaufman, 1972a) and Dewar (Dewar, 1973) concerning the role of non-resonant particles in demonstrating the energy-momentum conservation laws of quasilinear theory.

Eq. 2.25 from Kennel and Engelmann (Kennel and Engelmann, 1966) (see Supplementary Appendix A) can be expressed as the dyadic diffusion tensor

$$\mathsf{D} \equiv \sum_\ell \mathrm{Re}(-i\,\Delta_\ell)\,\tilde{\mathbf{v}}_\ell^*\,\tilde{\mathbf{v}}_\ell = \sum_\ell \mathrm{Re}(-i\,\Delta_\ell)\left[\left(\delta\tilde{V}_{\|\ell}^*\,\hat{z} + \delta\tilde{V}_{\perp\ell}^*\,\hat{\perp}\right)\left(\delta\tilde{V}_{\|\ell}\,\hat{z} + \delta\tilde{V}_{\perp\ell}\,\hat{\perp}\right)\right], \tag{2.29}$$

which is Hermitian since the term $-\,i\,\Delta_\ell$ is replaced with Re$(-i\,\Delta_\ell)$. Here, the perturbed velocity

$$\tilde{\mathbf{v}}_\ell = \delta\tilde{V}_{\|\ell}\,\hat{z} + \delta\tilde{V}_{\perp\ell}\,\hat{\perp} = \frac{q\,\delta\tilde{\mathbf{E}}}{M\Omega}\cdot\left[J_\ell(\lambda)\,\hat{z}\hat{z} + \mathbb{J}_{\perp\ell}(\lambda)\,\hat{\perp}\right] + \hat{z}\times\frac{\delta\tilde{\mathbf{B}}}{B_0}\cdot\mathbb{J}_{\perp\ell}(\lambda)\,\left(v_\|\,\hat{\perp} - v_\perp\,\hat{z}\right) \tag{2.30}$$

explicitly separates the electric and magnetic contributions to the quasilinear diffusion tensor Eq. 2.29. In particular, the role of the perturbed perpendicular magnetic field is clearly seen in the process of pitch-angle diffusion because of the presence of the terms $(v_\|\,\hat{\perp} - v_\perp\,\hat{z})$ associated with it. We also note that the parallel component of the perturbed magnetic field, $\delta\tilde{B}_\| = \hat{z}\cdot\delta\tilde{\mathbf{B}}$, does not contribute to quasilinear diffusion in a uniform magnetized plasma. The components of the perturbed electric field, on the other hand, involve the parallel component, $\delta\tilde{E}_\| = \hat{z}\cdot\delta\tilde{\mathbf{E}}$, as well as the right and left polarized components, $\delta\tilde{E}_R = \delta\tilde{\mathbf{E}}\cdot(\hat{x} - i\,\hat{y})/\sqrt{2}$ and $\delta\tilde{E}_L = \delta\tilde{\mathbf{E}}\cdot(\hat{x} + i\,\hat{y})/\sqrt{2}$, respectively, appearing through the definition Eq. 2.13.

Lastly, we note that the dyadic form Eq. 2.29 of the quasilinear diffusion tensor in the quasilinear diffusion Eq. 2.23 can be used to easily verify that the unperturbed entropy $\mathcal{S}_0 \equiv -\int f_0\,\ln f_0\,d^3v$ satisfies the H Theorem:

$$\frac{d\mathcal{S}_0}{dt} = -\epsilon^2\int\frac{\partial f_0}{\partial\tau}\,(\ln f_0 + 1)\,d^3v = \epsilon^2\sum_\ell\int \mathrm{Re}(-i\Delta_\ell)\,f_0\left|\tilde{\mathbf{v}}_\ell\cdot\frac{\partial\ln f_0}{\partial\mathbf{v}}\right|^2 d^3v > 0. \tag{2.31}$$

Once again, the energy-momentum conservation laws in quasilinear theory will not be discussed here. Instead the interested reader can consult earlier references (Kaufman, 1972a; Dewar, 1973), as well as Chapters 16–18 in the standard textbook by Stix (Stix, 1992).

2.3 Quasilinear diffusion in invariant velocity space

In preparation for Section 3, we note that a natural choice of velocity-space coordinates, suggested by guiding-center theory, involves replacing the parallel velocity $v_\|$ with the parallel momentum $p_\| = M\,v_\|$ and the perpendicular speed $v_\perp$ with the magnetic moment $\mu = Mv_\perp^2/(2B_0)$. We note that these two coordinates are independent dynamical invariants of the particle motion in a uniform magnetic field.

With this change of coordinates, the quasilinear diffusion Eq. 2.23 becomes

$$\frac{1}{\Omega}\frac{\partial f_0}{\partial\tau} \equiv \frac{\partial}{\partial p_\|}\left(D^{pp}\,\frac{\partial f_0}{\partial p_\|} + D^{p\mu}\,\frac{\partial f_0}{\partial\mu}\right) + \frac{\partial}{\partial\mu}\left(D^{\mu p}\,\frac{\partial f_0}{\partial p_\|} + D^{\mu\mu}\,\frac{\partial f_0}{\partial\mu}\right), \tag{2.32}$$

where the quasilinear diffusion coefficients are

$$\left.\begin{aligned} D^{pp} &= M^2\,\mathsf{D}^{\|\|} = \sum_\ell \mathrm{Re}(-i\,\Delta_\ell)\,|\delta\tilde{P}_{\|\ell}|^2 \\ D^{p\mu} &= (M^2 v_\perp/B_0)\,\mathsf{D}^{\|\perp} = \sum_\ell \mathrm{Re}(-i\,\Delta_\ell)\,\mathrm{Re}\left(\delta\tilde{P}_{\|\ell}^*\delta\tilde{\mu}_\ell\right) \\ D^{\mu\mu} &= (Mv_\perp/B_0)^2\,\mathsf{D}^{\perp\perp} = \sum_\ell \mathrm{Re}(-i\,\Delta_\ell)\,|\delta\tilde{\mu}_\ell|^2 \end{aligned}\right\}, \tag{2.33}$$

with the eikonal amplitudes

$$\delta\tilde{P}_{\|\ell} = \frac{q}{\Omega}\left(\delta\tilde{E}_\|\,J_\ell + \frac{v_\perp}{c}\,\mathbb{J}_{\perp\ell}\times\delta\tilde{\mathbf{B}}\cdot\hat{z}\right), \tag{2.34}$$

$$\delta\tilde{\mu}_\ell = \frac{q}{B_0\Omega}\left(\delta\tilde{\mathbf{E}} + \frac{v_\|\,\hat{z}}{c}\times\delta\tilde{\mathbf{B}}\right)\cdot v_\perp\,\mathbb{J}_{\perp\ell}, \tag{2.35}$$

and the symmetry $D^{\mu p} = D^{p\mu}$ follows from the assumption of a Hermitian diffusion tensor. Lastly, as expected, we note that the eikonal amplitude for the perturbed kinetic energy

$$\delta\tilde{\mathcal{E}}_\ell \equiv M\mathbf{v}\cdot\tilde{\mathbf{v}}_\ell = v_\|\,\delta\tilde{P}_{\|\ell} + \delta\tilde{\mu}_\ell\,B_0 = \frac{q}{\Omega}\,\delta\tilde{\mathbf{E}}\cdot\left(v_\|\,J_\ell\,\hat{z} + v_\perp\,\mathbb{J}_{\perp\ell}\right), \tag{2.36}$$

only involves the perturbed electric field. Hence, another useful representation of quasilinear diffusion in invariant velocity $(\mathcal{E},\mu)$ space is given by the quasilinear diffusion equation

$$\frac{1}{\Omega}\frac{\partial f_0}{\partial\tau} \equiv v_\|\,\frac{\partial}{\partial\mathcal{E}}\left[\frac{1}{v_\|}\left(D^{\mathcal{E}\mathcal{E}}\,\frac{\partial f_0}{\partial\mathcal{E}} + D^{\mathcal{E}\mu}\,\frac{\partial f_0}{\partial\mu}\right)\right] + v_\|\,\frac{\partial}{\partial\mu}\left[\frac{1}{v_\|}\left(D^{\mu\mathcal{E}}\,\frac{\partial f_0}{\partial\mathcal{E}} + D^{\mu\mu}\,\frac{\partial f_0}{\partial\mu}\right)\right], \tag{2.37}$$

where the quasilinear diffusion coefficients are

$$\left.\begin{aligned} D^{\mathcal{E}\mathcal{E}} &= \sum_\ell \mathrm{Re}(-i\,\Delta_\ell)\,|\delta\tilde{\mathcal{E}}_\ell|^2 \\ D^{\mathcal{E}\mu} &= \sum_\ell \mathrm{Re}(-i\,\Delta_\ell)\,\mathrm{Re}\left(\delta\tilde{\mathcal{E}}_\ell^*\delta\tilde{\mu}_\ell\right) \\ D^{\mu\mu} &= \sum_\ell \mathrm{Re}(-i\,\Delta_\ell)\,|\delta\tilde{\mu}_\ell|^2 \end{aligned}\right\}, \tag{2.38}$$

and the Jacobian $1/v_\|$ is a function of $(\mathcal{E},\mu)$: $|v_\|| = \sqrt{(2/M)(\mathcal{E} - \mu\,B_0)}$, while the sign of $v_\|$ is a constant of the motion in a uniform magnetic field.

3 Hamiltonian quasilinear diffusion equation

In Section 2, we reviewed the standard formulation of quasilinear theory in a uniform magnetized plasma (Kennel and Engelmann, 1966). In this Section, we introduce the Hamiltonian formulation of the Vlasov equation from which we will derive the Hamiltonian quasilinear diffusion equation, which will then be compared with the Kennel-Engelmann quasilinear diffusion Eq. 2.23.

In order to proceed with a Hamiltonian formulation, however, we will be required to express the perturbed electric and magnetic fields in terms of perturbed electric and magnetic potentials. We note that, despite the use of these potentials, the gauge invariance of the Hamiltonian quasilinear diffusion equation will be guaranteed in the formulation adopted here.

3.1 Non-adiabatic decomposition of the perturbed Vlasov distribution

The Hamiltonian formulation of quasilinear diffusion begins with the representation of the perturbed electric and magnetic fields in terms of the perturbed electric scalar potential $\delta\Phi$ and the perturbed magnetic vector potential $\delta\mathbf{A}$, where $\delta\mathbf{E} = -\nabla\delta\Phi - c^{-1}\partial\delta\mathbf{A}/\partial t$ and $\delta\mathbf{B} = \nabla\times\delta\mathbf{A}$. Hence, we find the identity

$$\begin{aligned}\delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\delta\mathbf{B} &= -\nabla\left(\delta\Phi - \frac{\mathbf{v}}{c}\cdot\delta\mathbf{A}\right) - \frac{1}{c}\frac{d\delta\mathbf{A}}{dt}\\ &\equiv -\nabla\delta\Psi - \frac{1}{c}\frac{d\delta\mathbf{A}}{dt},\end{aligned} \tag{3.1}$$

where d/dt denotes the unperturbed time derivative. We note that the gauge transformation

$$(\delta\Phi, \delta\mathbf{A}, \delta\Psi) \to \left(\delta\Phi - \frac{1}{c}\frac{\partial\delta\chi}{\partial t},\; \delta\mathbf{A} + \nabla\delta\chi,\; \delta\Psi - \frac{1}{c}\frac{d\delta\chi}{dt}\right) \tag{3.2}$$

guarantees the gauge invariance of the right side of Eq. 3.1.

Next, by removing the perturbed magnetic vector potential $\delta\mathbf{A}$ from the canonical momentum

$$\mathbf{P} = m\mathbf{v} + q\,(\mathbf{A}_0 + \epsilon\,\delta\mathbf{A})/c \to \mathbf{P}_0 = m\mathbf{v} + q\mathbf{A}_0/c,$$

the noncanonical Poisson bracket (which can also be expressed in divergence form)

$$\{f,\, g\} = \frac{1}{M}\left(\nabla f\cdot\frac{\partial g}{\partial\mathbf{v}} - \frac{\partial f}{\partial\mathbf{v}}\cdot\nabla g\right) + \frac{q\mathbf{B}_0}{M^2 c}\cdot\frac{\partial f}{\partial\mathbf{v}}\times\frac{\partial g}{\partial\mathbf{v}} \tag{3.3}$$

$$= \frac{\partial}{\partial\mathbf{v}}\cdot\left[\frac{1}{M}\left(\nabla f + \frac{q\mathbf{B}_0}{Mc}\times\frac{\partial f}{\partial\mathbf{v}}\right)g\right] - \nabla\cdot\left(\frac{\partial f}{\partial\mathbf{v}}\,\frac{g}{M}\right) \tag{3.4}$$

only contains the unperturbed magnetic field $\mathbf{B}_0$., where f and g are arbitrary functions of $(\mathbf{x}, \mathbf{v})$.

The removal of the perturbed magnetic vector potential $\delta\mathbf{A}$ from the noncanonical Poisson bracket Eq. 3.3, however, implies that the perturbed Vlasov distribution can be written as

$$\delta f = \frac{q}{c}\,\delta\mathbf{A}\cdot\frac{\partial f_0}{\partial\mathbf{v}} + \delta g \equiv \frac{q}{c}\,\delta\mathbf{A}\cdot\{\mathbf{x}, f_0\} + \{\delta s,\; f_0\}, \tag{3.5}$$

where the *non-adiabatic* contribution δg is said to be generated by the perturbation scalar field δs (Brizard, 1994; Brizard, 2018; Brizard and Chandre, 2020), which satisfies the first-order eikonal equation

$$i\,(\mathbf{k}\cdot\mathbf{v} - \omega)\delta\tilde{s} - \Omega\,\frac{\partial\delta\tilde{s}}{\partial\phi} = q\left(\delta\tilde{\Phi} - \frac{\mathbf{v}}{c}\cdot\delta\tilde{\mathbf{A}}\right) \equiv q\,\delta\tilde{\Psi}. \tag{3.6}$$

Hence, the eikonal solution for $\delta\tilde{s}$ is expressed with the same integrating factor used in Eq. 2.3:

$$\begin{aligned}\delta\tilde{s}(\mathbf{v}) &= -\frac{q}{\Omega}\,e^{i\Theta}\int^{\phi}\delta\tilde{\Psi}'\,e^{-i\Theta'}\,d\phi'\\ &= -\frac{q}{\Omega}\sum_{m,\ell} i\,\Delta_\ell\,J_m(\lambda)\,e^{i(m-\ell)(\phi-\psi)}\,\delta\tilde{\Psi}_\ell,\end{aligned} \tag{3.7}$$

where the gyroangle Fourier component of the effective perturbed potential is

$$\delta\tilde{\Psi}_\ell \equiv \left(\delta\tilde{\Phi} - \frac{v_\parallel}{c}\,\delta\tilde{A}_\parallel\right)J_\ell(\lambda) - \frac{v_\perp}{c}\,\delta\tilde{\mathbf{A}}\cdot\mathbb{J}_{\perp\ell}(\lambda), \tag{3.8}$$

and the eikonal amplitude of the non-adiabatic perturbed Vlasov distribution is

$$\begin{aligned}\delta\tilde{g} &= e^{-i\vartheta}\left\{\delta\tilde{s}\,e^{i\vartheta}, f_0\right\} = \frac{1}{M}\left(i\mathbf{k}\,\delta\tilde{s} + \Omega\,\hat{\mathbf{z}}\times\frac{\partial\delta\tilde{s}}{\partial\mathbf{v}}\right)\cdot\frac{\partial f_0}{\partial\mathbf{v}}\\ &= \frac{i\mathbf{k}}{M}\cdot\frac{\partial f_0}{\partial\mathbf{v}}\,\delta\tilde{s} - \frac{\Omega}{B_0}\,\frac{\partial\delta\tilde{s}}{\partial\phi}\,\frac{\partial f_0}{\partial\mu},\end{aligned} \tag{3.9}$$

where $\mu \equiv M|\mathbf{v}_\perp|^2/(2B_0)$ denotes the magnetic moment. We note that, under the gauge transformations Eq. 3.2, the scalar field δs transforms as $\delta s \to \delta s - (q/c)\,\delta\chi$ (Brizard, 1994; Brizard, 2018; Brizard and Chandre, 2020), and the expression Eq. 3.5 for the perturbed Vlasov distribution is gauge-invariant. Moreover, under the gauge transformation Eq. 3.2, the eikonal Fourier amplitude Eq. 3.8 transforms as

$$\delta\tilde{\Psi}_\ell \to \delta\tilde{\Psi}_\ell + \frac{i}{c}\,(\omega - k_\parallel v_\parallel - \ell\,\Omega)\,J_\ell\,\delta\tilde{\chi}, \tag{3.10}$$

which is consistent with Eq. 3.2.

Next, since the components of the Poisson bracket Eq. 3.3 are constant, the unperturbed time derivative of δf yields the linearized perturbed Vlasov equation

$$\begin{aligned}\frac{d\delta f}{dt} &= \frac{q}{c}\,\frac{d\delta\mathbf{A}}{dt}\cdot\{\mathbf{x}, f_0\} + \frac{q}{c}\,\delta\mathbf{A}\cdot\{\mathbf{v}, f_0\} + \left\{\frac{d\delta s}{dt},\; f_0\right\}\\ &= \frac{q}{c}\,\frac{d\delta\mathbf{A}}{dt}\cdot\{\mathbf{x}, f_0\} + \frac{q}{c}\,\delta\mathbf{A}\cdot\{\mathbf{v}, f_0\} + \{q\,\delta\Psi,\; f_0\}\\ &= \left(\frac{q}{c}\,\frac{d\delta\mathbf{A}}{dt} + q\,\nabla\delta\Psi\right)\cdot\{\mathbf{x}, f_0\}\\ &\equiv -\frac{q}{M}\left(\delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\delta\mathbf{B}\right)\cdot\frac{\partial f_0}{\partial\mathbf{v}},\end{aligned} \tag{3.11}$$

which implies that the non-adiabatic decomposition Eq. 3.5 is a valid representation of the perturbed Vlasov distribution.

3.2 Second-order perturbed Vlasov equation

In order to derive an alternate formulation of quasilinear theory for uniform magnetized plasmas, we begin with second-order evolution of the background Vlasov distribution

$$\frac{\partial f_0}{\partial\tau} = -\frac{q}{M}\left(\delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\delta\mathbf{B}\right)\cdot\frac{\partial\delta f}{\partial\mathbf{v}} = \left(q\,\nabla\delta\Psi + \frac{q}{c}\,\frac{d\delta\mathbf{A}}{dt}\right)\cdot\{\mathbf{x},\ \delta f\}, \tag{3.12}$$

where, once again, $\tau = \epsilon^2 t$ denotes the slow quasilinear diffusion time scale, we have ignored the second-order perturbed Vlasov distribution f_2, and we have inserted Eqs. 3.1, 3.3. The first term on the right side of Eq. 3.12 can be written as

$$\begin{aligned} q\nabla\delta\Psi\cdot\{\mathbf{x},\ \delta f\} &= \{q\,\delta\Psi,\ \delta f\} - q\,\frac{\partial\delta\Psi}{\partial\mathbf{v}}\cdot\{\mathbf{v},\ \delta f\} \\ &= \{q\,\delta\Psi,\ \delta f\} + \frac{q}{c}\,\delta\mathbf{A}\cdot\{\mathbf{v},\ \delta f\} \\ &= \left\{q\,\delta\Psi,\ \left(\frac{q}{c}\,\delta\mathbf{A}\cdot\{\mathbf{x}, f_0\} + \delta g\right)\right\} \\ &\quad + \frac{q}{c}\,\delta\mathbf{A}\cdot\{\mathbf{v},\ \delta f\}, \end{aligned} \tag{3.13}$$

where we have inserted the non-adiabatic decomposition Eq. 3.5, so that the first term can be written as

$$\begin{aligned} \left\{q\,\delta\Psi,\ \frac{q}{c}\,\delta\mathbf{A}\cdot\{\mathbf{x}, f_0\}\right\} &= \frac{q^2}{c}\left(\{\delta\Psi,\ \delta\mathbf{A}\}\cdot\{\mathbf{x},\ f_0\} + \delta\mathbf{A}\cdot\{\delta\Psi, \{\mathbf{x},\ f_0\}\}\right) \\ &= \frac{q^2}{Mc^2}\,\delta\mathbf{A}\cdot\nabla\delta\mathbf{A}\cdot\{\mathbf{x},\ f_0\} \\ &\quad + \frac{q}{c}\delta\mathbf{A}\cdot\{q\,\delta\Psi, \{\mathbf{x},\ f_0\}\}, \end{aligned} \tag{3.14}$$

where we used $\{\delta\Psi,\ \delta\mathbf{A}\} = (\delta\mathbf{A}/Mc)\cdot\nabla\delta\mathbf{A}$. The second term on the right side of Eq. 3.12, on the other hand, can be written as

$$\begin{aligned} \frac{q}{c}\,\frac{d\delta\mathbf{A}}{dt}\cdot\{\mathbf{x},\ \delta f\} &= \frac{q^2}{c^2}\,\frac{d\delta\mathbf{A}}{dt}\cdot\{\mathbf{x}, \{\mathbf{x}, f_0\}\}\cdot\delta\mathbf{A} \\ &\quad + \frac{d}{dt}\left(\frac{q}{c}\delta\mathbf{A}\cdot\{\mathbf{x},\ \delta g\}\right) - \frac{q}{c}\delta\mathbf{A}\cdot\{\mathbf{v}, \delta g\} \\ &\quad + \frac{q}{c}\delta\mathbf{A}\cdot\{\mathbf{x}, \{f_0,\ q\,\delta\Psi\}\}. \end{aligned} \tag{3.15}$$

Next, by using the Jacobi identity for the Poisson bracket (3.3):

$$\{f, \{g,\ h\}\} + \{g, \{h,\ f\}\} + \{h, \{f,\ g\}\} = 0, \tag{3.16}$$

which holds for arbitrary functions (f, g, h), we obtain

$$\begin{aligned} &\frac{q^2}{c}\,\delta\mathbf{A}\cdot\left(\{\delta\Psi, \{\mathbf{x},\ f_0\}\} + \{\mathbf{x}, \{f_0,\ \delta\Psi\}\}\right) \\ &= \frac{q^2}{c}\,\delta\mathbf{A}\cdot\{f_0, \{\mathbf{x},\ \delta\Psi\}\} \equiv -\{f_0,\ \delta H_2\}, \end{aligned} \tag{3.17}$$

where $\delta H_2 = q^2|\delta\mathbf{A}|^2/(2Mc^2)$ is the second-order perturbed Hamiltonian. We now look at the first term on the right side of Eq. 3.15, which we write as

$$\begin{aligned} \frac{q^2}{c^2}\,\frac{d\delta\mathbf{A}}{dt}\cdot\{\mathbf{x}, \{\mathbf{x}, f_0\}\}\cdot\delta\mathbf{A} &= \frac{d}{dt}\left(\frac{q^2}{c^2}\,\delta\mathbf{A}\cdot\{\mathbf{x}, \{\mathbf{x}, f_0\}\}\cdot\delta\mathbf{A}\right) \\ &\quad - \frac{q^2}{c^2}\,\delta\mathbf{A}\cdot\left(\{\mathbf{v}, \{\mathbf{x}, f_0\}\} + \{\mathbf{x}, \{\mathbf{v}, f_0\}\}\right) \\ &\quad \cdot\delta\mathbf{A} - \frac{q^2}{c^2}\,\delta\mathbf{A}\cdot\{\mathbf{x}, \{\mathbf{x}, f_0\}\}\cdot\frac{d\delta\mathbf{A}}{dt}. \end{aligned} \tag{3.18}$$

Because of the symmetry of the tensor $\{\mathbf{x}, \{\mathbf{x},\ f_0\}\}$, the last term on the right side (omitting the minus sign) is equal to the left side, so that we obtain

$$\begin{aligned} \frac{q^2}{c^2}\,\frac{d\delta\mathbf{A}}{dt}\cdot\{\mathbf{x}, \{\mathbf{x}, f_0\}\}\cdot\delta\mathbf{A} &= \frac{d}{dt}\left(\frac{q^2}{2c^2}\,\delta\mathbf{A}\cdot\{\mathbf{x}, \{\mathbf{x}, f_0\}\}\cdot\delta\mathbf{A}\right) \\ &\quad - \frac{q^2}{c^2}\,\delta\mathbf{A}\cdot\{\mathbf{v}, \{\mathbf{x}, f_0\}\}\cdot\delta\mathbf{A}, \end{aligned}$$

where we used the Jacobi identity Eq. 3.16 to find $\{\mathbf{x}, \{\mathbf{v}, f_0\}\} = \{\mathbf{v}, \{\mathbf{x}, f_0\}\}$, since $\{f_0,\ \{\mathbf{x},\ \mathbf{v}\}\} = 0$.

When these equations are combined into Eq. 3.12, we obtain the final Hamiltonian form of the second-order perturbed Vlasov equation

$$\begin{aligned} \frac{\partial f_0}{\partial\tau} &= \{\delta H, \delta g\} + \{\delta H_2, f_0\} + \frac{d}{dt}\left(\frac{q}{c}\,\delta\mathbf{A}\cdot\{\mathbf{x},\ \delta g\}\right. \\ &\quad \left. + \frac{q^2}{2c^2}\,\delta\mathbf{A}\ \cdot\{\mathbf{x}, \{\mathbf{x},\ f_0\}\}\cdot\delta\mathbf{A}\right), \end{aligned} \tag{3.19}$$

where $\delta H = q\ \delta\Psi = q\ \delta\Phi - q\ \delta\mathbf{A}\cdot\mathbf{v}/c$ and $\delta g = \{\delta s, f_0\}$.

3.3 Hamiltonian quasilinear diffusion equation

We now perform two separate averages of the second-order perturbed Vlasov Eq. 3.19: we first perform an average with respect to the wave phase ϑ, which will be denoted by an overbar, and, second, we perform an average with respect to the gyroangle ϕ. We begin by noting that the averaged second-order perturbed Hamiltonian $\delta\bar{H}_2 = q^2|\delta\tilde{\mathbf{A}}|^2/(2Mc^2)$ is a constant and, therefore, its contribution in Eq. 3.19 vanishes upon eikonal-phase averaging. Likewise, the total time derivative in Eq. 3.19 vanishes upon eikonal-phase averaging.

The Hamiltonian quasilinear diffusion equation is, therefore, defined as

$$\begin{aligned} \frac{\partial f_0}{\partial\tau} &\equiv \frac{1}{2}\left\langle\overline{\{\delta H, \delta g\}}\right\rangle = \frac{1}{2}\overline{\left[\nabla\cdot\left(\frac{q}{Mc}\,\delta\mathbf{A}\,\langle\delta g\rangle\right)\right]} \\ &\quad + \frac{1}{2}\left\langle\frac{\partial}{\partial\mathbf{v}}\cdot\overline{\left[\left(\nabla\delta H + \Omega\,\frac{q}{c}\,\delta\mathbf{A}\times\hat{\mathsf{z}}\right)\frac{\delta g}{M}\right]}\right\rangle \\ &= \frac{1}{2}\left\langle\frac{\partial}{\partial\mathbf{v}}\cdot\overline{\left[\left(\nabla\delta H + \Omega\,\frac{q}{c}\,\delta\mathbf{A}\times\hat{\mathsf{z}}\right)\frac{\delta g}{M}\right]}\right\rangle, \end{aligned} \tag{3.20}$$

where we used the divergence form Eq. 3.4 of the Poisson bracket and the eikonal average of the spatial divergence vanishes. Next, the eikonal average of the first term on the last line of the right side of Eq. 3.20 yields

$$\overline{(\nabla\delta H\delta g)} = i\,\mathbf{k}\left(\delta\tilde{H}\ \delta\tilde{g}^* - \delta\tilde{H}^*\ \delta\tilde{g}\right),$$

so that

$$\left\langle\frac{\partial}{\partial \mathbf{v}} \cdot \overline{\left(\nabla\delta H\,\frac{\delta g}{M}\right)}\right\rangle = \frac{\partial}{\partial p_\|}\left(ik_\|\langle\delta\tilde{H}\,\delta\tilde{g}^* - \delta\tilde{H}^*\,\delta\tilde{g}\rangle\right) + \frac{1}{B_0}\frac{\partial}{\partial\mu}\left[i\mathbf{k}\cdot\langle\mathbf{v}_\perp\left(\delta\tilde{H}\,\delta\tilde{g}^* - \delta\tilde{H}^*\,\delta\tilde{g}\right)\rangle\right], \tag{3.21}$$

where $p_\| = M\,v_\|$ and $\mu = M|\mathbf{v}_\perp|^2/2B_0$. The eikonal average of the second term on the last line of the right side of Eq. 3.20, on the other hand, yields

$$\frac{\Omega}{2\,v_\perp}\frac{\partial}{\partial v_\perp}\left[\frac{v_\perp}{M}\left(\frac{q}{c}\delta\tilde{\mathbf{A}}\cdot\langle\hat{\phi}\,\delta\tilde{g}^*\rangle + \frac{q}{c}\delta\tilde{\mathbf{A}}^*\cdot\langle\hat{\phi}\,\delta\tilde{g}\rangle\right)\right] \equiv \frac{\Omega}{B_0}\frac{\partial}{\partial\mu}\left[\mathrm{Re}\left\langle\left(\frac{q}{c}\delta\tilde{\mathbf{A}}\cdot\frac{\partial\mathbf{v}_\perp}{\partial\phi}\right)\delta\tilde{g}^*\right\rangle\right], \tag{3.22}$$

so that by combining Eqs. 3.21, 3.22 into Eq. 3.20, we find

$$\frac{1}{\Omega}\frac{\partial f_0}{\partial\tau} = \frac{\partial}{\partial p_\|}\left(\frac{k_\|}{\Omega}\,\mathrm{Re}\langle i\,\delta\tilde{H}\,\delta\tilde{g}^*\rangle\right) + \frac{1}{B_0}\frac{\partial}{\partial\mu}\left[\mathrm{Re}\left\langle\left(\frac{q}{c}\delta\tilde{\mathbf{A}}\cdot\frac{\partial\mathbf{v}_\perp}{\partial\phi} + i\,\frac{\mathbf{k}\cdot\mathbf{v}_\perp}{\Omega}\,\delta\tilde{H}\right)\delta\tilde{g}^*\right\rangle\right], \tag{3.23}$$

In order to evaluate the gyroangle averages in Eq. 3.23, we need to proceed with a transformation from particle phase space to guiding-center phase space, which is presented in the next Section.

4 Guiding-center Hamiltonian quasilinear diffusion equation

In this Section, we use the guiding-center transformation (Northrop, 1963) in order to simplify the calculations involved in obtaining an explicit expression for the Hamiltonian quasilinear diffusion Eq. 3.23 that can compared with the standard quasilinear diffusion Eq. 2.32 obtained from Kennel-Engelmann's work (Kennel and Engelmann, 1966).

4.1 Guiding-center transformation

In a uniform background magnetic field, the transformation from particle phase space to guiding-center phase space is simply given as $\mathbf{x} = \mathbf{X} + \boldsymbol{\rho}$, where the particle position $\mathbf{x}$ is expressed as the sum of the guiding-center position $\mathbf{X}$ and the gyroradius vector $\boldsymbol{\rho} \equiv \hat{\mathsf{z}}\times\mathbf{v}_\perp/\Omega$, while the velocity-space coordinates $(p_\|,\ \mu,\ \phi)$ remain unchanged (Cary and Brizard, 2009). Hence, the eikonal wave phase $\vartheta = \mathbf{k}\cdot\mathbf{x} - \omega\,t$ becomes

$$\vartheta = \mathbf{k}\cdot(\mathbf{X}+\boldsymbol{\rho}) - \omega\,t = \theta + \mathbf{k}\cdot\boldsymbol{\rho} \equiv \theta + \Lambda, \tag{4.1}$$

where θ denotes the guiding-center eikonal wave phase and $\Lambda \equiv \lambda\,\sin(\phi - \psi)$. Next, the particle Poisson bracket (Eq. 3.3) is transformed into the guiding-center Poisson bracket (Cary and Brizard, 2009)

$$\{F,\ G\}_{\mathrm{gc}} = \hat{\mathsf{z}}\cdot\left(\nabla F\,\frac{\partial G}{\partial p_\|} - \frac{\partial F}{\partial p_\|}\,\nabla G\right) - \frac{\Omega}{B_0}\left(\frac{\partial F}{\partial\phi}\frac{\partial G}{\partial\mu} - \frac{\partial F}{\partial\mu}\frac{\partial G}{\partial\phi}\right) - \frac{c\,\hat{\mathsf{z}}}{qB_0}\cdot\nabla F\times\nabla G, \tag{4.2}$$

where the last term vanishes in the case of a uniform background plasma since the guiding-center functions F and G depend on the guiding-center position only through the guiding-center wave phase θ (with $\nabla\theta = \mathbf{k}$).

4.2 First-order perturbed guiding-center Vlasov equation

The guiding-center transformation induces a transformation on particle phase-space functions f to a guiding-center phase-space function F through the guiding-center push-forward $\mathsf{T}_{\mathrm{gc}}^{-1}$: $F \equiv \mathsf{T}_{\mathrm{gc}}^{-1} f$. For a perturbed particle phase-space function $\delta g \equiv \delta\tilde{g}\,\exp(i\vartheta) + \mathrm{c.c.}$, we find the perturbed guiding-center phase-space function $\delta G \equiv \delta\tilde{G}\,\exp(i\theta) + \mathrm{c.c.}$, where the eikonal amplitude $\delta\tilde{G}$ is given by the push-forward expression as

$$\delta\tilde{G} = \delta\tilde{g}\,e^{-i\Lambda} = e^{-i\theta}\left\{\delta\tilde{S}\,e^{i\theta},\ f_0\right\}_{\mathrm{gc}} = i\,k_\|\,\frac{\partial f_0}{\partial p_\|}\,\delta\tilde{S} - \frac{\Omega}{B_0}\frac{\partial f_0}{\partial\mu}\frac{\partial\delta\tilde{S}}{\partial\phi}. \tag{4.3}$$

The eikonal amplitude of the guiding-center generating function $\delta\tilde{S} = \delta\tilde{s}\,\exp(-i\Lambda)$ satisfies an equation derived from the first-order eikonal Eq. 3.6:

$$i\,(k_\| v_\| - \omega)\,\delta\tilde{S} - \Omega\,\frac{\partial\delta\tilde{S}}{\partial\phi} = \delta\tilde{H}\,e^{-i\Lambda} \equiv \delta\tilde{H}_{\mathrm{gc}}. \tag{4.4}$$

The solution of the first-order guiding-center eikonal Eq. 4.4 makes use of the gyroangle expansion $\delta\tilde{S} = \sum_{\ell=-\infty}^{\infty}\delta\tilde{S}_\ell\exp[-i\ell(\phi-\psi)]$, which yields the Fourier component

$$\delta\tilde{S}_\ell = -\frac{i\,\Delta_\ell}{\Omega}\,\langle\delta\tilde{H}e^{-i\Lambda+i\ell(\phi-\psi)}\rangle = -\frac{i\,\Delta_\ell}{\Omega}\,q\,\delta\tilde{\Psi}_\ell. \tag{4.5}$$

Inserting this solution into Eq. 4.3, with the gyroangle expansion $\delta\tilde{G} = \sum_{\ell=-\infty}^{\infty}\delta\tilde{G}_\ell\exp[-i\ell(\phi-\psi)]$, yields

$$\delta\tilde{G}_\ell = i\left(k_\|\,\frac{\partial f_0}{\partial p_\|} + \frac{\ell\,\Omega}{B_0}\frac{\partial f_0}{\partial\mu}\right)\delta\tilde{S}_\ell = \frac{q}{\Omega}\,\delta\tilde{\Psi}_\ell\,\Delta_\ell\left(k_\|\,\frac{\partial f_0}{\partial p_\|} + \frac{\ell\,\Omega}{B_0}\frac{\partial f_0}{\partial\mu}\right). \tag{4.6}$$

Hence, the solution for the eikonal amplitude $\delta\tilde{g}$ appearing in Eq. 3.23 can be obtained from the guiding-center pull-back expression $\delta\tilde{g} = \delta\tilde{G}\,\exp(i\,\Lambda)$.

4.3 Guiding-center Hamiltonian quasilinear diffusion equation

Using the solution Eq. 4.6 for $\delta\tilde{G}_\ell$, we are now ready to calculate the quasilinear diffusion Eq. 3.23 and obtaina simple dyadic form for the quasilinear diffusion tensor.

4.3.1 Quasilinear diffusion in guiding-center ($p_\parallel$, μ)-space

Now that the solution for the eikonal amplitude δg is obtained in terms of the guiding-center phase-space function $\delta\tilde{g} = \delta\tilde{G}\,\exp(i\,\Lambda)$, we are now able to evaluate the gyroangle-averaged expressions in Eq. 3.23. We begin with the gyroangle-averaged quadratic term

$$\begin{aligned}\langle\delta\tilde{H}\,\delta\tilde{g}^*\rangle &= \langle\delta\tilde{H}\left(\delta\tilde{G}e^{i\Lambda}\right)^*\rangle = \langle\!\left(\delta\tilde{H}e^{-i\Lambda}\right)\delta\tilde{G}^*\rangle \\ &= \sum_{\ell=-\infty}^{\infty}\delta\tilde{G}_\ell^*\langle\delta\tilde{H}e^{-i\Lambda+i\,\ell\,(\phi-\psi)}\rangle \\ &= \sum_{\ell=-\infty}^{\infty} q\,\delta\tilde{\Psi}_\ell\,\delta\tilde{G}_\ell^* \\ &= \sum_{\ell=-\infty}^{\infty}\frac{q^2}{\Omega}\,|\delta\tilde{\Psi}_\ell|^2\,\Delta_\ell^*\left(k_\parallel\,\frac{\partial f_0}{\partial p_\parallel}+\frac{\ell\,\Omega}{B_0}\,\frac{\partial f_0}{\partial\mu}\right),\end{aligned} \tag{4.7}$$

so that

$$\frac{k_\parallel}{\Omega}\,\mathrm{Re}\langle i\,\delta\tilde{H}\delta\tilde{g}^*\rangle = \sum_{\ell=-\infty}^{\infty}k_\parallel\,\mathcal{D}_\ell\left(k_\parallel\,\frac{\partial f_0}{\partial p_\parallel}+\frac{\ell\,\Omega}{B_0}\,\frac{\partial f_0}{\partial\mu}\right), \tag{4.8}$$

where we introduced the quasilinear perturbation potential

$$\mathcal{D}_\ell = \mathrm{Re}(-i\,\Delta_\ell)\,|(q/\Omega)\,\delta\tilde{\Psi}_\ell|^2 \equiv \mathrm{Re}(-i\,\Delta_\ell)\,|\delta\tilde{\mathcal{J}}_\ell|^2, \tag{4.9}$$

and

$$\frac{\partial}{\partial p_\parallel}\left(\frac{k_\parallel}{\Omega}\,\mathrm{Re}\langle i\,\delta\tilde{H}\delta\tilde{g}^*\rangle\right) \equiv \frac{\partial}{\partial p_\parallel}\left(D_{\rm H}^{pp}\,\frac{\partial f_0}{\partial p_\parallel}+D_{\rm H}^{p\mu}\,\frac{\partial f_0}{\partial\mu}\right), \tag{4.10}$$

where $D_{\rm H}^{pp} = \sum_\ell k_\parallel^2\,\mathcal{D}_\ell$ and $D_{\rm H}^{p\mu} = \sum_\ell k_\parallel\,(\ell\,\Omega/B_0)\,\mathcal{D}_\ell$.

Next, we find

$$\begin{aligned}&\langle\left(\frac{q}{c}\delta\tilde{\mathbf{A}}\cdot\frac{\partial\mathbf{v}_\perp}{\partial\phi}+i\,\frac{\mathbf{k}\cdot\mathbf{v}_\perp}{\Omega}\,\delta\tilde{H}\right)\delta\tilde{g}^*\rangle \\ &= \langle\left(\frac{q}{c}\delta\tilde{\mathbf{A}}\cdot\frac{\partial\mathbf{v}_\perp}{\partial\phi}+i\,\frac{\mathbf{k}\cdot\mathbf{v}_\perp}{\Omega}\,\delta\tilde{H}\right)\left(\delta\tilde{G}e^{i\Lambda}\right)^*\rangle \\ &= \langle\left[\frac{q}{c}\delta\tilde{\mathbf{A}}\cdot\frac{\partial\mathbf{v}_\perp}{\partial\phi}\,e^{-i\Lambda}-i\,q\,\frac{\partial\Lambda}{\partial\phi}\left(\delta\tilde{\Phi}-\frac{v_\parallel}{c}\,\delta\tilde{A}_\parallel-\frac{\mathbf{v}_\perp}{c}\cdot\delta\tilde{\mathbf{A}}\right)\right. \\ &\quad\left.\times\,e^{-i\Lambda}\right]\delta\tilde{G}^*\rangle \\ &= -\langle\frac{\partial}{\partial\phi}\left(\delta\tilde{H}e^{-i\Lambda}\right)\delta\tilde{G}^*\rangle = \langle\delta\tilde{H}\,e^{-i\Lambda}\,\frac{\partial\delta\tilde{G}^*}{\partial\phi}\rangle \\ &= \sum_{\ell=-\infty}^{\infty}i\ell\,\delta\tilde{G}_\ell^*\langle\delta\tilde{H}e^{-i\Lambda+i\,\ell\,(\phi-\psi)}\rangle = \sum_{\ell=-\infty}^{\infty}i\ell\,q\,\delta\tilde{\Psi}_\ell\,\delta\tilde{G}_\ell^* \\ &= \sum_{\ell=-\infty}^{\infty}i\,\Delta_\ell^*\,\ell\Omega\,(q/\Omega)^2\,|\delta\tilde{\Psi}_\ell|^2\left(k_\parallel\,\frac{\partial f_0}{\partial p_\parallel}+\frac{\ell\,\Omega}{B_0}\,\frac{\partial f_0}{\partial\mu}\right),\end{aligned} \tag{4.11}$$

so that

$$\begin{aligned}&\frac{1}{B_0}\,\frac{\partial}{\partial\mu}\left[\mathrm{Re}\langle\left(\frac{q}{c}\delta\tilde{\mathbf{A}}\cdot\frac{\partial\mathbf{v}_\perp}{\partial\phi}+i\,\frac{\mathbf{k}\cdot\mathbf{v}_\perp}{\Omega}\,\delta\tilde{H}\right)\delta\tilde{g}^*\rangle\right] \\ &= \frac{\partial}{\partial\mu}\left[\sum_{\ell=-\infty}^{\infty}\frac{\ell\,\Omega}{B_0}\,\mathcal{D}_\ell\left(k_\parallel\,\frac{\partial f_0}{\partial p_\parallel}+\frac{\ell\,\Omega}{B_0}\,\frac{\partial f_0}{\partial\mu}\right)\right] \\ &\equiv \frac{\partial}{\partial\mu}\left(D_{\rm H}^{\mu p}\,\frac{\partial f_0}{\partial p_\parallel}+D_{\rm H}^{\mu\mu}\,\frac{\partial f_0}{\partial\mu}\right),\end{aligned} \tag{4.12}$$

where $D_{\rm H}^{\mu p} = \sum_\ell\,(\ell\,\Omega/B_0)\,k_\parallel\,\mathcal{D}_\ell$ and $D_{\rm H}^{\mu\mu} = \sum_\ell\,(\ell\,\Omega/B_0)^2\,\mathcal{D}_\ell$. We can now write the Hamiltonian quasilinear diffusion Eq. 3.23 as

$$\begin{aligned}\frac{1}{\Omega}\,\frac{\partial f_0}{\partial\tau} &= \frac{\partial}{\partial p_\parallel}\left(D_{\rm H}^{pp}\,\frac{\partial f_0}{\partial p_\parallel}+D_{\rm H}^{p\mu}\,\frac{\partial f_0}{\partial\mu}\right) \\ &\quad+\frac{\partial}{\partial\mu}\left(D_{\rm H}^{\mu p}\,\frac{\partial f_0}{\partial p_\parallel}+D_{\rm H}^{\mu\mu}\,\frac{\partial f_0}{\partial\mu}\right).\end{aligned} \tag{4.13}$$

This quasilinear diffusion equation will later be compared with the standard quasilinear diffusion Eq. 2.32 derived by Kennel and Engelmann (Kennel and Engelmann, 1966).

4.3.2 Quasilinear diffusion in guiding-center ($\mathrm{J_g}$, $\mathcal{E}$)-space

Before proceeding with this comparison, however, we consider an alternate representation for the Hamiltonian quasilinear diffusion Eq. 4.13, which will be useful in the derivation of a quasilinear diffusion equation for nonuniform magnetized plasmas. If we replace the guiding-center parallel momentum $p_\parallel$ with the guiding-center kinetic energy $\mathcal{E} = p_\parallel^2/2m + \mu\,B_0$, and the guiding-center magnetic moment μ with the gyroaction $\mathrm{J_g} = \mu B_0/\Omega$, the Fourier eikonal solution Eq. 4.6 becomes

$$\delta\tilde{G}_\ell = q\,\delta\tilde{\Psi}_\ell\,\frac{\partial f_0}{\partial\mathcal{E}}+\frac{q}{\Omega}\,\delta\tilde{\Psi}_\ell\,\Delta_\ell\left(\omega\,\frac{\partial f_0}{\partial\mathcal{E}}+\ell\,\frac{\partial f_0}{\partial\mathrm{J_g}}\right), \tag{4.14}$$

where the first term on the right side is interpreted as a guiding-center adiabatic contribution to the perturbed Vlasov distribution (Brizard, 1994), while the remaining terms (proportional to the resonant denominator Δ_ℓ) are non-adiabatic contributions.

By substituting this new solution in Eq. 4.8, we find

$$\frac{k_\parallel}{\Omega}\,\mathrm{Re}\langle i\,\delta\tilde{H}\delta\tilde{g}^*\rangle = \sum_{\ell=-\infty}^{\infty}k_\parallel\,\mathcal{D}_\ell\left(\omega\,\frac{\partial f_0}{\partial\mathcal{E}}+\ell\,\frac{\partial f_0}{\partial\mathrm{J_g}}\right), \tag{4.15}$$

while

$$\begin{aligned}&\mathrm{Re}\langle\left(\frac{q}{c}\delta\tilde{\mathbf{A}}\cdot\frac{\partial\mathbf{v}_\perp}{\partial\phi}+i\,\frac{\mathbf{k}\cdot\mathbf{v}_\perp}{\Omega}\,\delta\tilde{H}\right)\delta\tilde{g}^*\rangle \\ &= \sum_{\ell=-\infty}^{\infty}\ell\Omega\,\mathcal{D}_\ell\left(\omega\,\frac{\partial f_0}{\partial\mathcal{E}}+\ell\,\frac{\partial f_0}{\partial\mathrm{J_g}}\right),\end{aligned} \tag{4.16}$$

where the guiding-center adiabatic contribution has cancelled out. The guiding-center quasilinear diffusion Eq. 4.13 becomes

$$\frac{1}{\Omega}\frac{\partial f_0}{\partial \tau} = v_{\|}\frac{\partial}{\partial \mathcal{E}}\left[\frac{1}{v_{\|}}\left(D_{\rm H}^{\mathcal{E}\mathcal{E}}\frac{\partial f_0}{\partial \mathcal{E}} + D_{\rm H}^{\mathcal{E}\mathsf{J}}\frac{\partial f_0}{\partial \mathsf{J}_{\rm g}}\right)\right] + v_{\|}\frac{\partial}{\partial \mathsf{J}_{\rm g}}\left[\frac{1}{v_{\|}}\left(D_{\rm H}^{\mathsf{J}\mathcal{E}}\frac{\partial f_0}{\partial \mathcal{E}} + D_{\rm H}^{\mathsf{J}\mathsf{J}}\frac{\partial f_0}{\partial \mathsf{J}_{\rm g}}\right)\right], \tag{4.17}$$

where the guiding-center quasilinear diffusion tensor is represented in 2 × 2 matrix form as

$$\mathbf{D}_{\rm H} \equiv \sum_{\ell=-\infty}^{\infty}\begin{pmatrix}\ell^2 & \ell\omega \\ \omega\,\ell & \omega^2\end{pmatrix}\mathcal{D}_\ell. \tag{4.18}$$

We note that, because of the simple dyadic form of Eq. 4.18, other representations for the guiding-center quasilinear diffusion tensor $\mathbf{D}_{\rm H}$ can be easily obtained, e.g., by replacing the guiding-center gyroaction $\mathsf{J}_{\rm g}$ with the pitch-angle coordinate $\xi = \sqrt{1-\mu B_0/\mathcal{E}}$. We also note that the dyadic quasilinear tensor (4.18) has a simple modular form compared to the dyadic form Eq. 2.29.

4.4 Comparison with Kennel-Engelmann quasilinear theory

We can now compare the Kennel-Engelmann quasilinear diffusion Eq. 2.32 with the guiding-center Hamiltonian quasilinear diffusion Eq. 4.13. First, we express the perturbed fields Eqs. 2.34, 2.35 in terms of the perturbed potentials ($\delta\Phi$, $\delta\mathbf{A}$):

$$\begin{aligned}\delta\tilde{P}_{\|\ell} &= M\,\delta\tilde{V}_{\|\ell} \\ &= \frac{q}{\Omega}\left[J_\ell\left(-i\,k_{\|}\,\delta\tilde{\Phi} + i\,\frac{\omega}{c}\,\delta\tilde{A}_{\|}\right) - \frac{v_\perp}{c}\left(i\mathbf{k}\,\delta\tilde{A}_{\|} - i\,k_{\|}\,\delta\tilde{\mathbf{A}}\right)\cdot\mathbb{J}_{\perp\ell}\right] \\ &= -i\,k_{\|}\,\delta\tilde{\mathcal{J}}_\ell + i\left(\omega - k_{\|}v_{\|} - \ell\,\Omega\right)\frac{q\delta\tilde{A}_{\|}}{\Omega\,c}\,J_\ell,\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\delta\tilde{\mu}_\ell &= \frac{Mv_\perp}{B_0}\,\delta\tilde{V}_{\perp\ell} \\ &= \frac{q\,v_\perp\,\mathbb{J}_{\perp\ell}}{B_0\Omega}\cdot\left[-i\,\mathbf{k}\,\delta\tilde{\Phi} + i\,\frac{\omega}{c}\,\delta\tilde{\mathbf{A}} + \frac{v_{\|}}{c}\left(i\mathbf{k}\,\delta\tilde{A}_{\|} - i\,k_{\|}\,\delta\tilde{\mathbf{A}}\right)\right] \\ &= -i\,\frac{\ell\,\Omega}{B_0}\,\delta\tilde{\mathcal{J}}_\ell + i\left(\omega - k_{\|}v_{\|} - \ell\,\Omega\right)\frac{q\delta\tilde{\mathbf{A}}}{cB_0\Omega}\cdot v_\perp\mathbb{J}_{\perp\ell},\end{aligned} \tag{4.20}$$

which are both gauge invariant according to the transformation (Eq. 3.10). Hence, these perturbed fields are expressed in terms of a contribution from the perturbed action $\delta\tilde{\mathcal{J}}_\ell$ and a contribution that vanishes for resonant particles (i.e., $k_{\|}\,v_{\|{\rm res}} = \omega - \ell\Omega$). We note that, in the resonant-particle limit ($\Delta_\ell \to \infty$), the difference between the Kennel-Engelmann formulation and the Hamiltonian formulation vanishes. For example, the Kennel-Engelmann quasilinear diffusion coefficient $D^{pp} = \sum_\ell {\rm Re}(-i\Delta_\ell)\,|\delta\tilde{P}_{\|\ell}|^2$ is expressed as

$$D^{pp} = \sum_\ell {\rm Re}(-i\Delta_\ell)\left[k_{\|}^2\,|\delta\tilde{\mathcal{J}}_\ell|^2 + 2k_{\|}J_\ell{\rm Re}\left(\frac{\delta\tilde{\mathcal{J}}_\ell^*}{\Delta_\ell}\,\frac{q\delta\tilde{A}_{\|}}{\Omega c}\right) + \left(\frac{q}{\Omega c}\right)^2\frac{|\delta\tilde{A}_{\|}|^2 J_\ell^2}{|\Delta_\ell|^2}\right] \to D_{\rm H}^{pp}, \tag{4.21}$$

which yields $D_{\rm H}^{pp}$ in the resonant-particle limit ($\Delta_\ell \to \infty$).

In summary, we have shown that, in the resonant-particle limit ($\Delta_\ell \to \infty$), the Hamiltonian quasilinear diffusion Eq. 4.13 is identical to the standard quasilinear diffusion Eq. 2.32 derived by Kennel and Engelmann (Kennel and Engelmann, 1966) for the case of a uniform magnetized plasma. In the next Section, we will see how the Hamiltonian quasilinear formalism can be extended to the case of a nonuniform magnetized plasma.

5 Hamiltonian quasilinear formulations for nonuniform magnetized plasma

In this Section, we briefly review the Hamiltonian formulation for quasilinear diffusion in a nonuniform magnetized background plasma. In an axisymmetric magnetic-field geometry, the 2 × 2 quasilinear diffusion tensor in velocity space is generalized to a 3 × 3 quasilinear diffusion tensor that includes radial quasilinear diffusion. In a spatially magnetically-confined plasma, the process of radial diffusion is a crucial element in determining whether charged particles leave the plasma. A prime example is provided by the case of radial diffusion in Earth's radiation belt, which was recently reviewed by Lejosne and Kollmann (Lejosne and Kollmann, 2020).

We present two non-relativistic Hamiltonian formulations of quasilinear diffusion in a nonuniform magnetized plasmas. The first one based on the canonical action-angle formalism (Kaufman, 1972b; Mahajan and Chen, 1985; Mynick and Duvall, 1989; Schulz, 1996) and the second one based on a summary of our previous work (Brizard and Chan, 2004).

5.1 Canonical action-angle formalism

When a plasma is confined by a nonuniform magnetic field, the charged-particle orbits can be described in terms of 3 orbital angle coordinates $\boldsymbol{\theta}$ (generically referred to as the gyration, bounce, and precession-drift angles) and their canonically-conjugate 3 action coordinates $\mathbf{J}$ (generically referred to as the gyromotion, bounce-motion, and drift-motion actions). In principle, these action coordinates are adiabatic invariants of the particle motion and they are calculated according to standard methods of guiding-center theory (Tao et al., 2007; Cary and Brizard, 2009), which are expressed in terms of asymptotic expansions in powers of a small dimensionless parameter $\epsilon_B = \rho/L_B \ll 1$ defined as the ratio of a characteristic gyroradius (for a given particle species) and the gradient length scale L_B associated with the background magnetic field $\mathbf{B}_0$. When an asymptotic expansion for an adiabatic invariant $\mathsf{J} = \mathsf{J}_0 + \epsilon_B \mathsf{J}_1$ is truncated at first order, for example, we find $d\mathsf{J}/dt \sim \epsilon_B^2$ and the orbital angular average $\langle d\mathsf{J}/dt\rangle = 0$ is the necessary condition for the adiabatic invariance of J. The reader is referred to Refs. (Cary and Brizard, 2009) and (Tao et al., 2007) where explicit expansions for all three guiding-center adiabatic invariants are derived in the non-relativistic and relativistic limits, respectively, for arbitrary background magnetic geometry.

The canonical action-angle formulation of quasilinear theory assumes that, in the absence of wave-field perturbations, the action coordinates $\mathbf{J}$ are constants of motion $d\mathbf{J}/dt = -\partial H_0/\partial\boldsymbol{\theta} = 0$, which follows from the invariance of the unperturbed Hamiltonian $H_0(\mathbf{J})$ on the canonical orbital angles $\boldsymbol{\theta}$. In this case, the unperturbed Vlasov distribution $F_0(\mathbf{J})$ is a function of action coordinates only. We note that the action coordinates considered here are either exact invariants or adiabatic invariants (Kaufman, 1972b; Mynick and Duvall, 1989) of the particle motion, and it is implicitly assumed that any adiabatic action invariant used in this canonical action-angle formulation of quasilinear theory can be calculated to sufficiently high order in ϵ_B within a region of particle phase space that excludes non-adiabatic diffusion in action space (Bernstein and Rowlands, 1976). For example, see Ref. (Brizard and Markowski, 2022) for a brief discussion of the breakdown of the adiabatic invariance of the magnetic moment (on the bounce time scale) for charged particles trapped by an axisymmetric dipole magnetic field.

In the presence of wave-field perturbations, the perturbed Hamiltonian can be represented in terms of a Fourier decomposition in terms of a discrete wave spectrum ω_k and orbital angles (with Fourier-index vector $\mathbf{m}$):

$$\delta\mathcal{H}(\mathbf{J},\boldsymbol{\theta},t) = \sum_{\mathbf{m},k} \delta\tilde{\mathcal{H}}(\mathbf{J})\;\exp\left(i\mathbf{m}\cdot\boldsymbol{\theta} - i\,\omega_k t\right) + \text{c.c.}, \tag{5.1}$$

where the parametric dependence of $\delta\tilde{\mathcal{H}}$ on the Fourier indices ($\mathbf{m}$, k) is hidden. The perturbed Vlasov distribution δF is obtained from the perturbed Vlasov equation

$$\frac{\partial\delta F}{\partial t} + \frac{\partial\delta F}{\partial\boldsymbol{\theta}}\cdot\frac{\partial H_0}{\partial\mathbf{J}} = \frac{\partial\delta\mathcal{H}}{\partial\boldsymbol{\theta}}\cdot\frac{\partial F_0}{\partial\mathbf{J}}, \tag{5.2}$$

from which we obtain the solution for the Fourier component $\delta\tilde{f}$:

$$\delta\tilde{F} = -\left(\frac{\delta\tilde{\mathcal{H}}}{\omega_k - \mathbf{m}\cdot\boldsymbol{\Omega}}\right)\mathbf{m}\cdot\frac{\partial F_0}{\partial\mathbf{J}}, \tag{5.3}$$

where $\boldsymbol{\Omega}(\mathbf{J}) \equiv \partial H_0/\partial\mathbf{J}$ denotes the unperturbed orbital-frequency vector.

The quasilinear wave-particle interactions cause the Vlasov distribution $F_0(\mathbf{J},\tau)$ to evolve on a slow time scale $\tau = \epsilon^2 t$, represented by the quasilinear diffusion equation

$$\begin{aligned}\frac{\partial F_0(\mathbf{J},\tau)}{\partial\tau} &= \frac{1}{2}\langle\{\delta\mathcal{H},\delta F\}\rangle = \frac{1}{2}\frac{\partial}{\partial\mathbf{J}}\cdot\langle\frac{\partial\delta\mathcal{H}}{\partial\boldsymbol{\theta}}\;\delta F\rangle \\ &= \frac{\partial}{\partial\mathbf{J}}\cdot\left(\sum_{\mathbf{m},k}\mathbf{m}\;\text{Im}\langle\delta\tilde{\mathcal{H}}^*\delta\tilde{F}\rangle\right) \\ &= \frac{\partial}{\partial\mathbf{J}}\cdot\left[\text{Im}\left(\sum_{\mathbf{m},k}\frac{-\mathbf{mm}\;|\delta\tilde{\mathcal{H}}|^2}{\omega_k - \mathbf{m}\cdot\boldsymbol{\Omega}}\right)\cdot\frac{\partial F_0}{\partial\mathbf{J}}\right] \\ &\equiv \frac{\partial}{\partial\mathbf{J}}\cdot\left(\mathsf{D}_{\text{QL}}\cdot\frac{\partial F_0}{\partial\mathbf{J}}\right),\end{aligned} \tag{5.4}$$

where $\langle\;\rangle$ includes orbital-angle averaging and wave time-scale averaging, and the canonical quasilinear diffusion tensor

$$\mathsf{D}_{\text{QL}} \equiv \sum_{\mathbf{m},k}\mathbf{mm}\left[\pi\delta\left(\omega_k - \mathbf{m}\cdot\boldsymbol{\Omega}\right)\right]|\delta\tilde{H}|^2 \tag{5.5}$$

is expressed in terms of a dyadic Fourier tensor $\mathbf{mm}$, a wave-particle resonance condition obtained from the Plemelj formula

$$\text{Im}\left(\frac{-1}{\omega_k - \mathbf{m}\cdot\boldsymbol{\Omega}}\right) = \text{Re}\left(\frac{i}{\omega_k - \mathbf{m}\cdot\boldsymbol{\Omega}}\right) = \pi\,\delta\left(\omega_k - \mathbf{m}\cdot\boldsymbol{\Omega}\right),$$

and the magnitude squared of the perturbed Hamiltonian Fourier component $\delta\tilde{H}(\mathbf{J})$, which is an explicit function of the action coordinates $\mathbf{J}$ and the perturbation fields [see of Ref. (Brizard and Chan, 2004), for example]. We note that the perturbed Hamiltonian $\delta\tilde{H}(\mathbf{J})$ will, therefore, include terms that contain a product of an adiabatic action coordinate (such as the gyro action J_g) and a wave perturbation factor (such as $\delta B/B_0 \sim\epsilon_\delta$). This means that an expansion of an adiabatic action coordinate (e.g., $\mathsf{J}_\text{g} = \mathsf{J}_\text{g}^{(0)} + \epsilon_B\,\mathsf{J}_\text{g}^{(1)} + \cdots$) in the factor $|\delta\tilde{H}|^2$ in Eq. 5.5 results in a leading term of order ϵ_δ^2, followed by negligible terms of order $\epsilon_B\,\epsilon_\delta^2 \ll \epsilon_\delta^2$. Hence, only a low-order expansion (in ϵ_B) of the adiabatic action coordinates $\mathbf{J} \simeq \mathbf{J}_0$ is needed in an explicit evaluation of Eq. 5.5. In addition, we note that the form Eq. 5.4, with Eq. 5.5, guarantees that the Vlasov entropy $S_0 = -\int F_0\ln F_0\;d^3J$

$$\begin{aligned}\frac{dS_0}{dt} &= -\epsilon^2\int\frac{\partial F_0}{\partial\tau}\left(\ln F_0 + 1\right)d^3J \\ &= \epsilon^2\sum_{\mathbf{m},k}\int F_0\left(\mathbf{m}\cdot\frac{\partial\ln F_0}{\partial\mathbf{J}}\right)^2\pi\,\delta\left(\omega_k - \mathbf{m}\cdot\boldsymbol{\Omega}\right)|\delta\tilde{\mathcal{H}}|^2\,d^3J > 0\end{aligned} \tag{5.6}$$

satisfies the H Theorem. Lastly, we note that collisional transport in a magnetized plasma can also be described in terms of drag and diffusion in action space (Bernstein and Molvig, 1983).

5.2 Local and bounce-averaged wave-particle resonances in quasilinear theory

The canonical action-angle formalism presented in Section 5.1 unfortunately makes use of the bounce action $\mathsf{J}_\text{b} = \oint p_\parallel(s)\,ds$, which is a nonlocal quantity (Northrop, 1963), while the drift action $\mathsf{J}_\text{d} \equiv (q/2\pi c)\oint\psi\,d\varphi = q\psi/c$ is a local coordinate in an axisymmetric magnetic field $\mathbf{B} = \nabla\psi\times\nabla\varphi$, where the drift action is canonically conjugate to the toroidal angle φ. In our previous work (Brizard and Chan, 2001; Brizard and Chan, 2004), we replaced the bounce action with the guiding-center kinetic energy $\mathcal{E}$ in order to obtain a local quasilinear diffusion equation in three-dimensional $\mathsf{J}^i = (\mathsf{J}_b, \mathcal{E}, \mathsf{J}_d)$ guiding-center invariant space:

$$\frac{\partial F_0}{\partial \tau} = \frac{\partial}{\partial \mathbf{J}} \cdot \left(\mathbf{D}_{\rm QL} \cdot \frac{\partial F_0}{\partial \mathbf{J}} \right) = \frac{1}{\tau_b} \frac{\partial}{\partial \mathsf{J}^i} \left(\tau_b \, D_{\rm QL}^{ij} \, \frac{\partial F_0}{\partial \mathsf{J}^j} \right), \quad (5.7)$$

where the bounce period $\tau_b \equiv \oint ds/v_\parallel$ is the Jacobian. In addition, the 3×3 quasilinear diffusion tensor

$$\mathbf{D}_{\rm QL} = \sum_{\ell,k,m} \begin{pmatrix} \ell^2 & \ell\,\omega_k & \ell\, m \\ \omega_k \ell & \omega_k^2 & \omega_k m \\ m\ell & m\omega_k & m^2 \end{pmatrix} \Gamma_{\ell k m} \quad (5.8)$$

is defined in terms of the Fourier indices ℓ (associated with the gyroangle ζ) and m (associated with the toroidal angle φ) and the wave frequency ω_k, while the scalar $\Gamma_{\ell k m}$ was shown in Ref. (Brizard and Chan, 2004) to include the bounce-averaged wave-particle resonance condition

$$\omega_k = \ell \, \langle \omega_c \rangle_b + n\, \omega_b + m\, \langle \omega_d \rangle_b, \quad (5.9)$$

where $\langle \omega_c \rangle_b = (q/Mc)\, \langle B \rangle_b$ and $\langle \omega_d \rangle_b$ are the bounce-averaged cyclotron and drift frequencies, respectively, and $\omega_b = 2\pi/\tau_b$ is the bounce frequency. Here, the bounce-average operation is defined as

$$\langle \cdots \rangle_b \equiv \frac{1}{\tau_b} \sum_\sigma \int_{s_L}^{s_U} \frac{ds}{|v_\parallel|} \, (\cdots), \quad (5.10)$$

where $\sigma \equiv v_\parallel/|v_\parallel|$ denotes the sign of the parallel guiding-center velocity, and the points $s_{L,U}(\mathbf{J})$ along a magnetic field line are the bounce (turning) points where $v_\parallel$ changes sign (for simplicity, we assume all particles are magnetically trapped). In this Section, we present a brief derivation of the quasilinear diffusion Eq. 5.7, with the 3×3 quasilinear diffusion tensor Eq. 5.8 and the wave-particle resonance condition Eq. 5.9, based on our previous work (Brizard and Chan, 2004), which is presented here in the non-relativistic limit.

We begin with the linear guiding-center Vlasov equation in guiding-center phase space $(s, \varphi, \zeta; \mathbf{J})$:

$$\frac{d_0 \delta F}{dt} = \frac{\partial \delta F}{\partial t} + \{\delta F,\, \mathcal{E}\}_{\rm gc} = -\{F_0,\, \delta H\}_{\rm gc}, \quad (5.11)$$

where the perturbed Hamiltonian is a function of the guiding-center invariants $(\mathsf{J}_b, \mathcal{E}, \mathsf{J}_d)$ as well as the angle-like coordinates (s, φ, ζ). The unperturbed guiding-center Poisson bracket, on the other hand, is

$$\{F, G\}_{\rm gc} = \frac{\partial F}{\partial \zeta}\frac{\partial G}{\partial \mathsf{J}_g} - \frac{\partial F}{\partial \mathsf{J}_g}\frac{\partial G}{\partial \zeta} + \frac{\partial F}{\partial \varphi}\frac{\partial G}{\partial \mathsf{J}_d} - \frac{\partial F}{\partial \mathsf{J}_d}\frac{\partial G}{\partial \varphi} + \left(\frac{d_0 F}{dt} - \frac{\partial F}{\partial t} \right)\frac{\partial G}{\partial \mathcal{E}} - \frac{\partial F}{\partial \mathcal{E}}\left(\frac{d_0 G}{dt} - \frac{\partial G}{\partial t} \right), \quad (5.12)$$

and $d_0/dt = \partial/\partial t + v_\parallel\, \partial/\partial s + \omega_d\, \partial/\partial\varphi + \omega_c\, \partial/\partial\zeta$ denotes the unperturbed Vlasov operator (s denotes the local spatial coordinate along an unperturbed magnetic-field line). Since the right side of Eq. 5.11 is

$$-\{F_0,\, \delta H\}_{\rm gc} = \frac{\partial F_0}{\partial \mathsf{J}_g}\frac{\partial \delta H}{\partial \zeta} + \frac{\partial F_0}{\partial \mathsf{J}_d}\frac{\partial \delta H}{\partial \varphi} + \frac{\partial F_0}{\partial \mathcal{E}}\left(\frac{d_0 \delta H}{dt} - \frac{\partial \delta H}{\partial t} \right), \quad (5.13)$$

we can introduce the non-adiabatic decomposition (Chen and Tsai, 1983)

$$\delta F \equiv \delta H \, \frac{\partial F_0}{\partial \mathcal{E}} + \delta G, \quad (5.14)$$

where the non-adiabatic contribution δG satisfies the perturbed non-adiabatic Vlasov equation

$$\frac{d_0 \delta G}{dt} = \left(\frac{\partial F_0}{\partial \mathsf{J}_g}\frac{\partial}{\partial \zeta} + \frac{\partial F_0}{\partial \mathsf{J}_d}\frac{\partial}{\partial \varphi} - \frac{\partial F_0}{\partial \mathcal{E}}\frac{\partial}{\partial t} \right)\delta H \equiv \hat{\mathcal{F}}\, \delta H. \quad (5.15)$$

Next, since the background plasma is time independent and axisymmetric, and the unperturbed guiding-center Vlasov distribution is independent of the gyroangle, we perform Fourier transforms in (φ, ζ, t) so that Eq. 5.15 becomes

$$\begin{aligned} \left[v_\parallel \frac{\partial}{\partial s} - i\,(\omega_k - \ell\,\omega_c - m\,\omega_d) \right] \delta\tilde{G}(s, \sigma) &\equiv \hat{\mathcal{L}}\, \delta\tilde{G}(s,\sigma) \\ &= i\, \mathcal{F}\, \delta\tilde{H}(s,\sigma), \end{aligned} \quad (5.16)$$

where the amplitudes $(\delta\tilde{G}, \delta\tilde{H})$ depend on the spatial parallel coordinate s and the sign $\sigma = v_\parallel/|v_\parallel| = \pm 1$, as well as the invariants $\mathbf{J}$, while the operator $\hat{\mathcal{F}}$ becomes $i\,\mathcal{F}$, with

$$\mathcal{F} \equiv \omega_k \, \frac{\partial F_0}{\partial \mathcal{E}} + \ell\, \frac{\partial F_0}{\partial \mathsf{J}_g} + m\, \frac{\partial F_0}{\partial \mathsf{J}_d}. \quad (5.17)$$

In order to remove the dependence of the perturbed Hamiltonian $\delta\tilde{H}$ on σ (which appears through the combination $v_\parallel \delta\tilde{A}_\parallel$), we follow our previous work (Brizard and Chan, 2004) and introduce the gauge $\delta\tilde{A}_\parallel \equiv \partial\delta\tilde{\alpha}/\partial s$ and the transformation $(\delta\tilde{G}, \delta\tilde{H}) \rightarrow (\delta\tilde{G}', \delta\tilde{K})$, where $\delta\tilde{G}' = \delta\tilde{G} + i\,(q/c)\mathcal{F}\,\delta\tilde{\alpha}$ and $\delta\tilde{K} = \delta\tilde{H} + (q/c)\,\hat{\mathcal{L}}\,\delta\tilde{\alpha}$, so that Eq. 5.16 becomes $\hat{\mathcal{L}}\,\delta\tilde{G}'(s,\sigma) = i\,\mathcal{F}\,\delta\tilde{K}(s)$.

In order to obtain an integral solution for $\delta\tilde{G}'$, we now introduce the integrating factor

$$\begin{aligned} &\left[v_\parallel \frac{\partial}{\partial s} - i\,(\omega_k - \ell\,\omega_c - m\,\omega_d) \right] \delta\tilde{G}'(s,\sigma) \\ &\quad \equiv e^{i\sigma\theta} v_\parallel \, \frac{\partial}{\partial s}\left[e^{-i\sigma\theta} \delta\tilde{G}'(s,\sigma) \right] = i\, \mathcal{F}\, \delta\tilde{K}(s), \end{aligned} \quad (5.18)$$

where

$$\theta(s) \equiv \int_{s_L}^{s} \left(\omega_k - \ell\,\omega_c(s') - m\,\omega_d(s') \right) \frac{ds'}{|v_\parallel|} \quad (5.19)$$

is defined in terms of the lower (L) turning point $s_L(\mathbf{J})$. The solution of Eq. 5.18 is, therefore, expressed as

$$\delta\tilde{G}'(s,\sigma) = \delta\bar{G}'\, e^{i\sigma\theta} + i\,\sigma\, e^{i\sigma\theta}\left(\int_{s_L}^{s} \delta\tilde{K}(s')\, e^{-i\sigma\theta(s')}\frac{ds'}{|v_\||}\right)\mathcal{F} \tag{5.20}$$

where the constant amplitude $\delta\bar{G}'$ is determined from the matching conditions $\delta\tilde{G}'(s_L,+1) = \delta\tilde{G}'(s_L,-1)$ and $\delta\tilde{G}'(s_U,+1) = \delta\tilde{G}'(s_U,-1)$ at the two turning points. At the lower turning point, the matching condition implies that $\delta\bar{G}'$ is independent of σ. The matching condition at the upper turning point, on the other hand, is expressed as

$$e^{i\Theta}\,\delta\bar{G}' + \frac{i\tau_b}{2}\,\langle\delta\tilde{K}e^{-i\theta}\rangle_b\, e^{i\Theta}\,\mathcal{F} = e^{-i\Theta}\,\delta\bar{G}' - \frac{i\tau_b}{2}\,\langle\delta\tilde{K}e^{i\theta}\rangle_b \times e^{-i\Theta}\,\mathcal{F},$$

which yields

$$\delta\bar{G}' = -\frac{\tau_b}{2}\left(\cot\Theta\,\langle\delta\tilde{K}\cos\theta\rangle_b + \langle\delta\tilde{K}\sin\theta\rangle_b\right)\mathcal{F}, \tag{5.21}$$

where

$$\Theta \equiv \theta(s_U) = \frac{\tau_b}{2}\left(\omega_k - \ell\,\langle\omega_c\rangle_b - m\,\langle\omega_d\rangle_b\right). \tag{5.22}$$

We note that $\cot\Theta$ in Eq. 5.21 has singularities at $n\pi$, which immediately leads to the resonance condition Eq. 5.9.

Now that the solution $\delta\tilde{G}'$ has been determined, we can proceed with the derivation of the quasilinear diffusion equation, which has been shown by Brizard and Chan (Brizard and Chan, 2004) to be expressed as

$$\begin{aligned}\frac{\partial F_0}{\partial\tau} &= \frac{1}{\tau_b}\frac{\partial}{\partial\mathcal{E}}\left[\tau_b\left(\sum_{\ell,k,m}\omega_k\,\Gamma_{\ell km}\,\mathcal{F}\right)\right] \\ &+ \frac{1}{\tau_b}\frac{\partial}{\partial J_g}\left[\tau_b\left(\sum_{\ell,k,m}\ell\,\Gamma_{\ell km}\,\mathcal{F}\right)\right] \\ &+ \frac{1}{\tau_b}\frac{\partial}{\partial J_d}\left[\tau_b\left(\sum_{\ell,k,m}m\,\Gamma_{\ell km}\,\mathcal{F}\right)\right] \\ &\equiv \frac{1}{\tau_b}\frac{\partial}{\partial J^i}\left(\tau_b\, D_{QL}^{ij}\,\frac{\partial F_0}{\partial J^j}\right),\end{aligned} \tag{5.23}$$

which requires us to evaluate $\Gamma_{\ell km} \equiv \mathcal{F}^{-1}\mathrm{Im}\langle\delta\tilde{H}^*\delta\tilde{G}\rangle_b = \mathcal{F}^{-1}\mathrm{Im}\langle\delta\tilde{K}^*\delta\tilde{G}'\rangle_b$, which is found to be expressed as

$$\Gamma_{\ell km} = \frac{\tau_b}{2}\mathrm{Im}(-\cot\Theta)\,\left|\langle\delta\tilde{K}\cos\theta\rangle_b\right|^2, \tag{5.24}$$

where, using the Plemelj formula with the identity $\cot z = \sum_{n=-\infty}^{\infty}(z-n\pi)^{-1}$, we finally obtain

$$\Gamma_{\ell km} = \left|\langle\delta\tilde{K}\cos\theta\rangle_b\right|^2\sum_{n=-\infty}^{\infty}\pi\,\delta\left(\omega_k - \ell\,\langle\omega_c\rangle_b - n\,\omega_b - m\,\langle\omega_d\rangle_b\right). \tag{5.25}$$

This expression completes the derivation of the quasilinear diffusion tensor Eq. 5.8 and the perturbed Hamiltonian $\delta\tilde{K}$ is fully defined in Ref. (Brizard and Chan, 2004). We note that, in the limit of low-frequencies electromagnetic fluctuations, we also recover our previous work (Brizard and Chan, 2001) from Eq. 5.8.

We now make a few remarks concerning the bounce-averaged wave-particle resonance condition Eq. 5.9. First, in the case of a uniform magnetized plasma (with the drift frequency $\omega_d \equiv 0$), we substitute the eikonal representations $\delta\tilde{G} = \delta\bar{G}\,\exp(ik_\| s)$ and $\delta\tilde{H} = \delta\bar{H}\,\exp(ik_\| s)$ in Eq. 5.16 and we recover the uniform quasilinear diffusion Eq. 4.17. Second, the bounce-averaged wave-particle resonance condition Eq. 5.9 assumes that the waves are coherent on the bounce-time scale, which is not realistic for high-frequency (VLF), short-wavelength whistler waves (Stenzel, 1999; Allanson et al., 2021). We recover a local wave-particle resonance condition by introducing the bounce-angle coordinate $\xi(s)$ (Brizard, 2000), which is defined by the equation $d\xi/ds = \omega_b/v_\|$, so that $v_\|\,\partial/\partial s$ in Eq. 5.16 is replaced with $\omega_b\,\partial/\partial\xi$. Next, by introducing the bounce-angle Fourier series $\delta\tilde{G} = \sum_{n=-\infty}^{\infty}\delta\bar{G}\,\exp(in\xi)$ and $\delta\tilde{H} = \sum_{n=-\infty}^{\infty}\delta\bar{H}\,\exp(in\xi)$ in Eq. 5.16, the integral phase Eq. 5.19 is replaced by the new integral phase

$$\begin{aligned}\sigma\chi(s) &= \sigma\,\theta(s) - n\,\xi(s) \\ &= \sigma\int_{s_L}^{s}\left(\omega_k - \ell\,\omega_c(s') - m\,\omega_d(s') - n\,\omega_b\right)\frac{ds'}{|v_\||}.\end{aligned} \tag{5.26}$$

If we now evaluate this integral by stationary-phase methods (Stix, 1992), the dominant contribution comes from points s_0 along a magnetic-field line where

$$0 = \chi'(s_0) = |v_\|(s_0)|^{-1}\left(\omega_k - \ell\,\omega_c(s_0) - m\,\omega_d(s_0) - n\,\omega_b\right), \tag{5.27}$$

which yields the local wave-particle resonance condition, provided $v_\|(s_0) \neq 0$ (i.e., the local resonance does not occur at a turning point).

6 Summary

In the present paper, we have established a direct connection between the standard reference of quasilinear theory for a uniform magnetized plasma by Kennel and Engelmann (Kennel and Engelmann, 1966) and its Hamiltonian formulation in guiding-center phase space. We have also shown that the transition to a quasilinear theory for a nonuniform magnetized plasma is greatly facilitated within a Hamiltonian formulation. The main features of a Hamiltonian formulation of quasilinear theory is that the quasilinear diffusion tensor has a simple modular dyadic form in which a matrix of Fourier indices is multiplied by a single quasilinear scalar potential, which includes the resonant wave-particle delta function. This simple modular is observed in the case of a uniform magnetized plasma, as seen in Eq. 4.18, as well as in the case of a nonuniform magnetized plasma, as seen in Eq. 5.8. In particular, we note that the quasilinear diffusion tensor Eq. 5.8 naturally incorporates quasilinear radial diffusion as well as its synergistic connections to diffusion in two-dimensional invariant velocity space. These features are easily extended to the quasilinear diffusion of relativistic charged particles that are magnetically confined by nonuniform magnetic fields.

Author contributions

AB has written 90% of the manuscript. AC has added technical references as well as historical context.

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Conflict of interest

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Supplementary material

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References

Allanson, O., Watt, C. E. J., Allison, H. J., and Ratcliffe, H. (2021). Electron diffusion and advection during nonlinear interactions with whistler-mode waves. *JGR. Space Phys.* 126, 028793. doi:10.1029/2020ja028793

Bernstein, I. B., and Molvig, K. (1983). Lagrangian formulation of neoclassical transport theory. *Phys. Fluids* 26, 1488. doi:10.1063/1.864319

Bernstein, I., and Rowlands, G. (1976). Diffusion in velocity space associated with nonadiabatic changes in the magnetic moment of a charged particle. *Phys. Fluids* 19, 1546. doi:10.1063/1.861358

Brizard, A. (1994). Eulerian action principles for linearized reduced dynamical equations. *Phys. Plasmas* 1, 2460–2472. doi:10.1063/1.870574

Brizard, A. J., and Chan, A. A. (2001). Relativistic bounce-averaged quasilinear diffusion equation for low-frequency electromagnetic fluctuations. *Phys. Plasmas* 8, 4762–4771. doi:10.1063/1.1408623

Brizard, A. J., and Chan, A. A. (2004). Relativistic quasilinear diffusion in axisymmetric magnetic geometry for arbitrary-frequency electromagnetic fluctuations. *Phys. Plasmas* 11, 4220–4229. doi:10.1063/1.1773554

Brizard, A. J., and Chandre, C. (2020). Hamiltonian formulations for perturbed dissipationless plasma equations. *Phys. Plasmas* 27, 122111. doi:10.1063/5.0028471

Brizard, A. J. (2000). Nonlinear bounce-gyrocenter Hamiltonian dynamics in general magnetic field geometry. *Phys. Plasmas* 7, 3238–3246. doi:10.1063/1.874189

Brizard, A. J. (2018). Perturbative variational formulation of the Vlasov-Maxwell equations. *Phys. Plasmas* 25, 112112. doi:10.1063/1.5049570

Brizard, A., and Markowski, D. (2022). On the validity of the guiding-center approximation in a magnetic dipole field. *Phys. Plasmas* 29, 022101. doi:10.1063/5.0078786

Cary, J. R., and Brizard, A. J. (2009). Hamiltonian theory of guiding-center motion. *Rev. Mod. Phys.* 81, 693–738. doi:10.1103/revmodphys.81.693

Chen, L., and Tsai, S.-T. (1983). Linear oscillations in general magnetically confined plasmas. *Plasma Phys.* 25, 349–359. doi:10.1088/0032-1028/25/4/001

Crews, D., and Shumlak, U. (2022). On the validity of quasilinear theory applied to the electron bump-on-tail instability. *Phys. Plasmas* 29, 043902. doi:10.1063/5.0086442

Davidson, R. C. (1972). *Methods in nonlinear plasma theory*. Cambridge, Massachusetts, United States: Academic Press.

Dewar, R. (1973). Oscillation center quasilinear theory. *Phys. Fluids* 16, 1102. doi:10.1063/1.1694473

Dodin, I. (2022). Quasilinear theory for inhomogeneous plasma. *J. Plasma Phys.* 88, 905880407. doi:10.1017/s0022377822000502

Dupree, T. (1966). A perturbation theory for strong plasma turbulence. *Phys. Fluids* 9, 1773. doi:10.1063/1.1761932

Galeev, A., and Sagdeev, R. (1983). *Basic plasma physics*, 1. Amsterdam: North Holland.

Kaufman, A. N., and Cohen, B. I. (2019). Theoretical plasma physics. *J. Plasma Phys.* 85, 205850601. doi:10.1017/s0022377819000667

Kaufman, A. (1972). Quasilinear diffusion of an axisymmetric toroidal plasma. *Phys. Fluids* 15, 1063. doi:10.1063/1.1694031

Kaufman, A. (1972). Reformulation of quasi-linear theory. *J. Plasma Phys.* 8, 1–5. doi:10.1017/s0022377800006887

Kennel, C., and Engelmann, F. (1966). Velocity space diffusion from weak plasma turbulence in a magnetic field. *Phys. Fluids* 9, 2377. doi:10.1063/1.1761629

Krommes, J. A. (2002). Fundamental statistical descriptions of plasma turbulence in magnetic fields. *Phys. Rep.* 360, 1–352. doi:10.1016/s0370-1573(01)00066-7

Lejosne, S., and Kollmann, P. (2020). Radiation belt radial diffusion at Earth and beyond. *Space Sci. Rev.* 216, 19. doi:10.1007/s11214-020-0642-6

Lerche, I. (1968). Quasilinear theory of resonant diffusion in a magneto-active, relativistic plasma. *Phys. Fluids* 11, 1720. doi:10.1063/1.1692186

Mahajan, S., and Chen, C. Y. (1985). Plasma kinetic theory in action-angle variables. *Phys. Fluids* 28, 3538. doi:10.1063/1.865308

Mynick, H., and Duvall, R. (1989). Erratum: "A unified theory of tokamak transport via the generalized balescu–lenard collision operator" [phys. Fluids B **1**, 750 (1989)]. *Phys. Fluids B Plasma Phys.* 1, 1758. doi:10.1063/1.859212

Northrop, T. G. (1963). *The adiabatic motion of charged particles*. Geneva: Interscience Publishers.

Sagdeev, R. Z., and Galeev, A. A. (1969). *Nonlinear plasma theory*. California: W. A. Benjamin.

Schulz, M. (1996). Canonical coordinates for radiation-belt modeling. *Geophys. Monogr. - Am. Geophys. Union* 97, 153.

Stenzel, R. L. (1999). Whistler waves in space and laboratory plasmas. *J. Geophys. Res.* 104, 14379–14395. doi:10.1029/1998ja900120

Stix, T. (1992). *Waves in plasmas*. United States: American Institute of Physics.

Tao, X., Chan, A., and Brizard, A. (2007). Hamiltonian theory of adiabatic motion of relativistic charged particles. *Phys. Plasmas* 14, 092107. doi:10.1063/1.2773702