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Distributed Optimal Resource Allocation with Time-Varying Quadratic Cost Functions and Resources over Switching Agents

Amir-Salar Esteki and Solmaz S. Kia *Senior Member, IEEE*

***Abstract*—This paper proposes a distributed solution for an optimal resource allocation problem with a time-varying cost function and time-varying demand. The objective is to minimize a global cost, which is the summation of local quadratic time-varying cost functions, by allocating time-varying resources. A reformulation of the original problem is developed and is solved in a distributed manner using only local interactions over an undirected connected graph. In the proposed algorithm, the local state trajectories converge to a bounded neighborhood of the optimal trajectory. This bound is characterized in terms the parameters of the cost and topology properties. We also show that despite the tracking error, the trajectories are feasible at all times, meaning that the resource allocation equality constraint is met at every execution time. Our algorithm also considers the possibility of some generators going out of production from time to time and adjusts the solution so that the remaining generators can meet the demands in an optimal manner. Numerical examples demonstrate our results.**

***Keywords*—Resource Allocation, Distributed Optimization, Time-varying Optimization**

I. INTRODUCTION

This paper considers the well-known resource allocation problem where a group of agents seeks to solve a distributed constrained optimization problem where the equality constraint is met by the summation of local decision variables. This problem can either be solved centrally or in a distributed manner. Some centralized solutions can be found in [1] and [2] where one agent or a trusted third party collects all the information from each agent, solves the problem locally and then distributes the solution to other agents. However, in smart grid systems, due to the scalability and robustness of networked systems, it is more desirable to solve this problem in a distributed fashion [3], [4]. Economic dispatch (ED) is a practical example in smart grid systems where agents solve a resource allocation problem. For time-invariant local costs, [5] implements frequency control in an electric grid to solve an ED problem in continuous-time which is later extended to discrete-time. Considering the case where transmission line losses and generator constraints are present, in [5], agents use two consensus-based algorithms to reach the optimal load. Authors of [6]–[8] have also used a consensus-based approach to solve the ED problem. Particularly in [6], the incremental cost of each generator is found by using the first order optimality conditions, and then the agents use that to derive the optimal local load. Moreover, [9] employs consensus protocols and saddle point dynamics to propose a solution. In a time-invariant setting, [10] proposes two algorithms to solve the resource allocation problem that are free of initialization constraints regarding the distribution of resources. Using a control theoretic approach, [11] guarantees convergence even over switching topologies.

The works mentioned above consider the case of time-invariant costs and resources. We now seek to review optimal resource allocation problems where cost function parameters or resource constraints are time-varying. In this setting, instead of converging to an equilibrium point to reach the solution, agents track an optimal trajectory. Some recent works [12] and [13] introduce resource allocation algorithms in continuous-time. [12] studies two cases where the Hessian of local costs are common among the agents and, for otherwise, proposes a second algorithm where the average Hessian is estimated to find the solution. In both works, finite-time convergence is attained by implementing sign functions in the algorithm. In most applications, sign functions cause chattering and are impractical in a discrete-time setting. Note that [12] does not take into account time-varying resources. In contrast, [14] achieves asymptotic convergence by incorporating a correction and a prediction term. These former consist of a continuous-time version of Newton's method, and the latter assures a vanishing optimality gap. [15] also implements a prediction–correction scheme to achieve asymptotic convergence where time-varying costs are sampled in specific sampling periods.

In this paper, we propose to use a dynamic weighted average consensus algorithm to solve the resource allocation problem in a discrete-time manner. Cost parameters are time-varying and are sampled with a sampling rate over time. In addition, the total demand or resource is time-varying which suits more practical scenarios. Our algorithm can also take into account the possibility that agents might stop participating in the resource allocation problem. In smart grid systems, specifically in a time-varying fashion, network generators may stop responding at times and, therefore, cannot generate any load. The proposed algorithm is flexible to track the new optimal trajectory in a distributed manner. In our main result, we prove that the tracking error is bounded with respect to local parameters and topology properties. Our simulations show how the algorithm copes with switching agents and still tracks the optimal solution.

Notations: We follow [16] for graph theoretic terminologies. The interaction topology of N in-network agents is modeled by the undirected connected graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathbf{A})$ where $\mathcal{V}$ is the node set, $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ is the edge set and $\mathbf{A}=[\mathsf{a}_{ij}]$ is the

The authors are with the Department of Mechanical and Aerospace Engineering, University of California Irvine, Irvine, CA 92697, {aesteki,solmaz}@uci.edu. This work was supported by NSF award ECCS-1653838.

adjacency matrix defined such that $\mathbf{a}_{ij} > 0$, if $(i,j) \in \mathcal{E}$, otherwise $\mathbf{a}_{ij} = 0$. A graph is undirected if $\mathbf{a}_{ij} = \mathbf{a}_{ji}$ for all $i,j \in \mathcal{V}$. Moreover, a graph is connected if there is a directed path from every node to every other node. The degree of each node $i \in \mathcal{V}$ is $\mathsf{d}^i = \sum_{j=1}^N \mathbf{a}_{ij}$ and the Laplacian matrix of a graph $\mathcal{G}$ is $\mathbf{L} = \mathrm{Diag}(\mathsf{d}^1, \cdots, \mathsf{d}^N) - \mathbf{A}$. Furthermore, For a connected graph, we denote the eigenvalues of $\mathbf{L}$ by $\lambda_1, \cdots, \lambda_N$, where $\lambda_1 = 0$ and $\lambda_i \leq \lambda_j$, for $i < j$ and λ_2 and λ_N are, respectively, the smallest nonzero eigenvalue and maximum eigenvalue of $\mathbf{L}$. Finally, given an edge (i,j), i is called a neighbor of j, and vice versa. We let $\mathbf{1}_N$ denote the vector of N ones, and denote by $\mathbf{I}_N$ the $N \times N$ identity matrix. We also define $\mathfrak{r} = \frac{1}{\sqrt{N}}\mathbf{1}_N$, $\mathfrak{R} \in \mathbb{R}^{N\times(N-1)}$ and $\mathbf{T} = [\mathfrak{r} \quad \mathfrak{R}]$, such that $[\mathfrak{r} \quad \mathfrak{R}] \begin{bmatrix} \mathfrak{r}^\top \\ \mathfrak{R}^\top \end{bmatrix} = \begin{bmatrix} \mathfrak{r}^\top \\ \mathfrak{R}^\top \end{bmatrix} [\mathfrak{r} \quad \mathfrak{R}] = \mathbf{I}_N$. Note that $\mathbf{T}^\top\mathbf{T} = \mathbf{T}\mathbf{T}^\top = \mathbf{I}$, and for a connected graph, $\mathbf{T}^\top\mathbf{L}\mathbf{T} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^+ \end{bmatrix}$, where $\mathbf{L}^+ = \mathfrak{R}^\top\mathbf{L}\mathfrak{R}$. $\mathbf{L}^+$ is a positive definite matrix with eigenvalues $\{\lambda_i\}_{i=2}^N \in \mathbb{R}_{>0}$.

II. PROBLEM SETTING

In this section, we present the resource allocation problem with time-variant cost parameters and total demand where we then use its Lagrangian multiplier to form an unconstrained optimization problem. In central solvers, we show that using the gradient descent method as a solution results in maintaining a tracking error. We later in the section present solutions that utilize a prediction-correction term to avoid this tracking error.

A. Resource Allocation Problem

We study the resource allocation problem over an undirected connected graph $\mathcal{G}$. Agents of the network, denoted as $\mathcal{V} = \{1, 2, \cdots, N\}$, are endowed with a decision variable $p^i(t)$ and a corresponding local cost function

$$f^i(p^i(t), t) = \frac{1}{2\beta^i(t)}(p^i(t) + \alpha^i(t))^2, \quad i \in \mathcal{V}$$

where $\alpha^i(t) \in \mathbb{R}$ and $\beta^i(t) \in \mathbb{R}_{>0}$ are time-variant local cost parameters. Moreover, agents meet the total time-variant demand $D(t)$, i.e., $p^1(t) + p^2(t) + \cdots + p^N(t) = D(t)$. This optimal resource allocation problem is formulated as a constrained optimization problem, minimizing $\sum_{i=1}^N f^i(p^i(t), t)$ while maintaining a total demand $D(t)$, $\forall t \geq 0$, i.e.,

$$\min_{\mathbf{p}\in\mathbb{R}^N} \quad \sum\nolimits_{i=1}^N f^i(p^i(t), t), \tag{1a}$$

$$\text{s.t.} \quad p^1(t) + p^2(t) + \cdots + p^N(t) = \sum\nolimits_{i=1}^N d^i(t), \tag{1b}$$

$$\sum\nolimits_{i=1}^N d^i(t) = D(t), \tag{1c}$$

where $\mathbf{p}(t) = [p^1(t), p^2(t), \cdots, p^N(t)]^\top$. Note that to satisfy (1c), either each agent $i \in \mathcal{V}$ chooses $d^i(t) = \frac{1}{N}D(t)$ or without loss of generality, $d^1(t) = D(t)$ and $d^j(t) = 0$ for $j \in \mathcal{V}/1$. Let us now use the KKT conditions in [17] to obtain the dual form of (1) which is an unconstrained optimization problem. In this setting, agents of the network track a common signal, which is the solution to the unconstrained optimization problem, and then obtain $p^{i\star}$ locally by only using local information. We later show that this reformulation aids us to solve the unconstrained optimization problem by implementing a dynamic weighted average consensus algorithm. Let us consider the Lagrangian for the optimization problem in (1) as

$$L(\mathbf{p}(t), \mu(t), t) = \sum_{i=1}^N f^i(p^i(t), t) + \mu(\sum_{i=1}^N d^i(t) - \sum_{i=1}^N p^i(t)),$$

where $\mu(t)$ is the Lagrange multiplier corresponding to the equality constraint (1b). Using the first order optimality conditions, we have

$$\frac{\partial f^i}{\partial p^i} p^{i\star}(t) - \mu^\star(t) = 0, \quad i \in \mathcal{V} \tag{2a}$$

$$\sum\nolimits_{i=1}^N p^{i*}(t) - \sum\nolimits_{i=1}^N d^i(t) = 0, \tag{2b}$$

where by substituting (2b) in (2a), it is then concluded that

$$\mu^\star(t) = \frac{\sum_{i=1}^N d^i(t) + \sum_{i=1}^N \alpha^i(t)}{\sum_{i=1}^N \beta^i(t)}, \tag{3a}$$

$$p^{i\star}(t) = \beta^i(t)\mu^\star(t) - \alpha^i(t), \quad i \in \mathcal{V}. \tag{3b}$$

In section III, we propose an algorithm where agents track the common signal $\mu^\star$ and derive $p^{i\star}$ locally. This solution only applies when all agents are participating in the resource allocation problem, e.g., a smart grid system where all generators are generating power at all times. In a time-varying setting, agents may not participate in the network at all times. For example, a generator may shut down at specific periods of time and therefore, cannot produce any power due to its absence. Therefore, the solution presented in (3) is no longer valid since $p^i(t) = 0$ for $i \in \mathcal{V}_t^0$, where $\mathcal{V}_t^0$ denotes the set of absent agents at time t. Moreover, let us denote the set of agents participating in the network as $\mathcal{V}_t^1 = \mathcal{V}/\mathcal{V}_t^0$. The following lemma presents the solution to the resource allocation problem when the set $\mathcal{V}_t^0$ is not empty.

Lemma II.1. *Let us consider a group of agents over the network $\mathcal{G}$ solving the resource allocation problem (1). If at time t, only agents $i \in \mathcal{V}/\mathcal{V}_t^0$ are participating in the network, the optimal solution is*

$$\mu_p^\star(t) = \frac{\sum_{i=1}^N d^i(t) + \sum_{i\in\mathcal{V}_t^1} \alpha^i(t)}{\sum_{i\in\mathcal{V}_t^1} \beta^i(t)}, \tag{4a}$$

$$p_p^{i\star}(t) = \beta^i(t)\mu_p^\star(t) - \alpha^i(t), \quad i \in \mathcal{V}_t^1, \tag{4b}$$

$$p_p^{i\star}(t) = 0, \quad i \in \mathcal{V}_t^0. \tag{4c}$$

Proof: If $p^i(t) = 0$, for $i \in \mathcal{V}_t^1$, then the optimal resource allocation problem is formulated as

$$\begin{aligned} \min_{\mathbf{p}\in\mathbb{R}^N} \quad & \sum\nolimits_{i\in\mathcal{V}_t^1} f^i(p^i(t), t) + \sum\nolimits_{i\in\mathcal{V}_t^0} f^i(0, t), \\ \text{s.t.} \quad & \sum\nolimits_{i\in\mathcal{V}_t^1} p^i(t) = \sum\nolimits_{i=1}^N d^i(t), \\ & p^i(t) = 0, \quad i \in \mathcal{V}_t^0 \end{aligned}$$

where $f^i(0,t) = \frac{\alpha^i(t)}{2\beta^i(t)}$ for $i \in \mathcal{V}_t^0$. For brevity, we omit solving the new optimization problem and one can simply show that the solution is (4). ■

B. Centralized Solutions

In some scenarios, e.g., when the network is small or data is shared globally, centralized algorithms are practical. Thus, if local cost parameters α^i and β^i, and the total demand $D(t)$ are available to a central agent, one can propose a centralized solution. Consider the optimization problem

$$\mu^\star(t) = \arg\min_{\mu(t)\in\mathbb{R}} \sum_{i=1}^{N} \underbrace{\Big(\frac{1}{2}\beta^i(t)\mu^2(t) - (d^i(t)+\alpha^i(t))\mu(t)\Big)}_{g^i(\mu(t),t)}, \tag{6}$$

which induces the same solution as in (3a). The problem above with time-variant costs $g^i(\mu(t),t)$ can be interpreted as consecutive optimization problems with fixed costs. Note that $g^i(\mu(t),t)$ is convex for all $i \in \mathcal{V}$ and therefore, $\sum_{i=1}^N g^i(\mu(t),t)$ is also convex. If at time instants t_k with $k = 0,1,2,\cdots$ we sample the cost function, one can solve the time-invariant problem

$$\mu^\star(t_k) := \arg\min_{\mu\in\mathbb{R}} \sum\nolimits_{i=1}^{N} g^i(\mu, t_k)$$

where t_k is fixed [18]. However, this approach may not be practical in some applications [19], [20]. Thus, [21] proposes to implement the gradient descent algorithm to solve the time-varying optimization problem. In this work, they estimate an error bound of $\|x(t_k)-x^\star(t_k)\| = O(h)$, where $h = t_k - t_{k-1}$, with respect to the optimal trajectory. On the other hand, [18] proposes a prediction-correction scheme that incorporates the time derivative of the cost function and achieves an outperforming error bound $\|x(t_k)-x^\star(t_k)\| = O(h^2)$. In the next chapter, we address the distributed version of this problem, where cost parameters $\alpha^i(t)$ and $\beta^i(t)$ are only locally available to the agents, hence a central solution is impractical. Moreover, agents are not entitled to participate in the resource allocation problem at all times, which are denoted as switching agents. In this setting, due to large-scale networks and privacy concerns, cost parameters cannot be shared and we solve the constrained optimization problem (1) in a distributed fashion.

III. MAIN RESULT

In this section, we introduce a new scheme to solve the time-variant resource allocation problem with switching agents by casting (4a) as the solution of a dynamic weighted average consensus algorithm. To solve this problem in discrete-time, $\alpha^i(t)$, $\beta^i(t)$ and $d^i(t)$ for $i \in \mathcal{V}$ are considered continuous-time signals that are sampled by each agent $i \in \mathcal{V}$ at sampling times $t_l^s = l\delta_s \in \mathbb{R}_{\geq 0}, l \in \mathbb{Z}_{\geq 0}, \delta_s \in \mathbb{R}_{>0}$. Also, agents of the network communicate at discrete-times $t_k^c = k\delta_c \in \mathbb{R}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}, \delta_c \in \mathbb{R}_{>0}$. Here, we assume that if $\delta_s \neq \delta_c$, then $\beta^i(k) = \beta^i(\bar{l})$, $\alpha^i(k) = \alpha^i(\bar{l})$ and $d^i(k) = d^i(\bar{l})$ where $\bar{l}$ is the most up to date sampling time step such that $t_{\bar{l}}^s \leq t_k^c$. Moreover, $\mu_p^\star(k)$ is the discrete-time representation of (4a). Here, we propose the following dynamic weighted average consensus algorithm, originally introduced in [22], where agents track (4a) to obtain (4b)-(4c) locally.

$$p^i(k) = \beta^i(k)\mu^i(k) - \alpha^i(k) + \frac{1}{\delta_c}(z^i(k+1) - z^i(k)) \tag{7a}$$

$$\mu^i(k) = z^i(k) + d^i(k) + \alpha^i(k) \tag{7b}$$

$$\begin{aligned} z^i(k+1) = z^i(k) - \delta_c\big(&\beta^i(k)\mu^i(k) - d^i(k) - \alpha^i(k) \\ &+ \sum_{j=1}^{N} \mathsf{a}_{ij}(\mu^i(k)+v^i(k)-\mu^j(k)-v^j(k)\big), \end{aligned} \tag{7c}$$

$$v^i(k+1) = v^i(k) + \delta_c \sum\nolimits_{j=1}^{N} \mathsf{a}_{ij}(\mu^i(k) - \mu^j(k)), \tag{7d}$$

$$z^i(0), v^i(0) \in \mathbb{R},\ i \in \mathcal{V}. \tag{7e}$$

Here, (7b)-(7d) are inspired by the main algorithm in [22] where agents track $\mu_p^\star(k)$ and each agent $i \in \mathcal{V}$ uses (7a) to track the local decision variable $p^{i\star}(k)$. Note that compared to the formulation in (3b), we have an additional term $\frac{z^i(k+1)-z^i(k)}{\delta_c}$ in the distributed proposed formulation. This is to keep the total demand preserved at all times, i.e., $\sum_{i=1}^N p^i(k) = D(k)$ for all $k \geq 0$. We can show that by substituting (7c) in (7a) to obtain the equivalent formulation $p^i(k) = d^i(k) + \sum_{j=1}^N \mathsf{a}_{ij}(\mu^i(k)+v^i(k)-\mu^j(k)-v^j(k))$ where $\sum_{i=1}^N p^i(k) = D(k)$. In most resource allocation applications, it is important to maintain the total demand at the expected value in order to avoid overload or system collapse. In (7), if $\beta^i(k) \in \mathbb{R}_{>0}$, each local state μ^i tracks (3a), however, if we set $\beta^i(k) \equiv \alpha^i(k) \equiv 0$ for $i \in \mathcal{V}_k^0$, we obtain the solution $\mu_p^\star(k)$.

The objective of algorithm (7) is to track the optimal trajectory $p^{i\star}(k)$ in real-time considering the switching agents. To do that, we begin with reviewing some supporting results for the proof of convergence. Afterwards, the main result is presented which offers a bound on the convergence error $\|p^i(k)-p^{i\star}(k)\|$ with respect to the optimal trajectory. Let us first consider the following definitions. We define $\boldsymbol{\mu}^\star(k) = \mu^\star(k)\mathbf{1}_N$, $\mathbf{E}(k) = \text{diag}(\beta^1(k),\cdots,\beta^N(k))$, $\Delta\boldsymbol{\alpha}(k) = \boldsymbol{\alpha}(k+1) - \boldsymbol{\alpha}(k)$ where $\boldsymbol{\alpha}(k) = \big[\alpha^1(k),\cdots,\alpha^N(k)\big]^\top$, $\mathbf{w}(k) = \boldsymbol{\alpha}(k) - \mathbf{E}(k)\boldsymbol{\mu}^\star(k)$ and $\Delta\mathbf{w}(k) = \mathbf{w}(k+1)-\mathbf{w}(k)$. We now present the compact form of (7) as

$$\boldsymbol{\mu}(k) = \mathbf{z}(k) + \mathbf{d}(k) + \boldsymbol{\alpha}(k) \tag{8a}$$

$$\mathbf{p}(k) = \mathbf{d}(k) + \mathsf{L}(\boldsymbol{\mu}(k) + \mathbf{v}(k)) \tag{8b}$$

$$\begin{aligned} \mathbf{z}(k+1) = \mathbf{z}(k) - \delta_c\big(&\boldsymbol{\beta}(k)\boldsymbol{\mu}(k) - \mathbf{d}(k) - \boldsymbol{\alpha}(k) \\ &+ \mathsf{L}(\boldsymbol{\mu}(k)+\mathbf{v}(k)\big), \end{aligned} \tag{8c}$$

$$\mathbf{v}(k+1) = \mathbf{v}(k) + \delta_c \mathsf{L}\boldsymbol{\mu}(k), \tag{8d}$$

$$\mathbf{z}(0), \mathbf{v}(0) \in \mathbb{R}^N,\ i \in \mathcal{V}, \tag{8e}$$

which by using the change of variables $\bar{\mathbf{e}} = \mathsf{T}^\top(\boldsymbol{\mu} - \boldsymbol{\mu}^\star)$

and $\begin{bmatrix} q_1 & \mathbf{q}_{2:N}^\top \end{bmatrix}^\top = \mathbf{T}^\top(\mathbf{L}\mathbf{v} - \mathbf{w})$ is equivalent to

$$q_1(k+1) = q_1(k), \tag{9a}$$

$$\begin{bmatrix} \bar{\mathbf{e}}(k+1) \\ \mathbf{q}_{2:N}(k+1) \end{bmatrix} = (\mathbf{I}+\delta_c\bar{\mathbf{A}}(k))\begin{bmatrix} \bar{\mathbf{e}}(k) \\ \mathbf{q}_{2:N}(k) \end{bmatrix} + \bar{\mathbf{B}}\begin{bmatrix} \Delta\boldsymbol{\alpha}(k) - \Delta\boldsymbol{\mu}^\star(k) \\ \Delta\mathbf{w}(k) \end{bmatrix} \tag{9b}$$

where $\bar{\mathbf{A}}(k) = \begin{bmatrix} -\mathbf{T}^\top(\mathbf{E}(k)+\mathbf{L})\mathbf{T} & -\begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{N-1} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0} & \mathbf{L}^+\mathbf{L}^+ \end{bmatrix} & \mathbf{0} \end{bmatrix}$ and $\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{T}^\top & \mathbf{0} \\ \mathbf{0} & \mathfrak{R}^\top \end{bmatrix}$. In the next result, we prove that by choosing an admissible step size δ_c, every subsystem $\mathbf{I}+\delta_c\bar{\mathbf{A}}(k)$ is Schur under the following assumption.

Assumption 1. *Every $\mathbf{E}(k)$ for $k \in \mathbb{Z}_{\geq 0}$ is bounded, i.e., there exist $\bar{\beta} \in \mathbb{R}_{>0}$ such that $\beta^i(k) \leq \bar{\beta}$ for $i \in \mathcal{V}$. Thus, $\mathbf{0} < \mathbf{E}(k) \leq \bar{\beta}\mathbf{I}$, $k \in \mathbb{Z}_{\geq 0}$. Note that $\mathbf{E}(k) \neq \mathbf{0}$.*

Lemma III.1. *Let the agents of the network implement a connected undirected graph $\mathcal{G}$ to communicate. If $\delta_c \in (0, \bar{\delta})$ where*

$$\bar{\delta} = \min\Big\{\{-2\frac{\mathrm{Re}(\gamma_{i,k})}{|\gamma_{i,k}|^2}\}_{i=1}^{2N-1}\Big\}_{k\in\mathbb{Z}_{\geq 0}}$$

in which $\{\gamma_{i,k}\}_{i=1}^{2N-1}$ are the set of eigenvalues of $\bar{\mathbf{A}}(k)$, then every subsystem $\mathbf{I} + \delta_c\bar{\mathbf{A}}(k)$, $k \in \mathbb{Z}_{\geq 0}$ is Schur.

Proof: Let us implement the results in [22, Lemma 2], provided that by using the Lyapunov stability analysis and the LaSalle invariant principle, every subsystem $\bar{\mathbf{A}}(k)$, $k \in \mathbb{Z}_{\geq 0}$ is Hurwitz. Under the Assumption 1, since $\mathbf{0} < \mathbf{E}(k) \leq \bar{\beta}\mathbf{I}$, $k \in \mathbb{Z}_{\geq 0}$, the eigenvalues of $\bar{\mathbf{A}}(k)$, denoted as $\{\gamma_{i,k}\}_{i=1}^{2N-1}$ are upper bounded, i.e., there exist $\bar{\gamma} \in \mathbb{R}_{<0}$ such that $\bar{\mathbf{A}}(k) \leq \bar{\gamma}\mathbf{I} < 0$. Therefore, by the virtue of [22, Lemma 3], $\mathbf{I} + \delta_c\bar{\mathbf{A}}(k)$, $k \in \mathbb{Z}_{\geq 0}$ is Schur if we have $\delta_c \in (0, \bar{\delta})$. ■

Provided the result above, we find the error bound of local decision variables with respect to the optimal trajectory. Before the final statement, let us present the following assumption.

Assumption 2. *The rate of change of signals $\alpha^i(k)$ and $d^i(k)$, $i \in \mathcal{V}$ are bounded, i.e., there exist $\bar{\alpha}, \bar{d} \in \mathbb{R}_{\geq 0}$, such that $\Delta\boldsymbol{\alpha}(k) \leq \bar{\alpha}$ and $\Delta\mathbf{d}(k) \leq \bar{d}$ where $\Delta\mathbf{d}(k) = \begin{bmatrix} d^1(k+1) - d^1(k), \cdots, d^N(k+1) - d^N(k) \end{bmatrix}^\top$*

Lemma III.1 discusses internal stability of system (9) with no inputs. In the next theorem, we show that with bounded inputs, i.e., trajectories of α^i,β^i and d^i, local decision variables of algorithm (7) converge to a neighborhood of the optimal solution. Furthermore, we determine this bound by applying a Lyapunov analysis.

Theorem III.2. *Initialized at $z^i(0), v^i(0) \in \mathbb{R}$, $i \in \mathcal{V}$, let the agents implement algorithm (7) over an undirected connected graph $\mathcal{G}$. Under the Assumption 1 and 2, and provided that $\delta_c \in (0, \bar{\delta})$, the trajectories of local states satisfy*

$$|p^i(k) - p^{i\star}(k)| \leq (\beta^i(k) + \frac{2}{\delta_c})\bar{U}\frac{\phi+1}{1-\phi^2}, \tag{10}$$

where $\bar{U}$ and ϕ are defined respectively in (11b) and (12).

Proof: Let us note that under the assumptions 1 and 2, we conclude that $\Delta\boldsymbol{\mu}^\star(k)$ and $\Delta\mathbf{w}(k)$ are bounded. Therefore, there exist $\bar{\mu^\star} \in \mathbb{R}_{\geq 0}$ and $\bar{U} \in \mathbb{R}_{\geq 0}$ such that

$$\|\Delta\boldsymbol{\mu}^\star(k)\| \leq \bar{\mu^\star} \tag{11a}$$

$$\left\|\begin{bmatrix} \Delta\boldsymbol{\alpha}(k) - \Delta\boldsymbol{\mu}^\star(k) \\ \Delta\mathbf{w}(k) \end{bmatrix}\right\| \leq \bar{U}. \tag{11b}$$

Let us now consider the candidate Lyapunov function

$$V(k) = \boldsymbol{\mathcal{X}}^\top(k)\boldsymbol{\mathcal{X}}(k),$$

for the system (9) where $\boldsymbol{\mathcal{X}}(k) = \begin{bmatrix} \bar{\mathbf{e}}(k) \\ \mathbf{q}_{2:N}(k) \end{bmatrix}$. We can show that along the trajectories of (9), we have

$$\begin{aligned}\Delta V(k) = &\boldsymbol{\mathcal{X}}^\top(k)(\mathbf{I} + \delta_c\bar{\mathbf{A}}(k))^\top(\mathbf{I} + \delta_c\bar{\mathbf{A}}(k))\boldsymbol{\mathcal{X}}(k) \\ &+ 2\boldsymbol{\mathcal{X}}^\top(k)(\mathbf{I}+\delta_c\bar{\mathbf{A}}(k))^\top\bar{\mathbf{B}}\mathbf{U}(k) + \mathbf{U}^\top(k)\bar{\mathbf{B}}^\top\bar{\mathbf{B}}\mathbf{U}(k) \\ &- \boldsymbol{\mathcal{X}}^\top(k)\boldsymbol{\mathcal{X}}(k),\end{aligned}$$

where $\Delta V(k) = V(k+1) - V(k)$ and $\mathbf{U} = \begin{bmatrix} \Delta\boldsymbol{\alpha}(k) - \Delta\boldsymbol{\mu}^\star(k) \\ \Delta\mathbf{w}(k) \end{bmatrix}$. According to Lemma III.1, if we choose the admissible step size $\delta_c \in (0, \bar{\delta})$, then $\mathbf{I} + \delta_c\bar{\mathbf{A}}(k)$ is Schur. Therefore, $\|\mathbf{I} + \delta_c\bar{\mathbf{A}}(k)\| < 1$. Let us define

$$\phi = \max\{\|\mathbf{I} + \delta_c\bar{\mathbf{A}}(k)\|\}_{k\in\mathbb{Z}_{\geq 0}}, \tag{12}$$

where we know that $0 < \phi < 1$. Also, since $\|\mathbf{U}\| \leq \bar{U}$ and $\|\bar{\mathbf{B}}\| < 1$, it is then concluded that

$$\Delta V(k) \leq (\phi^2 - 1)\|\boldsymbol{\mathcal{X}}(k)\|^2 + 2\phi\bar{U}\|\boldsymbol{\mathcal{X}}(k)\| + \bar{U}^2.$$

If we prove that the right hand side of the inequality above is always negative, we can conclude that $\Delta V(k) < 0$ holds. Such as statement is true if $\|\boldsymbol{\mathcal{X}}(k)\| > \bar{U}\frac{\phi+1}{1-\phi^2}$ or $\|\boldsymbol{\mathcal{X}}(k)\| < \bar{U}\frac{\phi-1}{1-\phi^2}$, which guarantees that $\Delta V(k) < 0$ for $k \in \mathbb{Z}_{\geq 0}$. Note that since $\phi < 1$ and $\|\boldsymbol{\mathcal{X}}(k)\| \geq 0$, the second inequality above is infeasible. Therefore, as long as $\|\boldsymbol{\mathcal{X}}(k)\| > \bar{U}\frac{\phi+1}{1-\phi^2}$, $V(k)$ is decreasing and consequently $\|\boldsymbol{\mathcal{X}}(k)\|$ is decreasing. Considering this result, we have $\|\boldsymbol{\mathcal{X}}(k)\| \leq \bar{U}\frac{\phi+1}{1-\phi^2}$ by using the Lyapunov stability analysis. Given that $|\mu^i(k) - \mu^\star(k)| \leq \|\boldsymbol{\mathcal{X}}(k)\|$, and by using the definition of $p^i(k)$ for $i \in \mathcal{V}$, we can establish the tracking error in (10). ■

The following section considers an economic dispatch problem where algorithm (7) is implemented as a solution. We investigate two scenarios with time-invariant and time-variant cost parameters where in the latter some agents participate only at specific times.

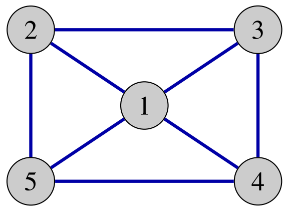


Fig. 1: An undirected connected graph with adjacency weights of $\mathbf{a}_{ij} = 1$, if $(i, j) \in \mathcal{E}$, otherwise $\mathbf{a}_{ij} = 0$. In second scenario of the economic dispatch problem, agent 5 stop generating any load and thus, $p^5(k) \equiv 0$ for $k \geq 2 \times 10^5$.

IV. NUMERICAL EXAMPLE

In this section, we illustrate the performance of the proposed algorithm and study its convergence. The resource allocation problem studied here is an economic dispatch problem where 5 generators are interacting over an undirected connected graph shown in Fig. 1 to track the optimal local load. In the first scenario, local cost parameters α and β, and the total demand D are time-invariant. In addition, no generator stops participating in the smart grid, i.e., all the generators are available to generate load at all times. However, in the second scenario, α, β, and D are all time-varying functions. Also, agent 1 stops participating after some time and therefore, cannot generate any load.

A. Time-invariant Scenario

In this example, agents are assigned with time-invariant costs. We use the IEEE bus 118 generator list to set the parameters of the local cost $f^i(p^i) = \frac{1}{2\beta_c^i}(p^i + \alpha_c^i)^2$, for $i \in \{1, 2, \cdots, 5\}$. The corresponding components are $\{\alpha_c^i\}_{i=1}^4 = \{188.3, 592.5, 2567.2, 1793.3, 2567.2\}$, $\{\beta_c^i\}_{i=1}^5 = \{7.17, 45.9, 208.2, 166.6, 208.2\}$ where the subscript c represents constant values. Here, the total demand to meet is $D_c = 25000$ and the distribution is equal among the agents, i.e., $d_c^i = 5000$ for each agent $i \in \mathcal{V}$. The optimal solution for the resource allocation problem is $p^{i\star} = \beta^i \frac{D + \sum_{i=1}^5 \alpha^i}{\sum_{i=1}^5 \beta^i} - \alpha^i$ which is asymptotically reached by agents of the network. We let the agents implement algorithm (7) to solve this problem. Fig. 2a indicates local trajectories $p^i(k)$ at $k \in \{0, \cdots, 5 \times 10^5$, represented by the blue lines. Moreover, the optimal trajectories $p^{i\star}$ are represented by the bred lines. Fig. 2b studies the convergence performance of the proposed algorithm by measuring the error $e^i(k) = \frac{|p^i(k) - p^{i\star}|}{|p^{i\star}|}$ with respect to the optimal local loads. As illustrated, the tracking error vanishes as $k \to \infty$ and the steady-state error reaches zero asymptotically. This is due to the fact that the cost and total demand are time-invariant, whereas we show in the next scenario that agents converge to a neighborhood of the optimal solution.

B. Time-variant Scenario

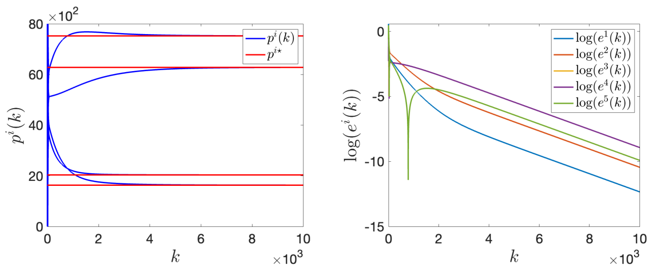
In the second scenario, we use time-varying functions for local cost parameters and demands as in the original setting of the problem. Let us also consider a situation where one of the generators shuts down at a specific time, and therefore cannot generate any load. This is a realistic scenario in smart grid systems where we plot it by setting $\alpha^5 \equiv \beta^5 \equiv 0$ at time step $k = 2 \times 10^5$. For local costs and demands, we perturb the constant values α, β, and D, introduced in the first scenario, by adding the following time-variant functions

$$\begin{aligned} \alpha_t^i(k) &= \frac{\alpha_c^i}{100}\sin(\omega i k), \\ \beta_t^i(k) &= \frac{\beta_c^i}{100}\cos(\omega i k), \\ d_t^i(k) &= \frac{d_c^i}{100}\sin(\omega k), \quad i \in \mathcal{V}, \end{aligned}$$

where $\omega = 10^{-5}$. Therefore, we have $\alpha^i(k) = \alpha_c^i + \alpha_t^i(k)$, $\beta^i(k) = \beta_c^i + \beta_t^i(k)$ and $d^i(k) = d_c^i + d_t^i(k)$. Note that β^i and d^i are positive at all times, and the rate of change of α^i and d^i are bounded. Let the agents of the network implement algorithm (7) to solve the resource allocation problem and track the optimal trajectories. Fig. 3a indicates the state trajectories $p^i(k)$ and the optimal trajectories $p^{i\star}(k)$, represented by the blue and the red lines, respectively. Provided by Theorem III.2, it is observed that the agents converge to a neighborhood of the optimal solution. The rate of change of cost parameters also affect the bound of the neighborhood. As seen in Fig. 3a, agents 1 and 2 track the optimal trajectory with less error compared to the other agents, since $\alpha^i(t)$ and $\beta^i(t)$ for $i = \{1, 2\}$ have lower frequencies. Note that $p^5(k) = 0$ for $k \geq 2 \times 10^5$ since agent 5 shuts down at a certain point. Fig. 3b shows the error $e^i(k) = \frac{|p^i(k) - p^{i\star}(k)|}{|p^{i\star}(k)|}$ with respect to the time-varying optimal trajectory. To avoid overfill, we omit to include the legends in this plot. As expected, our results show that by using the proposed algorithm, agents track the optimal solution when cost and demand are time-variant. Moreover, as in most smart grid systems where a generator is not necessarily available at all times, our algorithm is capable of handling such situations. This scenario is more noticeable in time-varying resource allocation problems since it is not a finite-time problem.

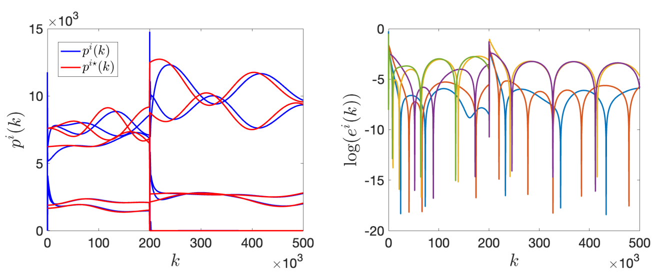
V. CONCLUSION

This paper studies the optimal resource allocation problem with time-varying costs and resources. The proposed algorithm tracks the optimal solution with a bounded tracking error. This bound is obtained by a Lyapunov stability analysis which is a function of local cost parameters and demands and the network topology. Moreover, agents can stop or start participating in the resource allocation problem at any time of the execution, as in most practical scenarios, e.g., in smart grid systems where a generator is out of order at some periods of time while the algorithm still tracks the optimal trajectory. Interestingly, despite the tracking error, the trajectories generated by our algorithm are feasible at all times, meaning that the resource allocation equality constraint is met at every execution time. Finally we demonstrated the algorithm's effectiveness in two scenarios, one



(a) (b)

Fig. 2: Trajectories of the proposed algorithm over the graph in Fig. 1 to obtain $p^{i\star}$ when local cost parameters and demands are time-invariant. The blue and the red lines represent the state trajectories $p^i(k)$ and optimal values $p^{i\star}$, respectively. The tracking error $|p^i(k) - p^{i\star}|$ is vanished asymptotically.



(a) (b)

Fig. 3: Execution of algorithm (7) over the graph in Fig. 1 while $\alpha^i(k)$, $\beta^i(k)$ and $d^i(k)$ are time-variant. As indicated, state trajectories $p^i(k)$ are represented by the red lines, whereas optimal trajectories $p^{i\star}(k)$ are represented by the red lines. The tracking error is bounded as seen in the figure.

in a time-invariant setting and the other with time-varying costs, demands, and switching agents. It is observed that the less frequent cost parameters change, the less tracking error we have.

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