A Study on Accelerating Average Consensus Algorithms Using Delayed Feedback

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***Abstract*—This article studies accelerating a Laplacian-based dynamic average consensus algorithm by splitting the conventional delay-free disagreement feedback into a weighted summation of current and outdated terms. When determining the weighted sum, there is a range of time delay that results in a higher convergence rate for the algorithm. For such weights, using the Lambert W function, we obtain the rate-increasing range of the time delay, obtain the maximum reachable rate, and characterize the value of the corresponding maximizer delay. We also study the effect of using the outdated feedback on the control effort of the agents. We show that only for some specific affine combination of the immediate and outdated feedback, the control effort of the agents does not go beyond that of the delay-free algorithm. In addition, we demonstrate that using outdated feedback does not increase the steady-state tracking error of the average consensus algorithm. Finally, we determine the optimum combination of the current and the outdated feedback weights to achieve the maximum increase in the rate of convergence without increasing the control effort of the agents. We demonstrate our results through a numerical example.**

***Index Terms*—Accelerated algorithms, average consensus, distributed systems.**

I. Introduction

THE average consensus problem for a group of N networked agents each endowed with a *reference input* r^i (a static reference value or a dynamic signal) is defined as designing a distributed interaction policy for each agent $i \in \{1, \dots, N\}$ such that its local agreement state $x^i \in \mathbb{R}$ converges asymptotically to the average of the reference inputs across the network. For this problem, in the continuous-time domain, when the reference inputs of all the agents are static, the well-known distributed solution is the Laplacian average consensus algorithm [1]–[4]. In the Laplacian average consensus, each agent i initializes its first-order integrator dynamics with its local reference input and uses the weighted sum of the difference between its local state and those of its neighbors, i.e., a weighted sum of $x^i - x^j$ with respect to every neighbor j, as the *disagreement feedback* to drive its local dynamics to the average of the reference signals across the network. When the reference inputs are dynamic signals, agents use a combination of the disagreement feedback and their local reference *input* signal and/or its derivative to drive their local integrator dynamics; see [5] for examples of dynamic average consensus algorithms. Average consensus algorithms are of interest in various multiagent applications such as sensor fusion [6], [7]; robot coordination [8]; formation control [9]; distributed optimal resource allocation [10], [11]; distributed optimization and learning [12], [13]; and distributed tracking [14]. For more details, see [5, pp. 47–55]. For these cooperative tasks, it is highly desired that the consensus among the agents is obtained fast, i.e., the consensus algorithm converges fast.

For a connected network with undirected communication, it is well understood that the convergence rate of the average consensus algorithms is associated with the connectivity of the graph [3], specified by the smallest nonzero eigenvalue of the Laplacian matrix [1], [5]. Given this connection, various efforts, such as optimal adjacency weight selection for a given topology to maximize the smallest nonzero eigenvalue of the Laplacian matrix [2], [15] or rewiring the graph to create topologies such as small-world networks [16], [17] with high connectivity, have been proposed in the literature to accelerate the Laplacian average consensus algorithm. In this article, we study the use of outdated disagreement feedback, i.e., delayed feedback, to increase the convergence rate of a dynamic average consensus algorithm. In our study, we pay careful attention to the effect of the use of outdated feedback on the agents' control effort and tracking performance. Our method can be applied together with the aforementioned weight and topology designs to maximize the convergence acceleration.

Our work is motivated by evidence in the literature on the positive effect of time delay on increasing stability margin and rate of convergence of time-delayed systems (see [18]–[23]). Specifically, the positive effect of time-delayed feedback in accelerating the convergence of the *static* Laplacian average consensus algorithm is reported in [21]–[23]. The study in [22] and [23] considers delaying the immediate disagreement feedback and shows that when the network topology is connected, there always exists a range of delay $(0, \tilde{\tau})$ such that the rate of convergence of the modified algorithm is faster. The technical results also include specifying $\tilde{\tau}$ and also showing that the maximum attainable convergence rate due to employing delayed feedback is the Euler number times the rate without delay. Moradian and Kia [22] also specify the delay for which

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maximum convergence rate is attained. However, the effect of using the outdated feedback on the control effort of the agents is left unexplored in [22] and [23]. An effective acceleration mechanism should not increase the control effort of the agents because one can always argue that the convergence rate of the original Laplacian algorithm can be increased simply by multiplying the agents' input with a gain greater than 1. On the other hand, the work in [21] studies a modified Laplacian algorithm, where the immediate disagreement feedback is split equally, and one-half is replaced by outdated feedback. For this modified algorithm, the result of [21] shows that it is possible to increase the convergence rate for some values of delay. However, [21] falls short of specifying the exact range of delay for which the convergence rate can be increased. Moreover, the study in [21] does not provide information on the maximum rate and its corresponding maximizer delay or the role of the relative size of the immediate and outdated feedback, i.e., splitting the disagreement feedback into other ratios of immediate and outdated components. This article studies the use of outdated disagreement feedback to increase the convergence rate of the *dynamic* average consensus algorithm of [24]. When the agents' reference input is all static, this algorithm becomes the static Laplacian average consensus algorithm [3]. In our study, we split the disagreement feedback into immediate and outdated feedback components. However, instead of equal contribution, we consider the affine combination of the current and outdated feedback to investigate the effect of the relative size of the outdated and immediate feedback terms on the induced acceleration. Our comprehensive study includes [21]–[23] as special cases and extends to a wider range of settings. We note that the analysis methods used in [21]–[23] do not generalize to study the case of the affine combination of the immediate and outdated disagreement feedback. This is due to the technical challenges involved with study of the variation of the infinite number of the roots of the characteristic equation of the linear time-delayed systems with delay,[1] which often are resolved via methods that conform closely to the specific algebraic structure of the system under study.

We start our study by characterizing the admissible range of delay for which the average consensus tracking is maintained. Dynamic average consensus algorithms, including the one we study, achieve their tracking with some nonzero steady-state error (see [5]). It is natural to expect that any acceleration method should not increase the tracking steady-state error. Therefore, we carefully study the tracking performance of our proposed accelerated algorithm. We show that for the delays in the admissible range, the ultimate tracking error of our modified average consensus algorithm of interest is not affected by the use of outdated disagreement feedback regardless of the affine combination's split factor. However, we show that the control effort of the agents does not increase only for a specific range of the split factor of the affine combination of the outdated and immediate feedback. Our results also specify: 1) for what values of the system parameters, the rate of convergence in the presence of delay can increase; 2) the exact values of delay for which the rate of convergence increases; and 3) the optimum value of τ corresponding to the maximum rate of convergence in the presence of delay. We close our study with a remark that discusses the tradeoff between the performance (maximum convergence rate) and the robustness of the algorithm to the delay and the level of control effort. Our study relies on using the Lambert W function [27], [28] to obtain the exact value of the characteristic roots of the internal dynamics of our dynamic consensus algorithm. Via the careful study of variation of the rightmost root in the complex plan with respect to delay, we then proceed to establish our results.

Our work is different than the vast literature on the study of distributed algorithms in the presence of delay (see, e.g., [1], [12], and [29]–[31]). This literature mainly focuses on evaluating the robustness of the algorithms against delay in communication and specifying the admissible delay bound; see [1] and [31] for robustness analysis of the average consensus algorithm against delay. Our study, however, centers around how the rate of convergence varies with delay and how one can take advantage of delay for accelerating the algorithm. Moreover, contrary to the abundance of results on determining the convergence rate of linear time-delayed systems for a given amount of time delay [32]–[38], there are few results on how the rate of convergence varies with time delay. Therefore, our study also contributes to expand the fundamental understanding of the internal dynamics of linear time-delay systems and how their stability margin is affected by nonzero time delay.

Organization: The rest of this article is organized as follows. Notations and preliminaries, including a brief review of the relevant properties of the Lambert W function and the graph-theoretic definitions, are given in Section II. Problem definitions and the objective statements are given in Section III, while the main results are given in Section IV. The proofs of the lemmas and the propositions of Section IV are given in Appendix A. Numerical simulations to illustrate our results are given in Section V. Finally, Section VI concludes this article. Appendixes B and C contain the auxiliary lemmas that we use to develop our main results.

II. Notation and Preliminaries

We let $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{C}$ denote the set of real, positive real, non-negative real, integer, and complex numbers, respectively. Given $i, j \in \mathbb{Z}$ with $i < j$, we define $\mathbb{Z}_i^j = \{i, i+1, \ldots, j\}$. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ represent, respectively, the real and imaginary parts of s. Moreover, $|s| = \sqrt{\mathrm{Re}(s)^2 + \mathrm{Im}(s)^2}$ and $\arg(s) = \mathrm{atan2}(\mathrm{Im}(s), \mathrm{Re}(s))$. For any vector $\mathbf{x} \in \mathbb{R}^n$, we let $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and $\|\mathbf{x}\|_\infty = \max\{x_i\}_{i=1}^n$. For a measurable locally essentially bounded function $\mathbf{u} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, we define $|\mathbf{u}|_\infty = \mathrm{ess}\sup\{\|\mathbf{u}(t)\|_\infty, t \geq 0\}$. For a matrix $\mathbf{A}$, its ith row is denoted by $[\mathbf{A}]_i$.

The Lambert W function specifies the solutions of $s\,\mathrm{e}^s = z$ for a given $z \in \mathbb{C}$, i.e., $s = W(z)$. This function is a multivalued function with an infinite number of solutions denoted by $W_k(z)$, $k \in \mathbb{Z}$, where W_k is called the kth branch of W function. For any

[1] Recall that the exact value of the worst convergence rate of a linear time-invariant system, with or without delay, is determined by the magnitude of the real part of the right most root of its characteristic equation [25], [26].

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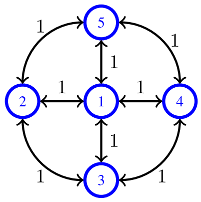

$$\mathbf{L} = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

Fig. 1. Connected graph of five nodes.

$z \in \mathbb{C}$, $W_k(z)$ can be evaluated in MATLAB or Mathematica. Below are some of the intrinsic properties of the Lambert W function, which we use (see [27] and [28])

$$\lim_{z\to 0} W_k(z)/z = 1 \tag{1a}$$

$$\mathrm{d}\, W_k(z)/\mathrm{d}\, z = 1/(z + \mathrm{e}^{W_k(z)}), \qquad \text{for } z \neq -1/\,\mathrm{e} \tag{1b}$$

for $k \in \mathbb{Z}$. For any $z \in \mathbb{R}$, the value of all the branches of the Lambert W function except for branches 0 and -1 is complex (nonzero imaginary part). Moreover, the zero branch satisfies $W_0(-1/\,\mathrm{e}) = -1$, $W_0(0) = 0$, and

$$W_0(z) \in \mathbb{R}, \qquad z \in [-1/\,\mathrm{e}, \infty) \tag{2a}$$

$$\mathrm{Im}(W_0(z)) \in (-\pi, \pi)\backslash\{0\}, \quad z \in \mathbb{C}\backslash[-1/\,\mathrm{e}, \infty) \tag{2b}$$

$$\mathrm{Re}(W_0(z)) > -1, \qquad z \in \mathbb{R}\backslash\{-1/\,\mathrm{e}\}. \tag{2c}$$

Lemma II.1 (Maximum real part of Lambert W function [27]): For any $z \in \mathbb{C}$, the following holds:

$$\mathrm{Re}(W_0(z)) \geq \max\{\mathrm{Re}(W_k(z)) | k \in \mathbb{Z}\backslash\{0\}\}. \tag{3}$$

The equality holds between branches 0 and -1 over $z \in (-\infty, -\frac{1}{\mathrm{e}})$, where we have $\mathrm{Re}(W_0(z)) = \mathrm{Re}(W_{-1}(z))$.

Lemma II.2 ($W_0(\mathsf{x})$ is an increasing function of $\mathsf{x} \in \mathbb{R}_{>0}$): For any $\mathsf{x}, \mathsf{y} \in \mathbb{R}_{>0}$ if $\mathsf{x} < \mathsf{y}$, then $W_0(\mathsf{x}) < W_0(\mathsf{y})$.

Proof: The proof follows from the fact that for $\mathsf{x} \in \mathbb{R}_{>0}$, $W_0(\mathsf{x}) \in \mathbb{R}_{>0}$. Therefore, $\frac{\mathrm{d}\, W_0(\mathsf{x})}{\mathrm{d}\, \mathsf{x}} = \frac{1}{\mathsf{x} + \mathrm{e}^{W_0(\mathsf{x})}} > 0$. ∎

We follow [39] to define our graph-related terminologies and notations. In a network of N agents, we model the inter-agent interaction topology by the undirected connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathsf{A})$, where $\mathcal{V}$ is the node set, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set, and $\mathsf{A} = [a_{ij}]$ is the adjacency matrix of the graph. Recall that $a_{ii} = 0$, $a_{ij} \in \mathbb{R}_{>0}$ if $j \in \mathcal{V}$ can send information to agent $i \in \mathcal{V}$, and zero otherwise. Moreover, a graph is undirected if the connection between the nodes is bidirectional and $a_{ij} = a_{ji}$ if $(i, j) \in \mathcal{E}$. Finally, an undirected graph is connected if there is a path from every agent to every other agent in the network (see, e.g., Fig. 1). The degree of node $i \in \mathcal{V}$ is $\mathsf{d}^i = \sum_{j=1}^N a_{ij}$, and the maximum degree of graph is $\mathsf{d}^{\max} = \max\{\mathsf{d}^i\}_{i=1}^N$. Here, $\mathbf{L} = \mathrm{Diag}(\mathsf{A}\mathbf{1}_N) - \mathsf{A}$ is the Laplacian matrix of the graph $\mathcal{G}$. The Laplacian matrix of a connected undirected graph is a symmetric positive-semidefinite matrix that has a simple $\lambda_1 = 0$ eigenvalue, and the rest of its eigenvalues satisfy $\lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_N$. Moreover, $\mathbf{L}\mathbf{1}_N = \mathbf{0}$. Since $\mathbf{L}$ of a connected undirected graph is a symmetric and real matrix, its normalized eigenvectors can be chosen to be mutually orthogonal. Let $v_1 = \frac{1}{\sqrt{N}}\mathbf{1}_N, v_2, \ldots, v_N$, be a set of such normalized eigenvectors. Then, for

$$\mathbf{T} = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & \mathbf{R} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} v_2 & \cdots & v_N \end{bmatrix} \tag{4}$$

we have $\mathbf{T}^\top \mathbf{L} \mathbf{T} = \mathbf{\Lambda} = \mathrm{Diag}(0, \lambda_2, \ldots, \lambda_N)$. We note that for any $\mathbf{q} \in \mathbb{R}^N$, we have $\|\mathbf{R}^\top \mathbf{q}\| = \|(\mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top)\,\mathbf{q}\|$.

III. Problem Definition

We consider a group of N agents each endowed with a one-sided time-varying measurable locally essentially bounded signal $\mathsf{r}^i : \mathbb{R}_{\geq 0} \to \mathbb{R}$, interacting over a connected undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathsf{A})$. To obtain the average of their reference inputs, $\mathsf{r}^{\text{avg}}(t) = \frac{1}{N}\sum_{i=1}^N \mathsf{r}^i(t)$, these agents implement the distributed algorithm

$$\begin{aligned} \dot{x}^i(t) &= -\alpha \sum_{j=1}^N a_{ij}(x^j(t) - x^i(t)) + \dot{\mathsf{r}}^i, i \in \mathcal{V} \\ x^i(0) &= \mathsf{r}^i(0) \end{aligned} \tag{5}$$

where $\alpha \in \mathbb{R}_{>0}$. When the reference inputs of the agents are all static, i.e., $\dot{\mathsf{r}}^i = 0$ for all $i \in \mathcal{V}$, (5) becomes the well-known Laplacian static average consensus algorithm that converges exponentially to $\mathsf{x}^{\text{avg}}(0) = \mathsf{r}^{\text{avg}} = \frac{1}{N}\sum_{j=1}^N \mathsf{r}^j$, with the rate of convergence $\rho_0 = \alpha\lambda_2$ (for details, see [3]). When one or more of the input signals are time varying, (5) is the dynamic average consensus algorithm of [24]. The convergence guarantee of (5) is as follows.

Lemma III.1 (Convergence of (5) over an undirected connected graph [5]): Let $\mathcal{G}$ be a connected undirected graph. Let $\|(\mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top)\dot{\mathsf{r}}\|_\infty = \kappa < \infty$. Then, for any $\alpha \in \mathbb{R}_{>0}$, the trajectories of algorithm (5) are bounded and satisfy

$$\lim_{t\to\infty} \left| x^i(t) - \mathsf{r}^{\text{avg}}(t) \right| \leq \epsilon_0, \qquad i \in \mathcal{V} \tag{6}$$

where $\epsilon_0 = \frac{\kappa}{\rho_0}$ and $\rho_0 = \alpha\lambda_2$. Moreover, the rate of convergence to this error neighborhood is no worse than ρ_0.

In this article, with the intention of using outdated information to accelerate the convergence, we alter the average consensus algorithm (5) to (compact representation)

$$\dot{\mathbf{x}}(t) = -\alpha\,(1 - \mathsf{k})\,\mathbf{L}\,\mathbf{x}(t) - \alpha\,\mathsf{k}\,\mathbf{L}\,\mathbf{x}(t - \tau) + \dot{\mathsf{r}} \tag{7a}$$

$$x^i(0) = \mathsf{r}^i(0), x^i(\eta) = 0 \text{ for } \eta \in [-\tau, 0), \quad i \in \mathcal{V} \tag{7b}$$

for $t \in \mathbb{R}_{\geq 0}$, where $\mathsf{k} \in \mathbb{R}$ and $\tau \in \mathbb{R}_{\geq 0}$. For $\mathsf{k} = 0$, (7) recovers the original algorithm (5). We refer to k as *split factor*.

To analyze convergence, we implement the change of variable

$$\mathbf{z}(t) = \mathbf{T}^\top(\mathbf{x}(t) - \mathsf{r}^{\text{avg}}(t)\mathbf{1}_N) \tag{8}$$

[recall (4)] to write (7) in equivalent form

$$\dot{z}_1(t) = 0, \qquad z_1(0) = 0 \tag{9a}$$

$$\begin{aligned} \dot{\mathbf{z}}_{2:N}(t) = &-\alpha(1 - \mathsf{k})\bar{\mathbf{\Lambda}}\,\mathbf{z}_{2:N}(t) - \alpha\mathsf{k}\,\bar{\mathbf{\Lambda}}\mathbf{z}_{2:N}(t - \tau) \\ &+ \mathbf{R}^\top\dot{\mathsf{r}}(t) \end{aligned} \tag{9b}$$

$$\mathbf{z}_{2:N}(0) = \mathbf{R}^\top\mathsf{r}(0), \mathbf{z}_{2:N}(\eta) = \mathbf{0}, \text{ for } \eta \in [-\tau, 0) \tag{9c}$$

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where $\bar{\mathbf{\Lambda}} = \mathrm{Diag}(\lambda_2, \ldots, \lambda_N)$. Under the given initial condition, the tracking error then is

$$\mathbf{x}(t) - \mathsf{r}^{\mathrm{avg}}(t)\mathbf{1}_N = \mathbf{R}\,\mathbf{z}_{2:N}(t), \quad t \in \mathbb{R}_{\geq 0}. \tag{10}$$

Using the method that specifies the solution of linear time-delayed systems [37], the trajectory of (9b) under initial condition (9c) is

$$\mathbf{z}_{2:N}(t) = \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathbf{S}_j t}\,\mathbf{C}_j\,\mathbf{z}_{2:N}(0) + \int_0^t \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathbf{S}_j (t-\zeta)}\,\mathbf{C}'_j \mathbf{R}^\top \dot{\mathsf{r}}(\zeta)\mathrm{d}\zeta \tag{11}$$

where

$$\mathbf{S}_j = \mathrm{Diag}(\mathbf{S}_j^1, \ldots, \mathbf{S}_j^{N-1}) \tag{12a}$$

$$\mathbf{S}_j^i = \frac{1}{\tau}\mathbf{W}_j(-\alpha\mathsf{k}\lambda_{i+1}\tau\,\mathrm{e}^{\alpha(1-\mathsf{k})\lambda_{i+1}\tau}) - \alpha(1-\mathsf{k})\lambda_{i+1} \tag{12b}$$

$$\mathbf{C}_j = \mathrm{Diag}(\mathbf{C}_j^1, \ldots, \mathbf{C}_j^{N-1}) \tag{12c}$$

$$\mathbf{C}_j^i = \frac{1}{1 - \alpha\mathsf{k}\lambda_{i+1}\tau\,\mathrm{e}^{-\mathbf{S}_j^i\tau}} \tag{12d}$$

and $\mathbf{C}'_j = \mathbf{C}_j$ because of the given initial conditions. When the reference input signals satisfy the condition given in Lemma III.1, it follows from (11) that $\mathbf{z}_{2:N}$ in the admissible delay range should converge exponentially to some neighborhood of zero, whose size is proportional to κ. Moreover, the rate of convergence[2] of algorithm (7) is $\rho_\tau(\mathsf{k}) = \min\{\{-\mathrm{Re}(\mathbf{S}_j^i)\}_{i=1}^{N-1}\}_{j=-\infty}^{\infty}$. By invoking Lemma II.1, $\rho_\tau(\mathsf{k})$ simplifies to $\rho_\tau(\mathsf{k}) = \min\{-\mathrm{Re}(\mathbf{S}_0^i)\}_{i=1}^{N-1}$, which reads as

$$\rho_\tau(\mathsf{k}) = \min\Big\{-\mathrm{Re}(\frac{1}{\tau}\mathbf{W}_0(-\alpha\mathsf{k}\lambda_i\tau\,\mathrm{e}^{\alpha(1-\mathsf{k})\lambda_i\tau})) + \alpha(1-\mathsf{k})\lambda_i\Big\}_{i=2}^{N}. \tag{13}$$

For further discussion about the convergence rate of linear time-delayed systems, see [26, Corollary 1].

Our objective in this article is to show that by splitting the disagreement feedback into current $-\alpha\,(1-\mathsf{k})\,\mathbf{L}\,\mathbf{x}(t)$ and outdated $-\alpha\,(1-\mathsf{k})\,\mathbf{L}\,\mathbf{x}(t-\tau)$ components, it is possible to increase the rate of convergence of algorithm (7). Specifically, we determine for what values of k, there exists ranges of time delay that the rate of convergence of (7) increases (ranges of delay for which decay rate of the transient response of (7) increases). We also specify the maximum reachable rate due to delay and its corresponding maximizer delay.

Remark III.1 (Constraints on choosing k and τ to accelerate algorithm (7)): One may argue that the rate of convergence of (5) can be increased by "cranking up" the gain α in (5). However, this choice leads to an increase in the control effort of the agents. Motivated by this observation, we set to identify values of split factor k for a fixed α for which the increase in the convergence rate of (7) due to delay in comparison to (5) does not lead to an increase in the control effort. On the other hand, it is also expected that accelerating delay value should not have an adverse effect on the tracking error of (7) to make it go beyond ϵ_0 in (6). Therefore, our study also includes a formal study of the ultimate tracking error of (7) for different values of k and admissible ranges of delay. These assertions increase the appeal of the modified average consensus algorithm (7) as an effective algorithm that yields faster convergence than the original algorithm (5).

We close this section by noting that following the change of variable method proposed in [5], algorithm (7a) can be implemented in the alternative way (recall that r^i is a one-sided signal)

$$\begin{aligned} \dot{y}^i(t) &= -\alpha\,(1-\mathsf{k})\,\mathbf{L}\,\mathbf{x}(t) - \alpha\,\mathsf{k}\,\mathbf{L}\,\mathbf{x}(t-\tau) \\ x^i(t) &= y^i(t) + \mathsf{r}^i(t) \\ y^i(0) &= 0,\ x^i(\eta) = 0, \quad \text{for } \eta \in [-\tau, 0), i \in \mathcal{V} \end{aligned}$$

which does not require the knowledge of the derivative of the reference input of the agents.

IV. Accelerating Average Consensus Using Outdated Feedback

To start our study, we identify the *admissible delay* range $(0, \bar{\tau})$ for algorithm (7) for different values of split factor k. Given the tracking error (10), the admissible delay bound is determined by the ranges of delay, for which the zero input dynamics of (9b) preserves its exponential stability. We recall that for linear time-delayed systems with exponentially stable dynamics when delay is set to zero, by virtue of the *continuity stability property* theorem [33, Proposition 3.1], the admissible delay range is a connected range $(0, \bar{\tau}) \subset \mathbb{R}_{>0}$, where $\bar{\tau} \in \mathbb{R}_{>0}$ is the *critical* delay bound, beyond which the system is always unstable.

Lemma IV.1 (Admissible range of delay for internal stability of algorithm (7)): The following assertions hold for the modified average consensus algorithm (7) over an undirected connected graph [recall (12b)].

a) For $\mathsf{k} \leq 0.5$, the modified average consensus algorithm (7) is internally stable for any $\tau \in \mathbb{R}_{\geq 0}$, i.e., $\bar{\tau} = \infty$.
b) For $\mathsf{k} > 0.5$, the modified average consensus algorithm (7) is internally stable if and only if $\tau \in [0, \bar{\tau})$, where

$$\bar{\tau} = \arccos(1 - 1/\mathsf{k})/(\alpha\,\lambda_N\sqrt{2\mathsf{k}-1}). \tag{14}$$

Also, for any $\tau \in [0, \bar{\tau})$, we have $\lim_{t\to\infty} \mathrm{e}^{\mathbf{S}_j^i t} = 0, i \in \mathbb{Z}_1^{N-1}$ and $j \in \mathbb{Z}$. Moreover, under the initial condition (7b), the trajectories of x^i, $i \in \mathcal{V}$, of the zero-input dynamics of algorithm (7) converges exponentially fast to $\mathsf{x}^{\mathrm{avg}}(0)$.

The results of Lemma IV.1 include the result in [31], which specifies the admissible range of delay for when $\mathsf{k} = 1$, as special case. Next, we study the ultimate tracking bound of the modified average consensus algorithm (7). We show that for delays in the

[2] We recall that for a given $\tau \in \mathbb{R}_{>0}$, a zero-input dynamical system with state $\mathbf{x}$ is said to be exponentially stable if and only if there exists a $k_\tau \in \mathbb{R}_{>0}$ and an $\rho_\tau \in \mathbb{R}_{>0}$ such that for the given initial conditions, the solution satisfies the inequality below $\|\mathbf{x}(t)\| \leq k_\tau \mathrm{e}^{-\rho_\tau t} \sup_{\eta\in[-\tau,0]} \|\mathbf{x}(\eta)\|, \quad t \in \mathbb{R}_{\geq 0}$. For linear systems, the tightest estimate on, ρ_τ, referred to as rate of convergence is determined by the magnitude of the real part of the rightmost root of its characteristic equation [25], [26].

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admissible delay bound, the ultimate tracking error is still ϵ_0, as defined in Lemma IV.1.

Proposition IV.1 (Convergence of (7) over connected graphs when $\tau \in [0, \bar{\tau})$): Let $\mathcal{G}$ be a connected undirected graph with communication delay in $\tau \in [0, \bar{\tau})$, where $\bar{\tau}$ is specified in Lemma IV.1. Let $\|(\mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top)\dot{\mathsf{r}}\|_\infty = \kappa < \infty$. Then, for any $\alpha \in \mathbb{R}_{>0}$, the trajectories of algorithm (7) for any $\mathsf{k} \in \mathbb{R}$ are bounded and satisfy (6). Moreover, the convergence rate to this error neighborhood is no worse than $\rho_\tau(\mathsf{k})$ defined in (13).

With the assertion that splitting the immediate disagreement feedback of (5) into current and outdated components as in (7) does not have adverse effect on the tracking performance, we set to identify the ranges of delay τ and split factor k, for which the rate of convergence of (7) is faster than (5). As we noted earlier, the rate of convergence of (7) is determined by its transient response that is governed by its zero-input dynamics. Consequently, we study the stability of the zero-input dynamics of the modified average consensus algorithm (7) and examine how its exponential rate of convergence to the average of its initial condition at time $t = 0$ changes due to delay at various values of $\mathsf{k} \in \mathbb{R}/\{0\}$. For any given value of k and τ, in what follows, we let $\rho_\tau(\mathsf{k})$ be the rate of convergence of (7) and $\mathbf{u}_{\tau,\mathsf{k}}(t) = -\alpha\,(1-\mathsf{k})\,\mathbf{L}\,\mathbf{x}(t) - \alpha\,\mathsf{k}\,\mathbf{L}\,\mathbf{x}(t-\tau)$ be the control effort to steer the zero-input dynamics of (7). Specifically, we show that for all $\mathsf{k} \in \mathbb{R}_{>0}$, there always exists a range of delay $(0, \tilde{\tau}_\mathsf{k})$ such that $\rho_\tau(\mathsf{k}) > \rho_0(0) = \rho_0 = \alpha\lambda_2$ for any $\tau \in (0, \tilde{\tau}_\mathsf{k})$. We show, however, that only for $\mathsf{k} \in (0, 1]$, we can guarantee $|\mathbf{u}_{\tau,\mathsf{k}}|_\infty \leq |\mathbf{u}_{0,0}|_\infty$, for $\tau \in (0, \tilde{\tau}_\mathsf{k})$. In what follows, we also investigate what the maximum value of $\rho_\tau(\mathsf{k})$ and the corresponding maximizer $\tau^\star_\mathsf{k} \in (0, \tilde{\tau}_\mathsf{k})$ are for a given $\mathsf{k} \in \mathbb{R}_{>0}$.

We start our analysis by defining the *delay gain function*

$$g(\gamma, \mathsf{x}) = \begin{cases} \frac{1}{\mathsf{x}}\,\mathrm{Re}(W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}})), & \mathsf{x} \in \mathbb{R}\backslash\{0\} \\ 1, & \mathsf{x} = 0 \end{cases} \tag{15}$$

with $\mathsf{x}, \gamma \in \mathbb{R}$, to write $\rho_\tau(\mathsf{k})$ in (13) as

$$\rho_\tau(\mathsf{k}) = \min\{\rho_{\tau,i}(\mathsf{k})\}_{i=2}^N \tag{16a}$$

$$\rho_{\tau,i}(\mathsf{k}) = (\mathsf{k}g(1 - \frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_i\alpha\tau) + (1-\mathsf{k}))\,\alpha\lambda_i. \tag{16b}$$

It follows from (1a) that $\lim_{\mathsf{x}\to 0} g(\gamma, \mathsf{x}) = 1$. Therefore, as expected, $\lim_{\tau\to 0}\rho_\tau(\mathsf{k}) = \rho_0 = \alpha\lambda_2$. When emphasis on k is not necessary, to simplify the notation, we write $\rho_\tau(\mathsf{k})$ as ρ_τ.

In what follows, we aim to determine ranges of delay $\tau \in \mathbb{R}_{>0}$ and k, for which we have $\rho_\tau(\mathsf{k}) > \alpha\lambda_2$. We also aim to identify the optimum value of the delay $\tau^\star$, for which ρ_τ has its maximum value for a given k, i.e., we identify the solution for

$$\tau^\star = \underset{\tau\in(0,\bar{\tau})}{\mathrm{argmax}}$$

$$\rho_\tau = \underset{\tau\in(0,\bar{\tau})}{\mathrm{argmax}}\ \min\{\rho_{\tau,i}\}_{i=2}^N. \tag{17}$$

According to (16b), for any given k, the variation of $\rho_{\tau,i}$ with τ is characterized by the variation of g with $\mathsf{x} = -\mathsf{k}\lambda_i\alpha\tau$. Our study of the properties of the delay gain function (15) in Appendix B shows that the variation of g with $\mathsf{x} \in \mathbb{R}_{\geq 0}$ for given values of γ is not monotone (see Lemma B.1 and Fig. 5). As a result, variation of $\rho_{\tau,i}$, $i \in \{2, \ldots, N\}$, with τ is not monotone, and thus, characterizing $\rho_\tau(\mathsf{k}) = \min\{\rho_{\tau,i}(\mathsf{k})\}_{i=2}^N$ with τ and solving the optimization problem (17) are not trivial. Nevertheless, as shown in the following, our comprehensive characterization of the variation of g versus $\mathsf{x} \in \mathbb{R}_{\geq 0}$ in Appendix B enables us to solve these problems.

In what follows, we set:

$$\tau_i^\star = \underset{\tau\in(0,\bar{\tau}_i)}{\mathrm{argmax}}\,\rho_{\tau,i}$$

$$\tilde{\tau}_i = \{\tau \in (0, \bar{\tau}_i)\,|\,g(1 - \frac{1}{\mathsf{k}}, -\mathsf{k}\alpha\lambda_i\tau)) = 1\}$$

where $\bar{\tau}_i$ is given in (22). With the notation defined, the following theorem examines the effect of outdated feedback on the rate of convergence of modified consensus algorithm (7) for different values of $\mathsf{k} \in \mathbb{R}/\{0\}$.

Theorem IV.1 (Effect of outdated feedback on the rate of convergence of average consensus algorithm (7)): The following assertions hold for the modified average consensus dynamics (7) over a connected graph whose rate of convergence is specified in (16a).

a) For $\mathsf{k} < 0$, the rate of convergence of the consensus algorithm (7) decreases by increasing $\tau \in \mathbb{R}_{\geq 0}$.

b) For $\mathsf{k} > 0$, $\rho_\tau > \rho_0$ if and only if $\tau \in [0, \hat{\tau}) \subset [0, \bar{\tau})$ where $\hat{\tau} = \min\{\hat{\tau}_i\}_{i=2}^N$ with $\hat{\tau}_i = \{\tau \in \mathbb{R}_{>0}|\rho_{\tau,i} = \rho_0\}$ and satisfies $\tilde{\tau}_N \leq \hat{\tau} \leq \min\{\tilde{\tau}_2, \bar{\tau}\}$. ρ_τ is a monotonically increasing function of $\tau \in (0, \tau_N^\star) \subset (0, \hat{\tau})$ and a decreasing function of τ for any $\tau > \tau_2^\star$, where

$$\tau_N^\star = \frac{1}{\alpha(1-\mathsf{k})\lambda_N}W_0(\frac{1-\mathsf{k}}{\mathsf{k}\,\mathrm{e}})$$

and

$$\tau_2^\star = \frac{1}{\alpha(1-\mathsf{k})\lambda_2}W_0(\frac{1-\mathsf{k}}{\mathsf{k}\,\mathrm{e}}).$$

Moreover, the optimum time delay $\tau^\star$ corresponding to the maximum rate of convergence of the consensus algorithm (7) satisfies $\tau^\star \in [\tau_N^\star, \min\{\tau_2^\star, \hat{\tau}\}]$ and is given by

$$\tau^\star = \{\tau \in [\tau_N^\star, \min\{\tau_2^\star, \hat{\tau}\}]\,|\,\rho_{\tau,2} = \min\{\rho_{\tau,i}\}_{i=3}^N\}. \tag{18}$$

Proof: Recall that the convergence rate of algorithm (7) is specified by (16a) [equivalent representation of (13)], which is the minimum of the rate of convergence of z_i, $i \in \{2, \ldots, N\}$, dynamics given in (21). Then, the proof of part (a) follows directly from statement (a) of Lemma B.2, which states that the rate of convergence of each z_i, $i \in \{2, \ldots, N\}$, dynamics decreases by increasing delay $\tau \in \mathbb{R}_{>0}$ (note that in Lemma B.2 each z_i dynamics reads as $\mathsf{a} = -\alpha\mathsf{k}\lambda_i > 0$ and $\mathsf{b} = -\alpha(1-\mathsf{k})\lambda_i < 0$). To prove statement (b), we proceed as follows. For $\mathsf{k} > 0$, because of statement (b) of Lemma B.2 for each z_i, $i \in \{2, \ldots, N\}$, dynamics ($\mathsf{a} = -\alpha\mathsf{k}\lambda_i < 0$), we are guaranteed that

$$\rho_{\tau,i} = (\mathsf{k}g(1 - \frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_i\alpha\tau) + (1-\mathsf{k}))\,\alpha\lambda_i > \rho_{0,i} \geq \rho_0$$

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for $\tau \in (0, \tilde{\tau}_i)$. Since $\alpha > 0$, $\lambda_N \geq \lambda_{N-1} \geq \cdots \geq \lambda_2 > 0$ and $\rho_{0,N} \geq \rho_{0,N-1} \geq \cdots \geq \rho_{0,2}$, we have

$$\tau_i^\star < \tilde{\tau}_i \leq \hat{\tau}_i < \bar{\tau}_i, \quad i \in \{3, \ldots, N\} \tag{19a}$$

$$\tilde{\tau}_N \leq \tilde{\tau}_{N-1} \leq \cdots \leq \tilde{\tau}_2 \tag{19b}$$

$$\bar{\tau} = \bar{\tau}_N \leq \bar{\tau}_{N-1} \leq \cdots \leq \bar{\tau}_2 \tag{19c}$$

$$\tau_N^\star \leq \tau_{N-1}^\star \leq \cdots \leq \tau_2^\star \tag{19d}$$

and $\hat{\tau}_2 = \tilde{\tau}_2$. Since $g(1 - \frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_i\alpha\tau)$ is a decreasing function of τ for any $\tau \in (\tilde{\tau}_i, \bar{\tau}_i) \subset (\tau_i^\star, \bar{\tau}_i)$ (recall Lemma B.1), it follows that for any $\tau \in [0, \hat{\tau}_j)$, we have $\rho_{\tau,j} > \rho_0$, and for any $\tau \in [\hat{\tau}_j, \bar{\tau})$, we have $\rho_{\tau,j} < \rho_0$. Because $\rho_\tau = \min\{\rho_{\tau,j}\}_{j=2}^N$, we have $\rho_\tau > \rho_0$, if and only if $\tau \in (0, \hat{\tau})$, where $\hat{\tau} = \min\{\hat{\tau}_j\}_{j=2}^N$. From (19a) and (19b), it follows that $\tilde{\tau}_N \leq \hat{\tau}$. Moreover, since $\rho_{\tau,2} > \rho_0$ for $\tau \in (0, \tilde{\tau}_2)$, we obtain $\hat{\tau} \leq \min\{\tilde{\tau}_2, \bar{\tau}\}$. This concludes the proof of the first part of statement (b).

To obtain $\tau^\star \in (0, \hat{\tau})$, which gives the maximum attainable $\rho_\tau^\star$, we proceed as follows. First, note that statement (b) of Lemma B.2 indicates that $\rho_{\tau,i}$, $i \in \{2, \ldots, N\}$ is a monotonically increasing (respectively, decreasing) function of $\tau \in (0, \tau_i^\star)$ (respectively, $\tau \in (\tau_i^\star, \bar{\tau}_i)$). Then, because of (19d), we have the guarantee that ρ_τ is a monotonically increasing function of $\tau \in (0, \tau_N^\star)$ and a decreasing function of τ for any $\tau > \tau_2^\star$. Therefore, the maximum value of ρ_τ should be attained at $\tau^\star \in ([\tau_N^\star, \tau_2^\star] \cap (0, \hat{\tau})) \subseteq [\tau_N^\star, \min\{\tau_2^\star, \hat{\tau}\}]$ with $\tau_2^\star = \frac{1}{\lambda_2} W_0(\frac{1-\mathsf{k}}{\mathsf{k}\,\mathrm{e}})$ and $\tau_N^\star = \frac{1}{\lambda_N} W_0(\frac{1-\mathsf{k}}{\mathsf{k}\,\mathrm{e}})$. Now, let $\mathsf{j} = \min\{i \in \{2, \ldots, N\} | \tau_i^\star \leq \tau^\star\}$. Then, given (19d), for any $i \in \{2, \ldots, N\}$ such that $i < \mathsf{j}$ (respectively, $i \geq \mathsf{j}$) by virtue of statement (e) of Lemma B.1, we know $\mathrm{d}g(1 - \frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_i\alpha\tau)/\mathrm{d}\tau > 0$ (respectively, < 0) and consequently $\mathrm{d}\rho_{\tau,i}/\mathrm{d}\tau > 0$ (respectively, < 0) at $\tau = \tau^\star$. Since $\rho_\tau = \min\{\rho_{\tau,i}\}_{i=2}^N$, the maximum value of ρ_τ is attained at $\tau = \tau^\star$ at which

$$\min\{\rho_{\tau,i}\}_{i=\mathsf{j}}^N = \min\{\rho_{\tau,i}\}_{i=2}^{\mathsf{j}-1}. \tag{20}$$

Since $\lambda_2\tau^\star \leq \cdots \leq \lambda_{\mathsf{j}-1}\tau^\star$ and $\mathrm{d}g(1 - \frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_i\alpha\tau^\star)/\mathrm{d}\tau > 0$ for $i \in \{2, \ldots, \mathsf{j}-1\}$, we have $g(1 - \frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_{\mathsf{j}-1}\alpha\tau^\star) \geq g(1 - \frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_{\mathsf{j}-2}\alpha\tau^\star) \geq \cdots \geq g(1 - \frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_2\alpha\tau^\star)$. As a result, it follows from (16a) that at $\tau = \tau^\star$ we have $\min\{\rho_{\tau,i}\}_{i=2}^{\mathsf{j}-1} = \rho_{\tau,2}$, which given (20) completes our proof. ∎

Remark IV.1 (Selecting the accelerating τ for a given $\mathsf{k} \in \mathbb{R}_{>0}$): According to Theorem IV.1, for $\mathsf{k} \in \mathbb{R}_{>0}$, there always exists a range of delay for which the rate of convergence of algorithm (7) increases with delay. When the network topology is fully known, and the eigenvalues of the Laplacian matrix are available, the best accelerating τ is computed from (18). The full knowledge of the topology is also assumed in other methods used to accelerate the average consensus algorithms, such as the optimal adjacency weight selection of [2]. On the other hand, when the network topology is not known fully but the maximum degree of the graph is available, we can use the upper bound $\lambda_N \leq 2\mathsf{d}^{\max}$ (see [3]) to write

$$\frac{1}{2\alpha(1-\mathsf{k})\mathsf{d}^{\max}} W_0(\frac{1-\mathsf{k}}{\mathsf{k}\,\mathrm{e}}) \leq \tau_N^\star \leq \tau^\star \leq \bar{\tau}$$

where $\bar{\tau} \geq \arccos(1 - 1/\mathsf{k})/(\alpha\, 2\mathsf{d}^{\max}\sqrt{2\mathsf{k} - 1})$ for $k > 0.5$ and $\bar{\tau} = \infty$ for $k \in [0, 0.5]$ (recall Lemma IV.1). Moreover, by virtue of Theorem IV.1, we know that for any $\tau \in (0, \frac{1}{2\alpha(1-\mathsf{k})\mathsf{d}^{\max}} W_0(\frac{1-\mathsf{k}}{\mathsf{k}\,\mathrm{e}})]$, we have the guaranteed $\rho_\tau > \rho_0$.

Next, our goal is to identify values of $\mathsf{k} \in \mathbb{R}_{>0}$ for which the maximum driving effort $\mathbf{u}_{\tau,\mathsf{k}}(t)$ does not exceed the one for the original algorithm (5) (for zero-input dynamics). However, before that, we make the following statement about the maximum attainable rate using outdated feedback.

Lemma IV.2 (Ultimate bound on the maximum attainable increase in the rate of convergence of (7)): For any $\mathsf{k} \in \mathbb{R}_{\geq 0}$, the ultimate bound on the maximum attainable rate of convergence for (7) by using outdated feedback is equal to $(1 - \mathsf{k})\left(1 + \frac{1}{W_0(\frac{1-\mathsf{k}}{\mathsf{k}\,\mathrm{e}})}\right)\rho_0$.

Next, we study how the maximum control effort of the agents while implementing the modified algorithm (7) compares to that of the original average consensus algorithm (5) for any $\mathsf{k} \in \mathbb{R}_{>0}$. The following theorem indicates that for any $\mathsf{k} \in (0, 1]$, using the outdated feedback does not increase the maximum control effort, while for $\mathsf{k} > 1$, the maximum control effort is greater than the one of the original algorithm (5).

Theorem IV.2 (The maximum control effort for steering the zero-input dynamics of the algorithm (7)): For a given $\alpha \in \mathbb{R}_{>0}$, let $\mathbf{u}_{0,0}$, and $\mathbf{u}_{\tau,\mathsf{k}}(t)$ be, respectively, the network aggregated control input of the zero-input dynamics of (5) and (7) for any $\mathsf{k} \in \mathbb{R}_{>0}$ and $\tau \in \mathbb{R}_{>0}$. Then, for any $\tau \in [0, \bar{\tau}]$, where admissible delay bound $\bar{\tau}$ is given in Lemma IV.1, the following assertions hold for $t \in \mathbb{R}_{\geq 0}$.

a) For $\mathsf{k} \in (0, 1]$, we have $|\mathbf{u}_{\tau,\mathsf{k}}(t)|_\infty \leq |\mathbf{u}_{0,0}(t)|_\infty$.

b) For $\mathsf{k} > 1$, we have $|\mathbf{u}_{\tau,\mathsf{k}}(t)|_\infty \geq \mathrm{e}^{(k-1)\alpha\lambda_2\tau}\,|\mathbf{u}_{0,0}(t)|_\infty$.

We close this section with a remark on how the split factor can be chosen based on the expectations on the convergence rate, robustness to delay, and managing the control effort.

Remark IV.2 (Selecting k in the algorithm (7)): Lemma IV.1 and Theorems IV.1 and IV.2 give insights on how we can choose the split factor $\mathsf{k} \in \mathbb{R}$ given expectations on the algorithm's acceleration, robustness to delay, and control effort. Theorem IV.2 certifies that for any $\mathsf{k} \in (0, 1]$, the rate of convergence we observe for any $\tau \in [0, \bar{\tau}]$ is attained without imposing any extra control effort on the agents. Therefore, assuming that the acceleration is expected without increasing the control effort, the split factor should be selected to satisfy $\mathsf{k} \in (0, 1]$. According to Lemma IV.2, the maximum attainable rate of convergence is an increasing function of $\mathsf{k} \in (0, 1]$. Moreover, as $\mathsf{k} \to 1$, the ultimate bound on the rate of convergence converges to $\mathrm{e}\,\rho_0$, which recovers the same bound established in [22, Th. 4.4]. On the other hand, as expected, as $\mathsf{k} \to 0$, the ultimate bound on the rate of convergence converges to ρ_0. Finally, we observe from Lemma IV.1 that for $\mathsf{k} > 0.5$, the admissible delay bound is finite, and thus, the robustness of the algorithm to delay is not strong. On the contrary, the algorithm is robust with respect to any perturbation in delay for $\mathsf{k} \in (0, 0.5]$, because the admissible delay range for such split factors is $\mathbb{R}_{\geq 0}$. These observations point to tradeoff between robustness to delay and achieving higher acceleration when it comes to choosing the split factor; $\mathsf{k} = 1$ gives the maximum rate of convergence with the corresponding optimum delay, while $\mathsf{k} = 0.5$ results in robustness as well as higher rate of convergence relative to the original system (5).

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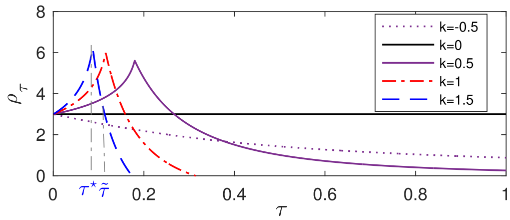

Fig. 2. Rate of convergence ρ_τ of the modified average consensus algorithm (7) over the graph in Fig. 1 for $\mathsf{k} \in \{-0.5, 0, 0.5, 1, 1.5\}$. For the example case of $\mathsf{k} = 1.5$, note that $\tilde{\tau} = 0.14$ and the maximum rate of convergence that can be achieved is $\rho_\tau^\star \approx 2\rho_0$ at $\tau^\star = 0.087$.

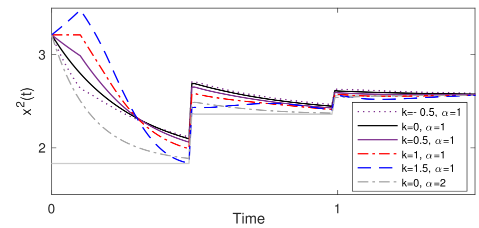

Fig. 3. Trajectory of local state of agent 2 executed by the algorithm (7) over the graph in Fig. 1 for $\tau = 0.1$, $\alpha = 1$, and $\mathsf{k} \in \{-0.5, 0, 0.5, 1, 1.5\}$ and for $\tau = 0$ and $\alpha = 2$.

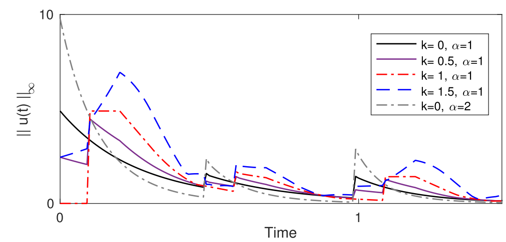

Fig. 4. Maximum control effort executed by the algorithm (7) over the graph in Fig. 1 for $\tau = 0.1$ and for $\mathsf{k} \in \{0, 0.5, 1, 1.5\}$ and for $\tau = 0$ and $\alpha = 2$.

V. Numerical Example

We consider the modified average consensus algorithm (7) over the graph depicted in Fig. 1. The reference input of each agent $i \in \{1, \ldots, 5\}$ is chosen according to the first numerical example in [5] to be the zero-order hold sampled points from the signal $\mathsf{r}^i(t) = a^i(2 + \sin(\omega(t)t + \phi(t)) + b^i$. The idea discussed in [5] is that the sensor agents sample the signal and should obtain the average of these sampled points before the next sampling arrives. The parameters a^i (the multiplicative sampling error) and b^i (additive bias), $i \in \{1, \ldots, 5\}$, are chosen as the ith element of [1.1,1,0.9,1.05,0.96] and $[-0.55, 1, 0.6, -0.9, -0.6]$, respectively. At each sampling time, ω and ϕ are chosen randomly according to $N(0, 5)$ and $N(0, (\pi/2)^2)$, where $N(\mu, \sigma)$ indicates the Gaussian distribution with mean μ and variance σ. We set the sampling rate at 2 Hz. This numerical example can be viewed as a simple abstraction for decentralized operations such as distributed sensor fusion, where a dynamic or static average consensus algorithm is used to create the additive fusion terms in a distributed manner (see, e.g., [6] and [40]). Since the convergence of the average consensus algorithm is asymptotic, there is always an error when the algorithm is terminated in the finite intersampling time. Faster convergence is desired to reduce the residual error.

For this example, in what follows, we study the response of the modified average consensus algorithm (7) for $\mathsf{k} \in \{-0.5, 0, 0.5, 1, 1.5\}$. We note that the case of $\mathsf{k} = 0$ gives the original (delay-free) dynamic average consensus algorithm (5) and, thus, is the baseline case that the rest of the cases should be compared to. For $\mathsf{k} \in \{-0.5, 0, 0.5, 1, 1.5\}$, the critical delay value $\bar{\tau}$ of the admissible delay range $(0, \bar{\tau})$ of (7), respectively, is $\{\infty, \infty, \infty, 0.32, 0.18\}$ s. Fig. 2 illustrates how ρ_τ changes with τ. First, we note that for $\mathsf{k} = -0.5$, the rate of convergence decreases with delay. However, for positive values of k, there is a range $(0, \tilde{\tau})$, for which $\rho_\tau > \rho_0$. For positive values of k, we also observe monotonic increase until reaching $\tau^\star$ and then the monotonic decrease afterward. The trend observed is in accordance with the results of Theorem IV.1. We can also observe that as k increases, the maximum achievable rate of convergence also increases. Fig. 3 shows the tracking response of agent 2 for $\mathsf{k} \in \{-0.5, 0, 0.5, 1, 1.5\}$ when the delay is $\tau = 0.1$ (similar trend is observed for the other agents). As seen, the convergence rate of (7) is different for each value of k. The fastest response is observed for $\mathsf{k} = 1.5$, while $\mathsf{k} = -0.5$ shows the lowest one. The decrease in the rate of convergence for $\mathsf{k} = -0.5$ and its increase for the positive values of k are in accordance with the trend certified by Theorem IV.1 (note that as seen in Fig. 2, $\tau = 0.1$ is in the rate increasing delay range of $(0, \tilde{\tau}$ of the cases corresponding to $\mathsf{k} \in \{0.5, 1, 1.5\}$). The desired effect of fast convergence is demonstrated by the smaller tracking error that is observed at the end of each sampling time, e.g., the tracking error in the first epoch for $\alpha = 1, \mathsf{k} \in \{0.5, 1, 1.5\}$ and for $\mathsf{k} = 0$, $\alpha = 2$ is, respectfully, 13%, 9%, 0.2%, and 2.5% that is an improvement over 15% that corresponds to $\mathsf{k} = 0$ (case of the original algorithm). We note here that as can be seen in Fig. 2, $\tau = 0.1$ is close to $\tau^\star$ of the case corresponding to $\mathsf{k} = 1.5$. The same fast rates of convergence can be achieved for the cases of $\mathsf{k} = 1$ and $\mathsf{k} = 0.5$ if one uses $\tau^\star$ corresponding to these split factors.

Fig. 4 shows the maximum control effort of zero-input dynamics of the algorithm (7) over time corresponding to $\tau = 0.1$ and $\mathsf{k} \in \{0, 0.5, 1, 1.5\}$ and $\alpha = 1$. For $\mathsf{k} = 1.5$, the maximum control effort exceeds the value for the original consensus algorithm (case of $\mathsf{k} = 0$). However, for $\mathsf{k} = 1$ and $\mathsf{k} = 0.5$, the maximum control effort is equal to or less than the case $\mathsf{k} = 0$. The trend observed above is in accordance with Theorem IV.2. We also observe that the control effort for the case of $\mathsf{k} = 0$ and $\alpha = 2$, which is equivalent to cranking up the control effort of the delay-free Laplacian average consensus algorithm two times, is much larger than control effort of the cases that we use outdated feedback for accelerating the algorithm.

VI. Conclusion

In this article, we analyzed the effect of using an affine combination of immediate and outdated disagreement feedback to increase the convergence rate of a dynamic average consensus algorithm. We showed that the modified algorithm has the same ultimate tracking accuracy but will have faster convergence

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with specific choices of the delay and the affine combination factor. Our study produced a set of closed-form expressions to specify the admissible delay range, the delay range for which the system experiences an increase in its rate of convergence, and a range in which the optimum time delay corresponding to the maximum rate of convergence lies. We also examined the range of affine combination factors, for which the outdated feedback can be used to improve the convergence of the algorithm without increasing the control effort. To develop our results, we used the Lambert W function to obtain the rate of convergence of our algorithm under study in the presence of the delay. Our future work includes extending our results for dynamic consensus algorithms over directed graphs and investigating the use of outdated feedback to increase the rate of convergence of other distributed algorithms for networked systems such as leader–follower algorithms.

Appendix A
Proofs of Lemmas and Propositions of Section IV

Proof of Lemma IV.1: Consider the zero-input dynamics of (9), the equivalent representation of zero dynamics of algorithm (7). It is evident that the delay tolerance of (9) is defined by the dynamics of states $\mathbf{z}_{2:N}$. Note that (9b) because of definition of $\bar{\mathbf{\Lambda}}$ also reads as

$$\dot{z}_i(t) = -\alpha(1-\mathsf{k})\lambda_i\, z_i(t) - \alpha\mathsf{k}\,\lambda_i z_i(t-\tau), i \in \mathbb{Z}_2^N. \tag{21}$$

When $\mathsf{k} \le 0.5$, we have $-\alpha\lambda_i(1-\mathsf{k}) \le |\alpha\lambda_i\mathsf{k}|$, while when $\mathsf{k} > 0.5$, we have $-\alpha\lambda_i\mathsf{k} < -|\alpha\lambda_i(1-\mathsf{k})|$. Therefore, the admissible delay ranges stated in statements (a) and (b) follow, respectively, from the first and second statements of [33, Proposition 3.15] as

$$\bar{\tau}_i = \frac{\arccos(1-\frac{1}{\mathsf{k}})}{\alpha\lambda_i\sqrt{2\mathsf{k}-1}}. \tag{22}$$

In admissible delay bound, the time-delayed systems (21) for $i \in \{2,\dots,N\}$ are exponentially stable, i.e., $z_i \to 0$ as $t \to \infty$, $i \in \{2,\dots,N\}$. As a result, $\lim_{t\to\infty} \mathrm{e}^{\mathbf{S}_j^i t} = 0$, $i \in \mathbb{Z}_1^{N-1}$ and $j \in \mathbb{Z}$, can be certified from (11) when the second term on the right-hand side is removed (zero-input response). Moreover, since $\mathbf{z}(t) = \mathbf{T}^\top \mathbf{x}(t)$ (in zero-input dynamics), we then obtain that in the stated admissible delay ranges in statements (a) and (b), $\mathbf{x}(t)$ converges exponentially fast to $\frac{1}{\sqrt{N}} z_1(0)\mathbf{1}_N = \frac{1}{\sqrt{N}}\left(\frac{1}{\sqrt{N}}\sum_{j=1}^N x^i(0)\right)\mathbf{1}_N = \mathsf{x}^{\text{avg}}(0)$, completing the proof. ∎

Proof of Proposition IV.1: To establish our proof, we consider (9), the equivalent representation of algorithm (7). Recall (9a), which along with the given initial condition gives $z_1(t) = 0$ for $t \in \mathbb{R}_{\ge 0}$. Also, given (11), the trajectories of $t \mapsto \mathbf{z}_{2:N}$ for $t \in \mathbb{R}_{\ge 0}$ satisfy

$$\begin{aligned}\|\mathbf{z}_{2:N}(t)\| \le & \Big\|\sum_{j\in\mathbb{Z}} \mathrm{Diag}\left(\mathbf{C}_j^1\,\mathrm{e}^{\mathbf{S}_j^1 t},\dots,\mathbf{C}_j^{N-1}\,\mathrm{e}^{\mathbf{S}_j^{N-1}t}\right) \\ & \Big\| \|\mathbf{z}_{2:N}(0)\| \\ & + \kappa \Big\|\sum_{j\in\mathbb{Z}} \mathrm{Diag}\left(\frac{\mathbf{C}_j^1}{\mathbf{S}_j^1}(1-\mathrm{e}^{\mathbf{S}_j^1 t}),\dots,\frac{\mathbf{C}_j^{N-1}}{\mathbf{S}_j^{N-1}}(1-\mathrm{e}^{\mathbf{S}_j^{N-1}t})\right)\Big\| \\ = & \max\left\{|\sum_{j\in\mathbb{Z}} \mathbf{C}_j^i\,\mathrm{e}^{\mathbf{S}_j^i t}|\right\}_{i=1}^{N-1} \|\mathbf{z}_{2:N}(0)\| \\ & + \kappa\max\left\{|\sum_{j\in\mathbb{Z}} \frac{\mathbf{C}_j^i}{\mathbf{S}_j^i}(1-\mathrm{e}^{\mathbf{S}_j^i t})|\right\}_{i=1}^{N-1}.\end{aligned} \tag{23}$$

Here, we used $\|\mathbf{R}^\top\dot{\mathsf{r}}\| \le \kappa$. Furthermore, using (8), we obtain $|x^i(t) - \mathsf{r}^{\text{avg}}(t)| \le \|\mathbf{x}(t) - \mathsf{r}^{\text{avg}}(t)\mathbf{1}_N\| = \|\mathbf{z}(t)\| = \sqrt{|z_1(t)|^2 + \|\mathbf{z}_{2:N}(t)\|^2} = \|\mathbf{z}_{2:N}(t)\|$. Then, it follows from (23) that $\lim_{t\to\infty}|x^i(t) - \mathsf{r}^{\text{avg}}(t)| \le \lim_{t\to\infty}\|\mathbf{z}_{2:N}(t)\| = \kappa\max\{|\sum_{j\in\mathbb{Z}}\frac{\mathbf{C}_j^i}{\mathbf{S}_j^i}|\}_{i=1}^{N-1}$. Next, we show that $\sum_{j\in\mathbb{Z}}\frac{\mathbf{C}_j^i}{\mathbf{S}_j^i} = \frac{1}{\alpha\lambda_{i+1}}$ for any $i \in \mathbb{Z}_1^{N-1}$. To this end, note that from zero-input response of (11), we have $z_i(t) = (\sum_{j\in\mathbb{Z}}\mathrm{e}^{\mathbf{S}_j^i t}\,\mathbf{C}_j^i)\, z_i(0)$, which gives $(\sum_{j\in\mathbb{Z}}\frac{\mathbf{C}_j^i}{\mathbf{S}_j^i}) z_i(0) = \int_0^\infty z_i(t)\mathrm{d}t$. On the other hand, using (21), for any $i \in \mathbb{Z}_1^{N-1}$, we have

$$\int_0^\infty \dot{z}_{i+1}(t)\mathrm{d}t = -\alpha\lambda_{i+1}\int_0^\infty z_{i+1}(t)\mathrm{d}t - \alpha\lambda_{i+1}\mathsf{k}\int_{-\tau}^0 z_{i+1}(t)\mathrm{d}t.$$

Recalling (9c), we get $\int_{-\tau}^0 z_{i+1}(t)\mathrm{d}t = 0$, which, along with the fact that under admissible range $\lim_{t\to\infty} z_{i+1}(t) = 0$, implies that $\int_0^\infty z_{i+1}(t)\mathrm{d}t = \frac{z_{i+1}(0)}{\alpha\lambda_i}$, which holds for any initial condition $z_{i+1}(0) \in \mathbb{R}$. Therefore, we get $\sum_{j\in\mathbb{Z}}\frac{\mathbf{C}_j^i}{\mathbf{S}_j^i} = \frac{1}{\alpha\lambda_{i+1}}$ and, consequently, $\lim_{t\to\infty}|x^i(t) - \mathsf{r}^{\text{avg}}(t)| \le \frac{\kappa}{\alpha\min\{\lambda_i\}_{i=2}^N} \le \frac{\kappa}{\rho_0}$. Moreover, the maximum rate of convergence corresponds to the worst rate of the exponential terms in (23) or equivalently $\min\{\{-\mathrm{Re}(\mathbf{S}_j^i)\}_{i=1}^{N-1}\}_{j\in\mathbb{Z}}$ given in (13). ∎

Proof of Lemma IV.2: It follows from part (f) of Lemma B.1 that $g(1-\frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_i\alpha\tau_i^\star) = \frac{1-\mathsf{k}}{\mathsf{k}W_0(\frac{1-\mathsf{k}}{\mathsf{k}\mathrm{e}})}$ for any $i \in \{2,\dots,N\}$. Then, given (16a), we have $\rho_\tau \le \rho_{\tau,2} \le \rho_{\tau^\star,2} = (\mathsf{k}g(1-\frac{1}{\mathsf{k}}, -\mathsf{k}\lambda_2\alpha\tau_2^\star) + (1-\mathsf{k}))\alpha\lambda_2 = (1-\mathsf{k})(1+\frac{1}{W_0(\frac{1-\mathsf{k}}{\mathsf{k}\mathrm{e}})})\rho_0$, which concludes our proof. ∎

Proof of Theorem IV.2: Consider the zero-input dynamics of (9), the equivalent representation of algorithm (7). For the maximum control effort of algorithm (7), we have

$$\begin{aligned}|\mathbf{u}_{\tau,k}(t)|_\infty &= |-\alpha\,(1-\mathsf{k})\,\mathbf{\Lambda}\,\mathbf{z}(t) - \alpha\,\mathsf{k}\,\mathbf{\Lambda}\,\mathbf{z}(t-\tau)|_\infty \\ &= \alpha\max\{|(1-\mathsf{k})\,\lambda_i\,z_i(t) + \mathsf{k}\,\lambda_i\,z_i(t-\tau)|_\infty\}_{i=2}^N.\end{aligned} \tag{24}$$

Here, we used the fact that $z_1(t) = 0$. Also, recalling (21), for $\tau = 0$ and any $i \in \{2,\dots,N\}$, we have $z_i(t) = \mathrm{e}^{-\lambda_i t}\, z_i(0)$, which gives $|\mathbf{u}_{0,0}(t)|_\infty = |\mathbf{u}_{0,0}(0)|_\infty = \alpha\max\{|\lambda_i z_i(0)|\}_{i=2}^N$.

Next, we show that for any $\tau \in (0,\bar{\tau})$ and $\mathsf{k} \in (0,1]$, $|\mathbf{u}_{\tau,\mathsf{k}}(t)|_\infty \le \alpha\{|\lambda_i z_i(0)|\}_{i=2}^N$. Notice that from (24), we have $|\mathbf{u}_{\tau,\mathsf{k}}(t)|_\infty \le \alpha(1-\mathsf{k})\max\{|\lambda_i z_i(t)|_\infty\} + \alpha\mathsf{k}\max\{|\lambda_i z(t-\tau)|_\infty\}$. Also, recall that for $t \in [0,\tau)$, we have $z_i(t-\tau) = 0$. Thus, to validate statement (a), it suffices to show that $|z_i(t)|_\infty = |z_i(0)|$. To this aim, consider the trajectories $t \to z_{2:N}$ of (21). Since set of dynamics (21) are exponentially stable with $-\alpha(1-\mathsf{k})\lambda_i \le 0$ and $-\alpha\mathsf{k}\lambda_i \le 0$, recalling Lemma B.3, for any delay in the admissible range, we have

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$|z_i(t)|_\infty = \max_{s\in[-\tau,2\tau]} |z_i(s)|$ for any $i \in \{2,\dots,N\}$. Also, note that from (21), we get

$$z_i(t) = 0, \qquad t \in [-\tau, 0) \tag{25a}$$
$$z_i(t) = \mathrm{e}^{-\alpha(1-\mathsf{k})\lambda_i t}\, z_i(0), \quad t \in [0,\tau) \tag{25b}$$
$$z_i(t) = \mathrm{e}^{-\alpha(1-\mathsf{k})\lambda_i t} z_i(0)\Big(1 + \frac{\mathsf{k}}{(1-\mathsf{k})}\big(\mathrm{e}^{-\alpha(1-\mathsf{k})\lambda_i (t-\tau)} - 1\big)\Big)$$
$$t \in [\tau, 2\tau] \tag{25c}$$

which results in $\max_{s\in[-\tau,2\tau]} |z_i(s)| = |z_i(0)|$ and, consequently, $|z_i(t)|_\infty = |z_i(0)|$, which concludes statement (a). To validate part (b), we proceed as follows. Recalling (24) for $\mathsf{k} > 1$, we have $|\mathbf{u}_{\tau,\mathsf{k}}(2\tau)|_\infty = \alpha \max\{|\mathsf{k}\lambda_i z_i(\tau) - (\mathsf{k}-1)\lambda_i z_i(2\tau)|\}_{i=2}^N \geq \alpha \max\{\mathsf{k}\lambda_i |z_i(\tau)| - (\mathsf{k}-1)\lambda_i |z_i(2\tau)|\}_{i=2}^N$. Also, from (25c), for $t \in [\tau, 2\tau)$, we have $|z_i(2\tau)| \leq |z_i(\tau)|$, which gives $|\mathbf{u}_{\tau,\mathsf{k}}(2\tau)|_\infty = \alpha \max\{\mathsf{k}\lambda_i |z_i(\tau)| - (\mathsf{k}-1)\lambda_i |z_i(\tau)|\}_{i=2}^N = \alpha(2\mathsf{k}-1)\max\{\lambda_i |z_i(\tau)|\}_{i=2}^N$. Moreover, (25b) implies that $z_i(\tau) = \mathrm{e}^{\alpha(\mathsf{k}-1)\lambda_i\tau} z_i(0)$, which deduces $|\mathbf{u}_{\tau,\mathsf{k}}(2\tau)| \geq \alpha(2\mathsf{k}-1)\,\mathrm{e}^{\alpha(\mathsf{k}-1)\lambda_i\tau} \max\{|\lambda_i z_i(0)|\}_{i=2}^N = (2\mathsf{k}-1)\,\mathrm{e}^{\alpha(\mathsf{k}-1)\lambda_2\tau}\, |\mathbf{u}_{0,0}|_\infty$. Knowing $2\mathsf{k}-1 \geq 1$ and $|\mathbf{u}_{\tau,\mathsf{k}}(t)|_\infty \geq |\mathbf{u}_{\tau,\mathsf{k}}(2\tau)|_\infty$, we can conclude the proof. ■

Appendix B
Delay Gain Function

The following lemma highlights some of the properties of the delay gain function $g(\gamma, \mathsf{x})$. Fig. 5 gives some graphical representation for the properties discussed in this lemma.

Lemma B.1 (Properties of $g(\gamma,\mathsf{x})$): The following assertions hold for the delay gain function (15) with $\gamma, \mathsf{x} \in \mathbb{R}$.

a) For any $\gamma \in \mathbb{R}$, we have $\lim_{\mathsf{x}\to 0} g(\gamma, \mathsf{x}) = 1$.
b) For any $\gamma > 1$ and $\mathsf{x} \in \mathbb{R}_{>0}$, we have $g(\gamma, \mathsf{x}) < \gamma$.
c) For any $\gamma > 1$ and $\mathsf{x} \in \mathbb{R}_{>0}$, $g(\gamma, \mathsf{x})$ is a strictly increasing function of x.
d) Let $\mathsf{x} \in (\bar{\mathsf{x}}, 0)$, where $\bar{\mathsf{x}} = \arccos(\gamma)/\sqrt{1-\gamma^2}$. Then, for any $\gamma < 1$ (respectively, $\gamma > 1$), we have $g(\gamma, \mathsf{x}) > \gamma$ (respectively, $g(\gamma, \mathsf{x}) < \gamma$).
e) For any $\gamma < 1$ and $\mathsf{x} \in \mathbb{R}_{<0}$, $g(\gamma, \mathsf{x})$ is a strictly decreasing function of x for any $\mathsf{x} \in [\mathsf{x}^\star, 0) \subset (\bar{\mathsf{x}}, 0)$ and a strictly increasing function of x for any $\mathsf{x} < \mathsf{x}^\star$, where $\mathsf{x}^\star = \frac{1}{\gamma} W_0(-\frac{\gamma}{\mathrm{e}})$ when $\gamma \neq 0$ and $\mathsf{x}^\star = -\frac{1}{\mathrm{e}}$ when $\gamma = 0$.
f) For any $\gamma < 1$ and $\mathsf{x} \in \mathbb{R}_{<0}$, the maximum value of $g(\gamma, \mathsf{x})$ occurs at $\mathsf{x}^\star = \frac{1}{\gamma} W_0(-\frac{\gamma}{\mathrm{e}})$, where $g(\gamma, \mathsf{x}^\star) = \frac{-\gamma}{W_0(-\frac{\gamma}{\mathrm{e}})}$ when $\gamma \neq 0$, and at $\mathsf{x}^\star = -\frac{1}{\mathrm{e}}$, where $g(\gamma, \mathsf{x}^\star) = \mathrm{e}$ when $\gamma = 0$.
g) For any $\gamma < 1$ and $\mathsf{x} \in \mathbb{R}_{<0}$, $g(\gamma, \mathsf{x}) > 1$ if and only if $\mathsf{x} \in (\tilde{\mathsf{x}}, 0)$, where $\tilde{\mathsf{x}}$ is the unique solution of $g(\gamma, \mathsf{x}) = 1$ in $(\bar{\mathsf{x}}, 0)$.

The proof of this lemma invokes various properties of the Lambert W function listed in Section II and is given in Appendix C. The next lemma, whose proof relies on the results of Lemma B.1 and is also given in Appendix C, characterizes the effect of delay on the rate of convergence of scalar time-delayed system

$$\dot{x}(t) = \mathsf{a}\, x(t-\tau) + \mathsf{b}\, x(t), \qquad t \in \mathbb{R}_{\geq 0}$$
$$x(\eta) \in \mathbb{R}, \qquad \eta \in [-\tau, 0]. \tag{26}$$

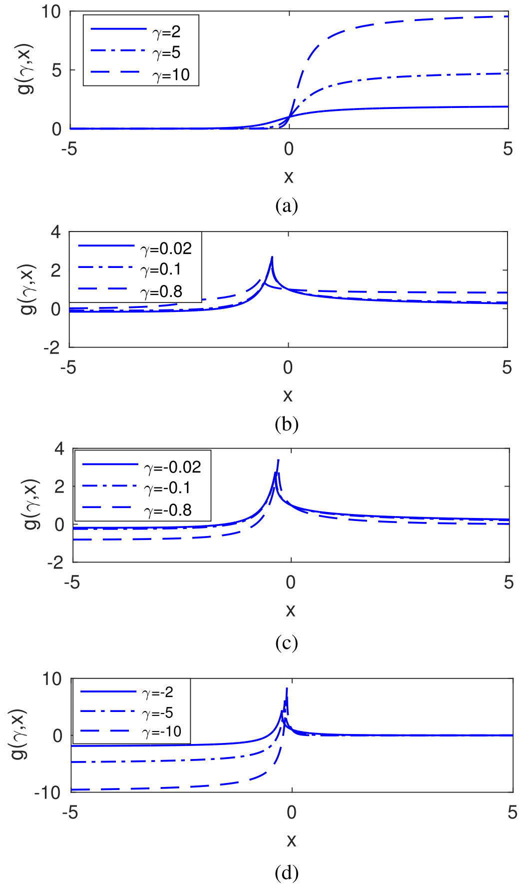

Fig. 5. Delay gain function for different values of x, γ. (a) $\gamma > 1$. (b) $0 < \gamma < 1$. (c) $-1 < \gamma < 0$. (d) $\gamma < -1$.

The tightest estimate of the rate of convergence of (26) is characterized by the magnitude of the real part of the rightmost root of its characteristic equation $s = \frac{1}{\tau} W_0(\alpha\,\tau\,\mathrm{e}^{-\tau\mathsf{b}}) + \mathsf{b}$ [recall Lemma II.1 and (2a)]. That is (see [26, Corollary 1])

$$\rho_\tau = -\frac{1}{\tau}\,\mathrm{Re}(W_0(\mathsf{a}\,\tau\,\mathrm{e}^{-\tau\mathsf{b}})) - \mathsf{b}. \tag{27}$$

Recalling (15), we write (27) as

$$\rho_\tau = -(g(\gamma, \mathsf{x})\,\mathsf{a} + \mathsf{b}) = -(g(\gamma, \mathsf{x}) - \gamma)\,\mathsf{a} \tag{28}$$

where $\mathsf{x} = \mathsf{a}\tau$ and $\gamma = -\frac{\mathsf{b}}{\mathsf{a}}$. It follows from (1a) that:

$$\lim_{\tau\to 0} g(\gamma, \mathsf{a}\tau) = 1. \tag{29}$$

Therefore, as expected, $\lim_{\tau\to 0} \rho_\tau = \rho_0$, where

$$\rho_0 = -(\mathsf{a} + \mathsf{b}) = -(1-\gamma)\mathsf{a}. \tag{30}$$

System (26) in terms of different values of $\mathsf{a}, \mathsf{b} \in \mathbb{R}$, $\mathsf{a} \neq 0$ satisfies $\mathsf{a} + \mathsf{b} < 0$.

Lemma B.2 (Effect of delay on the rate of convergence of delayed system (26)): Consider system (26) with $\mathsf{a} \in \mathbb{R}\backslash\{0\}$ and $\mathsf{b} \in \mathbb{R}$ such that $\mathsf{a} + \mathsf{b} < 0$, whose rate of convergence ρ_τ is specified by (28). Consider also the delay gain function (15) with $\gamma = -\frac{\mathsf{b}}{\mathsf{a}}$ and $\mathsf{x} = \mathsf{a}\tau$. Then, we have the following.

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a) For $\mathsf{a} > 0$ and $\mathsf{b} < 0$, the system (26) is exponentially stable for any $\tau \in \mathbb{R}_{\geq 0}$. Moreover, the rate of convergence decreases by increasing $\tau \in \mathbb{R}_{\geq 0}$.

b) For $\mathsf{a} < 0$ and $\mathsf{b} \in \mathbb{R}$, $\rho_\tau > \rho_0$ if and only if $\tau \in [0, \tilde{\tau}) \subset [0, \bar{\tau})$, where $\tilde{\tau}$ is the unique solution of $g(\gamma, \mathsf{a}\tau) = 1$ in $(0, \bar{\tau})$ and $\bar{\tau}$ is specified by

$$\bar{\tau} = \arccos(-\mathsf{b}/\mathsf{a})/\sqrt{\mathsf{a}^2 - \mathsf{b}^2}. \tag{31}$$

Moreover, ρ_τ is monotonically increasing (respectively, decreasing) with τ for any $\tau \in [0, \tau^\star) \subset [0, \bar{\tau})$ (respectively, $\tau \in (\tau^\star, \bar{\tau}) \subset [0, \bar{\tau})$), where $\tau^\star = -\frac{1}{\mathsf{b}} W_0(\frac{\mathsf{b}}{\mathsf{a}\,\mathrm{e}})$ when $\mathsf{b} \neq 0$ and $\tau^\star = -\frac{1}{\mathsf{a}\,\mathrm{e}}$ when $\mathsf{b} = 0$. Finally, the maximum rate of convergence of $\rho^\star_\tau = -(1 + \frac{1}{W_0(\frac{\mathsf{b}}{\mathsf{a}\mathrm{e}})})\mathsf{b}$ when $\mathsf{b} \neq 0$ and $\rho^\star_\tau = -\mathsf{a}\,\mathrm{e}$ when $\mathsf{b} = 0$ is obtained at $\tau = \tau^\star$.

In developing our results, we also invoke the following result.

Lemma B.3 (Maximum value of the trajectory of (26) [41, Th. 2.10]): For the time delay system (26) and any $\tau \in (0, \bar{\tau}]$ with $\mathsf{a}, \mathsf{b} \in \mathbb{R}_{<0}$, the following holds:

$$|x(t)|_\infty = \max_{s \in [-\tau, 2\tau]} |x(s)|. \tag{32}$$

Appendix C
Proofs of Lemmas B.1 and B.2

Proof of Lemma B.1: Part (a) can be readily deduced by invoking (1b) since $W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}}) \to \mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}}$ as $\mathsf{x} \to 0$. To prove statement (b), we proceed as follows. Let $q = \mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}}$. Since $x \in \mathbb{R}_{>0}$, then $q \in \mathbb{R}_{>0}$. As a result, given the properties of Lambert W function reviewed in Section II, we can write $\mathsf{x} = \frac{1}{\gamma} W_0(\gamma q)$ and $\mathrm{Re}(W_0(q)) = W_0(q)$, which allows us to represent $g(\gamma, \mathsf{x})$ as

$$g(\gamma, \mathsf{x}) = \frac{W_0(q)}{W_0(\gamma\, q)}\,\gamma, \qquad \text{for } x \in \mathbb{R}_{>0}. \tag{33}$$

Since for $\gamma > 1$, we have $q < \gamma\, q$, by invoking Lemma II.2, we obtain $\frac{W_0(q)}{W_0(\gamma\, q)} < 1$, which together with $W_0(q) \in \mathbb{R}_{>0}$ and $W_0(\gamma\, q) \in \mathbb{R}_{>0}$ validates statement (b) from (33).

Next, we validate statement (c). The derivative of $g(\gamma, \mathsf{x})$ with respect to $\mathsf{x} \in \mathbb{R}$ is

$$\begin{aligned}\frac{\mathrm{d}\, g(\gamma, \mathsf{x})}{\mathrm{d}\,\mathsf{x}} &= \frac{(1 + \gamma\,\mathsf{x})\,\mathrm{e}^{\gamma\,\mathsf{x}}}{\mathsf{x}}\,\mathrm{Re}\left(\frac{1}{\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}} + \mathrm{e}^{W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}})}}\right)\\ &- \frac{1}{\mathsf{x}^2}\,\mathrm{Re}(W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}})) = \frac{(1 + \gamma\,\mathsf{x})}{\mathsf{x}^2}\,\mathrm{Re}\left(\frac{W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}})}{W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}}) + 1}\right)\\ &- \frac{1}{\mathsf{x}^2}\,\mathrm{Re}(W_0(\mathsf{x}\,\mathrm{e}^{\gamma\mathsf{x}})) = \frac{1}{\mathsf{x}^2}\,\mathrm{Re}\left(\frac{(\gamma\mathsf{x} - W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}}))W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}})}{(W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}}) + 1)}\right)\end{aligned} \tag{34}$$

for $\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}} \neq -\frac{1}{\mathrm{e}}$. Recall (2c) that $\mathrm{Re}(W_0(\mathsf{z})) + 1 > 0$ for any $\mathsf{z} \in \mathbb{R}\backslash\{-\frac{1}{\mathrm{e}}\}$ and $\mathrm{Re}(W_0(\mathsf{z})) = W_0(\mathsf{z}) > 0$ for any $\mathsf{z} \in \mathbb{R}_{>0}$. Note also that we have already shown that for any $\gamma > 1$ and $\mathsf{x} > 0$, we have $g(\gamma, \mathsf{x}) < \gamma$, which gives $\gamma\,\mathsf{x} - W_0(\mathsf{x}\,\mathrm{e}^{\gamma\mathsf{x}}) > 0$. Therefore, for $\gamma > 1$ and $\mathsf{x} \in \mathbb{R}_{>0}$, from (34), we obtain $\frac{\mathrm{d}\, g(\gamma, \mathsf{x})}{\mathrm{d}\,\mathsf{x}} > 0$, which validates statement (c).

To validate statement (d), consider $x \in (\bar{x}, 0]$. For $x \to 0^-$, we have $g(\gamma, x) \to 1$. Therefore, for $\gamma < 1$ (respectively, $\gamma > 1$), we get $g(\gamma, x) > \gamma$ (respectively, $g(\gamma, x) < \gamma$) as $x \to 0^-$. Moreover, we know that the admissible bound $x = \bar{x}$ is the first point that $g(\gamma, x) = \gamma$ holds. Therefore, since $g(\gamma, x)$ is a continuous function, for any $x \in (\bar{x}, 0]$, we have $g(\gamma, x) > \gamma$ for $\gamma < 1$, and $g(\gamma, x) < \gamma$ for $\gamma > 1$.

For proof of statement (e), we proceed as follows. Recall the properties of Lambert W_0 function in (2). Note that for $0 < \gamma < 1$, we have $-1 < W_0(-\frac{\gamma}{\mathrm{e}}) < 0$, and for $\gamma < 0$, we have $W_0(-\frac{\gamma}{\mathrm{e}}) > 0$. Also, recall that $W_0(0) = 0$. Therefore, for $\gamma < 1$ and $\gamma \neq 0$, we have $\frac{1}{\gamma} W_0(-\frac{\gamma}{\mathrm{e}}) \in \mathbb{R}_{<0}$. Now, for $\gamma < 1$, consider $\mathsf{x} \in [\frac{1}{\gamma} W_0(-\frac{\gamma}{\mathrm{e}}), 0)$ for $\gamma \neq 0$ and $\mathsf{x} \in [-\frac{1}{\mathrm{e}}, 0)$ for $\gamma = 0$. For such x, we have $\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}} \in \mathbb{R}_{<0}$. For $f(\mathsf{x}) = \mathsf{x}\,\mathrm{e}^{\gamma\mathsf{x}}$, with $\mathsf{x}, \gamma \in \mathbb{R}$, we know $\frac{d\,f}{d\,\mathsf{x}} = (1 + \gamma\,\mathsf{x})\,\mathrm{e}^{\gamma\mathsf{x}} > 0$ for any $\mathsf{x} \in (-\frac{1}{\mathrm{e}}, 0]$ and $\gamma < 1$, i.e., $f(\mathsf{x})$ is a strictly increasing continuous function. Because the solutions of $\mathsf{z}\,\mathrm{e}^{\gamma\mathsf{z}} = -\frac{1}{\mathrm{e}}$ are $\mathsf{z} = \frac{1}{\gamma} W_l(-\frac{\gamma}{\mathrm{e}})$, $l = \{-1, 0\}$ for $\gamma \neq 0$ and $\mathsf{z} = -\frac{1}{\mathrm{e}}$ for $\gamma = 0$, for $\mathsf{x} \in [\frac{1}{\gamma} W_0(-\frac{\gamma}{\mathrm{e}}), 0)$, we have $\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}} \in (-\frac{1}{\mathrm{e}}, 0]$ and then $W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}}) \in \mathbb{R}_{<0}$ [recall (2a)]. Next, note that by statement (d), we have $\gamma\,\mathsf{x} - W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}}) = \mathsf{x}(\gamma - g(\mathsf{x}, \gamma)) > 0$ for $\mathsf{x} \in (\bar{\mathsf{x}}, 0]$. Therefore, $\frac{d\,g(\mathsf{x},\gamma)}{d\,\mathsf{x}} < 0$ can be inferred from (34). Next, for $\mathsf{x} < \frac{1}{\gamma} W_0(-\frac{\gamma}{\mathrm{e}})$, let $W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}}) = w + \mathrm{i}\,u$. Then, (34) can be written as $\frac{\mathrm{d}\,g(\gamma,\mathsf{x})}{\mathrm{d}\,\mathsf{x}} = \frac{1}{\mathsf{x}^2}\,\mathrm{Re}(\frac{(\gamma\,\mathsf{x} - (w + \mathrm{i}\,u))(w + \mathrm{i}\,u)}{((w + \mathrm{i}\,u) + 1)}) = \frac{1}{\mathsf{x}^2((w+1)^2 + u^2)}((\gamma\mathsf{x} - w)(w^2 + u^2 + w) + u^2)$. In addition, we have $w = -u\cot u$ since $\mathrm{Im}(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}}) = 0$, which gives $\frac{\mathrm{d}\,g(\gamma,\mathsf{x})}{\mathrm{d}\,\mathsf{x}} = \frac{1}{\mathsf{x}^2 u^2((\cot u + 1)^2 + 1)}((\gamma\mathsf{x} + u\cot u)(u^2\cot^2 u + u^2 - u\cot u) + u^2) > 0$. Here, we used $u^2\cot^2 u + u^2 - u\cot u = \frac{u}{\sin u}(\frac{u}{\sin u} - \cos u) > 0$, and $\gamma\mathsf{x} + u\cot u = \gamma\mathsf{x} - w = \gamma\mathsf{x} - \mathrm{Re}(W_0(\mathsf{x}\,\mathrm{e}^{\gamma\,\mathsf{x}})) = \mathsf{x}(\gamma - g(\gamma, x)) > 0$, which holds for any $\mathsf{x} \in [\bar{\mathsf{x}}, 0)$ [recall statement (d)], which finalize our proof for statement (e).

For proof of statement (f), notice that statement (e) explicitly implies that $\max(g(\gamma, \mathsf{x})) = g(\gamma, \mathsf{x}^\star)$ for any $x \in \mathbb{R}_{<0}$, where $\mathsf{x}^\star\,\mathrm{e}^{\gamma\,\mathsf{x}^\star} = -\frac{1}{\mathrm{e}}$, which is equivalent to $\mathsf{x}^\star = \frac{1}{\gamma} W_0(-\frac{\gamma}{\mathrm{e}})$ for $\gamma \neq 0$, and $\mathsf{x}^\star = -\frac{1}{\mathrm{e}}$ for $\gamma = 0$.

Proof of statement (g) is as follows. In statement (a), we showed that $g(\gamma, \mathsf{x}) \to 1$ as $\mathsf{x} \to 0^-$. Moreover, $g(\gamma, \mathsf{x})$ is a continuous ascending function in $\mathsf{x} \in (-\infty, \frac{1}{\gamma} W_0(-\frac{\gamma}{\mathrm{e}})]$ and a descending function in $\mathsf{x} \in [\frac{1}{\gamma} W_0(-\frac{\gamma}{\mathrm{e}}), 0)$. Therefore, continuity implies that there exists a $\tilde{\mathsf{x}} \in (\bar{\mathsf{x}}, \mathsf{x}^\star)$ such that $g(\gamma, \tilde{\mathsf{x}}) = 1$, or equivalently $\mathrm{Re}(W_0(\tilde{\mathsf{x}}\,\mathrm{e}^{\gamma\,\tilde{\mathsf{x}}})) = \tilde{\mathsf{x}}$, and also, $g(\gamma, \mathsf{x}) > 1$ holds for any $\mathsf{x} \in (\tilde{\mathsf{x}}, 0)$. ■

Proof of Lemma B.2: Because, by assumption, we have $\alpha + \mathsf{b} < 0$, $\mathsf{a} > 0$ implies that $\mathsf{b} < -\mathsf{a} < 0$, resulting in $\gamma > 1$ and $\mathsf{x} = \mathsf{a}\tau > 0$ for $\tau \in \mathbb{R}_{>0}$. Therefore, invoking statement (b) of Lemma B.1, we get $g(\gamma, \mathsf{x}) < \gamma$. Thus, (28) implies that system (26) is exponentially stable regardless of the value of $\tau \in \mathbb{R}_{\geq 0}$. Moreover, by taking derivative of ρ_τ with respect to τ, we obtain

$$\frac{\mathrm{d}\,\rho_\tau}{\mathrm{d}\,\tau} = \left(\frac{\mathrm{d}\,g(\gamma, \mathsf{x})}{\mathrm{d}\,\mathsf{x}}\right)\left(\frac{\mathrm{d}\,\mathsf{x}}{\mathrm{d}\,\tau}\right) = -\mathsf{a}\,\frac{\mathrm{d}\,g(\gamma, \mathsf{x})}{\mathrm{d}\,\mathsf{x}}. \tag{35}$$

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Statement (c) of Lemma B.1 states that $\frac{\mathrm{d}\, g(\gamma,\mathsf{x})}{\mathrm{d}\,\mathsf{x}} > 0$ for any $\gamma > 1$ and $\mathsf{x} > 0$. Hence, for $\mathsf{a} > 0$, we have $\frac{\mathrm{d}\,\rho_\tau}{\mathrm{d}\,\tau} < 0$, which concludes our proof of part (a).

For $\mathsf{a} < 0$ and $\mathsf{b} \in \mathbb{R}$, from (28), it follows that $\rho_\tau > \rho_0$ if and only if $g(\gamma, \mathsf{a}\tau) > 1$. In this case, because of $\mathsf{a} + \mathsf{b} < 0$, we have $\gamma < 1$ and $\mathsf{x} = \mathsf{a}\tau < 0$ for $\tau \in \mathbb{R}_{>0}$. Therefore, by virtue of statement (g) of Lemma B.1, we have $\rho_\tau > \rho_0$ if and only if $\tau \in [0, \tilde{\tau}) \subset [0, \bar{\tau})$, where $\tilde{\tau}$ is the unique solution of $g(\gamma, \mathsf{a}\tau) = 1$ in $(0, \bar{\tau})$. In addition, by virtue of part (e) of Lemma B.1, ρ_τ, whose rate of change with respect to τ is specified by (35), is monotonically increasing (respectively, decreasing) with τ for any $\tau \in [0, \tau^\star) \subset [0, \bar{\tau})$ (respectively, $\tau \in (\tau^\star, \bar{\tau}) \subset [0, \bar{\tau})$), where $\tau^\star = \frac{\mathsf{x}^\star}{\mathsf{a}} = \frac{1}{\gamma \mathsf{a}} W_0(-\frac{\gamma}{\mathrm{e}}) = -\frac{1}{\mathsf{b}} W_0(\frac{\mathsf{b}}{\mathsf{a}\,\mathrm{e}})$ for $\mathsf{b} \neq 0$ and $\tau^\star = \frac{\mathsf{x}^\star}{\mathsf{a}} = -\frac{1}{\mathsf{a}\,\mathrm{e}}$ for $\mathsf{b} = 0$. Moreover, by virtue of part (f) of Lemma B.1, we conclude that the maximum value of $g(\gamma, \mathsf{x})$ occurs at $\mathsf{x}^\star = \mathsf{a}\tau^\star$, where $g(\gamma, \mathsf{x}^\star) = \frac{-\gamma}{W_0(-\frac{\gamma}{\mathrm{e}})}$ for $\mathsf{b} \neq 0$, which gives $\rho^\star_\tau = -(1 + \frac{1}{W_0(\frac{\mathsf{b}}{\mathsf{a}\,\mathrm{e}})})\mathsf{b}$. For $\mathsf{b} = 0$, we have $\rho^\star_\tau = -\mathsf{a}\,\mathrm{e}$. ∎

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