HALL-HIGMAN TYPE THEOREMS FOR EXCEPTIONAL GROUPS OF LIE TYPE, I

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Abstract. The paper studies the minimum polynomial degrees of p-elements in cross-characteristic representations of simple groups of exceptional Lie type whose BN-pair rank is at most 2. Specifically, we prove that the degree in question equals the order of the element.

Dedicated to the memory of Jan Saxl

1. Introduction

This paper continues our earlier work [56] devoted to generalize the famous Hall-Higman theorem on the minimum polynomials of p-elements in representations of p-solvable groups to more general classes of groups. The bulk of the project is the case of almost simple groups. The paper [56] deals mainly with classical groups, and this paper completes the project for quasi-simple groups of BN-pair rank at most 2. Specifically, we prove the following result.

Theorem 1.1. *Let G be one of the groups ${}^2B_2(q)$ with $q>2$, ${}^2G_2(q)$ with $q>3$, ${}^2F_4(q)$, $G_2(q)$ with $q>2$, or ${}^3D_4(q)$. Let $g\in G$ be an element of prime power order coprime to q. Let ϕ be a non-trivial irreducible representation of G over a field F of characteristic ℓ coprime to q. Then the minimum polynomial degree of $\phi(g)$ equals $|g|$, unless possibly when $G={}^2F_4(8)$, $\ell=3$, $|g|=109$ and $\phi(1)<64692$.*

Observe that it suffices to prove Theorem 1.1 for F algebraically closed. Theorem 1.1 is valid for the Tits group ${}^2F_4(2)'$ (Lemma 9.1); also see Lemma 7.14 for $G=2\cdot G_2(4)$.

Let $|g|$ be a power of a prime p. Theorem 1.1 improves our earlier result [56, Theorem 4.6], stating that $\deg\phi(g)\geq|g|(1-1/p)$ whenever a Sylow p-subgroup of G is cyclic.

In some special cases the result of Theorem 1.1 was known earlier. These are

(i) Sylow p-subgroups of the quasi-simple group G are cyclic and $\ell\in\{0,p\}$ [61];
(ii) $\ell=0$, $p>2$ and $G\in\{G_2(q),{}^2F_4(q),{}^2F_4(2)',{}^3D_4(q)\}$ [63, Lemmas 4.11 and 4.14];
(iii) $G\cong G_2(q), q>2$, $p>2$ and g lies in a parabolic subgroup of G [63, Lemma 4.10].

Notation. Let G be a finite group. Then $|G|$ is the order of G, $Z(G)$ be the center of G and $O_p(G)$ the maximal normal p-subgroup of G for a prime p. We often use $|G|_p$ to denote the p-part of $|G|$. For $g\in G$ the order of g is denoted by $|g|$ and $o(g)$ is the order of g modulo $Z(G)$. A p'-element is one of order coprime to p.

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$\mathbb{F}_q$ means the finite field of q elements, $\overline{\mathbb{F}}_q$ its algebraic closure and $\mathbb{Q}, \mathbb{C}$ are the rational and complex number fields, respectively. $\mathbb{Z}$ denotes the set of integers.

Let F be an algebraically closed field of characteristic ℓ, and ϕ an F-representation of G. Then $\deg \phi(g)$ denotes the minimum polynomial degree of $\phi(g)$. We write $\phi \in \mathrm{Irr}_\ell\, G$ to indicate that ϕ is irreducible, and use this notation for the Brauer character of ϕ too. If $\ell = 0$, we drop the subscript ℓ. If χ is an ordinary (generalized) character of G, and with ℓ a fixed prime, then χ° is the restriction of χ to ℓ'-elements.

We denote by 1_G the trivial character of G (both ordinary and ℓ-modular), and by ρ_G^{reg} the regular representation of G or the (Brauer) character of it. The ordinary Steinberg representation and its character of a group of Lie type is denoted by St_G or St.

A Brauer character $\phi \in \mathrm{Irr}_\ell\, G$ is called *liftable* if there exists an ordinary character τ of G such that $\tau^\circ = \phi$. An irreducible representation or FG-module is called *liftable* if the Brauer character of it is liftable.

If H is a subgroup of G and η is a character or representation of H then η^G denotes the induced character. If ϕ is a character or representation of G then $\phi|_H$ stands for the restriction of ϕ to H.

If V is an FG-module and X a subset of G we write V^X or $C_V(X)$ for the subspace of elements of V fixed by all $x \in X$.

For integers $a, b > 0$ we write (a, b) for the greatest common divisor of a, b and $a|b$ means that a divides b. We also write (χ, ϕ) for the inner product of characters χ, ϕ of a group G. We write $\mathrm{diag}(x_1, \ldots, x_m)$ for a block-diagonal matrix with diagonal blocks $x_1, \ldots, x_m$.

A finite group of Lie type is that of shape $\mathbf{G}^{\mathsf{F}}$, where $\mathbf{G}$ is a connected reductive algebraic group and F a Frobenius endomorphism of it. For more information see [4] or [10].

2. Preliminaries

Lemma 2.1. *Let C be a cyclic group of order coprime to ℓ, and χ a non-trivial Brauer character of C such that $\chi(c) = a$ for every $1 \neq c \in C$ (so χ is constant on $C \smallsetminus \{1\}$). Then $\chi = \frac{\chi(1)-a}{|C|} \cdot \rho_C^{\mathrm{reg}} + a \cdot 1_C$. In particular, if $a \geq 0$ and $(\chi(1), a) \neq (|C|, 0)$, or if $a < 0$ and $-a\cdot(|C|-1) < \chi(1)$, then $\chi = \rho_C^{\mathrm{reg}} + \chi'$ for some proper character χ' of C.*

Proof. The first claim is obvious as $\chi - a \cdot 1_G$ vanishes at $C \smallsetminus \{1\}$.

Suppose $a \geq 0$. Then $\frac{\chi(1)-a}{|C|} \in \mathbb{Z}_{\geq 1}$ as this equals the multiplicity of any non-trivial irreducible character of C in χ.

Suppose $a < 0$. Then

$$\chi = \frac{\chi(1) - a}{|C|} \cdot \rho_C^{\mathrm{reg}} + a \cdot 1_C = \frac{\chi(1) + a(|C| - 1)}{|C|} \cdot \rho_C^{\mathrm{reg}} + (-a)\big(\rho_C^{\mathrm{reg}} - 1_C\big),$$

so $\chi(1) + a(|C| - 1) > 0$ implies the second claim in this case. □

Lemma 2.2. [33, Lemma IX.2.7] *Let p, r be primes and a, b positive integers such that $p^a = r^b + 1$. Then either*

(i) *$p = 2$, $b = 1$, and r is a Mersenne prime, or*
(ii) *$r = 2, a = 1$, and p is a Fermat prime, or*
(iii) *$p^a = 9$.* □

Lemma 2.3. *Let $G = A \rtimes H$ be a semidirect product, where A is an abelian normal subgroup of G and $H = \langle h \rangle$ is a p-group. Let ϕ be a ℓ-modular representation of G faithful on A. Suppose that $(p\ell, |A|) = 1$ and $C_H(A) \leq Z(G)$. Then $\deg \phi(h) \geq o(h)$.*

Proof. It suffices to prove the statement for ϕ a representation over an algebraically closed field F of characteristic ℓ. Next, we may assume that ϕ is completely reducible. Indeed, this follows from Maschke's theorem if $p \neq \ell$, so assume that $p = \ell$. Let τ be the direct sum of the irreducible constituents of ϕ. Then $\eta : \phi(G) \to \tau(G)$ is a group homomorphism, and $\ker \eta$ is a p-group. Since $\ker \phi \cap A = 1$, $\ker \phi$ is a p-group. Hence $\ker \tau$ is a p-group, and thus τ is faithful on A. Clearly, $\deg \phi(h) \geq \deg \tau(h)$. So it suffices to prove the lemma for τ in place of ϕ as claimed.

Set $g := h^{o(h)/p}$, so that $g^p \in Z(G)$ but $g \notin Z(G)$. Then $1 \neq [g, a] \in A$ for some $a \in A$. As τ is faithful on A, this implies that

$$[\lambda(g), \lambda(a)] = \lambda([g, a]) \neq \mathrm{Id}$$

for some irreducible constituent λ of ϕ. By Schur's lemma, $\lambda(g^p) = \lambda(h^{o(h)}) = \alpha \cdot \mathrm{Id}$ for some $\alpha \in F^\times$ Let $\beta \in F^\times$ be such that $\beta^{o(h)} = \alpha$, and let $\mu : G \to \mathrm{GL}_1(F)$ be a representation of G such that $\mu(A) = 1$ and $\mu(h) = \beta^{-1}$. Then $\nu := \lambda \otimes \mu \in \mathrm{Irr}_F(G)$, $\nu(g^p) = \nu(h^{o(h)}) = \mathrm{Id}$, but $[\nu(g), \nu(a)] = [\lambda(g), \lambda(a)] \neq \mathrm{Id}$. The last two facts imply that $C_{\langle \nu(h) \rangle}(\nu(A)) = 1$, and so $\deg \nu(h) = o(h)$ by Higman's lemma [33, Theorem IX.1.10]. As $\deg \tau(h) \geq \deg \lambda(h) = \deg \nu(h)$, the statement follows. □

Corollary 2.4. *Let $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q)$ with $n \geq 2$ and $(n, q) \neq (2, 2)$, $(2, 3)$, and let $h \in G$ be a non-central p-element with $p \nmid q$. Let ϕ be an ℓ-modular representation of G such that $\ker \phi \leq Z(G)$ and $\ell \nmid q$. Suppose that h is not irreducible on the natural $\mathrm{GL}_n(q)$-module $\mathbb{F}_q^n$. Then $\deg \phi(h) \geq o(h)$.*

Proof. Let $V = \mathbb{F}_q^n$ and let $W \neq 0$ be a proper h-stable subspace of V. Let

$$A := \{g \in G \mid gW = W \text{ and } g \text{ acts trivially on both } W \text{ and } V/W\}.$$

Then A is an abelian group, $(|A|, p\ell) = 1$ and $hAh^{-1} = A$. By [22, §13-2] or a direct calculation, we have $C_{\mathrm{GL}_n(q)}(A) = AZ(\mathrm{GL}_n(q))$, whence $C_{\langle h \rangle}(A) = \langle h \rangle \cap Z(G)$. The assumption $\ker \phi \leq Z(G)$ ensures that ϕ is faithful on A. So the result follows from Lemma 2.3. □

Lemma 2.5. *Let $G = \mathrm{Sp}_{2n}(q)$, q even, $n > 1$, and let $g \in G$ be a reducible p-element for $p \not| q$. Let $\phi \in \mathrm{Irr}_\ell\, G$ with $\dim \phi > 1$ and $\ell \neq 2$. Then $\deg \phi(g) = |g|$.*

Proof. If g belongs to a parabolic subgroup of G then the result is contained in [11]. Otherwise, $|g| > 3$ and $g \in H \cong H_1 \times H_2$, where $H_1 \cong \mathrm{Sp}_{2k}(q)$, $H_2 \cong \mathrm{Sp}_{2l}(q)$, $k + l = n$. Then $g = g_1 g_2$, where $g_1 \in H_1$, $g_2 \in H_2$ are p-elements. We may assume that $|g| = |g_1|$ and moreover that g_1 is irreducible in H_1. In addition, we may assume that $(k, q) \neq (1, 2)$ as otherwise $|g| = 3$.

(i) Suppose that $(l, q) \neq (1, 2)$. Then, by [56, Corollary 3.8], the restriction $\phi|_H$ contains an irreducible constituent $\tau = \tau_1 \otimes \tau_2$ with $\tau_1 \in \mathrm{Irr}_\ell\, H_1$, $\tau_2 \in \mathrm{Irr}_\ell\, H_2$ such that $\dim \tau_1, \dim \tau_2 > 1$. Then $\tau(g) = \tau_1(g_1) \otimes \tau_2(g_2)$. If $\deg \tau_1(g_1) = |g_1|$ then we are done. Otherwise, by [56, Lemma 3.3 and Prop. 5.7], $|g_1| = q^k + 1$ and $\deg \tau_1(g_1) \geq |g_1| - 2$ (in fact, $\deg \tau_1(g_1) \geq |g_1| - 1$ if $k > 1$ and $(k, q) \neq (3, 2)$). By Lemma 2.2, $|g_1| = q^k + 1$ implies that either $|g_1| = p = q^k + 1$, or $q^k = 8$, $|g_1| = 9$.

First suppose that $|g_1| = p > 3$. Then $\deg \tau_2(g_2) \geq 3$ by [56, Theorem 1.2], and hence $\deg \tau(g) = p$ by [56, Lemma 2.12].

Suppose that $|g| = 9, q = 8$. Then $\deg \tau_1(g_1) \geq 7$ and $\deg \tau_2(g_2) \geq 3$ (by [56, Theorem 1.3] unless possibly $|g_2| = 3$ and Sylow 3-subgroups of H_2 are cyclic; the latter implies $H_2 = \mathrm{SL}_2(8)$ and then $\deg \tau_2(g_2) = 3$ by [56, Lemma 3.3]). If $\ell > 3$ then [56, Lemma 2.12(i)] again yields the result. Let $\ell = 3$. Let J_i denote the Jordan block of size i over $\mathbb{F}_3$. Then the minimum polynomial degree of $J_7 \otimes J_3$ equals 9 [54, Lemma 2.11]); in addition, if $a \geq 7, b \geq 3$ then the minimum polynomial degree of $J_a \otimes J_b$ is at least 9 [54, Lemma 2.10]. Therefore, $\deg \tau(g) = 9$.

Let $k = 3, q = 2$, $|g| = 9$ and $l > 1$. Then $\deg \tau_1(g_1) \geq 7$. If $\deg \tau_2(g_2) \geq 3$, then the result follows as above. Suppose that $\deg \tau_2(g_2) = 2$. Then $|g_2| = 3$ by [56, Theorem 1.3]. As $l > 1$, one easily observes that g is contained in a parabolic subgroup of G.

(ii) We are left with the case where $q = 2, l = 1$. As $1 \neq g_2 \in \mathrm{Sp}_2(2)$, we have $p = 3$. Let τ be a non-trivial irreducible constituent of $\phi|_{H_1}$. By [56, Prop. 5.7] applied to $p = 3$, $\deg \tau(g_1) = |g_1|$, unless $k = 3$, $|g_1| = 9$, and either $\dim \tau = \deg \tau(g_1) = 7$, or $\ell = 3$, $\dim \tau = 21$ and $\deg \tau(g_1) \geq 7$. So the lemma follows unless $G = Sp_8(2)$.

In this case we show that $\phi|_{H_1}$ has an irreducible constituent of degree $d \neq 1, 7, 21$, which implies the result due to [56, Prop. 5.7(ii)]. Indeed, H_1 is contained in a parabolic subgroup P, the stabilizer of a line of the natural $\mathbb{F}_2G$-module. Let $Q = O_2(P)$. Then Q is an abelian group, and $\phi|_Q$ is a direct sum of linear representations λ of Q permuted by P when P acts on Q by conjugation. Note that there is λ whose $\langle g_1\rangle$-orbit is faithful (this follows from the equality $C_G(Q) = Z(G)Z(Q)$ [22, §13-2].) Let $\Lambda = H_1\lambda$ be the H_1-orbit of λ with point stabilizer $C_{H_1}(\lambda) \cong P_1$, where P_1 is the stabilizer of a nonzero vector of the natural $\mathbb{F}_2H_1$-module. This yields a permutational FH_1-module $L \cong 1_{P_1}^{H_1}$. The composition factors of L are the reduction modulo ℓ of those for the corresponding module over the complex numbers. The latter decomposes as $1_{H_1} + \chi_1 + \chi_2$, where $\chi_1(1) = 27$ and $\chi_2(1) = 35$ [6, p. 46]. Then χ_1 remains irreducible under restriction modulo $\ell = 3$, and χ_2 remains irreducible under restriction modulo $\ell \neq 3$, see [34]. Thus L has an irreducible constituent of degree 27 or 35, and the claim follows. □

Lemma 2.6. *Let $G = \mathrm{SL}_3(q)$, $q > 2$, and let $\phi \in \mathrm{Irr}_F(G)$, $\dim \phi > 1$, $\ell \nmid q$. Let $g \in (G \setminus Z(G))$ be a p-element. Then either $\deg \phi(g) = o(g)$, or $(3, q-1) = 1$ and $|g| = q^2 + q + 1$. Moreover, in the latter case $\dim \phi = q^2 + q - 1$ if $p = \ell$, whereas if $p \neq \ell$ then $\deg \phi(g) = \dim \phi = q^2 + q$ and 1 is not an eigenvalue of $\phi(g)$.*

Proof. If g is reducible in G then the result follows from Corollary 2.4.

Suppose that g is irreducible in G, and hence $|g|$ divides q^2+q+1. Observe that $p > 2$ as q^2+q+1 is odd. If Sylow p-subgroups are not cyclic then $p = 3$ and $3|(q-1)$, and then g is reducible by [63, Lemma 3.2], a contradiction.

So Sylow p-subgroups are cyclic. If $\ell = 0$ or p then the result is a special case of [61, Theorem 1.1], and the claim on eigenvalue 1 for $\ell = 0$ is contained in [61, Corollary 1.3(4)]. Let $\ell \neq p$. According to [56, Example 3.2(ii)], either ϕ lifts to characteristic 0 and the result follows from that for $\ell = 0$, or ℓ divides $q^2 + q + 1$ and the Brauer character χ of ϕ coincides on the ℓ'-elements with $\tau - 1_G$, where τ is the unipotent character of degree $q^2 + q$ of G. Let $C = \langle g\rangle \subseteq T$, where T is a cyclic group of order $q^2 + q + 1$. It is well known (and also follows from Lemma 3.2 below) that $\tau|_T = \rho_T^{\mathrm{reg}} - 1_T$. Then $\tau|_C = |T/C| \cdot \rho_C^{\mathrm{reg}} - 1_T$ and hence $\chi|_C = |T/C| \cdot \rho_C^{\mathrm{reg}} - 2 \cdot 1_C$. As $\ell \neq p$ and ℓ divides $|T|$, we have $\ell \neq 2$ so $|T/C| \geq 3$, Therefore, by Lemma 2.1, $\chi|_C = \rho_C^{\mathrm{reg}} + \chi'$, where χ' is a proper character of C. Therefore, $\deg \phi(g) = |g|$ in this case, whence the result. □

Lemma 2.7. (Borel-Tits, see [22, §13.1]) *Let H be a finite reductive group in characteristic r and $g \in G$. If g normalizes a non-trivial r-subgroup of H then g belongs to a proper parabolic subgroup of H. In particular, this holds if g is not regular.*

Let G be a finite quasi-simple group of Lie type in characteristic $r > 0$ of simply connected type. Let $\Phi_m(x)$ denote the cyclotomic polynomial for m-th roots of 1, and $\prod_m \Phi_m^{l_m}(x)$ a polynomial associated with G, see [22], pages $110 - -111$. Set $|G|_{r'} := |G|/|U|$ where U is a Sylow r-subgroup of G. Then $|G|_{r'} = \prod_m \Phi_m^{l_m}(q)$. If $G = {}^2B_2(q), {}^2F_4(q)$ we assume that $q = 2^{2a+1}$, and if $G = {}^2G_2(q)$ then $q = 3^{2a+1}$ which notation agrees with that in [22]. Throughout this section m_p denotes the multiplicative order of $q(\mathrm{mod}\ p)$, and e_p is the p-part of $\Phi_{m_p}(q)$. Observe that $\Phi_1(q) = q - 1$,

$\Phi_2(q) = q+1$, $\Phi_3(q) = q^2+q+1$, $\Phi_4(q) = q^2+1$, $\Phi_5(q) = q^4+q^3+q^2+q+1$, $\Phi_6(q) = q^2-q+1$, $\Phi_8(q) = q^4+1$, $\Phi_{10}(q) = q^4-q^3+q^2-q+1$, $\Phi_{12}(q) = q^4-q^2+1$.

Lemma 2.8. ([23, §4.10.2] and [2]) *With the above notation, let S be a Sylow p-subgroup p-subgroup of G. Then the following statements hold.*

(i) $|G|_{r'} = \prod_m \Phi_m^{l_m}(q)$;

(ii) *S is cyclic if and only if there is exactly one m such that p divides $\Phi_m(q)$ and $l_m = 1$ for this m.*

(iii) *For every factor $\Phi_m(x)$ of the above polynomial there is a torus T of G such that $|T| = \Phi_m^{l_m}(q)$. All tori of order $\Phi_m^{l_m}(q)$ are conjugate in G. In addition, T is a direct product of subtori of order $\Phi_m(q)$.*

(iv) *Let m_p be the multiplicative order of q (mod p) and let T be a torus in* (iii) *corresponding to $m = m_p$. Then $N_G(T)$ contains a conjugate of S. Furthermore, if $S \subset N_G(T)$ then the subgroup $A := T \cap S$ is homocyclic of rank l_{m_p} and of exponent e_p.* ☐

Lemma 2.9. *Let $\mathbf{G}$ be a simple simply connected algebraic group of rank $n > 0$, F a Frobenius endomorphism of $\mathbf{G}$, and $G := \mathbf{G}^{\mathsf{F}}$. Let A be as in Lemma* 2.8(iv)*.*

(i) [63, Proposition 4.8] *Let $p > 2$ be a prime dividing $|G|$ and $e_p =: |\Phi_{m_p}(q)|_p$, that is, e_p is the exponent of A. Then every p-element $g \in G$ of order at most e_p is conjugate to an element in A.*

(ii) *Let $\varepsilon \in \{\pm 1\}$ be such that $4|(q-\varepsilon)$, and let $q - \varepsilon = 2^e m$, where m is odd. Suppose that G has a maximal torus T of order $(q-\varepsilon)^n$. Then every 2-element of G of order at most 2^e is conjugate to an element of T.*

Proof. (ii) Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ such that $T = \mathbf{T}^{\mathsf{F}}$. Let $g \in G$ with $g^{2^e} = 1$. It is well-known that g is $\mathbf{G}$-conjugate to an element $g' \in \mathbf{T}$. Set $T_2 = \{t \in \mathbf{T} : t^{2^e} = 1\}$. Then $|T_2| = 2^{en}$. Therefore, T_2 coincides with the subgroup $\{x \in T : x^{2^e} = 1\}$, so $g' \in T$. As $\mathbf{G}$ is simply connected, $C_{\mathbf{G}}(g)$ is connected [53, Ch.II, 3.9]. By [53, Ch.I, 3.4], the elements $g, g' \in G$ are conjugate in G provided $C_{\mathbf{G}}(g)$ is connected. So the claim follows. ☐

3. Some observations on representations of groups of Lie type

Recall that $\mathrm{Irr}(G)$ partitions into (rational) Lusztig series denoted by $\mathcal{E}_s$, where s runs over the representatives of the conjugacy classes of semisimple elements of the dual group G^*. The characters in $\mathcal{E}_1$ are called unipotent.

Lemma 3.1. *Let T be a maximal torus of a finite reductive group $G = \mathbf{G}^{\mathsf{F}}$, and let $t_1, t_2 \in T$ be regular elements. If t_1, t_2 are conjugate in G then they are conjugate in $N_G(T)$.*

Proof. Let $\mathbf{T}$ be the maximal torus of $\mathbf{G}$ containing T. Then $\mathbf{T}$ is unique and t_1, t_2 are conjugate in $N_{\mathbf{G}}(\mathbf{T})$. Let $nt_1n^{-1} = t_2$ with $n \in N_{\mathbf{G}}(\mathbf{T})$. Then $\mathsf{F}(n)t_1\mathsf{F}(n^{-1}) = t_2$, whence $n^{-1}\mathsf{F}(n)t_1\mathsf{F}(n^{-1})n = t_1$, that is, $n^{-1}\mathsf{F}(n) \in C_{\mathbf{G}}(t_1) = \mathbf{T}$. By the Lang theorem, $n^{-1}\mathsf{F}(n) = t^{-1}\mathsf{F}(t)$ for some $t \in \mathbf{T}$. So $tn^{-1} = \mathsf{F}(t)\mathsf{F}(n^{-1}) = \mathsf{F}(tn^{-1})$, so $tn^{-1} \in G$ and $x := nt^{-1} \in G$. Clearly, $xt_1x^{-1} = t_2$ and $x \in N_{\mathbf{G}}(\mathbf{T}) \cap G = N_G(\mathbf{T})$. As $T = \mathbf{T} \cap G$, we have $xTx^{-1} = T$, as required. ☐

Lemma 3.2. *Let $\mathbf{G}$ be a simple algebraic group with a Frobenius endomorphism F, $G = \mathbf{G}^{\mathsf{F}}$, and let $S = \mathbf{S}^{\mathsf{F}}$ for an F-stable maximal torus of $\mathbf{G}$. Suppose that G is simple and every element $1 \neq t \in S$ is regular. Then, for any irreducible unipotent character χ of G, χ is constant on $S \smallsetminus \{1\}$ and $\chi(t) \in \{0, 1, -1\}$. Equivalently, $\chi|_S = \frac{\chi(1)-\eta}{|S|} \cdot \rho_S^{\mathrm{reg}} + \eta \cdot 1_S$, where $\eta \in \{0, 1, -1\}$.*

Proof. (i) Note that there is a simply connected algebraic group $\tilde{\mathbf{G}}$, a Frobenius endomorphism of $\tilde{\mathsf{F}}$ of $\tilde{\mathbf{G}}$ and a surjective homomorphism with finite kernel $h : \tilde{\mathbf{G}} \to \mathbf{G}$ such that $h(x^{\tilde{\mathsf{F}}}) = h(x)^{\mathsf{F}}$

for every $x \in \tilde{\mathbf{G}}$; moreover, $\mathbf{S} = h(\tilde{\mathbf{S}})$ for some $\tilde{\mathsf{F}}$-stable maximal torus of $\tilde{\mathbf{G}}$. Then $|\tilde{\mathbf{G}}^{\tilde{\mathsf{F}}}| = |\mathbf{G}^{\mathsf{F}}|$, $|\tilde{\mathbf{S}}^{\tilde{\mathsf{F}}}| = |\mathbf{S}^F|$, and the simple group G is a quotient of $\tilde{\mathbf{G}}^{\tilde{\mathsf{F}}}$. Hence $G \cong \tilde{\mathbf{G}}^{\tilde{\mathsf{F}}}$, and we may replace $\mathbf{G}$ by $\tilde{\mathbf{G}}$, $\mathbf{S}$ by $\tilde{\mathbf{S}}$, and therefore assume that $\mathbf{G}$ is simply connected. By assumption, every $1 \neq s \in S$ is regular, so $C_{\mathbf{G}}(s)$ is connected and a maximal torus, whence $C_{\mathbf{G}}(s) = \mathbf{S}$ and $C_G(s) = \mathbf{S}^{\mathsf{F}} = S$, (cf. [53, Ch. II, §3, Result 3.9]).

(ii) For a function f of G denote by $f^{\#}$ the restriction of f to the set of semisimple elements of G. By the Deligne-Lusztig theory, if $\chi \in \mathrm{Irr}(G)$ then $\chi^{\#}$ is a $\mathbb{Q}$-linear combination of $R^{\#}_{T_i,\theta_i}$, where R_{T_i,θ_i} are some Deligne-Lusztig characters, T_i is a maximal torus of G and θ_i is a linear character of T_i. Let a_i be the coefficient of $R^{\#}_{T_i,\theta_i}$ in the expression in question. The values $R_{T_i,\theta_i}(h)$ at the semisimple elements $h \in G$ are given by the formula

$$R_{T_i,\theta_i}(h) = \varepsilon(T_i)\varepsilon(G)\theta_i^G(h)/\mathsf{St}(h),$$

where St is the Steinberg character of G and $\varepsilon(T_i), \varepsilon(G) \in \{\pm 1\}$, see for instance [4, Prop. 7.5.4]. It is well known that a regular semisimple element of G lies in a unique maximal torus, so either T_i is conjugate to S or $R_{T_i,\theta_i}(t) = 0$ for every $t \in (S \setminus \{1\})$. Therefore, we conclude that either $\chi(t) = 0$ for all $t \in (S \setminus \{1\})$, or $\chi(t) = \sum a_i R_{S,\theta_i}(t)$, with some non-zero coefficient a_i. (Hence $\chi = \sum a_i R_{S,\theta_i} + f$, where f is a class function vanishing on $S \setminus \{1\}$.) Furthermore, $\mathsf{St}(h) = \varepsilon(T_i)$ whenever h is regular and $h \in T_i$. So $R_{S,\theta_i}(t) = \varepsilon(G)\theta_i^G(t)$, and hence $\chi(t) = \varepsilon(G) \cdot \sum a_i \theta_i^G(t)$. Furthermore, if $h, h' \in S$ are conjugate in G and regular, then h, h' are conjugate in $N = N_G(S)$ by Lemma 3.1. As $C_G(t) = S$, it follows that $\theta_i^G(t) = \theta_i^N(t)$.

If χ vanishes on $S \setminus \{1\}$ then $\chi|_S$ is a multiple of the regular character ρ_S^{reg}.

Suppose that χ is unipotent. Then $\theta_i = 1_S$ is the trivial character of S, so $\chi(t) = (\sum a_i)\varepsilon(G) \cdot 1_S^G(t) = a\varepsilon(G)|N/S|$, where $a = \sum a_i$. In particular, χ is constant on $S \setminus \{1\}$.

Let p be a prime dividing $|S|$. Then S contains a Sylow p-subgroup of G. As $C_G(t) = S$ for $1 \neq t \in S$, every p-singular element is conjugate to that in S. Therefore, χ is constant at the p-singular elements of G. Then, by [47, Theorem 1.3], χ belongs to the principal p-block of G and $\chi(t) = \eta \in \{\pm 1\}$. Therefore, $\chi(t) = \eta$ for every $1 \neq t \in S$. So we are done in this case. □

Remark 3.3. Suppose that χ is not unipotent. Then $\theta_i \neq 1_{T_i}$. Then $\theta_i^G(h)$ is the sum of $\theta_i(h')$, where h' runs over all elements of T_i that are conjugate to h. The number of them is $N_G(T_i)/T_i$ as T_i is a TI-set, and this does not depend on the choice of $1 \neq h \in T_i$. Then $\theta_i^G(h)$ is the sum of $|N_G(T_i)/T_i|$ non-trivial $|h|$-roots of unity.

An irreducible Brauer character ϕ for ℓ different from the defining characteristic of G is called unipotent if ϕ is a constituent of χ° for some unipotent ordinary character χ. Let G^* be the group dual to G. For a semisimple ℓ'-element $s \in G^*$, denote by $\mathcal{E}_{\ell,s}$ the union of the sets $\mathcal{E}_{ys}$, where $y \in G^*$, $ys = sy$ and $|y|$ is an ℓ-power. Then $\mathcal{E}_{\ell,s}$ is a union of ℓ-blocks ([8, Theorem 9.4.6]), so for every $\phi \in \mathrm{Irr}_\ell G$ there exists a semisimple ℓ'-element $s \in G^*$ such that ϕ is a constituent of χ° for some $\chi \in \mathcal{E}_{\ell,s}$. (Moreover, χ can be chosen in $\mathcal{E}_s$ [28, Theorem 3.1].) Therefore, it is meaningful to write $\phi \in \mathcal{E}_{\ell,s}$.

Lemma 3.4. *Let $\phi \in \mathcal{E}_{\ell,s}$ be a Brauer character. Then the restriction of ϕ to semisimple ℓ'-elements is a $\mathbb{Q}$-linear combination of the ordinary characters of $\mathcal{E}_s$ restricted to semisimple ℓ'-elements.*

Proof. Let $\chi \in \mathcal{E}_s$ be such that ϕ is a constituent of χ°. Every irreducible Brauer character of an ℓ-block is a $\mathbb{Z}$-linear combination of the ordinary characters of this block restricted to ℓ'-elements [45, Lemma 3.16], so ϕ is a $\mathbb{Z}$-linear combination of the ordinary characters of $\mathcal{E}_{\ell,s}$ restricted to ℓ'-elements. In turn, the ordinary characters of $\mathcal{E}_{\ell,s}$ restricted to the semisimple elements are $\mathbb{Q}$-linear combinations of the Deligne-Lusztig characters defining $\mathcal{E}_{\ell,s}$ restricted to semisimple elements (see [56, Lemma 4.1]). Therefore, ϕ is a $\mathbb{Q}$-linear combination of such Deligne-Lusztig characters restricted to semisimple ℓ'-elements. As every Deligne-Lusztig character from $\mathcal{E}_{ys}$ for $y \in C_G(s)$, $ys \neq 1$, restricted to semisimple ℓ'-elements coincides with some Deligne-Lusztig character from $\mathcal{E}_s$ restricted to semisimple ℓ'-elements [28, Prop 2.2], ϕ is a $\mathbb{Q}$-linear combination of the ordinary characters of $\mathcal{E}_s$ restricted to semisimple ℓ'-elements, the result follows. □

Remark 3.5. There is a conjecture that ϕ is a $\mathbb{Z}$-linear combination of the ordinary characters of $\mathcal{E}_s$ restricted to ℓ'-elements. This has been proven for many cases, see [21, Theorem 5.1] and [8, Section 9], in particular, this is true if $G = {}^2F_4(q)$ and ${}^3D_4(q)$, and if $G = \mathrm{SL}_3(q)$, $\mathrm{SU}_3(q)$ for $\ell \nmid |Z(G)|$.

Corollary 3.6. *Under the assumptions of Lemma* 3.2 *let ϕ be a unipotent ℓ-Brauer character of G. Then ϕ is constant on the set $S \setminus \{1\}$.*

Proof. This follows from Lemmas 3.2 and 3.4. □

The following lemma refines Theorem 4.2 in [56].

Lemma 3.7. *Under the assumptions of Lemma* 3.2 *let $\phi \in \mathrm{Irr}_\ell\, G$ and $\phi \in \mathcal{E}_{\ell,s}$. Then one of the following holds:*

(i) *$s = 1$, ϕ is unipotent and constant on the ℓ'-elements of $S \setminus \{1\}$;*
(ii) *$s \neq 1$, $|s|$ is coprime to $|S|$ and $\phi(t) = 0$ for all ℓ'-elements $t \in (S \setminus \{1\})$;*
(iii) *$s \neq 1$, $|s|$ divides $|S|$ and ϕ lifts to characteristic* 0*.*

Proof. If $s = 1$ then ϕ is unipotent, and we have (i) by Corollary 3.6. Let $s \neq 1$.

Suppose first that $|s|$ is coprime to $|S|$. Let $\chi \in \mathcal{E}_s$. Then, on restriction to semisimple elements, χ agrees with a $\mathbb{Q}$-linear combination of the Deligne-Lusztig characters R_{T_i,θ_i} defining $\mathcal{E}_s$ (see [56, Lemma 4.1]). Here, $|\theta_i| = |s|$ by [28, Lemma 2.1(a)], so $|s|$ divides $|T_i|$. Therefore, T_i is not conjugate to S, and hence $R_{T_i,\theta_i}(t) = 0$ for every $1 \neq t \in S$. So $\chi(s) = 0$ for every $\chi \in \mathcal{E}_s$. Now (ii) follows by Lemma 3.4.

Suppose that $|s| \neq 1$ divides $|S|$. As ys is a regular semisimple element for $y \in C_G(s)$, $ys \neq 1$, and hence $S = C_G(ys)$, the set $\mathcal{E}_{ys}$ consists of a single character of degree $d = |G|_{r'}/|S|$, where r is the defining characteristic of G. In addition, $\mathcal{E}_{\ell,s}$ is a union of ℓ-blocks, so, by [45, Lemma 3.16], every irreducible Brauer character ϕ in any of these blocks is a $\mathbb{Z}$-linear combination of ordinary characters of degree d, and ϕ itself is a constituent of η° for some irreducible character $\eta \in \mathcal{E}_{\ell,s}$, which is of degree d. Hence $\phi(1) = d$ and $\phi = \eta^\circ$. □

4. Unipotent elements in $\mathrm{GL}_n(F)$, char $F = 2$

Let $1 \neq g \in \mathrm{GL}_n(2) = \mathrm{GL}(V)$ be a 2-element and $z \in \langle g \rangle$ an involution. We set

$$j(g) = j(z) := \dim(\mathrm{Id} - z)V.$$

Note that $j(g)$ equals the number of blocks of size 2 in the Jordan canonical form of z.

Lemma 4.1. *Let $g \in \mathrm{GL}_n(2)$ be an element of order 2^{m+1}, $m > 0$, and $z := g^{2^m}$. Let $J(g) = (J_{n_1}, \ldots, J_{n_k})$ be the Jordan canonical form of g. Set $l_i = \max(0, n_i - 2^m)$ for $i = 1, \ldots, k$. Then $j(g) = \sum l_i$.*

Proof. It suffices to prove the statement in the case $k = 1$, where we have

$$j(g) = \dim(\mathrm{Id} - z)V = \dim(\mathrm{Id} - g)^{2^m}(V) = n - 2^m = l_1.$$

□

Lemma 4.2. *Let $0 < c < d < n$ be integers, and $n = kd + l$ for $0 \le l < d$. For a sequence $\lambda = (n_1 \ge \cdots \ge n_k \ge 0)$ set $\overline{\lambda} = (l_1, \dots, l_k)$ where $l_i = \max(0, n_i - c)$ for $i = 1, \dots, k$. Suppose that $\sum n_i = n$. Then*

$$\sum_i l_i \le k(d-c) + \max(0, l-c).$$

In addition, if $d < d' < n$ then

$$k(d-c) + \max(0, l-c) \le k'(d'-c) + \max(0, l'-c),$$

where $n = k'd' + l'$ with $0 \le l' < d'$.

Proof. This becomes clear if one views λ as a Young diagram of size n, and let $\mu = (d, \dots, d, l)$, where d is repeated k times. Then μ can be obtained from λ by moving down certain boxes of λ (note that $n_1 \ge d$). In addition, $\overline{\lambda}$ is a Young diagram of size $\sum_i l_i$, obtained from λ by deleting the first c columns. (Note that $\max(0, n_1 - c) \ge \cdots \ge \max(0, n_k - c) \ge 0$.) The first assertion in the lemma means that the size of $\overline{\lambda}$ is not greater than the size of $\overline{\mu} = (d-c, \dots, d-c, \max(0, l-c))$, which is again obtained from μ by removing the first c columns. It is clear that the number of boxes removed from μ to obtain $\bar{\mu}$ is at least the number of boxes removed from λ to obtain $\bar{\lambda}$, whence the assertion follows. Moreover, this number of removed boxes does not increase if one uses d' instead of d to form μ, whence the second assertion follows. □

Lemma 4.3. *Let $g \in \mathrm{GL}_n(2)$ be an element of order $|g| = 2^{m+1}$, $m > 0$. Let d be the minimum polynomial degree of g. Suppose that $d < |g|$. Then*

$$j(g) \le (n-l)(1 - \frac{|g|}{2d}) + \max(0, l - \frac{|g|}{2}),$$

where $n \equiv l \pmod{d}$ and $0 \le l < d$. If $d = |g| - 1$ then $1 - \frac{|g|}{2d} = \frac{|g|-2}{2(|g|-1)}$.

Proof. We first show that the bound is attained. Write $n = kd + l$ with $k \ge 0$. Let $x = \mathrm{diag}(J_d, \dots, J_d, J_l)$, where J_d occurs k times. By Lemma 4.1, $j(J_l) = \max(0, l - \frac{|g|}{2})$ and $j(J_d) = d - \frac{|g|}{2}$. So $j(g) = k(d - \frac{|g|}{2}) + \max(0, l - \frac{|g|}{2})$ and $k = \frac{n-l}{d}$. Then $j(g) = \frac{n-l}{d}(d - \frac{|g|}{2}) + \max(0, l - \frac{|g|}{2})$, as claimed.

The Jordan form of unipotent elements $g \in \mathrm{GL}_n(2)$ can be encoded by the Young diagrams, that is, $(n_1, \dots, n_k)$ corresponds to $\mathrm{diag}(J_{n_1}, \dots, J_{n_k})$, where we assume $n_1 \ge \cdots \ge n_k$, so n_1 is the minimum polynomial degree of g. Then the inequality in the lemma follows from Lemma 4.2 above. Indeed, if g is as in the statement then $n_1 = d > c =: \frac{|g|}{2}$. □

We specify the result of Lemma 4.3 as follows to make it more convenient for use in the next section.

Lemma 4.4. *Under the assumptions of Lemma 4.3, let q be an odd prime power such that $q+1$ is a multiple of $|g|$.*

(i) *Suppose that $|g| \le (q+1)/2$.*

$$j(g) \le \frac{n(q-3)}{2(q-1)}.$$

(ii) *Suppose that $|g| = q+1$. Then*

$$j(g) \le \begin{cases} \frac{n(q-1)}{2q}, & if\ q|n; \\ \frac{(n-1)(q-1)}{2q}, & if\ q|(n-1); \\ \frac{(n+1)(q-1)}{2q} - 1, & if\ q|(n+1). \end{cases}$$

Proof. Let l, d be as in Lemma 4.3. Observe first that $j(g) \leq n(1-\frac{|g|}{2d})$. Indeed, if $\max(0, l-\frac{|g|}{2})=0$ then this is true as $d > |g|/2$. Otherwise, $-l(1-\frac{|g|}{2d})+l-\frac{|g|}{2}=\frac{l|g|}{2d}-\frac{|g|}{2}<0$ as $l<d$. In addition, $j(g) \leq \frac{n(|g|-2)}{2(|g|-1)}$ as $d \leq |g|-1$.

(i) As $|g| \leq (q+1)/2$ we have $j(g) \leq n(1-\frac{|g|}{2d}) \leq n(1-\frac{|g|}{2(|g|-1)})=n(\frac{1}{2}-\frac{1}{q-1)})=\frac{n(q-3)}{2(q-1)}$.

(ii) Let $|g|=q+1=2^{m+1}$ and $d \leq |g|-1=q$.

(a) We have $j(g) \leq \frac{n(|g|-2)}{2(|g|-1)}=\frac{n(q-1)}{2q}$, in particular, this is true for $q|n$.

(b) Let $q|(n-1)$. Then $j(g) \leq \frac{n(q-1)}{2q}=\frac{(n-1)(q-1)}{2q}+\frac{1}{2q}$, whence the claim as $j(g) \in \mathbb{Z}$.

(c) Let $q|(n+1)$. Then $j(g) \leq \frac{(n(q-1)}{2q)}=\frac{(n+1)(q-1)}{2q}-\frac{1}{2q}$, whence the claim. ☐

5. The case of $SU_3(q)$

In this section we refine our results on minimal polynomials of elements of the group $SU_3(q)$ in its cross-characteristic irreducible representations. The main result is Proposition 5.2.

Lemma 5.1. *Let $S = SU_3(q) \leq G \leq H = GU_3(q)$, $2 \nmid q$, and let $g \in G$ be a non-central semisimple 2-element, and let $|g| = 2^\alpha$. Let $\phi \in IBr_\ell(G)$ with $(\ell, q) = 1$ and $\dim \phi > 1$. Suppose that $\deg \phi(g) < o(g)$. Then one of the following holds:*

(i) *$\ell = 2$, g is not a pseudoreflection and $|g|$ divides $q + 1$;*
(ii) *$q + 1 = 2^\alpha$, g is a pseudoreflection and $\deg \phi = o(g) - 1$,*
(iii) *$q + 1 = 2^{\alpha-1}$, g^2 is a pseudoreflection and $\deg \phi = o(g) - 2$.*
(iv) *$|g| = q + 1 = 4$, g is not a pseudoreflection and $\deg \phi = 3$;*

In addition, in cases (ii), (iii) *and* (iv), *ϕ is a Weil representation of G.*

Proof. Suppose first that g is contained in a parabolic subgroup of G. Then, applying the main result of [11, Theorem 13.2] (together with the corrigendum to [12]) and [24, Theorem 3.2] (together with the addendum), we conclude that (ii) or (iii) holds.

Suppose that g is not contained in any parabolic subgroup of G. Note that g is contained in a maximal torus T of H, and $|T| \in \{q^3+1, (q+1)(q^2-1), (q+1)^3\}$. The tori of order $(q+1)(q^2-1)$ lie in parabolic subgroups, and those of order q^3+1 contains no non-central 2-element. The torus of order $(q+1)^3$ is of exponent $q+1$, so $|g|$ divides $q+1$. If $\ell \neq 2$ or $|g| = q+1 = 4$ then the argument of the proof [56, Lemma 6.1] works (see page 653 there). So we have (i) and (iv). ☐

We improve the conclusion in (i) of Lemma 5.1 as follows:

Proposition 5.2. *Let $G = SU_3(q)$, q odd, and let $g \in G$ be a 2-element of order dividing $q + 1$. Let ϕ be a non-trivial irreducible 2-modular representation of G. Then the minimum polynomial of $\phi(g)$ is of degree $|g|$, unless possibly $\dim \phi = q(q-1)$, $|g| = q+1$ and $\deg \phi(g) = |g| - 1$. If $\deg \phi(g) = |g| - 1$ then the Jordan form of $\phi(g)$ consists of $q - 1$ blocks of size q.*

Before proving Proposition 5.2, we deduce a consequence of it:

Corollary 5.3. *Let $G = SU_3(q)$ with $q > 2$, and let $g \in G$ be a p-element with $p \nmid q$. Let $o(g)$ be the order of g modulo $Z(H)$ and let ϕ be a non-trivial irreducible ℓ-modular representation of G, $\ell \nmid q$, and $\dim \phi > 1$. Then $\deg \phi(g) = o(g)$, unless, possibly, ϕ is a Weil representation of G.*

Proof. Note that $o(g) = |g|$ if $(p, q + 1) \neq 3$, in particular, if $p \neq 3$. The result is contained in Proposition 5.2 if $p = 2$ (as $\dim \phi = q^2 - q$ implies ϕ to be Weil), and in [56, Lemma 6.1] for $p > 2$. ☐

The proof of Proposition 5.2 occupies the rest of this section. This is trivial if $|g| = 2$ so we assume $|g| \geq 4$. Then $4|(q+1)$. We start with some elementary observations.

Note that the involutions in $G = \mathrm{SU}_3(q)$ are conjugate. Denote by z an involution from a parabolic subgroup P of G. Let U be the unipotent radical of P. We can write

$$z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad U = \left\{ u = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right\}, \text{ and } zuz^{-1} = \begin{pmatrix} 1 & -a & c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

where $a \in \mathbb{F}_{q^2}$, $c \in \mathbb{F}_q$ and $b = a^q$. So $|U| = q^3$, $[z, Z(U)] = 1$. If ψ is an irreducible representation of $\langle z, U \rangle$ of dimension not 1 then $\dim \psi = q$, the eigenvalues of $\psi(z)$ are $1, -1$, and their multiplicities are $(q-1)/2$ and $(q+1)/2$, with the multiplicity of -1 being even. As $4|(q+1)$, we conclude that the multiplicity of 1 is $(q-1)/2$. In fact, if ψ is 2-modular then the Jordan form of $\psi(z)$ has $(q-1)/2$ blocks of size 2.

Lemma 5.4. *Let ϕ be an irreducible 2-modular Brauer character of $G = \mathrm{SU}_3(q)$, $4|(q+1)$. Let $z \in G$ be an involution. Denote by $j(\phi(z))$ the number of non-trivial Jordan blocks of $\phi(z)$. Then*

$$j(\phi(z)) \geq \frac{(q-1)\big(\phi(1) - (\phi|_{Z(U)}, 1_{Z(U)})\big)}{2q} + \frac{(\phi|_{Z(U)}, 1_{Z(U)}) - (\phi|_U, 1_U)}{2}.$$

Proof. Let V be the underlying module of ϕ. Then $V|_U = V^U \oplus V_1 \oplus V_2$, where $V_1 + V^U = V^{Z(U)}$ and $V_2 = [V, Z(U)]$. So $\dim V_2 = \dim V - \dim V^{Z(U)}$. Then V_2 is the direct sum of irreducible U-modules non-trivial on $Z(U)$, so V_2 is the sum of $\frac{\dim V_2}{q}$ irreducible U-modules of dimension q. In addition, V_1 is the sum of non-trivial one-dimensional U-modules. As $C_U(z) = Z(U)$, it follows that no non-trivial linear character of U is z-invariant; this implies the number of Jordan blocks of z on V_1 to be equal to $\dim V_1/2$. Therefore, the number in question equals

$$j(\phi(z)) = \frac{(q-1)\dim V_2}{2q} + \frac{\dim V_1}{2}.$$

As $\dim V^U = (\phi|_U, 1_U)$, $\dim V^{Z(U)} = (\phi|_{Z(U)}, 1_{Z(U)})$, and $\dim V_1 = \dim V^{Z(U)} - \dim V^U$, the statement follows. □

The terms of the formula in the lemma can be easily computed by using the character table of G and the decomposition matrix of G modulo 2. This is known to experts but the result is not explicitly written in literature. So we provide some detailed comments below.

There are three unipotent characters of G, of degree 1, q^3 and $q^2 - q$ [18]. The latter is irreducible modulo 2 [32, Proposition 9], and $\chi^\circ_{q^3}$ decomposes as $1_G + 2\chi^\circ_{q^2-q} + \phi_0$, where ϕ_0 is irreducible of degree $q^3 - 2q^2 + 2q - 1$ [31, Theorem 4.1]. This also shows that all these unipotent characters are in the principal block B_0. and they form a basic set for the union $\mathcal{E}_2(G, 1)$ of all $\mathcal{E}(G, s)$ with s a 2-element by [20, Theorem A]. It follows that this union is precisely B_0, and B_0 contains 3 irreducible Brauer characters.

Now we consider $\chi \in \mathcal{E}_2(G, s)$, where $s \in G^* = \mathrm{PGU}_3(q)$ is semisimple of odd order $|s| > 1$. Again by [20, Theorem A], the characters in $\mathcal{E}(G, s)$ form a basic set for $\mathcal{E}_2(G, s)$. The element s belongs to a maximal torus T^* of G^*, of order $(q+1)^2$, $q^2 - 1$, or $q^2 - q + 1$.

Suppose first that $|T^*| = q^2 - q + 1$. Then s is regular in G^*. If $|s| > 3$ then $\mathcal{E}(G, s)$ consists of a unique character of degree $(q+1)(q^2-1)$. If $|s| = 3$ then $3|(q+1)$ and $\mathcal{E}(G, s)$ consists of three characters of degree $(q+1)(q^2-1)/3$. These characters are all of 2-defect 0.

Suppose next that $|T^*| = q^2 - 1$ but s does not belong to any torus of order $(q+1)^2$. Then s is regular and $|C_{G^*}(s)| = q^2 - 1$. So $\mathcal{E}(G, s)$ consists of a unique character of degree $q^3 + 1$. As every

Brauer character is an integral linear combination of ordinary characters in its block, we conclude that its degree is q^3+1 too, and it is liftable.

Suppose now that $|T^*| = (q+1)^2$. Then $|s|$ divides $q+1$ and one of the following holds:

(i) s is regular. If $|s| > 3$ then $|C_{G^*}(s)| = (q+1)^2$ and $\mathcal{E}(G,s)$ consists of a unique character of degree $(q-1)(q^2-q+1)$, hence, as above, the unique Brauer character in the block has the same degree. If $|s| = 3$, then $|C_{G^*}(s)| = 3(q+1)^2$, and $\mathcal{E}(G,s)$ consists of 3 characters $\chi_{1,2,3}$ of the same degree $(q-1)(q^2-q+1)/3$. It follows that every Brauer character of $\mathcal{E}_2(G,s)$ is of degree divisible by $(q-1)(q^2-q+1)/3$. Hence, each χ_i° is irreducible, and, as χ_1, χ_2, χ_3 form a basic set, we again see that each irreducible Brauer character is liftable.

(ii) s is not regular. Then $C_{G^*}(s) \cong \mathrm{GU}_2(q)$, and $\mathcal{E}(G,s)$ consists of two characters, χ_1 (a Weil character) of degree q^2-q+1, and χ_2 of degree $q(q^2-q+1)$. It is well known that χ_1° is irreducible, see e.g. [32, Proposition 9]. We can represent s by a diagonal matrix $\mathrm{diag}(\alpha,\alpha,1)$ in $\mathrm{GU}_3(q)$ with $\alpha \neq 1$ of odd order dividing $q+1$. Then the element $t \in G^*$ represented by $\mathrm{diag}(\alpha,-\alpha,1)$ centralizes s and has s as its $2'$-part, and $\mathcal{E}(G,t)$ consists of a unique character ψ of degree $(q-1)(q^2-q+1)$. Note that $\psi(t) = 2q-1$, $\chi_1(t) = 1-q$, and $\chi_2(t) = q$ for a transvection $t \in G$, see [18, Table 3.1]. Since ψ° is a linear combination of χ_1° and χ_2°, it follows that $\chi_2^\circ = \psi^\circ + \chi_1^\circ$, and $\{\psi^\circ, \chi_1^\circ\}$ is a basic set for $\mathcal{E}_2(G,s)$. We claim that ψ° is irreducible. Indeed, $\psi^\circ = a\gamma + b\chi_1^\circ$ for some irreducible Brauer character γ and some integers $a, b \geq 0$. Inspecting the multiplicity of 1_U and of any nontrivial linear character ξ of U, using [18, Table 3.2], we obtain $(\psi|_U, 1_U) = 0 = a(\gamma|_U, 1_U) + b$ and $(\psi|_U, \xi) = 1 = a(\gamma|_U, \xi)$, whence $(a,b) = (1,0)$, i.e. $\psi^\circ = \gamma$, as claimed. Therefore, there are two irreducible Brauer characters in $\mathcal{E}_2(G,s)$, of degree q^2-q+1 and $(q-1)(q^2-q+1)$, and they both lift.

Thus, ϕ_0 is the only non-liftable 2-modular irreducible Brauer character, and it is of degree $q^3-1-2(q^2-q)$.

Let $1 \neq u \in Z(U)$ and $v \in (U \smallsetminus Z(U))$. Then $\phi(u), \phi(v)$ do not depend on the choice of u, v. Then

$$(\phi|_{Z(U)}, 1_{Z(U)}) = \frac{1}{q}(\phi(1) + (q-1)\phi(u)) \text{ and } (\phi|_U, 1_U) = \frac{1}{q^3}(\phi(1) + \phi(v)(q^3-q) + \phi(u)(q-1)).$$

For our purpose we could ignore irreducible representations of degree $(q+1)(q^2-1)$ as these are of 2-defect 0, so the restrictions of them to the Sylow 2-subgroup of G are the characters of projective modules.

Next, for every non-trivial irreducible 2-modular Brauer character ϕ of G of non-zero defect we compute the multiplicity of $1_{Z(U)}$ in $\phi|_{Z(U)}$ and the multiplicity of 1_U in $\phi|_U$. This can be easily done by using the character table of G. The results are summarized in Table 1.

Table 1

ϕ	$\phi(1)$	$\phi(u)$	$(\phi\|_{Z(U)}, 1_{Z(U)})$	$\phi(v)$	$(\phi\|_U, 1_U)$	$(\phi\|_{Z(U)}, 1_{Z(U)}) - (\phi\|_U, 1_U)$
ϕ_1	q^2-q	$-q$	0	0	0	0
ϕ_2	q^2-q+1	$-q+1$	1	1	1	0
ϕ_4	$(q-1)(q^2-q+1)$	$2q-1$	q^2-1	-1	0	q^2-1
ϕ_4^*	$(q-1)(q^2-q+1)/3$	$(2q-1)/3$	$(q^2-1)/3$	x, y, y	0	$(q^2-1)/3$
ϕ_5	q^3+1	1	q^2+1	1	2	q^2-1
ϕ_7	q^3	0	q^2	0	1	q^2-1
ϕ_0	q^3-2q^2+2q-1	$2q-1$	q^2-1	-1	0	q^2-1

In Table 1, $x = (2q-1)/3$, $y = -(q+1)/3$. Note that the character ϕ_4^* exists if and only if $3|(q+1)$; in this case the set $U \smallsetminus Z(U)$ is the union of three conjugacy classes, and $\phi_4^*(v) = y$ for v in two of them, and $\phi_4^*(v) = x$ when v lies in the remaining class. In fact, each irreducible representation of $U_3(q)$ of degree $(q-1)(q^2-q+1)$ restricts to G as the sum of three irreducible representations of degree $(q-1)(q^2-q+1)/3$. If $\phi_{4a}, \phi_{4b}, \phi_{4c}$ are the characters of these representations of G and v_1, v_2, v_3 are representatives of the three conjugacy classes in question then the corresponding fragment of the Brauer character table is

	v_1	v_2	v_3
ϕ_{4a}	x	y	y
ϕ_{4b}	y	x	y
ϕ_{4c}	y	y	x

This is irrelevant for computation of $(\phi_4^*|_U, 1_U)$. On the other hand, if 1_U does not occur as a constituent of $\phi_4|_U$ then 1_U does not occur as a constituent of $\phi_4^*|_U$.

Let

$$f_1(n) = \frac{n(q-3)}{2(q-1)} \quad \text{and} \quad f_2(n) = \begin{cases} n(q-1)/2q & if\ n \equiv 0 \pmod q \\ (n-1)(q-1)/2q & if\ n \equiv 1 \pmod q \\ \frac{(n+1)(q-1)}{2q} - 1 & if\ n \equiv -1 \pmod q. \end{cases}$$

be the functions defined in Lemma 4.4(i), (ii), respectively. Then we have (where $n = \phi(1)$):

Table 2

ϕ	$n = \phi(1)$	$j(\phi(z)) \geq$	$f_1(n)$	$f_2(n)$
ϕ_1	$q^2 - q$	$(q-1)^2/2$	$q(q-3)/2$	$(q-1)^2/2$
ϕ_2	$q^2 - q + 1$	$(q-1)^2/2$	$(q^2-q+1)(q-3)/2(q-1)$	
ϕ_4	$(q-1)(q^2-q+1)$	$(q-1)(q^2-2q+3)/2$	$(q^2-q+1)(q-3)/2$	$-1 + \frac{(q-1)(q^2-2q+2)}{2}$
ϕ_4^*	$(q-1)(q^2-q+1)/3$	$(q-1)(q^2-2q+3)/6$	$q^2-q+1)(q-3)/6$	
ϕ_5	q^3+1	$(q-1)(q^2+1)/2$	$\frac{q^3+1)(q-3)}{2(q-1)}$	$q^2(q-1)/2$
ϕ_0	$(q-1)(q^2-q+1)$	$(q-1)(q^2-2q+3)/2$	$(q^2-q+1)(q-3)/2$	$-1 + \frac{(q-1)(q^2-2q+2)}{2}$

In Table 2 we have left blank certain positions in the fifth column as this column is created under assumption that $|g| = q+1$. Recall that if $q+1$ is a 2-power then there are no irreducible 2-modular representations of degree q^2-q+1 and $(q-1)(q^2+q+1)/3$. Note that the third column gives the lower bound for $j(\phi(z))$ from Lemma 5.4 obtained using Table 1.

Proof of Proposition 5.2. Let $d = \deg \phi(g)$. Suppose the contrary, that $d < |g|$. Let $n = \dim \phi$. Suppose first that $|g| \neq q+1$. Then $|g| \leq (q+1)/2$. Let z be the involution in $\langle g \rangle$, so $j(\phi(g))$ is the number of non-trivial blocks in the Jordan form of $\phi(z)$. By Lemma 5.4, $j(\phi(g)) \geq t(n)$, where $t(n)$ is given in the 3-rd column of Table 2.

By Lemma 4.4, $d < |g|$ implies $j(\phi(g)) \leq f_1(n)$, so $t(n) \leq f_1(g)$. One easily observes that this is false, whence a contradiction.

Let $|g| = q+1$. Inspecting the second column of Table 2, one observes that $\phi_i(1) \pmod q \in \{1, 0, -1\}$ for $i = 1, 4, 5, 6$. So we can use Lemma 4.4(ii) to build the upper bound for $j(\phi(g))$, which is written in the 5-th column there. As above, we compare the entries of the 3rd and 5th columns of Table 2, to observe that these are not compatible for $i = 4, 5, 0$. (There is no contradiction for $i = 1$.)

Let $n = \phi(1) = q^2 - q$. Then $\mathrm{GL}_n(2)$ contains the matrix $J := (J_q, \dots, J_q)$, where J_q is repeated $q-1$ times. Then $j(J) = (q-1)^2/2$ by Lemma 4.1. Let $(J_{n_1}, \dots, J_{n_k})$ be the Jordan form of

$\phi(g)$, where $n_1 \geq \cdots \geq n_k$. If this is not J then, by Lemma 4.2, $j(\phi(g)) < (q-1)^2/2$, which is a contradiction (by Table 2). □

6. Groups of BN-pair rank 1

6.1. Groups ${}^2B_2(2^{2m+1})$ and ${}^2G_2(3^{2m+1})$.

Lemma 6.1. *Let $G = {}^2B_2(2^{2m+1})$, $m > 0$, be a Suzuki group. For $p > 2$ let $g \in G$ be a p-element. Let $1_G \neq \phi \in \mathrm{Irr}_\ell\, G$ and $\ell \neq 2$. Then $\deg \phi(g) = |g|$.*

Proof. Note that Sylow p-subgroups of G are cyclic. If $\ell = 0$ or $\ell = p$, the result follows from [61], see also [60, 2.8] for $\ell = 0$. Suppose $\ell \neq 0, p$. Let β be the Brauer character of ϕ. If β is liftable, the result follows from [61]. The decomposition numbers of G have been determined by Burkhardt [3]. Inspection of them in [3] shows that ϕ either liftable or $\mathsf{St}(x) - 1 = \phi(x)$ for every l'-element $x \in G$. As $\mathsf{St}(x) \in \{\pm 1\}$ and $\mathsf{St}(1) = 2^{2(2m+1)}$, the result easily follows from Lemma 2.1. □

Lemma 6.2. *Let $G = {}^2G_2(q)$, $q = 3^{2m+1}$, $m > 0$. Let $g \in G$ be a p-element for some prime $p \neq 3$ dividing $|G|$. Let $\ell \neq 3$, $\phi \in \mathrm{Irr}_\ell\, G$, $\dim \phi > 1$. Then $\deg \phi(g) = |g|$.*

Proof. The lemma is trivial for $p = 2$, as every 2-element of G is an involution.

Let $p > 2$. Then Sylow p-subgroups of G are cyclic. Let β be the Brauer character of ϕ. If $\ell \in \{0, p\}$ or β is liftable, the result follows from [61]. Suppose otherwise.

Let $\ell = 2$. By [39, p.104], β lies in the principal block and $\beta \in \{\beta_2, \beta_3\}$, where $\beta_2 = \xi_2 - 1_G$ and $\beta_3 = \mathsf{St}^\circ + 1_G - 2\xi_2^\circ - \xi_6^\circ - \xi_8^\circ$ in notation of [39].

Let $\ell > 2$. Then the Sylow ℓ-subgroups of G are cyclic. The Brauer tree for G for every $\ell > 2$ is determined by Hiss [30]. Inspection in [30, Section D.2] shows that β is either liftable or $\ell | (3^{2m+1}+1)$ and $\beta_1 = \mathsf{St}^\circ - 1_G - \xi_7^\circ - \xi_8^\circ$, or $\ell | (3^{2m+1} - 3^{m+1} + 1)$ and $\beta_4 = \mathsf{St}^\circ - 1_G$, in notation of [30, 59].

It suffices to show that the restriction $\beta_i|_C - \rho_C^{\mathrm{reg}}$ is either 0 or a proper character of C for every maximal cyclic p-subgroup C of G. There are 4 maximal tori of G; these are cyclic groups of order $|T_1| = q-1$, $|T_2| = q+1$, $|T_3| = q + \sqrt{3q} + 1$ and $|T_4| = q - \sqrt{3q} + 1$. We write $C = C_i$ if $|C|$ divides $|T_i|$.

Inspection of the character table of G in [59] shows that every character ξ_j ($j = 2, 5, 6, 7, 8$) as well as St is constant at $C \setminus \{1\}$. Therefore, the result follows by applying Lemma 2.1 to the values of these characters (given in [59]).

For reader's convenience in the following table we give (for $l > 2$) the values of the characters involved at $1 \neq t \in C_i$ for $i = 1, 2, 3, 4$.

	St	ξ_2	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	β_1	β_2	β_3	β_4
C_1	1	1	1	0	0	0	0	0	0	0	0
C_2	-1	3	-3	1	-1	1	-1	-4	2	-4	-2
C_3	-1	0	0	-1	0	-1	0	0	-1	0	-2
C_4	-1	0	0	0	1	0	1	-2	-1	-2	-2

Note that $\xi_4(1) = q(q^2-q+1)$, $\xi_2(1) = q^2-q+1$ and $\xi_7(1) = \xi_8(1) = (q-1)q(q+1-\sqrt{3q})/2\sqrt{3}$. So $\beta_1(1) = St(1) - 1 - \xi_7(1) - \xi_8(1) = (q-1)(q^2+1-(q+1)\sqrt{q/3})$.

7. The case of $G_2(q)$

In this section we prove Theorem 1.1 for $G = G_2(q)$, $q = r^a > 2$. The case of $G = G_2(2)$ is considered in Lemma 7.13 at the end of this section. (Also see Lemma 7.14 for group $2 \cdot G_2(4)$.)

Theorem 7.1. *Theorem 1.1 is true for $G = G_2(q)$, $q = r^a > 2$.*

Lemma 7.2. *Let $G = G_2(q)$ and $q \geq 3$. Then every semisimple element of G is contained in a subgroup isomorphic to* $\mathrm{SL}_3(q)$ *or* $\mathrm{SU}_3(q)$.

Proof. Every semisimple element is contained in a maximal torus of G. Also, G has 6 conjugacy classes of maximal tori, whose orders are q^2-q+1, q^2+q+1, $(q-1)^2$, $(q+1)^2$ and two classes of order q^2-1. In addition, G contains subgroups $L^+ \cong \mathrm{SL}_3(q)$ and $L^- \cong \mathrm{SU}_3(q)$, whose maximal tori are maximal in G. If $3|q$, then the statement follows from fusion of conjugacy classes of certain subgroups $L^+ \cong \mathrm{SL}_3(q)$ and $L^- \cong \mathrm{SU}_3(q)$ in G as given in [15]: every semisimple class of G intersects L^+ or L^-.

Assume now that $3 \nmid q$ and choose $\varepsilon \in \{+,-\}$ such that $3|(q-\varepsilon)$. It suffices to show that maximal tori $T^+ < L^+$ and $T^- < L^-$ (both cyclic groups of order q^2-1) are not conjugate in G. Assume the contrary: T^+ and T^- are conjugate in G. It is shown in [5] and [16] that G has two conjugacy classes of elements of order 3, with representatives u, v, where $C_G(u) \cong \mathrm{SL}_3^{\varepsilon}(q) = L^{\varepsilon}$, and $C_G(v) \cong \mathrm{GL}_2^{\varepsilon}(q)$. Then T^{ε} contains $Z(L^{\varepsilon})$, and so $u \in T^{\varepsilon}$. As $T^{-\varepsilon}$ is conjugate to T^{ε}, we may assume that $u \in T^{-\varepsilon} < L^{-\varepsilon}$.

Consider the case with $\varepsilon = -$ and $\chi \in \mathrm{Irr}(G)$ of degree q^3-1. Then $\chi(u) = -q(q-1)$, see [5] and [16]. On the other hand, if $\theta \in \mathrm{Irr}\,\mathrm{SL}_3(q)$ and $\theta(1) \leq q^3-1$, then $\theta(g) \geq 0$, or $\theta(1) = q^3-1$ and $\theta(g) = -2$ for any element g of order 3. Restricting χ to L^+, we arrive at a contradiction.

Now assume that $\varepsilon = +$, and take $\chi \in \mathrm{Irr}(G)$ of degree q^3+1. Then $\chi(u) = q(q+1)$, see [5] and [16]. We use the notation of [18], and decompose

$$\chi|_{L^-} = a\chi_1 + b\chi_{q^2-q} + c\chi_{q^3} + \sum_{i=1}^{q}(d_i\chi^{(i)}_{q^2-q+1} + e_i\chi^{(i)}_{q(q^2-q+1)}) + \sum_{i,j} f_{ij}\chi^{(i,j)}_{(q-1)(q^2-q+1)} + \sum_j g_j\chi^{(j)}_{q^3+1},$$

where $a, b, c, \ldots$ are non-negative integers. First, if some $g_j \geq 1$, then $\chi|_{L^-} = \chi^{(j)}_{q^3+1}$, and hence $|\chi(u)| \leq 2$, a contradiction. So $g_j = 0$ for all j. Likewise, if $c \geq 1$, then $c = 1$, $\chi|_{L^-} = \chi_1 + \chi_{q^3}$, yielding $\chi(u) = 2$, again a contradiction. Now, evaluating at an element of order q^2-q+1, we get $0 = a - b$, i.e. $b = a$. Comparing the degrees, we obtain

$$q^3+1 = (q^2-q+1)(a + \sum_i d_i + \sum_i qe_i + (q-1)\sum_{i,j} f_{ij}),$$

whence $a + \sum_i d_i + \sum_i qe_i \leq q+1$. Now, evaluating at u, we get

$$q(q+1) = \chi(u) = a + \sum_i d_i + \sum_i e_i \leq q+1,$$

again a contradiction. □

If $r \neq 3$ then all the subgroups of G isomorphic to $L^+ \cong \mathrm{SL}_3(q)$, respectively, to $L^- \cong \mathrm{SU}_3(q)$ are conjugate, as follows from the classification of maximal subgroups of G obtained in [7] and [36]. The case with $r = 3$ has features which force us to consider this separately (Lemma 7.10).

Lemma 7.3. [63, Lemma 4.10] *Let $p, q > 2$, $p \nmid q$ and let $g \in G \cong G_2(q)$ be a p-element contained in a proper parabolic subgroup of G. Let $1_G \neq \theta \in \mathrm{Irr}_\ell G$. Then $\deg\theta(g) = |g|$.*

Remark 7.4. In [63, Lemma 4.10] it is assumed that $\ell \neq p$, however, this is nowhere used in the prove of [63, Lemma 4.10]; so the claim is true for $\ell = p$ as well.

Lemma 7.5. *Let $G = G_2(q)$, $q > 2$, and let $\theta \in \mathrm{Irr}_F G$ with $\dim\theta > 1$. Let $g \in G$ be a semisimple p-element. Suppose that $g \in H = \mathrm{SL}_3(q)$. Then $\deg\theta(g) = |g|$.*

Proof. Suppose $\langle g\rangle \cap Z(H) \neq 1$. Then $p = 3|(q-1)$, and g is reducible in H by [62, Lemma 3.2]). Hence g normalizes a nontrivial unipotent subgroup of H and of G, and so it lies in a parabolic subgroup of G by Lemma 2.7. Then the result follows by Lemma 7.3.

Now we may assume that $\langle g\rangle \cap Z(H) = 1$, and hence $|g|$ is the same as its central order $o(g)$ computed in H. Assume the contrary that $\deg\theta(g) < |g|$. Then $|g| = q^2+q+1$ by Lemma 2.6, and either $p = \ell$, or $p \neq \ell$ and the irreducible constituents of $\theta|_H$ are of dimension 1 or q^2+q. In the former case the result follows from [61], which also contains the result for $\ell = 0$. In the latter case Lemma 2.6 additionally tells us that $\deg\phi(g) = |g|-1$ and 1 is not an eigenvalue of $\phi(g)$ whenever ϕ is an irreducible constituent of degree q^2+q of $\tau|_H$. If 1_H is a constituent of $\theta|_H$ then the result follows, otherwise $\dim\theta$ is a multiple of q^2+q. Then θ is liftable (this follows by inspection of Brauer character degrees of G and the decomposition numbers modulo ℓ available in [50, 51] for $\ell|(q^2+q+1)$ and in [30] for $\ell|(q^2-1)$), and we are back to the complex case. □

In view of Lemma 7.5 to prove Theorem 7.1 we can assume that $g \in H \cong \mathrm{SU}_3(q)$. Moreover, by [56, Lemma 6.1], if $p > 2$ then either $|g| = q+1$ or $|g| = q^2-q+1$. In the former case q is even. Let $p = 2$. Then, by Lemma 5.1 and Proposition 5.2, either $|g| = q+1$ or $2(q+1)$.

For $H = \mathrm{SU}_3(q)$ the arguments in the proof of Lemma 7.5 do not work. Indeed, if $p|(q+1)$ then we cannot use [61] for $p = \ell$ as Sylow p-subgroups of G are not cyclic. So we turn to another method. We show that the restriction to H of every non-trivial irreducible representation of G contains a non-trivial irreducible constituent which is not Weil. This will imply Theorem 7.1 in view of Proposition 5.2 for $p = 2$ and [56, Lemma 6.1]. In addition, to handle the case of ${}^3D_4(q)$ we need a similar result for $H = \mathrm{SL}_3(q)$ and $\ell = q^2+q+1$. In fact, for our use it suffices to consider the case where $Z(H) = 1$. To deal with $\ell|(q^4+q^2+1)$ we first prove the result for $\ell = 0$ and next use the decomposition numbers to deal with ℓ-Brauer characters. In our reasoning below a certain role is played by the Gelfand-Graev representation of H. This is the induced representation λ^G, where λ is a so-called non-degenerate linear character of the maximal unipotent subgroup of G.

Lemma 7.6. *Let $G = G_2(q)$, $q > 2$, and $H \cong \mathrm{SL}_3^\varepsilon(q) < G$, $(3, q-\varepsilon) = 1$. Let $1_G \neq \tau \in \mathrm{Irr}(G)$ (so $\ell = 0$ here). Then $\tau|_H$ contains a non-trivial irreducible constituent which is not Weil, except for the case with $q = 3$, $\varepsilon = 1$.*

Proof. Let U be a maximal unipotent subgroup of H and let $\Gamma = \Gamma^\varepsilon$ be the Gelfand-Graev character of H. Recall that $Z(H) = 1$ implies that the Gelfand-Graev representation of H is unique (this follows from [10, 14.28 and 14.29]). It is well known that neither 1_H nor any Weil character are constituents of Γ. (Indeed, if $r|q$ is a prime then the degrees of any irreducible constituent of Γ is $|C_H(s)|_r \cdot [H : C_H(s)]_{r'}$ for some semisimple element $s \in H$, see [4, Theorem 8.4.9]. One observes that 1, $q^2-\varepsilon q$, $q^2-\varepsilon q+1$ are not of this form.) Therefore, it suffices to show that

$$(\tau|_H, \Gamma) > 0 \tag{7.1}$$

for every non-trivial irreducible character τ of G. As the character table of G is known, this can be easily checked. Indeed, observe that $\Gamma(x) = 0$ if $x \in H$ is not unipotent. Let μ be the Weil character of H of degree $q^2+\varepsilon q$. Then $\mu(u) = 0$ and $\mu(t) = \varepsilon q$. Next,

$$(\Gamma, \mu) = 0 = \frac{1}{|H|}(\Gamma(1)\mu(1) + \frac{|H|}{|C_H(t)|}\cdot\Gamma(t)\mu(t)),$$

yielding $\Gamma(t) = -\varepsilon(q^2-1)$. Similarly,

$$(\Gamma, 1_H) = 0 = \frac{1}{|H|}(\Gamma(1) + \frac{|H|}{|C_H(u)|}\Gamma(u) + \frac{|H|}{|C_H(t)|}\Gamma(t)),$$

and so $\Gamma(u) = \varepsilon$. Therefore,

$$|H|(\tau|_H, \Gamma) = \tau(1)\frac{|H|}{|U|} + \tau(t)\Gamma(t)\frac{|H|}{|C_H(t)|} + \tau(u)\Gamma(u)\frac{|H|}{|C_H(u)|} = \frac{|H|}{|U|}\big(\tau(1) - (\varepsilon q + 1)\tau(t) + \varepsilon q\tau(u)\big).$$

So $(\tau|_H, \Gamma) > 0$ if and only if $\tau(1) > (\varepsilon q + 1)\tau(t) - \varepsilon q\tau(u)$. Inspection of the character table of G in [5] (for $(q, 6) = 1$), [15] (for $3|q$), and [16] (for $2|q$), yields the result, except for the case where $q = 3$, $\varepsilon = 1$.

In the latter case $H = \mathrm{SL}_3(3)$ and if $\lambda_d \in \mathrm{Irr}(H)$ with $d = \lambda_d(1) \leq 14$ then $d \in \{1, 12, 13\}$. Therefore, λ_d with $d \in \{12, 13\}$ is a constituent of $\tau|_H$, and this is a Weil representation of H. □

Lemma 7.7. *Let $H = \mathrm{SL}^\varepsilon_3(q)$ and let $\tau \in \mathrm{Irr}\, H$ with $\tau(1) > q^2 + \varepsilon q + 1$. Then τ° has an irreducible constituent of degree greater than $q^2 + \varepsilon q + 1$.*

Proof. The result can be easily deduced from the comments in [56, Examples 3.1 and 3.2]. □

Lemma 7.8. *Let G, H be as in Lemma 7.6. Let $1_G \neq \phi \in \mathrm{Irr}_\ell\, G$. Suppose that $(q, \varepsilon) \neq (3, 1)$ and $\phi + m \cdot 1_G$ is liftable for some integer $m \geq 0$. Then $\phi|_H$ contains a non-trivial irreducible constituent which is not Weil.*

Proof. Let $\tau \in \mathrm{Irr}(G)$ be a character of G such that $\tau^\circ = \phi + m \cdot 1_G$. Then the non-trivial irreducible Brauer characters of H occurring in $\phi|_H$ are the same as those in $(\tau|_H)^\circ$. By Lemma 7.6, there is an irreducible constituent ν in $\tau|_H$ such that $\nu(1) > q^2 + \varepsilon q + 1$. By Lemma 7.7, ν° has an irreducible constituent of degree greater than $q^2 + \varepsilon q + 1$, whence the lemma follows. □

Lemma 7.9. *Let G, H be as in Lemma 7.6, and let $1_G \neq \phi \in \mathrm{Irr}_\ell\, G$, where $(q, \varepsilon) \neq (3, 1)$ and $\ell|(q^4 + q^2 + 1)$. Then $\phi|_H$ contains a non-trivial irreducible constituent which is not a Weil character, unless $q = 3$ and $\varepsilon = 1$.*

Proof. (i) Recall that $\Gamma = \lambda^H$ for the Gelfand-Graev character Γ of H, where λ is a linear character of U. Therefore, $\Gamma^\circ = (\lambda^\circ)^H$, so Γ° is the Brauer character of a projective FH-module. In particular, if α is any class function on H, then the inner product $(\alpha^\circ, \Gamma^\circ)'$ over ℓ'-elements of H is equal to the usual inner product (α, Γ) over all elements in H. Also, as Γ is multiplicity free, it follows that Γ° is a sum of Brauer characters of indecomposable projective modules P_{ψ_i} of H, each of them occurs with multiplicity 1. Therefore, $(\phi|_H, \Gamma^\circ)'$ equals the sum of multiplicities of the irreducible constituents ν of $\phi|_H$ such that $(\nu, \Gamma^\circ)' > 0$.

Note that $(1_H, \Gamma^\circ)' = (1_H, \Gamma) = 0$ and hence $\nu \neq 1^\circ_H$. Let μ be an irreducible Weil character of H; in particular, $(\mu, \Gamma^\circ)' = (\mu, \Gamma) = 0$ as mentioned above. It is well known [32, §6] that $\mu^\circ = \mu' + a \cdot 1_H$, for some $\mu' \in \mathrm{Irr}_\ell\, H$ and $a \geq 0$. Therefore, $(\mu', \Gamma^\circ)' = (\mu, \Gamma^\circ)' = 0$, so $\mu' \neq \nu$. It follows that $\phi|_H$ has an irreducible constituent that is neither trivial nor Weil if and only if $(\phi|_H, \Gamma^\circ)' > 0$.

(ii) We will now prove that $(\phi|_H, \Gamma^\circ)' > 0$. Denote by $\mathrm{Irr}^*_\ell\, G$ the set of characters $\phi \in \mathrm{Irr}_\ell\, G$ that are not of the form $\tau^\circ - a \cdot 1_G$ for some $\tau \in \mathrm{Irr}(G)$ and $a \geq 0$. Suppose first that $\phi \notin \mathrm{Irr}^*_\ell\, G$, so that $\phi = \tau^\circ - a \cdot 1_G$ for some $\tau \in \mathrm{Irr}(G)$ and $a \geq 0$. Then $(\phi, \Gamma^\circ)' = ((\tau|_H)^\circ, \Gamma^\circ)' = (\tau|_H, \Gamma)$, and (7.1) implies that $(\tau|_H, \Gamma) > 0$.

From now on, assume that $\phi \in \mathrm{Irr}^*_\ell\, G$. As in Lemma 7.6, observe that $(\phi, \Gamma^\circ)' > 0$ if and only if

$$\phi(1) - (\varepsilon q + 1)\phi(t) + \varepsilon q\phi(u) > 0. \tag{7.2}$$

To verify the inequality (7.2) we need the character values of ϕ at u, t, where $t \in H$ is a transvection and u is a regular unipotent element of H. These can be computed from the decomposition numbers of G modulo ℓ, which are available in Shamash [50, 51] for $3 \neq \ell|(q^2 + \varepsilon q + 1)$ and [30] for $\ell|(q^2 - 1)$.

Suppose first that $3 \neq \ell|(q^2 + \varepsilon q + 1)$.

In notation of [30] the characters in $\mathrm{Irr}_\ell^* G$ that are in the principal block are $(X_3 - X_{18})^\circ$ if $\varepsilon = 1$, and $(X_{12} - X_{16} + 1_G)^\circ$ if $\varepsilon = -1$. From this one easily checks (7.2).

Suppose that ϕ is not in the principal block. Then either ϕ is liftable, or $\ell|(q^2 - \varepsilon q + 1)$ and $3|(q - \varepsilon)$. The latter case does not hold by assumption.

For $(q, 6) \neq 1$ and $\ell|(q^2 + \varepsilon q + 1)$ the decomposition numbers are determined by Shamash [51]. The decomposition numbers for the characters in the principal block and, for $2|q$, in the other blocks containing non-liftable characters are the same as for q with $(q, 6) = 1$. If $3|q$ then all non-liftable characters are in the principal block. The character tables of $G_2(q)$ with $(q, 6) > 0$ can be found in [15] for $3|q$ and [16] for $2|q$. Inspection of [16] shows that $X_i(1)$ and $X_i(t)$ for non-liftable characters X_i are the same polynomials in q as for q with $(q, 6) = 1$. In addition, the absolute value of $X_i(u)$ is small enough to satisfy the inequality (7.2). □

We are left with the cases where $\ell|(q^2 - 1)$. As the character table of $G_2(q)$ with $3|q$ differs from that for q with $(q, 3) = 1$, we consider this case separately.

Lemma 7.10. *Theorem* 1.1 *is true for* $q = 3^m$.

Proof. By Lemma 7.5 and the comments following it, we may assume that $p = 2$ and $q + 1$ is a 2-power. We assume $|g| > 2$ as the case $|g| = 2$ is trivial. Observe that $|q + 1|_2 = 4$ if m is odd and $|q + 1|_2 = 2$ otherwise. Therefore $q = 3$. If $\ell \neq 2$ then the result follows by the Brauer character table of $G_2(3)$ [34]. Let $\ell = 2$. Let $H \cong \mathrm{SU}_3(3) < G$. We show that $\phi|_H$ has a non-trivial irreducible constituent which is not Weil. As above, it suffices to check that $(\phi, \Gamma) > 0$, where Γ is the Gelfand-Graev character of H. As above, this holds as $14 = \phi(1) > (\varepsilon q + 1)\phi(t) - \varepsilon q \phi(u) = -2\phi(t) + 3\phi(u) = -10 + 6 = -4$. □

From now on, until the end of this subsection we assume $r \neq 3$. To complete the proof of Theorem 7.1 it suffices to show that $\phi|_H$ has a non-trivial constituent which is not Weil (here $\phi \in \mathrm{Irr}_\ell G$, $3 \neq \ell|(q^2 - 1)$). We can do this by the method used in the proof of Lemma 7.9, however, there is a more conceptual approach to do this.

Lemma 7.11. *Let* $G = G_2(q)$, $q > 2$, $3 \nmid q$, *and let* P *be a long-root parabolic subgroup of* G. *Let* U *be the unipotent radical of* P, V *be a non-trivial irreducible* FG*-module and* $V_0 = C_V(Z(U))$. *Then* $U/Z(U)$ *acts on* V_0 *faithfully.*

Proof. Assume the contrary: $U/Z(U)$ acts non-faithfully on V_0. It is well known that for $q > 2$, $(q, 3) = 1$ the group $U/Z(U)$ has no non-trivial proper P-invariant subgroup. (See [1] or [43, Theorem 17.6] for $(q, 6) = 1$ and [38] for q even; note that this fails for the excluded case $q = 2$). As V_0 is a FP-module, it follows that U acts trivially on V_0, i.e. $V_0 = C_V(U)$ and thus $V = [V, Z(U)] \oplus V_0$.

It is also well known that U is generated by the root subgroups U_β, $U_{\alpha+\beta}$, $U_{2\alpha+\beta}$, $U_{3\alpha+\beta}$, $U_{3\alpha+2\beta}$ and $[U, U] = U_{3\alpha+2\beta} = [U_\beta, U_{3\alpha+\beta}] = Z(U)$. Moreover, for every $1 \neq y \in (U \setminus Z(U))$ we have $[U, y] = Z(U)$. It follows that the character of every irreducible representation of U non-trivial on $Z(U)$ vanishes on y.

Note that there exists some $y \in (U \setminus Z(U))$ so that the G-conjugacy class of y meets $Z(U)$, say at a (long-root) element x; indeed, both $U_{3\alpha+\beta}$, $U_{3\alpha+2\beta}$ are long root subgroups of G, and hence G-conjugate. Let φ_0 and φ_1 denote the Brauer characters of V_0 and $[V, Z(U)]$. Then $\varphi_0(x) = \varphi_0(y) = \dim V_0$, and $\varphi_1(y) = 0$. On the other hand, all elements of $Z(U)$ are conjugate in G, so all non-trivial irreducible representations of $Z(U)$ occur in $[V, Z(U)]|_{Z(U)}$ with the same multiplicity, whence $\varphi_1(x) = -\varphi_1(1)/(q - 1) < 0$. Thus $(\varphi_0 + \varphi_1)(x) \neq (\varphi_0 + \varphi_1)(y)$, a contradiction. □

Lemma 7.12. *Let $G = G_2(q)$, $q > 2$, $3 \nmid q$, and let V be a non-trivial irreducible FG-module. Let $H \cong \mathrm{SU}_3(q) < G$. Then the restriction of V to H has a non-trivial composition factor which is not an irreducible Weil module.*

Proof. Recall that G has a single conjugacy class of subgroups isomorphic to $\mathrm{SU}_3(q)$ if $3 \nmid q$, see [36] and [7].

We use notation of Lemma 7.11. Let r be a prime dividing q. Let P be a parabolic subgroup specified in Lemma 7.11 and U the unipotent radical of P.

Let $R \in \mathrm{Syl}_r(H)$; then $R' = Z(R)$. Observe first that the elements of R' are conjugate with those in $Z(U)$. This follows from a result of Kantor [35, p. 377], stated that H is generated by some subgroups of G conjugate to $Z(U) = U_{3\alpha+2\beta}(q)$. Indeed, if $r \neq 2$ then in a 7-dimensional representation ρ, say, of G over $\overline{\mathbb{F}}_r$ the elements of $Z(U)$ satisfy the equation $(x - \mathrm{Id})^2 = 0$; this is the case only for elements of R' in R as $\rho(M)$ is a direct sum of two representations of degree 3 and the trivial one [35]. If $r = 2$ then the elements of $Z(R)$ are involutions, whereas all involutions of R lie in R'. It follows that a subgroup of H conjugate with $Z(U)$ is conjugate in H to $Z(R)$.

Therefore, we may assume that $R' = Z(U)$. Let $B = N_H(R')$, so that B is a Borel subgroup of H, and $|B| = q^3(q^2 - 1)$. One observes that B, acting on R/R' by conjugation, either permutes transitively the non-identity elements of R/R', or has 3 orbits of the same length $(q^2 - 1)/3$ (in the latter case $Z(H) \cong C_3$). In particular, the only B-invariant subgroups of R that contain R' are R' or R. Now, as $U = O_r(P)$ and $B = P \cap H$, we have that $U \cap B \leq O_r(B) = R$ and $U \cap B \lhd B$. If moreover $U \cap B = R'$, then $|BU/U| = |B/R'|$ has order $q^2(q^2 - 1)$, where Sylow r-subgroups of P/U have order q, a contradiction. Hence $U \cap B = R$.

Suppose the contrary, that every irreducible constituent of $V|_H$ is either trivial module or a Weil module. By [24, Lemma 11.1], for a Weil module L, say, the restriction $L|_R$ contains no *nontrivial* linear character of R. So the same is true for $V|_R$ and thus R acts trivially on $C_V(Z(U))$. However, this contradicts Lemma 7.11 as $Z(U) < R < U$. □

Even though $G = G_2(2)$ is not simple, we still consider it for completeness. Note that it contains a normal subgroup $H \cong \mathrm{SU}_3(3)$ of index 2.

Lemma 7.13. *Let $G = G_2(2)$ and let $g \in G$ be a p-element for an odd prime p. Let $\phi \in \mathrm{Irr}_\ell G$ for $\ell > 2$. Suppose that $\deg \phi(g) < |g|$. Then $|g| \in \{3, 7\}$, $\dim \phi = 6$ and $\deg \phi(g) = |g| - 1$.*

Proof. Note that $|G| = 2^6 \cdot 3^3 \cdot 7$ so $p = 7$ or 3. If $p \neq \ell$ then the result follows by inspection of the Brauer character tables. More precisely, $|g| = 3$ implies $g \in 3A$ in notation of [6].

Let $p = \ell$. If $p = 7$ then Sylow 7-subgroups of G are cyclic. By [61, Lemma 3.3(v)], if $\deg \phi(g) < |g|$ then $\dim \phi \leq 6$. As $g \in H \cong \mathrm{SU}_3(3)$, the result follows from the main theorem of [61].

Let $|g| = 3$ and $\ell = 3$. If $\dim \phi = 6$ then $\phi|_H$ is a direct sum of two irreducible representations of degree 3, which are Galois conjugate to each other. It follows that the Jordan form of $\phi(g)$ is $\mathrm{diag}(J_2, J_2, 1, 1)$. Suppose that $\dim \phi > 6$. By Clifford's theorem, $\phi|_H$ is either irreducible or a direct sum of two irreducible representations of equal degrees. It follows that there exists a 3-modular irreducible representation τ of H of degree $d > 3$ such that $\deg \tau(g) = 2$. This contradicts a result of [48, Theorem 1]. □

Lemma 7.14. *Let $G = 2 \cdot G_2(4)$, $1_G \neq \phi \in \mathrm{Irr}_\ell G$ for $\ell \neq 2$ and let $g \in G$ be a p-element of G for $p > 2$. Suppose that $\deg \phi(g) < |g|$. Then $\dim \phi = 12$, $|g| \in \{3, 5, 7, 13\}$ and $\deg \phi(g) = |g| - 1$.*

Proof. If $\ell \neq p$ then the result follows from the Brauer character table of G [34]. More precisely, $|g| = 3$, respectively, 5 implies $g \in 3A$, respectively, $g \in 5C \cup 5D$ in notation of [6].

Let $\ell = p$. If $|g| \in \{7, 13\}$ then Sylow p-subgroups of G are cyclic and the result is contained in [61]. We are left with $p \in \{3, 5\}$ and $|g| = p$. Let $|g| = 3$. If $g \in 3B$ then g is contained in a subgroup $X \cong \mathrm{PSL}_2(13)$, so $\deg \phi(g) = 3$ for every $1_G \neq \phi \in \mathrm{Irr}_3\, G$ as this holds for X. Suppose that $g \in 3A$. As g has just two distinct eigenvalues in an irreducible representation τ of G of degree 12 over $\mathbb{C}$, the minimum polynomial degree of g in the reduction of this modulo 3 equals 2 as well.

Suppose that $\dim \phi > 12$. Note that $G/Z(G)$ contains a subgroup $Y \cong \mathrm{SU}_3(3) = G_2(2)'$. As the Schur multiplier of Y is trivial, we can assume that Y is a subgroup of G. As $|Y|_3 = |G|_3 = 27$, we observe that g is conjugate to an element of Y. We claim that the non-trivial composition factors of $\phi|_Y$ are of dimension 3. Indeed, if $\lambda \in \mathrm{Irr}_3(Y)$ is such a factor then $\deg \lambda(g) = 2$. Then λ extends to a representation $\overline{\lambda}$ of $\mathrm{SL}_3(\overline{\mathbb{F}}_3)$. By [48], $\overline{\lambda}$ is a Frobenius twist of an irreducible representation with highest weight ω_1 or ω_2, which are the fundamental weights of the weight system of $\mathrm{SL}_3(\overline{\mathbb{F}}_3)$. These representations are of dimension 3, whence the claim. Note that Y itself has two irreducible representations over $\overline{\mathbb{F}}_3$ dual to each other, and we denote their Brauer characters by λ_1, λ_2.

Let $h \in G$ be of order 7. There is a single conjugacy class of such elements, so h is rational and hence $\beta(h)$ is an integer, where β is the Brauer character of ϕ. It follows that $\beta|_Y = a(\lambda_1+\lambda_2)+b\cdot 1_Y$, so $\beta(1) = 6a + b$. Therefore, $\beta(h) = -a + b$, whence $a = (\beta(1) - \beta(h))/7$, $b = a + \beta(h)$.

In view of Theorem 7.1, ϕ is faithful. Furthermore, Y has a unique conjugacy class of involution z, say, and $\lambda_1(z) = \lambda_2(z) = -1$. Therefore, $\beta(z) = -2a + b$. We use the Brauer character table of G for $\ell = 3$ in [34, p. 274]. Note that z is in class $2A$ in $G/Z(G)$, as $\beta(z) = -4$ if $\beta(1) = 12$. By [34, p. 274], we have

$\beta(1)$	12	104	352	1260	1364	1800	2016	3744	3888
$\beta(h)$	-2	-1	2	0	-1	1	0	-1	3
$\beta(z)$	-4	8	-32	-36	-28	40	96	32	-16
$\beta(1) - \beta(h)$	14	105	350	1260	1365	1799	2016	3745	3885
a	2	15	50	180	195	257	288	535	555
b	0	14	52	180	194	258	288	534	558

It is obvious that the relation $\beta(z) = -2a + b$ holds only for $\beta(1) = 12$.

Let $|g| = 5$. Note that a subgroup $H \cong \mathrm{SU}_3(4)$ contains a Sylow 5-subgroup of G. The irreducible representation of G of degree 12 (in any characteristic) remains irreducible under restriction to H, and this is a Weil representation of H. By [56, Lemma 6.1], $\deg \phi(g) = 4$ if ϕ is a Weil representation of degree 12.

Let $\phi \in \mathrm{Irr}_5(G)$ and $\phi(1) > 12$. Then $\phi(1) \neq 1800, 3600, 3900$ as these characters are of 5-defect 0. In notation of [34], we are to inspect the characters ϕ_i with $i = 22, 23, 24, 25, 28, 29, 30$.

By [56, Lemma 6.1], every non-trivial irreducible constituent of $\phi|_H$ is of degree 12. So $\phi|_H = a \cdot \tau + b \cdot 1_G$, where $\tau \in \mathrm{Irr}_5(H)$ with $\tau(1) = 12$. Then $\phi(1) = 12a + b$. Let $g_{13} \in H$ be of order 13. Then $\tau(g_{13}) = -1$, so $\phi(g_{13}) = b - a$. In particular, $\phi(g_{13})$ is an integer, and the Brauer character table for $\ell = 5$ shows that $\phi(g_{13}) \in \{0, 1, -1\}$. Let g_2 be of order 2 then $\tau(g_2) = -4$, whence $\phi(g_2) = -4a + b$.

If $a = b$, then $\phi(g_{13}) = 0$ and $\phi(1) = 13a$, so $13|\phi(1)$, whence $(i, a) = (23, 28), (28, 148), (30, 252)$. In addition, $\phi(g_2) = -3a$, whence $(i, a) = (23, 12), (28, 28), (30, 20)$. This is a contradiction.

Let $b = a + 1$. Then $\phi(g_{13}) = 1$, whence $i = 22, 24, 29$. As $\phi(1) = 13a + 1$, we have $a = 7, 43, 167$, respectively. In addition, $\phi(g_2) = -3a + 1 = -20, -128, -500$, which is false.

Let $b = a - 1$. Then $\phi(g_{13}) = -1$, whence $i = 25$. Then $\phi(1) = 1260 = 13a - 1$, whence $a = 97$. Then $\phi(g_2) = -3a - 1 = -292$, which is false again. □

8. The case of ${}^3D_4(q)$

In this section we consider the groups $G = {}^3D_4(q)$ and prove the following result.

Theorem 8.1. *Theorem* 1.1 *is true for groups of type* $^3D_4(q)$.

We first consider the case where Sylow p-subgroups are cyclic and next the remaining cases.

8.1. **The case of cyclic Sylow p-subgroups.** Note that G contains a cyclic torus of order $q^4 - q^2 + 1$. As $|T|$ is coprime to $|G|/|T|$, the Sylow p-subgroups of T and of G are cyclic for p dividing $|T|$, and $C_G(t) = T$ for every $1 \neq t \in T$ so t is regular. So the assumptions of Lemmas 3.2 and 3.7 hold. Therefore, if $\phi \in \mathrm{Irr}_\ell\, G$ then either $\phi|_T = k \cdot \rho_T^{\mathrm{reg}}$ with $k > 0$ or ϕ is liftable or ϕ is unipotent. If $\ell \in \{0, p\}$ then Theorem 8.1 follows from [61], so we assume $\ell \neq 0, p$. In addition, we can assume that ϕ is not liftable. By Lemma 3.7, we are left with ℓ-modular Brauer irreducible characters from the unipotent blocks, that is, we assume that ϕ is a constituent of a unipotent character modulo ℓ.

We first specify Lemma 2.1 for our situation. Note that the degree of any non-trivial Brauer character of G (provided ℓ is coprime to q) is at least $q^5 - q^3$ [49, Table 1].

Lemma 8.2. *In the notation of Lemma* 2.1, *let* $C = T < G$ *so that* $|C| = q^4 - q^2 + 1$. *Let* $1_G \neq \phi \in \mathrm{Irr}_\ell\, G$. *Suppose that* $\phi(g) = c < 0$ *for all* $g \in C$, *and that* $-c < q$. *Then* $\phi|_C - \rho_C^{\mathrm{reg}}$ *is a proper character of* C.

Proof. We have $-c(|C| - 1) < q^5 - q^3 \leq \phi(1)$, so the result follows from Lemma 2.1. □

The ℓ-decomposition matrix of G is determined by Geck [19] for $\ell > 2$ and Himstedt [27] for $\ell = 2$, but a few entries for which only partial information has been obtained. For $\ell > 3$ Dudas [13] has determined some of those entries. For undetermined entries we need upper bounds; for $\ell = 2$ these are available from [27], and for $\ell > 2$ these can be read off from the proof given in [19]. We acknowledge Dr. Himstedt's help with this matter.

There are 8 unipotent characters of G, denoted by $\mathbf{1}, [\varepsilon_1], [\varepsilon_2], [\rho_1], [\rho_2], \mathsf{St}, {}^3D_4[1], {}^3D_4[-1]$ in [19] and elsewhere. We simplify this notation below by setting $D^+ = {}^3D_4[1]$, $D^- = {}^3D_4[-1]$ and using $1_G, \varepsilon_1, \varepsilon_2, \rho_1, \rho_2$ in place of $\mathbf{1}, [\varepsilon_1], [\varepsilon_2], [\rho_1], [\rho_2]$.

We have $\varepsilon_1(1) = q(q^4 - q^2 + 1)$, $\varepsilon_2(1) = q^7(q^4 - q^2 + 1)$, $\mathsf{St}(1) = q^{12}$, $\rho_1(1) = q^3(q^3 + 1)^2/2$, $\rho_2(1) = q^3(q + 1)^2(q^4 - q^2 + 1)/2$, $D^+(1) = q^3(q - 1)^2(q^4 - q^2 + 1)/2$, $D^-(1) = q^3(q^3 - 1)^2/2$.

Furthermore $\varepsilon_1(1) \equiv 0 \pmod{T}$, $\varepsilon_2(1) \equiv 0 \pmod{T}$, $\rho_2(1) \equiv 0 \pmod{T}$, $D^+(1) \equiv 0 \pmod{T}$, $\rho_1(1) \equiv -1 \pmod{T}$, $\mathsf{St}(1) \equiv 1 \pmod{T}$, $D^-(1) \equiv 1 \pmod{T}$. This implies $\varepsilon_1(t) = \varepsilon_2(t) = \rho_2(t) = D^+(t) = 0$, $\rho_1(t) = -1$, $\mathsf{St}(t) = D^-(t) = 1$ for $1 \neq t \in T$.

Himstedt [27] identifies the ℓ-modular irreducible representations of G of degree $< (q^5 - q^3 + q - 1)^2$. As a consequence of this, we have

Lemma 8.3. *Let* $1_G \neq \phi \in \mathrm{Irr}_\ell\, G$. *Then either* $\phi(1) > q^8 + q^4$ *or* ϕ *is an irreducible constituent of* ε_1° *and* $\phi(1) = \varepsilon_1(1)$ *or* $\varepsilon_1(1) - 1$.

Let T be a torus of order $q^4 - q^2 + 1$ and $1 \neq t \in T$ is an arbitrary ℓ'-element. Using the data from [19] and [27], we will show that either $\phi(t) \geq 0$ or $-\phi(t) \cdot |T| < \phi(1)$ whenever $\phi \neq 1_G$ is a unipotent Brauer character of G. For this we first obtain an upper bound for $\phi(t)$.

Recall that $\mathrm{Irr}^0_\ell(G)$ denotes the set of non-liftable unipotent Brauer characters of G, and use ϕ_i with $1 \leq i \leq |\mathrm{Irr}^0_\ell(G)|$ to denote the Brauer characters in $\mathrm{Irr}^0_\ell(G)$. (This notation for ϕ_i does not coincide with that used in [27].)

If $2, 3 \neq \ell | (q - 1)$ then $|\mathrm{Irr}^0_\ell(G)| = 0$, so every irreducible ℓ-modular character is liftable.

Let $\ell | (q^4 - q^2 + 1)$. By [19, p. 3265], $|\mathrm{Irr}^0_\ell(G)| = 2$, and $\phi_1 = \rho_1 - 1_G$, $\phi_2 = \mathsf{St} - \rho_1$. So $\phi_1(t) = -2$ and $\phi_2(t) = 2$.

Let $2, 3 \neq \ell | (q + 1)$. Then $|\mathrm{Irr}^0_\ell(G)| = 3$ and $\phi_1 = \varepsilon_1 - 1_G$, $\phi_2 = \varepsilon_2 - 1_G$, $\phi_3 = \mathsf{St} - \varepsilon_1 - \varepsilon_2 - aD^- - bD^+$, where $1 \leq a, b \leq (q - 1)/2$. (In fact, $a = b = 2$ unless possibly $\ell = 5$ and $q + 1$ is not a multiple of 25 [13, Theorem 2.3]). Then $\phi_1(t) = \phi_2(t) = -1$, $\phi_3(t) = 1 - a$ so $-\phi_3(t) < q$.

Let $3 \neq \ell|(q^2+q+1)$. Then $|\operatorname{Irr}^0_\ell(G)| = 4$ and $\phi_1 = \rho_1 - \varepsilon_1$, $\phi_2 = \rho_2 - 1_G$, $\phi_3 = \varepsilon_2 - \rho_1 + \varepsilon_1 - aD^+$, $\phi_4 = \mathsf{St} - c\phi_3 - bD^+ - \phi_2 = \mathsf{St} - c\varepsilon_2 + c\rho_1 - c\varepsilon_1 + acD^+ - \rho_2 + 1_G - bD^+ = \mathsf{St} - c\varepsilon_2 + c\rho_1 - c\varepsilon_1 - (b-ac)D^+ - \rho_2 + 1_G$.

So $\phi_1(t) = \phi_2(t) = -1$, $\phi_3(t) = 1$, $\phi_4(t) = 2-c$. In fact, $c = 2$ by [13, Theorem 2.4], so $\phi_4(t) = 0$.

Let $3 \neq \ell|(q^2-q+1)$. Then $|\operatorname{Irr}^0_\ell(G)| = 3$ and $\phi_1 = \rho_2 - \varepsilon_1 - 1_G$, $\phi_2 = \varepsilon_2 - aD^- - \phi_1 - 1_G = \varepsilon_2 - aD^- - \rho_2 + \varepsilon_1$, $\phi_3 = \mathsf{St} - d\phi_2 - cD^+ - bD^- - \phi_1 - \varepsilon_1 = \mathsf{St} - d(\varepsilon_2 - aD^- + \varepsilon_1 + 1_G) - cD^+ - bD^- - (\varepsilon_1 - 1_G) - \varepsilon_1 = \mathsf{St} - d\phi_2 - cD^+ - bD^-\phi_1 - \varepsilon_1 = \mathsf{St} - d\varepsilon_2 + (ad-b)D^- + d\varepsilon_1 + (1-d).1_G - cD^+$.

So $\phi_1(t) = -1$, $\phi_2(t) = -a$, $\phi_3(t) = 2 + ad - b$. Here $a = d = 0$, $b = 2$ by [13, Theorem 2.5], so $\phi_3(t) = 0$.

Let $\ell = 3|(q-1)$. Then $|\operatorname{Irr}^0_\ell(G)| = 4$ and $\phi_1 = \rho_1 - \varepsilon_1$, $\phi_2 = \rho_2 - 1_G$, $\phi_3 = \varepsilon_2 - \phi_1 - aD^+ = \varepsilon_2 - \rho_2 + \varepsilon_1 - aD^+$, $\phi_4 = \mathsf{St} - c\phi_3 - bD^+ - \phi_2 = \mathsf{St} - c(\varepsilon_2 - \rho_2 + \varepsilon_1 - aD^+) - bD^+ - \rho_2 + 1_G = \mathsf{St} - c(\varepsilon_1 + \varepsilon_2) + (c-1)\rho_2 + (ac-b)D^+ + 1_G$.

So $\phi_1(t) = \phi_2(t) = -1$, $\phi_3(t) = 1$, $\phi_4(t) = 2-c$. Here $c \leq q$, so $-\phi_4(t) < q$.

Let $\ell = 3|(q+1)$. Then $|\operatorname{Irr}^0_\ell(G)| = 4$ and $\phi_1 = \varepsilon_1 - 1_G$, $\phi_2 = \rho_2 - \phi_1 - 2\cdot 1_G$, $\phi_3 = \varepsilon_2 - \phi_2 - aD^- - 1_G$, $\phi_4 = \mathsf{St} - d\phi_3 - cD^+ - bD^- - \phi_2 - \phi_1 + 1_G = \mathsf{St} - d\phi_3 - cD^+ - bD^- - \rho_2(1)$, as $\phi_1 + \phi_2 = \rho_2 - 2\cdot 1_G$.

So $\phi_1(t) = \phi_2(t) = -1$, $\phi_3(t) = -a$, $\phi_4(t) = 2 + ad - b$.

In this case $0 \leq a \leq 1$, $a+1 \leq b \leq 3(q+1)/2$, $c \leq (q-1)/2$ and $1 \leq d \leq q$ [19]. So $-\phi_4(t) \leq \frac{3q-1}{2}$.

Let $\ell = 2$. Then by [27, Theorem 3.1, p.572], we have $|\operatorname{Irr}^0_\ell(G)| = 5$ and $\phi_1 = \varepsilon_1 - 1_G$, $\phi_2 = \rho_1 - D^+$, $\phi_3 = \rho_2 - \phi_2$, $\phi_4 = \varepsilon_2 - 1_G$, $\phi_5 = \mathsf{St} - \phi_4 - aD^+ - b\phi_3 - \phi_1 - 1_G$, where $0 \leq a, b \leq q$. We have $\phi_1(t) = \phi_2(t) = \phi_4(t) = -1$, $\phi_3(t) = 1$, $\phi_5(t) = 2 - b$.

Lemma 8.4. *Let $1_G \neq \phi \in \operatorname{Irr}^0_\ell G$ and T be a torus of order $q^4 - q^2 + 1$ of G. Then either $\phi(t) > 0$ or $-\phi(t) \cdot |T| < \phi(1)$ for every $1 \neq t \in T$. In particular,* $\deg \phi(t) = |t|$ *for any p-element $t \in T$ with $p \neq \ell$.*

Proof. Suppose first that ϕ is a non-trivial irreducible constituent of ε_1. Then $\phi(1) = \varepsilon_1(1) - 1$ (and either $\ell = 2$ or $\ell = 3|(q+1)$). Then $\phi(t) = -1$ and $\phi(1) = q(q^4 - q^2 + 1) - 1$, whence the claim.

Suppose that ϕ is not a constituent of ε_1. Then $\phi(1) > q^8 + q^4$ by Lemma 8.3, and either $\phi(t) > 0$ or $-\phi(t) \leq (3q-1)/2$. Then $|T| \cdot (3q-1)/2 = (q^4 - q^2 + 1)(3q-1)/2 < q^8 + q^4$.

For the last claim, observe by Lemma 3.7 that ϕ takes a constant value c on $\langle t \rangle \smallsetminus \{1\}$. Now apply Lemma 2.1. ☐

Remark 8.5. In [27] there are weaker bounds for a, b, c, d for $3 = \ell|(q+1)$, specifically $a \leq q(q-1)$, $b \leq (q^3-1)/2$, $c, d \leq (q-1)/2$. These are sufficient for our purpose, as either $\phi(t) > 1$ or $-\phi_4(t) \leq ((q^3-1)/2) - 2$, and again we have $q^3|T| < \phi_4(1)$.

8.2. **The case where Sylow p-subgroups are not cyclic.** For uniformity we denote by $\mathrm{SO}_{2n}(\overline{\mathbb{F}}_q)$ the subgroup of index 2 of $O_{2n}(\overline{\mathbb{F}}_q)$ if q is odd and of $O_{2n}(\overline{\mathbb{F}}_q)$ if q is even. Then $\mathrm{SO}_{2n}(\overline{\mathbb{F}}_q)$ is a connected simple algebraic group of type D_n. If q is even then $SO_{2n}(\overline{\mathbb{F}}_q)$ is formed by elements of quasi-determinant 1 in $O_{2n}(\overline{\mathbb{F}}_q)$. Also, let $\mathbf{G} = Spin_{2n}(\overline{\mathbb{F}}_q)$ denote the simply connected simple algebraic group of type D_n, so that $\mathbf{G} = \mathrm{SO}_{2n}(\overline{\mathbb{F}}_q)$ when $2|q$ and $\mathbf{G}/C_2 = \mathrm{SO}_{2n}(\overline{\mathbb{F}}_q)$ when $2 \nmid q$. Taking $n = 4$, we can view $G = \mathbf{G}^F$ for some Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$. By [37, p. 33], $G \cong {}^3D_4(q)$ has maximal subgroups isomorphic to $G_2(q)$ and $(C_{q^2+q+1} \circ \mathrm{SL}_3(q)).\gcd(3, q-1)$ (where C_{q^2+q+1} is cyclic of order q^2+q+1). If $g \in G$ lies in the $G_2(q)$-subgroup then we can use our result on $G_2(q)$ (Theorem 7.1). So our first goal is to establish Lemma 8.9 below.

Lemma 8.6. *Let $\mathbf{G}$ be a simple, simply connected algebraic group of type D_4 in defining characteristic r, and let V be the standard $\mathbf{G}$-module. Let $G \cong {}^3D_4(q) < \mathbf{G}$, where $r|q$ and $q > 2$. Let*

$M_1 \cong (C_{q^2+\varepsilon q+1} \circ \mathrm{SL}_3^{\varepsilon}(q)).\gcd(3, q-\varepsilon)$ and $M_2 \cong G_2(q)$ be maximal subgroups of G and $H_1 < M_1$ a subgroup isomorphic to $\mathrm{SL}_3^{\varepsilon}(q)$. Then H_1 is $\mathbf{G}$-conjugate to a subgroup of M_2.

Proof. Note that V is self-dual. Our strategy is to show that for some subgroup $H_2 \cong \mathrm{SL}_3^{\varepsilon}(q)$ of $M_2 = G_2(q)$ for every $i = 1, 2$ there are H_i-stable subspaces V_1^i, V_2^i, V_3^i such that $V = V_1^i \oplus V_2^i \oplus V_3^i$, V_1^i, V_2^i are totally singular of dimension 3, and V_3^i is non-degenerate and trivial on H_i. Then V_1^i, V_2^i are dual FH_i-modules, and the result will follow from Witt's theorem. Indeed, by Witt's theorem applied to $\mathrm{O}(V)$, there is some $x \in \mathrm{O}(V)$ that sends V_j^1 to V_j^2 for $j = 1, 2, 3$. In our case V_3^i is a trivial H_i-module of dimension 2 and so we can find an element $y \in (\mathrm{O}(V_3^2) \setminus \mathrm{SO}(V_3^2))$ that commutes with H_2. Replacing x by yx if necessary, we may assume $x \in \mathrm{SO}(V)$ and sends V_j^1 to V_j^2. Since all 3-dimensional nontrivial representations of H_i are irreducible and quasi-equivalent to the natural representation, we now have that x sends the image of H_1 in $\mathrm{SO}(V)$ to the image of H_2 in $\mathrm{SO}(V)$, and we are done if $r = 2$. If $r > 2$, then an inverse image of x conjugates the full inverse image $C_2 \times H_1$ to $C_2 \times H_2$, hence H_1 to H_2.

If $r \neq 3$ then the subgroups of M_2 isomorphic to $\mathrm{SL}_3^{\varepsilon}(q)$ are conjugate so choose for H_2 any of them. If $r = 3$ then G_2 has two conjugacy classes of subgroups isomorphic to $\mathrm{SL}_3^{\varepsilon}(q)$, one of which is reducible on the irreducible FM_2-modules of dimension 7, and the other is irreducible [36, Theorem A]. In this case we choose H_2 from the former one.

It suffices to show that the composition factors of $V|_{H_i}$, $i = 1, 2$ are of dimension 1 or 3. Indeed, as V is self-dual and the simple H_i-modules of dimension 3 are not self-dual, there are two composition factors of dimension 3 and they are dual to each other. Let N be a composition factor of dimension 3. It suffices to observe that $\mathrm{Ext}^1_{H_i}(N, N^*) = 0$ and $\mathrm{Ext}^1_{H_i}(N, N_0) = 0 = \mathrm{Ext}^1_{H_i}(N_0, N)$, where N_0 is the trivial FH_i-module and N^* is dual to N. If $r > 2$ then every FH_i-module of dimension at most 6 is completely reducible by [44, Theorem 1.1]. If $r = 2$ then this follows by [52].

Suppose first that $i = 2$. As the dimensions at most 8 of nontrivial simple $G_2(q)$-modules are 6 if $r = 2$ and 7 if $r \neq 2$ [41, p. 167], the fixed point subspace L of M_2 on V is non-zero. Next, if $r \neq 2$ then K must be non-degenerate, and $V = L \oplus V'$, where V' is an irreducible FM_2-module. If $r = 2$ then as $\dim V/L = 1$, V/L is reducible, and has a composition factor V_1' of degree 6. Note that the irreducible constituents of the restrictions of these modules V', respectively V_1' to H_2 are 3-dimensional and dual to each other. (Indeed, if $r = 2$ then all nontrivial simple H_2-modules of dimension ≤ 6 are of dimension 3 and non-self-dual, see [41, p. 149]. If $r \neq 2$, then by the choice of H_2, H_2 is reducible on V'. If moreover it has a composition factor W of dimension $\neq 1, 3$ on V', then $\dim W = 6$ by [41, p. 149] and W is not self-dual. The other composition factors of the H_2-modules are all trivial, and this contradicts $V \cong V^*$.)

Suppose that $i = 1$. Let K be an irreducible FH_1-submodule of V of maximal dimension. Then $K \neq V$ by Schur's lemma. Suppose that $\dim K = 7$. By [41, p. 149], we have $r = 3$ and V has a composition factor K' of dimension 1. As V is self-dual, we conclude that V is completely reducible over H_1, so K is non-degenerate, $V = K \oplus K^{\perp}$ and $\dim K^{\perp} = 1$. Now the odd order subgroup $C_{q^2+\varepsilon q+1}$ must act trivially on both K and $K^{\perp}$, a contradiction. Suppose that $\dim K = 6$. Then $r > 2$ and K is not self-dual [41, p.149], so V_{H_1} has a composition factor dual to K, which is not the case by dimension reason. So $\dim K \leq 5$ and hence $\dim K = 3$ by [41, p. 149]. □

Lemma 8.7. *Let $H \cong \mathrm{SL}_3^{\varepsilon}(q) \subset \mathbf{G} = \mathrm{Spin}_8(\overline{\mathbb{F}}_q)$ and V the natural module for $\mathrm{SO}_8(\overline{\mathbb{F}}_q)$. Suppose that $V|_H = V_1 \oplus V_2 \oplus V_3$ with V_1, V_2 totally singular of dimension 3 and V_3 a non-degenerate subspace on which H acts trivially. Then $Y := C_{\mathbf{G}}(H)$ is connected.*

Proof. By assumption, V_1, V_2 are dual to each other and V_3 is trivial as H-modules. Hence Y stabilizes these modules; let $\bar{Y}$ denote the image of Y in $\mathrm{SO}(V)$. Then $\bar{Y} \leq \{\mathrm{diag}(y_1, y_2, t)\}$, where $y_1 \in \mathrm{GL}_3(\overline{\mathbb{F}}_q)$ is a scalar (3×3)-matrix, $y_2 = y_1^{-1}$ and $t \in \mathrm{O}(V_3)$. In fact, $t \in \mathrm{SO}(V_3)$. This is obvious if q is odd; if q is even then, since $\mathrm{GL}_3(\overline{\mathbb{F}}_q)$ has no subgroup of index 2, so the matrix $\mathrm{diag}(y_1, y_1^{-1}, \mathrm{Id}_2)$ lies in $\mathrm{SO}(V)$, whence $\mathrm{diag}(\mathrm{Id}_6, t) \in \mathrm{SO}(V)$, which implies the claim. Note that $C_{\mathrm{SO}(V)}(H)$ contains the subgroup $\mathrm{diag}(\mathrm{Id}_3, \mathrm{Id}_3, \mathrm{SO}(V_3))$. Hence $C_{\mathrm{SO}(V)}(H)$ is isomorphic to the direct product of $Z(\mathrm{GL}_3(\overline{\mathbb{F}}_q)) \cong \overline{\mathbb{F}}_q^\times$ and $\mathrm{SO}_2(\overline{\mathbb{F}}_q)$, which is a connected group. Now, if $r = 2$ then $Y = \bar{Y} = C_{\mathrm{SO}(V)}(H)$, and we are done. If $r > 2$, then these two subgroups lift to a one-dimensional torus T and $S = \mathrm{Spin}_2$, respectively, which centralize each other modulo $C_2 = Z(\mathrm{Spin}(V))$. Since S is perfect, we have that $[T, S] = [T, [S, S]]$ is contained in $[[T, S], S] = 1$, so the full inverse image of $C_{\mathrm{SO}(V)}(H)$ is a central product of two connected subgroups, and so is connected. Each of T and S centralizes the perfect subgroup H modulo $Z(\mathrm{Spin}(V))$, so the same argument shows that Y is the full inverse image of $C_{\mathrm{SO}(V)}(H)$, and so Y is connected. □

Lemma 8.8. *Let $\mathbf{G}$ be a connected algebraic group, F a Frobenius map, and let $G = \mathbf{G}^{\mathsf{F}} = \{g \in \mathbf{G} \mid \mathsf{F}(g) = g\}$. Let H be a subgroup of G and $H_1 = xHx^{-1} \leq G$ for some $x \in \mathbf{G}$. Suppose that $C_{\mathbf{G}}(H)$ is connected. Then H, H_1 are conjugate in G.*

Proof. Let $h \in H$. Then $xhx^{-1} \in G$, so $xhx^{-1} = \mathsf{F}(x)h\mathsf{F}(x)^{-1}$. Then $\mathsf{F}(x)^{-1}x \in C_{\mathbf{G}}(h)$ for all $h \in H$. Therefore, $\mathsf{F}(x)^{-1}x \in C_{\mathbf{G}}(H)$. By Lang's theorem, there exists $c \in C_{\mathbf{G}}(H)$ such that $\mathsf{F}(x)^{-1}x = \mathsf{F}(c)^{-1}c$, so $xc^{-1} \in G$ and $H_1 = (xc^{-1})H(xc^{-1})^{-1}$. □

Lemma 8.9. *Let H_1 be as in Lemma 8.6. Then H_1 is G-conjugate to a subgroup of $M_2 \cong G_2(q)$.*

Proof. By Lemma 8.6, $xH_1x^{-1} < M_2 < G$ for some $x \in \mathbf{G}$, so Lemma 8.8 yields the result. □

Remark 8.10. Lemma 8.9 justifies the claim in [63, p. 2520, line 6] stated therein with no proof. Thus, this fixes a gap in the proof of [63, Lemma 4.14] which gives a proof of Theorem 8.1 for $\ell = 0$ and $p > 2$.

Proposition 8.11. *Let $g \in G \cong {}^3D_4(q)$ be a semisimple p-element. Suppose that Sylow p-subgroups of G are not cyclic. Let $\theta \in \mathrm{Irr}_F\, G$ with $\dim\theta > 1$. Then $\deg\theta(g) = |g|$.*

Proof. (i) The case with $q = 2$ can be settled by a computer computation. (Note that when $p \neq \ell$ one can also use the Brauer character table, and the $p = \ell > 3$ case follows from [61]; so if $q = 2$ then it suffices to deal with the case $p = \ell = 3$.) Let $q > 2$.

If g is contained in a subgroup isomorphic to $G_2(q)$ then the lemma follows from Theorem 7.1. Suppose the opposite. Then p divides the r'-part of $|G|/|G_2(q)| = q^6(q^8+q^4+1)$, i.e. $p|(q^8+q^4+1)$. As we assume the Sylow p-subgroups of G are non-cyclic, we have $p|(q^4+q^2+1)$, so p divides $q^2+\varepsilon q+1$ for some $\varepsilon \in \{\pm 1\}$. In particular, $p > 2$.

(ii) Here we consider the case $p > 3$. By [37], G contains a subgroup H isomorphic to $X \circ Y$, where $X \cong \mathrm{SL}_3^\varepsilon(q)$ and $Y \cong C_\varepsilon$, a cyclic subgroup of order $q^2 + \varepsilon q + 1$. Then

$$|G|/|H| = \gcd(3, q - \varepsilon)q^9(q^3 + \varepsilon)(q^8 + q^4 + 1)/(q^2 + \varepsilon q + 1),$$

whence $(p, |G|/|H|) = 1$. Therefore, H contains a Sylow p-subgroup of G. By Lemma 8.9, X is contained in a subgroup $D \cong G_2(q)$.

Express $g = xy$ for $x \in X$ and $y \in Y$. Then $y \neq 1$ as otherwise $g \in X < D \cong G_2(q)$. In addition, by [63, Lemma 4.13], we may assume that $|x| = |y| = |g|$.

Suppose that $\deg\theta(g) < |g|$. Let τ be any composition factor of $\theta|_H$ such that $\tau = \phi \otimes \lambda$ with $1_X \neq \phi \in \mathrm{Irr}_F\, X$ and $\lambda \in \mathrm{Irr}_F\, Y$. Then $\deg\phi(x) < |g| = |x|$. This implies that ϕ is a Weil representation of X and $|g| = |C_\varepsilon|$, see Lemma 2.6 if $\varepsilon = 1$ and [56, Proposition 6.1] if $\varepsilon = -1$. Therefore, every non-trivial composition factor of $\theta|_X$ is a Weil representation of X. If $\varepsilon = -1$ and $(3, q) = 1$ then this contradicts Lemma 7.12. If $3 \nmid (q - \varepsilon)$ and $\ell|(q^4 + q^2 + 1)$ then this contradicts Lemma 7.9. Thus, if $\varepsilon = -1$, then we may assume $3|q$ and $\ell \nmid (q^4 + q^2 + 1)$, whence $p \neq \ell$. If

$\varepsilon = 1$, then, as $p > 3$ and q^2+q+1 is a p-power, we have $3 \nmid (q-1)$, and so we may assume that $\ell \nmid (q^4+q^2+1)$, whence $p \neq \ell$. In both the cases, we have $Z(X) = 1$ and $X \circ Y = X \times Y$.

We can write $\tau|_{(X\times Y)} = (\chi_1 \otimes \lambda_1) + \ldots + (\chi_s \otimes \lambda_s)$ as a sum of Brauer characters, where $\chi_i \in \mathrm{Irr}_F\, X$ and $\lambda_i \in \mathrm{Irr}_F\, Y$ for $i = 1, \ldots, s$. We order the summands so that $\chi_i(1) > 1$ for $i = 1, \ldots, t \leq s$ and $\chi_i(1) = 1$ for $i > t$. Let ϕ_i be the representation afforded by χ_i, and let λ_i also denote the respective representation of Y as this character is linear. Then $\deg \phi_i(x) < |g| = |x|$. As $Z(X) = 1$ and $\ell \neq p | (q^2 + q\varepsilon + 1)$, every non-trivial p-element is regular in X, so, by Lemma 3.7, either ϕ is liftable or constant on the non-identity p-elements of X. By Lemma 2.6 and [56, Proposition 6.1], if $1 \leq i \leq t$ then $\deg \phi_i(x) = |x| - 1 = \dim \phi_i(1)$, ϕ_i is real, and hence $\deg \theta(g) = |g| - 1$. Then $1 \notin \mathrm{Spec}\, \theta(g)$. (Indeed, otherwise some other $|g|$-root ν of unity is not in the spectrum of $\theta(g)$; as $p > 2$ and g is conjugate to g^{-1} (see for instance [55, Theorem 1(vi)]), $\nu^{-1} \notin \mathrm{Spec}\, \theta(g)$, and hence $\deg \theta(g) \leq |g| - 2$, which is a contradiction). Since ϕ_i is real, we also have that $1 \notin \mathrm{Spec}\, \phi_i(x)$, and hence $\mathrm{Spec}\, \phi_i(x)$ consists of all nontrivial $|g|$-roots of unity when $1 \leq i \leq t$. It follows that $\lambda_i(y) = 1$ for $1 \leq i \leq t$. As $y \notin \ker(\tau)$ and $1 \notin \mathrm{Spec}\, \theta(g)$, we have $s > t$, and none of $\lambda_{t+1}(y), \ldots, \lambda_s(y)$ equals 1. It follows that 1 is not an eigenvalue of $\theta(x^k y^l)$ whenever k, l are coprime to p, whereas $1 \in \mathrm{Spec}\, \theta(x)$. However, by [63, Lemma 4.13], x is conjugate to some $x^k y^l$ with $|x^k| = |y^l| = |x|$ (the key point used in [63, Lemma 4.13] is that $N_G(T_\varepsilon)$ acts primitively on a maximal torus $T_\varepsilon \cong C^2_{q^2+\varepsilon q+1}$). This is a contradiction.

(iii) Let $p = 3$. Then $|g|$ does not divide $q^2 - 1$. (Indeed, otherwise $|g|$ divides $q - \varepsilon$ for some $\varepsilon \in \{1, -1\}$, and, by Lemma 2.9(i), g is contained in a torus of G of order $(q-\varepsilon)^2$. By Lemma 2.8(iii), the tori of order $(q-\varepsilon)^2$ are conjugate in G. A subgroup isomorphic to G_2 contains a torus of this order, so it contains a conjugate of g.)

It is known (see claim (*) in the proof of [63, Proposition 4.8, p. 2517]) that g is contained in a subgroup $H \cong X \circ Y$, where $X \cong \mathrm{SL}_2(q)$ and $Y \cong \mathrm{SL}_2(q^3)$. Express $g = xy$ with $x \in X$, $y \in Y$ to be 3-elements. Let τ be an irreducible constituent of $\theta|_H$. Then $\tau = \phi \otimes \eta$, where ϕ, η are irreducible representations of X, Y, respectively. Hence $\tau(g) = \phi(x) \otimes \eta(y)$. Suppose that $\deg \theta(g) < |g|$. Then $\deg \tau(g) < |g|$ implies $\deg \phi(x) < |g|$ and $\deg \eta(y) < |g|$. If $|y| \leq |x|$ then $|g|$ divides $q^2 - 1$; by the above this is not the case. So $|y| > |x|$, and hence $|g| = |y|$. In this case, choose τ so that $\dim \eta > 1$. Then $\deg \eta(y) < |g| = |y|$ implies by [11, Theorem 1.1] that $3|(q+1)$. By [56, Lemma 3.3] applied to Y, it follows that $q^3 + 1$ is a 3-power if q is even and $(q^3+1)/2$ is a 3-power if q is odd. The former case is ruled out by Lemma 2.2 (as $q > 2$). In the latter case $(q^3+1)/2 = (q^2-q+1)(q+1)/2$ is a 3-power implies $q^2 - q + 1$ and $(q+1)/2$ to be 3-powers, which is false as $\gcd(q^2 - q + 1, (q+1)/2) \in \{1, 3\}$. □

Proof of Theorem 8.1. The result follows from Lemma 8.4 (and the discussion at the beginning of the section) if $p|(q^4 - q^2 + 1)$, and from Proposition 8.11 if $p \nmid (q^4 - q^2 + 1)$. □

9. The case of ${}^2F_4(q)$

In this section we prove Theorem 1.1 for $G = {}^2F_4(q)$, $q = 2^{2k+1}$.

Lemma 9.1. *Theorem* 1.1 *is true for* $G = {}^2F_4(2)'$ *or* ${}^2F_4(2)$.

Proof. If $\ell \neq p$ then the result follows by inspection of the Brauer character table of G [34]. If $p = \ell = 13$ then this follows from [61]. So we are left with $p = 3, 5$.

Note that G has no element of order 9 or 25 and all elements of order 3 and of order 5 are conjugate. In addition, G contains a subgroup $H \cong \mathrm{PSL}_2(25)$, so we can assume that $g \in H$. If $|g| = 3$ then the result follows from [56, Lemma 3.3].

Let $|g| = 5$. Then we can assume that g is contained in a subgroup $K \cong \mathrm{PSL}_2(9)$. If $1_K \neq \tau \in \mathrm{Irr}_5(K)$ and $\deg \tau(g) < 5$ then $\dim \tau = 4$ ([56, Lemma 3.3]). Therefore, if the lemma is false then

$\phi|_K = a\tau + b \cdot 1_G$. Let $x \in K$ be of order 3. Then $\tau(x) = 2$, so $\phi(x) = 2a + b$, where $a \geq 1$. This implies $\phi(1) \leq 2\phi(x)$, which contradicts the data in the Brauer character table of G for $\ell = 5$. □

So in what follows we assume $q > 2$. Observe that if Sylow p-subgroups of $G = {}^2F_4(q)$ are cyclic, then p divides $q^2 - q + 1$ or $q^4 - q^2 + 1$.

Lemma 9.2. *Theorem* 1.1 *is true if is true if Sylow p-subgroups of G are not cyclic.*

Proof. Suppose the contrary, that is, $\deg \theta(g) < |g|$. As a Sylow p-subgroup is not cyclic, one observes that $p|(q^4 - 1)$. If $p|(q - 1)$ or $p|(q^2 + 1)$ then g is contained in a direct product of two copies of ${}^2B_2(q)$ (see [63, Table 1]). In this case the result follows from that for ${}^2B_2(q)$.

Suppose that $p|(q + 1)$. If $p = 3$ then g is contained in a subgroup $H \cong \mathrm{SU}(3, q)$ [63, Table 1]. Then [56, Lemma 6.1] implies $q = 8$, $|g| = 9$ and g is contained in a maximal torus of H of order $(q + 1)^2$. This is also true if $p > 3$. This torus is also a maximal torus of a subgroup $H_1 \cong \mathrm{Sp}_4(q)$. So we can assume $g \in H_1$, and the result follows from Lemma 2.5. □

Lemma 9.3. *Theorem* 1.1 *is true if $p|(q^2 - q + 1)$.*

Proof. Suppose the contrary. As $q^2 - q + 1$ divides the order of a subgroup $H \cong \mathrm{SU}_3(q)$, it follows that H contains a Sylow p-subgroup of G. Therefore, we assume that $g \in H$. By [56, Prop. 6.1(ii)], $3 \not| (q + 1)$, which is impossible since $q = 2^{2m+1}$. □

The above analysis reduces the proof of Theorem 1.1 to the case where $p|(q^4-q^2+1)$; in particular, the Sylow p-subgroups are cyclic. In view of [61] and [63], we may assume that $\ell \neq p$ and that ϕ is not liftable. Note that

$$q^4 - q^2 + 1 = (q^2 + q\sqrt{2q} + q + \sqrt{2q} + 1)(q^2 - q\sqrt{2q} + q - \sqrt{2q} + 1),$$

in fact, G contains maximal tori T_1, T_2 of these orders so we may assume that $g \in T_1$ or $g \in T_2$. The rest of the section is therefore devoted to the proof of

Proposition 9.4. *Theorem* 1.1 *is true if $p|(q^4 - q^2 + 1)$ unless possibly when $\ell = 3$, $q = 8$, $p = 109$ and $\phi = \phi_{21}$ in notation of* [27].

First we recall

Lemma 9.5. *Let d_ℓ be the minimum degree of a nontrivial ℓ-modular irreducible representation of $G = {}^2F_4(q)$ with $q > 2$.*

(i) [27, Theorem 6.1] *If $\ell > 3$ then $d_\ell \geq (q - 1)(q + 1)^2(q^2 - q + 1)\sqrt{q/2}$.*
(ii) [57, Theorem 1.4] *If $\ell = 3$ then $d_3 \geq (q - 1)(q^4 + q^3 + q)\sqrt{q/2}$.*

Lemma 2.1 implies the following:

Lemma 9.6. *Let ϕ be a non-trivial unipotent Brauer character of G, and $T \in \{T_1, T_2\}$. Let T' be the subgroup of ℓ'-elements of T and $1 \neq t \in T'$. If $\phi(t) \geq 0$ or $a = \phi(t) < 0$ and $-a(|T| - 1) < \phi(1)$ then $\phi|_T$ contains every $\nu \in \mathrm{Irr}\, T$. In particular, this holds if $q > 3$, $a < 0$ and $-a < q(q^3+q^2+1)/12$.*

Proof. We have $|T| - 1 \leq q^2 + q + (q + 1)\sqrt{2q}$, so

$$-a < \frac{(q - 1)(q^4 + q^3 + q)\sqrt{q/2}}{(q + 1)(q + \sqrt{2q})} = \frac{(q - 1)(q^4 + q^3 + q)}{(q + 1)(2 + \sqrt{2q})}$$

and

$$\frac{q - 1}{(q + 1)(2 + \sqrt{2q})} > \frac{7}{9(2 + \sqrt{16})} = \frac{7}{54} > 1/12,$$

as $q \geq 8$. □

Recall that we have to deal only with the cases where $\ell \neq p$ and ϕ is not liftable. Therefore, Lemma 3.7 together with Lemma 9.6 reduces the proof to the case where ϕ is unipotent. Our strategy in proving Proposition 9.4 is to show, using the ℓ-decomposition numbers of G, that either $\phi(t) \geq 0$ or $\phi(t) < q(q^3+q^2+1)/12$, and then use Lemma 9.6.

Hiss [29] has determined the decomposition numbers of $G = {}^2F_4(q)$ modulo $\ell|(q^4-q^2+1)$ and $\ell|(q^2-q+1)$; Himstedt [26] has computed these for remaining $\ell \neq 2$. Note that Himstedt's tables involve some indetermined values; this leads to certain difficulties below, in particular, for $q = 8$.

The degrees of the unipotent characters are available in Malle [42]. In his notation these are χ_k with $k = 1, \ldots, 21$. Note that $G = {}^2F_4(q)$ has exactly two maximal tori T_1, T_2 (up to conjugation) that satisfies the assumption of Lemma 3.2. These are of the aforementioned orders $|T_1| = q^2 + q\sqrt{2q} + q + \sqrt{2q} + 1$ and $|T_2| = q^2 - q\sqrt{2q} + q - \sqrt{2q} + 1$. Let η be as in Lemma 3.2. Then η can be computed by taking the congruences of $\chi_k(1)$ modulo $|T_i|$ for $i = 1, 2$. The result is recorded in Hiss [26, p. 886 and p. 884], and we display it in Table 3 below. Note that fourth column lists the characters of p_i-defect 0 for $i = 1, 2$.

Table 3: Unipotent character values at $1 \neq t \in (T_1 \cup T_2)$

	$\chi_i(t) = 1$	$\chi_i(t) = -1$	$\chi_i(t) = 0$
$1 \neq t \in T_1$	$i = 1, 4, 5, 6, 13, 15, 16, 19, 20$	$i = 7, 8, 10$	$i = 2, 3, 9, 11, 12, 14, 17, 18, 21$
$1 \neq t \in T_2$	$i = 1, 4, 7, 8, 12, 17, 18, 19, 20$	$i = 5, 6, 9$	$i = 2, 3, 10, 11, 13, 14, 15, 16, 21$

In what follows, we will use data of Table 3 without further referring. Also, denote

$$\phi_1 = q-1,\ \phi_2 = q+1,\ \phi_8' = q+\sqrt{2q}+1,\ \phi_8'' = q-\sqrt{2q}+1,\ \phi_8 = q^2+1,\ \phi_{12} = q^2-q+1,$$

$$\phi_{24}' = q^2+q\sqrt{2q}+q+\sqrt{2q}+1 = |T_1|,\ \phi_{24}'' = q^2-q\sqrt{2q}+q-\sqrt{2q}+1 = |T_2|,$$

and $\phi_{24} = q^4-q^2+1$, so that $\phi_{24} = \phi_{24}'\phi_{24}'' = |T_1| \cdot |T_2|$.

The notation of irreducible characters is as in [42]. Note that their parametrization is the same as in Hiss [29] (where ξ_j is used for χ_j).

9.1. **Brauer characters.** Note that we only need to deal with non-liftable irreducible Brauer characters as $\ell \neq p$. The set of such characters is denoted by $\mathrm{Irr}_\ell^0(G)$. Note that tori T_1, T_2 satisfy the assumption of Lemma 3.2. By Lemma 3.7, every character in $\mathrm{Irr}_\ell^0(G)$ is unipotent and constant on the ℓ'-elements of T_i, $i = 1, 2$. The result is trivial if the constant in question equals 0 (for $i \in \{1, 2\}$), as in this case $\phi|_{T_i}$ is a multiple of the regular character $\rho_{T_i}^{\mathrm{reg}}$. So below we only consider the cases where $\phi(t) \neq 0$ for $1 \neq t \in T_i$. If ℓ divides $|T_i|$) then we write T_i' for the subgroup of ℓ'-elements of T_i, $i = 1, 2$.

Every irreducible Brauer character agrees on ℓ'-elements with some integral linear combination of ordinary characters. If an ℓ-modular character τ, say, agrees on ℓ'-elements with $\sum a_i\chi_i$, where the χ_i's are ordinary characters, we simply write $\tau = \sum a_i\chi_i$. (or $\tau =_{\ell'} \sum a_i\chi_i$.)

Lemma 9.7. *Let $\phi \in \mathrm{Irr}_\ell^0(G)$, $T \in \{T_1, T_2\}$, and let $t \in T$ be any element. Then either $\phi(t) \geq 0$, or $-\phi(t) < q(q^3+q^2+1)/8$, or $\ell = 3$, $q = 8$ and $T = T_1$.*

Proof. (i) We start with primes ℓ for which Sylow ℓ-subgroups are cyclic. This means that ℓ divides either q^2-q+1 or $|T_1|$ or $|T_2|$. These are the cases (a), (b), (c) below.

(a) Suppose that $\ell|(q^2-q+1)$. Then, by Hiss [29, Theorem 4.5], $|\,\mathrm{Irr}_\ell^0(G)| = 2$, and for $\phi_1, \phi_2 \in \mathrm{Irr}_\ell^0(G)$ we have $\phi_1 = \chi_{11} - 1_G$, $\phi_2 = \chi_4 - \chi_{11} - \chi_{19} - \chi_{20} + 1_G$. Then $\phi_1(t) = -1$ and $\phi_2(t) = 0$ for every $1 \neq t \in T_1 \cup T_2$. So the result follows from Lemma 9.6.

(b) Let ℓ divide $|T_1|$. Then, by Hiss [29, Theorem 4.7], $|\operatorname{Irr}^0_\ell(G)| = 4$, and we have

$$\phi_1 = \chi_{10} - 1_G,\ \phi_2 = \chi_7 - \chi_5 - \chi_{15} - \chi_{19},\ \phi_3 = \chi_8 - \chi_6 - \chi_{16} - \chi_{20},\ \phi_4 = \chi_4 - \phi_1 - \phi_2 - \phi_3.$$

Then for every $1 \neq t \in T_2$ we have $\phi_1(t) = -1$, $\phi_2(t) = \phi_3(t) = 1$ and $\phi_4(t) = 0$.

In turn, for every $1 \neq t \in T_1'$ we have $\phi_1(t) = -2$, $\phi_2(t) = \phi_3(t) = -4$, and $\phi_4(t) = 11$. So the result follows from Lemma 9.6.

(c) Let ℓ divide $|T_2|$. Then, by Hiss [29, Theorem 4.6], $|\operatorname{Irr}^0_\ell(G)| = 4$, and we have

$$\phi_1 = \chi_9 - 1_G,\ \phi_2 = \chi_7 - \chi_5,\ \phi_3 = \chi_8 - \chi_6,\ \phi_4 = \chi_4 - \chi_9 + 1_G.$$

Then for every $1 \neq t \in T_1$ we have $\phi_1(t) = -1$, $\phi_2(t) = \phi_3(t) = -2$, $\phi_4(t) = 2$, and for every $1 \neq t \in T_2'$ we have $\phi_1(t) = -2$, $\phi_2(t) = \phi_3(t) = 2$, and $\phi_4(t) = 3$. So the result follows from Lemma 9.6.

(ii) Next we consider the cases where Sylow ℓ-subgroups are not cyclic. Our main reference is [27]. Our ordering of the unipotent characters is as in [29] and [42], which is different from those in [27]. So we indicate by arrows the correspondence of our characters to those in [27]:

$$1_G = \chi_1 \to \chi_1,\ \chi_2 \to \chi_4,\ \chi_3 \to \chi_{18},\ \chi_4 \to \chi_{21},\ \chi_5 \to \chi_2,\ \chi_6 \to \chi_3,\ \chi_7 \to \chi_{19},$$
$$\chi_8 \to \chi_{20},\ \chi_9 \to \chi_5,\ \chi_{10} \to \chi_6,\ \chi_{11} \to \chi_7,\ \chi_{12} \to \chi_8,\ \chi_{13} \to \chi_9,\ \chi_{14} \to \chi_{10},$$
$$\chi_{15} \to \chi_{11},\ \chi_{16} \to \chi_{12},\ \chi_{17} \to \chi_{13},\ \chi_{18} \to \chi_{14},\ \chi_{19} \to \chi_{15},\ \chi_{20} \to \chi_{16},\ \chi_{21} \to \chi_{17}.$$

(d) $\ell|(q^2-1)$. Here $|\operatorname{Irr}^0_\ell(G)| = 0$, see [27, Table C.1].

(*e*) $3 \neq \ell|(q^2+1)$. Then $\operatorname{Irr}^0_\ell(G) = \{\phi_i : i = 4, 7, 18, 21\}$ in notation of [27]. We have
$\phi_4 = \chi_2 - 1_G$ so $\phi_4(t) = -1$ and for all $1 \neq t \in T_1 \cup T_2$.
$\phi_7 = \chi_{11} - \chi_2$ and $\phi_{18} = \chi_3 - a\chi_{21} - \chi_{11} + \chi_2$, where $2 \leq a \leq (q^2-2)/3$, see [27, Theorem 3.2]. Then $\phi_7(t) = \phi_{18}(t) = 0$ for all $1 \neq t \in T_1 \cup T_2$.

Furthermore, we have

$$\phi_{21} = \chi_4 - e\phi_{18} - d\chi_{21} - c\chi_{13} - b\chi_{12} - \phi_7 - \phi_4,$$

where $1 \leq b \leq (q+\sqrt{2q})/4$, $1 \leq c \leq (q-\sqrt{2q})/4$, $2 \leq d \leq (q^2+2)/3$, $2 \leq e \leq (q+2)/2$, so $\phi_{21}(t) = 2 - c$ for all $1 \neq t \in T_1$ and $\phi_{21}(t) = 2 - b$ for all $1 \neq t \in T_2$. In both the cases $\phi_{21}(t) < q(q^3+q^2+1)/12$, so the result follows by Lemma 9.6.

(f) $\ell = 3$. Then $(3, q^4 - q^2 + 1) = 1$ as $3|(q^2-1)$. In this case there are 22 unipotent Brauer characters and $\operatorname{Irr}^0_\ell(G) = \{\phi_{5,1}, \phi_i : i \in \{4, 7, 8, 10, 15, 18, 21\}$.

In this case the expressions of ϕ_i in terms of ordinary characters χ_j depend of parameters which are not determined in full but satisfy certain inequalities. These are

$$2 \leq a \leq q,\ 0 \leq b \leq (q+\sqrt{2q})/4,\ 0 \leq c \leq (q-\sqrt{2q})/4,\ 2 \leq d \leq q^2, 1 \leq e \leq (q+2)/2,$$
$$0 \leq x_7 \leq q/2,\ 0 \leq x_8 \leq (q+3\sqrt{2q}+4)/12,\ 0 \leq x_{10} \leq (q-2)/6,$$
$$1 \leq x_{15} \leq (q+1)/3,\ 0 \leq x_{18} \leq q(q-1),\ 1 \leq x_{21} \leq q^3.$$

We have

$\phi_4 = \chi_2 - 1_G$, so $\phi_4(t) = -1$ for $1 \neq t \in T_1 \cup T_2$;

$\phi_{5,1} = \chi_{20} - \chi_{21}$, whence $\phi_{5,1}(t) = 1$ for $1 \neq t \in T_1 \cup T_2$.

$\phi_8 = \chi_{12} - x_8\phi_{5,1}$. So $\phi_8(t) = 1 - x_8$ if $T = T_1$ and $-x_8$ if $T = T_2$. As $x_8 < q(q^3+q^2+1)/12$, Lemma 9.6 applies.

$\phi_7 = \chi_{11} - 1_G - \phi_4 - x_7\phi_{5,1} = \chi_{11} - \chi_2 - x_7\phi_{5,1}$. Note that $\chi_{11}(t) = 0$ for $1 \neq t \in T_1 \cup T_2$. So $\phi_7(t) = -x_7$, and $x_7 < q(q^3+q^2+1)/12$.

$\phi_{10} = \chi_{14} - \phi_8 - x_{10}\phi_{5,1}$. Note that $\chi_{14}(t) = 0$ for $1 \neq t \in T_1 \cup T_2$. So $\phi_{10}(t) = -x_8 - x_{10}$ if $t \in T_2$ and $1 - x_8 - x_{10}$ if $t \in T_1$. Here $x_8 + x_{10} \leq< (q + \sqrt{2q})/4 < q(q^3 + q^2 + 1)/12$.

$\phi_{15} = \chi_{19} - x_{15}\phi_{5,1}$, where $1 \leq x_{15} \leq (q+1)/3$. As $\chi_{19}(t) = 1$ for $1 \neq t \in T_1 \cup T_2$, we have $\phi_{15} = 1 - x_{15}$. As above, $x_{15} < q(q^3 + q^2 + q)/12$ yields the result.

$\phi_{18} = \chi_3 - \phi_7 - x_{18}\phi_{5,1} - a\phi_{15}$. So $\phi_{18}(t) = x_7 - x_{18} + a(x_{15} - 1)$ for $1 \neq t \in T_1 \cup T_2$. As $x_{18} + a \leq q^2 < q(q^3 + q^2 + 1)/12$, the result follows from Lemma 9.6.

$\phi_{21} = \chi_4 - \phi_4 - \phi_7 - x_{21}\phi_{5,1} - b\phi_8 - c\phi_{10} - d\phi_{15} - e\phi_{18}$. So $\phi_{21}(t) = 2 + x_7 + d(x_{15} - 1) - x_{21} - e(x_7 - x_{18} + a(x_{15} - 1)) - b\phi_8(t) - c\phi_{10}(t) = 2 + (1-e)x_7 + (d - ae)x_{15} - x_{21} + ex_{18} - a - d - b\phi_8(t) - c\phi_{10}(t)$. In addition, $\phi_8(t) = 1 - x_8$ if $T = T_1$ and $-x_8$ if $T = T_2$, and $\phi_{10}(t) = 1 - x_8 - x_{10}$ if $1 \neq t \in T_1$ and $-(x_8 + x_{10})$ if $t \in T_2$. So $-b\phi_8(t) - c\phi_{10}(t) = -b(1 - x_8) - c(1 - x_8 - x_{10}) = (b + c)(x_8 - 1) + cx_{10}$ if $1 \neq t \in T_1$, and $-b(-x_8) - c(-x_8 - x_{10}) = (b + c)x_8 + cx_{10}$ if $1 \neq t \in T_2$. So

$$\phi_{21}(t) = \begin{cases} 2 - a - d + (1-e)x_7 + (d - ae)x_{15} - x_{21} + ex_{18} + (b+c)(x_8 - 1) + cx_{10} & 1 \neq t \in T_1, \\ 2 - a - d + (1-e)x_7 + (d - ae)x_{15} - x_{21} + ex_{18} + (b+c)x_8 + cx_{10} & 1 \neq t \in T_2. \end{cases}$$

Therefore

$$-\phi_{21}(t) \leq a + d + (e-1)x_7 + (ae - d)x_{15} + x_{21} \leq q + q^2 + \frac{q^2}{4} + \frac{q(q-2)(q+3)}{6} + q^3 < 2q^3.$$

If $q > 8$ then $2q^3 < q(q^3 + q^2 + q)/12$, so the result follows by Lemma 9.6.

Let $q = 8$. If $t \in T_2$ then

$$-\phi_{21}(t) \cdot (|T| - 1) \leq 688 \cdot 36 = 24768 < q(q-1)(q^3 + q^2 + 1)\sqrt{q/2} = 112 \cdot 577 = 64624,$$

so Lemma 9.6 yields the result.

As ϕ_{21} is constant on $T_1 \smallsetminus \{1\}$, all non-trivial $|t|$-roots of unity are eigenvalue of $\phi_{21}(t)$. □

Remark 9.8. Observe that $|T_1| = 109$, $|T_2| = 37$. By [27, Corollary 4.2], $c = 1$ for $q = 8$. Then $-\phi_{21}(t) \leq -2 + a + d + (e-1)x_7 + (ae - d)x_{15} + x_{21} \leq -2 + q + q^2 + \frac{q^2}{2} + \frac{q^2+2q+4)(q+1)}{6} + q^3 = 6 + 64 + 32 + 111 + 512 = 725$ and $-\phi_{21}(t)(|T_1| - 1) \leq 78300$. The lower bound for $\dim\phi$ suggested in [57], see Lemma 9.5, is $q(q-1)(q^3 + q^2 + 1)\sqrt{q/2}$ for $q = 8$ yields $\dim\phi \geq 64624$. Note that if 1_{T_1} is not a constituent of $\phi|_{T_1}$ then $\dim\phi$ is a multiple of $|T_1| - 1 = 108$. As 64624 is not a multiple of 108, we conclude that $\dim\phi \geq 64692$. So if $t \in T_1$ then the question remains open.

Proof of Theorem 1.1. The result follows from Lemma 6.1 when $G = {}^2B_2(q)$, Lemma 6.2 when $G = {}^2G_2(q)$, Theorem 7.1 and Lemma 7.13 when $G = G_2(q)$, Theorem 8.1 when $G = {}^3D_4(q)$, and from Lemmas 9.2, 9.3, Proposition 9.4, and Remark 9.8 when $G = {}^2F_4(q)$. □

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