A question of Katz and Tiep on representations of finite general unitary groups

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Abstract

Katz and Tiep proved a characterization of the total Weil character of finite special unitary groups $\mathrm{SU}_n(q)$ for any integer $n \geq 3$ and any prime power q other than $(n, q) = (3, 2)$, in terms of the character values and irreducible constituents. They asked whether a similar characterization can be done for finite general unitary groups $\mathrm{GU}_n(q)$. In this article, we prove that such characterization for $\mathrm{GU}_n(q)$ can be done up to endomorphisms and tensoring by real linear characters.

Keywords Weil representations; Finite general unitary groups.
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1 Introduction

Fix a primitive $(q+1)$th root of unity $\epsilon \in \mathbb{C}$, where q is a prime power. Let τ be a generator of $\mathbb{F}_{q^6}^{\times}$. Also, let $\sigma = \tau^{q^4+q^2+1}$ and $\rho = \sigma^{q-1}$. Note that σ is a generator of $\mathbb{F}_{q^2}^{\times}$, and ρ is an element of $\mathbb{F}_{q^2}^{\times}$ of order $q+1$. Let $G = \mathrm{GU}_n(q)$ be the finite general unitary group, and let $\mathrm{Irr}(G)$ be the set of complex irreducible characters of G. G has a linear character $\lambda \in \mathrm{Irr}(G)$ of order $q+1$, defined as $\lambda(g) = \epsilon^t$ when $\det g = \rho^t$. Also, for each $i \in \mathbb{Z}$, there is an *irreducible Weil character* $\zeta_{n,q}^i \in \mathrm{Irr}(G)$ defined as

$$\zeta_{n,q}^i(g) = \frac{(-1)^n}{q+1} \sum_{l=0}^{q} \epsilon^{il} (-q)^{\dim \ker(g - \rho^l 1_W)}$$

where we identify G with $\mathrm{GU}(W)$ for a unitary vector space W. Note that $\zeta_{n,q}^i = \zeta_{n,q}^{i+q+1}$, so we can take i from $Q := \{0, 1, \ldots, q\}$ instead of $\mathbb{Z}$. The sum of all $q+1$ distinct irreducible Weil characters $\zeta_{n,q}^i$ is called the *total Weil character*

$$\zeta_{n,q}(g) = (-1)^n (-q)^{\dim_{\mathbb{F}_{q^2}} \ker(g - 1_W)}.$$

If we restrict the irreducible Weil characters $\zeta_{n,q}^i$ to $\mathrm{SU}_n(q)$, we obtain the $q+1$ pairwise distinct *irreducible Weil characters* ζ_n^i, $i \in Q$, of $\mathrm{SU}_n(q)$ if $n \geq 3$. For the proofs of these facts, see [4].

Katz and Tiep [3] proved the following characterization of the total Weil character of $\mathrm{SU}_n(q)$:

Theorem 1.1. [3, Theorem 16.6] Let $L = \mathrm{SU}_n(q)$ with $n \geq 3$ and $(n, q) \neq (3, 2)$. Suppose ψ is a (not necessarily irreducible) complex character of L such that

(a) $\psi(1) = q^n$;

(b) $\psi(g) \in \{0, \pm q^i \mid 0 \leq i \leq n\}$ for all $g \in L$; and

(c) every irreducible constituent of ψ is among the $q+1$ irreducible Weil characters ζ_n^u, $u \in Q$, of L.

Then ψ is the total Weil character, that is, $\psi = \sum_{u=0}^{q} \zeta_n^u$.

They proved [3, Theorem 16.11] that the geometric monodromy group of certain hypergeometric sheaves over $\mathbb{G}_m/\mathbb{F}_p$ are isomorphic to $\mathrm{GU}_n(q)$, and the corresponding q^n-dimensional representations can be viewed as the total Weil character $\zeta_{n,q}$ up to automorphisms. Theorem 1.1 is used in the proof of this theorem to show that certain representations of $\mathrm{SU}_n(q)$ are total Weil representations up to automorphisms.

In the same paper [3], they posed the following question which extends Theorem 1.1 to $\mathrm{GU}_n(q)$:

Question. [3, Remark 16.7] Is the total Weil character $\zeta_{n,q} = \sum_{i=0}^{q} \zeta_{n,q}^i$ the only character of $\mathrm{GU}_n(q)$, whose irreducible constituents are among the $(q+1)^2$ irreducible Weil characters $\zeta_{n,q}^i \lambda^j$, $0 \leq i, j \leq q$, and which takes values only among $0, \pm q^l$, $0 \leq l \leq n$?

They gave a negative answer to this question by finding a family of $q+1$ endomorphisms (not necessarily isomorphisms) of $\mathrm{GU}_n(q)$, whose compositions with the total Weil representation produce representations which afford characters of the form $\sum_{i=0}^{q} \zeta_{n,q}^{i} \lambda^{ei}$, $e \in Q$. Such characters, and the characters obtained by tensoring them with a real linear character, have the properties in the question. However, they can still be considered to be close enough to the total Weil character. One might ask whether there are other examples of characters with these properties which are, in some sense, far from being a total Weil character.

In this paper, we prove that these examples are the only characters which have the properties in the question.

Theorem 1.2. Let $n \geq 3$ be an integer and q be any prime power with $(n,q) \neq (3,2)$. Suppose that ψ is a character of $G = \mathrm{GU}_n(q)$ with the following properties.

(a) For every $g \in G$, $\psi(g) \in \mathcal{V} = \{0\} \cup \{\pm q^i : 0 \leq i \leq n\}$.

(b) Every irreducible constituent of ψ is among the $(q+1)^2$ irreducible Weil characters $\zeta_{n,q}^{i} \lambda^{j}$.

Then $\psi = \sum_{i=0}^{q} \zeta_{n,q}^{i} \lambda^{ei}$ (or possibly $\psi = \sum_{i=0}^{q} \zeta_{n,q}^{i} \lambda^{ei+(q+1)/2}$ if q is odd) for some $0 \leq e \leq q$.

This theorem tells that although the question of Katz and Tiep does not have a completely positive answer, such characters are reasonably close to the total Weil characters. This might help, as Theorem 1.1 did, identifying the total Weil character and its variants in the study of monodromy groups in situations similar to ones studied in [3] .

2 Preliminary Results

One of the key ingredients of the proof of Theorem 1.1 given in [3] is the following branching formula, which allows us to compute the values of irreducible Weil characters of $\mathrm{GU}_n(q)$ at certain conjugacy classes in terms of the values of irreducible Weil characters of $\mathrm{GU}_3(q)$.

Lemma 2.1. [3, Lemma 16.5(i)] Let $n = m + l$ with $m, l \in \mathbb{Z}_{\geq 1}$. Then the restriction of $\zeta_{n,q}^{i}$ to the natural subgroup $\mathrm{GU}_m(q) \times \mathrm{GU}_l(q)$ of $\mathrm{GU}_n(q)$ is

$$\sum_{\substack{0 \leq r,s \leq q, \\ (q+1) \mid (r+s-i)}} \zeta_{m,q}^{r} \boxtimes \zeta_{l,q}^{s}$$

where $\boxtimes$ denotes the (outer) tensor product.

According to [2], the conjugacy classes of $\mathrm{GU}_3(q)$ and their values of irreducible Weil characters are as listed in Table 1 and Table 2 below.

Class	Canonical Matrix Form	Parameters
$C_1^{(k)}$	$\rho^k I$	$0 \le k \le q$
$C_2^{(k)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^k \end{pmatrix}$	$0 \le k \le q$
$C_3^{(k)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & 1 & \rho^k \end{pmatrix}$	$0 \le k \le q$
$C_4^{(k,l)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^l \end{pmatrix}$	$0 \le k, l \le q,\ k \ne l$
$C_5^{(k,l)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^l \end{pmatrix}$	$0 \le k, l \le q,\ k \ne l$
$C_6^{(k,l,m)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^l & \\ & & \rho^m \end{pmatrix}$	$0 \le k < l < m \le q$
$C_7^{(k,l)}$	$\begin{pmatrix} \rho^k & & \\ & \sigma^l & \\ & & \sigma^{-ql} \end{pmatrix}$	$0 \le k \le q,\ 1 \le l \le q^2-2$, $l \not\equiv 0 \mod q-1$; if $l' \equiv -ql \mod q^2-1$ then $C_7^{(k,l)} = C_7^{(k,l')}$.
$C_8^{(k)}$	$\begin{pmatrix} \tau^{k(q^3-1)} & & \\ & \tau^{q^2k(q^3-1)} & \\ & & \tau^{q^4k(q^3-1)} \end{pmatrix}$	$1 \le k \le q^3,\ k \not\equiv 0 \mod q^2-q+1$; if $k' \equiv q^2k$ or $q^4k \mod q^3+1$ then $C_8^{(k)} = C_8^{(k')}$.

Table 1: The conjugacy classes of $\mathrm{GU}_3(q)$, as described in [2].

For a monic polynomial $p(x) = \sum_{i=0}^m \alpha_i x^i \in \mathbb{F}_{q^2}[x]$ with $\alpha_0 \ne 0$, define

$$\widetilde{p}(x) = a_0^{-q} \sum_{i=0}^m \alpha_{m-i}^q x^i.$$

Note that $\widetilde{(\widetilde{p})} = p$. We will make use of the following characterization of

Class	$\zeta_{3,q}^0\lambda^j$	$\zeta_{3,q}^i\lambda^j$
$C_1^{(k)}$	$(q^2-q)\epsilon^{3jk}$	$(q^2-q+1)\epsilon^{ik+3jk}$
$C_2^{(k)}$	$-q\epsilon^{3jk}$	$-(q-1)\epsilon^{ik+3jk}$
$C_3^{(k)}$	0	ϵ^{ik+3jk}
$C_4^{(k,l)}$	$-(q-1)\epsilon^{2jk+jl}$	$-(q-1)\epsilon^{ik+2jk+jl}+\epsilon^{2jk+il+jl}$
$C_5^{(k,l)}$	ϵ^{2jk+jl}	$\epsilon^{ik+2jk+jl}+\epsilon^{2jk+il+jl}$
$C_6^{(k,l,m)}$	$2\epsilon^{j(k+l+m)}$	$\epsilon^{jk+jl+jm}(\epsilon^{ik}+\epsilon^{il}+\epsilon^{im})$
$C_7^{(k,l)}$	0	$\epsilon^{ik+jk-jl}$
$C_8^{(k)}$	$-\epsilon^{jk}$	0

Table 2: The values of characters $\zeta_{3,q}^i\lambda^j$, as described in [2].

the unitary matrices by Ennola [1] to show the existence of some conjugacy class.

Theorem 2.2. [1, p. 12] Let $A \in \mathrm{GL}_n(q^2)$ be a matrix with the characteristic polynomial $f_1^{k_1}f_2^{k_2}\cdots f_N^{k_N}$, where $f_1,\dots,f_N$ are distinct monic irreducible polynomials over $\mathbb{F}_{q^2}$ and $k_1,\dots,k_N \in \mathbb{Z}_{\geq 1}$. For each $i=1,\dots,N$, let ν_i be the (unordered) partition of the positive integer k_i, which can be obtained by computing a generalized Jordan canonical form of A and looking at the sizes of diagonal blocks corresponding to the irreducible factor f_i. Then, A is similar to a matrix of $\mathrm{GU}_n(q)$ if and only if for every index r, $1 \leq r \leq N$, there is exactly one index s, $1 \leq s \leq N$ such that $f_r = \widetilde{f_s}$ and $\nu_r = \nu_s$.

We will not need precise descriptions of the partitions ν_i and the generalized Jordan canonical form, as we will use this theorem only for the cases where $k_i = 1$ for all i.

3 Proof of Theorem 1.2

The main idea of the proof of Theorem 1.2 is the following theorem.

Theorem 3.1. Let $j_0 = 0, j_1, \dots, j_q \in Q$, and for each $a, b, i \in Q$, let $c_{a,b,i} \in Q$ be the unique number such that $c_{a,b,i} \equiv ai + bj_i \mod q+1$. Suppose that for each pair $a, b \in Q$, either

(i) there exists an integer $d_{a,b} \geq 1$ dividing $q+1$ such that

$$\sum_{i=0}^{q} t^{c_{a,b,i}} = d_{a,b}(t^{q+1}-1)/(t^{d_{a,b}}-1) = d_{a,b}(t^{q+1-d_{a,b}}+t^{q+1-2d_{a,b}}+\cdots+1),$$

or

(ii) $\sum_{i=0}^{q} t^{c_{a,b,i}} = q+1$, i.e. $c_{a,b,i} = 0$ for every $i = 1, \ldots, q$.

Then there exists $e \in Q$ such that $j_i \equiv ei \mod q+1$ for every $i \in Q$.

Proof. We divide the proof into several steps.

(1) We can assume $j_1 = 0$.

For each $i \in Q$, define j'_i to be the unique element of Q such that $j'_i \equiv j_i - ij_1 \mod q+1$. Fix a pair $a, b \in Q$. For each $i \in Q$, let $c'_{a,b,i} \in Q$ be the unique element of Q such that $c'_{a,b,i} \equiv ai + bj'_i \mod q+1$. Then

$$c'_{a,b,i} \equiv ai + bj'_i = ai + bj_i - bij_1 = (a - bj_1)i + bj_i \equiv c_{a',b,i} \mod q+1,$$

where $a' \in Q$ is the number such that $a' \equiv a - bj_1 \mod q+1$. Since $(c_{a',b,0}, \ldots, c_{a',b,q})$ satisfies either (i) or (ii), so does $(c'_{a,b,0}, \ldots, c'_{a,b,q})$. Therefore, the numbers j'_i satisfy the assumption in the statement of this theorem, so it is enough to show that if we have an additional assumption $j_1 = 0$, then $j_i = 0$ for every $i \in Q$.

(2) We can assume that there exists a prime p such that every j_i is a multiple of $d = (q+1)/p$.

Suppose that $j_1 = 0$ but $j_i \neq 0$ for at least one $i \in Q$. Let M be the smallest positive integer such that for every $i \in Q$, $Mj_i \equiv 0 \mod q+1$. Since we assumed that $j_i \neq 0$ for at least one i, we can see that $M > 1$, so $M = mp$ for some integer $m \geq 1$ and a prime p. For each i, let $j'_i \in Q$ be the unique element such that $j'_i \equiv mj_i \mod q+1$. Note that every j'_i is a multiple of $d = (q+1)/p$, but some j'_i is nonzero by the minimality of M. For each $a, b, i \in Q$, let $c'_{a,b,i} \in Q$ be the unique element such that $c'_{a,b,i} \equiv ai + bj'_i \mod q+1$. Then

$$c'_{a,b,i} \equiv ai + bj'_i \equiv ai + bmj_i \equiv c_{a,b',i} \mod q+1$$

where $b' \equiv mb \mod q+1$. By the same reason as in (1), the numbers j'_i satisfy the assumption. If we can prove that these j'_i are all zero, then this contradicts our earlier observation that some of them are nonzero. Therefore, it is enough to prove the following:

If $j_0 = j_1 = 0, j_2, \ldots, j_q \in Q$ satisfy the condition of the theorem, and if there exists some prime p such that every j_i is a multiple of $d = (q+1)/p$, then $j_i = 0$ for all i.

For each $k \in \mathbb{Z}_{\geq 0}$, let P_k be the condition

$$P_k : p^k < d,\ p^k \mid d \text{ and } j_{\frac{n_1 d}{p^{k-1}}+r} = j_{\frac{n_2 d}{p^{k-1}}+r} \text{ for every } 0 \leq n_1, n_2 \leq p^k - 1 \text{ and } 0 \leq r \leq \frac{d}{p^{k-1}} - 1.$$

Assume that $d > 1$ (I will prove this in part (4)). Then P_0 says nothing more than $d > p^0 = 1$ since $0 \leq n_1, n_2 \leq p^0 - 1 = 0$ forces $n_1 = n_2$, so P_0 is true. I will use induction on k to prove that P_k is true for every k. Assume that $k \geq 1$ and P_{k-1} is true.

(3) $j_{n_1 d/p^{k-1}+r} = j_{n_2 d/p^{k-1}+r}$ for all $n_1, n_2 \in \{0, \ldots, p^k - 1\}$ and $0 \leq r \leq d/p^{k-1} - 1$.

Suppose that $n_1 \neq n_2 \in \{0, \ldots, p^k - 1\}$ and $r \in \{0, \ldots, d/p^{k-1} - 1\}$. Let $k_1 = j_{n_1 d/p^{k-1}+r}/d$, $k_2 = j_{n_2 d/p^{k-1}+r}/d$, and suppose that $k_1 \neq k_2$. Note that $0 \leq k_1 = j_{n_1 d/p^{k-1}+r}/d < (q+1)/d = p$ and similarly $0 \leq k_2 < p$, so $k_1 \neq k_2$ implies $k_1 \not\equiv k_2 \mod p$. Therefore, there exists $u \in Q$ such that $u(k_1 - k_2) \equiv 1 \mod p$. Consider the pair $(a, b) = (p^{k-1}, b)$ where $b \equiv -u(n_1 - n_2) \mod q+1$. Then $c_{p^{k-1},b,1} = p^{k-1} \neq 0$, so condition (i) holds for this pair. Moreover, $d_{p^{k-1},b}$ divides $c_{p^{k-1},b,1} = p^{k-1}$. On the other hand, by P_{k-1}, p^{k-1} divides d, which divides j_i for every $i \in Q$. Since

$$c_{p^{k-1},b,i} \equiv p^{k-1} i - u(n_1 - n_2) j_i \mod q+1,$$

$c_{p^{k-1},b,i}$ is also divisible by p^{k-1}. Therefore, $d_{p^{k-1},b} = p^{k-1}$ and every multiple of p^{k-1} in Q appears exactly p^{k-1} times among $c_{p^{k-1},b,0}, \ldots, c_{p^{k-1},b,q}$. However,

$$\begin{aligned} c_{p^{k-1},b,\frac{n_1 d}{p^{k-1}}+r} - c_{p^{k-1},b,\frac{n_2 d}{p^{k-1}}+r} &\equiv (n_1 - n_2)d + b(j_{\frac{n_1 d}{p^{k-1}}+r} - j_{\frac{n_2 d}{p^{k-1}}+r}) \\ &\equiv (n_1 - n_2)d - u(n_1 - n_2)(k_1 - k_2)d \\ &\equiv 0 \mod q+1. \end{aligned}$$

Since they are both in Q, $c_{p^{k-1},b,n_1 d/p^{k-1}+r} = c_{p^{k-1},b,n_2 d/p^{k-1}+r}$. Moreover, if $n_1 = pm_1 + l_1$ and $n_2 = pm_2 + l_2$ for $m_1, m_2 \in \{0, \ldots, p^{k-1} - 1\}$ and $l_1, l_2 \in \{0, \ldots, p-1\}$, then by P_{k-1},

$$c_{p^{k-1},b,\frac{md}{p^{k-2}}+\frac{l_1 d}{p^{k-1}}+r} = c_{p^{k-1},b,\frac{n_1 d}{p^{k-1}}+r} = c_{p^{k-1},b,\frac{n_2 d}{p^{k-1}}+r} = c_{p^{k-1},b,\frac{m' d}{p^{k-2}}+\frac{l_2 d}{p^{k-1}}+r}$$

for every $m, m' \in \{0, \dots, p^{k-1}-1\}$. Also, since $k_1 \neq k_2$, by P_{k-1} we must have $l_1 d/p^{k-1} \neq l_2 d/p^{k-1}$. Hence, this number appears at least $2p^{k-1}$ times among $c_{p^{k-1},b,0}, \dots, c_{p^{k-1},b,q}$, which is impossible. Therefore $k_1 = k_2$, so $j_{nd/p^{k-1}+r}$ does not depend on the choice of n.

(4) d is divisible by p^k, and $d > p^k$.

If we choose $a = 0$ and $b = 1$, we get $c_{0,1,i} = j_i$. Since $j_i \neq 0$ for some i, this pair $(a, b) = (0, 1)$ satisfies the condition (i), so

$$\sum_{i=0}^{q} t^{j_i} = d_{0,1}(t^{q+1-d_{0,1}} + t^{q+1-2d_{0,1}} + \cdots + 1).$$

In particular, there are exactly $d_{0,1}$ copies of each multiple of $d_{0,1}$ among $j_0, \dots, j_q$. Since they are multiples of d and $p = (q+1)/d$ is a prime, it follows that $d_{0,1} = d$. Moreover, by part (3), whenever we have $j_r = 0$ for some $0 \leq r \leq d/p^{k-1} - 1$, we also have $j_{nd/p^{k-1}+r} = 0$ for every $n \in \{0, \dots, p^k - 1\}$. In particular, the number of 0 among j_is, which is d, is precisely p^k times $|\{r : 0 \leq r \leq d/p^{k-1} - 1, j_r = 0\}|$. By the assumption $j_0 = j_1 = 0$, the number of such r is at least 2, so d is strictly larger than p^k. Note that by only using the facts $j_0 = j_1 = 0$ and $d = |\{i \in Q \mid j_i = 0\}|$, we can see that $d \geq 2$ as we assumed before part (3).

By parts (3) and (4), P_{k-1} implies P_k. By induction, P_k is true for all $k \in \mathbb{Z}_{\geq 0}$. In particular, d is divisible by p^k for all k, which is impossible. Therefore, our assumption that $j_i \neq 0$ for some i cannot be true. By parts (1) and (2), this completes the proof. $\square$

We will apply Theorem 3.1 on the exponents of λ appearing in the $q + 1$ irreducible constituents $\zeta_{n,q}^i \lambda^j$ of ψ. To do this, we need to first check whether these exponents satisfy the conditions of Theorem 3.1. This will be done by applying the following lemma for the case $r = q$.

Lemma 3.2. Let r be a positive integer. Suppose that a polynomial $f(t) = \sum_{i=0}^{r} c_i t^i \in \mathbb{Z}[t]$ has the following properties:

(1) $c_0 > 0$ and $c_i \geq 0$ for every $i \in Q$.

(2) $f(1) = r + 1$.

(3) There is an integer a relatively prime to $r+1$, such that if ϵ is a primitive $(r+1)$th root of unity in $\mathbb{C}$, then $f(\epsilon^n) \in \{0, r+1\} \cup \{\pm a + (r+1)m \mid m \in \mathbb{Z}\}$ for every integer n.

Then either $f(t) = r+1$ or $f(t) = d(t^{r+1-d} + \cdots + t^d + 1) = d(t^{r+1} - 1)/(t^d - 1)$ for some $d > 0$ dividing $r+1$.

Proof. We first show that there is no integer n such that $f(\epsilon^n) \equiv \pm a \mod r+1$. Consider the cyclic group $G = \langle \epsilon \rangle \leq \mathbb{C}^*$. Let $\lambda_i \in \mathrm{Irr}(G)$ be the linear character defined by $\lambda_i(\epsilon) = \epsilon^i$. Then the restriction of f to G is the character $\chi = \sum_{i=0}^r c_i \lambda_i$ of G. Note that since the irreducible characters are orthonormal with respect to the usual inner product $(\cdot, \cdot)$,

$$\frac{\sum_{j=0}^r |\chi(\epsilon^j)|^2}{r+1} = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 = (\chi, \chi) = (\sum_{i=0}^r c_i \lambda_i, \sum_{i=0}^r c_i \lambda_i) = \sum_{i=0}^r c_i^2 \in \mathbb{Z}.$$

In particular, the integer $\sum_{j=0}^r |\chi(\epsilon^j)|^2$ is divisible by $r+1$. Since $\chi(\epsilon^j)^2 \equiv 0$ or $a^2 \mod r+1$ and a^2 is relatively prime to $r+1$, the number $|\{j \mid j \in \{0, \ldots, r\}, \chi(\epsilon^j) \equiv \pm a \mod r+1\}|$ must be divisible by $r+1$. Since $\chi(1) = f(1) = r+1 \not\equiv \pm a \mod r+1$, so the above number is not $r+1$, hence it must be 0. Therefore, $f(\epsilon^n) \in \{0, r+1\}$ for every integer n.

Assume $f(t) \neq r+1$. Let $1 \leq s \leq r+1$ be an integer such that

$$f(\epsilon^s) = c_r \epsilon^{rs} + \cdots + c_0 = r+1.$$

We also know that

$$|c_r \epsilon^{rs}| + \cdots + |c_0| = c_r + \cdots + c_0 = f(1) = r+1.$$

Therefore, $c_i \epsilon^{is} = c_i$ for every $i \in Q$. Hence, $c_i = 0$ unless is is divisible by $r+1$. This happens if and only if i is a multiple of d, where $d = (r+1)/gcd(r+1, s)$. We may assume that our choice of s maximizes this d. Note that $f(\epsilon^{(r+1)/d}) = r+1$, so we may assume that $s = (r+1)/d$.

If s' is another integer such that $f(\epsilon^{s'}) = 0$ and $d' = (r+1)/gcd(r+1, s')$, then $c_i = 0$ unless it is a multiple of $lcm(d, d')$. But then $s'' = (r+1)/lcm(d, d')$ satisfies $f(\epsilon^{s''}) = 0$ and $(r+1)/s'' = lcm(d, d')$, so by the maximality of d, d' must divide d. Therefore, $f(\epsilon^n) = r+1$ if and only if n is a multiple of s. In particular, $f(t) - (r+1)$ is divisible by $t^d - 1$ and $f(t)$ is divisible by $(t^{r+1} - 1)/(t^d - 1) = t^{r+1-d} + t^{r+1-2d} + \cdots + 1$.

If we write $f(t) = g(t)(t^{r+1-d} + t^{r+1-2d} + \cdots + 1)$, then since $\deg g = \deg f - (r+1-d) \le r - (r+1-d) = d-1$, we can compare the terms of degree $\le d-1$ in both sides to see that $g(t) = c_{d-1}t^{d-1} + \cdots + c_0 = c_0$. Also, $r+1 = f(1) = c_0(r+1)/d$, so $g(t) = c_0 = d$ and $f(t) = d(t^{r+1}-1)/(t^d-1)$. □

Now we are ready to prove Theorem 1.2, which we restate here.

Theorem. Let $n \ge 3$ be an integer and q be any prime power with $(n,q) \ne (3,2)$. Suppose that ψ is a character of $G = \mathrm{GU}_n(q)$ with the following properties.

(a) For every $g \in G$, $\psi(g) \in \mathcal{V} = \{0\} \cup \{\pm q^i : 0 \le i \le n\}$.

(b) Every irreducible constituent of ψ is among $\zeta_{n,q}^i \lambda^j$.

Then $\psi = \sum_{i=0}^q \zeta_{n,q}^i \lambda^{ei}$ (or possibly $\psi = \sum_{i=0}^q \zeta_{n,q}^i \lambda^{ei+(q+1)/2}$ if q is odd) for some $0 \le e \le q$.

Proof. We first assume that $q \ge 4$. The cases $q \le 3$ will be proved later.

(1) $\psi = \sum_{i=0}^q \zeta_{n,q}^i \lambda^{j_i}$ for some $0 \le j_i \le q$.

To prove this, we mimic the proof of Theorem 1.1 given in [3]. In fact, this could be done by merely restricting ψ to $\mathrm{SU}_n(q)$ and applying Theorem 1.1. We included the details in order to fix notations and do some computations which will be used later.

By assumption (b), we can write $\psi = \sum_{i=0}^q \sum_{j=0}^q a_{i,j}\zeta_{n,q}^i \lambda^j$ for some non-negative integers $a_{i,j}$. Define $b_i = \sum_{j=0}^q a_{i,j}$. To find b_0, note that

$$\begin{aligned} q^n \ge \psi(1) &= \sum_{i=0}^q \sum_{j=0}^q a_{i,j}\zeta_{n,q}^i \lambda^j(1) = \sum_{i=0}^q b_i \zeta_{n,q}^i(1) \\ &= b_0 \frac{q^n + q(-1)^n}{q+1} + \sum_{i=1}^q b_i \frac{q^n - (-1)^n}{q+1}. \end{aligned}$$

The above number is a power of q by condition (a), while the degrees of the characters $\zeta_{n,q}^i$ are not powers of q, so we must have $\sum_{i=0}^q b_i > 1$. Then $\psi(1) \ge 2\frac{q^n-q}{q+1} > q^{n-1}$, which forces $\psi(1) = q^n$. It follows that

$$b_0 - 1 = \frac{q^n - (-1)^n}{q+1}(-1)^n \left(q + 1 - \sum_{i=0}^q b_i \right).$$

In particular, $b_0 - 1$ is divisible by $(q^n - (-1)^n)/(q+1)$. On the other hand,

$$-1 \leq b_0 - 1 \leq \frac{\psi(1)}{\zeta^0_{n,q}(1)} - 1 = \frac{q^n(q+1)}{q^n + q(-1)^n} - 1 < \frac{q^n - (-1)^n}{q+1}$$

since $(n,q) \neq (3,2)$ and $n \geq 3$. Therefore $b_0 = 1$ and $\sum_{i=0}^{q} b_i = q+1$.

Let $G = GU(\mathbb{F}^n_{q^2})$ be acting on a n-dimensional $\mathbb{F}_{q^2}$-vector space with a G-invariant nondegenerate Hermitian form. Let $H \cong \mathrm{GU}_3(q)$ be a subgroup of G acting trivially on a nondegenerate $(n-3)$-dimensional subspace. By Lemma 2.1,

$$\begin{aligned}
\psi_H &= \sum_{i=0}^{q}\sum_{j=0}^{q} a_{i,j}(\zeta^i_{n,q}\lambda^j)_H \\
&= \sum_{i=0}^{q}\sum_{j=0}^{q} a_{i,j}\left(\lambda^j_H \sum_{\substack{0\leq r,s\leq q\\ (q+1)|(r+s-i)}} (\zeta^r_{3,q} \boxtimes \zeta^s_{n-3,q})_H\right) \\
&= \sum_{i=0}^{q}\sum_{j=0}^{q} a_{i,j}\left(\lambda^j_H \sum_{\substack{0\leq r,s\leq q\\ (q+1)|(r+s-i)}} \zeta^s_{n-3,q}(1)\zeta^r_{3,q}\right) \\
&= \sum_{r=0}^{q}\sum_{j=0}^{q}\left(\sum_{i=0}^{q} a_{i,j}\zeta^{i-r}_{n-3,q}(1)\right)\zeta^r_{3,q}\lambda^j_H \\
&= \sum_{r=0}^{q}\sum_{j=0}^{q}\left(\sum_{i=0}^{q} a_{i,j}\frac{q^{n-3}-(-1)^{n-3}}{q+1} + (-1)^{n-3}a_{r,j}\right)\zeta^r_{3,q}\lambda^j_H \\
&= \sum_{r=0}^{q}\sum_{j=0}^{q} c_{r,j}\zeta^r_{3,q}\lambda^j_H
\end{aligned}$$

where $c_{r,j} = \sum_{i=0}^{q} a_{i,j}\frac{q^{n-3}-(-1)^{n-3}}{q+1} + (-1)^{n-3}a_{r,j}$. Let $d_i = \sum_{j=0}^{q} c_{i,j}$. Note that since $b_0 = 1$ and $\sum_{i=1}^{q} b_i = q$, we have

$$\begin{aligned}
d_0 = \sum_{j=0}^{q} c_{0,j} &= \sum_{j=0}^{q}\sum_{i=0}^{q} a_{i,j}\frac{q^{n-3}-(-1)^{n-3}}{q+1} + (-1)^{n-3}\sum_{j=0}^{q} a_{0,j} \\
&= q^{n-3} - (-1)^{n-3} + (-1)^{n-3} = q^{n-3}
\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{q} d_i =& \sum_{i=1}^{q}\sum_{j=0}^{q} c_{i,j}\\ =& \sum_{i=1}^{q}\sum_{j=0}^{q}\left(\sum_{r=0}^{q} a_{r,j}\frac{q^{n-3}-(-1)^{n-3}}{q+1}+(-1)^{n-3}a_{i,j}\right)\\ =& q^{n-2}-(-1)^{n-3}q+(-1)^{n-3}q\\ =& q^{n-2}.\end{aligned}$$

By evaluating ψ_H at $g'_k \in C_7^{(k,k)}$ (or $C_7^{(k,k+q+1)}$ if $k \equiv 0 \mod q-1$), we get:

$$\psi(g'_k) = \sum_{i=0}^{q}\sum_{j=0}^{q} c_{i,j}\zeta_{3,q}^{i}(g'_k)\lambda^j(g'_k) = \sum_{i=1}^{q}\sum_{j=0}^{q} c_{i,j}\epsilon^{ik} = \sum_{i=1}^{q} d_i\epsilon^{ik}.$$

On the other hand, by evaluating ψ_H at $g_k \in C_6^{(0,k,-k)}$ for $0 < k < (q+1)/2$ (note that $\det g = 1$), we get:

$$\begin{aligned}\psi(g_k) &= \sum_{i=0}^{q}\sum_{j=0}^{q} c_{i,j}\zeta_{3,q}^{i}\lambda^j(g_k)\\ &= \sum_{j=0}^{q} 2c_{0,j}\epsilon^{j(0+k+(-k))} + \sum_{i=1}^{q}\sum_{j=0}^{q} c_{i,j}\epsilon^{j(0+k+(-k))}(\epsilon^{i0}+\epsilon^{ik}+\epsilon^{-ik})\\ &= \sum_{j=0}^{q} 2c_{0,j} + \sum_{i=1}^{q}\sum_{j=0}^{q} c_{i,j}(1+\epsilon^{ik}+\epsilon^{-ik})\\ &= 2d_0 + \sum_{i=1}^{q} d_i + \sum_{i=1}^{q} d_i\epsilon^{ik} + \sum_{i=1}^{q} d_i\epsilon^{-ik}\\ &= 2q^{n-3}+q^{n-2}+\psi(g'_k)+\overline{\psi(g'_k)}.\end{aligned}$$

By assumption (a), $\psi(g_k), \psi(g'_k) \in \mathcal{V}$ (so $\psi(g'_k) = \overline{\psi(g'_k)}$). Moreover, from the above observations, we know that $|\psi(g'_k)| \leq \sum_{i=1}^{q} d_i = q^{n-2}$ and $|\psi(g_k)| \leq 2q^{n-3}+3q^{n-2} < q^{n-1}$ (we assumed $q \geq 4$). Therefore, $\sum_{i=1}^{q} d_i\epsilon^{ik} = \psi(g'_k) = -q^{n-3}$ and $\psi(g_k) = q^{n-2}$. In particular, the polynomial

$$\sum_{i=0}^{q} d_i t^i = \sum_{i=0}^{q}\sum_{j=0}^{q} c_{i,j}t^i = \sum_{i=0}^{q}\sum_{j=0}^{q}(q^{n-3}+(-1)^n(1-a_{i,j}))t^i$$

has a zero at each ϵ^k for $0 < k < q+1$, $k \neq (q+1)/2$. Since $\sum_{i=0}^{q}\sum_{j=0}^{q}(q^{n-3}+(-1)^n)t^i$ also vanishes at each ϵ^k, the same is true for the polynomial

$$\sum_{i=0}^{q}\sum_{j=0}^{q} a_{i,j}t^i = \sum_{i=0}^{q} b_i t^i.$$

If $q+1$ is odd, then this polynomial is divisible by $(t^{q+1}-1)/(t-1) = t^q + t^{q-1} + \cdots + 1$. Comparing the degrees and the value at 1 (=sum of coefficients), we get $b_i = 1$ for $i = 0, \ldots, q$. If $q+1$ is even, then the polynomial is divisible by $(t^{q+1}-1)/(t^2-1) = t^{q-1}+t^{q-3}+\cdots+1$. Again, by comparing the degrees and the values at 0 and 1, we get $b_i = 1$ for $i = 0, \ldots, q$. The nonnegativity of $a_{i,j}$'s now implies that for each $i = 0, \ldots, q$, there exists j_i such that $a_{i,j_i} = 1$ and $a_{i,j} = 0$ for all $j \neq j_i$, so that

$$\psi = \sum_{i=0}^{q} \zeta_{n,q}^i \lambda^{j_i}.$$

(2) $j_0 = 0$ or $(q+1)/2$.

We will find a monic polynomial $f \in \mathbb{F}_{q^2}[x]$ of degree n, which is either an irreducible polynomial such that $\tilde{f} = f$, or a product of two irreducible monic polynomials f_1, f_2 such that $\tilde{f_1} = f_2$. We also require its roots to be distinct elements none of which is a power of ρ and whose product is ρ. Such an f, if it exists, satisfies the condition of Theorem 2.2, so there exists a matrix $B \in G$ whose characteristic polynomial is f. By looking at $\psi(B)$, we will be able to see that $j_0 = 0$ or $(q+1)/2$. The following constructions of f are motivated by [5, Proposition 7].

If n is odd, let $\theta \in \mathbb{F}_{q^{2n}}$ be a primitive (q^n+1)th root of unity such that $\rho = \theta^{(q^n+1)/(q+1)}$. Let $f(x) \in \mathbb{F}_{q^2}[x]$ be the minimal polynomial of θ over $\mathbb{F}_{q^2}$. Then $\deg f = |\mathbb{F}_{q^{2n}} : \mathbb{F}_{q^2}| = n$ since q^n+1 does not divide $q^m - 1$ for any $m < 2n$, so that θ is not contained in any proper subfield of $\mathbb{F}_{q^{2n}}$. Also, every other root of f must be another primitive (q^n+1)th root of unity. Moreover, $z \mapsto z^{q^2}$ is an automorphism of $\mathbb{F}_{q^{2n}}$ which acts trivially on $\mathbb{F}_{q^2}$, so $\theta^{q^{2d}}$ for $d = 0, 1, \ldots, n-1$ are also roots of f. They are distinct, since if $\theta^{q^{2d_1}} = \theta^{q^{2d_2}}$ for some $0 \leq d_1 < d_2 \leq n-1$, then $\theta^{(q^{2d_2-2d_1}-1)q^{2d_1}} = 1$, so $\theta^{q^{2d_2-2d_1}-1} = 1$, which is impossible since q^n+1 does not divide q^m-1 for any $m < 2n$. Hence,

$$f = \prod_{d=0}^{n-1}(x - \theta^{q^{2d}}).$$

Note that $\theta^{q^n} = \theta^{-1}$, so $\theta^{-q^{2d+1}} = \theta^{q^{n+2d+1}} = \theta^{q^{-n+2d+1}}$. From this, we can see that

$$\begin{aligned}
\tilde{f}(x) &= \left(\prod_{d=0}^{n-1}(-\theta^{q^{2d}})^{-q}\right) x^n \prod_{d=0}^{n-1}(\frac{1}{x} + (-\theta^{q^{2d}})^q) \\
&= x^n \prod_{d=0}^{n-1}(1 + \frac{(-\theta^{q^{2d}})^{-q}}{x}) \\
&= \prod_{d=0}^{n-1}(x - \theta^{-q^{2d+1}}) \\
&= \left(\prod_{d=0}^{(n-1)/2-1}(x - \theta^{q^{n+2d+1}})\right)\left(\prod_{d=(n-1)/2}^{n-1}(x - \theta^{q^{-n+2d+1}})\right) = f(x).
\end{aligned}$$

Let $A \in GL_n(q^2)$ be the companion matrix of f. By Theorem 2.2, this matrix is similar to an element $B \in G$. Moreover, its eigenvalues are precisely the roots of f, which are $\theta^{q^{2d}}$ for $d = 0, \ldots, n-1$. Then none of these is a power of ρ, and

$$\begin{aligned}
\det B &= \theta^{q^{2(n-1)}+q^{2(n-2)}+\cdots+1} = \theta^{(q^{2n}-1)/(q^2-1)} = \theta^{(q^n+1)(q^n-1)/(q+1)(q-1)} \\
&= \rho^{q^{n-1}+q^{n-2}+\cdots+1} = \rho,
\end{aligned}$$

so

$$\psi(B) = \sum_{i=0}^{q} \zeta_{n,q}^i(B)\lambda^{j_i}(B) = (-1)^n \epsilon^{j_0} \in \mathcal{V} \subset \mathbb{R}.$$

Therefore $\epsilon^{j_0} \in \mathbb{R}$, so $j_0 = 0$ (or $(q+1)/2$, if q is odd).

If n is even, let $\theta \in \mathbb{F}_{q^n}$ be a primitive (q^n-1)th root of unity such that $\theta^{(q^n-1)/(q+1)} = \rho^{-1}$, and let $f(x) \in \mathbb{F}_{q^2}[x]$ be the minimal polynomial of θ over $\mathbb{F}_{q^2}$. By the same logic, $\theta^{q^{2d}}$ for $d = 0, \ldots, n/2-1$ are distinct roots of f, and $\deg f = |\mathbb{F}_{q^n} : \mathbb{F}_{q^2}| = n/2$, so $f(x) = \prod_{d=0}^{n/2-1}(x - \theta^{q^{2d}})$. Also,

$$\begin{aligned}\tilde{f}(x) &= \left(\prod_{d=0}^{n/2-1}(-(\theta^{q^{2d}})^{-q})\right) x^{n/2} \prod_{d=0}^{n/2-1}(\frac{1}{x} + (-\theta^{q^{2d}})^q) \\ &= x^{n/2} \prod_{d=0}^{n/2-1}(1 + \frac{(-\theta^{q^{2d}})^{-q}}{x}) \\ &= \prod_{d=0}^{n/2-1}(x - \theta^{-q^{2d+1}}) \\ &= \prod_{d=0}^{n/2-1}(x - \theta^{q^{n+2d-1}}) \neq f(x)\end{aligned}$$

since $\theta^{q^{n-1}}$ is not among $\theta, \theta^{q^2}, \ldots, \theta^{q^{n-2}}$. Let $h = f\tilde{f}$. Then h satisfies the condition of Theorem 2.2, so there is a matrix $B \in G$ which is similar to the direct sum of companion matrices of f and $\tilde{f}$ in $GL_n(q^2)$. In particular, the eigenvalues of B are $\theta^{q^{2d}}$ and $\theta^{-q^{2d+1}}$, where $d = 0, \ldots, n/2-1$. None of them is a power of ρ, and

$$\det B = \theta^{q^{n-2}+q^{n-4}+\cdots+1-q-q^3-\cdots-q^{n-1}} = \theta^{(-q^n+1)/(q+1)} = \rho.$$

Hence,

$$\psi(B) = \sum_{i=0}^{q} \zeta_{n,q}^i(B)\lambda^{j_i}(B) = (-1)^n\epsilon^{j_0} \in \mathcal{V} \subset \mathbb{R}.$$

Therefore, $\epsilon^{j_0} \in \mathbb{R}$, so $j_0 = 0$ (or $(q+1)/2$, if q is odd).

If q is odd, then the assumptions (a) and (b) about ψ also holds for $\psi\lambda^{(q+1)/2}$. Therefore, we can safely assume that $j_0 = 0$.

(3) Applying Lemma 3.2 and Theorem 3.1.

Our plan is to show that for each $a, b, k \in Q$,

$$\sum_{i=0}^{q} \epsilon^{aik+bj_ik} \in \{0, 1, -1, q, -q, q+1\}. \qquad (*)$$

Once we have this, we can apply Lemma 3.2 to the polynomial

$$f_{a,b}(t) = \sum_{i=0}^{q} t^{c_{a,b,i}}$$

where $c_{a,b,i} \in Q$ is the unique element such that $c_{a,b,i} \equiv ai + bj_i \mod q+1$. This polynomial clearly satisfies conditions (1) and (2) of Lemma 3.2, and the condition (3) is also satisfied since

$$\begin{aligned} f_{a,b}(\epsilon^k) = \sum_{i=0}^{q} \epsilon^{aik+bj_ik} &\in \{0, 1, -1, q, -q, q+1\} \\ &\subseteq \{0, q+1\} \cup \{\pm 1 + (q+1)m \mid m \in \mathbb{Z}\}. \end{aligned}$$

Therefore, the numbers $j_0, \dots, j_q$ satisfy the condition of Theorem 3.1, so there exists some integer $0 \le e \le q$ such that $j_i \equiv ei \mod q+1$ for every $i \in Q$, and

$$\psi = \sum_{i=0}^{q} \sum_{j=0}^{q} a_{i,j} \zeta_{n,q}^i \lambda^j = \sum_{i=0}^{q} \zeta_{n,q}^i \lambda^{ei}$$

which is the conclusion of this theorem.

To show $(*)$, we will use some relations between the values of ψ_H at certain conjugacy classes of H: $C_8^{(bk)}$, $C_2^{(ak)}$, $C_3^{(ak)}$, $C_7^{(ak,(a-b)k)}$, $C_7^{((b-2a)k,-2ak)}$, $C_4^{(ak,(b-2a)k)}$, and $C_5^{(ak,(b-2a)k)}$. Here, when the parameters for some conjugacy classes are out of the ranges given in Table 1, then add some (possibly negative) integer multiples of $q+1$ to those parameters so that each of the new parameters is in the range given in Table 1 for the corresponding conjugacy class. This modification will not affect what follows.

Recall that in part (1), we saw that

$$a_{i,j} = \delta_{j,j_i}$$

and

$$\psi_H = \sum_{i=0}^{q} \sum_{j=0}^{q} c_{i,j} \zeta_{3,q}^i \lambda_H^j$$

where

$$c_{i,j} = \sum_{r=0}^{q} a_{r,j} \frac{q^{n-3} - (-1)^{n-3}}{q+1} + (-1)^{n-3} a_{i,j}.$$

Let $b, k \in Q$, and let $g_0 \in C_8^{(bk)}$. We will deal with the cases where $\psi_H(g_0) = 0$ and $\psi_H(g_0) \neq 0$ separately.

(4) Proof of $(*)$ for the cases where $\psi_H(g_0) = 0$.

Suppose that $\psi_H(g_0) = 0$ for $g_0 \in C_8^{(bk)}$. For $a \in Q$, let $h_1 \in C_7^{(ak,(a-b)k)}$. Then according to Table 2 and the equalities we just recalled,

$$\begin{aligned} 0 = \psi_H(g_0) &= \sum_{i=0}^{q}\sum_{j=0}^{q} c_{i,j}\zeta_{3,q}^{i}(g_0)\lambda_H^{j}(g_0) \\ &= \sum_{j=0}^{q} -c_{0,j}\epsilon^{bjk} \\ &= -\sum_{j=0}^{q}(\sum_{r=0}^{q} a_{r,j}\frac{q^{n-3}-(-1)^{n-3}}{q+1} + (-1)^{n-3}a_{0,j})\epsilon^{bjk} \\ &= -(-1)^{n-3} - \frac{q^{n-3}-(-1)^{n-3}}{q+1}\sum_{i=0}^{q}\epsilon^{bj_ik}. \end{aligned}$$

When $n = 3$, then $\frac{q^{n-3}-(-1)^{n-3}}{q+1} = 0$, so this cannot happen. So $n > 3$ and

$$-(-1)^{n-3}\frac{q+1}{q^{n-3}-(-1)^{n-3}} = \sum_{i=0}^{q}\epsilon^{bj_ik}.$$

This is an algebraic integer which is also a rational number, so it is an integer. If $n \geq 5$, then $|q^{n-3} - (-1)^{n-3}| \geq q^2 - 1 > q + 1$, so the number cannot be an integer. Therefore,

$$n = 4 \text{ and } \sum_{i=0}^{q}\epsilon^{bj_ik} = -(-1)^{4-3}\frac{q+1}{q^{4-3}-(-1)^{4-3}} = 1.$$

Also,

$$\begin{aligned}
\psi_H(h_1) &= \sum_{i=1}^{q}\sum_{j=0}^{q} c_{i,j}\epsilon^{aik+bjk} \\
&= \sum_{i=1}^{q}\sum_{j=0}^{q}\left(\sum_{r=0}^{q} a_{r,j}\frac{q^{4-3}-(-1)^{4-3}}{q+1} + (-1)^{4-3}a_{i,j}\right)\epsilon^{aik+bjk} \\
&= \sum_{i=1}^{q}\sum_{j=0}^{q}\sum_{r=0}^{q} a_{r,j}\epsilon^{aik+bjk} - \sum_{i=1}^{q}\sum_{j=0}^{q} a_{i,j}\epsilon^{aik+bjk} \\
&= \sum_{i=1}^{q}\sum_{r=0}^{q}\epsilon^{aik+bj_rk} - \sum_{i=1}^{q}\epsilon^{aik+bj_ik} \\
&= (\sum_{i=1}^{q}\epsilon^{aik})(\sum_{r=0}^{q}\epsilon^{bj_rk}) - \sum_{i=1}^{q}\epsilon^{aik+bj_ik}.
\end{aligned}$$

By plugging in $\sum_{i=0}^{q}\epsilon^{bj_ik} = 1$, we get

$$\begin{aligned}
\sum_{i=0}^{q}\epsilon^{aik+bj_ik} &= 1 + \sum_{i=1}^{q}\epsilon^{aik+bj_ik} \\
&= 1 + \sum_{i=1}^{q}\epsilon^{aik} - \psi_H(h_1) \\
&= \sum_{i=0}^{q}\epsilon^{aik} - \psi_H(h_1) \\
&= (0 \text{ or } q+1) \pm (0 \text{ or power of } q) \in \mathbb{Z}.
\end{aligned}$$

On the other hand, since $|\sum_{i=1}^{q}\epsilon^{aik+bj_ik}| \leq \sum_{i=1}^{q}|\epsilon^{aik+bj_ik}| = q$,

$$\sum_{i=0}^{q}\epsilon^{aik+bj_ik} = 1 + \sum_{i=1}^{q}\epsilon^{aik+bj_ik} \in [-q+1, q+1].$$

Therefore, the possible values of $\sum_{i=0}^{q}\epsilon^{aik+bj_ik}$ are $\{0, 1, -1, q, q+1\}$, so this case satisfies $(*)$.

(5) Proof of $(*)$ for the cases where $\psi_H(g_0) \neq 0$.

Now consider the pairs (b,k) such that $\psi_H(g_0) \neq 0$ for $g_0 \in C_8^{(bk)}$. I claim that for every $a \in Q$,

$$\sum_{i=1}^{q}\sum_{j=0}^{q} c_{i,j}\epsilon^{aik+bjk} \in \{0, -\frac{1}{q}\psi_H(g_0), \psi_H(g_0), -q\psi_H(g_0)\}. \qquad (**)$$

First, consider the case where $3ak \equiv bk \mod q+1$. Let $g_1 \in C_2^{(ak)}$ and $g_2 \in C_3^{(ak)}$. Then by Table 2,

$$\begin{aligned}
\psi_H(g_0) &= \sum_{j=0}^{q} -c_{0,j}\epsilon^{bk} \in \mathcal{V} \setminus \{0\},\\
\psi_H(g_2) &= \sum_{i=1}^{q}\sum_{j=0}^{q} c_{i,j}\epsilon^{aik+3ajk} = \sum_{i=1}^{q}\sum_{j=0}^{q} c_{i,j}\epsilon^{aik+bjk} \in \mathcal{V},\\
\psi_H(g_1) &= \sum_{j=0}^{q} -qc_{0,j}\epsilon^{3ajk} + \sum_{i=1}^{q}\sum_{j=0}^{q} -(q-1)c_{i,j}\epsilon^{aik+3ajk}\\
&= q\psi_H(g_0) - (q-1)\psi_H(g_2) \in \mathcal{V}.
\end{aligned}$$

Note that $\psi_H(g_2)$ is exactly the number appearing in $(**)$.

If $\psi_H(g_1) = 0$, then $q\psi_H(g_0) = (q-1)\psi_H(g_2)$. Since $\psi_H(g_0)$ is a nonzero element of $\mathcal{V}$, $q\psi_H(g_0)$ is not divisible by $q-1$, so this is impossible. If $\psi_H(g_2) = 0$, this already satisfies $(**)$. So assume that they are all nonzero. We may write $\psi_H(g_\ell) = (-1)^{s_\ell}q^{t_\ell}$ for some $s_0, s_1, s_2 \in \{0,1\}$ and $t_0, t_1, t_2 \in \mathbb{Z}_{\geq 0}$.

If $t_0 \geq t_2$, then

$$\begin{aligned}
(-1)^{s_1}q^{t_1} &= \psi_H(g_1) = q\psi_H(g_0) - (q-1)\psi_H(g_2)\\
&= \left((-1)^{s_0}q^{t_0+1-t_2} - (-1)^{s_2}q + (-1)^{s_2}\right)q^{t_2}.
\end{aligned}$$

Hence, $(-1)^{s_0}q^{t_0+1-t_2} - (-1)^{s_2}q + (-1)^{s_2} = (-1)^{s_1}q^{t_1-t_2}$. Since $t_0+1-t_2 \geq 1$, the left hand side is not divisible by q, so the right hand side must be ± 1. Then the first two terms of the left hand side must cancel each other, so $t_0 = t_2$ and $s_0 = s_2$, hence $\psi_H(g_2) = \psi_H(g_0)$.

If $t_0 < t_2$, then

$$\begin{aligned}
(-1)^{s_1}q^{t_1} &= \psi_H(g_1) = q\psi_H(g_0) - (q-1)\psi_H(g_2)\\
&= \left((-1)^{s_0} - (-1)^{s_2}q^{t_2-t_0} + (-1)^{s_2}q^{t_2-t_0-1}\right)q^{t_0+1}.
\end{aligned}$$

Hence, $(-1)^{s_0} - (-1)^{s_2} q^{t_2-t_0} + (-1)^{s_2} q^{t_2-t_0-1} = (-1)^{s_1} q^{t_1-t_0-1}$. The left hand side cannot be ± 1 since $q^{t_2-t_0} > q^{t_2-t_0-1} + 1$. Hence, this number must be divisible by q, so $(-1)^{s_0}$ and $(-1)^{s_2} q^{t_2-t_0-1}$ must cancel each other. Therefore, $s_0 = -s_2$ and $t_2 - t_0 - 1 = 0$, so $\psi_H(g_2) = (-1)^{s_2} q^{t_2} = -(-1)^{s_0} q^{t_0+1} = -q\psi_H(g_0)$.

The above results together shows that if $3ak \equiv bk \mod q+1$, then

$$\sum_{i=1}^{q} \sum_{j=0}^{q} c_{i,j} \epsilon^{aik+bjk} = \psi_H(g_2) \in \{0, \psi_H(g_0), -q\psi_H(g_0)\}.$$

Therefore, such a satisfies $(**)$.

Now consider $a \in Q$ with $3ak \not\equiv bk \mod q+1$. In this case, let $h_1 \in C_7^{(ak,(a-b)k)}$ (or $C_7^{(ak,(a-b)k+q+1)}$), $h_2 \in C_7^{((b-2a)k,-2ak)}$ (or $C_7^{((b-2a)k,-2ak+q+1)}$), $h_3 \in C_4^{(ak,(b-2a)k)}$, and $h_4 \in C_5^{(ak,(b-2a)k)}$, where the alternative parameters are used when the original parameters are not in the ranges given in Table 1. Then by Table 2 and the previous observations,

$$\begin{aligned}
\psi_H(g_0) &= \sum_{j=0}^{q} -c_{0,j}\epsilon^{bjk} \in \mathcal{V} \setminus \{0\}, \\
\psi_H(h_1) &= \sum_{i=1}^{q} \sum_{j=0}^{q} c_{i,j}\epsilon^{aik+bjk} \in \mathcal{V}, \\
\psi_H(h_2) &= \sum_{i=1}^{q} \sum_{j=0}^{q} c_{i,j}\epsilon^{(b-2a)ik+bjk} \in \mathcal{V}, \\
\psi_H(h_3) &= -(q-1)\left(\sum_{j=0}^{q} c_{0,j}\epsilon^{bjk} + \sum_{i=1}^{q} \sum_{j=0}^{q} c_{i,j}\epsilon^{aik+bjk}\right) + \sum_{i=1}^{q} \sum_{j=0}^{q} c_{i,j}\epsilon^{(b-2a)ik+bjk} \\
&= -(q-1)(-\psi_H(g_0) + \psi_H(h_1)) + \psi_H(h_2) \in \mathcal{V}, \\
\psi_H(h_4) &= \sum_{j=0}^{q} c_{0,j}\epsilon^{bjk} + \sum_{i=1}^{q} \sum_{j=0}^{q} c_{i,j}\epsilon^{aik+bjk} + \sum_{i=1}^{q} \sum_{j=0}^{q} c_{i,j}\epsilon^{(b-2a)ik+bjk} \\
&= -\psi_H(g_0) + \psi_H(h_1) + \psi_H(h_2) \in \mathcal{V}.
\end{aligned}$$

Note that $\psi_H(h_1)$ is the number appearing in $(**)$. Also, since ϵ is a primitive $(q+1)$th root of unity, the choices of parameters in the definitions of h_1 and h_2 does not change the above values.

If $\psi_H(h_1) = 0$, then this a satisfies $(**)$. If $\psi_H(h_2) = 0$, then $\psi_H(h_3) = -(q-1)(-\psi_H(g_0) + \psi_H(h_1)) = -(q-1)\psi_H(h_4)$. Since the only element of

$\mathcal{V}$ which is divisible by $q-1$ is 0, it follows that $\psi_H(h_3) = \psi_H(h_4) = 0$, so $\psi_H(h_1) = \psi_H(g_0)$. This also satisfies $(**)$.

If $\psi_H(h_4) = 0$, then $\psi_H(g_0) = \psi_H(h_1) + \psi_H(h_2)$. A sum or difference of two powers of q is never a power of q (since $q \geq 4$). However, $\psi_H(g_0) \neq 0$, so it is a power of q. Therefore, either $\psi_H(h_1) = 0$ or $\psi_H(h_2) = 0$, and we already checked these cases. Similarly, if $\psi_H(h_3) = 0$, then $(q-1)(-\psi_H(g_0) + \psi_H(h_1)) = \psi_H(h_2)$, so $\psi_H(h_2) = 0$.

The remaining cases are where these character values are all nonzero. As before, we can write $\psi_H(g_0) = (-1)^{s_0} q^{t_0}$ and $\psi_H(h_\ell) = (-1)^{s_\ell} q^{t_\ell}$ for some $s_0, \dots, s_4 \in \{0, 1\}$ and $t_0, \dots, t_4 \in \mathbb{Z}_{\geq 0}$.

Suppose that there is exactly one largest number among t_0, t_1 and t_2, so that the other two are strictly less than the largest one. Then since $q \geq 4$,

$$\begin{aligned} q^{t_4} = |\psi_H(h_4)| &= |-\psi_H(g_0) + \psi_H(h_1) + \psi_H(h_2)| \\ &= |-(-1)^{s_0} q^{t_0} + (-1)^{s_1} q^{t_1} + (-1)^{s_2} q^{t_2}| \\ &\in (q^{\max(t_0,t_1,t_2)-1}, 3q^{\max(t_0,t_1,t_2)}) \\ &\subset (q^{\max(t_0,t_1,t_2)-1}, q^{\max(t_0,t_1,t_2)+1}) \end{aligned}$$

so it must be exactly $q^{\max(t_0,t_1,t_2)}$, and the two terms in $\psi_H(h_4) = -\psi_H(g_0) + \psi_H(h_1) + \psi_H(h_2)$ with smaller absolute value must cancel each other.

If $t_0 = \max(t_0, t_1, t_2)$, then $(-1)^{s_1} q^{t_1} + (-1)^{s_2} q^{t_2} = \psi_H(h_1) + \psi_H(h_2) = 0$, so $t_0 > t_1 = t_2$ and $s_1 \neq s_2$. Since

$$\begin{aligned} q^{t_3} = |\psi_H(h_3)| &= |-(q-1)(-\psi_H(g_0) + \psi_H(h_1)) + \psi_H(h_2)| \\ &= |(q-1)(-1)^{s_0} q^{t_0} - (q-1)(-1)^{s_1} q^{t_1} + (-1)^{s_2} q^{t_2}| \\ &= |(-1)^{s_0} q^{t_0+1} - (-1)^{s_0} q^{t_0} - (-1)^{s_1} q^{t_1+1}| \\ &\in (q^{t_0+1} - q^{t_0} - q^{t_1+1}, q^{t_0+1} + q^{t_0} + q^{t_1+1}) \\ &\subset (q^{t_0}, q^{t_0+2}) \end{aligned}$$

this number is exactly q^{t_0+1}, so it follows that $(-1)^{s_0} q^{t_0} + (-1)^{s_1} q^{t_1+1} = 0$. Therefore $\psi_H(h_1) = (-1)^{s_1} q^{t_1} = -\psi_H(g_0)/q$.

If $t_1 = \max(t_0, t_1, t_2)$, then $-(-1)^{s_0} q^{t_0} + (-1)^{s_2} q^{t_2} = -\psi_H(g_0) + \psi_H(h_2) = 0$

and $t_1 > t_0 = t_2$, so

$$\begin{aligned} q^{t_3} = |\psi_H(h_3)| &= |-(q-1)(-\psi_H(g_0) + \psi_H(h_1)) + \psi_H(h_2)| \\ &= |(q-1)(-1)^{s_0} q^{t_0} - (q-1)(-1)^{s_1} q^{t_1} + (-1)^{s_2} q^{t_2}| \\ &= |(-1)^{s_0} q^{t_0+1} - (-1)^{s_1} q^{t_1+1} + (-1)^{s_1} q^{t_1}| \\ &\in (q^{t_1+1} - q^{t_1} - q^{t_0+1}, q^{t_1+1} + q^{t_1} + q^{t_0+1}) \\ &\subset (q^{t_1}, q^{t_1+2}). \end{aligned}$$

Therefore, this number must be q^{t_1+1}, so $(-1)^{s_0} q^{t_0+1} + (-1)^{s_1} q^{t_1} = 0$, hence $\psi_H(h_1) = (-1)^{s_1} q^{t_1} = -(-1)^{s_0} q^{t_0+1} = -q\psi_H(g_0)$.

If $t_2 = \max(t_0, t_1, t_2)$, then $-(-1)^{s_0} q^{t_0} + (-1)^{s_1} q^{t_1} = -\psi_H(g_0) + \psi_H(h_1) = 0$, so $\psi_H(h_1) = \psi_H(g_0)$.

Suppose that there are exactly two largest numbers among t_0, t_1, t_2. Then

$$q^{t_4} = |\psi_H(h_4)| = |-(-1)^{s_0} q^{t_0} + (-1)^{s_1} q^{t_1} + (-1)^{s_2} q^{t_2}|.$$

If $t_0 = t_1 > t_2$ and $s_0 \neq s_1$, then the above number becomes $2q^{t_0} \pm q^{t_2}$, which cannot be a power of q. Therefore $s_0 = s_1$ and $\psi_H(h_1) = (-1)^{s_1} q^{t_1} = (-1)^{s_0} q^{t_0} = \psi_H(g_0)$.

Similarly, if $t_0 = t_2 > t_1$, then $s_0 = s_2$ and $\psi_H(h_2) = \psi_H(g_0)$. In this case,

$$\begin{aligned} q^{t_3} = |\psi_H(h_3)| &= |-(q-1)(-\psi_H(g_0) + \psi_H(h_1)) + \psi_H(h_2)| \\ &= |(q-1)(-1)^{s_0} q^{t_0} - (q-1)(-1)^{s_1} q^{t_1} + (-1)^{s_2} q^{t_2}| \\ &= |(-1)^{s_0} q^{t_0+1} - (-1)^{s_1} q^{t_1+1} + (-1)^{s_1} q^{t_1}| \\ &\in (q^{t_0+1} - q^{t_1+1} - q^{t_1}, q^{t_0+1} + q^{t_1+1} + q^{t_1}) \\ &\subset (q^{t_0}, q^{t_0+2}). \end{aligned}$$

Therefore this number is q^{t_0+1}, so $-(-1)^{s_1} q^{t_1+1} + (-1)^{s_1} q^{t_1} = 0$. There is no such t_1, so this case cannot happen.

If $t_1 = t_2 > t_0$, then by the same reason, $s_1 \neq s_2$ and $\psi_H(h_1) = -\psi_H(h_2)$. In this case,

$$\begin{aligned} q^{t_3} = |\psi_H(h_3)| &= |-(q-1)(-\psi_H(g_0) + \psi_H(h_1)) + \psi_H(h_2)| \\ &= |(q-1)(-1)^{s_0} q^{t_0} - (q-1)(-1)^{s_1} q^{t_1} + (-1)^{s_2} q^{t_2}| \\ &= |(-1)^{s_0} q^{t_0+1} - (-1)^{s_0} q^{t_0} - (-1)^{s_1} q^{t_1+1}| \\ &\in (q^{t_1+1} - q^{t_0+1} - q^{t_0}, q^{t_1+1} + q^{t_0+1} + q^{t_0}) \\ &\subset (q^{t_1}, q^{t_1+2}). \end{aligned}$$

Therefore, this number is q^{t_1+1}, so $(-1)^{s_0}q^{t_0+1} - (-1)^{s_0}q^{t_0} = 0$. There is no such t_0, so this case also cannot happen.

Finally, suppose that $t_0 = t_1 = t_2$. If $\psi_H(g_0) \neq \psi_H(h_1)$, then $\psi_H(g_0) = -\psi_H(h_1) = \pm\psi_H(h_2)$, so

$$\begin{aligned}(-1)^{s_3}q^{t_3} = \psi_H(h_3) &= -(q-1)(-\psi_H(g_0) + \psi_H(h_1)) + \psi_H(h_2) \\ &= 2(q-1)\psi_H(g_0) \pm \psi_H(g_0) \\ &= (2(q-1) \pm 1)(-1)^{s_0}q^{t_0}\end{aligned}$$

which is impossible. Therefore $\psi_H(g_0) = \psi_H(h_1)$ in this case.

In all of the above cases, we always got

$$\sum_{i=1}^{q}\sum_{j=0}^{q} c_{i,j}\epsilon^{aik+bjk} = \psi_H(h_1) \in \{0, -\frac{1}{q}\psi_H(g_0), \psi_H(g_0), -q\psi_H(g_0)\}.$$

Therefore, such a also satisfies $(**)$, so $(**)$ is true for every $a \in Q$.

Since $\psi_H(g_0) = \sum_{j=0}^{q} -c_{0,j}\epsilon^{bjk}$, by $(**)$,

$$\begin{aligned}\sum_{i=0}^{q}\sum_{j=0}^{q} \frac{c_{i,j}}{\psi_H(g_0)}\epsilon^{aik+bjk} &= \sum_{j=0}^{q} \frac{c_{0,j}}{\psi_H(g_0)}\epsilon^{bjk} + \sum_{i=1}^{q}\sum_{j=0}^{q} \frac{c_{i,j}}{\psi_H(g_0)}\epsilon^{aik+bjk} \\ &= -1 + \frac{\sum_{i=1}^{q}\sum_{j=0}^{q} c_{i,j}\epsilon^{aik+bjk}}{\psi_H(g_0)} \\ &\in \{-1, -\frac{q+1}{q}, 0, -(q+1)\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sum_{i=0}^{q}\sum_{j=0}^{q} \frac{c_{i,j}}{\psi_H(g_0)}\epsilon^{aik+bjk} \\ =&\frac{\sum_{i=0}^{q}\sum_{j=0}^{q} c_{i,j}\epsilon^{aik+bjk}}{\sum_{s=0}^{q} -c_{0,s}\epsilon^{bsk}} \\ =&\frac{\sum_{i=0}^{q}\sum_{j=0}^{q}(\sum_{r=0}^{q} a_{r,j}\frac{q^{n-3}-(-1)^{n-3}}{q+1} + (-1)^{n-3}a_{i,j})\epsilon^{aik+bjk}}{\sum_{s=0}^{q} -(\sum_{u=0}^{q} a_{u,s}\frac{q^{n-3}-(-1)^{n-3}}{q+1} + (-1)^{n-3}a_{0,s})\epsilon^{bsk}} \\ =&-\frac{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{i=0}^{q}\sum_{r=0}^{q}\epsilon^{aik+bj_rk}) + (-1)^{n-3}\sum_{i=0}^{q}\epsilon^{aik+bj_ik}}{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{u=0}^{q}\epsilon^{bj_uk}) + (-1)^{n-3}}.\end{aligned}$$

Note that the denominator does not depend on the choice of a. When $ak \equiv 0 \mod q+1$, this becomes

$$
\begin{aligned}
&-\frac{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{i=0}^{q}\sum_{r=0}^{q}\epsilon^{aik+bj_rk})+(-1)^{n-3}\sum_{i=0}^{q}\epsilon^{aik+bj_ik}}{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{u=0}^{q}\epsilon^{bj_uk})+(-1)^{n-3}} \\
=&-\frac{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{i=0}^{q}\sum_{r=0}^{q}\epsilon^{bj_rk})+(-1)^{n-3}\sum_{i=0}^{q}\epsilon^{bj_ik}}{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{u=0}^{q}\epsilon^{bj_uk})+(-1)^{n-3}} \\
=&-\frac{\frac{q^{n-3}-(-1)^{n-3}}{q+1}((q+1)\sum_{r=0}^{q}\epsilon^{bj_rk})+(-1)^{n-3}\sum_{i=0}^{q}\epsilon^{bj_ik}}{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{u=0}^{q}\epsilon^{bj_uk})+(-1)^{n-3}} \\
=&-(q+1)-\frac{-(-1)^{n-3}(q+1)+(-1)^{n-3}\sum_{i=0}^{q}\epsilon^{bj_ik}}{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{u=0}^{q}\epsilon^{bj_uk})+(-1)^{n-3}}.
\end{aligned}
$$

If $v \in \{-1, -\frac{q+1}{q}, 0, -(q+1)\}$ is the value of the above number, then

$$
\begin{aligned}
&(v+q+1)\left(\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{u=0}^{q}\epsilon^{bj_uk})+(-1)^{n-3}\right) \\
=&(-1)^{n-3}(q+1)-(-1)^{n-3}\sum_{i=0}^{q}\epsilon^{bj_ik}.
\end{aligned}
$$

Solve this for $\sum_{i=0}^{q}\epsilon^{bj_ik}$. Then we get

$$
\sum_{i=0}^{q}\epsilon^{bj_ik}=\frac{(-1)^{n-3}(q+1)-(-1)^{n-3}(v+q+1)}{(v+q+1)\frac{q^{n-3}-(-1)^{n-3}}{q+1}+(-1)^{n-3}}=\frac{-(q+1)v}{(v+q+1)(-q)^{n-3}-v}
$$

When $v=-1$, this becomes

$$
\sum_{i=0}^{q}\epsilon^{bj_ik}=\frac{q+1}{q(-q)^{n-3}+1}.
$$

This number is a rational number which is also an algebraic integer, so it must be an integer. This is possible only when $n=3$, and in this case $\sum_{i=0}^{q}\epsilon^{bj_ik}=1$.

If $v = -\frac{q+1}{q}$, then

$$\sum_{i=0}^{q} \epsilon^{bj_ik} = \frac{-(q+1)(-\frac{q+1}{q})}{(-\frac{q+1}{q}+q+1)(-q)^{n-3}+\frac{q+1}{q}} = \frac{(q+1)^2}{(q^2-1)(-q)^{n-3}+q+1}$$
$$= \frac{q+1}{(q-1)(-q)^{n-3}+1}.$$

This is also an integer by the same reason as the previous case, but there is no such n. Therefore, this case is impossible.

If $v = -(q+1)$, then

$$\sum_{i=0}^{q} \epsilon^{bj_ik} = \frac{-(q+1)(-(q+1))}{0(-q)^{n-3}+q+1} = q+1.$$

Finally, if $v = 0$, then

$$\sum_{i=0}^{q} \epsilon^{bj_ik} = 0.$$

This shows that for those $a \in Q$ such that $ak \equiv 0 \mod q+1$, we have

$$\sum_{i=0}^{q} \epsilon^{aik+bj_ik} = \sum_{i=0}^{q} \epsilon^{bj_ik} \in \{0, 1, q+1\}.$$

In particular, these (a, b, k) satisfy $(*)$. Also, with these values, we can compute the values of the denominator of $\sum_{i=0}^{q}\sum_{j=0}^{q} \frac{c_{ij}}{\psi_H(g_0)}\epsilon^{aik+bjk}$:

$$\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{u=0}^{q}\epsilon^{bj_uk}) + (-1)^{n-3} = \begin{cases} q^{n-3} & \text{when } v = (-q+1), \\ (-1)^{n-3} & \text{when } v = -1 \text{ or } 0. \end{cases}$$

For those $a \in Q$ with $ak \not\equiv 0 \mod q+1$, we know that $\sum_{i=0}^{q} \epsilon^{aik} = 0$, so by the previous observations,

$$\begin{aligned}
&\sum_{i=0}^{q}\sum_{j=0}^{q}\frac{c_{i,j}}{\psi_H(g_0)}\epsilon^{aik+bjk} \\
&= -\frac{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{i=0}^{q}\sum_{r=0}^{q}\epsilon^{aik+bj_rk}) + (-1)^{n-3}\sum_{i=0}^{q}\epsilon^{aik+bj_ik}}{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{u=0}^{q}\epsilon^{bj_uk}) + (-1)^{n-3}} \\
&= -\frac{\frac{q^{n-3}-(-1)^{n-3}}{q+1}\left((\sum_{i=0}^{q}\epsilon^{aik})(\sum_{r=0}^{q}\epsilon^{bj_rk})\right) + (-1)^{n-3}\sum_{i=0}^{q}\epsilon^{aik+bj_ik}}{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{u=0}^{q}\epsilon^{bj_uk}) + (-1)^{n-3}} \\
&= -\frac{(-1)^{n-3}\sum_{i=0}^{q}\epsilon^{aik+bj_ik}}{\frac{q^{n-3}-(-1)^{n-3}}{q+1}(\sum_{u=0}^{q}\epsilon^{bj_uk}) + (-1)^{n-3}} \in \left\{-1, -\frac{q+1}{q}, 0, -(q+1)\right\}
\end{aligned}$$

Since we know the possible values of the denominator, we can solve this for $\sum_{i=0}^{q}\epsilon^{aik+bj_ik}$ and get:

$$\sum_{i=0}^{q}\epsilon^{aik+bj_ik} \in \left\{1, \frac{q+1}{q}, 0, q+1, (-q)^{n-3}, (-q)^{n-3}\frac{q+1}{q}, (-q)^{n-3}(q+1)\right\}.$$

On the other hand, as we saw before, $\sum_{i=0}^{q}\epsilon^{aik+bj_ik} = 1 + \sum_{i=1}^{q}\epsilon^{aik+bj_ik}$ is 1 plus an algebraic integer whose absolute value does not exceed q. Therefore, it cannot be $\frac{q+1}{q}$. If it is $(-q)^{n-3}$, then $n \leq 4$ and the values are $-q$ or 1. If it is $(-q)^{n-3}\frac{q+1}{q}$, then $n \leq 4$; $n \neq 3$ since it is $\frac{q+1}{q}$ when $n = 3$, and when $n = 4$, it is $-(q+1)$, which is not a sum of 1 plus some number of absolute value at most q. Therefore, it is never of the form $(-q)^{n-3}\frac{q+1}{q}$. Finally, if it is $(-q)^{n-3}(q+1)$, then $n = 3$ and the value becomes $q+1$. Therefore,

$$\sum_{i=0}^{q}\epsilon^{aik+bj_ik} \in \{0, 1, q+1, -q\}$$

so these (a, b, k) satisfy $(*)$. Therefore, $(*)$ is true for all possible triples (a, b, k).

(6) The case $q = 2, 3$.

Here, we again mimic the proof of Theorem 1.1 given in [3]. Note that first few arguments of part (1) did not assume that $q \geq 4$. In particular,

$\psi(1) = q^n$, $\sum_{j=0}^{q} a_{0,j} = b_0 = 1$ and $\sum_{i=0}^{q} \sum_{j=0}^{q} a_{i,j} = \sum_{i=0}^{q} b_i = q+1$ still holds in these cases (we still need $(n,q) \neq (3,2)$).

First, suppose that $q = 2$ and $n \geq 4$. By condition (a), ψ is real-valued. Note that $\overline{\zeta_{n,2}^1} = \zeta_{n,2}^2$ and $\overline{\lambda} = \lambda^2$. Therefore,

$$\sum_{i=0}^{2} \sum_{j=0}^{2} a_{i,j} \zeta_{n,2}^i \lambda^j = \psi = \overline{\psi} = \sum_{i=0}^{2} \sum_{j=0}^{2} a_{i,j} \overline{\zeta_{n,2}^i \lambda^j}$$
$$= \sum_{j=0}^{2} a_{0,j} \zeta_{n,2}^0 \lambda^{2j} + \sum_{i=1}^{2} \sum_{j=0}^{2} a_{i,j} \zeta_{n,2}^{3-i} \lambda^{2j}.$$

By comparing the coefficients and using the linear independence of irreducible characters, we obtain $a_{0,1} = a_{0,2}$, $a_{1,0} = a_{2,0}$, $a_{1,1} = a_{2,2}$, and $a_{2,1} = a_{1,2}$. The observation in part (1) that $\sum_{i=0}^{2} \sum_{j=0}^{2} a_{i,j} = \sum_{i=0}^{2} b_i = 2+1 = 3$ and $\sum_{j=0}^{2} a_{0,j} = b_0 = 1$, together with the fact that $a_{i,j}$ are nonnegative integers, forces $a_{0,0} = 1$, $a_{0,1} = a_{0,2} = 0$, and that one of the three pairs $(a_{1,0}, a_{2,0}), (a_{1,1}, a_{2,2}), (a_{2,1}, a_{1,2})$ is $(1,1)$ and the other two pairs are $(0,0)$. Therefore, this theorem is valid for $q = 2$ when $n \geq 4$.

Suppose that $q = 3$. Again, ψ is real-valued, $\overline{\zeta_{n,3}^1} = \zeta_{n,3}^3$, and $\overline{\lambda} = \lambda^3$. Also, $\zeta_{n,3}^2$ and λ^2 are real-valued. As in the previous case, we get

$$\sum_{i=0}^{3} \sum_{j=0}^{3} a_{i,j} \zeta_{n,3}^i \lambda^j = \psi = \overline{\psi} = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{i,j} \overline{\zeta_{n,3}^i \lambda^j}$$
$$= \sum_{i=0,2} \sum_{j=0}^{3} a_{i,j} \zeta_{n,3}^i \lambda^{3j} + \sum_{i=1,3} \sum_{j=0}^{3} a_{i,j} \zeta_{n,3}^{4-i} \lambda^{3j}.$$

Hence $a_{0,1} = a_{0,3}$, $a_{2,1} = a_{2,3}$, $a_{1,1} = a_{3,3}$, $a_{1,3} = a_{3,1}$, $a_{1,0} = a_{3,0}$, and $a_{1,2} = a_{3,2}$.

Since $b_0 = 1$, exactly one of $a_{0,0}$ and $a_{0,2}$ is 1 and the other one is 0. Also, exactly one of $a_{2,0}$ and $a_{2,2}$ is 1 or 3 and the other one is 0, since the sum of all $a_{i,j}$ is 4, while all $a_{i,j}$s other than $a_{0,0}, a_{0,2}, a_{2,0}, a_{2,2}$ appear in the above pairs so that their sum must be even. Note that each of these pairs, except $(a_{0,1}, a_{0,3})$ and $(a_{2,1}, a_{2,3})$, consists of $a_{i,j}$ and $a_{i',j'}$ with $i \neq i'$. Moreover, $a_{0,1} = a_{0,3} = 0$ since $b_0 = 1$. Hence, for each i, there is unique j_i such that $a_{i,j} = \delta_{j,j_i}$, unless $a_{2,1} = a_{2,3} = 1$ or one of $a_{2,0}, a_{2,2}$ is 3. Recall that in part

(1) we saw that

$$\psi_H = \sum_{i=0}^{3}\sum_{j=0}^{3} c_{i,j}\zeta_{3,3}^i \lambda_H^j$$

where

$$c_{i,j} = \sum_{r=0}^{3} a_{r,j}\frac{3^{n-3}-(-1)^{n-3}}{4} + (-1)^{n-3}a_{i,j}.$$

Hence, for $g \in H$ with $\det g = 1$, we get

$$\begin{aligned}
\psi_H(g) &= \sum_{i=0}^{3}\sum_{j=0}^{3} c_{i,j}\zeta_{3,3}^i(g)\lambda_H^j(g) \\
&= \sum_{i=0}^{3}\sum_{j=0}^{3}(\sum_{r=0}^{3} a_{r,j}\frac{3^{n-3}-(-1)^{n-3}}{4} + (-1)^{n-3}a_{i,j})\zeta_{3,3}^i(g) \\
&= (3^{n-3}-(-1)^{n-3})\sum_{i=0}^{3}\zeta_{3,3}^i(g) + (-1)^{n-3}\sum_{i=0}^{3}\sum_{j=0}^{3} a_{i,j}\zeta_{3,3}^i(g).
\end{aligned}$$

When $g \in C_4^{(2,0)}$, it has $\det g = 1$, and the irreducible Weil characters has values

$$\zeta_{3,3}^0(g) = -2,\ \zeta_{3,3}^1(g) = 3,\ \zeta_{3,3}^2(g) = -1,\ \zeta_{3,3}^3(g) = 3.$$

Therefore,

$$\begin{aligned}
\psi_H(g) &= (3^{n-3}-(-1)^{n-3})(3) + (-1)^{n-3}\left(-2 + 3\sum_{j=0}^{3}(a_{1,j}+a_{3,j}) - \sum_{j=0}^{3} a_{2,j}\right) \\
&= 3^{n-2} + (-1)^{n-3}\left(-5 + 3\sum_{j=0}^{3}(a_{1,j}+a_{3,j}) - \sum_{j=0}^{3} a_{2,j}\right)
\end{aligned}$$

$\sum_{j=0}^{3}(a_{1,j}+a_{3,j})$ is either 0 or 2, and in these cases, $\sum_{j=0}^{3} a_{2,j} = 3$ and 1, respectively. If $\sum_{j=0}^{3}(a_{1,j}+a_{3,j}) = 0$, then $\psi_H(g) = 3^{n-2}+(-1)^{n-3}(-5-3) = 3^{n-2} - 8(-1)^{n-3}$. This is not an element of $\mathcal{V}$, so $\sum_{j=0}^{3}(a_{1,j}+a_{3,j}) = 2$. In particular, $a_{2,1} = a_{2,3} = 0$ and for each i, there exists unique j_i such that

$a_{i,j} = \delta_{j,j_i}$ for all j. It follows that for general $g \in H$,

$$\begin{aligned}
\psi_H(g) &= \sum_{i=0}^{3}\sum_{j=0}^{3} c_{i,j}\zeta_{3,3}^i(g)\lambda_H^j(g) \\
&= \sum_{i=0}^{3}\sum_{j=0}^{3}\left(\sum_{r=0}^{3} a_{r,j}\frac{3^{n-3}-(-1)^{n-3}}{4} + (-1)^{n-3}a_{i,j}\right)\zeta_{3,3}^i(g)\lambda_H^j(g) \\
&= \frac{3^{n-3}-(-1)^{n-3}}{4}\left(\sum_{i=0}^{3}\zeta_{3,3}^i(g)\right)\left(\sum_{r=0}^{3}\lambda_H^{j_r}(g)\right) + (-1)^{n-3}\sum_{i=0}^{3}\zeta_{3,3}^i(g)\lambda_H^{j_i}(g)
\end{aligned}$$

Let $h_1 \in C_1^{(1)}$ and $h_8 \in C_8^{(3)}$. Then

$$\begin{aligned}
\psi_H(h_1) =& \frac{-3^{n-3}+(-1)^{n-3}}{4}\left(\epsilon^{3j_0}+\epsilon^{3j_1}+\epsilon^{3j_2}+\epsilon^{3j_3}\right) \\
&+ (-1)^{n-3}(6\epsilon^{3j_0} + 7(\epsilon^{1+3j_1}+\epsilon^{2+3j_2}+\epsilon^{3+3j_3})), \\
\psi_H(h_8) =& \frac{-3^{n-3}+(-1)^{n-3}}{4}(\epsilon^{3j_0}+\epsilon^{3j_1}+\epsilon^{3j_2}+\epsilon^{3j_3}) + (-1)^{n-3}(-\epsilon^{3j_0}).
\end{aligned}$$

With these formulas, we can compute the values for given j_0, j_1, j_2, j_3. We saw at the end of part (2) that we only need to check the cases where $j_0 = 0$.

Suppose that $(j_0, j_1, j_2, j_3) = (0, 1, 0, 3)$. Then by the above formulas,

$$\psi_H(h_1) = \frac{-3^{n-3}+27(-1)^{n-3}}{2}, \qquad \psi_H(h_8) = \frac{-3^{n-3}-(-1)^{n-3}}{2}.$$

Hence, $\psi_H(h_8) = \psi_H(h_1) - 14(-1)^{n-3}$. No elements of $\mathcal{V}$ satisfy these relations, so $(j_0, j_1, j_2, j_3) \neq (0, 1, 0, 3)$.

Similarly, if $(j_0, j_1, j_2, j_3) = (0, 3, 0, 1)$, then

$$\psi_H(h_1) = \frac{-3^{n-3}-29(-1)^{n-3}}{2}, \qquad \psi_H(h_8) = \frac{-3^{n-3}-(-1)^{n-3}}{2}.$$

Hence, $\psi_H(h_8) = \psi_H(h_1) + 14(-1)^{n-3}$. Again, there is no such elements in $\mathcal{V}$, so this case is also impossible.

If $(j_0, j_1, j_2, j_3) = (0, 0, 2, 0)$, then

$$\psi_H(h_1) = \frac{-3^{n-3}+27(-1)^{n-3}}{2}, \qquad \psi_H(h_8) = \frac{-3^{n-3}-(-1)^{n-3}}{2}.$$

This is the same as in the case $(j_0, j_1, j_2, j_3) = (0, 1, 0, 3)$, so it is impossible.

If $(j_0, j_1, j_2, j_3) = (0, 2, 2, 2)$, then

$$\psi_H(h_1) = \frac{3^{n-3} + 25(-1)^{n-3}}{2}, \qquad \psi_H(h_8) = \frac{3^{n-3} - 3(-1)^{n-3}}{2}.$$

Hence, $\psi_H(h_8) = \psi_H(h_1) - 14(-1)^{n-3}$. Again, this is impossible. All remaining choices of (j_0, j_1, j_2, j_3), where $j_0 = 0$, are $(0, 0, 0, 0)$, $(0, 2, 0, 2)$, $(0, 1, 2, 3)$, and $(0, 3, 2, 1)$. Each of these are of the form $j_i \equiv ei \mod 4$ for some $e \in \mathbb{Z}$. Therefore, the theorem holds for $q = 3$. □

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