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Calculus of Variations

Bourgain–Brezis–Mironescu convergence via Triebel-Lizorkin spaces

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Abstract
We study a convergence result of Bourgain–Brezis–Mironescu (BBM) using Triebel-Lizorkin spaces. It is well known that as spaces $W^{s,p} = F^s_{p,p}$, and $H^{1,p} = F^1_{p,2}$. When $s \to 1$, the $F^s_{p,p}$ norm becomes the $F^1_{p,p}$ norm but BBM showed that the $W^{s,p}$ norm becomes the $H^{1,p} = F^1_{p,2}$ norm. Naively, for $p \neq 2$ this seems like a contradiction, but we resolve this by providing embeddings of $W^{s,p}$ into $F^s_{p,q}$ for $q \in \{p, 2\}$ with sharp constants with respect to $s \in (0, 1)$. As a consequence we obtain an $\mathbb{R}^N$-version of the BBM-result, and obtain several more embedding and convergence theorems of BBM-type that to the best of our knowledge are unknown.

1 Introduction and main results

1.1 Previous results

For $s \in (0, 1)$, $p \in (1, \infty)$ and an open set $\Omega \subset \mathbb{R}^N$ the $\dot{W}^{s,p}$-Gagliardo-seminorm is defined as

$$[f]_{\dot{W}^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{N+sp}}\, dx\, dy\right)^{\frac{1}{p}} \equiv \left\| \frac{f(x) - f(y)}{|x - y|^{\frac{N}{p}+s}} \right\|_{L^p(\Omega\times\Omega)}.$$

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For $s = 1$ we denote the usual $\dot{H}^{1,p}$-Sobolev space seminorm by

$$[f]_{\dot{H}^{1,p}(\Omega)} = \|\nabla f\|_{L^p(\Omega)}$$

and write $H^{1,p}$ for the inhomogeneous Sobolev space so that

$$H^{1,p}(\Omega) := \{f \in L^p(\Omega) : \nabla f \in L^p(\Omega)\}.$$

In the influential paper [4] Bourgain–Brezis–Mironescu showed that for any smooth bounded domain $\Omega \subset \mathbb{R}^N$ and any $f \in H^{1,p}(\Omega)$ we have

$$\|\nabla f\|_{L^p(\Omega)} = \left(\frac{p}{k(p,N)}\right)^{1/p} \lim_{s\to 1^-} (1-s)^{\frac{1}{p}} [f]_{\dot{W}^{s,p}(\Omega)}, \tag{BBM1}$$

where $k(p, N) := \int_{\mathbb{S}^{N-1}} |e \cdot \omega|^p d\omega$ and e is any unit vector in $\mathbb{R}^N$. Even more crucially, Bourgain–Brezis–Mironescu established the following convergence result.

Theorem 1.1 (Bourgain–Brezis–Mironescu [4]) *Let $\Omega \subset \mathbb{R}^N$ be open and bounded with smooth boundary, and $p \in (1, \infty)$.*

(BBM2) *Assume that $f_k \in C_c^\infty(\Omega)$ such that*

$$f_k \rightharpoonup f \quad \text{weakly in } L^p(\Omega) \text{ as } k \to \infty.$$

Let $(s_k)_{k\in\mathbb{N}} \subset (0, 1)$ such that $s_k \uparrow 1$ and assume that

$$\Lambda := \sup_k \left(\|f_k\|_{L^p(\Omega)} + (1-s_k)^{\frac{1}{p}} [f_k]_{\dot{W}^{s_k,p}(\Omega)}\right) < \infty.$$

Then $f \in H^{1,p}(\Omega)$ and we have

$$\|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)} \le C\,\Lambda.$$

The constant C depends only p and N. Also, $f_k \xrightarrow{k\to\infty} f$ strongly in $L^p_{loc}(\Omega)$.

See also [5, 8, 22] for related results, [15, 21] for an interpretation via interpolation space, and [19, 20] for the regime $s \to 0$.

1.2 Questions on $\mathbb{R}^N$

In this paper, we explore what happens when the bounded domain Ω above is replaced by the whole space $\mathbb{R}^N$. It is relatively easy to show that (BBM1) holds with Ω replaced by $\mathbb{R}^N$; we provide a short proof in Appendix A. Our main result will be an analog of Theorem 1.1 on $\mathbb{R}^N$. In fact, from the point of view of Harmonic Analysis, Theorem 1.1 seems like a surprising result, as we shall explain here. Denote the homogeneous Triebel-Lizorkin norm $[\cdot]_{\dot{F}^s_{p,p}(\mathbb{R}^N)}$ by

$$[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \sum_{j\in\mathbb{Z}} 2^{sjp} |\Delta_j f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Here $\Delta_j f$ are the Littlewood-Paley projections (see Sect. 2.3 for their definitions). It is well-known that for $s \in (0, 1)$, $p \in (1, \infty)$,

$$[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \approx [f]_{\dot{W}^{s,p}(\mathbb{R}^N)},$$

whenever $f \in \mathscr{S}(\mathbb{R}^N)$, where $\mathscr{S}(\mathbb{R}^N)$ denotes the set of Schwartz functions on $\mathbb{R}^N$. However, since $\|f\|_{L^p(\mathbb{R}^N)} \approx \|f\|_{\dot{F}^0_{p,2}}$ we have

$$[f]_{\dot{F}^1_{p,2}(\mathbb{R}^N)} \approx \|\nabla f\|_{L^p(\mathbb{R}^N)}.$$

From the definition of Triebel-Lizorkin spaces, it easily follows (cf. Lemma 2.7)

$$\lim_{s\to 1}[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} = [f]_{\dot{F}^1_{p,p}(\mathbb{R}^N)}.$$

So if Theorem 1.1 holds true on $\mathbb{R}^N$, it then seems to suggest that in some way $\dot{W}^{s,p} \approx_{s,p} \dot{F}^s_{p,p}$ "converges to" $\dot{H}^{1,p} \approx_p \dot{F}^1_{p,2}$, which appears to be a contradiction to the above, because for $p \neq 2$ we have that $\dot{F}^1_{p,2} \not\approx \dot{F}^1_{p,p}$. These statements, of course, do not make any sense, because spaces do not converge, but norms. The aim of this note is to clarify the effects we are seeing here, which we achieve by clarifying various relationships between the $\dot{W}^{s,p}$, $\dot{F}^s_{p,p}$ and $\dot{F}^s_{p,2}$ seminorms for $0 < s < 1$.

1.3 Results about $\dot{F}^s_{p,p}$

Our first main theorem is the following quantitative comparison between the $\dot{W}^{s,p}$ and the $\dot{F}^s_{p,p}$ seminorms.

Theorem 1.2 *Let $N \geq 1$, $p \in (1,\infty)$. Then there exists $C = C(N,p) > 0$, such that for every $s \in (0,1)$ and $f \in \mathscr{S}(\mathbb{R}^N)$,*

(1) if $1 < p \leq 2$:

$$C^{-1}\left(\frac{1}{s^{\frac{1}{2}}} + \frac{1}{(1-s)^{\frac{1}{2}}}\right)[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \leq [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \leq C\left(\frac{1}{s^{\frac{1}{p}}} + \frac{1}{(1-s)^{\frac{1}{p}}}\right)[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)}. \tag{1.1}$$

(2) if $2 \leq p < \infty$:

$$C^{-1}\left(\frac{1}{s^{\frac{1}{p}}} + \frac{1}{(1-s)^{\frac{1}{p}}}\right)[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \leq [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \leq C\left(\frac{1}{s^{\frac{1}{2}}} + \frac{1}{(1-s)^{\frac{1}{2}}}\right)[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)}. \tag{1.2}$$

The upper bounds in (1.1) and (1.2) have been proven by Gu and the third author in [13].

As an immediate corollary we obtain the following Sobolev-type inequality for $p = 2$. It is well-known and elementary to show that

$$[f]_{\dot{W}^{s,2}(\mathbb{R}^N)} \leq C_{s,t}\left(\|f\|_{L^2(\mathbb{R}^N)} + [f]_{\dot{W}^{t,2}(\mathbb{R}^N)}\right), \quad \text{for } 0 < s \leq t < 1.$$

The main nontriviality in the corollary below is the prefactor $\min\{s,(1-s)\}^{\frac{1}{2}}$ on the left-hand side and $\min\{t,(1-t)\}^{\frac{1}{2}}$ on the right-hand side. We do not know if a similar statement is true for any $p \in (1,\infty)$, see Question 1.10.

Corollary 1.3 *Let $N \geq 1$. Then there exists $C = C(N) > 0$, such that for all $0 < s \leq t < 1$ and $f \in \mathscr{S}(\mathbb{R}^N)$,*

$$\min\{s,(1-s)\}^{\frac{1}{2}}[f]_{\dot{W}^{s,2}(\mathbb{R}^N)} \leq C\left(\|f\|_{L^2(\mathbb{R}^N)} + \min\{t,(1-t)\}^{\frac{1}{2}}[f]_{\dot{W}^{t,2}(\mathbb{R}^N)}\right).$$

For the convenience of the reader, we give the details of the proof Corollary 1.3 in Appendix B.

Remark 1.4 (Sharpness of the constants) To some extent the constants in Theorem 1.2 are sharp, as can be shown using the results of [4].

(1) Observe that in general for $p < 2$

$$\left(\frac{1}{s^{\frac{1}{p}}} + \frac{1}{(1-s)^{\frac{1}{p}}}\right) [f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \not\lesssim C[f]_{\dot{W}^{s,p}(\mathbb{R}^N)}$$

for $C = C(N, p) > 0$. Indeed, if that was true for all $s \in (0, 1)$, we could pick a function $f \in H^{1,p}(\mathbb{R}^N)$ with compact support that does not belong to $F^1_{p,p}(\mathbb{R}^N)$. From [4] we would then obtain that

$$\limsup_{s \to 1^-} (1-s)^{\frac{1}{p}} [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} < \infty,$$

however we have

$$\lim_{s \to 1^-} [f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} = [f]_{\dot{F}^1_{p,p}(\mathbb{R}^N)} = \infty.$$

(2) Similarly, for $p > 2$ in general

$$[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \not\lesssim C \left(\frac{1}{s^{\frac{1}{p}}} + \frac{1}{(1-s)^{\frac{1}{p}}}\right) [f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)}$$

for $C = C(N, p) > 0$. To obtain a counterexample in this case take, $f \in F^1_{p,p}(\mathbb{R}^N)$ with compact support and $f \notin H^{1,p}(\mathbb{R}^N)$. Then, again by [4]

$$\limsup_{s \to 1^-} (1-s)^{\frac{1}{p}} [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} = \infty,$$

however

$$\liminf_{s \to 1^-} [f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} = [f]_{\dot{F}^1_{p,p}(\mathbb{R}^N)} < \infty.$$

1.4 Results about $\dot{F}^s_{p,2}$

Next we explore relationships between the $\dot{W}^{s,p}$ and the $\dot{F}^s_{p,2}$ seminorms. Observe that while Theorem 1.2 is a nice characterization, and we obtain some convergence for functions with uniformly bounded $(1-s)^{\frac{1}{p}} [f]_{\dot{W}^{s,p}(\mathbb{R}^N)}$-norms, we do not recover Theorem 1.1 yet. For this we need a different space. Namely, we obtain the following $\dot{F}^s_{p,2}$-estimate and the main focus should be on how changing from $\dot{F}^s_{p,p}$ to $\dot{F}^s_{p,2}$ improves the dependency on s and $(1-s)$.

Theorem 1.5 *Let $N \geq 1$, $p \in (1, \infty)$. Then there exists $C = C(N, p) > 0$, such that for all $s \in (0, 1)$ and $f \in \mathscr{S}(\mathbb{R}^N)$,*

(1) if $1 < p \leq 2$:

$$C^{-1} \left(\frac{1}{s^{\frac{1}{p}}} + \frac{1}{(1-s)^{\frac{1}{p}}}\right) [f]_{\dot{F}^s_{p,2}(\mathbb{R}^N)} \leq [f]_{\dot{W}^{s,p}(\mathbb{R}^N)}. \tag{1.3}$$

(2) *if* $2 \leq p < \infty$:

$$[f]_{\dot W^{s,p}(\mathbb{R}^N)} \leq C \left(\frac{1}{s^{\frac{1}{p}}} + \frac{1}{(1-s)^{\frac{1}{p}}} \right) [f]_{\dot F^s_{p,2}(\mathbb{R}^N)}. \tag{1.4}$$

The upper bound for $[f]_{\dot F^s_{p,2}(\mathbb{R}^N)}$ in (1.3) in Theorem 1.5 provides a full, $\mathbb{R}^N$-version of Theorem 1.1 if $p \leq 2$, see Corollary 1.9 below. For $p \geq 2$ the desired upper bound for $[f]_{\dot F^s_{p,2}(\mathbb{R}^N)}$ will be provided by the following Sobolev-type estimate: see (1.6).

Theorem 1.6 (Sobolev-Estimate) *Let* $N \geq 1$, $p \in (1,\infty)$.

(1) *Then there exists* $C = C(N,p) > 0$, *such that for* $0 \leq r < s < t \leq 1$ *and* $f \in \mathscr{S}(\mathbb{R}^N)$,

$$[f]_{\dot W^{s,p}(\mathbb{R}^N)} \leq C \left(\frac{1}{(s-r)^{\frac{1}{p}}} [f]_{\dot F^r_{p,2}} + \frac{1}{(t-s)^{\frac{1}{p}}} [f]_{\dot F^t_{p,2}} \right). \tag{1.5}$$

(2) *Let* $\Lambda > 1$. *Then there exists* $C = C(N,p,\Lambda) > 0$, *such that the following holds: Let* $s \in [1 - \frac{1}{2\Lambda}, 1)$. *Let* $\bar r \in (0,s)$ *such that* $(1-\bar r) = \Lambda(1-s)$. *Pick* $r \in [0,\bar r]$. *Then for any* $f \in \mathscr{S}(\mathbb{R}^N)$,

$$[f]_{\dot F^r_{p,2}(\mathbb{R}^N)} \leq C \left(\|f\|_{L^p(\mathbb{R}^N)} + (1-s)^{\frac{1}{p}} [f]_{\dot W^{s,p}(\mathbb{R}^N)} \right). \tag{1.6}$$

Applying [4, Theorem 1] to $\rho(x) = |x|^{-N-(1-s)p}$ one obtains for a bounded set Ω

$$\sup_{s \in (0,1)} (1-s)^{\frac{1}{p}} [f]_{\dot W^{s,p}(\Omega)} \leq C(N,\Omega,p) \|\nabla f\|_{L^p(\Omega)}.$$

As an immediate corollary of Theorem 1.6, we find a variant of this inequality on $\mathbb{R}^N$ and even obtain a fractional version of it. In the following we denote for non-integral values of s

$$\dot H^{s,p}(\mathbb{R}^N) \equiv \dot F^s_{p,2}(\mathbb{R}^N),$$

whose seminorm $[f]_{\dot H^{s,p}(\mathbb{R}^N)} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^N)}$ is defined via the fractional Laplacian. Observe that $\|(-\Delta)^{\frac{1}{2}} f\|_{L^p(\mathbb{R}^N)} \approx \|\nabla f\|_{L^p(\mathbb{R}^N)}$ for any $p \in (1,\infty)$ by the L^p-boundedness of the Riesz transforms, so for our purposes it does not really matter whether we defined $[f]_{\dot H^{1,p}}$ to be $\|\nabla f\|_{L^p}$ or $\|(-\Delta)^{1/2} f\|_{L^p}$. From (1.5) it is easy to deduce

Corollary 1.7 *Let* $N \geq 1$, $p \in (1,\infty)$, $0 < \theta < 1$. *Then there exists* $C = C(N,p,\theta) > 0$, *such that for* $s \in (\theta, 1]$ *and* $f \in \mathscr{S}(\mathbb{R}^N)$,

$$\sup_{r \in [\theta,s)} (s-r)^{\frac{1}{p}} [f]_{\dot W^{r,p}(\mathbb{R}^N)} \leq C \left(\|f\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^N)} \right). \tag{1.7}$$

In particular, setting $s = 1$, *we obtain*

$$\sup_{r \in [\theta,1)} (1-r)^{\frac{1}{p}} [f]_{\dot W^{r,p}(\mathbb{R}^N)} \leq C \left(\|f\|_{L^p(\mathbb{R}^N)} + \|\nabla f\|_{L^p(\mathbb{R}^N)} \right).$$

Remark 1.8 Barring the independence of the constant on s, the case $s < 1$ in (1.7) is only interesting for the case $p < 2$. For $p \geq 2$ and $s < 1$ it is an obvious (and non-optimal) estimate. Indeed, if $f \in \mathscr{S}(\mathbb{R}^N)$, $p \geq 2$ and $r \in (0,1)$, we have

$$[f]_{\dot W^{r,p}(\mathbb{R}^N)} \approx [f]_{\dot F^r_{p,p}} \leq [f]_{\dot F^r_{p,2}} \approx \|(-\Delta)^{\frac{r}{2}} f\|_{L^p(\mathbb{R}^N)},$$

where the implicit constants depend on r, p and N. If r is in a compact subinterval of $(0,1)$, then the constants can be taken independent of r. Since

$$(s-r)^{\frac{1}{p}}[f]_{\dot{W}^{r,p}(\mathbb{R}^N)} \leq C[f]_{\dot{W}^{r,p}(\mathbb{R}^N)}$$

and

$$\|(-\Delta)^{\frac{r}{2}} f\|_{L^p(\mathbb{R}^N)} \leq C\left(\|f\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^N)}\right)$$

for $r \in [0,s]$ (with constants independent of r and s), if $s \leq 1-\theta'$ for some $\theta' > 0$, then whenever $\theta > 0$

$$\sup_{r\in[\theta,s)} (s-r)^{\frac{1}{p}}[f]_{\dot{W}^{r,p}(\mathbb{R}^N)} \leq C\left(\|f\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^N)}\right)$$

with a constant depending on θ, θ', p and N but not on r and s.

1.5 Back to Bourgain–Brezis–Mironescu's convergence result

From (1.6) and Rellich–Kondrachov theorem we recover in particular (BBM2) of Theorem 1.1 – actually with a stronger convergence than is commonly considered in the literature.

Corollary 1.9 *Let $p \in (1,\infty)$, assume that $f_k \in \mathscr{S}(\mathbb{R}^N)$ such that*

$$f_k \rightharpoonup f \quad \text{weakly in } L^p(\mathbb{R}^N) \text{ as } k \to \infty.$$

Let $(s_k)_{k\in\mathbb{N}} \subset (0,1)$ such that $s_k \uparrow 1$ and assume that

$$\Lambda := \sup_k \left(\|f_k\|_{L^p(\mathbb{R}^N)} + (1-s_k)^{\frac{1}{p}}[f_k]_{\dot{W}^{s_k,p}(\mathbb{R}^N)}\right) < \infty. \tag{1.8}$$

Then $f \in H^{1,p}(\mathbb{R}^N)$ and we have

$$\|f\|_{L^p(\mathbb{R}^N)} + \|\nabla f\|_{L^p(\mathbb{R}^N)} \leq C\,\Lambda.$$

The constant C depends on p and N.

Also, $f_k \xrightarrow{k\to\infty} f$ strongly in $H^{t,p}_{loc}(\mathbb{R}^N)$ for any $t \in [0,1)$, that is

$$\lim_{k\to\infty} \|(-\Delta)^{\frac{t}{2}} f_k - (-\Delta)^{\frac{t}{2}} f\|_{L^p(K)} = 0 \quad \forall \text{compact sets } K \subset \mathbb{R}^N, \tag{1.9}$$

and for any $t \in (0,1)$

$$\lim_{k\to\infty} [f_k - f]_{\dot{W}^{t,p}(K)} = 0 \quad \forall \text{ compact sets } K \subset \mathbb{R}^N. \tag{1.10}$$

We give the details of the proof in Sect. 7. The above strong convergence may not be global in $\mathbb{R}^N$ (even strong convergence in $L^p(\mathbb{R}^N)$ may be false). A counterexample is given by a standard counterexample to the global Rellich-Kondrachov Theorem for $W^{1,p}(\mathbb{R}^N)$ (which shows that $W^{1,p}(\mathbb{R}^N)$ does not embed compactly into $L^p(\mathbb{R}^N)$): for instance, if $f \in C_c^\infty(\mathbb{R}^N)$ and $\{f_k\}_k$ is a sequence of translates of f that escapes off to infinity, then f_k converges weakly to 0 in $W^{1,p}(\mathbb{R}^N)$, (1.8) is satisfied by Corollary 1.7, but f_k does not converge strongly in $L^p(\mathbb{R}^N)$.

1.6 Open questions and further directions

Question 1.10 Let $p \in (1,\infty)$, $0 < \theta < s < t < 1$ and $f \in \mathscr{S}(\mathbb{R}^N)$. Is it true that there exists $C = C(N,p,\theta) > 0$, such that

$$\min\{s,(1-s)\}^{\frac{1}{p}}[f]_{\dot W^{s,p}(\mathbb{R}^N)} \le C\left(\|f\|_{L^p(\mathbb{R}^N)} + \min\{t,(1-t)\}^{\frac{1}{p}}[f]_{\dot W^{t,p}(\mathbb{R}^N)}\right)?$$

An indication that the above might be true, is the case $p = 2$, Corollary 1.3. Also, of course, Question 1.10 holds asymptotically for $s = t$ and – in view of [4] – it holds if we first let $t \uparrow 1$ and *then* take $s \uparrow 1$.

Let us remark that a very rough toy-case for Question 1.10 are characteristic functions – and indeed the inequality from Question 1.10 holds in that case: for $A \subset \mathbb{R}^N$ measurable we have $|\chi_A(x) - \chi_A(y)|^p = |\chi_A(x) - \chi_A(y)|^2$, and $|x-y|^{N+sp} = |x-y|^{N+\frac{sp}{2}2}$. Thus,

$$[\chi_A]_{\dot W^{s,p}(\mathbb{R}^N)} = [\chi_A]^{\frac{2}{p}}_{\dot W^{\frac{sp}{2},2}(\mathbb{R}^N)},$$

so

$$\begin{aligned}
\min\{s,(1-s)\}^{\frac{1}{p}}[\chi_A]_{\dot W^{s,p}(\mathbb{R}^N)} &= \min\{s,(1-s)\}^{\frac{1}{p}}[\chi_A]^{\frac{2}{p}}_{\dot W^{\frac{sp}{2},2}(\mathbb{R}^N)}\\
&= \left(\min\{s,(1-s)\}^{\frac{1}{2}}[\chi_A]_{\dot W^{\frac{sp}{2},2}(\mathbb{R}^N)}\right)^{\frac{2}{p}}\\
&\lesssim_p \|\chi_A\|^{\frac{2}{p}}_{L^2} + \left(\min\{t,(1-t)\}^{\frac{1}{2}}[\chi_A]_{\dot W^{\frac{tp}{2},2}(\mathbb{R}^N)}\right)^{\frac{2}{p}}\\
&= \|\chi_A\|_{L^p} + \min\{t,(1-t)\}^{\frac{1}{p}}[\chi_A]_{\dot W^{t,p}(\mathbb{R}^N)}.
\end{aligned}$$

Moving on to the next question, the estimate (1.7) hints towards the possibility that there might be a Brezis–Bourgain–Mironescu-type result for $s < 1$, namely it establishes an $H^{s,p}$-type (BBM1)-estimate. It is unclear to us if the convergence result is also true.

Question 1.11 Let $p \in (1,\infty)$, assume that $f_k \in \mathscr{S}(\mathbb{R}^N)$ such that

$$f_k \rightharpoonup f \quad \text{weakly in } L^p(\mathbb{R}^N) \text{ as } k \to \infty.$$

Let $t \in (0,1)$ and $(s_k)_{k\in\mathbb{N}} \subset (0,t)$ such that $s_k \uparrow t$ and assume that

$$\Lambda := \sup_k \left(\|f_k\|_{L^p(\mathbb{R}^N)} + (t-s_k)^{\frac{1}{p}}[f_k]_{\dot W^{s_k,p}(\mathbb{R}^N)}\right) < \infty.$$

Is it true that $f \in H^{t,p}(\mathbb{R}^N)$ and that there exists $C = C(N,p,t) > 0$, such that

$$\lim_{\tilde t \uparrow t} \limsup_{k\to\infty} \|(-\Delta)^{\frac{\tilde t}{2}} f_k\|_{L^p(\mathbb{R}^N)} \le C\Lambda?$$

Our next question concerns an extension to other Triebel-Lizorkin spaces. It is known that for $p > \frac{Nq}{N+sq}$

$$[f]_{\dot W^{s,p}_q(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^q}{|x-y|^{N+sq}}dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \approx [f]_{\dot F^s_{p,q}(\mathbb{R}^N)}, \tag{1.11}$$

see [1, 25] for $q = 2$, [28, Section 2.5.10] for $s \ge \frac{N}{\min\{p,q\}}$ and [23] for the general $p > \frac{Nq}{N+sq}$. Unless $q = 2$, which was treated in [11], the case of equality $p = \frac{Nq}{N+sq}$ seems to be open.

Question 1.12 What is the dependency on s and $(1-s)$ as $s \downarrow 0$ or $s \uparrow 1$ in the equivalence (1.11)?

Question 1.12 is related to works by Spector–Leoni, [17, 18]. Of course, the limit cases $p = 1$ and $p = \infty$ would also be an interesting direction, cf. [7].

Lastly let us mention that Bourgain–Brezis–Mironescu [4] (see also [22]) the singular kernel $|x-y|^{-N-sp}$ is only one special case considered. In general they work with family of kernels ρ_n that suitably approximate $|x-y|^{-p}\delta_{x,y}$. It might be possible to adapt our methods to treat this case as well, in the sense that as $n \to \infty$ the corresponding ρ_n-seminorm controls more and more frequencies estimated in $\dot{F}^s_{p,2}$.

The paper will be organized as follows. In Sect. 2 we collect a few basic results about Triebel-Lizorkin spaces. In Sect. 3 we give a simple proof of Theorem 1.2 in the special case $p = 2$. In Sect. 4 we prove the upper bounds for $[f]_{\dot{W}^{s,p}}$ in Theorem 1.2 and Theorem 1.5, i.e. the second inequalities of (1.1) and (1.2), and the inequality (1.4). In Sect. 5 we prove the upper bound (1.5) for $[f]_{\dot{W}^{s,p}}$ in Theorem 1.6. In Sect. 6 we prove the lower bounds for $[f]_{\dot{W}^{s,p}}$ in Theorem 1.2, Theorem 1.5 and Theorem 1.6, i.e. the first inequalities of (1.1) and (1.2), and the inequalities (1.3) and (1.6). In Sect. 7 we prove Corollary 1.9. Finally, in Appendix A we prove (BBM1) with Ω replaced by $\mathbb{R}^N$, and in Appendix B we give a short proof of Corollary 1.3.

Recent progress in [10]

After finishing this manuscript, Domínguez and Milman [10] settled Question 1.10 and Question 1.11, using heavy interpolation machinery. They also provide alternative proofs of our main theorems via these interpolation and extrapolation techniques.

2 Preliminaries

In this section we gather preliminary results that most likely are all widely known. Throughout the paper we use the notation $A \lesssim B$ whenever there is a multiplicative constant $C > 0$ such that $A \le CB$. $A \approx B$ means $A \lesssim B$ and $B \lesssim A$. The constant C can change from line to line and depends on dimension and exponent, but unless otherwise noted does not depend on s, t etc.

2.1 Mixed measure spaces

Let $p, q \in (1, \infty)$, and consider the space $L^p(\ell^q)$ given by sequence $(f_j)_{j\in\mathbb{Z}} \subset L^p(\mathbb{R}^N)$ with finite norm

$$\|f_j\|_{L^p(\ell^q)} := \left\| \left(\sum_j |f_j(x)|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^N, dx)}.$$

By a slight abuse of notation, we will have the same notation when considering finite sequences $(f_j)_{j=-K}^K \subset L^p(\mathbb{R}^N)$.

From [2, Theorem 1] we obtain that $L^p(\ell^q)$ is a Banach space, and more importantly its dual space is $L^{p'}(\ell^{q'})$ in the following way: any linear functional $J \in (L^p(\ell^q))^*$ is given by an element $g \in L^{p'}(\ell^{q'})$ such that

$$J(f) = \int_{\mathbb{R}^N} \sum_{j\in\mathbb{Z}} f_j(x) g_j(x)\, dx.$$

In particular, from Hahn-Banach theorem, we have

Proposition 2.1 *Let $p, q \in (1, \infty)$. Let $(f_j)_{j\in\mathbb{Z}} \in L^p(\ell^q)$. Then there exists $(g_j)_{j\in\mathbb{Z}} \subset L^{p'}(\mathbb{R}^N)$ with*

$$\|(g_j)_{j\in\mathbb{Z}}\|_{L^{p'}(\ell^{q'})} \le 1$$

and

$$\|f_j\|_{L^p(\ell^q)} = \int_{\mathbb{R}^N} \sum_{j\in\mathbb{Z}} f_j(x) g_j(x)\, dx.$$

An analogous statement holds for sequences $(f_j)_{j=-K}^K$.

2.2 Fractional Laplacian

For $s > 0$ denote by $(-\Delta)^{\frac{s}{2}}$ the operator with Fourier symbol $|\xi|^s$, that is

$$\mathcal{F}((-\Delta)^{\frac{s}{2}} f)(\xi) := |\xi|^s \mathcal{F} f(\xi).$$

It is well-known that there is an integral formula for the fractional Laplacian when $s \in (0, 2)$, cf. [9]. We need the following estimate on the constant that appears there.

Lemma 2.2 *Let $f \in \mathscr{S}(\mathbb{R}^N)$ and $s \in (0, 1)$. Then*

$$(-\Delta)^s f(x) = c_{N,s} \int_{\mathbb{R}^N} \frac{2f(x) - f(x+z) - f(x-z)}{|z|^{N+2s}}\, dz,$$

where

$$c_{N,s} \approx \min\{s, 1-s\}.$$

Proof For $s \in (0, 1)$ we have (see e.g. [9])

$$(-\Delta)^s f(x) = c_{N,s} \int_{\mathbb{R}^N} \frac{2f(x) - f(x+z) - f(x-z)}{|z|^{N+2s}}\, dz, \quad c_{N,s} = \frac{1}{2} \frac{4^s \Gamma\left(\frac{N}{2} + s\right)}{\pi^{\frac{N}{2}} |\Gamma(-s)|}.$$

Since

$$\Gamma(-s) = -\frac{\pi}{\sin(\pi s)} \frac{1}{\Gamma(1+s)},$$

for $s \in (0, 1)$ we get $c_{N,s} \approx |\sin(\pi s)| \approx \min\{s, 1-s\}$. □

2.3 Littlewood-Paley projections and Triebel-Lizorkin spaces

Below we will need to understand the space $L^p(\mathbb{R}^N)$ and the inhomogeneous Sobolev space

$$H^{1,p}(\mathbb{R}^N) := \{f \in L^p(\mathbb{R}^N) : \nabla f \in L^p(\mathbb{R}^N)\}$$

for $1 < p < \infty$, via Bessel potentials and Triebel-Lizorkin spaces.

First, recall the Bessel potential $(I-\Delta)^{s/2}$, given by the Fourier multiplier $(1+|\xi|^2)^{s/2}$ for $s \in \mathbb{R}$. We have $(I-\Delta)^{s/2} : \mathscr{S}(\mathbb{R}^N) \to \mathscr{S}(\mathbb{R}^N)$ continuously, thus $(I-\Delta)^{s/2}$ extends by duality to a map that acts on tempered distributions $\mathscr{S}'(\mathbb{R}^N)$. For $1 < p < \infty$, it is known that $f \in \mathscr{S}'(\mathbb{R}^N)$ with $(I-\Delta)^{1/2} f \in L^p(\mathbb{R}^N)$ if and only if $f \in L^p(\mathbb{R}^N)$ with distributional gradient $\nabla f \in L^p(\mathbb{R}^N)$. This motivates one to define, for $s \in \mathbb{R}$ and $1 < p < \infty$, the space $H^{s,p}(\mathbb{R}^N)$, as the space of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^N)$ for which

$$\|f\|_{H^{s,p}(\mathbb{R}^N)} := \|(I-\Delta)^{s/2} f\|_{L^p(\mathbb{R}^N)} < \infty.$$

When $s = 1$, the definition of $H^{s,p}(\mathbb{R}^N)$ agrees with our earlier definition in the previous paragraph using distributional gradients. We also have

$$\|f\|_{H^{1,p}(\mathbb{R}^N)} \approx_{p,N} \|f\|_{L^p(\mathbb{R}^N)} + \|\nabla f\|_{L^p(\mathbb{R}^N)}$$

for $f \in H^{1,p}(\mathbb{R}^N)$.

Next, for a function $f \in L^p(\mathbb{R}^N)$, the j-th Littlewood-Paley projection is defined as

$$\Delta_j f(x) := f * [2^{jN} \eta(2^j \cdot)](x).$$

Here $\eta \in \mathscr{S}(\mathbb{R}^N)$ is a Schwartz function such that its Fourier transform $\mathcal{F}\eta$ satisfies

$$\sum_{j \in \mathbb{Z}} (\mathcal{F}\eta)(2^j \xi) = 1 \quad \forall \xi \neq 0. \tag{2.1}$$

It is customary to assume $\eta \in \mathscr{S}(\mathbb{R}^N)$ is real-valued and symmetric in the sense that

$$\eta(-z) = \eta(z), \tag{2.2}$$

so that Δ_j is a self-adjoint operator with respect to the $L^2(\mathbb{R}^N)$-scalar product. We can and will also assume that $\mathcal{F}\eta(\xi) = \mathcal{F}\eta_0(\xi) - \mathcal{F}\eta_0(2\xi)$ for some Schwartz function η_0 with $\mathcal{F}\eta_0(\xi) = 1$ for $|\xi| \le 1$ and $\mathcal{F}\eta_0(\xi) = 0$ for $|\xi| \ge 2$. In particular

$$\int_{\mathbb{R}^N} \eta(x)\, dx = c\, \mathcal{F}(\eta)(0) = 0. \tag{2.3}$$

Also we have

$$\operatorname{supp} \mathcal{F}(\Delta_j f) \subset \left\{ \xi \in \mathbb{R}^N : \quad \frac{1}{2} \le |2^{-j}\xi| \le 2 \right\}. \tag{2.4}$$

In particular, $\mathcal{F}\eta(2^j\xi)\mathcal{F}\eta(2^{j+\ell}\xi) = 0$ whenever $|\ell| \ge 2$, and thus

$$\Delta_j f(x) = \sum_{\ell=j-1}^{j+1} \Delta_j \Delta_\ell f(x). \tag{2.5}$$

Then we have for any $f \in \mathscr{S}(\mathbb{R}^N)$, see [12, Exercise 1.1.4],

$$f(x) = \sum_{j \in \mathbb{Z}} \Delta_j f(x) \quad \forall x \in \mathbb{R}^N, \tag{2.6}$$

and the convergence is in $L^p(\mathbb{R}^N)$ for any $p \in (1, \infty]$. In particular, the set of all Schwartz functions whose Fourier transform is supported in a compact subset of $\mathbb{R}^N \setminus \{0\}$ is dense in $L^p(\mathbb{R}^N)$ if $1 < p < \infty$. For further reading on Littlewood-Paley projection we refer to [12, 6.2.2]. Below we write $\Delta_{\le 0} f$ for $f * \eta_0$.

Definition 2.3 (Triebel-Lizorkin space) Let $s \in \mathbb{R}$, $p, q \in (1, \infty)$. Then the inhomogeneous Triebel-Lizorkin space $F^s_{p,q}$ is defined as the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^N)$ such that

$$\|f\|_{F^s_{p,q}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \left(|\Delta_{\le 0} f(x)|^q + \sum_{j \ge 1} 2^{jsq} |\Delta_j f(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty.$$

For a tempered distribution $f \in \mathscr{S}'(\mathbb{R}^N)$ we also define its homogeneous Triebel-Lizorkin semi-norm as follows:

$$[f]_{\dot{F}^s_{p,q}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \left(\sum_{j \in \mathbb{Z}} 2^{jsq} |\Delta_j f(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

It is known, for instance, that if $s > 0$, $p, q \in (1, \infty)$ and $f \in F^s_{p,q}(\mathbb{R}^N)$, the homogeneous Triebel-Lizorkin semi-norm $[f]_{\dot{F}^s_{p,q}(\mathbb{R}^N)}$ is finite and $[f]_{\dot{F}^s_{p,q}(\mathbb{R}^N)} \lesssim_{s,p,q,N} \|f\|_{F^s_{p,q}(\mathbb{R}^N)}$.

The class of Triebel-Lizorkin spaces and Besov spaces (where the role of integral and sum are reversed) contains several classical function spaces, we refer e.g. to [24, § 2.1.2, p.14] or [28, § 2.3.5]. A well-known function space that is of Triebel-Lizorkin type is $L^p(\mathbb{R}^N)$ for $1 < p < \infty$: the theory of Hörmander-Mikhlin multipliers implies, for $1 < p < \infty$, that $L^p(\mathbb{R}^N) = F^0_{p,2}(\mathbb{R}^N)$ and $H^{1,p}(\mathbb{R}^N) = F^1_{p,2}(\mathbb{R}^N)$ with equivalence of norms. Furthermore:

Lemma 2.4 (Littlewood-Paley) *Let $p \in (1, \infty)$. Then for every $f \in L^p(\mathbb{R}^N)$ it holds*

$$\|f\|_{L^p(\mathbb{R}^N)} \approx [f]_{\dot{F}^0_{p,2}(\mathbb{R}^N)}.$$

Similarly, a function $f \in L^p(\mathbb{R}^N)$ is in $H^{1,p}(\mathbb{R}^N)$, if and only if $[f]_{\dot{F}^1_{p,2}(\mathbb{R}^N)} < \infty$, in which case

$$\|\nabla f\|_{L^p(\mathbb{R}^N)} \approx [f]_{\dot{F}^1_{p,2}(\mathbb{R}^N)}.$$

The implicit constants in these equivalences depend only on p and N.

For $0 < s < 1$ and $1 < p < \infty$, we also have $H^{s,p}(\mathbb{R}^N) = F^s_{p,2}(\mathbb{R}^N)$, with

$$\|f\|_{H^{s,p}(\mathbb{R}^N)} \approx \|f\|_{F^s_{p,2}(\mathbb{R}^N)}$$

where the constants depend on p and N (and uniform over $s \in (0, 1)$).

Recall the Gagliardo semi-norm

$$[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{1/p}$$

from the Introduction. It is also known that the following identification holds for $s \in (0, 1)$, $p \in (1, \infty)$:

$$[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \approx_{s,p,N} [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \quad \forall s \in (0, 1), \ f \in \mathscr{S}(\mathbb{R}^N),$$

and it is the objective of the present work to understand the dependency of the constant on s. It is important to observe that $F^1_{p,p}$ does *not* correspond to the classical Sobolev space $H^{1,p}(\mathbb{R}^N) = F^1_{p,2}(\mathbb{R}^N)$, unless $p = 2$.

We will need the following well-known vector-valued estimate for Littlewood-Paley projections, which follows from a vector-valued singular integral estimate (see e.g. [27, Chapter II.5.4]):

Lemma 2.5 *For any $1 < p < \infty$ and any $(f_j)_{j\in\mathbb{Z}} \in L^p(\ell^2)$, we have*

$$\left(\int_{\mathbb{R}^N}\left(\sum_{j\in\mathbb{Z}}|\Delta_j f_j(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \precsim \left(\int_{\mathbb{R}^N}\left(\sum_{j\in\mathbb{Z}}|f_j(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}$$

For $s > 0$ and $1 < p < \infty$, the Fourier multiplier $|\xi|^s(1+|\xi|^2)^{-s/2}$ defines a bounded linear map on $L^p(\mathbb{R}^N)$ (see [26, Chapter V]). Thus one can define a bounded linear map $(-\Delta)^{s/2}\colon H^{s,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$. It is known that $(-\Delta)^{\frac{s}{2}} : \dot{F}^{t+s}_{p,q} \to \dot{F}^t_{p,q}$ is an isomorphism, see [24, 2.6.2, Proposition 2] and [28, 5.2.3], [28, 2.3.8]. Their argument (basically a vector-valued multiplier theorem) implies:

Lemma 2.6 *Let $p, q \in (1,\infty)$, $\Theta > 0$. Then for any $s \in [0,\Theta]$ and any $f \in \mathscr{S}(\mathbb{R}^N)$ we have*

$$[f]_{\dot{F}^s_{p,q}} \approx [(-\Delta)^{\frac{s}{2}} f]_{\dot{F}^0_{p,q}}.$$

Also

$$[f]_{\dot{F}^s_{p,q}} \precsim [f]_{\dot{F}^0_{p,q}} + \left\|\left(\sum_{j\geq 0}|\Delta_j(-\Delta)^{\frac{s}{2}} f|^q\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^N)}.$$

The constant depends on p, q, N and Θ, and is otherwise independent of s.

Next we need the following result about the Triebel-Lizorkin norm of a weak limit in L^p:

Lemma 2.7 *Let $f_k \in L^p(\mathbb{R}^N)$ weakly converge to $f \in L^p(\mathbb{R}^N)$, and assume that for some $s_k \uparrow t \in (0,\infty)$ we have*

$$\sup_k [f_k]_{\dot{F}^{s_k}_{p,q}(\mathbb{R}^N)} < \infty.$$

Then

$$[f]_{\dot{F}^t_{p,q}(\mathbb{R}^N)} \leq \sup_k\, [f_k]_{\dot{F}^{s_k}_{p,q}(\mathbb{R}^N)}.$$

Proof For each fixed M and R,

$$\begin{aligned}\left\|\|2^{jt}\Delta_j f\|_{\ell^q(-M\leq j\leq M)}\right\|_{L^p(B(0,R))} &= \lim_{k\to\infty}\left\|\|2^{js_k}\Delta_j f_k\|_{\ell^q(-M\leq j\leq M)}\right\|_{L^p(B(0,R))}\\ &\leq \sup_k [f_k]_{\dot{F}^{s_k}_{p,q}(\mathbb{R}^N)}.\end{aligned}$$

In the estimate above, the middle equality follows from the fact that

$$\begin{aligned}\left(\sum_{j=-M}^{M}\left|2^{js_k}\Delta_j f_k - 2^{jt}\Delta_j f\right|^q\right)^{\frac{1}{q}} &\precsim \max_{j\in\{-M,\dots,M\}}|2^{js_k}-2^{jt}|\max_{j\in\{-M,\dots,M\}}|\Delta_j f|\\ &\quad + \max_{j\in\{-M,\dots,M\}} 2^{js_k}\max_{j\in\{-M,\dots,M\}}|\Delta_j f_k - \Delta_j f|\end{aligned}$$

which as $k \to \infty$ converges to 0 pointwise almost everywhere in $B(0, R)$, and hence in $L^p(B(0, R))$, if f_k converges weakly to f on $L^p(\mathbb{R}^N)$. □

2.4 A duality characterization for Triebel-Lizorkin spaces

The following duality statement must be known to experts – we did not find it in this precise form in the literature, and thus repeat the proof.

Theorem 2.8 (Duality) *Let $s \geq 0$, $p, q \in (1, \infty)$. For any $f \in \mathscr{S}(\mathbb{R}^N)$ there exist $g \in \mathcal{F}^{-1}(C_c^\infty(\mathbb{R}^N \backslash \{0\}))$ such that*

$$[g]_{\dot{F}^s_{p',q'}(\mathbb{R}^N)} \leq 1$$

and

$$[f]_{\dot{F}^s_{p,q}(\mathbb{R}^N)} \approx \left| \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} f(x) \, (-\Delta)^{\frac{s}{2}} g(x) dx \right|. \tag{2.7}$$

The constants depend on s, p, q, N, however if for some $\theta > 0$ we have $s \in [0, \theta)$, $p, q \in (1 + \frac{1}{\theta}, \theta)$, then the constant can be chosen only to depend on θ and N.

Observe that in (2.7), $(-\Delta)^{\frac{s}{2}} g$ belongs to the Schwartz class, since $\mathcal{F}g \in C_c^\infty(\mathbb{R}^N \backslash \{0\})$, consequently $\mathcal{F}(-\Delta)^{\frac{s}{2}} g \in C_c^\infty(\mathbb{R}^N \backslash \{0\})$, which implies $(-\Delta)^{\frac{s}{2}} g \in \mathscr{S}(\mathbb{R}^N)$. Moreover, it is easy to check that $f \in \mathscr{S}(\mathbb{R}^N)$ implies that $(-\Delta)^{\frac{s}{2}} f \in L^\infty(\mathbb{R}^N)$, so the integral on the right hand side of (2.7) makes sense.

Proof of Theorem 2.8 Once g is found, the $\gtrsim$-direction follows from two applications of Hölder's inequality and definition of the associated spaces.

So we focus on the $\lesssim$-direction. Let $f \in \mathscr{S}(\mathbb{R}^N)$. Then by Lemma 2.6

$$[f]_{\dot{F}^s_{p,q}(\mathbb{R}^N)} \approx [(-\Delta)^{\frac{s}{2}} f]_{\dot{F}^0_{p,q}(\mathbb{R}^N)}.$$

In particular $(\Delta_j (-\Delta)^{\frac{s}{2}} f)_{j \in \mathbb{Z}} \in L^p(\ell^q)$. In the case that $[f]_{\dot{F}^s_{p,q}} = 0$ we have f is zero since the only polynomial in $\mathscr{S}(\mathbb{R}^N)$ is zero, and thus (2.7) is trivially true for any g.

Consequently, from now own we assume $[(-\Delta)^{\frac{s}{2}} f]_{\dot{F}^0_{p,q}(\mathbb{R}^N)} > 0$. By monotone convergence theorem, there must be $K \in \mathbb{N}$, depending on f, such that

$$[(-\Delta)^{\frac{s}{2}} f]_{\dot{F}^0_{p,q}(\mathbb{R}^N)} \leq 2 \left(\int_{\mathbb{R}^N} \left(\sum_{j=-K}^{K} |\Delta_j (-\Delta)^{\frac{s}{2}} f(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

Applying Proposition 2.1, there exists $(\bar{h}_j)_{j=-K}^K \subset L^{p'}(\mathbb{R}^N)$ with

$$\left(\int_{\mathbb{R}^N} \left(\sum_{j=-K}^{K} |\bar{h}_j(x)|^{q'} \right)^{\frac{p'}{q'}} dx \right)^{\frac{1}{p'}} \leq 1,$$

such that

$$[(-\Delta)^{\frac{s}{2}} f]_{\dot{F}^0_{p,q}(\mathbb{R}^N)} \leq 2 \left| \int_{\mathbb{R}^N} \sum_{j=-K}^{K} \Delta_j (-\Delta)^{\frac{s}{2}} f(x) \, \bar{h}_j(x) \, dx \right|.$$

By density of $C_c^\infty(\mathbb{R}^N)$ in $L^{p'}(\mathbb{R}^N)$, there exists $h_{-K}, \dots, h_K \in C_c^\infty(\mathbb{R}^N)$ such that

$$\left\| \left(\sum_{j=-K}^{K} |h_j - \bar{h}_j|^{q'} \right)^{\frac{1}{q'}} \right\|_{L^{p'}(\mathbb{R}^N)} \leq \frac{1}{4}.$$

Consequently,

$$\left(\int_{\mathbb{R}^N} \left(\sum_{j=-K}^{K} |h_j(x)|^{q'} \right)^{\frac{p'}{q'}} dx \right)^{\frac{1}{p'}} \leq \frac{5}{4} \tag{2.8}$$

and

$$[(-\Delta)^{\frac{s}{2}} f]_{\dot{F}^0_{p,q}(\mathbb{R}^N)} \leq 2 \left| \int_{\mathbb{R}^N} \sum_{j=-K}^{K} \Delta_j (-\Delta)^{\frac{s}{2}} f(x)\, h_j(x)\, dx \right| + \frac{1}{2} [(-\Delta)^{\frac{s}{2}} f]_{\dot{F}^0_{p,q}(\mathbb{R}^N)},$$

which implies

$$[(-\Delta)^{\frac{s}{2}} f]_{\dot{F}^0_{p,q}(\mathbb{R}^N)} \leq 4 \left| \int_{\mathbb{R}^N} \sum_{j=-K}^{K} \Delta_j (-\Delta)^{\frac{s}{2}} f(x)\, h_j(x)\, dx \right|.$$

With an integration by parts (in this case this is just Fubini's theorem, using also symmetry (2.2)),

$$\int_{\mathbb{R}^N} \Delta_j (-\Delta)^{\frac{s}{2}} f(x) h_j(x)\, dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} f(x)\, \Delta_j h_j(x)\, dx.$$

Now we set

$$h := \sum_{j=-K}^{K} \Delta_j h_j(x), \quad \text{and} \quad g := (-\Delta)^{-\frac{s}{2}} h.$$

Then clearly $g \in \mathcal{F}^{-1}[C_c^\infty(\mathbb{R}^N \setminus \{0\})] \subset \mathscr{S}(\mathbb{R}^N)$, and the above shows that

$$[(-\Delta)^{\frac{s}{2}} f]_{\dot{F}^0_{p,q}} \leq 4 \left| \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} f(x)\, h(x)\, dx \right| = 4 \left| \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} f(x)\, (-\Delta)^{\frac{s}{2}} g(x)\, dx \right|.$$

Furthermore,

$$[g]_{\dot{F}^s_{p,q}} \approx [h]_{\dot{F}^0_{p,q}} \approx \max_{\ell=-1,0,1} \left(\int_{\mathbb{R}^N} \left(\sum_{j=-K}^{K} |\Delta_{j+\ell} \Delta_j h_j(x)|^{q'} \right)^{\frac{p'}{q'}} dx \right)^{\frac{1}{p'}}$$

By Lemma 2.5 and (2.8), we then have $[g]_{\dot{F}^s_{p,q}} \lesssim 1$. This completes the proof of this theorem. □

We also obtain the inhomogeneous version of Theorem 2.8.

Theorem 2.9 (Inhomogeneous Duality Estimate) *Let $s \geq 0$, $p, q \in (1, \infty)$. For any $f \in \mathscr{S}(\mathbb{R}^N)$ there exist $g \in \mathscr{S}(\mathbb{R}^N)$ with $\mathcal{F}g$ supported on $\{|\xi| \geq 1/4\}$ such that*

$$[g]_{\dot{F}^s_{p',q'}} \leq 1$$

and

$$[f]_{\dot F^s_{p,q}(\mathbb{R}^N)} \lesssim [f]_{\dot F^0_{p,q}(\mathbb{R}^N)} + \left| \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} f(x)\, (-\Delta)^{\frac{s}{2}} g(x)\, dx \right|.$$

The constants depend on s, p, q, N, however if for some $\theta > 0$ we have $s \in [0, \theta)$, $p, q \in (1 + \frac{1}{\theta}, \theta)$, then the constant can be chosen only to depend on θ and N.

Proof We have

$$[f]_{\dot F^s_{p,q}(\mathbb{R}^N)} \le [f]_{\dot F^0_{p,q}(\mathbb{R}^N)} + [\tilde f]_{\dot F^s_{p,q}(\mathbb{R}^N)} \quad \text{where} \quad \tilde f := \sum_{k \ge 0} \Delta_k f.$$

Following the proof of Theorem 2.8, one can find $\tilde g \in \mathscr{S}(\mathbb{R}^N)$, with $\mathcal{F}\tilde g$ supported on $\{|\xi| \ge 1/4\}$, such that $[\tilde g]_{\dot F^s_{p',q'}} \le 1$ and

$$[\tilde f]_{\dot F^s_{p,q}(\mathbb{R}^N)} \lesssim \left| \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \tilde f(x)\, (-\Delta)^{\frac{s}{2}} \tilde g(x)\, dx \right| = \left| \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} f(x)\, (-\Delta)^{\frac{s}{2}} \sum_{k \ge 0} \Delta_k \tilde g(x)\, dx \right|.$$

It remains to check that $g := \sum_{k \ge 0} \Delta_k \tilde g$ satisfies the conclusion of the theorem. □

3 An easy proof for the estimates for $\dot W^{s,p}$ when $p = 2$

As a curiosity we give now a simple proof of the equivalence between $\dot W^{s,2}$ and $\dot F^s_{2,2}$ semi-norms.

Proposition 3.1 *Let $s \in (0, 1)$ and $f \in \mathscr{S}(\mathbb{R}^N)$. Then it holds with constants independent of s and f,*

$$\min\{s, (1-s)\}^{\frac{1}{2}} [f]_{\dot W^{s,2}(\mathbb{R}^N)} \approx [f]_{\dot F^s_{2,2}(\mathbb{R}^N)}.$$

Proof Let $f \in \mathscr{S}(\mathbb{R}^N)$. Then by Lemma 2.6 and Fubini's theorem,

$$[f]^2_{\dot F^s_{2,2}(\mathbb{R}^N)} \approx [(-\Delta)^{\frac{s}{2}} f]^2_{\dot F^{0,2}_2(\mathbb{R}^N)} = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \Delta_j f(x)\, \overline{(-\Delta)^{\frac{s}{2}} \Delta_j f(x)}\, dx.$$

Integrating by parts (via the Fourier transform) we have

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \Delta_j f(x)\, \overline{(-\Delta)^{\frac{s}{2}} \Delta_j f(x)}\, dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{2s}{2}} \Delta_j f(x)\, \overline{\Delta_j f(x)}\, dx.$$

With the integral characterization of the fractional Laplacian, Lemma 2.2, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} (-\Delta)^{\frac{2s}{2}} \Delta_j f(x)\, \overline{\Delta_j f(x)}\, dx \\ &= c_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(2\Delta_j f(x) - \Delta_j f(x+z) - \Delta_j f(x-z))\, \overline{\Delta_j f(x)}}{|z|^{N+2s}}\, dz\, dx. \end{aligned} \tag{3.1}$$

We note that by a change of variables $x \mapsto x + z$, we have

$$\int_{\mathbb{R}^N} (\Delta_j f(x) - \Delta_j f(x-z)) \overline{\Delta_j f(x)} dx = \int_{\mathbb{R}^N} (\Delta_j f(x+z) - \Delta_j f(x)) \overline{\Delta_j f(x+z)} dx. \tag{3.2}$$

Hence (3.1) is equal to

$$c_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\Delta_j f(x+z)-\Delta_j f(x)|^2}{|z|^{N+2s}}\,dx\,dz = c_{N,s}\int_{\mathbb{R}^N}\frac{\|\Delta_j f(\cdot+z)-\Delta_j f(\cdot)\|^2_{L^2(\mathbb{R}^N)}}{|z|^{N+2s}}dz.$$

We obtain via Lemma 2.4

$$\sum_{j\in\mathbb{Z}}\|\Delta_j f(\cdot+z)-\Delta_j f(\cdot)\|^2_{L^2(\mathbb{R}^N)} = \|f(\cdot+z)-f(\cdot)\|^2_{L^2(\mathbb{R}^N)},$$

from which we deduce

$$[f]_{\dot{F}^s_{2,2}(\mathbb{R}^N)} \approx \left(c_{N,s}\int_{\mathbb{R}^N}\frac{\|f(\cdot+z)-f(\cdot)\|^2_{L^2(\mathbb{R}^N)}}{|z|^{N+2s}}dz\right)^{\frac{1}{2}} = c_{N,s}^{\frac{1}{2}}[f]_{\dot{W}^{s,2}(\mathbb{R}^N)}.$$

The proposition then follows from the estimate $c_{N,s}\approx\min\{s,(1-s)\}$ in Lemma 2.2. □

4 The upper bounds for $[f]_{\dot{W}^{s,p}}$ in Theorems 1.2 and 1.5

In this section we prove the upper bounds for $[f]_{\dot{W}^{s,p}}$ in Theorems 1.2 and 1.5, namely we show

Theorem 4.1 *Let $p\in(1,\infty)$, $s\in(0,1)$ and $f\in\mathscr{S}(\mathbb{R}^N)$. Then*

$$[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \lesssim \left(\frac{1}{s^{\frac{1}{p}}}+\frac{1}{(1-s)^{\frac{1}{p}}}\right)[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \quad \text{if } 1<p\le 2, \tag{4.1}$$

$$[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \lesssim \left(\frac{1}{s^{\frac{1}{2}}}+\frac{1}{(1-s)^{\frac{1}{2}}}\right)[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \quad \text{if } 2\le p<\infty, \tag{4.2}$$

and

$$[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \lesssim \left(\frac{1}{s^{\frac{1}{p}}}+\frac{1}{(1-s)^{\frac{1}{p}}}\right)[f]_{\dot{F}^s_{p,2}(\mathbb{R}^N)} \quad \text{if } 2\le p<\infty. \tag{4.3}$$

(4.1) and (4.2) have been proven by Gu and the third author in [13], and (4.3) is a slight adaptation of their argument. We still present it for the sake of completeness.

Below we repeatedly use the following estimate for geometric sums: for $1<p<\infty$,

$$\sum_{j\ge 0}2^{-jsp} = \frac{1}{1-2^{-sp}} \approx \frac{1}{s} \quad \text{for } s>0 \tag{4.4}$$

and

$$\sum_{j\le 0}2^{j\sigma p} = \frac{1}{1-2^{-\sigma p}} \approx \frac{1}{\sigma} \quad \text{for } \sigma>0. \tag{4.5}$$

The first step for (4.1), (4.2) and (4.3) is the following estimate.

Lemma 4.2 *Let $p \in (1, \infty)$ and $s \in (0, 1)$. Then*

$$[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \lesssim \left(\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^N} \left(\sum_{j \geq 0} |2^{ks} \Delta_{k+j} f(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} + \left(\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^N} \left(\sum_{j \leq 0} \left| 2^j 2^{ks} \Delta_{k+j} f(x) \right|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.$$

Proof We have

$$[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \frac{\|f(\cdot + z) - f(\cdot)\|^p_{L^p(\mathbb{R}^N)}}{|z|^{N+sp}} dz \right)^{\frac{1}{p}} \lesssim \left(\sum_{k \in \mathbb{Z}} 2^{ksp} \sup_{|z| \approx 2^{-k}} \|f(\cdot + z) - f(\cdot)\|^p_{L^p(\mathbb{R}^N)} \right)^{\frac{1}{p}}.$$

But for $|z| \approx 2^{-k}$, Littlewood-Paley implies

$$\|f(\cdot + z) - f(\cdot)\|_{L^p(\mathbb{R}^N)} \lesssim \left(\int_{\mathbb{R}^N} \left(\sum_{j \in \mathbb{Z}} |\Delta_{k+j} f(x+z) - \Delta_{k+j} f(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}$$

which by the triangle inequality is

$$\lesssim \left(\int_{\mathbb{R}^N} \left(\sum_{j \geq 0} |\Delta_{k+j} f(x+z) - \Delta_{k+j} f(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N} \left(\sum_{j < 0} |\Delta_{k+j} f(x+z) - \Delta_{k+j} f(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.$$

The first term above is bounded by the triangle inequality by

$$2 \left(\int_{\mathbb{R}^N} \left(\sum_{j \geq 0} |\Delta_{k+j} f(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.$$

For the second term, the fundamental theorem of calculus implies

$$|\Delta_{k+j} f(x+z) - \Delta_{k+j} f(x)| \lesssim |z| \int_0^1 |\nabla \Delta_{k+j} f(x + tz)| dt,$$

so Minkowski's inequality implies

$$\left(\int_{\mathbb{R}^N}\left(\sum_{j<0}|\Delta_{k+j}f(x+z)-\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}$$
$$\lesssim |z|\int_0^1\left(\int_{\mathbb{R}^N}\left(\sum_{j<0}|\nabla\Delta_{k+j}f(x+tz)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}dt$$

which is

$$\approx 2^{-k}\left(\int_{\mathbb{R}^N}\left(\sum_{j<0}|\nabla\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\lesssim\left(\int_{\mathbb{R}^N}\left(\sum_{j<0}|2^j\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}$$

by the Littlewood-Paley inequality again. Altogether, we get

$$2^{ksp}\sup_{|z|\approx 2^{-k}}\|f(\cdot+z)-f(\cdot)\|^p_{L^p(\mathbb{R}^N)}\lesssim\int_{\mathbb{R}^N}\left(\sum_{j\geq 0}|2^{ks}\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}}dx$$
$$+\int_{\mathbb{R}^N}\left(\sum_{j<0}|2^j2^{ks}\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}}dx$$

which implies the desired estimate. □

Now (4.1) is a consequence of Lemma 4.2 and the following proposition.

Proposition 4.3 *Let $1<p\leq 2$. Then*

$$\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\left(\sum_{j\geq 0}|2^{ks}\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\lesssim\frac{1}{s^{\frac{1}{p}}}[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)}.$$

and

$$\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\left(\sum_{j<0}\left|2^j2^{ks}\Delta_{k+j}f(x)\right|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\lesssim\frac{1}{(1-s)^{\frac{1}{p}}}[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)}.$$

Proof Since $p\in(1,2]$, we have $\left|\sum_j F_j\right|^{\frac{p}{2}}\leq\sum_j|F_j|^{\frac{p}{2}}$. Thus

$$\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\left(\sum_{j\geq 0}|2^{ks}\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\leq\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\sum_{j\geq 0}|2^{ks}\Delta_{k+j}f(x)|^pdx\right)^{\frac{1}{p}},$$

and we conclude by noting that for $s > 0$, (4.4) gives

$$\sum_{k\in\mathbb{Z}}\sum_{j\geq 0}|2^{ks}\Delta_{k+j}f(x)|^p = \sum_{j\geq 0}2^{-jsp}\sum_{k\in\mathbb{Z}}|2^{(k+j)s}\Delta_{k+j}f(x)|^p \approx \frac{1}{s}\sum_{k\in\mathbb{Z}}|2^{ks}\Delta_k f(x)|^p;$$

similarly

$$\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\left(\sum_{j<0}\left|2^j2^{ks}\Delta_{k+j}f(x)\right|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}} \leq \left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\sum_{j<0}|2^j2^{ks}\Delta_{k+j}f(x)|^p dx\right)^{\frac{1}{p}},$$

and we conclude by noting that for $s < 1$, (4.5) with $\sigma = 1 - s$ gives

$$\sum_{k\in\mathbb{Z}}\sum_{j<0}|2^j2^{ks}\Delta_{k+j}f(x)|^p = \sum_{j<0}2^{j(1-s)p}\sum_{k\in\mathbb{Z}}|2^{(k+j)s}\Delta_{k+j}f(x)|^p \approx \frac{1}{1-s}\sum_{k\in\mathbb{Z}}|2^{ks}\Delta_k f(x)|^p.$$

□

Next, (4.2) is a consequence Lemma 4.2 and the following

Proposition 4.4 *Let $2 \leq p < \infty$. Then*

$$\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\left(\sum_{j\geq 0}|2^{ks}\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}} \lesssim \frac{1}{s^{\frac{1}{2}}}[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)}$$

and

$$\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\left(\sum_{j<0}\left|2^j2^{ks}\Delta_{k+j}f(x)\right|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}} \lesssim \frac{1}{(1-s)^{\frac{1}{2}}}[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)}.$$

Proof Since $p \geq 2$ we can apply Minkowski inequality for $\ell^{\frac{p}{2}}(L^{\frac{p}{2}}(\mathbb{R}^N))$ and get

$$\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\left(\sum_{j\geq 0}|2^{ks}\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}} \leq \left(\sum_{j\geq 0}\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}|2^{ks}\Delta_{k+j}f(x)|^p dx\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}$$

which for $s > 0$ is equal to

$$\left(\sum_{j\geq 0}2^{-2js}\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}|2^{(k+j)s}\Delta_{k+j}f(x)|^p dx\right)^{\frac{2}{p}}\right)^{\frac{1}{2}} = \left(\sum_{j\geq 0}2^{-2js}[f]^2_{\dot{F}^s_{p,p}}\right)^{\frac{1}{2}} \approx \frac{1}{s^{\frac{1}{2}}}[f]_{\dot{F}^s_{p,p}}$$

using (4.4) with $p = 2$. Similarly,

$$\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\left(\sum_{j<0}\left|2^j2^{ks}\Delta_{k+j}f(x)\right|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}} \leq \left(\sum_{j<0}\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\left|2^j2^{ks}\Delta_{k+j}f(x)\right|^p dx\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}$$

which for $s<1$ is equal to

$$\left(\sum_{j<0} 2^{2j(1-s)}\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}|2^{(k+j)s}\Delta_{k+j}f(x)|^p\,dx\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}$$

$$=\left(\sum_{j<0} 2^{2j(1-s)}[f]^2_{\dot{F}^s_{p,p}}\right)^{\frac{1}{2}} \approx \frac{1}{(1-s)^{\frac{1}{2}}}[f]_{\dot{F}^s_{p,p}}$$

using (4.5) with $\sigma=1-s$ and $p=2$. □

Lastly, (4.3) is a consequence of Lemma 4.2 and the following proposition.

Proposition 4.5 *Let $2\le p<\infty$. Then*

$$\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\left(\sum_{j\ge 0}|2^{ks}\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}} \lesssim \frac{1}{s^{\frac{1}{p}}}[f]_{\dot{F}^s_{p,2}}$$

and

$$\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^N}\left(\sum_{j<0}\left|2^j2^{ks}\Delta_{k+j}f(x)\right|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}} \lesssim \frac{1}{(1-s)^{\frac{1}{p}}}[f]_{\dot{F}^s_{p,2}}.$$

Proof Fix $x\in\mathbb{R}^N$. We have for any $k\in\mathbb{Z}$

$$\sum_{j\ge 0}|2^{ks}\Delta_{k+j}f(x)|^2 \le \sum_{j\ge 0}|2^{(k+j)s}\Delta_{k+j}f(x)|^2 \le \sum_{\ell\in\mathbb{Z}}|2^{\ell s}\Delta_\ell f(x)|^2.$$

Consequently, since $p\ge 2$ we have

$$\sum_{k\in\mathbb{Z}}\left(\sum_{j\ge 0}|2^{ks}\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}} \le \sum_{k\in\mathbb{Z}}\sum_{j\ge 0}|2^{ks}\Delta_{k+j}f(x)|^2\left(\sum_{\ell\in\mathbb{Z}}|2^{\ell s}\Delta_\ell f(x)|^2\right)^{\frac{p}{2}-1}$$

which is

$$=\sum_{j\ge 0}2^{-2js}\sum_{k\in\mathbb{Z}}|2^{(k+j)s}\Delta_{k+j}f(x)|^2\left(\sum_{\ell\in\mathbb{Z}}|2^{\ell s}\Delta_\ell f(x)|^2\right)^{\frac{p}{2}-1}$$

$$\approx \frac{1}{s}\left(\sum_{\ell\in\mathbb{Z}}|2^{\ell s}\Delta_\ell f(x)|^2\right)^{\frac{p}{2}}$$

using (4.4) with $p=2$. Integrating this with respect to x gives the first inequality. Similarly, for any $k\in\mathbb{Z}$

$$\sum_{j<0}|2^j2^{ks}\Delta_{k+j}f(x)|^2 \le \sum_{j<0}|2^{(k+j)s}\Delta_{k+j}f(x)|^2 \le \sum_{\ell\in\mathbb{Z}}|2^{\ell s}\Delta_\ell f(x)|^2.$$

Consequently, since $p \geq 2$ we have

$$\sum_{k\in\mathbb{Z}} \left(\sum_{j<0} |2^{ks} \Delta_{k+j} f(x)|^2 \right)^{\frac{p}{2}} \leq \sum_{k\in\mathbb{Z}} \sum_{j<0} |2^{ks} \Delta_{k+j} f(x)|^2 \left(\sum_{\ell\in\mathbb{Z}} |2^{\ell s} \Delta_\ell f(x)|^2 \right)^{\frac{p}{2}-1}$$

which is

$$= \sum_{j<0} 2^{2j(1-s)} \sum_{k\in\mathbb{Z}} |2^{(k+j)s} \Delta_{k+j} f(x)|^2 \left(\sum_{\ell\in\mathbb{Z}} |2^{\ell s} \Delta_\ell f(x)|^2 \right)^{\frac{p}{2}-1}$$

$$\approx \frac{1}{1-s} \left(\sum_{\ell\in\mathbb{Z}} |2^{\ell s} \Delta_\ell f(x)|^2 \right)^{\frac{p}{2}}$$

using (4.5) with $\sigma = 1 - s$ and $p = 2$. Integrating this with respect to x gives the second inequality. $\square$

5 The upper bound for $[f]_{\dot{W}^{s,p}}$ in of Theorem 1.6

In this section we prove the first part of Theorem 1.6, which provides an upper bound for $[f]_{\dot{W}^{s,p}}$ in terms of $[f]_{\dot{F}^r_{p,2}}$ and $[f]_{\dot{F}^t_{p,2}}$ when $0 \leq r < s < t \leq 1$ and $1 < p < \infty$. Namely, we show that for any such r, s, t, p and $f \in \mathscr{S}(\mathbb{R}^N)$,

$$[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \lesssim \frac{1}{(s-r)^{\frac{1}{p}}} [f]_{\dot{F}^r_{p,2}(\mathbb{R}^N)} + \frac{1}{(t-s)^{\frac{1}{p}}} [f]_{\dot{F}^t_{p,2}(\mathbb{R}^N)}. \tag{1.5}$$

The constant C depends on p and N only. In view of Lemma 4.2, (1.5) is then a consequence of the following four lemmata.

Lemma 5.1 *Let $p \in (1, \infty)$, $0 \leq r < s$. Then*

$$\left(\sum_{k<0} \int_{\mathbb{R}^N} \left(\sum_{j\geq 0} |2^{ks} \Delta_{k+j} f(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \lesssim \frac{1}{(s-r)^{\frac{1}{p}}} [f]_{\dot{F}^r_{p,2}(\mathbb{R}^N)}.$$

Proof We have for any $0 \leq r < s$

$$\sum_{k<0} \left(\sum_{j\geq 0} \left|2^{ks} \Delta_{k+j} f(x)\right|^2 \right)^{\frac{p}{2}} \leq \sum_{k<0} \left(\sum_{j\geq 0} \left|2^{jr} 2^{ks} \Delta_{k+j} f(x)\right|^2 \right)^{\frac{p}{2}}$$

$$= \sum_{k<0} 2^{k(s-r)p} \left(\sum_{j<0} \left|2^{(k+j)r} \Delta_{k+j} f(x)\right|^2 \right)^{\frac{p}{2}}.$$

We extend the sum over j to all integers, and use (4.5) with $s - r$ in place of σ to evaluate the sum over k. This gives

$$\sum_{k<0} \left(\sum_{j\geq 0} \left|2^{ks} \Delta_{k+j} f(x)\right|^2 \right)^{\frac{p}{2}} \lesssim \frac{1}{s-r} \left(\sum_{\ell\in\mathbb{Z}} \left|2^{\ell r} \Delta_\ell f(x)\right|^2 \right)^{\frac{p}{2}},$$

which gives the conclusion of the lemma upon integrating in x. $\square$

Lemma 5.2 *Let $p \in (1, \infty)$ and $s < t \leq 1$. Then*

$$\left(\sum_{k \geq 0} \int_{\mathbb{R}^N} \left(\sum_{j<0} \left| 2^j 2^{ks} \Delta_{k+j} f(x) \right|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \lesssim \frac{1}{(t-s)^{\frac{1}{p}}} [f]_{\dot{F}^t_{p,2}(\mathbb{R}^N)}.$$

Proof We have for any $s < t \leq 1$

$$\begin{aligned} \sum_{k \geq 0} \left(\sum_{j<0} \left| 2^j 2^{ks} \Delta_{k+j} f(x) \right|^2 \right)^{\frac{p}{2}} &\leq \sum_{k \geq 0} \left(\sum_{j<0} \left| 2^{jt} 2^{ks} \Delta_{k+j} f(x) \right|^2 \right)^{\frac{p}{2}} \\ &= \sum_{k \geq 0} 2^{-k(t-s)p} \left(\sum_{j<0} \left| 2^{(k+j)t} \Delta_{k+j} f(x) \right|^2 \right)^{\frac{p}{2}}. \end{aligned}$$

We extend the sum over j to all integers, and use (4.4) with $t-s$ in place of s to evaluate the sum over k. This gives

$$\sum_{k \geq 0} \left(\sum_{j<0} \left| 2^j 2^{ks} \Delta_{k+j} f(x) \right|^2 \right)^{\frac{p}{2}} \lesssim \frac{1}{t-s} \left(\sum_{\ell \in \mathbb{Z}} \left| 2^{\ell t} \Delta_\ell f(x) \right|^2 \right)^{\frac{p}{2}},$$

which gives the conclusion of the lemma upon integrating in x. $\square$

Lemma 5.3 *Let $p \in (1, \infty)$, $r < s$ and $r \leq 1$. Then*

$$\left(\sum_{k \leq 0} \int_{\mathbb{R}^N} \left(\sum_{j<0} \left| 2^j 2^{ks} \Delta_{k+j} f(x) \right|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \lesssim \frac{1}{(s-r)^{\frac{1}{p}}} [f]_{\dot{F}^r_{p,2}(\mathbb{R}^N)}.$$

Proof We have for any $r \leq 1$ and $s > r$

$$\begin{aligned} \sum_{k \leq 0} \left(\sum_{j<0} \left| 2^j 2^{ks} \Delta_{k+j} f(x) \right|^2 \right)^{\frac{p}{2}} &\leq \sum_{k \leq 0} \left(\sum_{j<0} \left| 2^{jr} 2^{ks} \Delta_{k+j} f(x) \right|^2 \right)^{\frac{p}{2}} \\ &= \sum_{k \leq 0} 2^{k(s-r)p} \left(\sum_{j<0} \left| 2^{(k+j)r} \Delta_{k+j} f(x) \right|^2 \right)^{\frac{p}{2}}. \end{aligned}$$

We extend the sum over j to all integers, and use (4.5) with $s-r$ in place of σ to evaluate the sum over k. This gives

$$\sum_{k \leq 0} \left(\sum_{j<0} \left| 2^j 2^{ks} \Delta_{k+j} f(x) \right|^2 \right)^{\frac{p}{2}} \lesssim \frac{1}{s-r} \left(\sum_{\ell \in \mathbb{Z}} \left| 2^{\ell r} \Delta_\ell f(x) \right|^2 \right)^{\frac{p}{2}},$$

which gives the conclusion of the lemma upon integrating in x. $\square$

Lemma 5.4 *Let $p \in (1,\infty)$, $s < t$ and $t \geq 0$. Then*

$$\left(\sum_{k\geq 0}\int_{\mathbb{R}^N}\left(\sum_{j\geq 0}|2^{ks}\Delta_{k+j}f(x)|^2\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}} \lesssim \frac{1}{(t-s)^{\frac{1}{p}}}[f]_{\dot{F}^{t,p}_2(\mathbb{R}^N)}.$$

Proof We have for any $t \geq 0$ and $s < t$

$$\begin{aligned}\sum_{k\geq 0}\left(\sum_{j\geq 0}\left|2^{ks}\Delta_{k+j}f(x)\right|^2\right)^{\frac{p}{2}} &\leq \sum_{k\geq 0}\left(\sum_{j\geq 0}\left|2^{jt}2^{ks}\Delta_{k+j}f(x)\right|^2\right)^{\frac{p}{2}}\\ &= \sum_{k\geq 0}2^{-k(t-s)p}\left(\sum_{j<0}\left|2^{(k+j)t}\Delta_{k+j}f(x)\right|^2\right)^{\frac{p}{2}}.\end{aligned}$$

We extend the sum over j to all integers, and use (4.4) with $t-s$ in place of s to evaluate the sum over k. This gives

$$\sum_{k\geq 0}\left(\sum_{j\geq 0}\left|2^{ks}\Delta_{k+j}f(x)\right|^2\right)^{\frac{p}{2}} \lesssim \frac{1}{t-s}\left(\sum_{\ell\in\mathbb{Z}}\left|2^{\ell t}\Delta_{\ell}f(x)\right|^2\right)^{\frac{p}{2}},$$

which gives the conclusion of the lemma upon integrating in x. □

6 The lower bounds for $[f]_{\dot{W}^{s,p}}$: proof via duality

We obtain the lower bounds for $[f]_{\dot{W}^{s,p}}$ in Theorem 1.2, Theorem 1.5 and Theorem 1.6 from the corresponding upper bounds by a duality argument, and using the integral representation of the fractional Laplacian, adapting the proof of Proposition 3.1.

Our main ingredient is the following duality estimate.

Proposition 6.1 *Let $p, q \in (1,\infty)$, $s \in (0,1)$. Let $t_1, t_2 > 0$, such that $t_1 + t_2 = 2s$. Let $p_1, p_2 \in (1,\infty)$, such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then for any $f \in \mathscr{S}(\mathbb{R}^N)$ we have*

$$[f]_{\dot{F}^s_{p,q}(\mathbb{R}^N)} \lesssim \min\{s,(1-s)\}\,[f]_{\dot{W}^{t_1,p_1}(\mathbb{R}^N)} \sup_{[g]_{\dot{F}^s_{p',q'}}\leq 1}[g]_{\dot{W}^{t_2,p_2}(\mathbb{R}^N)},$$

where the supremum on the right-hand side is over Schwartz functions $g \in \mathcal{F}^{-1}(C_c^\infty(\mathbb{R}^N\backslash\{0\}))$.

We also have

$$[f]_{\dot{F}^s_{p,q}(\mathbb{R}^N)} \lesssim [f]_{\dot{F}^0_{p,q}(\mathbb{R}^N)} + \min\{s,(1-s)\}\,[f]_{\dot{W}^{t_1,p_1}(\mathbb{R}^N)} \sup_{\substack{[g]_{\dot{F}^s_{p',q'}}\leq 1\\ supp\mathcal{F}g\subset\{|\xi|\geq 1/4\}}}[g]_{\dot{W}^{t_2,p_2}(\mathbb{R}^N)}$$

where this time the supremum on the right-hand side is over Schwartz functions g with the indicated constraints.

Proof By duality, Theorem 2.8, for any $f \in \mathscr{S}(\mathbb{R}^N)$ there exists $g \in \mathcal{F}^{-1}(C_c^\infty(\mathbb{R}^N\backslash\{0\}))$ with $[g]_{\dot{F}^s_{p',q'}} \leq 1$ and

$$[f]_{\dot{F}^s_{p,q}(\mathbb{R}^N)} \lesssim \left|\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}f(x)(-\Delta)^{\frac{s}{2}}g(x)\,dx\right|. \tag{6.1}$$

Integrating by parts the operator $(-\Delta)^{\frac{s}{2}}$ (this can be done via Fourier transform and Plancherel, since $f, g \in \mathscr{S}(\mathbb{R}^N)$) we find for a constant $C_{\mathcal{F}}$ depending on the precise choice of Fourier transform

$$\begin{aligned}\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} f (-\Delta)^{\frac{s}{2}} g =& C_{\mathcal{F}} \int_{\mathbb{R}^N} |\xi|^s \mathcal{F} f(\xi)\, |\xi|^s \mathcal{F} g(-\xi)\, d\xi \\ =& C_{\mathcal{F}} \int_{\mathbb{R}^N} |\xi|^{2s} \mathcal{F} f(\xi)\, \mathcal{F} g(-\xi)\, d\xi = \int_{\mathbb{R}^N} (-\Delta)^s f\, g.\end{aligned}$$

From Lemma 2.2 and symmetry arguments we then find with a constant $c_{N,s}$ such that $c_{N,s} \approx \min\{s, 1-s\}$ and

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} f (-\Delta)^{\frac{s}{2}} g = c_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(2f(x) - f(x+z) - f(x-z))g(x)}{|z|^{N+2s}}\, dz dx$$

Now since $f, g \in \mathscr{S}(\mathbb{R}^N)$ and $2s \in (0, 2)$, we may apply Fubini's theorem to interchange the z and the x integral, and use a similar change of variable as in (3.2). Then

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} f (-\Delta)^{\frac{s}{2}} g = c_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(f(x+z) - f(x))(g(x+z) - g(x))}{|z|^{N+2s}}\, dx dz.$$

Hence using the bound for $c_{N,s}$, and writing $N + 2s = \frac{N}{p_1} + t_1 + \frac{N}{p_2} + t_2$, we obtain

$$[f]_{\dot{F}^s_{p,p}} \lesssim \min\{s, (1-s)\} \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(f(x+z) - f(x))}{|z|^{\frac{N}{p_1}+t_1}} \frac{(g(x+z) - g(x))}{|z|^{\frac{N}{p_2}+t_2}}\, dx dz \right|.$$

Applying twice the integral Hölder inequality yields

$$[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \lesssim \min\{s, (1-s)\} [f]_{\dot{W}^{t_1,p_1}(\mathbb{R}^N)}\, [g]_{\dot{W}^{t_2,p_2}(\mathbb{R}^N)}.$$

This concludes the proof of the first inequality.

The proof of the second inequality is very similar. Instead of Theorem 2.8 we use Theorem 2.9 and obtain instead of (6.1)

$$[f]_{\dot{F}^s_{p,q}} \lesssim [f]_{\dot{F}^0_{p,q}(\mathbb{R}^N)} + \left| \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} f(x)\, (-\Delta)^{\frac{s}{2}} g(x)\, dx \right|,$$

where this time we have $g \in \mathscr{S}(\mathbb{R}^N)$, $\mathcal{F} g$ supported on $\{|\xi| \geq 1/4\}$, and $[g]_{\dot{F}^s_{p',q'}(\mathbb{R}^N)} \leq 1$. The remaining arguments are the same. $\square$

With Proposition 6.1 we obtain the lower bound of (1.2).

Proposition 6.2 *Let $p \in [2, \infty)$, $s \in (0, 1)$ and $f \in \mathscr{S}(\mathbb{R}^N)$. Then*

$$\left(\frac{1}{s^{\frac{1}{p}}} + \frac{1}{(1-s)^{\frac{1}{p}}} \right) [f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \lesssim [f]_{\dot{W}^{s,p}(\mathbb{R}^N)}.$$

Proof Since $p \in [2, \infty)$, we have $p' = \frac{p}{p-1} \in (1, 2]$. So from the upper bound (1.1) for $g \in \mathscr{S}(\mathbb{R}^N)$ we have

$$[g]_{\dot{W}^{s,p'}(\mathbb{R}^N)} \lesssim \left(\frac{1}{s^{\frac{1}{p'}}} + \frac{1}{(1-s)^{\frac{1}{p'}}} \right) [g]_{\dot{F}^s_{p',p'}(\mathbb{R}^N)}.$$

From Proposition 6.1, we thus obtain

$$[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \lesssim \min\{s,(1-s)\}\left(\frac{1}{s^{\frac{1}{p'}}}+\frac{1}{(1-s)^{\frac{1}{p'}}}\right)[f]_{\dot{W}^{s,p}(\mathbb{R}^N)}.$$

Now we can conclude since for any $s \in (0,1)$

$$\left(\min\{s,(1-s)\}\left(\frac{1}{s^{\frac{1}{p'}}}+\frac{1}{(1-s)^{\frac{1}{p'}}}\right)\right)^{-1} \approx \min\{s,(1-s)\}^{-\frac{1}{p}} \approx \left(\frac{1}{s^{\frac{1}{p}}}+\frac{1}{(1-s)^{\frac{1}{p}}}\right) \tag{6.2}$$

□

We also obtain the lower bound of (1.1):

Proposition 6.3 *Let $p \in (1,2]$, $s \in (0,1)$ and $f \in \mathscr{S}(\mathbb{R}^N)$. Then*

$$\left(\frac{1}{s^{\frac{1}{2}}}+\frac{1}{(1-s)^{\frac{1}{2}}}\right)[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \leq C[f]_{\dot{W}^{s,p}(\mathbb{R}^N)}$$

Proof Since $p \in (1,2]$, we have $p' = \frac{p}{p-1} \in [2,\infty)$. So from the upper bound of (1.2) for $g \in \mathscr{S}(\mathbb{R}^N)$ we have

$$[g]_{\dot{W}^{s,p'}(\mathbb{R}^N)} \lesssim \left(\frac{1}{s^{\frac{1}{2}}}+\frac{1}{(1-s)^{\frac{1}{2}}}\right)[g]_{\dot{F}^s_{p',p'}(\mathbb{R}^N)}.$$

From Proposition 6.1, we thus obtain

$$[f]_{\dot{F}^s_{p,p}(\mathbb{R}^N)} \lesssim \min\{s,(1-s)\}\left(\frac{1}{s^{\frac{1}{2}}}+\frac{1}{(1-s)^{\frac{1}{2}}}\right)[f]_{\dot{W}^{s,p}(\mathbb{R}^N)}.$$

Now we can conclude since for any $s \in (0,1)$

$$\left(\min\{s,(1-s)\}\left(\frac{1}{s^{\frac{1}{2}}}+\frac{1}{(1-s)^{\frac{1}{2}}}\right)\right)^{-1} \approx \min\{s,(1-s)\}^{-\frac{1}{2}} \approx \left(\frac{1}{s^{\frac{1}{2}}}+\frac{1}{(1-s)^{\frac{1}{2}}}\right).$$

□

Next is the proof of (1.3).

Proposition 6.4 *Let $p \in (1,2]$, $s \in (0,1)$ and $f \in \mathscr{S}(\mathbb{R}^N)$. Then*

$$\left(\frac{1}{s^{\frac{1}{p}}}+\frac{1}{(1-s)^{\frac{1}{p}}}\right)[f]_{\dot{F}^s_{p,2}(\mathbb{R}^N)} \lesssim [f]_{\dot{W}^{s,p}(\mathbb{R}^N)}.$$

Proof Since $p \in (1,2]$, we have $p' \in [2,\infty)$. So from the upper bound (1.4) for $g \in \mathscr{S}(\mathbb{R}^N)$ we have

$$[g]_{\dot{W}^{s,p'}(\mathbb{R}^N)} \lesssim \left(\frac{1}{s^{\frac{1}{p'}}}+\frac{1}{(1-s)^{\frac{1}{p'}}}\right)[g]_{\dot{F}^s_{p',2}(\mathbb{R}^N)}.$$

From Proposition 6.1, we thus obtain

$$[f]_{\dot F^s_{p,2}(\mathbb{R}^N)} \lesssim \min\{s,(1-s)\}\left(\frac{1}{s^{\frac{1}{p'}}}+\frac{1}{(1-s)^{\frac{1}{p'}}}\right)[f]_{\dot W^{s,p}(\mathbb{R}^N)}$$

We conclude by using (6.2). □

The lower bound of Theorem 1.6 is contained in the following statement:

Proposition 6.5 *Let $\Lambda > 1$ such that $(1-s) \le \frac{1}{2\Lambda}$. Let $\bar r \in (0,s)$ such that $(1-\bar r) = \Lambda(1-s)$. Let $r \in [0,\bar r]$. Then*

$$[f]_{\dot F^r_{p,2}(\mathbb{R}^N)} \lesssim \left(\|f\|_{L^p(\mathbb{R}^N)} + (1-s)^{\frac{1}{p}}[f]_{\dot W^{s,p}(\mathbb{R}^N)}\right).$$

Proof Since for $r \in [0,\bar r]$,

$$[f]_{\dot F^r_{p,2}} \le [f]_{\dot F^0_{p,2}} + [f]_{\dot F^{\bar r}_{p,2}} \lesssim \|f\|_{L^p} + [f]_{\dot F^{\bar r}_{p,2}}$$

it suffices to prove the proposition when $r = \bar r$. From Proposition 6.1 we have

$$[f]_{\dot F^{\bar r}_{p,2}(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)} + \min\{\bar r,(1-\bar r)\}[f]_{\dot W^{s,p}(\mathbb{R}^N)} \sup_{\substack{[g]_{\dot F^{\bar r}_{p',2}}\le 1\\ \mathrm{supp}\mathcal{F}g\subset\{|\xi|\ge 1/4\}}} [g]_{\dot W^{2\bar r-s,p'}(\mathbb{R}^N)}.$$

Now from (1.5), since $\frac{1}{2}s < \bar r < s$, which implies $0 < 2\bar r - s < \bar r$, we find for any $g \in \mathscr{S}(\mathbb{R}^N)$ with $\mathcal{F}g$ supported in $\{|\xi| \ge 1/4\}$,

$$\begin{aligned}[g]_{\dot W^{2\bar r-s,p'}(\mathbb{R}^N)} &\lesssim \frac{1}{(2\bar r-s)^{\frac{1}{p}}}[g]_{\dot F^0_{p',2}(\mathbb{R}^N)} + \frac{1}{(s-\bar r)^{\frac{1}{p}}}[g]_{\dot F^{\bar r}_{p',2}(\mathbb{R}^N)}\\ &\le \left(\frac{1}{(2\bar r-s)^{\frac{1}{p'}}}+\frac{1}{(s-\bar r)^{\frac{1}{p'}}}\right)[g]_{\dot F^{\bar r}_{p',2}(\mathbb{R}^N)}.\end{aligned}$$

Here the support condition on $\mathcal{F}g$ guarantees that $[g]_{\dot F^0_{p',2}} \lesssim [g]_{\dot F^{\bar r}_{p',2}}$. So we arrive at

$$[f]_{\dot F^{\bar r}_{p,2}(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)} + \min\{\bar r,(1-\bar r)\}\left(\frac{1}{(2\bar r-s)^{\frac{1}{p'}}}+\frac{1}{(s-\bar r)^{\frac{1}{p'}}}\right)[f]_{\dot W^{s,p}(\mathbb{R}^N)}$$

Now we have

$$\min\{\bar r,(1-\bar r)\}\lesssim(1-s),\quad s-\bar r=(\Lambda-1)(1-s)\quad\text{and}\quad 2\bar r-s=2-2\Lambda(1-s)-s\ge 1-s.$$

So

$$\min\{\bar r,(1-\bar r)\}\left(\frac{1}{(2\bar r-s)^{\frac{1}{p'}}}+\frac{1}{(s-\bar r)^{\frac{1}{p'}}}\right)\lesssim \frac{1-s}{(1-s)^{\frac{1}{p'}}} = (1-s)^{\frac{1}{p}}.$$

This establishes the claim of the proposition. □

7 Strong convergence as $s \to 1$: Proof of Corollary 1.9

Proof of Corollary 1.9 Let $p \in (1, \infty)$, assume that $f_k \in \mathscr{S}(\mathbb{R}^N)$ such that

$$f_k \rightharpoonup f \quad \text{weakly in } L^p(\mathbb{R}^N) \text{ as } k \to \infty.$$

Let $(s_k)_{k \in \mathbb{N}} \subset (0,1)$ such that $s_k \uparrow 1$ and assume that

$$\Lambda := \sup_k \left(\|f_k\|_{L^p(\mathbb{R}^N)} + (1-s_k)^{\frac{1}{p}} [f_k]_{\dot{W}^{s_k,p}(\mathbb{R}^N)} \right) < \infty. \tag{1.8}$$

First we claim that

$$\limsup_{k\to\infty} \|f_k\|_{L^p(\mathbb{R}^N)} + [f_k]_{\dot{F}^r_{p,2}(\mathbb{R}^N)} \precsim \Lambda \quad \forall r \in (0,1). \tag{7.1}$$

with constant independent of r. If $p \leq 2$ this follows easily from (1.3), but the following proof, using (1.6) instead, works for all $p \in (1,\infty)$. Up to removing finitely many sequence elements, we may assume that $(1-s_k) < \frac{1}{4}$ for all $k \in \mathbb{N}$. From (1.6) we have for any $r < 1 - 2(1-s_k)$,

$$[f_k]_{\dot{F}^r_{p,2}(\mathbb{R}^N)} \leq C \left(\|f_k\|_{L^p(\mathbb{R}^N)} + (1-s_k)^{\frac{1}{p}} [f_k]_{\dot{W}^{s_k,p}(\mathbb{R}^N)} \right) \leq C\,\Lambda.$$

Since $s_k \xrightarrow{k\to\infty} 1$, this proves (7.1).

In view of Lemma 2.7 we deduce from (7.1) that $f \in L^p(\mathbb{R}^N)$ and $[f]_{\dot{F}^1_{p,2}(\mathbb{R}^N)} < \infty$ with

$$\|f\|_{L^p(\mathbb{R}^N)} + [f]_{\dot{F}^1_{p,2}(\mathbb{R}^N)} \precsim \Lambda.$$

In view of Lemma 2.4, we conclude that $f \in H^{1,p}(\mathbb{R}^N)$ and

$$\|f\|_{L^p(\mathbb{R}^N)} + \|\nabla f\|_{L^p(\mathbb{R}^N)} \precsim \Lambda.$$

The locally strong convergence of $f_k \to f$ in $H^{t,p}$ for any $t \in (0,1)$ follows from Rellich's Theorem. More precisely, fix $0 < t < r < 1$ and a ball $B(0,R)$ for some $R > 0$. Denote by $\eta \in C_c^\infty(B(0,2R))$, $\eta \equiv 1$ in $B(0,R)$ any usual bump function. We then have by (7.1)

$$\sup_{k\in\mathbb{N}} \|\eta f_k\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{r}{2}}(\eta f_k)\|_{L^p(\mathbb{R}^N)} \precsim \sup_{k\in\mathbb{N}} \|f_k\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{r}{2}} f_k\|_{L^p(\mathbb{R}^N)} \precsim \Lambda < \infty.$$

Here we have used the Coifman–McIntosh–Meyer commutator estimate for

$$[\eta, (-\Delta)^{\frac{\tilde{t}}{2}}](g) := \eta (-\Delta)^{\frac{\tilde{t}}{2}} g - (-\Delta)^{\frac{\tilde{t}}{2}}(\eta g),$$

which implies that for any $\tilde{t} \in (0,1)$

$$\|[\eta, (-\Delta)^{\frac{\tilde{t}}{2}}](g)\|_{L^p(\mathbb{R}^N)} \precsim \left(\|\eta\|_{L^\infty} + [\eta]_{\mathrm{Lip}} \right) \|g\|_{L^p(\mathbb{R}^N)}.$$

For an overview of these commutator estimates see, e.g., [16].

Then, ηf_k has uniformly compact support and is uniformly bounded in $H^{r,p}(\mathbb{R}^N)$ and thus, up to taking a subsequence, converges strongly in $H^{t,p}(\mathbb{R}^N)$ (this can be either proven via the usual Rellich–Kondrachov argument, or by interpolation theory). That is, after passing to a subsequence (which we denote by f_{n_k})

$$\lim_{k\to\infty} \|\eta f_{n_k} - \eta f\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{t}{2}}(\eta f_{n_k} - \eta f)\|_{L^p(\mathbb{R}^N)} = 0. \tag{7.2}$$

Repeating this argument for different balls $B(0,\Gamma R)$ (extracting subsequence again if necessary) we obtain that

$$\lim_{k\to\infty} \|f_{n_k} - f\|_{L^p(B(0,\Gamma R))} = 0 \quad \forall \Gamma > 0. \tag{7.3}$$

Now we have

$$\begin{aligned}
&\|(-\Delta)^{\frac{t}{2}} f_{n_k} - (-\Delta)^{\frac{t}{2}} f\|_{L^p(B(0,R))}\\
&\quad \le \|\eta(-\Delta)^{\frac{t}{2}}(f_{n_k} - f)\|_{L^p(B(0,R))}\\
&\quad \le \|(-\Delta)^{\frac{t}{2}}(\eta f_{n_k} - \eta f)\|_{L^p(B(0,R))} + \|[\eta,(-\Delta)^{\frac{t}{2}}](f_{n_k} - f)\|_{L^p(B(0,R))}\\
&\quad \le \|(-\Delta)^{\frac{t}{2}}(\eta f_{n_k} - \eta f)\|_{L^p(B(0,R))} + \|[\eta,(-\Delta)^{\frac{t}{2}}](\chi_{B(0,\Gamma R)} f_{n_k} - \chi_{B(0,\Gamma R)} f)\|_{L^p(B(0,R))}\\
&\qquad + \left\|[\eta,(-\Delta)^{\frac{t}{2}}](\chi_{B(0,\Gamma R)^c} f_{n_k} - \chi_{B(0,\Gamma R)^c} f)\right\|_{L^p(B(0,R))}.
\end{aligned}$$

By (7.2) we have

$$\lim_{k\to\infty} \|(-\Delta)^{\frac{t}{2}}(\eta f_{n_k} - \eta f)\|_{L^p(B(0,R))} = 0.$$

By the Coifman–McIntosh–Meyer estimate and then (7.3) we have

$$\begin{aligned}
&\lim_{k\to\infty} \|[\eta,(-\Delta)^{\frac{t}{2}}](\chi_{B(0,\Gamma R)} f_{n_k} - \chi_{B(0,\Gamma R)} f)\|_{L^p(B(0,R))}\\
&\quad \le C(\eta) \lim_{k\to\infty} \|f_{n_k} - f\|_{L^p(B(0,\Gamma R)} = 0.
\end{aligned}$$

Lastly, observe that since $\eta\chi_{B(0,\Gamma R)^c} \equiv 0$,

$$\begin{aligned}
&\left\|[\eta,(-\Delta)^{\frac{t}{2}}](\chi_{B(0,\Gamma R)^c} f_{n_k} - \chi_{B(0,\Gamma R)^c} f)\right\|_{L^p(B(0,R))}\\
&\quad \le \left\|(-\Delta)^{\frac{t}{2}}\left(\chi_{B(0,\Gamma R)^c}(f_{n_k} - f)\right)\right\|_{L^p(B(0,R))}.
\end{aligned}$$

for $t \in (0,1)$, $\Gamma > 2$, and $x \in B(0,R)$ from the integral representation of the fractional Laplacian $(-\Delta)^{\frac{t}{2}}$ we find

$$\begin{aligned}
&|(-\Delta)^{\frac{t}{2}}\left(\chi_{B(0,\Gamma R)^c}(f_{n_k} - f)\right)(x)| \le C(t)\int_{B(0,\Gamma R)^c} \frac{|f_{n_k}(y) - f(y)|}{|x-y|^{N+t}}\,dy\\
&\quad \lesssim (\Gamma R)^{-t-\frac{N}{p}} \|f_{n_k} - f\|_{L^p(\mathbb{R}^N)}
\end{aligned}$$

Consequently, for any $\Gamma > 2$,

$$\lim_{k\to\infty} \left\|[\eta,(-\Delta)^{\frac{t}{2}}](\chi_{B(0,\Gamma R)^c} f_{n_k} - \chi_{B(0,\Gamma R)^c} f)\right\|_{L^p(B(0,R))} \lesssim R^{-t}\Gamma^{-t-\frac{N}{p}}\Lambda.$$

We conclude that for any $\Gamma > 2$,

$$\lim_{k\to\infty} \|(-\Delta)^{\frac{t}{2}} f_{n_k} - (-\Delta)^{\frac{t}{2}} f\|_{L^p(B(0,R))} \lesssim R^{-t}\Gamma^{-t-\frac{N}{p}}\Lambda.$$

Taking $\Gamma \to \infty$ we conclude

$$\lim_{k\to\infty} \|(-\Delta)^{\frac{t}{2}} f_{n_k} - (-\Delta)^{\frac{t}{2}} f\|_{L^p(B(0,R))} = 0.$$

This holds for any $R > 0$ and thus in particular for any compact set $K \subset \mathbb{R}^N$

$$\lim_{k\to\infty} \|(-\Delta)^{\frac{t}{2}} f_{n_k} - (-\Delta)^{\frac{t}{2}} f\|_{L^p(K)} = 0.$$

Since the weak limit f is unique, we can apply this argument to any subsequence of $(f_k)_{k\in\mathbb{N}}$, and find that actually

$$\lim_{k\to\infty} \|(-\Delta)^{\frac{t}{2}} f_k - (-\Delta)^{\frac{t}{2}} f\|_{L^p(K)} = 0.$$

This implies (1.9). As for (1.10), from Sobolev embedding one finds that for any $0 < \tilde{t} < t$ and $\tilde{K} \subset K$ both compact with dist $(\tilde{K}, \partial K) > 0$ we have

$$[f_k - f]_{\dot{W}^{\tilde{t},p}(\tilde{K})} \le C(t, \tilde{t}, p, K, \tilde{K}, N) \left(\|(-\Delta)^{\frac{t}{2}}(f_k - f)\|_{L^p(K)} + \|f_k - f\|_{L^p(K)} \right).$$

So we conclude (1.10) as well. □

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Appendix A. Proof of the BBM formula on $\mathbb{R}^N$

For the convenience of the reader we give here the proof of the following BBM formula on $\mathbb{R}^N$:

Theorem A.1 *For $1 < p < \infty$ and $f \in L^p(\mathbb{R}^N)$, one has*

$$\|\nabla f\|_{L^p(\mathbb{R}^N)} = \left(\frac{p}{k(p,N)}\right)^{1/p} \lim_{s\to 1^-} (1-s)^{\frac{1}{p}} [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \tag{A.1}$$

in the sense that the left hand side of the equality is finite if and only if the right hand side is finite, in which case the two sides are equal. In fact, we have $f \in H^{1,p}(\mathbb{R}^N)$ as long as $\liminf_{s\to 1^-} (1-s)^{\frac{1}{p}} [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} < \infty$.

Proof **Step 1.** First, we establish (A.1) for $f \in C^1 \cap H^{1,p}(\mathbb{R}^N)$. Let $R \ge 100$ and $s \in [\frac{1}{2}, 1)$. Then

$$\begin{aligned}
&\left|(1-s)^{\frac{1}{p}} [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} - (1-s)^{\frac{1}{p}} [f]_{\dot{W}^{s,p}(B(R))}\right| \\
&\lesssim (1-s)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N\setminus B(R)} \int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{\frac{1}{p}} \\
&\lesssim (1-s)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N\setminus B(R)} \int_{|x-y|\le \frac{1}{4}R} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{\frac{1}{p}} \\
&\quad + (1-s)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N\setminus B(R)} \int_{|x-y|\ge \frac{1}{4}R} \frac{|f(x)|^p + |f(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{\frac{1}{p}}.
\end{aligned}$$

We observe for the second term

$$(1-s)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N\setminus B(R)} \int_{|x-y|\ge \frac{1}{4}R} \frac{|f(x)|^p + |f(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{\frac{1}{p}} \lesssim \left(\frac{(1-s)}{s}\right)^{\frac{1}{p}} R^{-s} \|f\|_{L^p(\mathbb{R}^N)}.$$

For the first term, we use

$$|f(x)-f(y)| \lesssim |x-y| \left(\mathcal{M}_{2|x-y|}|\nabla f(x)| + \mathcal{M}_{2|x-y|}|\nabla f(y)|\right),$$

where

$$\mathcal{M}_r g(x) := \sup_{\sigma\in(0,r)} \fint_{B(x,\sigma)} |g(z)|\,dz$$

is the centered maximal function, cf. [3, 14]. Then we have

$$\begin{aligned}
&(1-s)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N\setminus B(R)} \int_{|x-y|\le\frac{1}{4}R} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{1}{p}}\\
&\lesssim (1-s)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N\setminus B(3R/4)} \left(\mathcal{M}_{R/2}|\nabla f(z)|\right)^p dz \int_{|w|\le R} \frac{1}{|w|^{N+(s-1)p}}\,dw\right)^{\frac{1}{p}}\\
&\lesssim R^{1-s} \left(\int_{\mathbb{R}^N\setminus B(R/4)} |\nabla f(z)|^p dz\right)^{\frac{1}{p}}
\end{aligned}$$

In the last step we used the maximal theorem. That is, we have shown that for any $s\in[\frac{1}{2},1)$

$$\begin{aligned}
&\left|(1-s)^{\frac{1}{p}}[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} - (1-s)^{\frac{1}{p}}[f]_{\dot{W}^{s,p}(B(R))}\right|\\
&\quad\lesssim R^{1-s}\|\nabla f\|_{L^p(\mathbb{R}^N\setminus B(R/4))} + (1-s)^{\frac{1}{p}}\ R^{-s}\|f\|_{L^p(\mathbb{R}^N)}.
\end{aligned}\tag{7.2}$$

Now we can conclude from the local case in [4]; recall that in [4, Corollary 2] it is proven that for any $R>0$

$$\|\nabla f\|_{L^p(B(R))} = \left(\frac{p}{k(p,N)}\right)^{1/p} \lim_{s\to1^-} (1-s)^{\frac{1}{p}}[f]_{\dot{W}^{s,p}(B(R))},\tag{7.3}$$

where $k(p,N) := \int_{\mathbb{S}^{N-1}} |e\cdot\omega|^p d\omega$ and e is any unit vector in $\mathbb{R}^N$. Fix $\varepsilon>0$. Since $\nabla f\in L^p(\mathbb{R}^N)$ there must be a large radius $R>0$ such that

$$\|\nabla f\|_{L^p(\mathbb{R}^N\setminus B(R/4))} < \varepsilon.\tag{7.4}$$

Then

$$\begin{aligned}
&\left|\left(\frac{p}{k(p,N)}\right)^{1/p} (1-s)^{\frac{1}{p}}[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} - \|\nabla f\|_{L^p(\mathbb{R}^N)}\right|\\
&\le \varepsilon + \left|\left(\frac{p}{k(p,N)}\right)^{1/p} (1-s)^{\frac{1}{p}}[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} - \|\nabla f\|_{L^p(B(R))}\right|\\
&\lesssim \varepsilon + R^{1-s}\|\nabla f\|_{L^p(\mathbb{R}^N\setminus B(R/4))} + (1-s)^{\frac{1}{p}}\ R^{-s}\|f\|_{L^p(\mathbb{R}^N)}\\
&\quad + \left|\left(\frac{p}{k(p,N)}\right)^{1/p} (1-s)^{\frac{1}{p}}[f]_{\dot{W}^{s,p}(B(R))} - \|\nabla f\|_{L^p(B(R))}\right|\\
&\le (R^{1-s}+1)\varepsilon + (1-s)^{\frac{1}{p}}\ R^{-s}\|f\|_{L^p(\mathbb{R}^N)}\\
&\quad + \left|\left(\frac{p}{k(p,N)}\right)^{1/p} (1-s)^{\frac{1}{p}}[f]_{\dot{W}^{s,p}(B(R))} - \|\nabla f\|_{L^p(B(R))}\right|
\end{aligned}$$

where the first and the third inequality follows from (7.4) and the second inequality follows from (7.2). Since R is fixed once ε is fixed, we may let $s \to 1^-$ and use (7.3). This shows

$$\limsup_{s\to 1^-} \left| \left(\frac{p}{k(p,N)}\right)^{1/p} (1-s)^{\frac{1}{p}} [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} - \|\nabla f\|_{L^p(\mathbb{R}^N)} \right| \lesssim \varepsilon,$$

but since $\varepsilon > 0$ is arbitrary, this proves

$$\left(\frac{p}{k(p,N)}\right)^{1/p} \lim_{s\to 1^-} (1-s)^{\frac{1}{p}} [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} = \|\nabla f\|_{L^p(\mathbb{R}^N)}.$$

Step 2. Next, assume $f \in H^{1,p}(\mathbb{R}^N)$. We show that

$$\limsup_{s\to 1^-} (1-s)^{1/p} [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \le \left(\frac{k(p,N)}{p}\right)^{1/p} \|\nabla f\|_{L^p(\mathbb{R}^N)}. \tag{7.5}$$

We first observe that for $f \in H^{1,p}(\mathbb{R}^N)$ and $s \in [1/2, 1)$,

$$\begin{aligned}
[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} &= \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x+h)-f(x)|^p}{|h|^{N+sp}} dx dh \right)^{1/p} \\
&\le \left(\int_{|h|\le 1} \int_{\mathbb{R}^N} \frac{|f(x+h)-f(x)|^p}{|h|^p} dx \frac{|h|^{(1-s)p}}{|h|^N} dh \right)^{1/p} \\
&\quad + \left(\int_{|h|>1} \int_{\mathbb{R}^N} (|f(x+h)|^p + |f(x)|^p) dx \frac{1}{|h|^{N+sp}} dh \right)^{1/p}.
\end{aligned}$$

We appeal to the facts that

$$\int_{\mathbb{R}^N} |f(x+h)-f(x)|^p dx \le |h|^p \|\nabla f\|^p_{L^p(\mathbb{R}^N)}$$

(see [6, Proposition 9.3]) and that $\int_{\mathbb{R}^N} |f(x+h)|^p + |f(x)|^p dx = 2\|f\|^p_{L^p(\mathbb{R}^N)}$. Thus

$$\begin{aligned}
[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} &\lesssim \left(\int_{|h|\le 1} \frac{|h|^{(1-s)p}}{|h|^N} dh \right)^{1/p} \|\nabla f\|_{L^p(\mathbb{R}^N)} + \left(\int_{|h|>1} \frac{1}{|h|^{N+sp}} dh \right)^{1/p} \|f\|_{L^p(\mathbb{R}^N)} \\
&\lesssim \frac{1}{(1-s)^{1/p}} \|\nabla f\|_{L^p(\mathbb{R}^N)} + \frac{1}{s^{1/p}} \|f\|_{L^p(\mathbb{R}^N)}
\end{aligned}$$

which implies

$$\sup_{1/2\le s<1} (1-s)^{1/p} [f]_{\dot{W}^{s,p}(\mathbb{R}^N)} \le C \|f\|_{H^{1,p}(\mathbb{R}^N)} \tag{7.6}$$

for some constant $C = C_{p,N}$, whenever $f \in H^{1,p}(\mathbb{R}^N)$ (note that (7.6) strengthens the second conclusion of Corollary 1.7 by weakening the hypothesis on f from $f \in \mathscr{S}(\mathbb{R}^N)$ to $f \in H^{1,p}(\mathbb{R}^N)$). To proceed further, for any $\varepsilon > 0$, pick $g \in C^1 \cap H^{1,p}(\mathbb{R}^N)$ so that $\|f-g\|_{H^{1,p}(\mathbb{R}^N)} < \varepsilon$. Then

$$\begin{aligned}
(1-s)^{1/p}[f]_{\dot{W}^{s,p}(\mathbb{R}^N)} &\le (1-s)^{1/p}[g]_{\dot{W}^{s,p}(\mathbb{R}^N)} + (1-s)^{1/p}[f-g]_{\dot{W}^{s,p}(\mathbb{R}^N)} \\
&\le (1-s)^{1/p}[g]_{\dot{W}^{s,p}(\mathbb{R}^N)} + C\varepsilon,
\end{aligned}$$

where in the last inequality we applied (7.6) to $f-g \in H^{1,p}(\mathbb{R}^N)$ in place of f. Now recall (A.1) has already been proved for $g \in C^1 \cap H^{1,p}(\mathbb{R}^N)$. As a result, letting $s \to 1^-$, we obtain

$$\limsup_{s\to 1^-}(1-s)^{1/p}[f]_{\dot W^{s,p}(\mathbb{R}^N)} \le \left(\frac{k(p,N)}{p}\right)^{1/p}\|\nabla g\|_{L^p(\mathbb{R}^N)}+\varepsilon$$
$$\le \left(\frac{k(p,N)}{p}\right)^{1/p}(\|\nabla f\|_{L^p(\mathbb{R}^N)}+\varepsilon)+C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (7.5) follows.
Step 3. Finally, assume $f \in L^p(\mathbb{R}^N)$ and

$$A := \liminf_{s\to 1^-}(1-s)^{1/p}[f]_{\dot W^{s,p}(\mathbb{R}^N)} < \infty.$$

It is known that then $f \in H^{1,p}(\mathbb{R}^N)$ and

$$\|\nabla f\|_{L^p(\mathbb{R}^N)} \le \left(\frac{p}{k(p,N)}\right)^{1/p} A. \tag{7.7}$$

In fact, then for every bounded smooth domain $\Omega \subset \mathbb{R}^N$, we have

$$\liminf_{s\to 1^-}(1-s)^{1/p}[f]_{\dot W^{s,p}(\Omega)} \le A < \infty,$$

so [4, Theorem 2] (and its proof) shows that $f \in H^{1,p}(\Omega)$ with

$$\|\nabla f\|_{L^p(\Omega)} \le \left(\frac{p}{k(p,N)}\right)^{1/p} A.$$

Since Ω is an arbitrary bounded smooth domain in $\mathbb{R}^N$, this shows $f \in H^{1,p}(\mathbb{R}^N)$ and that (7.7) holds.

Appendix B. Proof of Corollary 1.3

Proof of Corollary 1.3 Let $0 < s \le t < 1$ and $f \in \mathscr{S}(\mathbb{R}^N)$. From Theorem 1.2 we have

$$\min\{s,(1-s)\}^{\frac12}[f]_{\dot W^{s,2}(\mathbb{R}^N)} \lesssim [f]_{\dot F^s_{2,2}}$$

and

$$[f]_{\dot F^t_{2,2}} \lesssim \min\{t,(1-t)\}^{\frac12}[f]_{\dot W^{t,2}(\mathbb{R}^N)}.$$

The result now follows from the inequality $[f]_{\dot F^s_{2,2}} \lesssim \|f\|_{L^2} + [f]_{\dot F^t_{2,2}}$ when $0 \le s \le t$. □

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