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Birds on a wire

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PAPER: Interdisciplinary statistical mechanics

Birds on a wire

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Abstract. We investigate the occupancy statistics of birds on a wire. Birds land one by one on a wire and rest where they land. Whenever a newly arriving bird lands within a fixed distance of already resting birds, these resting birds immediately fly away. We determine the steady-state occupancy of the wire, the distribution of gaps between neighboring birds, and other basic statistical features of this process. We briefly discuss conjectures for corresponding observables in higher dimensions.

Keywords: stochastic processes

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1. Introduction

Statistical mechanics provides us with the ‘eyes’ to appreciate collective phenomena in quantitative and insightful ways. Figure 1 illustrates this synergy between phenomenology and analysis: birds alight one at a time to rest at random positions on a wire. We postulate that birds are sociable but skittish—if a newly arriving bird lands within a specified distance of any resting birds, they immediately fly away. A first question to address is: what is the dynamics of this process? Eventually, a steady state is reached in which the average arrival and departure rates are equal and this prompts several questions. For example, what is the steady-state density of birds on the wire? What are the separations between adjacent birds?

While much is known about the spatial patterns of moving animal groups [1–8], the spatial organization of static groups is less studied (see, however, [9, 10]). We formulate the ‘pushy birds’ (PB) model (see figure 2) to mimic the spatial organization that results from repeated landings and departures of birds. This idealized model is similar in spirit to models of flocking and schooling [1–8]. While our model focuses on the one-dimensional geometry with local interactions, it naturally extends to longer-range interactions that may lead to self-organized cooperative behavior, as in forest-fire models [11–16]. A generalization to higher dimensions leads to a dynamic version of the famous sphere packing problems in arbitrary dimensions (see, e.g. [17–24]) for which many open questions still exist.

Our PB model also resembles random sequential adsorption (RSA) [25–33], where fixed-shape particles impinge on open regions of a substrate and stick irreversibly. One example of RSA that is close to the PB model is the ‘unfriendly seating arrangement’ problem [34, 35], where people arrive one at a time at a luncheonette and sit at a counter. People are all mutually unfriendly so they choose seats at random but never next to another person. The luncheonette reaches a static jammed state of density $\rho_{\text{jam}} = \frac{1}{2}(1 - e^{-2}) \approx 0.432$, after which additional patrons cannot be accommodated. In contrast, the PB model reaches a steady state that is constantly changing locally, but its global properties are stationary and independent of the initial conditions.

While our model is couched in terms of birds, it should not be taken literally as a description of real birds. There are many other influences that determine how birds organize themselves on a spatially restricted landing spot, such as a wire. Nevertheless, the behavior of our admittedly unrealistic model is non trivial and perhaps this study provides some initial steps to understand the organizational dynamics of more realistic



Figure 1. Birds on wires.

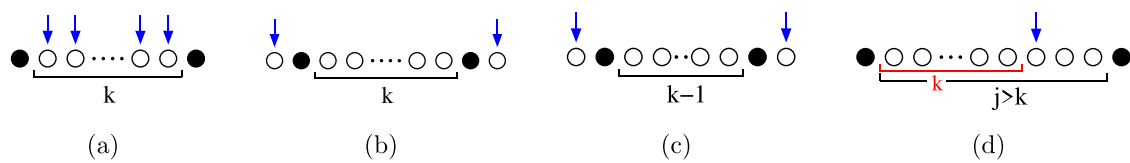


Figure 2. Processes that contribute to changes in the void densities in the PB model of equation (1). The vertical arrows indicate the possible locations for a bird to land. In (d) only one of the two possible landing spots that creates a void of length k is shown.

models of the arrival and departure of birds at some resting spot. We view the PB model as being akin to some of the idealized forest-fire models that were proposed long ago in the statistical physics literature [11–14]. These abstract models miss many features of real forest fires; nevertheless, the phenomenology that arises from this class of models is extremely rich and led to many advances about self-organized criticality [36]. It is in this impressionistic spirit that we investigate the PB model.

2. One-dimensional lattice

It is conceptually simplest to formulate a discrete version of the PB model in which birds land on empty sites of a one-dimensional lattice; we later treat a continuous version. Each landing event of a bird scares away birds on adjacent lattice sites (if they are present) so that they fly away. Our analysis of the PB model focuses on V_k , defined as the number of voids of length k divided by the total number of lattice points on the wire; this is just the density of voids of length k . A void of length k is defined as the following arrangement of birds and vacancies

$$\circ \bullet \underbrace{\circ \dots \circ}_k \bullet \circ,$$

where an occupied site is denoted by \bullet and an empty site by \circ . Since birds cannot be adjacent, the sites next to each bird outside any void must also be empty.

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2.1. The void densities

The void densities change in time according to the following rate equations:

$$\dot{V}_k = -kV_k - 2V_k + 2V_{k-1} + 2 \sum_{j \geq k+1} V_j = -(4+k)V_k + 2 \sum_{j \geq k-1} V_j, \quad (1)$$

where the overdot denotes time derivative. Each of the terms on the right corresponds to one of the processes shown in figure 2. The first term accounts for the loss of a k -void due to a bird landing anywhere within this void (figure 2(a)). The second term accounts for the loss of the k -void when a bird lands in either of the two sites just outside this void. Immediately afterward, the adjacent bird at the edge of the k -void flies away, so that a k -void disappears (figure 2(b)). The third term accounts for the gain of a k -void when a bird lands on either of the two sites just outside a void of length $k-1$; this ultimately causes an increase in the number of k -voids (figure 2(c)). The last term accounts for the gain of k -voids when a bird lands within a j -void, with $j > k$, such that a k -void is created. If $j \neq 2k+1$, there are two possible landing sites (figure 2(d)), each of which creates one k -void. If $j = 2k+1$, there is a unique landing site in the middle of the j -void that creates two k -voids.

The void distribution also satisfy the following basic conditions that will be useful in solving the model:

$$V_0 = 0, \quad \sum_{k \geq 0} V_k = \rho, \quad \sum_{k \geq 0} (k+1)V_k = 1. \quad (2)$$

The first equality states that voids of length 0 cannot exist because this corresponds to two birds being adjacent. The one-to-one correspondence between each void and exactly one bird on a wire with periodic boundary conditions leads to the second equality between void densities V_k and the overall density ρ . The last equality states that the length of all voids plus the bird at one end of each void equals the total length.

Summing equation (1) over all $k \geq 1$ and using the sum rules (2), we obtain the closed equation for the density, $\dot{\rho} = 1 - 3\rho$. For an initially empty system, the solution is

$$\rho = \frac{1}{3}(1 - e^{-3t}). \quad (3)$$

Thus, the approach to the steady-state density of $\rho = \frac{1}{3}$ is purely exponential. We now recast equation (1) as

$$\begin{aligned} \dot{V}_1 &= -5V_1 + 2\rho \\ \dot{V}_2 &= -6V_2 + 2\rho \\ \dot{V}_3 &= -7V_3 - 2V_1 + 2\rho \\ \dot{V}_4 &= -8V_4 - 2V_1 - 2V_2 + 2\rho, \end{aligned}$$

etc, which we can solve recursively to give

$$\begin{aligned} V_1 &= \frac{1}{15}(2 - 5e^{-3t} + 3e^{-5t}) \\ V_2 &= \frac{1}{9}(1 - e^{-3t})^2 \\ V_3 &= \frac{1}{35}(2 - 7e^{-5t} + 5e^{-7t}) \\ V_4 &= \frac{1}{45}(1 + 4e^{-3t} - 6e^{-5t} - 5e^{-6t} + 6e^{-8t}), \end{aligned} \tag{4}$$

etc, for an initially empty system. Since each void density approaches its steady-state value exponentially quickly, we now focus on the steady state, where equation (1) reduces to

$$(k + 4)V_k = 2 \sum_{j \geq k-1} V_j. \tag{5}$$

Introducing the cumulative distribution $F_k \equiv \sum_{j \geq k} V_j$, (5) becomes

$$F_{k+1} - F_{k+2} = \frac{2}{k+5} F_k. \tag{6}$$

The first two of equation (2) give $F_0 = F_1 = \rho = \frac{1}{3}$; these serve as the initial conditions that allow us to generate all the F_k one by one: $F_2 = \frac{1}{5}$, $F_3 = \frac{4}{45}$, $F_5 = \frac{2}{63}$, etc.

To find the general solution of equation (6) we employ the generating function technique [37]. The factor $(k+5)^{-1}$ on the right-hand side of (6) suggests that it is expedient to define the generating function as

$$F(z) \equiv \sum_{k \geq 0} F_k z^{k+4}.$$

Multiplying equation (6) by z^{k+5} and summing over all $k \geq 0$, we transform the recurrence (6) into the integral equation

$$F(z) - \rho z^4 - \frac{F(z) - \rho z^4 - \rho z^5}{z} = 2 \int_0^z dw F(w). \tag{7}$$

We now define

$$\Phi(z) = \int_0^z dw F(w) = \sum_{k \geq 0} \frac{F_k}{k+5} z^{k+5},$$

and after some elementary manipulations, we may express (7) as the ordinary differential equation

$$(1 - z) \frac{d\Phi}{dz} + 2z\Phi = \rho z^4. \tag{8}$$

Integrating (8) subject to $\Phi(0) = 0$ yields

$$\Phi = \rho(1-z)^2 e^{2z} \int_0^z dw \frac{w^4 e^{-2w}}{(1-w)^3} = \frac{1}{4} \rho [3(1-z)^2 e^{2z} - 3 + 3z^2 + 2z^3].$$

Finally, we differentiate Φ to give the generating function

$$F(z) = \frac{3}{2} \rho [z + z^2 - z(1-z)e^{2z}]. \quad (9)$$

We now expand $F(z)$ in a power series to extract the F_k :

$$F_k = 2^{k+1} \frac{k+1}{(k+3)!},$$

from which the density of voids of length k is

$$V_k = F_k - F_{k+1} = 2^{k+1} \frac{k(k+3)}{(k+4)!}. \quad (10)$$

The average void length $\langle k \rangle = \sum k V_k / \sum V_k = 2$, which accords both with $\rho = \frac{1}{3}$ and with the conditions (2). Higher moments of the void length are less simple: $\langle k^2 \rangle = 3e^2 - 17 \approx 5.167$, $\langle k^3 \rangle = 83 - 9e^2 \approx 16.499$, etc.

A basic question about the steady state is: how many birds fly away after each landing event? According to our model definition, either 0, 1, or 2 birds can fly away when a bird lands. The probabilities q_n that $n \leq 2$ birds fly away after each landing event satisfy the sum rules

$$q_0 + q_1 + q_2 = 1, \quad 0 \times q_0 + 1 \times q_1 + 2 \times q_2 = 1.$$

The first equation imposes normalization. The second equation states that in the steady state, the average number of birds that leave upon each landing event must equal the number of birds that arrive, and the latter equals 1. These lead to $q_0 = q_2$. The probabilities q_n are determined by

$$q_0 = \sum_{k \geq 3} \frac{(k-2)V_k}{1-\rho} \quad q_1 = \sum_{k \geq 2} \frac{2V_k}{1-\rho} \quad q_2 = \frac{V_1}{1-\rho}.$$

The first term accounts for a bird that lands in the interior of a gap of length $k \geq 3$ so that no bird leaves. The second term accounts for a bird that lands at either end of a gap of length $k > 2$ so that a single bird leaves. The last term accounts for a bird that lands in a vacancy between two birds so that both these birds leave. The denominator $(1-\rho)$ is the probability for a bird to land on any vacancy. Using $\rho = \frac{1}{3}$ and equation (10), we find $q_0 = q_2 = \frac{1}{5}$, $q_1 = \frac{3}{5}$.

We can also readily extend our approach to treat the situation in which all birds within a range $b > 1$ fly away when a bird lands on an unoccupied site. While the qualitative features of this generalization are the same as that for the case $b = 1$ given above, some quantitative differences arise. The solution for general $b > 1$ is given in appendix A.

3. The pair correlation function

The spatial distribution of birds may be characterized by the pair correlation function $C_j \equiv \langle n_0 n_j \rangle$, where n_j is the occupancy indicator function at site j . That is, $n_j = 0$ if site j is empty and $n_j = 1$ if j is occupied. If the locations of the birds are spatially uncorrelated, then $\langle n_0 n_j \rangle = \langle n_0 \rangle \langle n_j \rangle$. This implies that the connected correlation function, $C_j = \langle n_0 n_j \rangle - \langle n_0 \rangle \langle n_j \rangle$ would equal zero. Our calculations below seem to suggest that this is the case. The connected correlation functions C_1 and C_2 are non zero, while $C_3 = 0$ by its very definition, and we show that C_4 , and C_5 are zero. These calculations become tedious for C_4 and C_5 and we can only conjecture that $C_j = 0$ for $j > 5$.

The steady-state pair correlation function C_j for $j \leq 3$ can be deduced directly from our results for the density and the void densities. Indeed, $C_0 = \langle n_0^2 \rangle = \langle n_0 \rangle = \frac{1}{3}$, while $C_1 = V_0$, $C_2 = V_1$ and $C_3 = V_2$, from which

$$C_0 = \frac{1}{3}, \quad C_1 = 0, \quad C_2 = \frac{2}{15}, \quad C_3 = \frac{1}{9}. \tag{11}$$

We now derive $C_4 = C_5 = \frac{1}{9}$. As we show, determining these correlation functions requires various multi-void distributions. The formal expressions for the first few correlation functions C_j , with $j \geq 4$, are:

$$\begin{aligned} C_4 &= \text{Prob}[\bullet \circ \bullet \circ \bullet] + \text{Prob}[\bullet \circ \circ \circ \bullet] = V_{1,1} + V_3 \\ C_5 &= \text{Prob}[\bullet \circ \bullet \circ \circ \bullet] + \text{Prob}[\bullet \circ \circ \bullet \circ \bullet] + \text{Prob}[\bullet \circ \circ \circ \circ \bullet] = 2V_{1,2} + V_4 \\ C_6 &= \text{Prob}[\bullet \circ \bullet \circ \bullet \circ \bullet] + \text{Prob}[\bullet \circ \circ \bullet \circ \circ \bullet] + \text{Prob}[\bullet \circ \bullet \circ \circ \circ \bullet] \\ &\quad + \text{Prob}[\bullet \circ \circ \circ \bullet \circ \bullet] + \text{Prob}[\bullet \circ \circ \circ \circ \circ \bullet] \\ &= V_{1,1,1} + V_{2,2} + 2V_{1,3} + V_5, \end{aligned}$$

where

$$\begin{aligned} V_k &\equiv \text{Prob} \left[\bullet \underbrace{\circ \dots \circ}_k \bullet \right] \\ V_{i,j} &\equiv \text{Prob} \left[\bullet \underbrace{\circ \dots \circ}_i \bullet \underbrace{\circ \dots \circ}_j \bullet \right] \\ V_{i,j,k} &\equiv \text{Prob} \left[\bullet \underbrace{\circ \dots \circ}_i \bullet \underbrace{\circ \dots \circ}_j \bullet \underbrace{\circ \dots \circ}_k \bullet \right], \end{aligned}$$

denote the single-void, two-void, and three-void distributions. The subscripts on the multi-void distributions account for the number of sites in the adjacent empty strings.

The void distributions V_{i_1, \dots, i_p} satisfy rate equations that are natural extensions of the rate equation (1) for V_k . Consider first the distribution $V_{i,j}$. Using the same reasoning as that given in figure 2 to write equation (1), the rate equation for $V_{i,j}$ is

$$\dot{V}_{i,j} = -(2 + i + j)V_{i,j} + \sum_{\ell \geq i+1} V_{\ell,j} + \sum_{\ell \geq j+1} V_{i,\ell}$$

$$+ V_{i+j+1} + V_{i-1,j+1} + V_{i+1,j-1} + V_{i-1,j} + V_{i,j-1}, \quad (12)$$

subject to the boundary conditions

$$V_{i,0} = 0 = V_{0,j} \quad (i, j \geq 0), \quad (13a)$$

and the sum rules

$$\sum_{\ell \geq 1} V_{\ell,j} = V_j, \quad \sum_{\ell \geq 1} V_{i,\ell} = V_i. \quad (13b)$$

In the steady state, (12) reduces to the recurrence

$$(4 + i + j)V_{i,j} = \sum_{\ell \geq i} V_{\ell,j} + \sum_{\ell \geq j} V_{i,\ell} + V_{i+j+1} + V_{i-1,j+1} + V_{i+1,j-1} + V_{i-1,j} + V_{i,j-1}. \quad (14)$$

Specializing (14) and (13b) to $(i, j) = (1, 1)$ and additionally using (13a) we obtain

$$6V_{1,1} = 2V_1 + V_3. \quad (15)$$

Recalling that $V_1 = \frac{2}{15}$ and $V_3 = \frac{2}{35}$ from equation (10), we obtain $V_{1,1} = \frac{17}{315}$, which finally gives $C_4 = V_{1,1} + V_3 = \frac{1}{9}$.

Next, we specialize (14) and (13b) to $(i, j) = (1, 2)$, from which we obtain

$$6V_{1,2} = V_2 + V_1 + V_4. \quad (16)$$

Using the known results $V_1 = \frac{2}{15}$, $V_2 = \frac{1}{9}$, $V_4 = \frac{1}{45}$ we obtain $V_{1,2} = \frac{2}{45}$ and then $C_5 = 2V_{1,2} + V_4 = \frac{1}{9}$. It seems unlikely that we can determine the correlation functions C_j for arbitrary j via this straightforward, but laborious method.

We mention that we can also determine the full time dependence of the low-order pair correlation functions. The behaviors of C_j with $j = 0, 1, 2$, and 3 follow directly from the relation between these correlation functions and the appropriate void densities. Namely, $C_0(t) = \rho(t)$, $C_1(t) = V_0(t)$, $C_2(t) = V_1(t)$ and $C_3(t) = V_2(t)$. To derive $C_4(t) = V_{1,1}(t) + V_3(t)$ we must find $V_{1,1}(t)$. From (12) the rate equation for $V_{1,1}$ is

$$\dot{V}_{1,1}(t) = -6V_{1,1}(t) + 2V_1(t) + V_3(t),$$

with solution, for an initially empty system,

$$V_{1,1}(t) = \frac{1}{315}(17 - 70e^{-3t} + 63e^{-5t} + 35e^{-6t} - 45e^{-7t}). \quad (17)$$

Using $C_4(t) = V_{1,1}(t) + V_3(t)$ with $V_3(t)$ from (4) and $V_{1,1}(t)$ from (17) we have

$$C_4(t) = V_{1,1}(t) + V_3(t) = \frac{1}{9}(1 - e^{-3t})^2. \quad (18)$$

To derive $C_5(t)$, we must find $V_{1,2}(t)$. Again from (12), the rate equation for $V_{1,2}(t)$ is

$$\dot{V}_{1,2}(t) = -6V_{1,2}(t) + V_1(t) + V_2(t) + V_4(t),$$

whose solution is

$$V_{1,2}(t) = \frac{1}{45}(2 - 7e^{-3t} + 3e^{-5t} + 5e^{-6t} - 3e^{-8t}). \quad (19)$$

Using $C_5(t) = 2V_{1,2}(t) + V_4(t)$ with $V_4(t)$ from (4) and $V_{1,2}(t)$ from (19) we thus find

$$C_5(t) = 2V_{1,2}(t) + V_4(t) = \frac{1}{9}(1 - e^{-3t})^2. \quad (20)$$

4. One-dimensional continuum

A more natural scenario for the dynamics is that each birds can land anywhere along a wire. Within the RSA framework, the analogous process is the famous Rényi car parking model [38] in which fixed-length cars attempt to park anywhere along a one-dimensional line until there are no gaps remaining that can accommodate a car. Without loss of generality we set the interaction range between birds equal to one. Thus if a bird lands within a unit distance of one (or two) birds, this bird (or these birds) immediately fly away.

Instead of voids of integer length, the basic dynamical variable is $V(x)$, the density of voids of length x . Following the same reasoning as that which led to equation (1), the evolution equation for the void distribution is now (see also figure 3)

$$\dot{V}(x, t) = -(2 + x)V(x, t) + \begin{cases} 2 \int_1^\infty dy V(y, t) & 1 < x < 2, \\ 2 \int_{x-1}^\infty dy V(y, t) & x > 2. \end{cases} \quad (21)$$

In close analogy with equation (2), the void distribution $V(x)$ must now satisfy the sum rules: (a) $V(x) = 0$ for $x < 1$, (b) the density of birds is $\rho = \int_1^\infty dx V(x)$, and (c) $\int_1^\infty dx x V(x) = 1$. As a result of condition (b), the first of equation (21) can be re-expressed as $\dot{V}(x, t) = -(2 + x)V(x, t) + 2\rho(t)$.

Integrating (21) over all x , the density

$$\rho(t) = \int_1^2 dx V(x, t) + \int_2^\infty dx V(x, t)$$

obeys the rate equation $\dot{\rho} = 1 - 2\rho$. For an initially empty system, the solution is simply $\rho = \frac{1}{2}(1 - e^{-2t})$. We now use this result ρ to solve $\dot{V}(x, t) = -(2 + x)V(x, t) + 2\rho(t)$ in the range $1 < x < 2$ to give

$$V(x, t) = \frac{1 - e^{-(2+x)t}}{2 + x} - \frac{e^{-2t} - e^{-(2+x)t}}{x}. \quad (22)$$

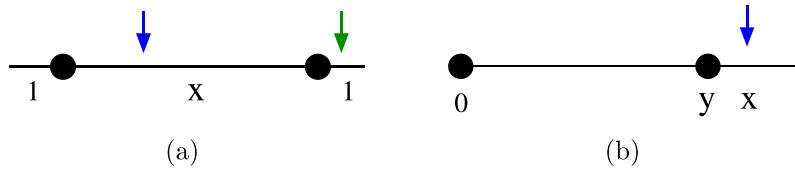


Figure 3. Processes that contribute to changes in the void densities in equation (21). (a) An x -void disappears if a bird lands anywhere inside the void (blue arrow) or within a unit distance of either bird outside the void (green arrow), (b) an x -void is created when a new bird lands a distance x from an existing bird. Another bird may be anywhere in the range $[1, \infty]$ for $1 < x < 2$ or in the range $[x - 1, \infty]$ for $x > 2$.

Using $\rho = \frac{1}{2}(1 - e^{-2t})$ in the second of (21), we may rewrite this equation as

$$\dot{V}(x, t) = -(2 + x)V(x, t) - 2 \int_1^{x-1} dy V(y, t) + 1 - e^{-2t}. \quad (23)$$

We now substitute the solution for $V(x, t)$ in the range $1 < x < 2$ in equation (23) to solve this equation in the interval $2 < x < 3$. Continuing this procedure we can recursively solve (23) for each interval $n < x < n + 1$ using the previously determined solutions for $x < n$. While this procedure is straightforward in principle, it quickly becomes tedious as x increases.

To obtain the large- x behavior of the void distribution, we first rely on the fact that the approach to the steady state again occurs exponentially quickly. Thus, we henceforth focus on the steady-state properties of the continuum case. In this case, the void density is determined by

$$(2 + x)V(x) = \begin{cases} 2 \int_1^{\infty} dy V(y) & 1 < x < 2, \\ 2 \int_{x-1}^{\infty} dy V(y) & x > 2. \end{cases} \quad (24)$$

One way to solve equation (24), in parallel with the approach to solve the discrete equation (5) for the void densities V_k , is to introduce the Laplace transform $\widehat{V}(s) \equiv \int_1^{\infty} dx e^{-xs} V(x)$. Then the Laplace transform of the left-hand side of equation (24) is

$$\int_1^{\infty} dx e^{-xs} (2 + x)V(x) = 2\widehat{V} - \frac{d\widehat{V}}{ds}.$$

The Laplace transform of the right-hand side of the first of (24) is, after accounting for the constraint $1 < x < 2$,

$$2\rho \int_1^2 dx e^{-xs} = \frac{2\rho}{s} (e^{-s} - e^{-2s}).$$

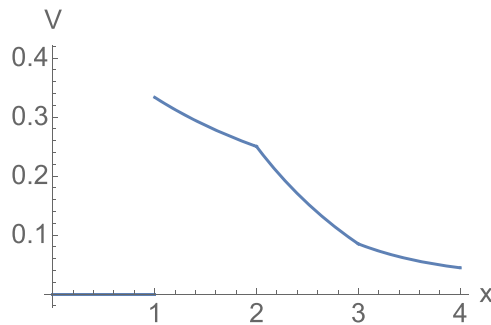


Figure 4. The void density $V(x)$ for $0 < x < 4$, showing the jump at $x = 1$ and singularities in the first derivative at $x = 2$ and $x = 3$.

Similarly, the Laplace transform of the right-hand side of the second of (24) is, after accounting for the constraint $x > 2$,

$$\int_2^\infty dx e^{-xs} \int_{x-1}^\infty dy V(y) = \int_1^\infty dy V(y) \int_2^{y+1} dx e^{-xs} = \frac{e^{-s}}{s} (\rho e^{-s} - \widehat{V}).$$

Using these results, the Laplace transform satisfies

$$2(1 + s^{-1}e^{-s})\widehat{V} - \frac{d\widehat{V}}{ds} = 2\rho s^{-1}e^{-s}. \quad (25)$$

Integrating (25) and using the steady-state density $\rho = \frac{1}{2}$ yields

$$\widehat{V}(s) = \frac{1}{2} - \mathcal{E}(s) \int_s^\infty \frac{d\sigma}{\mathcal{E}(\sigma)}. \quad (26)$$

where we define $\mathcal{E}(s) \equiv e^{2s-2E_1(s)}$ and E_1 is the exponential integral [39]

$$E_1(s) = \int_s^\infty \frac{d\sigma}{\sigma} e^{-\sigma}.$$

The large- x behavior of $V(x)$ is in principle encoded in the Laplace transform $\widehat{V}(s)$. While the Laplace transform solution is compact, it is not in a form that one can readily extract the asymptotic form of the gap distribution. It is easier to extract this asymptotic behavior from the derivative of equation (24), namely, from

$$[(2+x)V(x)]' = -2V(x-1), \quad (27)$$

where the prime denotes differentiation with respect to x . We will find that $V(x)$ decays super-exponentially with x for large x . Thus, a Taylor expansion of $V(x)$ is not justified. Instead we seek a solution of the form $V(x) = e^{-w(x)}$, where it is justifiable to expand $w(x-1)$ as $w(x) - w'(x)$. Doing so in equation (27) gives $xw' = 2e^{w'}$ to leading order. The solution to this equation is

$$w = x[\ln x + \ln(\ln x) - 1 - \ln 2] + \dots \quad (28)$$

Thus the void density $V(x) = e^{-w(x)}$ exhibits essentially a factorial (faster than exponential) decay. This mirrors the discrete solution for V_k given in equation (10).

We can use the result $\rho = \frac{1}{2}$ to directly find $V(x)$ in the successive intervals $1 < x < 2$, $2 < x < 3$, etc, from (24) without recourse to the Laplace transform method. From the first of (24), we obtain

$$V(x) = \begin{cases} \frac{1}{2+x} & 1 < x < 2, \\ \frac{[1 - 2 \ln((x+1)/3)]}{2+x} & 2 < x < 3. \end{cases} \quad (29)$$

For $x > 2$, we recast the first of equation (24) into

$$(2+x)V(x) = 1 - 2 \int_1^{x-1} dy V(y), \quad (30)$$

from which the density, for $3 < x < 4$, is

$$(2+x)V(x) = 1 - \ln(4/9) - 2\text{Li}_2(-3) + 2\text{Li}_2(-x) - (1 + 2 \ln 3 - 2 \ln x) \ln(1+x),$$

where $\text{Li}_2(-x) = \sum_{j \geq 1} (-x)^j / j^2$ is the dilogarithm function [39]. One may continue this iterative procedure to obtain explicit expressions for $V(x)$ for $n < x < n+1$ for positive integer n . These calculations quickly become tedious, so we do not extend them beyond $x = 4$. The resulting function $V(x)$ is singular (figure 4) with a slope discontinuity at every integer $x \geq 2$; thus inversion of the Laplace transform (26) in terms of a compact formula is also not possible. The main features of the void distribution $V(x)$ is that it is a piecewise smooth function, with increasingly cumbersome expressions for $V(x)$ for $n < x < n+1$, and which decays as x^{-x} for large x .

In analogy to the argument that led to the probabilities q_n for n birds to fly away at each landing event in the lattice model, in the $1d$ continuum version the corresponding probabilities are

$$\begin{aligned} q_0 &= \int_2^\infty dx (x-2)V(x) \\ q_1 &= 2 \int_2^\infty dx V(x) + 2 \int_1^2 dx (x-1)V(x) \\ q_2 &= \int_1^2 dx (2-x)V(x). \end{aligned} \quad (31)$$

Using $\rho = \frac{1}{2}$ and (29) we find $q_0 = q_2 = 4 \ln(4/3) - 1 \approx 0.151$ and $q_1 = 3 - 8 \ln(4/3) \approx 0.699$.

5. Higher dimensions

Our PB model naturally extends to the realistic situation of multiple wires, as in figure 1, and to higher dimensions. On hyper-cubic lattices \mathbb{Z}^d , we posit that all resting birds that are one lattice spacing from the newly arriving bird fly away. In the continuum \mathbb{R}^d , all resting birds within a unit distance of the newly arriving bird fly away. Simulations of the PB model on various substrates show that an initially empty system quickly reaches a steady state, and the steady-state densities are $\rho \approx \frac{1}{5}$ and $\rho \approx \frac{1}{7}$, respectively, for the square and cubic lattices. These results lead to conjectural steady-state densities on d -dimensional hyper-cubic lattices

$$\rho = \frac{1}{2d+1}. \quad (32)$$

The derivation of this result is left to future work.

It is also instructive to construct a mean-field theory for the steady-state density of the PB model on hypercubic lattices. This theory is based on neglecting correlations in the spatial positions of the birds. In this approximation, the density of birds on a d -dimensional hypercubic lattice obeys the rate equation

$$\frac{d\rho}{dt} = -(1-\rho) \sum_{n=0}^{2d} (n-1) \binom{2d}{n} (1-\rho)^{2d-n} \rho^n. \quad (33)$$

The $n=0$ term in this sum is positive corresponds to the case where the bird lands on an empty site and all neighbors of this site are also empty, so that no birds fly away and ρ increases. The terms with $n \geq 1$ are non-negative and correspond to the situations where at least one resting bird flies away when the bird lands. Equation (33) simplifies to $\frac{d\rho}{dt} = (1-\rho)(1-2\rho d)$. This gives the steady-state density $\rho = \frac{1}{2d}$, which approaches the exact steady state (32) in the limit $d \rightarrow \infty$. From this same mean-field argument, the probabilities q_n for n birds to fly away, with $0 \leq n \leq 2d$, after each landing event is

$$q_n = \binom{2d}{n} (1-\rho)^{2d-n} \rho^n. \quad (34)$$

Using the mean-field steady-state density $\rho = \frac{1}{2d}$, the above expression reduces to $q_n = e^{-1}/n!$ as $d \rightarrow \infty$. This is a rapidly decaying distribution, so that the average size of the ‘avalanche’ that is nucleated when a bird lands is small: $\langle n \rangle = 1 - e^{-1}$.

6. Concluding comments

Our PB model is inspired by natural observations and seamlessly leads to a simple non-equilibrium statistical physics model of competing adsorption/desorption. We solved for the time-dependent and steady-state properties of the model analytically. An appealing challenge is to determine the steady-state properties of the PB model in general dimensions, both on lattices and on a continuum. Another potentially fruitful direction is to extend to realistic longer-range interactions between birds. In such a scenario, when a

bird lands, it may drive a large groups of birds to fly away. This type of slow driving and sudden large ‘avalanches’ is reminiscent of the size of fires in self-organized forest fire models [12, 13], as well as the size of mass rearrangements in the random organization model [40, 41].

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Appendix A. Birds with interaction range $b > 1$

We outline some basic steady-state properties of the PB model on a discrete one-dimensional lattice in which, after each landing event, all birds that are within a distance b of the incident bird fly away. While the solution for the generating function can again be obtained by following the steps from equations (5)–(9), this calculation becomes cumbersome as b increases. However, the steady-state density $\rho = 1/(2b + 1)$ can be extracted fairly easily without the complete solution for the void distribution.

Let us first treat the case $b = 2$; the extension for $b > 2$ then readily follows. In the steady state, the generalization of equation (5) for the void densities V_k is

$$(k + 6)V_k = 2 \sum_{j \geq k-2} V_j = 2F_{k-2}. \quad (\text{A.1})$$

We use the initial conditions $V_0 = V_1 = 0$, as well as $\rho = \sum_{k \geq 0} V_k$ to solve (A.1) recursively and obtain

$$V_2 = \frac{1}{4}\rho, \quad V_3 = \frac{2}{9}\rho, \quad V_4 = \frac{1}{5}\rho, \quad V_5 = \frac{3}{22}\rho, \quad (\text{A.2})$$

etc. By using the generating function technique, we can fix ρ and then determine V_k for arbitrary k . However, if we merely want to find the steady-state density, we adopt the following approach. We first rewrite (A.1) as

$$(k + 8)[F_{k+2} - F_{k+3}] = 2F_k, \quad (\text{A.3})$$

and then sum over all $k \geq 0$ to yield

$$7F_2 + \sum_{k \geq 2} F_k = 2 \sum_{k \geq 0} F_k. \quad (\text{A.4})$$

The initial conditions $V_0 = V_1 = 0$ leads to $F_0 = F_1 = F_2 = \rho$, which then allows us to reduce (A.4) to

$$5\rho = \sum_{k \geq 0} F_k. \quad (\text{A.5})$$

Using the normalization condition $\sum_{k \geq 0} (k+1) V_k = \sum_{k \geq 0} F_k = 1$ we arrive at the basic result

$$\rho = \frac{1}{5}. \quad (\text{A.6})$$

We now determine the probabilities q_n that n birds fly away after each landing event. First note that q_2 is given by

$$q_2 = \frac{5}{4}(2V_2 + V_3). \quad (\text{A.7})$$

The factor $\frac{5}{4}$ accounts for that fact that the fraction of successful landing events in the steady state is $\frac{4}{5}$. The term $2V_2$ accounts for the two landing spots inside a vacancy of length 2 that leads to two birds flying away, while the term V_3 accounts for the fact that the landing must be at the center of a gap of length 3 to trigger two departures. Using (A.2) and (A.6), the remaining probabilities q_n are

$$q_0 = q_2 = \frac{13}{72}, \quad q_1 = \frac{23}{36}. \quad (\text{A.8})$$

For the case of arbitrary b . The analog of equation (A.3) is

$$(k + 3b + 2)[F_{k+b} - F_{k+b+1}] = 2F_k. \quad (\text{A.9})$$

Summing over all $k \geq 0$ we obtain

$$(3b + 1)F_b + \sum_{k \geq b} F_k = 2 \sum_{k \geq 0} F_k. \quad (\text{A.10})$$

The initial condition $F_0 = F_1 = \dots = F_b = \rho$ yields $\sum_{k \geq b} F_k = \sum_{k \geq 0} F_k + b\rho$. Using this in (A.10), we obtain

$$(2b + 1)\rho = \sum_{k \geq 0} F_k. \quad (\text{A.11})$$

Now using the normalization condition $\sum_{k \geq 0} F_k = 1$, the steady-state density is

$$\rho = \frac{1}{2b + 1}. \quad (\text{A.12})$$

From (A.9) and (A.12), and using the initial condition $F_j = \rho$ for $j \leq b$ as well as the definition of V_k in terms of F_k , we find

$$V_{b+j} = \frac{2}{2b + 1} \frac{1}{3b + 2 + j}. \quad (\text{A.13})$$

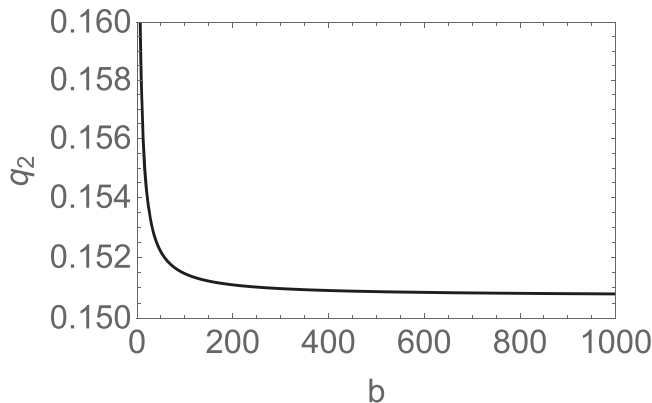


Figure A1. The probability q_2 as a function of b for $b \leq 1000$.

Let us now determine the probabilities q_n for arbitrary b . The generalization of (A.7) is

$$q_2 = \frac{2b+1}{2b} \sum_{j=0}^b (b-j)V_{b+j}. \quad (\text{A.14})$$

The meaning of each term in the sum is the same as the two terms in equation (A.7): we are counting the number of ways that a bird can land within a gap of length $b+j$ such that exactly two birds fly away. Substituting in (A.13) into (A.14) and computing the sum, we obtain

$$q_2 = 2(2+b^{-1})(H_{4b+2} - H_{3b+1}) - 1 - b^{-1}, \quad (\text{A.15})$$

where $H_n = \sum_{1 \leq j \leq n} j^{-1}$ is the n th harmonic number. Again, $q_0 = q_2$ and q_1 is fixed by normalization, $q_1 = 1 - 2q_2$. For $b \rightarrow \infty$, $q_2 \rightarrow 4 \ln(4/3) - 1 \approx 0.15073$, which reproduces the continuum result of equation (31), as it must. The dependence of q_2 on b is shown in figure A1.

Now we extend the above result to find the time-dependent behavior. For general $b > 1$, the void densities V_k with $k \geq b$ evolve according to

$$\dot{V}_k = -(2b+2+k)V_k + 2 \sum_{\ell \geq k-b} V_\ell, \quad (\text{A.16})$$

subject to the constraint that $V_0 = \dots = V_{b-1} = 0$. Summing equation (A.16) over $k \geq b$ and using the above constraint, as well as equation (2), we obtain the simple equation for the density

$$\dot{\rho} = 1 - (2b+1)\rho,$$

from which

$$\rho(t) = \frac{1 - e^{-(2b+1)t}}{2b+1}. \quad (\text{A.17})$$

The first non-trivial void density V_b satisfies

$$\dot{V}_b(t) = -(3b + 2)V_b + 2\rho, \quad (\text{A.18})$$

from which

$$V_b(t) = \frac{2}{(2b + 1)(3b + 2)} - \frac{2 e^{-(2b+1)t}}{(b + 1)(2b + 1)} + \frac{2 e^{-(3b+2)t}}{(b + 1)(3b + 2)}. \quad (\text{A.19})$$

The density V_{b+1} satisfies

$$\dot{V}_{b+1} = -(3b + 3)V_{b+1} + 2\rho, \quad (\text{A.20})$$

from which

$$V_{b+1}(t) = \frac{2}{(2b + 1)(3b + 3)} - \frac{2 e^{-(2b+1)t}}{(b + 2)(2b + 1)} + \frac{2 e^{-(3b+3)t}}{(b + 2)(3b + 3)}. \quad (\text{A.21})$$

When $b \leq k \leq 2b$, the rate equation for V_k has a form

$$\dot{V}_k = -(2b + 2 + k)V_{b+j} + 2\rho \quad (\text{A.22})$$

similar to (A.18) and (A.20). Solving (A.22) yields

$$V_k(t) = \frac{2}{(2b + 1)(2b + 2 + k)} - \frac{2 e^{-(2b+1)t}}{(k + 1)(2b + 1)} + \frac{2 e^{-(2b+2+k)t}}{(k + 1)(2b + 2 + k)} \quad (\text{A.23})$$

for $b \leq k \leq 2b$.

Appendix B. Higher-order correlation functions

The pattern in the equations for C_4 , C_5 , and C_6 generalizes in straightforward way and we merely write the equations for the next three correlation functions in the steady state:

$$\begin{aligned} C_7 &= 2V_{1,1,2} + V_{1,2,1} + 2V_{1,4} + 2V_{2,3} + V_6 \\ C_8 &= V_{1,1,1,1} + 2V_{1,1,3} + V_{1,3,1} + 2V_{1,5} + 2V_{2,4} + V_{3,3} + V_7 \\ C_9 &= 2V_{1,1,1,2} + 2V_{1,1,2,1} + 2V_{1,1,4} + V_{1,4,1} + 2V_{2,1,3} + 2V_{1,2,3} + 2V_{1,3,2} + V_{2,2,2} \\ &\quad + 2V_{1,6} + 2V_{2,5} + 2V_{3,4} + V_8. \end{aligned}$$

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