KHOVANOV HOMOLOGY DETECTS SPLIT LINKS

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Abstract. Extending ideas of Hedden-Ni, we show that the module structure on Khovanov homology detects split links. We also prove an analogue for untwisted Heegaard Floer homology of the branched double cover. Technical results proved along the way include two interpretations of the module structure on untwisted Heegaard Floer homology in terms of twisted Heegaard Floer homology and the fact that the module structure on the reduced Khovanov complex of a link is well-defined up to quasi-isomorphism.

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1. Introduction

Since the Jones polynomial and Khovanov homology are somewhat mysterious invariants, there has been substantial interest in understanding their geometric content. Much progress along these lines has been finding detection results. Grigsby-Wehrli showed that the Khovanov homology of nontrivial cables detects the unknot [GW10]. (See also [Hed09].) Kronheimer-Mrowka showed that Khovanov homology itself detects the unknot [KM11]. So, by work of Hedden-Ni, Khovanov homology also detects the 2-component unlink [HN10].

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Hedden-Ni went on to show that the module structure on Khovanov homology detects the n-component unlink [HN13]. Batson-Seed refined this to show that Khovanov homology as a bi-graded abelian group detects the unlink [BS15]. Recently, Baldwin-Sivek showed that Khovanov homology detects the trefoils [BS] and Baldwin-Sivek-Xie showed that Khovanov homology detects the Hopf links [BSX19]. Even more recently, Xie-Zhang classified n-component links with Khovanov homology of dimension 2^n [XZ].

The detection problem for Heegaard Floer homology has also received considerable attention. Ozsváth-Szabó showed that knot Floer homology detects the genus (and Heegaard Floer homology detects the Thurston norm) [OSz04a], and hence the unknot. Ghiggini showed that knot Floer homology detects the trefoils and figure 8 knot [Ghi08]. Ni showed that knot Floer homology detects fibered knots in general and Heegaard Floer homology detects 3-manifolds that fiber over the circle with fiber of genus > 1 [Ni07, Ni09]. Ai-Peters and Ai-Ni showed that twisted Heegaard Floer homology detects fibered 3-manifolds with genus 1 fibers [AP10, AN09]. Ni showed that Heegaard Floer homology detects the Borromean knots [Ni14], and Hedden-Ni classified manifolds with small Heegaard Floer ranks [HN10]. Building slightly on these results, Alishahi-Lipshitz showed that bordered Heegaard Floer homology detects homologically essential compressing disks, bordered-sutured Heegaard Floer homology detects boundary-parallel tangles, and twisted Heegaard Floer homology detects homologically essential 2-spheres [AL19]. (This last detection theorem will be used below.)

Indeed, all of the detection results for Khovanov homology come from comparing Khovanov homology to some gauge-theoretic invariant, like Heegaard Floer homology. This paper will be no exception. Extending ideas of Hedden-Ni's, we will use the fact that the branched double cover of a link L is irreducible if and only if L is prime and non-split to show:

Theorem 1. *Let L be a 2-component link in S^3. Fix basepoints p, q on the two components of L. Let $\widetilde{Kh}(L;\mathbb{F}_2)$ be the reduced Khovanov homology of L with respect to the basepoint p, viewed as an $\mathbb{F}_2[X]/(X^2)$-module with respect to the basepoint q. Then, $\widetilde{Kh}(L;\mathbb{F}_2)$ is a free module if and only if L is a split link.*

More generally, for a link L with k components and basepoints p, q on L, there is a 2-sphere in $S^3 \setminus L$ separating p from q if and only if $\widetilde{Kh}(L;\mathbb{F}_2)$ is a free module over $\mathbb{F}_2[X]/(X^2)$.

We give a refined version of Theorem 1, and a version for unreduced Khovanov homology, below, after recalling some algebra.

Definition 1.1. Let C be a bounded chain complex over a ring R or, more generally, an A_∞-module over R. We say that C is *quasi-free* if C is (A_∞) quasi-isomorphic to a bounded chain complex of free R-modules.

Definition 1.2. Let $\mathbb{F}_2[Y^{-1}, Y]]$ denote the ring of Laurent series. Let (C, ∂_C) be a differential $\mathbb{F}_2[X]/(X^2)$-module (e.g., a chain complex over $\mathbb{F}_2[X]/(X^2)$). By the *unrolling* of C we mean the differential $\mathbb{F}_2[Y^{-1}, Y]]$-module $C^{\mathrm{un}} = C \otimes_{\mathbb{F}_2} \mathbb{F}_2[Y^{-1}, Y]]$ with differential

$$\partial(z \otimes Y^n) = \partial_C(z) \otimes Y^n + zX \otimes Y^{n+1}.$$

This is a completion of the total complex of the bicomplex

$$\cdots \xrightarrow{X} C \xrightarrow{X} C \xrightarrow{X} C \xrightarrow{X} \cdots.$$

More generally, if C is a strictly unital A_∞-module over $\mathbb{F}_2[X]/(X^2)$ then the unrolled complex of C is $C \otimes_{\mathbb{F}_2} \mathbb{F}_2[Y^{-1}, Y]]$ with differential

$$\partial(z \otimes Y^n) = \sum_{m \geq 0} \mu_{1+m}(z, \overbrace{X, \cdots, X}^{m}) \otimes Y^{n+m}.$$

This is an honest differential module over $\mathbb{F}_2[Y^{-1}, Y]]$. (The notion of strict unitality is recalled in Definition 2.6.)

We will refer to the homology of C^{un}, $H_*(C^{\mathrm{un}})$, as the *unrolled homology* of C.

Theorem 2. *Let L be a link in S^3 and $p, q \in L$. Let $\widetilde{\mathcal{C}}_{Kh}(L; \mathbb{F}_2)$ be the reduced Khovanov complex with respect to p, which is a module over $\mathbb{F}_2[X]/(X^2)$ via the basepoint q. Then, the following are equivalent:*

(1) There is a 2-sphere in $S^3 \setminus L$ separating p from q.
(2) $\widetilde{Kh}(L; \mathbb{F}_2)$ is a free module.
(3) $\widetilde{\mathcal{C}}_{Kh}(L; \mathbb{F}_2)$ is quasi-free.
(4) $\widetilde{\mathcal{C}}_{Kh}(L; \mathbb{F}_2)^{\mathrm{un}}$ is acyclic.

Corollary 1.3. *Let L be a link in S^3 and p, q points in L. There is a 2-sphere in $S^3 \setminus L$ separating p from q if and only if $Kh(L; \mathbb{F}_2)$ is a free module over $\mathbb{F}_2[W, X]/(W^2, X^2)$ where the action of W corresponds to p and the action of X corresponds to q.*

Remark 1.4. In Theorem 2, the implication (1) $\Longrightarrow$ (2) is a result of Shumakovitch [Shu14, Corollary 3.2.B]; see Lemma 5.3. (This also follows from an argument in odd Khovanov homology [ORSz13, Proposition 1.8].) The implication (1) $\Longrightarrow$ (3) is obvious, modulo knowing that the basepoint action is well-defined, up to quasi-isomorphism, on the reduced Khovanov complex. The implication (2) $\Longrightarrow$ (4) follows from an easy spectral sequence argument. The implication (3) $\Longrightarrow$ (4) is Lemma 2.12, which again follows from an easy spectral sequence argument. Most of the work is in proving the implication (4) $\Longrightarrow$ (1), which uses the Ozsváth-Szabó spectral sequence [OSz05], a nontriviality result for twisted Heegaard Floer homology of Ozsváth-Szabó and Hedden-Ni, and a computation of the A_∞ module structure on $\widehat{HF}(Y)$ in terms of the twisted Floer homology. In particular, the restriction to characteristic 2 is because of the corresponding restriction for the Ozsváth-Szabó spectral sequence.

As in Hedden-Ni's work, the key to proving Theorem 2 is tracking the module structure through the Ozsváth-Szabó spectral sequence $\widetilde{Kh}(m(L)) \Rightarrow \widehat{HF}(\Sigma(L))$. The Heegaard Floer homology $\widehat{HF}(Y)$ is a module over the exterior algebra $\Lambda^*(H_1(Y)/tors)$ [OSz04c]. In the case $Y = \Sigma(L)$, the pair of points $p, q \in L$ specifies an element $X \in H_1(\Sigma(L))$ so, by restriction of scalars, $\widehat{HF}(\Sigma(L))$ is a module over $\mathbb{F}_2[X]/(X^2)$.

Proving Theorem 2 requires working at the chain level. As Hedden-Ni note, at the chain level, the action of X on $\widehat{CF}(\Sigma(L))$ is only associative up to homotopy. In fact, $\widehat{CF}(\Sigma(L))$ is naturally an A_∞-module over $\mathbb{F}_2[X]/(X^2)$; see Section 3. By homological perturbation theory, $\widehat{HF}(\Sigma(L))$ inherits the structure of an A_∞-module. Similarly, the action of $\mathbb{F}_2[X]/(X^2)$ on $\widetilde{\mathcal{C}}_{Kh}(L)$ induces an A_∞-module structure on $\widetilde{Kh}(L)$.

We have the following Heegaard Floer analogue of Theorem 2:

Theorem 3. *Let $L \subset S^3$ be a link and $p, q \in L$. Consider the induced A_∞-module structures on $\widehat{CF}(\Sigma(L);\mathbb{F}_2)$ and $\widehat{HF}(\Sigma(L);\mathbb{F}_2)$ over $\mathbb{F}_2[X]/(X^2)$. We can also view $\widehat{HF}(\Sigma(L);\mathbb{F}_2)$ as an ordinary module over $\mathbb{F}_2[X]/(X^2)$, by forgetting the higher A_∞ operations. Then, the following are equivalent:*
(1) There is a 2-sphere in $S^3 \setminus L$ separating p and q.
(2) $\widehat{HF}(\Sigma(L);\mathbb{F}_2)$, viewed as an ordinary module over $\mathbb{F}_2[X]/(X^2)$, is a free module.
(3) the A_∞-module $\widehat{CF}(\Sigma(L);\mathbb{F}_2)$ is quasi-free.
(4) $\widehat{CF}(\Sigma(L);\mathbb{F}_2)^{\mathrm{un}}$ is acyclic.

Note that, for Heegaard Floer homology with appropriate twisted coefficients, some of these equivalences were essentially proved by Hedden-Ni [HN13, Corollary 5.2].

Remark 1.5. This project stems from thinking about Eisermann's result [Eis09] that the reduced Jones polynomial of a 2-component ribbon link is divisible by $(q + q^{-1})$. Among his prescient comments, Eisermann [Eis09, Section 7.3] notes that it is not true that the reduced Khovanov homology of such a ribbon link is divisible by the Khovanov homology of the unknot. We thought that perhaps, instead, the reduced Khovanov complex of a ribbon link might be quasi-free over $\mathbb{F}_2[X]/(X^2)$, which would recover Eisermann's result after decategorification. Theorem 2 shows that this is definitely not the case, at least in characteristic 2. In fact, for Eisermann's example $L10^n_{36}$, a 2-component ribbon knot, the Khovanov complex is not quasi-free in any characteristic: the reduced Khovanov homology of $L10^n_{36}$, as computed by sKnotJob [Sch], is:

$q \setminus h$	-5	-4	-3	-2	-1	0	1	2	3	4	5
9											$\mathbb{Z}$
7										$\mathbb{Z}$	
5									$\mathbb{Z}$		
3							$\mathbb{Z}$	$\mathbb{Z}^2$			
1						$\mathbb{Z}^2$	$\mathbb{Z}$				
-1					$\mathbb{Z}$	$\mathbb{Z}^2$					
-3				$\mathbb{Z}^2$	$\mathbb{Z}$						
-5			$\mathbb{Z}$								
-7		$\mathbb{Z}$									
-9	$\mathbb{Z}$										

Considering the bi-gradings, this implies that the unrolled homology is nontrivial.

Remark 1.6. The restriction to $\mathbb{F}_2$-coefficients in Theorem 3 is presumably unnecessary. The additional work required to generalize Theorem 3 to arbitrary field coefficients is adding signs to Section 3.2.

Remark 1.7. An analogue of Corollary 1.3 for link Floer homology was recently proved by Wang [Wan21].

This paper is organized as follows. In Section 2 we collect some algebraic definitions and results. Section 3 recalls Heegaard Floer homology with twisted coefficients and the (A_∞) $\Lambda^*(H_1(Y)/tors)$-module structure on Heegaard Floer homology, and relates them. The relations are in Section 3.2; much of this works more generally for complexes over $\mathbb{F}_2[t^{-1}, t]$ and A_∞-modules over $\mathbb{F}_2[X]/(X^2)$, and may be of independent interest. Section 4 recalls the module structure on the Khovanov complex and reduced Khovanov complex, and proves these are invariants up to quasi-isomorphism. Finally, Section 5 combines these ingredients to prove the detection theorems.

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2. Algebraic background

Throughout this section, for convenience and because it suffices for our application, we work in characteristic 2. Many of the results have easy extensions to arbitrary characteristic.

2.1. Ungraded chain complexes.

The Heegaard Floer complexes are cyclically graded. Since the homological algebra of cyclically graded chain complexes behaves differently in some cases, we note some properties that hold for ungraded chain complexes and, consequently, for cyclically graded ones.

Definition 2.1. Let R be a ring. An *ungraded chain complex* over R or *differential R-module* is an R-module C and a homomorphism $\partial\colon C \to C$ with $\partial^2 = 0$. The *homology* $H(C, \partial)$ of (C, ∂) is $\ker(\partial)/\operatorname{im}(\partial)$.

Given ungraded chain complexes (C, ∂_C) and (D, ∂_D) over R, an R-module homomorphism $f\colon C \to D$ is a *chain map* if $\partial_D \circ f = f \circ \partial_C$. A chain map induces a map on homology. A chain map is a *quasi-isomorphism* if the induced map on homology is an isomorphism.

We will also be interested in ungraded A_∞-modules:

Definition 2.2. Let R be an $\mathbb{F}_2$-algebra. An *ungraded A_∞-module* over R is an $\mathbb{F}_2$-vector space M together with maps

$$\mu_{1+n}\colon M \otimes R^{\otimes n} \to M$$

satisfying

$$\sum_{i+j=n} \mu_{1+i}(\mu_{1+j}(z, a_1, \dots, a_j), a_{j+1}, \dots, a_n) + \sum_{i=1}^{n-1} \mu_n(z, a_1, \dots, a_{i-1}, a_i a_{i+1}, \dots, a_n) = 0$$

for each $n \geq 0$, $z \in M$, and $a_1, \dots, a_n \in R$.

Given ungraded A_∞-modules (M, μ^M) and (N, μ^N) over R, an *A_∞-module homomorphism* $f\colon (M, \mu^M) \to (N, \mu^N)$ is a collection of $\mathbb{F}_2$-vector space homomorphisms

$$f_{1+n}\colon M \otimes R^{\otimes n} \to N$$

satisfying

$$\sum_{i+j=n} f_{1+i}(\mu^M_{1+j}(z, a_1, \dots, a_j), a_{j+1}, \dots, a_n) + \sum_{i+j=n} \mu^N_{1+i}(f_{1+j}(z, a_1, \dots, a_j), a_{j+1}, \dots, a_n) + \sum_{i=1}^{n-1} f_n(z, a_1, \dots, a_{i-1}, a_i a_{i+1}, \dots, a_n) = 0$$

for each $n \geq 0$, $z \in M$, and $a_1, \dots, a_n \in R$. An A_∞-module homomorphism f is a *quasi-isomorphism* if the map $f_1\colon (M, \mu^M_1) \to (N, \mu^N_1)$ is a quasi-isomorphism.

Given A_∞ homomorphisms $f, g\colon (M,\mu^M) \to (N,\mu^N)$, a *homotopy* from f to g is a collection of $\mathbb{F}_2$-vector space homomorphisms $k_{1+n}\colon M \otimes R^{\otimes n} \to N$ so that for all n,

$$\sum_{i+j=n} k_{1+i}(\mu^M_{1+j}(z, a_1, \dots, a_j), a_{j+1}, \dots, a_n) + \sum_{i+j=n} \mu^N_{1+i}(k_{1+j}(z, a_1, \dots, a_j), a_{j+1}, \dots, a_n) + \sum_{i=1}^{n-1} k_n(z, a_1, \dots, a_{i-1}, a_i a_{i+1}, \dots, a_n) = f_{1+n} + g_{1+n}.$$

Given A_∞ homomorphisms $f\colon (M,\mu^M) \to (N,\mu^N)$ and $g\colon (N,\mu^N) \to (P,\mu^P)$, define $(g \circ f)\colon M \to P$ by

$$(g \circ f)_{1+n} = \sum_{i+j=n} g_{1+i}(f_{1+j}(z, a_1, \dots, a_j), a_{j+1}, \dots, a_n).$$

The *identity homomorphism* of M is defined by $\mathrm{Id}_1(x) = x$ and $\mathrm{Id}_{1+n} = 0$ for $n > 0$.

An A_∞ homomorphism $f\colon M \to N$ is a *homotopy equivalence* if there is an A_∞ homomorphism $g\colon N \to M$ so that $f \circ g$ and $g \circ f$ are homotopic to the identity maps.

(Of course, these definitions generalize to the case that R is an A_∞-algebra, but we will not need this generalization.)

The universal coefficient theorem holds in the ungraded setting:

Lemma 2.3. *Let R be a principal ideal domain, (C,∂) an ungraded chain complex over R, and M an R-module. Assume that C is a projective R-module. Then, there is a natural short exact sequence*

$$0 \to H(C,\partial) \otimes_R M \to H(C \otimes_R M, \partial) \to \mathrm{Tor}^1_R(H(C,\partial), M) \to 0$$

which splits (unnaturally).

Proof. From C, construct an ordinary, bounded below, $\mathbb{Z}$-graded chain complex $\widetilde{C}$ by setting

$$\widetilde{C}_n = \begin{cases} C & n \geq 0 \\ 0 & n < 0. \end{cases}$$

and letting $\partial_n\colon \widetilde{C}_n \to \widetilde{C}_{n-1}$ be the map ∂ for all $n \geq 1$. Then, for any $i > 0$, $H_i(\widetilde{C}) \cong H(C)$. Applying the usual universal coefficient theorem for homology to $\widetilde{C}$ for any $i > 0$ gives the result. ◻

Proposition 2.4. *Let R be a principal ideal domain and let C be a free chain complex over R. If C is graded, assume that C is finitely generated in each grading; if C is ungraded, assume that C is finitely generated. View the homology $H(C)$ as an honest R-module, i.e., with trivial higher operations μ_{1+n} ($n > 1$). Then, there is a quasi-isomorphism of R-modules $f\colon C \to H(C)$.*

Proof. In the graded case, this is well-known; we observe that the proof also works for ungraded complexes (C,∂). Let $K = \ker(\partial)$. We claim that C/K is a free module. Since C/K is finitely generated, from the classification of modules over a PID it suffices to show that C/K is torsion-free; but if $[\alpha] \in C/K$ satisfies $r[\alpha] = 0$ for some $r \in R$ then $r\alpha \in K$ so either $\alpha \in K$ or $r = 0$.

Hence, the short exact sequence $0 \to K \to C \to C/K \to 0$ splits. So, we can extend an ordered basis $x_1, \dots, x_k$ for K to an ordered basis $x_1, \dots, x_k, y_1, \dots, y_\ell$ for C. With respect to this basis, the matrix for ∂ has the form

$$\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$$

where A is some $k \times \ell$ matrix. By changing basis among the x_i and y_j we can assume A is in Smith normal form. So, assume that A has entries $r_1, \dots, r_j$ on the diagonal ($j = \min\{k, \ell\}$) and 0s off the diagonal. Then,

$$H(C) \cong R/(r_1) \oplus \cdots \oplus R/(r_j)$$

and the homomorphism $C \to H(C)$ sending x_i to $1 \in R/(r_i)$ and y_j to 0 is a quasi-isomorphism. $\square$

Remark 2.5. In this paper, we make use of A_∞-modules over $\mathbb{F}_2[X]/(X^2)$ and chain complexes of honest modules over $\mathbb{F}_2[t^{-1}, t]$. One might wonder why A_∞-modules over $\mathbb{F}_2[t^{-1}, t]$ do not also make an appearance. This is because of Proposition 2.4, which shows that no interesting A_∞ operations over $\mathbb{F}_2[t^{-1}, t]$ arise. (In particular, there are no interesting A_∞ operations on $\widehat{HF}(Y; \mathbb{F}_2[t^{-1}, t])$.)

2.2. **Further notions for A_∞-modules.**

In this section, we recall a few more definitions and results regarding A_∞-modules.

Definition 2.6. Let R be a ring with unit 1. A (graded or ungraded) A_∞-module $(M, \{\mu_{1+i}\})$ over R is *strictly unital* if:

- $\mu_2(x, 1) = x$ for all $x \in M$, and
- $\mu_{1+n}(x, a_1, \dots, a_n) = 0$ if $n > 1$ and some $a_i = 1$.

Similarly, a morphism $\{f_{1+n} \colon M \otimes R^{\otimes n} \to N\}$ of strictly unital A_∞-modules is *strictly unital* if $f_{1+n}(m, a_1, \dots, a_n) = 0$ if some $a_i = 1$.

Convention 2.7. Throughout this paper, all A_∞-modules and maps are strictly unital.

Example 2.8. A strictly unital A_∞-module over $\mathbb{F}_2[X]/(X^2)$ is determined by the operations $\mu_{1+n}(\cdot, X, \dots, X)$.

There are several advantages of working with A_∞-modules; we highlight two (related) ones. First, A_∞-module structures transfer nicely under maps; results of this kind for A_∞ objects are often called *homological perturbation theory*:

Proposition 2.9. *Let R be an $\mathbb{F}_2$-algebra and (M, μ^M) an A_∞-module over R. Let (N, μ_1^N) be a chain complex over $\mathbb{F}_2$ and $f_1 \colon (M, \mu_1^M) \to (N, \mu_1^N)$ a homotopy equivalence of chain complexes over $\mathbb{F}_2$. Then, there is an A_∞ structure $\{\mu_{1+n}^N\}$ on N extending μ_1^N and an A_∞ homotopy equivalence $f \colon (M, \mu^M) \to (N, \mu^N)$ extending f_1.*

The corresponding statement also holds for A_∞ (R, S)-bimodules.

See, e.g., Keller's survey [Kel01, Section 4.3], or [LOT14, Lemma 9.6]. In particular, the former reference has a nice description of the history of such results, and the latter does not rely on gradings.

Second, for differential modules or chain complexes of modules, there is an important distinction between homotopy equivalence and quasi-isomorphism. This distinction does not exist for A_∞-modules:

Proposition 2.10. *Let R be an algebra over $\mathbb{F}_2$, M and N A_∞ modules over R, and $f\colon M \to N$ a quasi-isomorphism. Then, f is a homotopy equivalence. Further, two ordinary differential modules M, N over R are A_∞ quasi-isomorphic (or homotopy equivalent) if and only if M and N are quasi-isomorphic in the usual sense.*

See, e.g., Keller's paper [Kel01, Section 4.3], or [LOT15, Proposition 2.4.1]. (Again, the latter reference does not rely on gradings.) The point is that the map from the bar resolution of M to M is an A_∞ homotopy equivalence, and the bar resolution is a projective module. Hence, the quasi-isomorphism to the bar resolution is invertible up to homotopy, and hence any quasi-isomorphism is invertible up to homotopy.

2.3. **The unrolled homology.** Recall that given an A_∞-module C over $\mathbb{F}_2[X]/(X^2)$, in Definition 1.2 we defined the unrolled complex C^{un} of C.

Lemma 2.11. *Let $(C, \{\mu^C_{1+n}\})$ and $(D, \{\mu^D_{1+n}\})$ be finitely generated, graded or ungraded A_∞-modules over $S = \mathbb{F}_2[X]/(X^2)$. A homomorphism of A_∞-modules $f\colon C \to D$ induces a homomorphism $F\colon C^{\mathrm{un}} \to D^{\mathrm{un}}$, and if f is a quasi-isomorphism then so is F.*

Proof. Given a collection of maps $f_{1+n}\colon C \otimes_{\mathbb{F}_2} S^{\otimes n} \to D$ define a map

$$F\colon C^{\mathrm{un}} \to D^{\mathrm{un}}$$

by

$$F(z \otimes Y^n) = \sum_{m \geq 0} f_{1+m}(z, \overbrace{X, \cdots, X}^{m}) \otimes Y^{n+m}.$$

It is immediate from the construction that:

- If f is the identity map (i.e., $f_1 = \mathrm{Id}$ and $f_n = 0$ for $n > 1$) then the induced map F is also the identity map.
- The map F associated to a collection of maps $f = \{f_{1+n}\}$ is well-defined. (In particular, this uses the fact that we have completed with respect to Y.)
- The map F associated to a collection of maps $f = \{f_{1+n}\}$ is an $\mathbb{F}_2[Y^{-1}, Y]]$-module homomorphism.
- If $f = \{f_{1+n}\}$ is an A_∞-module homomorphism then F is a chain map. (In fact, F is a chain map if and only if f is an A_∞ module homomorphism.)
- If k is a homotopy between A_∞-module homomorphisms f and g then the induced map K is a chain homotopy between F and G.
- The map associated to $g \circ f$ is the composition of the maps G associated to g and F associated to f.

It follows that homotopy equivalent A_∞-modules have homotopy equivalent unrolled complexes. Since by Proposition 2.10, quasi-isomorphism and homotopy equivalence agree for A_∞-modules (over an algebra over a field), this proves the result. ☐

Lemma 2.12. *Let C be a (graded or ungraded) chain complex over $\mathbb{F}_2[X]/(X^2)$, not necessarily free. If C is quasi-free then C^{un} is acyclic.*

Proof. By Lemma 2.11 it suffices to prove the result when C is a finite-dimensional free module (with a differential).

As a warm-up, we start with the graded case when X has grading 0. Consider the spectral sequence associated to the vertical filtration on C^{un}_*, where the d^0-differential is multiplication by X. Since C_* is free, this d^0-differential is exact. Hence, for this spectral sequence, $E^1 = 0$. Since C_* is bounded, this implies that $H_*(C_*^{\mathrm{un}}) = 0$, as well. This proves the result.

Essentially the same argument works in the general (ungraded) case. Choose an ordered basis $[e_1, \dots, e_N, f_1, \dots, f_N]$ for C over $\mathbb{F}_2$, where $Xe_i = f_i$. Write the differential on C as a block matrix $\left(\begin{smallmatrix} A & D \\ B & E \end{smallmatrix}\right)$ where each block is $N \times N$. Since $\partial(f_i) = X\partial(e_i)$, we have $D = 0$ and $A = E$, so the differential actually has the form $\left(\begin{smallmatrix} A & 0 \\ B & A \end{smallmatrix}\right)$.

The complex C^{un} is a vector space over $\mathbb{F}_2[Y^{-1}, Y]]$ with ordered basis $[e_1 \otimes 1, \dots, e_N \otimes 1, f_1 \otimes 1, \dots, f_N \otimes 1]$. The differential on C^{un} has the form

$$\begin{pmatrix} A & 0 \\ B + IY & A \end{pmatrix}$$

where I denotes the $N \times N$ identity matrix. Since $\partial^2 = 0$, the differential on C^{un} has rank at most N, so since

$$\det(B + IY) = Y^N + \text{lower order terms} \neq 0,$$

$(B + IY)$ is invertible, the differential on C^{un} has rank equal N. Hence, since $\mathbb{F}_2[Y^{-1}, Y]]$ is a field, C^{un} is acyclic, as claimed. ☐

The spectral sequence in the (graded case of the) proof of Lemma 2.12 is only well-behaved under restrictive hypotheses: for unbounded chain complexes, convergence becomes a problem. (Consider, for example, the chain complex $0 \leftarrow \mathbb{F}_2[X]/(X^2) \xleftarrow{X} \mathbb{F}_2[X]/(X^2) \xleftarrow{X} \cdots$.) On the other hand, because we have completed with respect to Y, the *horizontal filtration* of C_*^{un}, by the power of Y, induces a spectral sequence that is well-behaved even for C_* unbounded or ungraded. For this spectral sequence, the d^0-differential is the differential on C_*, the d^1-differential is the action of X on the homology of C_*, and the higher differentials are induced from the A_∞ operations on the homology of C_*.

Remark 2.13. In the language of bordered Floer theory [LOT11, Section 8], there is a rank 1 type *DD* bimodule over $\mathbb{F}_2[X]/(X^2)$ and $\mathbb{F}_2[Y]$ defined by $P = \langle \iota \rangle$ and $\delta^1(\iota) = (X \otimes Y) \otimes \iota$. The bimodule P witnesses the Koszul duality between $\mathbb{F}_2[X]/(X^2)$ and $\mathbb{F}_2[Y]$. The unrolled complex is obtained by taking the box tensor product with P, modulifying the result, and extending scalars from $\mathbb{F}_2[[Y]]$ to $\mathbb{F}_2[Y^{-1}, Y]]$. The appearance of power series in Y relates to operational boundedness (cf. [LOT, Section 9]).

3. Two views of the module structure on Heegaard Floer homology

3.1. **Geometry: holomorphic curves with point constraints.**

Fix a commutative ring k.

Let Y be a closed, oriented 3-manifold and let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a weakly admissible pointed Heegaard diagram for Y. Given an abelian group G, a *G-valued additive assignment* is a function $A\colon \pi_2(x, y) \to G$ for each pair of points $x, y \in T_\alpha \cap T_\beta$ so that for all $w, x, y \in T_\alpha \cap T_\beta$, $\phi \in \pi_2(w, x)$, and $\psi \in \pi_2(x, y)$, $A(\phi * \psi) = A(\phi) + A(\psi)$. Given a G-valued additive assignment A, there is an associated twisted Floer complex with coefficients in the group ring $\mathbb{F}_2[G]$,

$$\widehat{CF}(Y; \mathbb{F}_2[G]_A) = \widehat{CF}(\mathcal{H}; \mathbb{F}_2[G]_A) = \bigoplus_{x \in T_\alpha \cap T_\beta} \mathbb{F}_2[G],$$

with differential

$$\underline{\partial}(x) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1,\ n_z(\phi)=0}} \big(\#\mathcal{M}^\phi(x,y)\big) t^{A(\phi)} y.$$

Here, we are writing elements of $\mathbb{F}_2[G]$ as linear combinations $\sum n_i t^{g_i}$ with $n_i \in \mathbb{F}_2$ and $g_i \in G$, and $\mathcal{M}^\phi(x,y)$ denotes the moduli space of holomorphic Whitney disks connecting x to y in the homotopy class ϕ (modulo the action of $\mathbb{R}$ on the source), with respect to a sufficiently generic family of almost complex structures. It turns out that there is a universal, *totally twisted coefficient* Floer homology $\underline{\widehat{CF}}(Y; \mathbb{F}_2[H_2(Y)]_A)$, where A is any $H_2(Y)$-valued additive assignment which is bijective on $\{\phi \in \pi_2(x,x) \mid n_z(\phi) = 0\}$, and any other twisted Floer complex is obtained from the totally twisted coefficient Floer complex by extension of scalars. (In particular, Ozsváth-Szabó originally defined Heegaard Floer homology with twisted coefficients via the totally twisted Floer complex and extension of scalars [OSz04b, Section 8].)

Recall that each homotopy class $\phi \in \pi_2(x,y)$ is represented by a cellular 2-chain in $(\Sigma, \boldsymbol{\alpha} \cup \boldsymbol{\beta})$, called its *domain* $D(\phi)$. Let $\partial_\alpha D(\phi)$ be the part of $\partial D(\phi)$ lying in the α-circles. Fix an embedded, oriented 1-manifold $\zeta \subset \Sigma$ which intersects $\boldsymbol{\alpha}$ transversely and is disjoint from $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$. There is a corresponding $\mathbb{Z}$-valued additive assignment

$$\phi \mapsto \zeta \cdot \partial_\alpha D(\phi),$$

the algebraic intersection number of ζ with $\partial_\alpha D(\phi)$. This additive assignment gives a twisted coefficient complex $\underline{\widehat{CF}}(\mathcal{H}; \mathbb{F}_2[t^{-1}, t]_\zeta)$ with differential

$$\underline{\partial}(x) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1,\ n_z(\phi)=0}} \big(\#\mathcal{M}^\phi(x,y)\big) t^{\zeta \cdot \partial_\alpha D(\phi)} y. \tag{3.1}$$

It is not hard to show that, up to quasi-isomorphism, the complex $\underline{\widehat{CF}}(\mathcal{H}; \mathbb{F}_2[t^{-1}, t]_\zeta)$ depends on ζ only through the homology class $[\zeta] \in H_1(Y)/tors = \mathrm{Hom}(H_2(Y), \mathbb{Z})$ it represents.

Of course, there is also an untwisted Heegaard Floer homology group

$$\widehat{CF}(Y; \mathbb{F}_2) = \widehat{CF}(\mathcal{H}; \mathbb{F}_2) = \bigoplus_{x \in T_\alpha \cap T_\beta} \mathbb{F}_2$$

with differential

$$\partial(x) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1,\ n_z(\phi)=0}} \big(\#\mathcal{M}^\phi(x,y)\big) y. \tag{3.2}$$

As Ozsváth-Szabó noted [OSz04c, Section 4.2.5], the untwisted Heegaard Floer complex $\widehat{CF}(Y; \mathbb{F}_2)$ inherits an action of $H_1(Y)/tors$ via the formula

$$\zeta \cdot x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1,\ n_z(\phi)=0}} \big(\#\mathcal{M}^\phi(x,y)\big)\big(\zeta \cdot \partial_\alpha D(\phi)\big) y. \tag{3.3}$$

(The action of ζ lowers the Maslov grading by 1.) As they show, at the level of homology this endows $\widehat{HF}(Y)$ with the structure of a module over the exterior algebra $\Lambda^*(H_1(Y)/tors)$. (There is a tiny but relevant omission in Ozsváth-Szabó's argument [OSz04c, Proof of Proposition 4.17]: they dropped the homotopy term which is discussed below. See also [Lip06, Proof of Proposition 8.6].)

This statement can be refined slightly to make $\widehat{CF}(\mathcal{H})$, and hence $\widehat{HF}(\mathcal{H})$, into an A_∞-module over $\Lambda^*(H_1(Y)/tors)$. This is a special case of the quantum cap product in Floer (co)homology, as sketched, say, by Seidel [Sei08, Section 8l] or Perutz [Per08, Section 3.9]. Rather than describe the general case, we will focus on the action by a single element of $H_1(Y)$, where tracking perturbations is less cumbersome; this is sufficient for our applications.

So, fix an oriented multicurve $\zeta \subset \Sigma$ representing an element of $H_1(Y)$, such that $\zeta \pitchfork \boldsymbol{\alpha}$ and $\zeta \cap \boldsymbol{\alpha} \cap \boldsymbol{\beta} = \varnothing$. Let $A_1 = \zeta \cap \boldsymbol{\alpha}$. The orientations of ζ and Σ induce a coorientation of ζ; let $A_i \subset \boldsymbol{\alpha}$ be a small pushoff of A_1 so that each point of A_{i+1} is in the negative direction of the coorientation of ζ from the corresponding point of A_i.

There are corresponding subsets

$$C_i = \{(x_1, \dots, x_g) \in T_\alpha \mid x_k \in A_i \text{ for some } k\}$$
$$C_{i,j} = \{(x_1, \dots, x_g) \in T_\alpha \mid x_k \in A_i,\ x_\ell \in A_j \text{ for some } k \neq \ell\}.$$

The sets C_i and $C_{i,j}$ are finite unions of submanifolds of T_α, of codimension 1 and 2, respectively.

Given integers $i_1, \cdots, i_k$, consider the moduli space

$$\mathcal{M}^\phi(x, y; C_{i_1}, \dots, C_{i_k}) \tag{3.4}$$

of holomorphic Whitney disks

$$u\colon ([0,1] \times \mathbb{R}, \{1\} \times \mathbb{R}, \{0\} \times \mathbb{R}) \to (\mathrm{Sym}^g(\Sigma), T_\alpha, T_\beta)$$

together with points $(1, t_1), \dots, (1, t_k) \in \{1\} \times \mathbb{R}$ with $t_1 < \cdots < t_k$ with $u(1, t_j) \in C_{i_j}$. There is also a moduli space

$$\mathcal{M}^\phi(x, y; C_{i_1}, \dots, C_{i_{\ell-1}}, C_{i_\ell, i_{\ell+1}}, C_{i_{\ell+2}}, \dots, C_{i_k}) \tag{3.5}$$

defined similarly except with $u(1, t_\ell) \in C_{i_\ell, i_{\ell+1}}$.

Choose ζ so that for every disk u with Maslov index 1, $C_1 \pitchfork u|_{\{1\}\times\mathbb{R}}$. (This is possible since there are finitely many disks u with Maslov index 1.) Let U be a neighborhood of $A_1 = \zeta \cap \boldsymbol{\alpha}$ small enough that for all Maslov index 1 disks u and all $a \in U$,

$$\{(x_1, \dots, x_g) \in T_\alpha \mid x_k = a \text{ for some } k\} \pitchfork u|_{\{1\}\times\mathbb{R}}.$$

Choose the perturbations A_i to be entirely contained in U. Then, these perturbations have the following two properties:

(M-1) The moduli spaces in Equations (3.4) and (3.5) are transversely cut out.
(M-2) The moduli spaces $\mathcal{M}^\phi(x, y; C_1, \dots, C_k)$ and $\mathcal{M}^\phi(x, y; C_{i+1}, \dots, C_{i+k})$ are identified for all i.

Now, define the operation

$$\mu_{1+n}\colon \widehat{CF}(\mathcal{H}) \otimes \mathbb{F}_2[X]/(X^2)^{\otimes n} \to \widehat{CF}(\mathcal{H})$$

by

$$\mu_{1+n}(x, X, \dots, X) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1,\ n_z(\phi)=0}} \big(\#\mathcal{M}^\phi(x, y; C_1, \dots, C_n)\big) y. \tag{3.6}$$

Define the operation μ_1 to be the differential on $\widehat{CF}(\mathcal{H})$. Observe that the operation μ_2 is the restriction of the $H_1(Y)/tors$ action.

Lemma 3.1. *The operations μ_{1+n} satisfy the A_∞ relations, so $\widehat{CF}(\mathcal{H})$ inherits the structure of an A_∞-module.*

Proof. Consider the boundary of the moduli space

$$\bigcup_{\substack{\phi\in\pi_2(x,y)\\ \mu(\phi)=2,\ n_z(\phi)=0}} \mathcal{M}^\phi(x,y;C_1,\dots,C_n).$$

This moduli space has two kinds of boundary points: points in

$$\bigcup_{w\in T_\alpha\cap T_\beta}\ \bigcup_{\substack{\phi_1\in\pi_2(x,w),\phi_2\in\pi_2(w,y)\\ \mu(\phi_i)=1,\ n_z(\phi_i)=0}} \mathcal{M}^{\phi_1}(x,w;C_1,\dots,C_k)\times\mathcal{M}^{\phi_2}(w,y;C_{k+1},\dots,C_n)$$

and points in

$$\bigcup_{\substack{\phi\in\pi_2(x,y)\\ \mu(\phi)=2,\ n_z(\phi)=0}} \mathcal{M}^\phi(x,y;C_1,\dots,C_{m,m+1},\dots,C_n).$$

By Condition (M-2), points of the first kind correspond to the term

$$\mu_{n-k+2}(\mu_{k+1}(x,X,\dots,X),X,\dots,X)$$

in the A_∞ relation. Points in the second kind of terms come in pairs: An element $u\in\mathcal{M}^\phi(x,y;C_1,\dots,C_{m,m+1},\dots,C_n)$ with $u(1,t_m)=v\in C_{m,m+1}$ with $v_k\in A_m\cap\alpha_k$ and $v_\ell\in A_{m+1}\cap\alpha_\ell$ is paired with a nearby $u'\in\mathcal{M}^\phi(x,y;C_1,\dots,C_{m,m+1},\dots,C_n)$ with $u'(1,t_m)=v'\in C_{m,m+1}$ with $v'_k\in A_{m+1}\cap\alpha_k$ and $v'_\ell\in A_m\cap\alpha_\ell$, using the condition (M-2) on these types of moduli spaces. This proves the result. □

We will show next that the counts of the moduli spaces $\mathcal{M}^\phi(x,y;C_{i_1},\dots,C_{i_k})$ are completely determined by the moduli spaces $\mathcal{M}^\phi(x,y)$ and the homotopy classes ϕ. (A key point is that the sets $C_i\subset T_\alpha$ have codimension 1.) As a first step, given a curve $u\in\mathcal{M}^\phi(x,y)$, let $N_k(u)$ be the number of tuples $t_1<t_2<\cdots<t_k$ so that $u(1,t_i)\in C_i$ or, equivalently, so that $u(1,t_i)\cap A_i\neq\varnothing$. Then

$$\mu_{1+n}(x,X,\dots,X)=\sum_{y\in T_\alpha\cap T_\beta}\ \sum_{\substack{\phi\in\pi_2(x,y)\\ \mu(\phi)=1,\ n_z(\phi)=0}}\ \sum_{u\in\mathcal{M}^\phi(x,y)} N_n(u)y.$$

Recall that given integers m,n with $n\geq 0$ there is an integer $\binom{m}{n}=m(m-1)\cdots(m-n+1)/n!\in\mathbb{Z}$, which reduces to an element $\binom{m}{n}\in\mathbb{F}_2$.

Lemma 3.2. *The A_∞ operation μ_{1+n} from Formula (3.6) is given by*

$$\mu_{1+n}(x,X,\dots,X)=\sum_{y\in T_\alpha\cap T_\beta}\ \sum_{\substack{\phi\in\pi_2(x,y)\\ \mu(\phi)=1,\ n_z(\phi)=0}} \binom{\zeta\cdot\partial_\alpha D(\phi)}{n}\big(\#\mathcal{M}^\phi(x,y)\big)y.$$

where $\zeta\cdot\partial_\alpha D(\phi)$ denotes the algebraic intersection number of ζ with the part of $\partial D(\phi)$ lying in $\boldsymbol{\alpha}$.

Proof. Consider a holomorphic curve $u\in\mathcal{M}^\phi(x,y)$ so that $(u|_{\{1\}\times\mathbb{R}})^{-1}(C_1)$ consists of $a+b$ points, a of which are positive and b of which are negative. (In other words, the boundary of u, viewed as a smooth 1-chain in Σ, intersects ζ a times positively and b times negatively.)

We claim that $N_n(u) \equiv \binom{a-b}{n} \pmod 2$. In particular, this implies that $N_n(u)$ depends only on the intersection number $a-b$ of $\partial_\alpha D(\phi)$ and ζ:

$$N_n(u) \equiv \binom{a-b}{n} = \binom{\zeta \cdot \partial_\alpha D(\phi)}{n} \pmod 2.$$

This, then, will immediately imply the result.

To see that $N_n(u) \equiv \binom{a-b}{n} \pmod 2$, suppose that

$$(u|_{\{1\}\times\mathbb{R}})^{-1}(C_1) = \{(1,t_1),(1,t_2),\dots,(1,t_\ell)\}$$

where $t_1 < \cdots < t_\ell$. (In the notation of the previous paragraph, $\ell = a+b$.) Let $s_i \in \{\pm 1\}$ be the sign of $(1,t_i) \in (u|_{\{1\}\times\mathbb{R}})^{-1}(C_1)$, with respect to the coorientation of ζ and the orientation of $\{1\}\times\mathbb{R}$. The inverse function theorem implies that there are small, positive real numbers $\epsilon_{2,1},\dots,\epsilon_{n,\ell}$ so that

$$(u|_{\{1\}\times\mathbb{R}})^{-1}(C_2) = \big\{(1,t_1 - s_1\epsilon_{2,1}),(1,t_2 - s_2\epsilon_{2,2}),\dots,(1,t_\ell - s_\ell\epsilon_{2,\ell})\big\}$$
$$(u|_{\{1\}\times\mathbb{R}})^{-1}(C_3) = \big\{\big(1,t_1 - s_1(\epsilon_{2,1}+\epsilon_{3,1})\big),\big(1,t_2 - s_2(\epsilon_{2,2}+\epsilon_{3,2})\big),\dots,\big(1,t_\ell - s_\ell(\epsilon_{2,\ell}+\epsilon_{3,\ell})\big)\big\}$$
$$(u|_{\{1\}\times\mathbb{R}})^{-1}(C_4) = \big\{\big(1,t_1 - s_1(\epsilon_{2,1}+\epsilon_{3,1}+\epsilon_{4,1})\big),\dots,\big(1,t_\ell - s_\ell(\epsilon_{2,\ell}+\epsilon_{3,\ell}+\epsilon_{4,\ell})\big)\big\}$$

and so on. In particular, suppose $j < k$. The preimage $(1,t_i)$ of C_1 gives preimages of C_j and C_k that occur in order if $s_i < 0$ and out of order if $s_i > 0$. It follows that $N_n(u)$ is the number $N_n(a,b)$ of ways of choosing n points among the $a+b$ intersection points, possibly with repetitions, subject to the restriction that positive intersection points cannot be repeated

It remains to prove that $N_n(a,b) \equiv \binom{a-b}{n} \pmod 2$. The number $N_n(a,b)$ is the coefficient of s^n in $(1+s)^a(1+s+s^2+\cdots)^b$: the a $(1+s)$ factors represent the a positive intersections, which can be chosen 0 or 1 times, and the the b $(1+s+s^2+\cdots)$ factors represent the b negative intersections, which can be chosen any number of times. Since $1+s+s^2+\cdots \equiv 1-s+s^2-\cdots \equiv (1+s)^{-1} \pmod 2$, this equals $(1+s)^a(1+s)^{-b} = (1+s)^{a-b} = \sum_{n\in\mathbb{N}}\binom{a-b}{n}s^n$, and so $N_n(a,b) \equiv \binom{a-b}{n} \pmod 2$, as desired. □

Theorem 3.3. *Up to quasi-isomorphism, the A_∞-modules $\widehat{CF}(\mathcal{H})$ and $\widehat{HF}(\mathcal{H})$ over the ring $\mathbb{F}_2[X]/(X^2)$ are independent of the multi-curve ζ representing $[\zeta] \in H_1(Y)/tors$, the perturbations, the Heegaard diagram, and the almost complex structure in their construction.*

Proof. This is a simple adaptation of the usual invariance proof for Heegaard Floer homology [OSz04c], and is left to the reader. The result also follows from invariance of $\underline{\widehat{HF}}(Y;\mathbb{Z}[t^{-1},t]_\zeta)$ and Theorem 3.12 below (whose proof does not depend on this theorem). □

Remark 3.4. We have suppressed Spin^c-structures from the discussion above. All of the complexes decompose as direct sums over the Spin^c-structures on Y, and the A_∞ action respects this decomposition.

3.2. **Algebra: twisted coefficients and Koszul duality.** Michael Hutchings pointed out to the first author around 2004 that one can recover the $H_1/tors$-action on $\widehat{HF}(Y)$ from $\underline{\widehat{CF}}(Y)$. As he may also have explained, this extends to the A_∞-module structure. In this section, we give two formulations of this construction.

To motivate the first formulation, consider the relations between Equations (3.1), (3.2), and (3.3): the operation ∂ is obtained from $\underline{\partial}$ by setting $t = 1$, while the operation $\zeta\cdot$ is obtained from $\underline{\partial}$ by differentiating once with respect to t and then setting $t = 1$. (Compare the operators Φ_w and Ψ_z [Sar15, Zem17].) Of course, the second derivative vanishes in characteristic 2, but the right generalization of this relation was introduced by Hasse and Schmidt [SH37]:

Definition 3.5. Let k be a commutative ring with unit and let $k[t^{-1}, t]$ be the ring of Laurent polynomials over k. Given $m, n \in \mathbb{Z}$, $n \geq 0$, the element $\binom{m}{n} \in \mathbb{Z}$ induces an element $\binom{m}{n} = \binom{m}{n}1 \in k$. The n^{th} *Hasse derivative* $D^n \colon k[t^{-1}, t] \to k[t^{-1}, t]$ is the k-linear map which satisfies

$$D^n(t^m) = \binom{m}{n} t^{m-n}.$$

Over a field of characteristic 0,

$$D^n = \frac{1}{n!}\frac{d^n}{dt^n}.$$

Proposition 3.6. *The Hasse derivatives satisfy the Leibniz rule*

$$D^n\big(fg\big) = \sum_{i=0}^{n} \big(D^i(f)\big)\big(D^{n-i}(g)\big).$$

Further, for any Laurent polynomial $p(t)$ and any $a \neq 0$, if $(D^i p(t))|_{t=a} = 0$ for all i then $p(t) = 0$.

Proof. For a proof of the first statement, see, for example, Conrad [Con00, Section 4]. For the second, suppose $(D^i p(t))|_{t=a} = 0$ for all i. By the first statement, we also have $(D^i(t^N p(t)))|_{t=a} = 0$ for all i, so we may assume that $p(t) \in k[t]$. If the highest degree term in $p(t)$ is $b_n t^n$, $b_n \neq 0$, then $D^n(p(t))|_{t=a} = b_n$. □

See Jeong [Jeo11] for a recent, more thorough discussion of what is known about Hasse derivatives, and further references.

Corollary 3.7. *Let A and B be $m \times n$ and $n \times p$ matrices over $k[t^{-1}, t]$, respectively. Define $D^j(A)$ to be the result of taking the j^{th} Hasse derivative of each entry of A. Then,*

$$D^n(AB) = \sum_{i=0}^{n} \big(D^i(A)\big)\big(D^{n-i}(B)\big).$$

Proof. This is immediate from Proposition 3.6 and the formula for matrix multiplication. □

Definition 3.8. Let (C_*, ∂) be a (graded or ungraded) freely generated chain complex over $\mathbb{F}_2[t^{-1}, t]$, with a choice of distinguished basis. View $\mathbb{F}_2$ as an $\mathbb{F}_2[t^{-1}, t]$-algebra via the homomorphism sending $t \mapsto 1$ and let $C_*^{t=1} = C_* \otimes_{\mathbb{F}_2[t^{-1}, t]} \mathbb{F}_2$. Define an A_∞-module structure on $C_*^{t=1}$ over $\mathbb{F}_2[X]/(X^2)$,

$$\mu_{1+n} \colon C_*^{t=1} \otimes_{\mathbb{F}_2} (\mathbb{F}_2[X]/(X^2))^{\otimes n} \to C_*^{t=1},$$

by declaring that:

- $\mu_1(c)$ is the differential on C_*, with t evaluated at 1, and
- $\mu_{1+n}(c, X, \dots, X) = \big(D^n \partial(\overline{c})\big)|_{t=1}$. Here, $\overline{c}$ is the image of c under the inclusion $C_*^{t=1} \hookrightarrow C_*$ induced by the inclusion $\mathbb{F}_2 \hookrightarrow \mathbb{F}_2[t^{-1}, t]$ as the constant polynomials.

We will say that $(C_*^{t=1}, \{\mu_{1+n}\})$ is the *A_∞-module induced by C_*.*

Note that the Hasse derivatives of $\partial(\overline{c})$ here depend on the choice of basis for C_* over $\mathbb{F}_2[t^{-1},t]$, used to represent $\partial(\overline{c})$ as a vector or ∂ as a matrix. The Leibniz rule (Proposition 3.6) also implies a Leibniz rule for matrix multiplication. In particular, this Hasse derivative is respected by changing basis by a matrix over $\mathbb{F}_2$ (i.e., consisting of constant polynomials), though we will not use this fact directly.

Lemma 3.9. *Let (C_*,∂) be a freely generated chain complex over $\mathbb{F}_2[t^{-1},t]$, with a distinguished basis. Then the A_∞-module induced by C_* satisfies the A_∞ relations. Further, if $f\colon C_*\to E_*$ is a quasi-isomorphism of freely generated chain complexes over $\mathbb{F}_2[t^{-1},t]$ then there is an induced quasi-isomorphism of A_∞-modules $F\colon C_*^{t=1}\to E_*^{t=1}$.*

Proof. This follows from Corollary 3.7. For the first statement, we need to check that

$$\sum_{i+j=n}\mu_{1+i}(\mu_{1+j}(c,X,\dots,X),X,\dots,X)=0 \tag{3.7}$$

for all n and all c. Consider the n^{th} Hasse derivative of the matrix equation $\partial^2=0$. By Corollary 3.7, this gives

$$\sum_{i+j=n}D^i(\partial)\circ D^j(\partial)=0.$$

Setting $t=1$ gives Equation (3.7).

For the second statement, define

$$F_{1+n}\colon C_*^{t=1}\otimes_{\mathbb{F}_2}(\mathbb{F}_2[X]/(X^2))^{\otimes n}\to E_*^{t=1}$$

by

$$F_{1+n}(c,X,\dots,X)=\big(D^nf(\overline{c})\big)|_{t=1}.$$

To see that F is an A_∞ homomorphism, we need to check that

$$\sum_{i+j=n}F_{1+i}(\mu_{1+j}(c,X,\dots,X),X,\dots,X)+\mu_{1+i}(F_{1+j}(c,X,\dots,X),X,\dots,X)=0.$$

This follows from the equation $f\circ\partial+\partial\circ f=0$ by taking the n^{th} Hasse derivative and using Corollary 3.7. Now, it follows from the universal coefficient theorem (see Lemma 2.3 above for the ungraded case) and the 5-lemma that F is a quasi-isomorphism: the map F_1 is just the map $f\otimes\mathrm{Id}\colon C_*\otimes_{\mathbb{F}_2[t^{-1},t]}\mathbb{F}_2\to E_*\otimes_{\mathbb{F}_2[t^{-1},t]}\mathbb{F}_2$ induced by f, and we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H(C_*)\otimes_{\mathbb{F}_2[t^{-1},t]}\mathbb{F}_2 & \longrightarrow & H(C_*^{t=1}) & \longrightarrow & \mathrm{Tor}^1_{\mathbb{F}_2[t^{-1},t]}(H(C_*),\mathbb{F}_2) & \longrightarrow & 0\\
 & & \cong\big\downarrow & & \big\downarrow (F_1)_* & & \cong\big\downarrow & & \\
0 & \longrightarrow & H(E_*)\otimes_{\mathbb{F}_2[t^{-1},t]}\mathbb{F}_2 & \longrightarrow & H(E_*^{t=1}) & \longrightarrow & \mathrm{Tor}^1_{\mathbb{F}_2[t^{-1},t]}(H(E_*),\mathbb{F}_2) & \longrightarrow & 0.
\end{array}$$

This proves the result. ☐

A priori, the isomorphism type of the A_∞-module $C_*^{t=1}$ depends on the basis for C_* we are working with. Lemma 3.9 implies that this dependence is superficial:

Corollary 3.10. *Up to quasi-isomorphism, the A_∞-module $C_*^{t=1}$ is independent of the choice of basis for C_*. That is, if C_* is isomorphic to E_* then $C_*^{t=1}$ is quasi-isomorphic to $E_*^{t=1}$.*

(In fact, inspecting the proof a little more shows that the corollary holds up to isomorphism of A_∞-modules, not just quasi-isomorphism.)

Let $R = \mathbb{F}_2[t^{-1}, t]$. An element $\zeta \in H_1(Y)/tors$ induces a homomorphism $\mathbb{F}_2[H_2(Y)] \to R$, making R into an $\mathbb{F}_2[H_2(Y)]$-algebra. When thinking of R as an $\mathbb{F}_2[H_2(Y)]$-algebra, we will denote it R_ζ. When we only want to think of R as a ring, we will drop the subscript ζ.

Proposition 3.11. *Let Y be a closed 3-manifold and $\zeta \in H_1(Y)/tors$. Let $\underline{\widehat{CF}}(Y; R_\zeta)$ be the twisted Floer complex of Y with respect to ζ and $\widehat{CF}(Y)$ the untwisted Floer complex, with A_∞-module structure over $\mathbb{F}_2[X]/(X^2)$ induced by ζ. Then, there is a quasi-isomorphism of A_∞-modules*

$$\widehat{CF}(Y) \simeq \underline{\widehat{CF}}(Y; \mathbb{F}_2[t^{-1}, t]_\zeta)^{t=1}.$$

Proof. This is immediate from Formula (3.1) and Lemma 3.2. ☐

Theorem 3.12. *Let $\underline{\widehat{HF}}(Y; R_\zeta)$ be the Heegaard Floer homology of Y with (twisted) coefficients in R_ζ. As R-modules, let*

$$\underline{\widehat{HF}}(Y; R_\zeta) \cong R^m \oplus R/(p_1(t)) \oplus \cdots \oplus R/(p_n(t))$$

where each $p_i(t) \neq 0$. Assume that $p_1(1), \dots, p_k(1) = 0$ and $p_{k+1}(1), \dots, p_n(1) \neq 0$. Then, there is an isomorphism of strictly unital A_∞-modules over $\mathbb{F}_2[X]/(X^2)$

$$\widehat{HF}(Y) \cong \mathbb{F}_2^m \oplus \mathbb{F}_2\langle z_1, \dots, z_k, w_1, \dots, w_k\rangle$$

where

- *The A_∞-module structure on $\mathbb{F}_2^m$ and on $\mathbb{F}_2\langle w_1, \dots, w_k\rangle$ is trivial, i.e., for $y \in \mathbb{F}_2^m \oplus \mathbb{F}_2\langle w_1, \dots, w_k\rangle$ and any $n \geq 0$,*
$$\mu_{1+n}(y, X, \dots, X) = 0.$$
- *We have*
$$\mu_{1+n}(z_i, X, \dots, X) = \big(D^n p_i(t)\big)|_{t=1} w_i.$$

Proof. First, observe that $\underline{\widehat{CF}}(Y; R_\zeta)$ decomposes as a direct sum of 1-step and 2-step complexes. That is, we can find a basis $b_1, \cdots, b_p, c_1, \dots, c_p, d_1, \dots, d_m$ for $\underline{\widehat{CF}}(Y; R_\zeta)$ so that $\underline{\partial}(b_i) = q_i(t)c_i$ and $\underline{\partial}(c_i) = \underline{\partial}(d_i) = 0$. That such a basis exists follows from the proof of Proposition 2.4. Further, we can arrange that $q_i(t) = p_i(t)$ for $i \leq n$ and $q_i(t)$ is a unit for $i > n$.

By Corollary 3.10, $\underline{\widehat{CF}}(Y; R_\zeta)^{t=1}$ can be computed using this basis, and by Proposition 3.11, $\underline{\widehat{CF}}(Y; R_\zeta)^{t=1} \simeq \widehat{CF}(Y)$, as A_∞-modules. So, it suffices to consider a single summand d_i or $b_i \xrightarrow{q_i(t)} c_i$ of $\underline{\widehat{CF}}(Y; R_\zeta)^{t=1}$.

It is immediate from the definitions that the summand generated by d_i gives a summand R of $\underline{\widehat{HF}}(Y; R_\zeta)$ and a summand $\mathbb{F}_2$ of $\underline{\widehat{CF}}(Y; R_\zeta)^{t=1}$.

A summand of the form $b_i \xrightarrow{q_i(t)} c_i$ gives a copy of $R/(q_i(t))$ of $\underline{\widehat{HF}}(Y; R_\zeta)$ (which is trivial if $q_i(t)$ is a unit). From Definition 3.8, a summand of the form $b_i \xrightarrow{q_i(t)} c_i$ gives a summand of $\underline{\widehat{CF}}(Y; R_\zeta)^{t=1}$ with trivial homology if $q_i(1) \neq 0$. If $q_i(1) = 0$ then the corresponding summand of $\underline{\widehat{CF}}(Y; R_\zeta)^{t=1}$ is 2-dimensional, generated by z_i and w_i, say, has trivial differential, and has

$$\mu_{1+n}(z_i, X, \dots, X) = \big(D^n p_i(t)\big)|_{t=1} w_i,$$

as claimed. □

Corollary 3.13. *With notation as in Theorem 3.12, the unrolled homology of* $\widehat{CF}(Y)$ *is isomorphic to* $\mathbb{F}_2[Y^{-1}, Y]]^m$.

Proof. Since the unrolled homology is invariant under A_∞ quasi-isomorphism, the unrolled homology of $\widehat{CF}(Y)$ is isomorphic to the unrolled homology of $\widehat{HF}(Y)$. Clearly, the $\mathbb{F}_2^m \subset \widehat{HF}(Y)$ survives to give a copy of $\mathbb{F}_2[Y^{-1}, Y]]^m$ in the unrolled homology. It remains to see that the other summands of $\widehat{HF}(Y)$ do not contribute to the unrolled homology.

Consider the summand of $\widehat{HF}(Y)$ generated by z_i and w_i. By Proposition 3.6, since $p_i(t) \neq 0$, there is some integer $k \geq 1$ so that $\big(D^k p_i(t)\big)|_{t=1} \neq 0$. Let k be the first such integer. Consider the spectral sequence computing the unrolled homology of $\widehat{HF}(Y)$, associated to the horizontal filtration. Then, on this summand, the first nontrivial differential in this spectral sequence is $d_k(z_i) = \alpha Y^k w_i$, where $\alpha = \big(D^k p_i(t)\big)|_{t=1}$. The homology of this summand with respect to this differential vanishes. □

Corollary 3.14. *If* $\underline{\widehat{HF}}(Y; R_\zeta)$ *has an* $\mathbb{F}_2[t^{-1}, t]$*-summand then the unrolled homology of* $\widehat{CF}(Y)$ *with respect to the action by* ζ *is nontrivial.*

Corollary 3.15. *The unrolled homology of* $\widehat{CF}(Y)$ *is isomorphic to the completed twisted coefficient homology* $\underline{\widehat{HF}}(Y; \mathbb{F}_2[t^{-1}, t]])$.

Proof. This is immediate from Corollary 3.13, which computes the unrolled homology in terms of $\underline{\widehat{HF}}(Y; R_\zeta)$, and the universal coefficient theorem, which says that

$$\underline{\widehat{HF}}(Y; \mathbb{F}_2[t^{-1}, t]]) \cong \underline{\widehat{HF}}(Y; R_\zeta) \otimes_R \mathbb{F}_2[t^{-1}, t]].$$

(Recall that $\mathbb{F}_2[t^{-1}, t]]$ is flat over $\mathbb{F}_2[t^{-1}, t]$.) □

While we will not need it for our application, we conclude this section by noting a more homological-algebraic interpretation of Proposition 3.11. View $\mathbb{F}_2$ as an $R = \mathbb{F}_2[t^{-1}, t]$-module in the usual way, by letting t act by 1. Then, $\mathbb{F}_2$ has a 2-step free resolution over R:

$$R \xrightarrow{1-t} R.$$

From this, it is straightforward to compute that $\mathrm{Ext}_R(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[X]/(X^2)$ (and, in fact, $\mathrm{RHom}_R(\mathbb{F}_2, \mathbb{F}_2)$ is quasi-isomorphic to $\mathbb{F}_2[X]/(X^2)$).

For any chain complex C_* over R, there is an A_∞ action of $\mathrm{Ext}_R(\mathbb{F}_2, \mathbb{F}_2)$ on $\mathrm{Tor}_R(C_*, \mathbb{F}_2)$. Explicitly, $\mathrm{Tor}_R(C_*, \mathbb{F}_2)$ is the homology of the total complex of the bicomplex

$$0 \longrightarrow C \xrightarrow{1-t} C \longrightarrow 0. \tag{3.8}$$

The element X shifts this bicomplex one unit to the right, i.e., sends the first copy of C to the second by the identity map and sends the second copy of C to 0. So, this total complex is a differential module over $\mathbb{F}_2[X]/(X^2)$, and its homology $\mathrm{Tor}_R(C_*, \mathbb{F}_2)$ inherits the structure of an A_∞-module over $\mathbb{F}_2[X]/(X^2)$.

Theorem 3.16. *For any finitely generated, free chain complex* C_* *over* R*, there is a quasi-isomorphism of* A_∞*-modules over* $\mathbb{F}_2[X]/(X^2)$

$$C_*^{t=1} \simeq \mathrm{Tor}_R(C_*, \mathbb{F}_2).$$

In particular, as A_∞-modules over $\mathbb{F}_2[X]/(X^2)$,

$$\widehat{HF}(Y) \simeq \mathrm{Tor}_{\mathbb{F}_2[t^{-1},t]}(\underline{\widehat{HF}}(Y), \mathbb{F}_2).$$

Proof. As in the proof of Theorem 3.12, it suffices to prove the result when C_* consists of a single generator d or a pair of generators b, c with differential $b \xrightarrow{p(t)} c$. In the first case, it is straightforward to see that both $C_*^{t=1}$ and $\mathrm{Tor}_R(C_*, \mathbb{F}_2)$ are isomorphic to $\mathbb{F}_2$ with trivial A_∞-module structure. In the second case, if $p(1) \neq 0$ then $C_*^{t=1}$ is acyclic and $\mathrm{Tor}_R(C_*, R) = 0$. So, it remains to verify the second case under the assumption that $p(1) = 0$.

Let E_* be the total complex of the bicomplex (3.8). We will construct an A_∞ quasi-isomorphism $f\colon C_*^{t=1} \to E_*$.

To fix notation, write $E_* = C \otimes \mathbb{F}_2[X]/(X^2)$, with differential

$$\begin{bmatrix} \partial_C & (1-t) \\ 0 & \partial_C \end{bmatrix}.$$

That is, the complex E_* is the total complex of the square

$$\begin{array}{ccc} R & \xrightarrow{1-t} & R \\ {\scriptstyle p(t)}\downarrow & & \downarrow{\scriptstyle p(t)} \\ R & \xrightarrow[1-t]{} & R. \end{array} \quad = \quad \begin{array}{ccc} R\langle b\rangle & \xrightarrow{1-t} & R\langle Xb\rangle \\ {\scriptstyle p(t)}\downarrow & & \downarrow{\scriptstyle p(t)} \\ R\langle c\rangle & \xrightarrow[1-t]{} & R\langle Xc\rangle. \end{array}$$

Define Laurent polynomials $q_n(t)$ inductively by

$$q_1(t) = p(t)/(1-t)$$
$$q_{n+1}(t) = (q_n(1) - q_n(t))/(1-t).$$

The fact that $q_1(t)$ is a Laurent polynomial follows from the restriction that $p(1) = 0$.

We claim that

$$q_n(1) = \big(D^n p(t)\big)|_{t=1}.$$

By induction, we have

$$p(t) = (t-1)q_1(1) + (t-1)^2 q_2(1) + \cdots + (t-1)^{n-1} q_{n-1}(1) + (t-1)^n q_n(t)$$

(cf. Taylor's theorem). Hence,

$$D^n p(t) = (D^n (t-1)^n) q_n(t) + (t-1)r(t) = q_n(t) + (t-1)r(t).$$

Evaluating at 1 verifies the claim.

Now, define:

$$\begin{aligned} f_1(c) &= Xc \\ f_{1+n}(c, X, \dots, X) &= 0 && n > 0 \\ f_1(b) &= Xb + q_1(t)c \\ f_{1+n}(b, X, \dots, X) &= q_{n+1}(t)c && n > 0. \end{aligned}$$

It is straightforward to see that f_1 is a quasi-isomorphism. We claim that the f_i satisfy the A_∞ homomorphism relations; this finishes the proof. We must check that for $y \in \{b, c\}$,

$$\sum_{i+j=n} \mu^E_{1+i}(f_{1+j}(y, X, \dots, X), X, \dots, X) + f_{1+i}(\mu^{C^{t=0}}_{1+j}(y, X, \dots, X), X, \dots, X) = 0.$$

Recall that $\mu^E_{1+i} = 0$ for $i > 1$. In the case $y = c$, each term in the equation vanishes. For $y = b$ and $n = 0$, the left side of the equation is

$$\partial^E(f_1(b)) + f_1(\partial^{C^{t=0}}(b)) = \partial^E(Xb + q_1(t)c) = (p(t) + (1-t)q_1(t))Xc = 0.$$

For $y = b$ and $n > 0$, the left side is equal to

$$\begin{aligned}
&\partial^E(f_{1+n}(b, X, \dots, X) + \mu_2^E(f_n(b, X, \dots, X), X) + \sum_{i+j=n} f_{1+i}((D^j p(t))|_{t=1} c, X, \dots, X) \\
&= \partial^E(q_{n+1}(t)c) + \mu_2^E(q_n(t)c, X) + (D^n p(t))|_{t=1} Xc \\
&= q_{n+1}(t)(1-t)Xc + q_n(t)Xc + q_n(1)Xc \\
&= (q_n(1) - q_n(t) + q_n(t) + q_n(1))Xc \\
&= 0,
\end{aligned}$$

as desired. □

Remark 3.17. Presumably, one can give a direct proof of Theorem 3.16, without relying on the classification of finitely generated modules over a PID, but the computations required seem involved.

Remark 3.18. For simplicity, we have worked in characteristic 2 and focused on the action of a single element $\zeta \in H_1(Y)/tors$, but we expect that the results in this section generalize to the entire A_∞-module structure over $\Lambda^* H_1(Y)/tors$ over $\mathbb{Z}$ (though some of the proofs do not).

4. The module structure on Khovanov homology and the Ozsváth-Szabó spectral sequence

4.1. Definition and invariance of the basepoint action on Khovanov homology. Fix a link diagram L and a basepoint $q \in L$ not at any of the crossings. (From here on, *basepoint* means "basepoint not at a crossing.") The Khovanov complex $\mathcal{C}_{Kh}(L)$ of L inherits the structure of a module over $\mathbb{F}_2[X]/(X^2)$ as follows. A generator of $\mathcal{C}_{Kh}(L)$ is a complete resolution of L and a decoration of each component of the resolution by 1 or X. Multiplication by X on a generator of $\mathcal{C}_{Kh}(L)$:

- is zero if the generator labels the circle containing q by X and
- changes the label on the circle containing q to X, if the generator labels the circle containing q by 1.

It is straightforward to check that multiplication by X is a chain map. The action of X preserves the homological grading and decreases the quantum grading by 2.

Given two basepoints $p, q \in L$, the actions at p and q commute, and hence make $\mathcal{C}_{Kh}(L)$ into a differential bimodule over $\mathbb{F}_2[W]/(W^2)$ and $\mathbb{F}_2[X]/(X^2)$ or, equivalently, a differential module over $\mathbb{F}_2[W, X]/(W^2, X^2)$. Note that while $\mathcal{C}_{Kh}(L)$ is free over $\mathbb{F}_2[X]/(X^2)$, it is typically not free over $\mathbb{F}_2[W, X]/(W^2, X^2)$.

Let $\Sigma^{a,b}$ denote shifting the homological grading up by a and the quantum grading up by b.

Given a basepoint p on L, the *reduced Khovanov complex* $\widetilde{\mathcal{C}}_{Kh}(L)$ is the subcomplex of $\Sigma^{0,1}\mathcal{C}_{Kh}(L)$ where the circle containing p is labeled X or, equivalently, the quotient complex of $\Sigma^{0,-1}\mathcal{C}_{Kh}(L)$ where the circle containing p is labeled 1. Given a second basepoint q on L, $\widetilde{\mathcal{C}}_{Kh}(L)$ inherits a module structure over $\mathbb{F}_2[X]/(X^2)$.

We will use the following lemma, to avoid writing the same proof twice in Theorem 4.2.

Lemma 4.1. *Let L be a link and p, q basepoints on L. Write $\mathcal{C}_{Kh}(L)_{p,q}$ for the Khovanov complex $\mathcal{C}_{Kh}(L)$ viewed as a bimodule over $\mathbb{F}_2[W]/(W^2)$ and $\mathbb{F}_2[X]/(X^2)$ via the basepoints p and q. Write $\widetilde{\mathcal{C}}_{Kh}(L)_{p,q}$ for the reduced Khovanov complex, reduced at p and viewed as a module over $\mathbb{F}_2[X]/(X^2)$ via the basepoint q. View $\mathbb{F}_2$ as a module over $\mathbb{F}_2[W]/(W^2)$ where W acts by 0. Then, there is a chain isomorphism of $\mathbb{F}_2[X]/(X^2)$-modules*

$$\widetilde{\mathcal{C}}_{Kh}(L)_{p,q} \cong \Sigma^{0,-1}\mathcal{C}_{Kh}(L)_{p,q} \otimes_{\mathbb{F}_2[W]/(W^2)} \mathbb{F}_2.$$

Further, $\mathcal{C}_{Kh}(L)$ is a free module over $\mathbb{F}_2[W]/(W^2)$, so $\widetilde{\mathcal{C}}_{Kh}(L)$ is quasi-isomorphic to the A_∞ tensor product of the A_∞-bimodule $Kh(L)$ and the $\mathbb{F}_2[W]/(W^2)$ module $\mathbb{F}_2$.

Proof. The first statement is immediate from the definitions. The second follows from the facts that the A_∞ tensor product is invariant under A_∞ homotopy equivalence and that any A_∞-(bi)module is A_∞ homotopy equivalent to its homology (Proposition 2.9). ☐

Theorem 4.2. *Let L be a link and p, p', q, q' points on L so that p, p' lie on the same component of L and q, q' lie on the same component of L.*

(1) Write $\mathcal{C}_{Kh}(L)_{p,q}$ (respectively $\mathcal{C}_{Kh}(L)_{p',q'}$) for the Khovanov complex $\mathcal{C}_{Kh}(L)$ viewed as a module over $\mathbb{F}_2[W,X]/(W^2,X^2)$ via the basepoints p, q (respectively p', q'). Then, $\mathcal{C}_{Kh}(L)_{p,q}$ is quasi-isomorphic to $\mathcal{C}_{Kh}(L)_{p',q'}$.

(2) Write $\widetilde{\mathcal{C}}_{Kh}(L)_{p,q}$ (respectively $\mathcal{C}_{Kh}(L)_{p',q'}$) for the reduced Khovanov complex $\mathcal{C}_{Kh}(L)$, reduced at p (respectively p'), and viewed as a module over $\mathbb{F}_2[X]/(X^2)$ via the basepoint q (respectively q'). Then, $\widetilde{\mathcal{C}}_{Kh}(L)_{p,q}$ is quasi-isomorphic to $\widetilde{\mathcal{C}}_{Kh}(L)_{p',q'}$.

The fact that the Khovanov homology has a well-defined structure of a bimodule appears in Hedden-Ni [HN13, Proposition 1], via a similar argument. The fact that the action of $\mathbb{Z}[X]/(X^2)$ at a single basepoint is well-defined is due to Khovanov [Kho03, Section 3], by a different argument which apparently does not generalize.

Proof. For the first half of the theorem, by Proposition 2.10, it suffices to show that moving q past a crossing C gives an A_∞ quasi-isomorphic bimodule over $\mathbb{F}_2[W]/(W^2)$ and $\mathbb{F}_2[X]/(X^2)$. An A_∞-bimodule map from M to N consists of maps

$$f_{m,1,n} \colon (\mathbb{F}_2[W]/(W^2))^{\otimes m} \otimes M \otimes (\mathbb{F}_2[X]/(X^2))^{\otimes n} \to N.$$

Our quasi-isomorphism f will have $f_{0,1,0} = \mathrm{Id}$ and $f_{m,1,n} = 0$ if $m > 0$ or $n > 1$. To define $f_{0,1,1}$, we need some more notation.

Write a generator of the Khovanov complex as a pair (v, x) where $v \in \{0,1\}^c$ (where c is the number of crossings of L) and x is a labeling of the circles of the v-resolution L_v by 1 or X. That is, $x \in \{1, X\}^{\pi_0(L_v)}$. Let $|v| = \sum v$ be the height of the vertex v, and let $\leq$ denote the partial order on the cube $\{0,1\}^c$. Given v, w with $|v| - |w| = \pm 1$ and $v < w$ or $w < v$; $x \in \{1, X\}^{\pi_0(L_v)}$; and $y \in \{1, X\}^{\pi_0(L_w)}$, let $n_{x,y}$ be 1 if the following conditions are satisfied:

- For $Z \in L_v \cap L_w$, $x(Z) = y(Z)$;
- If there are two circles Z_1, Z_2 in L_v which are merged into a circle Z in L_w then $y(Z) = x(Z_1)x(Z_2)$ (where the multiplication is in $\mathbb{F}_2[X]/(X^2)$); and
- If there is one circle Z in L_v which is split into two circles Z_1, Z_2 in L_w then $y(Z_1) \otimes y(Z_2) = \Delta(x(Z))$ where $\Delta \colon \mathbb{F}_2[X]/(X^2) \to \mathbb{F}_2[X]/(X^2) \otimes \mathbb{F}_2[X]/(X^2)$ is the comultiplication

$$\Delta(1) = 1 \otimes X + X \otimes 1 \qquad\qquad \Delta(X) = X \otimes X.$$

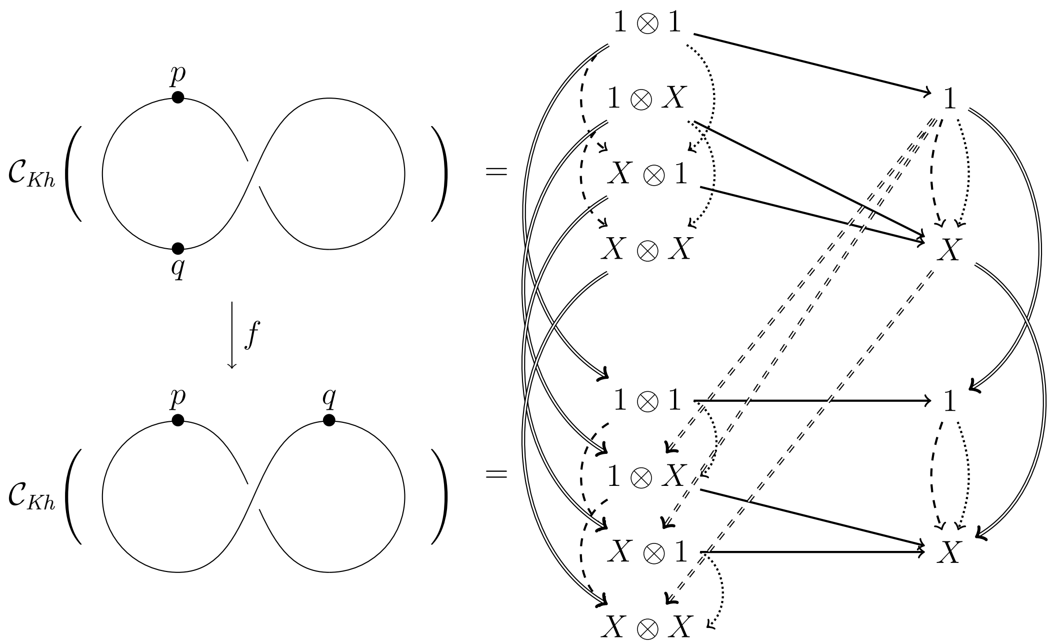


FIGURE 4.1. **Example of the map f.** The solid arrows are the differential, the dashed arrows are the action of W, the dotted arrows are the action of X, the double arrows are the map $f_{0,1,0}$, and the dashed double arrows are the map $f_{0,1,1}(\cdot, X)$. The A_∞ relations correspond to certain ways of getting from one vertex to another in two steps.

Define $n_{x,y} = 0$ otherwise. Then, the Khovanov differential is

$$\delta(x, v) = \sum_{(y,w) | w \geq v, \ |v|+1=|w|} n_{x,y} \cdot (y, w).$$

Define the map $f_{0,1,1}$ by

$$f_{0,1,1}((y, w), X) = \sum_{(x,v) | w \geq v, \ |v|+1=|w|, \ v(C) \neq w(C)} n_{y,x} \cdot (x, v).$$

Equivalently, $f_{0,1,1}$ comes from performing the differential at C backwards. That is, if $\overline{C}$ is the opposite crossing to C then $f_{0,1,1}$ is the part of the differential associated to changing $\overline{C}$. (Compare [HN13, Lemma 2.3], [BS15, Section 2.2].) See Figures 4.1 and 4.2.

The nontrivial A_∞ relations to verify are the following:

$$\mu_{0,1,0}(f_{0,1,0}((v, x))) + f_{0,1,0}(\mu_{0,1,0}((v, x))) = 0 \tag{4.1}$$

$$\mu_{1,1,0}(W, f_{0,1,0}((v, x))) + f_{0,1,0}(\mu_{1,1,0}(W, (v, x))) = 0 \tag{4.2}$$

$$\begin{aligned} \mu_{0,1,1}(f_{0,1,0}((v, x)), X) + f_{0,1,0}(\mu_{0,1,1}((v, x), X)) \\ + \mu_{0,1,0}(f_{0,1,1}((v, x), X)) + f_{0,1,1}(\mu_{0,1,0}((v, x)), X) = 0 \end{aligned} \tag{4.3}$$

$$\mu_{1,1,0}(W, f_{0,1,1}((v, x), X)) + f_{0,1,1}(\mu_{1,1,0}(W, (v, x)), X) = 0 \tag{4.4}$$

$$\mu_{0,1,1}(f_{0,1,1}((v, x), X), X) + f_{0,1,1}(\mu_{0,1,1}((v, x), X), X) = 0 \tag{4.5}$$

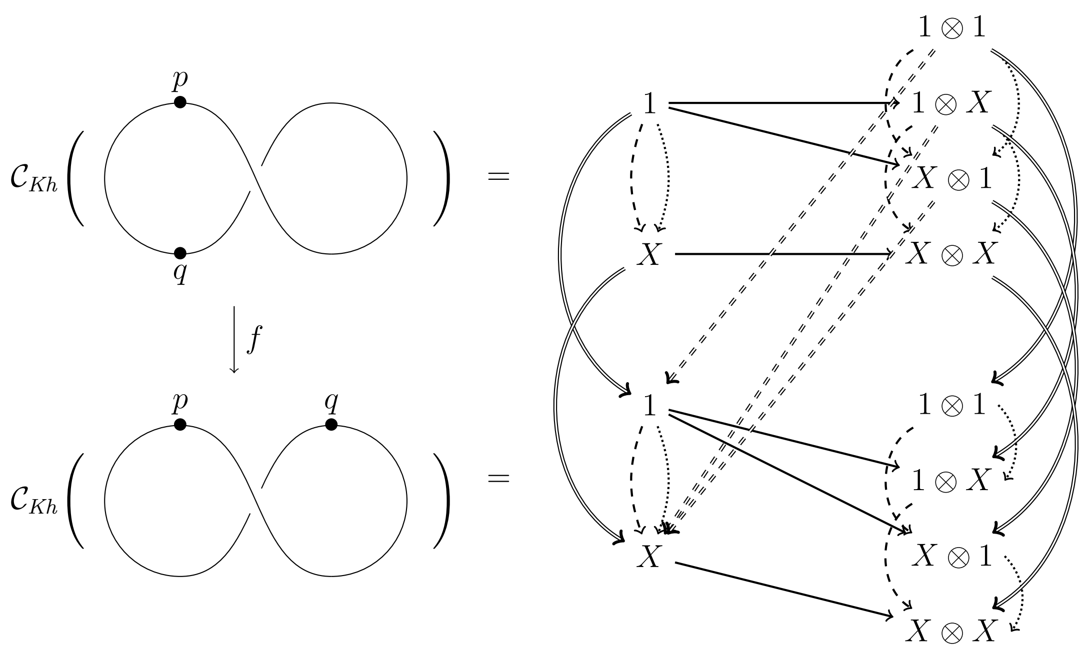


FIGURE 4.2. **A second example of the map f.** Notation is the same as in Figure 4.1.

All other A_∞ relations are automatically satisfied because all of the terms vanish. In these equations, we have dropped terms which automatically vanish, like $\mu_{0,1,0}(f_{1,1,0}(W,(v,x)))$, to keep the expressions shorter.

Since $f_{0,1,0}$ is the identity map, Equations (4.1) and (4.2) are obviously satisfied. Equation (4.3) was checked in Hedden-Ni's paper [HN13, Equation (1)]. For Equation (4.4), let L' be the result of replacing the crossing C in L by the opposite crossing C'; then, the equation follows from the fact that the differential on the Khovanov complex of L' respects the $\mathbb{F}_2[W]/(W^2)$-module structure.

For Equation (4.5), consider the coefficient of (w, y). For the coefficient to be non-zero, v must be obtained from w by changing the entry corresponding to C from 0 to 1. Suppose first that two circles Z_1, Z_2 in L_w merge into one circle Z in L_v. Note that one circle, say Z_1, must contain q and the other circle Z_2 must contain q'. Then, recording only the labels of the circles Z_1, Z_2, and Z, $\mu_{0,1,1}(f_{0,1,1}((v,x),X),X)$ is given by

$$1 \mapsto 1 \otimes X + X \otimes 1 \mapsto X \otimes X$$
$$X \mapsto X \otimes X \mapsto 0,$$

while $f_{0,1,1}(\mu_{0,1,1}((v,x),X),X)$ is given by

$$1 \mapsto X \mapsto X \otimes X$$
$$X \mapsto 0 \mapsto 0.$$

So, these two terms cancel. Next, suppose that one circle Z in L_w splits into two circles Z_1, Z_2 in L_v, where Z_1 contains q and Z_2 contains q'. Then, $\mu_{0,1,1}(f_{0,1,1}((v,x),X),X)$ is given by

$$1 \otimes 1 \mapsto 1 \mapsto X \qquad 1 \otimes X \mapsto X \mapsto 0$$
$$X \otimes 1 \mapsto X \mapsto 0 \qquad X \otimes X \mapsto 0 \mapsto 0$$

while $f_{0,1,1}(\mu_{0,1,1}((v,x),X),X)$ is given by

$$1\otimes 1\mapsto X\otimes 1\mapsto X \qquad 1\otimes X\mapsto X\otimes X\mapsto 0$$
$$X\otimes 1\mapsto 0\mapsto 0 \qquad X\otimes X\mapsto 0\mapsto 0.$$

So, again, these two terms cancel. These two cases are illustrated in Figures 4.1 and 4.2. (An alternative way to view this identity is as follows: The map $f_{0,1,1}\colon \mathcal{C}_{Kh}(L_v)\to\mathcal{C}_{Kh}(L_w)$ is induced by an elementary saddle cobordism, while the X-actions on $\mathcal{C}_{Kh}(L_v)$ and $\mathcal{C}_{Kh}(L_w)$ are induced by identity cobordisms decorated with a single 'dot'; then, either term of Equation (4.5) corresponds to an elementary saddle cobordism decorated with a single dot on the saddle component.)

This proves the first half of the theorem. The second half of the theorem follows from the first and Lemma 4.1 (and the fact that the A_∞ tensor product is invariant under A_∞ quasi-isomorphisms). ☐

Corollary 4.3. *Let L be a link and p,p',q,q' points on L so that p,p' lie on the same component of L and q,q' lie on the same component of L. Up to A_∞-isomorphism, the A_∞-modules $Kh(L)_p$ and $Kh(L)_{p'}$ (respectively $\widetilde{Kh}(L)_{p,q}$ and $\widetilde{Kh}(L)_{p',q'}$) over $\mathbb{F}_2[X]/(X^2)$ are isomorphic. In fact, the isomorphism classes of these A_∞-modules are invariants of the isotopy classes of the triple (L,p) and (L,p,q), respectively.*

Proof. The first statement is immediate from Theorem 4.2 and homological perturbation theory. For the second statement, it suffices to verify invariance under Reidemeister moves disjoint from the basepoints and moving a strand across a basepoint. Invariance under Reidemeister moves disjoint from the basepoints was proved by Khovanov [Kho00], and invariance under moving a strand across a basepoint is a special case of the first half of the corollary. ☐

4.2. **The Ozsváth-Szabó spectral sequence respects the A_∞-module structure.** Let L be a link in S^3 and $p,q\in L$. Choose an arc $\gamma\subset S^3\setminus L$ from p to q. The preimage $\zeta\subset\Sigma(L)$ of γ is a simple closed curve, representing an element of $H_1(\Sigma(L))$. The homology class represented by ζ is independent of the choice of γ since isotoping γ across L changes ζ by the preimage of a meridian of L, which bounds a disk in $\Sigma(L)$.

The homology class $[\zeta]$ makes $\widehat{CF}(\Sigma(L);\mathbb{F}_2)$ into a module over $\mathbb{F}_2[X]/(X^2)$, as described in Section 3. (Of course, if $[\zeta]\in H_1(\Sigma(L))$ is torsion—for example, if p and q lie on the same component of L or if $\Sigma(L)$ is a rational homology sphere—then this module structure is trivial.)

In the following proposition, by a *filtration* we mean a descending filtration, i.e., a sequence of submodules $C=F^0\supset F^1\supset F^2\supset\cdots$.

Proposition 4.4. *Let L be a link in S^3 and p,q points on L. There is an ungraded, filtered A_∞-module $(C,\{\mu_n\})$ over $\mathbb{F}_2[X]/(X^2)$ with the following properties:*

(1) Forgetting the filtration, C is quasi-isomorphic to $\widehat{CF}(\Sigma(L))$ as an A_∞-module over $\mathbb{F}_2[X]/(X^2)$.
(2) The differential μ_1 on C strictly increases the filtration.
(3) There is an isomorphism of $\mathbb{F}_2$-modules $g\colon C\xrightarrow{\cong}\widetilde{\mathcal{C}}_{Kh}(m(L))$, taking the filtration on C to the homological grading on $\widetilde{\mathcal{C}}_{Kh}(m(L))$.
(4) To first order, μ_1 agrees with the Khovanov differential. That is,

$$\mu_1 - g^{-1}\circ\partial_{\mathcal{C}_{Kh}}\circ g$$

increases the filtration by at least 2.

(5) *To zeroth order, the operation* $\mu_2(\cdot, X)$ *on* C *agrees with the action of* X *on the Khovanov complex. That is,*

$$y \mapsto \mu_2(y, X) - g^{-1}(g(y)\cdot X)$$

increases the filtration by at least 1. *(Note that the Khovanov multiplication actually respects the homological grading.)*

Proof. This is essentially Hedden-Ni's refinement [HN13, Theorem 4.5] of Ozsváth-Szabó's construction of the spectral sequence for a branched double cover [OSz05, Theorem 6.3]. The only additional assertions are that there is an A_∞-module structure on C and the quasi-isomorphism between C and $\widehat{CF}(\Sigma(L))$ extends to an A_∞ homomorphism. So, we will only explain the additional steps required to adapt Hedden-Ni's proof, and will adopt much of their notation without re-introducing it.

Throughout this proof, Floer complexes are with $\mathbb{F}_2$-coefficients (which we suppress from the notation).

Let c be the number of crossings of L and, given $I \in \{0, 1, \infty\}^c$, let

$$\mathcal{H}^I = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^I, z)$$

be the Heegaard diagram considered by Ozsváth-Szabó and Hedden-Ni. Fix

- a curve ζ in Σ representing the homology class of a lift of an arc from p to q,
- small pushoffs $A_2, A_3, \dots$ of $A_1 = \zeta \cap \boldsymbol{\alpha}$ as in Section 3.1, and
- a collection of sufficiently generic almost complex structures.

Given a sequence $I_0 < I_1 < \cdots < I_m$ of immediate successors in $\{0, 1, \infty\}^c$ and an integer $n \geq 0$, define a map

$$\mu_{1+n}^{I_0<\cdots<I_m}(\cdot, \overbrace{X, \cdots, X}^{n})\colon \widehat{CF}(\mathcal{H}^{I_0}) \to \widehat{CF}(\mathcal{H}^{I_m})$$

by counting rigid holomorphic $(m+2)$-gons in the Heegaard multi-diagram

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^{I_0}, \dots, \boldsymbol{\beta}^{I_m}, z)$$

with point constraints along the α-boundary coming from $A_1, \dots, A_n$ and corners at some generator $x \in T_\alpha \cap T_{\beta^{I_0}}$, some generator $y \in T_\alpha \cap T_{\beta^{I_m}}$, and the top generators $\Theta_1, \dots, \Theta_m$ for $(\boldsymbol{\beta}^{I_0}, \boldsymbol{\beta}^{I_1}), \dots, (\boldsymbol{\beta}^{I_{m-1}}, \boldsymbol{\beta}^{I_m})$.

Let

$$\mu_{1+n}\colon \bigoplus_{I\in\{0,1,\infty\}^c} \widehat{CF}(\mathcal{H}^I) \otimes \big(\mathbb{F}_2[X]/(X^2)\big)^n \to \bigoplus_{I\in\{0,1,\infty\}^c} \widehat{CF}(\mathcal{H}^I)$$

be

$$\mu_{1+n}(\cdot, X, \dots, X) = \sum_{I_0<\cdots<I_m} \mu_{1+n}^{I_0<\cdots<I_m}(\cdot, \overbrace{X, \cdots, X}^{n}).$$

Note that in the special case $n = 0$, μ_1 is the map D introduced by Ozsváth-Szabó, and in the case $n = 1$, $\mu_2(\cdot, X)$ is the map a^ζ introduced by Hedden-Ni.

We claim that the μ_{1+n} make $M = \bigoplus_{I\in\{0,1,\infty\}^c} \widehat{CF}(\mathcal{H}^I)$ into an A_∞-module. This follows by considering the ends of the 1-dimensional moduli spaces of polygons with point constraints. With notation as in the rigid case discussed above, the ends of the 1-dimensional moduli spaces are:

- Ends where the polygon degenerates as a polygon on $(\boldsymbol{\alpha}, \boldsymbol{\beta}^0, \dots, \boldsymbol{\beta}^k)$ with p point constraints and a polygon on $(\boldsymbol{\alpha}, \boldsymbol{\beta}^k, \dots, \boldsymbol{\beta}^m)$ with $n-p$ point constraints. These degenerations correspond to the terms
$$\mu_{1+n-p}(\mu_{1+p}(\cdot, X, \dots, X), X, \dots, X)$$
in the A_∞ relation. (This uses the fact that the pushoffs A_i are chosen consistently, as in Section 3.1, so that the count of curves constrained by $A_1, \dots, A_{n-p}$ and the count of curves constrained by $A_{p+1}, \dots, A_n$ are the same.)
- Ends where the polygon degenerates as a polygon on $(\boldsymbol{\alpha}, \boldsymbol{\beta}^0, \dots, \boldsymbol{\beta}^i, \boldsymbol{\beta}^j, \dots, \boldsymbol{\beta}^k)$ with n point constraints and a polygon on $(\boldsymbol{\beta}^i, \boldsymbol{\beta}^{i+1}, \dots, \boldsymbol{\beta}^k)$. These contributions vanish because the count of rigid polygons on $(\boldsymbol{\beta}^i, \boldsymbol{\beta}^{i+1}, \dots, \boldsymbol{\beta}^k)$ with corners at the Θ^i is zero [OSz05, Lemma 4.5].
- Ends where a pair of constrained points collide. These cancel in pairs, as in the proof of Lemma 3.1

(Compare [OSz05, Section 4.2], [HN13, Theorem 3.4 and Lemma 4.4].)

By construction, given a crossing C_0 of L, M is the mapping cone of an A_∞-module homomorphism
$$\bigoplus_{\{I \in \{0,1,\infty\}^c \mid I(C_0) \in \{0,1\}\}} \widehat{CF}(\mathcal{H}^I) \to \bigoplus_{\{I \in \{0,1,\infty\}^c \mid I(C_0)=\infty\}} \widehat{CF}(\mathcal{H}^I),$$
and the surgery exact triangle for $\widehat{HF}$ implies that this homomorphism is an isomorphism. So, it follows from the same inductive argument as in Ozsváth-Szabó's case [OSz05, Proof of Theorem 4.1] that
$$\widetilde{C} := \bigoplus_{\{I \in \{0,1\}^c\}} \widehat{CF}(\mathcal{H}^I)$$
is quasi-isomorphic, as an A_∞-module, to
$$\widehat{CF}(\mathcal{H}^{\infty,\infty,\dots,\infty}) = \widehat{CF}(\Sigma(L)).$$

Let
$$C := \bigoplus_{\{I \in \{0,1\}^c\}} \widehat{HF}(\mathcal{H}^I).$$
The complex C is filtered by $|I| = \sum I$, the *cube filtration*. That is,
$$F^i = \bigoplus_{\{I \in \{0,1\}^c \mid |I| \geq i\}} \widehat{HF}(\mathcal{H}^I).$$

Choose a homotopy equivalence, over $\mathbb{F}_2$, between each $\widehat{CF}(\mathcal{H}^I)$ and $\widehat{HF}(\mathcal{H}^I)$, so that the composition $\widehat{HF}(\mathcal{H}^I) \to \widehat{CF}(\mathcal{H}^I) \to \widehat{HF}(\mathcal{H}^I)$ is the identity map. These homotopy equivalences induce maps $f\colon C \to \widetilde{C}$ and $g\colon \widetilde{C} \to C$ with $g \circ f = \mathrm{Id}_C$. Define the operation μ_1 on C by
$$\mu_1(x) = g(\mu_1(f(x))).$$
By homological perturbation theory (Proposition 2.9), C inherits the structure of an A_∞-module over $\mathbb{F}_2[X]/(X^2)$. (Equivalently, C is obtained from $\widetilde{C}$ by canceling all differentials which do not change the cube filtration.) Further:

- The operations μ_n on C all respect the cube filtration.

- The differential μ_1 on C increases the cube filtration by at least 1. Further, by construction, μ_1 agrees with the differential on Ozsváth-Szabó's cube [OSz05, Proposition 6.2], and hence the first-order part of μ_1 agrees with the Khovanov differential.
- The zeroth-order part of μ_2 is induced by the $H_1/tors$-action on the groups $\widehat{HF}(\mathcal{H}^I) = \widehat{HF}(\#^{k(I)}(S^2 \times S^1))$. Hence, by Hedden-Ni [HN13, Theorem 4.5] (or inspection), the zeroth-order part of μ_2 agrees with the X-action on Khovanov homology.

This completes the proof. □

5. Proof of the detection theorems

We will use the following:

Proposition 5.1. *[HN10, Proposition 5.1] Let L be a link in S^3. If $L = L_1 \# L_2$ then $\Sigma(L) = \Sigma(L_1)\#\Sigma(L_2)$. If $L = L_1 \amalg L_2$ then $\Sigma(L) = \Sigma(L_1)\#\Sigma(L_2)\#(S^2 \times S^1)$. If L is a non-split prime link then $\Sigma(L)$ is irreducible. If L is a non-split link then $\Sigma(L)$ has no homologically essential 2-spheres (i.e., no $S^2 \times S^1$ summands).*

Corollary 5.2. *Let L be a link in S^3 and p, q points in L. Let γ be a path in S^3 from p to q with the interior of γ disjoint from L, and let $\zeta \subset \Sigma(L)$ be the preimage of γ. If there is an embedded sphere $\widetilde{S} \subset \Sigma(L)$ so that $\gamma \cdot \widetilde{S}$ is nonzero then there is an embedded sphere $S \subset S^3 \setminus L$ separating p and q.*

Proof. This follows from the same argument used to prove Proposition 5.1, but we can also deduce it from Proposition 5.1.

We will prove the contrapositive. Assume there is no sphere separating p and q. Write $L = L_1 \amalg \cdots \amalg L_k$ as a (split) disjoint union of links, so that each L_i is non-split. Let $B_1, \dots, B_k$ be disjoint balls around $L_1, \dots, L_k$.

Reordering the L_i, suppose that $p, q \in L_1$. As shown in Section 4.2, the homology class of ζ is independent of the choice of γ. So, we can assume that γ is contained in B_1. By Proposition 5.1 we have

$$\Sigma(L) = \Sigma(L_1)\# \cdots \#\Sigma(L_k)\#(S^2 \times S^1)^{\#(k-1)},$$

and each $\Sigma(L_i)$ has no homologically essential 2-spheres. The curve γ lies in $\Sigma(L_1)$, so is disjoint from all of the homologically essential 2-spheres. This proves the result. □

5.1. Khovanov homology of split links.

Lemma 5.3. *Let L be a link and p, q points on L. Write $\widetilde{\mathcal{C}}_{Kh}(L)_{p,q}$ for the reduced Khovanov complex of L, reduced at p and viewed as a module over $\mathbb{F}_2[X]/(X^2)$ via the basepoint q. If there is a 2-sphere in $S^3 \setminus L$ separating p and q then $\widetilde{Kh}(L; \mathbb{F}_2)$ is a free module.*

Proof. By Corollary 4.3, we may assume that we are computing $\widetilde{Kh}(L; \mathbb{F}_2)$ from a split diagram, i.e., the disjoint union of a link diagram L_1 containing p and a link diagram L_2 containing q. Then, $\widetilde{Kh}(L; \mathbb{F}_2) \cong \widetilde{Kh}(L_1; \mathbb{F}_2) \otimes Kh(L_2; \mathbb{F}_2)$ as $\mathbb{F}_2[X]/(X^2)$-modules. By a result of Shumakovitch [Shu14, Corollary 3.2.B], $Kh(L_2; \mathbb{F}_2)$ is a free module, so $\widetilde{Kh}(L; \mathbb{F}_2)$ is, as well. □

5.2. **Detection of split links by Khovanov homology.** Let Λ denote the universal Novikov field over $\mathbb{F}_2$, consisting of formal sums $\sum_i f_i t^{r_i}$ where the $f_i \in \mathbb{F}_2$, $r_i \in \mathbb{R}$, and $\lim_{i\to\infty} r_i = \infty$. An element $\omega \in H^2(Y;\mathbb{R})$ induces a map $H_2(Y;\mathbb{Z}) \to \mathbb{R}$ and hence a ring homomorphism $\mathbb{F}_2[H_2(Y;\mathbb{Z})] \to \Lambda$. This makes Λ into a module Λ_ω over $\mathbb{F}_2[H_2(Y;\mathbb{Z})]$.

Theorem 5.4. *[AL19, Theorem 1.1] Let Y be a closed, oriented 3-manifold and $\omega \in H^2(Y;\mathbb{R})$. Then, $\underline{\widehat{HF}}(Y;\Lambda_\omega) = 0$ if and only if Y contains a 2-sphere S so that $\int_S \omega \neq 0$.*

Corollary 5.5. *Suppose that $\omega \in \mathrm{Hom}(H_2(Y;\mathbb{Z}),\mathbb{Z})$, and let $\mathbb{F}_2[t^{-1},t]_\omega$ be $\mathbb{F}_2[t^{-1},t]$, viewed as an $\mathbb{F}_2[H_2(Y;\mathbb{Z})]$-module via ω. Then, $\underline{\widehat{HF}}(Y;\mathbb{F}_2[t^{-1},t]_\omega)$ is a torsion $\mathbb{F}_2[t^{-1},t]$-module if and only if Y contains a 2-sphere S so that $\omega([S]) \neq 0$.*

Proof. In the case $\omega = 0$, this is the well-known statement that $\widehat{HF}(Y) \neq 0$ (see [AL19, Theorem 1.2]). So, assume $\omega \neq 0$.

Let $\mathbb{F}_2(t)$ denote the field of rational functions in t; this is also the field of fractions of $\mathbb{F}_2[t,t^{-1}]$. The module $\underline{\widehat{HF}}(Y;\mathbb{F}_2[t^{-1},t]_\omega)$ is torsion if and only if $\underline{\widehat{HF}}(Y;\mathbb{F}_2[t^{-1},t]_\omega) \otimes_{\mathbb{F}_2[t^{-1},t]} \mathbb{F}_2(t) = 0$. It follows from the universal coefficient theorem that

$$\underline{\widehat{HF}}(Y;\mathbb{F}_2(t)_\omega) \cong \underline{\widehat{HF}}(Y;\mathbb{F}_2[t^{-1},t]_\omega) \otimes_{\mathbb{F}_2[t^{-1},t]} \mathbb{F}_2(t)$$
$$\underline{\widehat{HF}}(Y;\Lambda_\omega) \cong \underline{\widehat{HF}}(Y;\mathbb{F}_2(t)_\omega) \otimes_{\mathbb{F}_2(t)} \Lambda.$$

(Compare [AL19, Formula (2.1)].) So, since Λ and $\mathbb{F}_2(t)$ are fields,

$$\dim_\Lambda \underline{\widehat{HF}}(Y;\Lambda_\omega) = \dim_{\mathbb{F}_2(t)} \underline{\widehat{HF}}(Y;\mathbb{F}_2(t)_\omega).$$

Hence, $\dim_\Lambda \underline{\widehat{HF}}(Y;\Lambda_\omega) = 0$ if and only if $\underline{\widehat{HF}}(Y;\mathbb{F}_2[t^{-1},t]_\omega)$ is torsion, so the result follows from Theorem 5.4. □

Lemma 5.6. *Let $\mathbb{F}_2$ be any field. If Y is a 3-manifold with $H_1(Y) \cong \mathbb{Z}$ and no homologically essential 2-spheres in Y then the unrolled homology of $\widehat{CF}(Y;\mathbb{F}_2)$ is nontrivial. More generally, for any 3-manifold Y, if $\zeta \in H_1(Y)$ is such that the intersection number $\zeta \cdot S = 0$ for all 2-spheres $S \subset Y$ then the unrolled homology of $\widehat{CF}(Y;\mathbb{F}_2)$ with respect to ζ is nontrivial.*

Proof. We prove the more general statement. Let $\omega \in \mathrm{Hom}(H_2(Y),\mathbb{Z})$ be intersection with ζ. By Corollary 5.5, $\underline{\widehat{HF}}(Y;\mathbb{F}_2[t^{-1},t]_\omega)$ has an $\mathbb{F}_2[t^{-1},t]$-summand. So, by Corollary 3.14, the unrolled homology of $\widehat{CF}(Y;\mathbb{F}_2)$ is nontrivial. □

Corollary 5.7. *If L is a non-split, 2-component link then the unrolled homology of the complex $\widehat{CF}(\Sigma(L);\mathbb{F}_2)$ is nontrivial.*

More generally, suppose $L = L_1 \cup L_2$ is a union of two disjoint sublinks, $p \in L_1$, and $q \in L_2$. Endow $\widehat{CF}(\Sigma(L);\mathbb{F}_2)$ with the A_∞-module structure over $\mathbb{F}_2[X]/(X^2)$ coming from a lift of a path from p to q (§4.2). If there is no 2-sphere separating L_1 and L_2 then the unrolled homology of $\widehat{CF}(\Sigma(L);\mathbb{F}_2)$ is nontrivial.

Here, if $\Sigma(L)$ is a rational homology sphere then we view $\widehat{CF}(\Sigma(L))$ as a module over $\mathbb{F}_2[X]/(X^2)$ trivially, i.e., $\mu_{1+n}(y,a_1,\dots,a_n) = 0$ if $n > 1$ or if $n = 1$ and $a_1 = X$.

Proof. This is immediate from Lemma 5.6 and Proposition 5.1 (for the first statement) or Corollary 5.2 (for the second statement). □

Proof of Theorem 2. (1) $\Longrightarrow$ (2) This is Lemma 5.3.

(1) $\Longrightarrow$ (3) By Theorem 4.2, we may assume the diagram for L is itself split. Then, the reduced Khovanov complex is itself a complex of free modules.

(2) $\Longrightarrow$ (4) This follows by considering the horizontal filtration on $\mathcal{C}_{Kh}^{\mathrm{un}}$. The E^1-page of the associated spectral sequence is the unrolled complex for $\widetilde{Kh}(L;\mathbb{F}_2)$, and the unrolled complex of a free module is acyclic.

(3) $\Longrightarrow$ (4) This is immediate from Lemma 2.12.

(4) $\Longrightarrow$ (1) Suppose that L is a link and $p, q \in L$ are such that $\widetilde{\mathcal{C}}_{Kh}(L;\mathbb{F}_2)^{\mathrm{un}}$ is acyclic. Let C_* be the complex from Proposition 4.4. By Lemma 2.11, the unrolled homology of C_* is isomorphic to the unrolled homology of $\widehat{CF}(\Sigma(m(L));\mathbb{F}_2)$. Let $\mathcal{F}$ be the sum of the cube filtration on C_* and the horizontal filtration on C_*^{un}. (That is, for an element $x \in C_*^{\mathrm{un}}$, $\mathcal{F}(x)$ is the sum of the horizontal filtration of x and the minimal cube filtration of any term in x.)

Consider the spectral sequence associated to the filtration $\mathcal{F}$. Since μ_1 strictly raises the cube filtration, the d_0-differential vanishes. The differential on the E^1-page is the sum of the first-order part of μ_1 and the differential on the unrolled complex coming from the zeroth order part of $\mu_2(\cdot, X)$. By Properties (4) and (5) in Proposition 4.4, this is exactly the differential on $\widetilde{\mathcal{C}}_{Kh}(m(L))^{\mathrm{un}}$. Hence, the E^1-page is acyclic. Since the complex C_*^{un} is complete in the filtration $\mathcal{F}$, this implies that $C_*^{\mathrm{un}} \simeq \widehat{CF}(\Sigma(m(L));\mathbb{F}_2)^{\mathrm{un}}$ is acyclic. Hence, by Corollary 5.7, there is a 2-sphere in $S^3 \setminus L$ separating p an q. ☐

Proof of Theorem 1. This is immediate from the equivalence of parts (1) and (2) in Theorem 2. ☐

Proof of Corollary 1.3. Suppose there is a sphere in $S^3 \setminus L$ separating p and q. By Theorem 4.2, we may assume that $Kh(L)$ is computed from a split diagram $L_p \amalg L_q$ with $p \in L_p$ and $q \in L_q$. Then, $Kh(L) \cong Kh(L_p) \otimes_{\mathbb{F}_2} Kh(L_q)$ as a module over $\mathbb{F}_2[W, X]/(W^2, X^2)$. As in the proof of Lemma 5.3, $Kh(L_p)$ is a free module over $\mathbb{F}_2[W]/(W^2)$ and $Kh(L_q)$ is a free module over $\mathbb{F}_2[X]/(X^2)$. So, $Kh(L)$ is a free module over $\mathbb{F}_2[W, X]/(W^2, X^2)$, as desired.

Conversely, suppose that $Kh(L)$ is a free module over $\mathbb{F}_2[W, X]/(W^2, X^2)$. We claim that $\widetilde{Kh}(L)$ is a free module over $\mathbb{F}_2[X]/(X^2)$. If we knew that all higher A_∞ operations on $Kh(L)$ vanished then this would be immediate, since the reduced Khovanov homology is the A_∞ tensor product of the $(\mathbb{F}_2[W]/(W^2), \mathbb{F}_2[X]/(X^2))$-bimodule $Kh(L)$ over $\mathbb{F}_2[W]/(W^2)$ with $\mathbb{F}_2$ (Lemma 4.1). In fact, the result follows from homological algebra nonetheless. The A_∞ tensor product is

$$0 \longleftarrow Kh(L) \longleftarrow Kh(L) \longleftarrow Kh(L) \longleftarrow Kh(L) \longleftarrow \cdots$$

where an arrow of length n comes from the operation $m_{1+n}(\cdot, W, \dots, W)$. (More generally, an A_∞-bimodule operation $\mu_{k,1,\ell}$ contributes an A_∞-module operation $\mu_{1+\ell}$ which goes k steps to the left.)

Consider the spectral sequence associated to the obvious horizontal filtration. (This is a formulation of the universal coefficient spectral sequence.) Since $Kh(L)$ is finitely generated and the d^i differential changes the homological grading by $i-1$, the spectral sequence converges. The E^2-page is

$$0 \longleftarrow Kh(L)/\big(W \cdot Kh(L)\big) \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

so the spectral sequence collapses. Thus, as an (ordinary) module, the E^∞-page is $Kh(L)/\big(W \cdot Kh(L)\big)$. Further, the form of the E^∞-page implies that the module structure on the E^∞-page is the same as the module structure on the homology of the total complex. So, the reduced Khovanov homology is isomorphic to $Kh(L)/\big(W \cdot Kh(L)\big)$, a free module over $\mathbb{F}_2[X]/(X^2)$. Hence, by Theorem 1, there is a sphere in $S^3 \setminus L$ separating p and q. ☐

Finally, we note that Hedden-Ni's result that the Khovanov homology module detects the unlink follows from Theorem 1 (and [KM11]). (Of course, the techniques we used to prove Theorem 1 are similar to the ones they used.)

Theorem 5.8. *[HN13, Theorem 2] Let L be an n-component link and U the n-component unlink. If $Kh(L) \cong Kh(U) = \mathbb{F}_2[X_1, \dots, X_n]/(X_1^2, \dots, X_n^2)$, as modules over the ring $\mathbb{F}_2[X_1, \dots, X_n]/(X_1^2, \dots, X_n^2)$, then $L \sim U$.*

Proof. By Corollary 1.3, there is a sphere in $S^3 \setminus L$ separating each pair of components of L. It follows that L is a disjoint union of n knots. By the Künneth theorem for Khovanov homology, each of these knots has Khovanov homology $\mathbb{F}_2[X]/(X^2)$. So, by Kronheimer-Mrowka's theorem [KM11] that Khovanov homology detects the unknot, each component is an unknot, and so L is an n-component unlink. ☐

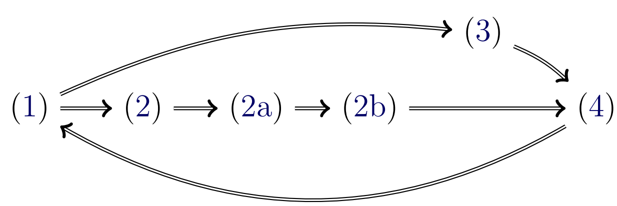
5.3. **Detection of split links by Heegaard Floer homology.**

Proof of Theorem 3. We will prove the properties stated in the theorem are equivalent to the following two additional properties, as well:

(2a) $\widehat{HF}(\Sigma(L); \mathbb{F}_2)^{\mathrm{un}}$ is acyclic, where $\widehat{HF}(\Sigma(L); \mathbb{F}_2)$ is viewed as an ordinary module over the ring $\mathbb{F}_2[X]/(X^2)$.

(2b) $\widehat{HF}(\Sigma(L); \mathbb{F}_2)^{\mathrm{un}}$ is acyclic, where $\widehat{HF}(\Sigma(L); \mathbb{F}_2)$ is viewed as an A_∞-module over the ring $\mathbb{F}_2[X]/(X^2)$.

The logic of the proof is:



(1) $\Longrightarrow$ (2) By Proposition 5.1, there is a decomposition $\Sigma(L) \cong \Sigma(L_1) \# \Sigma(L_2) \# (S^2 \times S^1)$, where the loop ζ induced by p and q intersects the 2-sphere $S^2 \times \{\mathrm{pt}\} \subset S^2 \times S^1$ algebraically once. So, the result follows from the Künneth theorem for $\widehat{HF}$ and a model computation of the $H_1/tors$ action on $\widehat{HF}(S^2 \times S^1)$ [OSz04c].

(2) $\Longrightarrow$ (2a) This is immediate from the definition of the unrolled complex.

(2a) $\Longrightarrow$ (2b) This follows from the spectral sequence associated to the horizontal filtration on $\widehat{CF}(\Sigma(K))^{\mathrm{un}}$: the E^1-page is the unrolled complex for $\widehat{HF}(\Sigma(K))$ (viewed as an honest module over $\mathbb{F}_2[X]/(X^2)$), which is acyclic by assumption.

(2b) $\Longrightarrow$ (4) This follows from invariance of the unrolled homology under A_∞ quasi-isomorphism (Lemma 2.11) and the fact that any A_∞-module is quasi-isomorphic to its homology.

(1) $\Longrightarrow$ (3) By Corollary 5.2, we can factor $\Sigma(L) = (S^2 \times S^1)\#Y'$ where the loop ζ induced by p and q is the circle $\{\mathrm{pt}\} \times S^1$ in the $S^2 \times S^1$. Choose a Heegaard diagram $\mathcal{H} = \mathcal{H}_1\#\mathcal{H}_2$ which witnesses this splitting, where $\mathcal{H}_1$ is the standard Heegaard diagram for $S^2 \times S^1$ and the connected sum happens in the region containing the basepoint z. Then, the chain complex $\widehat{CF}(\mathcal{H})$ is a free module over $\mathbb{F}_2[X]/(X^2)$ (with trivial higher A_∞-operations).

(3) $\Longrightarrow$ (4) This is immediate from Lemma 2.12.

(4) $\Longrightarrow$ (1) This is Corollary 5.7. ☐

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