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We show that under a suitable oddness condition, for $p \gg_F n$ irreducible n-dimensional mod p representations of the absolute Galois group of an arbitrary number field F have characteristic zero lifts which are unramified outside a finite set of primes and trianguline at all primes of F dividing p. We also prove a variant of this result under some extra hypotheses for representations into connected reductive groups.

1. Introduction

For any field F we let Γ_F be its absolute Galois group. In [13] and [14] we constructed geometric (in the sense of Fontaine–Mazur) lifts for odd representations $\bar{\rho} : \Gamma_F \to G(k)$, where F is a totally real number field, and G is a split reductive group (possibly disconnected) over the ring of integers $\mathcal{O}$ of a p-adic field E with residue field k, when $\bar{\rho}$ satisfies certain additional conditions. For example, if G is connected, $\bar{\rho}|_{\Gamma_{F(\zeta_p)}}$ is absolutely irreducible, and lifts (which are regular de Rham at primes dividing p) exist locally at all primes of F, then global lifts exist whenever p is sufficiently large by [13, Theorem A]. In [14] we constructed lifts for reducible representations under some more technical hypotheses. The oddness assumption is crucial, but if $G = \mathrm{GL}_n$ and $n > 2$ then no $\bar{\rho}$ is odd in the sense of [13, Definition 1.2], so the results of the latter paper and [14] cannot be used to construct geometric lifts.[1] In fact, Calegari has proved [6, Theorem 5.1] that without any oddness condition whatsoever, geometric lifts need not exist even when $n = 2$ and $F = \mathbb{Q}$. The goal of this note is to show that if we weaken the requirement that the lift be geometric to being unramified outside a finite set of primes and trianguline[2] at all primes above p, then the methods of [13] and [14] can be adapted to apply

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[1] The results of these papers do apply — and have interesting consequences — when $G = \mathrm{GL}_n$ and F is a global function field, and even for number fields if we do not impose any conditions at primes dividing p.

[2] The reader may consult [2] for a survey of trianguline representations; we give a brief sketch of the definition in Section 2B.

even when $G = \mathrm{GL}_n$ and F is an arbitrary number field, if we assume the following weaker oddness condition:

Definition 1.1. (1) An involution c in $\mathrm{GL}_n(K)$, where K is a field of characteristic $\neq 2$, is said to be GL_n-*odd* if $|n^+(c) - n^-(c)| \leq 1$, where $n^+(c)$ (resp. $n^-(c)$) is the number of eigenvalues of $\rho(c)$ which are equal to $+1$ (resp. -1).

(2) Let F be a number field and $\rho : \Gamma_F \to \mathrm{GL}_n(K)$ a continuous representation, where K is any topological field of characteristic $\neq 2$. We say that ρ is GL_n-*odd*[3] if for every real place v of F with $c_v \in \Gamma_{F_v}$ the corresponding complex conjugation, the involution $\rho(c_v)$ is odd as in (1).

All Galois representations corresponding to cohomological automorphic forms or the cohomology (possibly with torsion coefficients) of locally symmetric spaces for GL_n/F are GL_n-odd when F is totally real by results of Caraiani and Le Hung [7], so there is a plethora of such representations.

Trianguline representations were introduced by Colmez [9], motivated by work of Kisin [17] on Galois representations attached to overconvergent modular forms, and are closely related to the theory of eigenvarieties. For example, Hansen has conjectured [15, Conjecture 1.2.3] that any (semisimple) GL_n-odd representation $\rho : \Gamma_F \to \mathrm{GL}_n(E)$ which is unramified outside a finite set of primes and is trianguline at all primes above p arises as the representation associated (in [16, Theorem B]) to a point on one of the eigenvarieties for GL_n/F constructed in [15], so the condition imposed on the lifts is a natural weakening[4] of geometricity. We also mention that conjecturally the set of isomorphism classes of geometric representations is a countable set, but (global) trianguline representations vary in p-adic families, e.g., the families of representations associated to eigenvarieties are expected to be trianguline, and our constructions do produce nontrivial families of trianguline lifts (see Remark 3.11).

For F any number field and $\mathcal{S}$ a finite set of finite places of F, we let $\Gamma_{F,\mathcal{S}}$ denote the Galois group of the maximal extension of F (inside a fixed algebraic closure) unramified outside all places in $\mathcal{S}$ (and the infinite places). The following is a special case of the main result of this note.

Theorem 1.2. *Let F be an arbitrary number field and $\bar{\rho} : \Gamma_{F,\mathcal{S}} \to \mathrm{GL}_n(k)$ a GL_n-odd representation. If $p \gg_{n,F} 0$ and $\bar{\rho}|_{\Gamma_{F(\zeta_p)}}$ is absolutely irreducible, then there exists a finite set of places $\mathcal{S}' \supset \mathcal{S}$ and a finite extension E' of E with ring of integers $\mathcal{O}'$ such that $\bar{\rho}$ lifts to a GL_n-odd representation $\rho : \Gamma_{F,\mathcal{S}'} \to \mathrm{GL}_n(\mathcal{O}')$*

[3] This is often simply called *odd* in the literature, but we have added GL_n here to distinguish this notion from the stronger notion of oddness used in [13] for representations into general reductive groups G.

[4] Almost: all crystalline, and even semistable, representations are trianguline, but not all de Rham representations are trianguline, see [2].

which is regular trianguline at all primes of F above p. Furthermore, one can also ensure that $\rho(\Gamma_F)$ contains an open subgroup of $\mathrm{SL}_n(\mathcal{O}')$.

In the main text this is Corollary 3.12. It is deduced from Theorem 3.8 which allows one to construct trianguline lifts under weaker hypotheses, e.g., for $\bar{\rho}$ which might not be irreducible.

We recall that for odd irreducible mod p representations over totally real fields the Khare–Wintenberger method allows one to construct geometric lifts (for many G) without the need for enlarging $\mathcal{S}$. However, this depends crucially on potential modularity results, but since (global) trianguline representations do not in general correspond to classical automorphic forms, it is not possible to apply this method to construct trianguline lifts. In other words, in the trianguline setting there is no R and no $\mathbb{T}$ for which a potential $R = \mathbb{T}$ theorem could lead to the construction of trianguline lifts.

Recently, a definition of trianguline representations with values in general (connected) reductive groups G has been given by Vincent de Daruvar in his thesis [11]; one expects that these are related to eigenvarieties for groups that are not forms of GL_n. Using his results we can extend the above theorem to general (connected) reductive groups, see Theorem 3.16, with the caveat that the main result of loc. cit. depends on two assumptions: first, the existence of a sufficiently general trianguline lift of $\bar{\rho}$, and second, that G should have no factors of type G_2, F_4 and E_8. One expects that the first assumption always holds and the second is unnecessary.

It is an interesting question to extend the definition of trianguline representations to disconnected groups G and to prove analogues of the results of [11] in this case. If this is done, the methods of [13] and [14] should allow one to construct global trianguline lifts also for such G.

1A. Our proofs are based on the methods developed in [13] and [14] and as far as the global arguments go there are no essential changes, except that in Section 3 we axiomatise in Section 3A and Section 3B the conditions under which the methods of loc. cit. lead to the construction (in Theorem 3.6) of lifts, unramified outside a finite set of primes, of a very general class of Galois representations. The main improvements are in the local arguments, so we describe these briefly here.

In the original lifting arguments of Ramakrishna [19], which are generalised in [13] and [14], it is assumed that one is given smooth local conditions at all primes at which $\bar{\rho}$ is ramified, i.e., smooth quotients of the framed local deformation rings, and these should have sufficiently large dimension. One of the key improvements of this method in [13] was that we were able to dispense with the smoothness condition, but the dimension condition cannot be relaxed if we want our lifts to be geometric; in particular, the geometric lifting results there require the field F to be totally real. However, if we only require that our local lifts be trianguline at primes dividing p, then the dimension condition can indeed be relaxed to overcome the first obstacle to constructing lifts for arbitrary n and F, but we face the further problem that trianguline lifts are not parametrised by a quotient of the universal framed deformation ring. Section 2 is devoted to showing how we can avoid this difficulty: in Section 2A we explain how the method of [13, §4] can be modified to deal with more general local conditions, and in Section 2B we show that this can be applied in the setting of trianguline lifts using the properties of the "trianguline variety" of Breuil, Hellmann and Schraen [5, théorème 2.6] and the generalisation thereof in [11]. Once these local improvements are in place the proofs of the main results follow by specialising Theorem 3.6 to the case where the local lifts are assumed to be trianguline.

2. Cocycle constructions

2A. ***A general cocycle construction.*** Let E be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ its ring of integers, $m_{\mathcal{O}} = (\varpi)$ its maximal ideal and $k = \mathcal{O}/m_{\mathcal{O}}$ its residue field. Let G be a reductive group scheme over $\mathcal{O}$ (possibly disconnected) and $\mathfrak{g}$ be its Lie algebra. Let G^{der} be the derived group of the identity component of G and $\mathfrak{g}^{\mathrm{der}}$ its Lie algebra.

Let Γ be any topologically finitely generated profinite group. Given a continuous homomorphism $\bar{\rho} : \Gamma \to G(k)$, there exists a complete local Noetherian $\mathcal{O}$-algebra $R^{\square,\mathrm{univ}}$ with residue field k and a homomorphism $\rho^{\mathrm{univ}} : \Gamma \to G(R^{\square,\mathrm{univ}})$ representing the functor of lifts of $\bar{\rho}$ on the category of complete local Noetherian $\mathcal{O}$-algebras with residue field k. For any such algebra A, we set $\widehat{G}(A) := \ker(G(A) \to G(k))$. The conjugation action of $\widehat{G}(\mathcal{O})$ on $G(R^{\square,\mathrm{univ}})$ and the universal property of ρ^{univ} induces an action of $\widehat{G}(\mathcal{O})$ on $R^{\square,\mathrm{univ}}$.

Let R be a quotient of $R^{\square,\mathrm{univ}}$ by an ideal which is invariant under this action. We assume that R is reduced and flat over $\mathcal{O}$. We also assume that R corresponds to "fixed-multiplier" liftings of $\bar{\rho}$, by which we mean that the map $\Gamma \to G/G^{\mathrm{der}}(R)$ induced from $\rho^{\square,\mathrm{univ}}$ by the maps $R^{\square,\mathrm{univ}} \to R$ and $G \to G/G^{\mathrm{der}}$ factors through a fixed map $\Gamma \to G/G^{\mathrm{der}}(\mathcal{O})$.[5]

For any representation $\rho : \Gamma \to G(\mathcal{O})$ we will denote by ρ_r the reduction of ρ modulo ϖ^r. Also, for any representation $\rho_r : \Gamma \to G(\mathcal{O}/\varpi^r)$, $\rho_r(\mathfrak{g}^{\mathrm{der}})$ will denote $\mathfrak{g}^{\mathrm{der}} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^r$ equipped with the $\mathrm{Ad} \circ \rho_r$ action.

The following result is essentially contained in [13, §4].

Proposition 2.1. *Assume that* $\mathrm{Spec}(R)$ *has an* $\mathcal{O}$*-valued point* y *such that the corresponding point of* $\mathrm{Spec}(R[1/\varpi])$ *is contained in the smooth locus, and let* $\rho : \Gamma \to G(\mathcal{O})$ *be the corresponding lift of* $\bar{\rho}$*. Then there is an open* (*in the* p*-adic*

[5]This is not essential, but it is convenient for our applications to impose this condition. The results below hold in general if one removes all occurrences of the superscript "der".

topology) subset $Y \subset \mathrm{Spec}(R)(\mathcal{O})$ *with* $y \in Y$ *having the properties below, where* Y_n *is the image of* Y *in* $\mathrm{Spec}(R)(\mathcal{O}/\varpi^n)$ *and for integers* $n, r \geq 0$, *the natural maps are denoted* $\pi^Y_{n,r} : Y_{n+r} \to Y_n$.

(1) *Given* $r_0 > 0$ *there exists* $n_0 > 0$ *such that for all* $n \geq n_0$ *and* $0 \leq r \leq r_0$ *the fibres of* $\pi^Y_{n,r}$ *are nonempty principal homogeneous spaces over a submodule* $Z_r \subset Z^1(\Gamma, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ *which is free of rank* d *over* $\mathcal{O}/\varpi^r$, *where* d *is the dimension of* $\mathrm{Spec}(R[1/\varpi])$ *at* y.

(2) $B^1(\Gamma, \rho_r(\mathfrak{g}^{\mathrm{der}})) \subset Z_r$.

(3) *The* $\mathcal{O}$*-module inclusions* $\mathcal{O}/\varpi^{r-1} \to \mathcal{O}/\varpi^r$, *mapping* 1 *to* ϖ, *and the surjections* $\mathcal{O}/\varpi^r \to \mathcal{O}/\varpi^{r-1}$ *induce inclusions* $Z_{r-1} \to Z_r$ *and surjections* $Z_r \to Z_{r-1}$.

(4) *Let* L_r *be the image of* Z_r *in* $H^1(\Gamma, \rho(\mathfrak{g}^{\mathrm{der}}) \otimes \mathcal{O}/\varpi^r)$. *The groups* L_r *are compatible with the maps on cohomology induced by the inclusions* $\mathcal{O}/\varpi^{r-1} \to \mathcal{O}/\varpi^r$ *and the surjections* $\mathcal{O}/\varpi^r \to \mathcal{O}/\varpi^{r-1}$.

(5) $|L_r| = |\mathcal{O}/\varpi^r|^d \cdot |\rho_r(\mathfrak{g}^{\mathrm{der}})^\Gamma| \cdot |\rho_r(\mathfrak{g}^{\mathrm{der}})|^{-1}$.

Proof. This is proved in [13, Proposition 4.7 together with Lemma 4.5 for (2)] for Γ the absolute Galois group of a local field and some specific choices of rings R for which the dimension d is also known, but the proof given there works without any changes under the conditions that we have given. In particular, the proof of the formula for $|L_r|$ there extends to give the formula in (5). □

The goal of this subsection is to formulate a generalisation of Proposition 2.1 which will be the key to our construction of trianguline lifts in Section 3D, allowing us to circumvent the fact that there is no quotient of $R^{\square,\mathrm{univ}}$ parametrising the trianguline lifts of $\bar{\rho}$. In the next proposition, R is as above, but we emphasise that the manifold U will no longer be assumed open in $\mathrm{Spec}(R[1/\varpi])(E)$ $(= \mathrm{Spec}(R)(\mathcal{O}))$. For $m > 0$, let $(\widehat{G^{\mathrm{der}}})^{(m)}(\mathcal{O}) = \ker(G^{\mathrm{der}}(\mathcal{O}) \to G^{\mathrm{der}}(\mathcal{O}/\varpi^m))$.

Proposition 2.2. *Let* U *be a compact* E*-adic manifold of some dimension* d *contained in the smooth locus of* $\mathrm{Spec}(R[1/\varpi])(E)$ *with* $v \in U$, *and let* ρ *be the lift of* $\bar{\rho}$ *corresponding to* v. *Assume that there exists* $m \gg 0$ *such that the* $(\widehat{G^{\mathrm{der}}})^{(m)}(\mathcal{O})$*-orbit of* v *is contained in* U. *Then there is an open* (*in the* p*-adic topology*) *subset* $V \subset U$ *with* $v \in V$ *having the properties below, where* V_n *is the image of* V *in* $\mathrm{Spec}(R)(\mathcal{O}/\varpi^n)$ *and for integers* $n, r \geq 0$, *the natural maps are denoted* $\pi^V_{n,r} : V_{n+r} \to V_n$.

(1) *Given* $r_0 > 0$ *there exists* $n_0 > 0$ *such that for all* $n \geq n_0$ *and* $0 \leq r \leq r_0$ *the fibres of* $\pi^V_{n,r}$ *are nonempty principal homogeneous spaces over a submodule* $Z_r \subset Z^1(\Gamma, \rho(\mathfrak{g}^{\mathrm{der}}) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^r)$ *which is free of rank* d *over* $\mathcal{O}/\varpi^r$.

(2) $B^1(\Gamma, \rho(\mathfrak{g}^{\mathrm{der}}) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^n) \subset Z_r$.

(3) *The $\mathcal{O}$-module inclusions $\mathcal{O}/\varpi^{r-1} \to \mathcal{O}/\varpi^r$, mapping 1 to ϖ, and the surjections $\mathcal{O}/\varpi^r \to \mathcal{O}/\varpi^{r-1}$ induce inclusions $Z_{r-1} \to Z_r$ and surjections $Z_r \to Z_{r-1}$.*

(4) *Let L_r be the image of Z_r in $H^1(\Gamma, \rho(\mathfrak{g}^{\mathrm{der}}) \otimes \mathcal{O}/\varpi^r)$. The groups L_r are compatible with the maps on cohomology induced by the inclusions $\mathcal{O}/\varpi^{r-1} \to \mathcal{O}/\varpi^r$ and the surjections $\mathcal{O}/\varpi^r \to \mathcal{O}/\varpi^{r-1}$.*

(5) $|L_r| = |\mathcal{O}/\varpi^r|^d \cdot |\rho_r(\mathfrak{g}^{\mathrm{der}})^\Gamma| \cdot |\rho_r(\mathfrak{g}^{\mathrm{der}})|^{-1}$.

Proof. Fix r_0. Then apply Proposition 2.1 by taking $y = v$ and get an open set $Y \subset \mathrm{Spec}(R[1/\varpi])(\mathcal{O})$ with $v \in Y$, an integer n_0^Y and cocycles Z_r^Y satisfying all its conclusions.

Any open subset of an E-adic manifold is an E-adic submanifold, so replacing U by $U \cap Y$ we may assume that $U \subset Y$. By applying the result of Serre [20, Proposition 11] to the inclusion $v \in U$ in the same way as in the proof of [13, Lemma 4.3] we get an open submanifold $V \subset U$ with $v \in V$ and free $\mathcal{O}/\varpi^r$ submodules $Z_r \subset Z_r^Y$ with the property in (1), with the proviso that the n_0 associated to r_0 might be greater than the n_0^Y obtained as an output of Proposition 2.1. (The inclusion $Z_r \subset Z_r^Y$ follows from the definitions, since $U \subset Y$ so each $U_n \subset Y_n$.)

Using the assumption on the $(\widehat{G^{\mathrm{der}}})^{(m)}(\mathcal{O})$-orbit of v we see that (2) holds in the same way as in the proof of [13, Lemma 4.5].

The surjectivity part of (3) is clear from the defining property of the Z_r. To prove the injectivity part, we must examine the proof of [13, Lemma 4.3] (since there is no analogue of Lemma 4.4 of that paper in the present setting) from which we obtain the Z_r. That proof proceeds by first choosing a formally smooth complete Noetherian $\mathcal{O}$-algebra A and a surjection of $\mathcal{O}$-algebras $A \to R$. The Z_r (corresponding to U) are then constructed as certain submodules of $\mathrm{Hom}_{\mathcal{O}}(\Omega_{A/\mathcal{O}} \otimes_{A,y} \mathcal{O}, \mathcal{O}/\varpi^r)$ by choosing local equations for U in the E-adic manifold $\mathrm{Spec}(A)(\mathcal{O})$ and reducing to the case where U corresponds to the E-valued points of a formally smooth quotient R' of A. It is an immediate consequence of this construction that the map

$$\mathrm{Hom}_{\mathcal{O}}(\Omega_{A/\mathcal{O}} \otimes_{A,y} \mathcal{O}, \mathcal{O}/\varpi^{r-1}) \to \mathrm{Hom}_{\mathcal{O}}(\Omega_{A/\mathcal{O}} \otimes_{A,y} \mathcal{O}, \mathcal{O}/\varpi^r)$$

induced by the inclusion $\mathcal{O}/\varpi^{r-1} \to \mathcal{O}/\varpi^r$ maps Z_{r-1} injectively into Z_r (since this is true when A is replaced by R'). The map $A \to R$ induces compatible (with respect to $\mathcal{O}/\varpi^{r-1} \to \mathcal{O}/\varpi^r$) inclusions

$$\mathrm{Hom}_{\mathcal{O}}(\Omega_{R/\mathcal{O}} \otimes_{R,y} \mathcal{O}, \mathcal{O}/\varpi^r) \to \mathrm{Hom}_{\mathcal{O}}(\Omega_{A/\mathcal{O}} \otimes_{A,y} \mathcal{O}, \mathcal{O}/\varpi^r)$$

from which (3) follows once we identify $\mathrm{Hom}_{\mathcal{O}}(\Omega_{R/\mathcal{O}} \otimes_{R,y} \mathcal{O}, \mathcal{O}/\varpi^r)$ with a submodule of $Z^1(\Gamma, \rho(\mathfrak{g}^{\mathrm{der}}) \otimes \mathcal{O}/\varpi^r)$ as in [13, §4.2].

Finally, (4) follows immediately from (3) and the definition of L_r and (5) follows as in Proposition 2.1. □

2B. ***Cocycles for trianguline deformations.*** Let F be a finite extension of $\mathbb{Q}_p$ and let $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$ be its absolute Galois group. In this subsection we recall some facts about trianguline representations of Γ_F and carry out the local analysis needed in order to be able to construct the cocycles used to prove the existence of global lifts of GL_n-odd representations which are trianguline at primes above p. For a general introduction to trianguline representations the reader may consult [2] (and also [11] for the G-valued case), but we will only use the existence and some properties of the rigid-analytic variety of (local) trianguline lifts from [5] (and [11] for the G-valued case). However, to orient the reader not familiar with this notion, we mention briefly that trianguline representations are defined by embedding the category of E-adic representations of Γ_F (for $E/\mathbb{Q}_p$ a finite extension) fully faithfully into a larger category, the category of (ϕ, Γ)-modules, which are modules over a (Robba) ring $\mathcal{R}$ with some extra structure. A Galois representation is said to be trianguline if the corresponding (ϕ, Γ)-module has a filtration by subobjects such that the successive subquotients are free of rank one as $\mathcal{R}$-modules; the parameter δ occurring below corresponds to the ordered tuple of these subquotients. The condition for a G-valued representation to be trianguline is formulated in Tannakian terms using the tensor structure on the category of (ϕ, Γ)-modules or in the language of principal G-bundles over $\mathcal{R}$ (with extra structure).

2B1. We will first consider the case of GL_n since the results for general G are slightly weaker and partly conditional. Let $G = \mathrm{GL}_n$ and $\bar{\rho} : \Gamma_F \to G(k)$ be as in Section 2A. Let $\mathfrak{X}_{\bar{\rho}}^{\square}$ be the rigid analytic space over E corresponding to the formal scheme $\mathrm{Spf}(R^{\square,\mathrm{univ}})$, where $R^{\square,\mathrm{univ}}$ is the universal lifting ring of $\bar{\rho}$; its E-valued points are canonically equal to the $\mathcal{O}$-valued points of $\mathrm{Spec}(R^{\square,\mathrm{univ}})$ and similarly for all finite extensions of E. Let $\mathcal{T}$ denote the rigid analytic space over $\mathbb{Q}_p$ parametrising the continuous characters of $F^\times$. Let $X_{\mathrm{tri}}^{\square}(\bar{\rho}) \subset \mathfrak{X}_{\bar{\rho}}^{\square} \times \mathcal{T}_E^n$ be the space of trianguline deformations of $\bar{\rho}$ as in [5, définition 2.4]; our notation is however slightly different. It is a Zariski closed rigid analytic subvariety of $\mathfrak{X}_{\bar{\rho}}^{\square} \times \mathcal{T}_E^n$ defined as the Zariski closure of a subset $U_{\mathrm{tri}}^{\square}(\bar{\rho})^{\mathrm{reg}}$. The points (x, δ) of the latter consist of certain trianguline lifts x of $\bar{\rho}$ together with a system of parameters δ of a triangulation of the associated (ϕ, Γ)-module; the assumption is that δ is regular in the sense explained in the paragraph before [5, définition 2.4]. For our purposes what is important is that x corresponds to a trianguline lift, and the precise nature of δ plays no explicit role. We call the trianguline lifts of $\bar{\rho}$ which correspond to points of $X_{\mathrm{tri}}^{\square}(\bar{\rho})$ *good*; we do not know whether all trianguline lifts are good.

The following result of Breuil, Hellmann and Schraen is the key input for the construction of cocycles for trianguline lifts.

Lemma 2.3. (1) *The space $X_{\mathrm{tri}}^{\square}(\bar{\rho})$ is nonempty and equidimensional of dimension $n^2 + \frac{1}{2}[F : \mathbb{Q}_p]n(n+1)$.*

(2) *$U^{\square}_{\mathrm{tri}}(\bar\rho)^{\mathrm{reg}}$ is a smooth Zariski open subvariety of $X^{\square}_{\mathrm{tri}}(\bar\rho)$ which is Zariski dense.*

(3) *The projection map $\pi_1 : X^{\square}_{\mathrm{tri}}(\bar\rho) \to \mathfrak{X}^{\square}_{\bar\rho}$ is an immersion at all points of $U^{\square}_{\mathrm{tri}}(\bar\rho)^{\mathrm{reg}}$.*

Proof. This is a part of [5, théorème 2.6]. The third part is not contained in the statement there but is used in the proof, where it is deduced from results of Bellaïche and Chenevier [1, §2.3]. We note that the proof of nonemptiness of $U^{\square}_{\mathrm{tri}}(\bar\rho)^{\mathrm{reg}}$ in [5] uses the existence of regular crystalline lifts of $\bar\rho$; this is a highly nontrivial result for general $\bar\rho$ and was proved only recently in [12]. □

The group $\widehat{\mathrm{GL}_n}(\mathcal{O})$ acts on $\mathfrak{X}^{\square}_{\bar\rho} \times \mathcal{T}^n_E$ via its action on the first factor, and this action preserves $U^{\square}_{\mathrm{tri}}(\bar\rho)^{\mathrm{reg}}$ and $X^{\square}_{\mathrm{tri}}(\bar\rho)$.

For our application we need to work with lifts of $\bar\rho$ which have a fixed determinant. To arrange this we consider the morphism $\det : \mathfrak{X}^{\square}_{\bar\rho} \to \mathfrak{X}^{\square}_{\det(\bar\rho)}$ which sends any lift of $\bar\rho$ to its determinant and the induced morphism $\det_1 := \det \circ \pi_1 : X^{\square}_{\mathrm{tri}}(\bar\rho) \to \mathfrak{X}^{\square}_{\det(\bar\rho)}$. The space $\mathfrak{X}^{\square}_{\det(\bar\rho)}$ has dimension $1 + [F : \mathbb{Q}_p]$ and the morphism $\det_1$ commutes with the action of $\widehat{\mathrm{GL}_n}(\mathcal{O})$, where the action on $\mathfrak{X}^{\square}_{\det(\bar\rho)}$ is taken to be the trivial action.

If χ is a character of Γ_F with values in a finite extension E' of E with trivial reduction modulo its maximal ideal, then taking the tensor product with χ induces an automorphism of $X^{\square}_{\mathrm{tri}}(\bar\rho)_{E'}$ which preserves $U^{\square}_{\mathrm{tri}}(\bar\rho)^{\mathrm{reg}}$. This is compatible with the morphism $\det_1$ if we let χ act on $\mathfrak{X}^{\square}_{\det(\bar\rho)}$ by tensoring with $\chi^{\otimes n}$.

If $p \nmid n$, then any character χ as above has an n-th root. This implies that in this case all the fibres of $\det_1$ over $\mathfrak{X}^{\square}_{\det(\bar\rho)}(E')$ are isomorphic, for any finite extension E' of E. It follows that for any point y of $\mathfrak{X}^{\square}_{\det(\bar\rho)}$, $U^{\square}_{\mathrm{tri}}(\bar\rho)^{\mathrm{reg}} \cap (\det_1)^{-1}(y)$ is Zariski dense in $(\det_1)^{-1}(y)$.

Lemma 2.4. *Let $\bar\rho : \Gamma_F \to \mathrm{GL}_n(k)$ be a continuous homomorphism, and also let $\rho : \Gamma_F \to \mathrm{GL}_n(\mathcal{O})$ be a lift of $\bar\rho$ corresponding to a point $(x, \delta) \in X^{\square}_{\mathrm{tri}}(\bar\rho)(E)$. Assume $p \nmid n$. Then, after replacing E by a finite extension if necessary, the following holds:*

(1) *There is a quotient R of $R^{\square,\mathrm{univ}}$ by an ideal invariant under the action of $\widehat{\mathrm{GL}_n}(\mathcal{O})$ which is reduced and flat over $\mathcal{O}$.*

(2) *There is a compact E-adic submanifold U of $\mathrm{Spec}(R[1/\varpi])(E)$ of dimension $d = n^2 - 1 + [F : \mathbb{Q}_p]\big(\frac{1}{2}n(n+1) - 1\big)$ such that all the points of U correspond to regular trianguline lifts ρ' of $\bar\rho$ with $\det(\rho) = \det(\rho')$.*

(3) *Given any integer $N > 0$, U can be chosen such that all ρ' corresponding to points of U are congruent to ρ modulo ϖ^N.*

(4) *There is a point $v \in U$ and an integer $m > 0$ such that the $(\widehat{\mathrm{GL}_n})^{(m)}(\mathcal{O})$-orbit of v is contained in U.*

Proof. By replacing E with a finite extension if necessary, we may ensure that $U^{\square}_{\mathrm{tri}}(\bar\rho)^{\mathrm{reg}}(E) \neq \varnothing$, so by Lemma 2.3 it has a natural structure of an E-adic manifold of dimension $n^2 + \frac{1}{2}[F:\mathbb{Q}_p]n(n+1)$. Since the space $\mathfrak{X}^{\square}_{\det(\bar\rho)}$ has dimension $1+[F:\mathbb{Q}_p]$ and all (nonempty) fibres of $\det_1$ are geometrically isomorphic since $p \nmid n$, it follows from the results of [21, Part II, Chapter III, Section 10] that $\mathcal{F} := (\det_1)^{-1}(\det_1((x,\delta)))$ is of pure dimension $d = n^2 - 1 + [F:\mathbb{Q}_p]\big(\frac{1}{2}n(n+1)-1\big)$ and has a smooth Zariski dense open subset $\mathcal{F}'$ consisting of points in $U^{\square}_{\mathrm{tri}}(\bar\rho)^{\mathrm{reg}}$.

Now using [13, Lemma 4.9][6] — this might again require replacing E with a finite extension — we can find a sequence of points $(x'_N, \delta'_N) \in \mathcal{F}'(E)$, $N = 1, 2, \ldots$, such that the lift of $\bar\rho$ corresponding to x'_N is congruent to ρ modulo ϖ^N. For any such N, let $U' \subset \mathcal{F}'(E)$ be an E-adic neighbourhood of (x'_N, δ'_N) such that all the lifts of $\bar\rho$ corresponding to points of U' are congruent to ρ modulo ϖ^N. By shrinking U' further using (3) of Lemma 2.3, we may assume that π_1 induces an embedding of U' onto a compact E-adic submanifold U of $\mathrm{Spec}(R^{\square,\mathrm{univ}}[1/\varpi])(E)$.

Let $\widetilde{U}' = \widehat{\mathrm{GL}_n}(\mathcal{O}) \cdot U' \subset \mathcal{F}'(E)$ and let R be the reduced quotient of $R^{\square,\mathrm{univ}}$ corresponding to the Zariski closure in $\mathrm{Spec}(R^{\square,\mathrm{univ}})$ of $\pi_1(\widetilde{U}')$. Since $\pi_1(\widetilde{U}')$ is preserved by the action of $\widehat{\mathrm{GL}_n}(\mathcal{O})$, the kernel of the quotient map $R^{\square,\mathrm{univ}} \to R$ is also preserved by this group. By the Zariski density of $\pi_1(\widetilde{U}')$ in $\mathrm{Spec}(R)$ and the reducedness of $\mathrm{Spec}(R)$, we may find a point $v \in \pi_1(\widetilde{U}')$ which is a smooth point of $\mathrm{Spec}(R[1/\varpi])$. The $\widehat{\mathrm{GL}_n}(\mathcal{O})$-equivariance of π_1 implies that we may take $v = \pi_1(v')$ for some $v' \in U'$. Replacing U' by a compact open neighbourhood of v' we may assume that $\pi_1(U')$ is contained in the smooth locus of $\mathrm{Spec}(R[1/\varpi])(E)$.

Letting $U = \pi_1(U')$ and $v = \pi_1(v')$, it is clear from the above that (1), (2) and (3) hold. To prove (4), we note that $\widehat{\mathrm{GL}_n}(\mathcal{O})$ acts continuously on $\mathcal{F}'(E)$. Since U' is an open neighbourhood of v' in $\mathcal{F}'(E)$ and the sequence of subgroups $(\widehat{\mathrm{GL}_n})^{(m)}(\mathcal{O}) := \ker(\mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}/\varpi^m))$ of $\widehat{\mathrm{GL}_n}(\mathcal{O})$ forms a neighbourhood basis of the identity, for all $m \gg 0$ the $(\widehat{\mathrm{GL}_n})^{(m)}(\mathcal{O})$-orbit of v' is contained in U'. The $\widehat{\mathrm{GL}_n}(\mathcal{O})$-equivariance of π_1 then implies that for all $m \gg 0$ the $(\widehat{\mathrm{GL}_n})^{(m)}(\mathcal{O})$-orbit of v is contained in U. □

Remark 2.5. If $\bar\rho$ is upper triangular with respect to some basis, then the points of $U^{\square}_{\mathrm{tri}}(\bar\rho)^{\mathrm{reg}}$ corresponding to pairs (x,δ) with x being upper triangular with respect to some basis form a nonempty $\widehat{\mathrm{GL}_n}(\mathcal{O})$-invariant open subspace, as a consequence of the inductive nature of the proof of existence of regular crystalline lifts in [12]. In this case one may arrange that all the points of the U constructed in Lemma 2.4 correspond to lifts of $\bar\rho$ which are upper triangular with respect to some basis.

2B2. Suppose G is a connected split reductive group over $\mathcal{O}$. Then the notion of trianguline representations of Γ_F valued in $G(E)$ is defined in [11] using (ϕ,Γ)-modules with G_E-structure, these being defined by Tannakian methods. We do not recall the details of the construction, but only mention that this definition generalises the usual definition of trianguline representations in a natural way. The

[6] The lemma as stated does not quite apply here, but its proof gives what we claim.

main result of [11], Theorem 6.22, is an analogue for G-valued representations of [5, théorème 2.6], with some modifications and conditions which we briefly explain.

For $\bar{\rho} : \Gamma_F \to G(k)$, we let $R^{\square,\mathrm{univ}}$ and $\mathfrak{X}_{\bar{\rho}}^{\square}$ be as before. For T a maximal split torus of G contained in a Borel subgroup B, we let $T^\vee$ be the dual torus and we replace $\mathcal{T}_E^n$ by $\widehat{T^\vee}$, the rigid analytic space over E parametrising (continuous) characters of $T^\vee(F)$. Then $X_{\mathrm{tri}}^{\square}(\bar{\rho}) \subset \mathfrak{X}_{\bar{\rho}}^{\square} \times \widehat{T^\vee}$ is defined as the Zariski closure of a subset $U_{\mathrm{tri}}^{\square}(\bar{\rho})^{\mathrm{vreg}}$ of *very regular* lifts. The points of the latter correspond to pairs (x, δ) where x is a trianguline lift of $\bar{\rho}$ in the sense of [11, Definition 4.9] with a triangulation such that the associated parameter δ is very regular in the sense of [11, Definition 6.1]. We then have the following analogue of Lemma 2.3.

Lemma 2.6. *Assume G has no factors of type* G_2, F_4 *and* E_8 *and $\bar{\rho}$ has a very regular trianguline lift. Then*:

(1) *The space $X_{\mathrm{tri}}^{\square}(\bar{\rho})$ is nonempty and equidimensional, and its dimension equals* $\dim(G_k) + [F : \mathbb{Q}_p] \dim(B_k)$.

(2) $U_{\mathrm{tri}}^{\square}(\bar{\rho})^{\mathrm{vreg}}$ *is a smooth Zariski open subvariety of $X_{\mathrm{tri}}^{\square}(\bar{\rho})$ which is Zariski dense.*

(3) *The projection map $\pi_1 : X_{\mathrm{tri}}^{\square}(\bar{\rho}) \to \mathfrak{X}_{\bar{\rho}}^{\square}$ is an immersion at all points of* $U_{\mathrm{tri}}^{\square}(\bar{\rho})^{\mathrm{vreg}}$.

It is expected that the condition on G is unnecessary, the condition on $\bar{\rho}$ is always satisfied, and “very regular” can be replaced by “regular”.

Proof. This is part of [11, Theorem 6.22], the last part being a consequence of the proof using [11, Proposition 6.6]. □

As in the case of GL_n, we would like to work with “fixed determinant” lifts, where now by “determinant” we mean the map $\det : G \to G/G^{\mathrm{der}}$, with G^{der} being the derived group of G. This induces a map $\det : \mathfrak{X}_{\bar{\rho}}^{\square} \to \mathfrak{X}_{\det(\bar{\rho})}^{\square}$ which sends any lift of $\bar{\rho}$ to its determinant, inducing a morphism $\det_1 := \det \circ \pi_1 : X_{\mathrm{tri}}^{\square}(\bar{\rho}) \to \mathfrak{X}_{\det(\bar{\rho})}^{\square}$. Let n be the order of the kernel of the isogeny $Z(G) \to G/G^{\mathrm{der}}$, where $Z(G)$ is the connected centre of G.

Lemma 2.7. *Assume G has no factors of type* G_2, F_4 *and* E_8, *and that $p \nmid n$. Let $\bar{\rho} : \Gamma_F \to G(k)$ be a continuous homomorphism and let $\rho : \Gamma_F \to G(\mathcal{O})$ be a lift of $\bar{\rho}$ corresponding to a point $(x, \delta) \in X_{\mathrm{tri}}^{\square}(\bar{\rho})(E)$. Then, after replacing E by a finite extension if necessary, the following holds*:

(1) *There is a quotient R of $R^{\square,\mathrm{univ}}$ by an ideal invariant under the action of $\widehat{G}(\mathcal{O})$ which is reduced and flat over $\mathcal{O}$.*

(2) *There is a compact E-adic submanifold U of* $\mathrm{Spec}(R[1/\varpi])(E)$ *of dimension* $d = \dim(G^{\mathrm{der}}) + [F : \mathbb{Q}_p](\dim(B) - \dim(Z(G)))$ *such that all the points of U correspond to very regular trianguline lifts ρ' of $\bar{\rho}$ with* $\det(\rho) = \det(\rho')$.

(3) *Given any integer* $N > 0$, U *can be chosen such that all* ρ' *corresponding to points of* U *are congruent to* ρ *modulo* ϖ^N.

(4) *There is a point* $v \in U$ *and an integer* $m > 0$ *such that the* $\widehat{G}^{(m)}(\mathcal{O})$*-orbit of* v *is contained in* U.

Proof. This is proved in essentially the same way as Lemma 2.4 using Lemma 2.6 instead of Lemma 2.3. □

3. Trianguline lifting theorems

In this section we formulate a general lifting theorem for representations of global fields.

3A. ***Selmer and relative Selmer groups.*** Let F be a global field, and let $E, \mathcal{O}, \varpi$ be as before. Let G be a reductive group scheme over $\mathcal{O}$, G^{der} the derived group of its identity component and $\mathfrak{g}^{\mathrm{der}}$ the Lie algebra of G^{der}. For some $n \geq 1$ let $\rho_n : \Gamma_F \to G(\mathcal{O}/\varpi^n)$ be a continuous homomorphism. Let $\mathcal{S}$ be a finite set of primes containing all primes at which ρ_n is ramified and also all primes dividing p. We first recall some definitions and results from [13].

In what follows, when we have an integer $0 < r < n$, we will write ρ_r for the reduction $\rho_n \pmod{\varpi^r}$. For each prime $v \in \mathcal{S}$ we assume that for $0 < r \leq n$ we have subgroups $Z_{r,v} \subset Z^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ such that

- each $Z_{r,v}$ contains the group of boundaries $B^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$;
- as r varies, the inclusion and reduction maps induce short exact sequences
$$0 \to Z_{a,v} \to Z_{a+b,v} \to Z_{b,v} \to 0$$
as in Proposition 2.2.

We let $L_{r,v} \subset H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ be the image of $Z_{r,v}$, and $L_{r,v}^{\perp} \subset H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*)$ be the annihilator of $L_{r,v}$ under the local duality pairing. Let $\mathcal{L}_r = \{L_{r,v}\}_{v \in \mathcal{S}}$, $0 < r \leq n$, and similarly define $\mathcal{L}_r^{\perp}$. The Selmer group $H^1_{\mathcal{L}_r}(\Gamma_{F,\mathcal{S}}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ is defined to be
$$\ker\left(H^1(\Gamma_{F,\mathcal{S}}, \rho_r(\mathfrak{g}^{\mathrm{der}})) \to \bigoplus_{v \in \mathcal{S}} \frac{H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))}{L_{r,v}} \right)$$
and the dual Selmer group $H^1_{\mathcal{L}_r^{\perp}}(\Gamma_{F,\mathcal{S}}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*)$ is defined analogously.

The definition below is a variant of the definition of *balanced* in [13, Definition 6.2], which is the condition when $a = 0$.

Definition 3.1. We say that the local conditions $\mathcal{L}_r$, for $0 < r \leq n$, are *semibalanced* if there exists a nonnegative integer a such that
$$|H^1_{\mathcal{L}_r}(\Gamma_{F,\mathcal{S}}, \rho_r(\mathfrak{g}^{\mathrm{der}}))| = |\mathcal{O}/\varpi^r|^a \cdot |H^1_{\mathcal{L}_r^{\perp}}(\Gamma_{F,\mathcal{S}}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*)|$$

for all $0 < r \le n$.

The basic objects that we need to control in order to prove lifting results using the methods of [13] and [14] are the relative (dual) Selmer groups of [13, Definition 6.2], so we recall them here:

Definition 3.2. For $0 < r \le n$, we define the r-th relative Selmer group to be

$$\overline{H^1_{\mathcal{L}_r}(\Gamma_{F,\mathcal{S}}, \rho_r(\mathfrak{g}^{\mathrm{der}}))} := \mathrm{Im}\big(H^1_{\mathcal{L}_r}(\Gamma_{F,\mathcal{S}}, \rho_r(\mathfrak{g}^{\mathrm{der}})) \to H^1_{\mathcal{L}_1}(\Gamma_{F,\mathcal{S}}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}}))\big)$$

and the r-th relative dual Selmer group to be

$$\overline{H^1_{\mathcal{L}_r^\perp}(\Gamma_{F,\mathcal{S}}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*)} := \mathrm{Im}\big(H^1_{\mathcal{L}_r^\perp}(\Gamma_{F,\mathcal{S}}, \rho_r(\mathfrak{g}^{\mathrm{der}})^*) \to H^1_{\mathcal{L}_1^\perp}(\Gamma_{F,\mathcal{S}}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)\big).$$

3B. *Adequate cocycles and semibalancedness.* We continue with the notation from Section 3A, but now assume that F is a number field. We set $\bar{\rho} = \rho_1$ and for each infinite place v of F we let $c(\bar{\rho}, v) = \dim(\mathfrak{g}^{\mathrm{der}})^{\Gamma_{F_v}}$.

The definition below is a modification of [13, Definition 1.4]; it is useful for lifting representations which are more general than the odd representations considered there.

Definition 3.3. We say that the $\mathcal{O}$-submodules $L_{r,v} \subset H^1(\Gamma_{F_v}, \rho_r(\mathfrak{g}^{\mathrm{der}}))$ are *adequate* if for each finite place v there exists an integer a_v such that $a_v = 0$ for all but finitely many v,

$$|L_{r,v}| = |\rho_r(\mathfrak{g}^{\mathrm{der}})^{\Gamma_{F_v}}| \cdot |\mathcal{O}/\varpi^r|^{a_v}, \tag{3-1}$$

and $\sum_{v \text{ finite}} a_v \ge \sum_{v|\infty} c(\bar{\rho}, v)$.

The following lemma is a variant of [13, Lemma 6.3].

Lemma 3.4. *If the collection of $\mathcal{O}/\varpi^r$-modules $\mathcal{L}_r$ is adequate and the spaces of invariants $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^{\Gamma_F}$ and $(\bar{\rho}(\mathfrak{g}^{\mathrm{der}})^*)^{\Gamma_F}$ are both zero, then the relative Selmer and dual Selmer groups are also semibalanced in the sense that*

$$\dim\big(\overline{H^1_{\mathcal{L}_n}(\Gamma_{F,\mathcal{S}}, \rho_n(\mathfrak{g}^{\mathrm{der}}))}\big) \ge \dim\big(\overline{H^1_{\mathcal{L}_n^\perp}(\Gamma_{F,\mathcal{S}}, \rho_n(\mathfrak{g}^{\mathrm{der}})^*)}\big).$$

Proof. The hypotheses together with the Greenberg–Wiles formula [10, Theorem 2.18] imply that the Selmer and dual Selmer groups are semibalanced: the integer a of Definition 3.1 is $\sum_{v \text{ finite}} a_v - \sum_{v|\infty} c(\bar{\rho}, v)$. From this the proof of semibalancedness of the relative Selmer and dual Selmer group follows from [13, Lemma 6.1] as in the proof of [13, Lemma 6.3]. □

3C. *The general lifting theorem.* As previously, let F be a global field and let $\bar{\rho} : \Gamma_F \to G(k)$ be a continuous representation, where k is a finite field of characteristic p and G is a split reductive group over the ring of integers $\mathcal{O}$ of a finite extension $E/\mathbb{Q}_p$. The group G need not be connected, and if this is the case we assume that the component group of G has order prime to p and G is a semidirect product of its identity component G^0 and the component group. Let $\widetilde{F}$ be the smallest extension of F such that $\bar{\rho}(\Gamma_{\widetilde{F}}) \subset G^0(k)$ and let $K = F(\bar{\rho}, \mu_p)$.

We let $\bar{\mu} : \Gamma_F \to (G/G^{\text{der}})(k)$ be the map induced by $\bar{\rho}$ and the quotient map $G \to G/G^{\text{der}}$, and we fix a lift $\mu : \Gamma_F \to G(\mathcal{O})$ of $\bar{\mu}$.

Assumption A. (1) $\bar{\rho}$ and μ are unramified outside a finite set of places $\mathcal{S}$ of F containing all places of F over p if F is a number field.

(2) If F is a function field we assume that $p \neq \text{char}(F)$.

(3) $H^1(\text{Gal}(K/F), \bar{\rho}(\mathfrak{g}^{\text{der}})^*) = 0$.

(4) $\bar{\rho}(\mathfrak{g}^{\text{der}})$ and $\bar{\rho}(\mathfrak{g}^{\text{der}})^*$ do not contain the trivial representation as a submodule.

(5) There is no surjection of $\mathbb{F}_p[\Gamma_F]$-modules from $\bar{\rho}(\mathfrak{g}^{\text{der}})$ onto any $\mathbb{F}_p[\Gamma_F]$-module subquotient of $\bar{\rho}(\mathfrak{g}^{\text{der}})^*$.

We also need some local assumptions on $\bar{\rho}$ which we formulate separately.

Assumption B. There exists a finite extension E' of E with ring of integers O' such that for each finite place $v \in \mathcal{S}$ there exists a lift ρ_v of $\bar{\rho}|_{\Gamma_{F_v}}$ to a continuous homomorphism $\Gamma_{F_v} \to G(\mathcal{O}')$ with fixed determinant $\mu|_{\Gamma_{F_v}}$. Furthermore, ρ_v is an element of an E-adic manifold U_v contained in $\text{Spec}(R^{\square,\text{univ}}_{\bar{\rho}|_{\Gamma_{F_v}}})(\mathcal{O}')$ $(= \text{Spec}(R^{\square,\text{univ}}_{\bar{\rho}|_{\Gamma_{F_v}}}[1/\varpi])(E'))$, with fixed determinant as above, and which is invariant under conjugation by $\widehat{G^{\text{der}}}(O')$ and of dimension d_v, with $d_v = \dim_k(\mathfrak{g}^{\text{der}})$, for each finite $v \nmid p$. Finally,

$$\sum_{v\ \text{finite}} (d_v - \dim(\mathfrak{g}^{\text{der}})) \geq \sum_{v|\infty} c(\bar{\rho}, v). \tag{3-2}$$

Remark 3.5. (1) We usually choose U_v to lie in the $\mathcal{O}'$-points of an irreducible component of $\text{Spec}(R)$, where R is a $G(\mathcal{O}')$-equivariant quotient of the universal local lifting ring of $\bar{\rho}|_{\Gamma_{F_v}}$.

(2) The question of the existence of U_v is most delicate for primes $v \mid p$, but for arbitrary groups G it is still unknown whether a lift ρ_v always exists even for other (ramified) primes. In most applications, if $v \nmid p$ then we choose U_v so that $d_v = \dim_k(\mathfrak{g}^{\text{der}})$ (see [13, Proposition 4.7]; this is the maximal possible), in which case $a_v = 0$.

The following theorem is an easy consequence of the results of [13] and [14].

Theorem 3.6. *Let $\bar{\rho} : \Gamma_F \to G(k)$ satisfy Assumptions A and B and assume that $p \gg_G 0$. Then there exists a finite set of places $\mathcal{S}' \supset \mathcal{S}$ such that $\bar{\rho}$ lifts to a continuous representation $\rho : \Gamma_{F,\mathcal{S}'} \to G(\mathcal{O}')$ such that*

(1) *the image $\rho(\Gamma_{F,\mathcal{S}'})$ intersects $G^{\text{der}}(\mathcal{O}')$ in an open subgroup*;

(2) $\rho|_{\Gamma_{F_v}}$ *is an element of* U_v *for all* $v \in \mathcal{S}$ *and* ρ *can be chosen so that* $\rho|_{\Gamma_{F_v}}$ *is, modulo* $\widehat{G^{\mathrm{der}}}(\mathcal{O}')$*-conjugation, congruent to* ρ_v *modulo any prespecified power of* ϖ.

Proof. Assumption A and the existence of ρ_v in Assumption B imply that all the arguments of Sections 3 and 4 of [14] go through without any changes to construct lifts ρ_n as in Theorem 4.4 of loc. cit. The point is that neither the archimedean primes, nor the dimensions d_v, play any role in the proof of this theorem. To complete the proof we must explain how to carry out the "relative deformation theory" arguments of [14, Section 6].

To do this, we apply Proposition 2.2 to the sets U_v for each $v \in \mathcal{S}$ to produce cocycles $Z_{r,v}$ for all $r \leq n$. The condition on d_v in Assumption B and (5) of Proposition 2.2 implies that the $L_{r,v}$ constructed from the $Z_{r,v}$ are adequate in the sense of Definition 3.3. Then by Lemma 3.4, the relative Selmer and dual Selmer groups are semibalanced. The first part of Assumption 5.1 of [14] is part of (4) of Assumption A above, and this together with the existence of local lifts giving rise to semibalanced relative Selmer and dual Selmer groups is exactly what is needed for the proof of [14, Theorem 5.1] to go through to produce a lift ρ satisfying all the claimed properties. □

3D. *Triangulline lifts of* GL_n*-odd representations of number fields.* Before proving our main theorem we need the following:

Lemma 3.7. *If* $\bar{\rho} : \Gamma_F \to \mathrm{GL}_n(k)$ *is a* GL_n*-odd representation, then for every real place* v *of* F,

$$c(\bar{\rho}, v) = \begin{cases} 2m^2 - 1 & \text{if } n = 2m \text{ is even,} \\ 2m^2 + 2m & \text{if } n = 2m+1 \text{ is odd.} \end{cases} \tag{3-3}$$

Proof. This is an elementary computation. □

We now restate and prove our main theorem.

Theorem 3.8. *Let* F *be a number field and* $\bar{\rho} : \Gamma_{F,\mathcal{S}} \to \mathrm{GL}_n(k)$ *a* GL_n*-odd representation. If* $p \gg_n 0$ *and* $\bar{\rho}$ *satisfies all the conditions in Assumption A then there exists a finite set of places* $\mathcal{S}' \supset \mathcal{S}$ *and a finite extension* E' *of* E *with ring of integers* $\mathcal{O}'$ *such that* $\bar{\rho}$ *lifts to a* GL_n*-odd representation* $\rho : \Gamma_{F,\mathcal{S}'} \to \mathrm{GL}_n(\mathcal{O}')$ *which is regular trianguline at all primes of* F *above* p. *Furthermore, one can also ensure that* $\rho(\Gamma_{F,\mathcal{S}'})$ *contains an open subgroup of* $\mathrm{SL}_n(\mathcal{O}')$.

Proof. The proof is an application of Theorem 3.6.

We begin by choosing any lift μ of $\bar{\mu}$. The conditions of Assumption A hold tautologically, so we only need to show that those of Assumption B also hold.

Since the group G in this case is GL_n, for $v \nmid p$ this holds by [8, §2.4.4] with $d_v = \dim_k(\mathfrak{g}^{\mathrm{der}})$. The main change that we need to make is in the local arguments for primes $v \mid p$. For each such prime, we apply Lemma 2.4 (with the F there being F_v) to obtain a set U_v of trianguline lifts of $\bar{\rho}|_{\Gamma_{F_v}}$. For these U_v we have by (2) of Lemma 2.4 $d_v = n^2 - 1 + [F_v : \mathbb{Q}_p]\big(\frac{1}{2}n(n+1) - 1\big)$.

Since $\sum_{v|p}[F_v : \mathbb{Q}_p] = [F : \mathbb{Q}]$, we get

$$\sum_{v|p} d_v = \sum_{v|p}(n^2 - 1) + [F : \mathbb{Q}]\big(\tfrac{1}{2}n(n+1) - 1\big).$$

On the other hand $\sum_{v|\infty} c(\bar{\rho}, v) = \sum_{v \text{ real}} c(\bar{\rho}, v) + \sum_{v \text{ complex}} c(\bar{\rho}, v)$. Using Lemma 3.7 we see that if v is a real place then $c(\bar{\rho}, v) \le \big(\frac{1}{2}n(n+1) - 1\big)$ and if v is a complex place then $c(\bar{\rho}, v) = n^2 - 1 \le 2\big(\frac{1}{2}n(n+1) - 1\big)$. Using this it follows that (3-2) holds, so the proof is complete. □

Remark 3.9. It seems reasonable to expect that Assumption A and the assumption that $p \gg_n 0$ are superfluous and trianguline lifts as in Theorem 3.8 exist as long as $\bar{\rho}$ is GL_n-odd.

Remark 3.10. If $\bar{\rho}|_{\Gamma_{F_v}}$ is upper triangular for a prime v above p, using Remark 2.5 one may arrange that the same holds for $\rho|_{\Gamma_{F_v}}$.

Remark 3.11. The proof of Theorem 3.8 actually gives a family of (fixed-determinant) lifts of $\bar{\rho}$, unramified outside a fixed finite set $\mathcal{S}'$ and trianguline at all primes above p, parametrised by a $\big([F : \mathbb{Q}]\big(\frac{1}{2}n(n+1) - 1\big) - \sum_{v|\infty} c(\bar{\rho}, v)\big)$-dimensional E-adic manifold.

Corollary 3.12. *Let F be an arbitrary number field and $\bar{\rho} : \Gamma_{F,\mathcal{S}} \to \mathrm{GL}_n(k)$ a GL_n-odd representation. If $p \gg_n 0$ and $\bar{\rho}|_{\Gamma_{F(\zeta_p)}}$ is absolutely irreducible, then there exists a finite set of places $\mathcal{S}' \supset \mathcal{S}$ and a finite extension E' of E with ring of integers $\mathcal{O}'$ such that $\bar{\rho}$ lifts to a GL_n-odd representation $\rho : \Gamma_{F,\mathcal{S}'} \to \mathrm{GL}_n(\mathcal{O}')$ which is regular trianguline at all primes of F above p. Furthermore, one can also ensure that $\rho(\Gamma_F)$ contains an open subgroup of $\mathrm{SL}_n(\mathcal{O}')$.*

Proof. This follows immediately from Theorem 3.8 once we note that the irreducibility assumption implies that (3), (4) and (5) of Assumption A automatically hold; see [13, Corollary A.7] for a more general statement that holds for arbitrary G. □

Remark 3.13. The recent results of [3] show that the second bulleted assumption in [13, Theorem 6.21] is always satisfied for $G = \mathrm{GL}_n$. Consequently, if F is an arbitrary number field, and we drop the GL_n-odd assumption in Theorem 3.8, we can construct characteristic zero lifts which are unramified outside a finite set of primes, but without any control at primes dividing p.

Remark 3.14. As far as we are aware, it is not known whether or not every GL_n-odd representation has a geometric lift, even when F is $\mathbb{Q}$ — the examples of Calegari mentioned in the introduction are not GL_n-odd — and this question is closely related to the question whether torsion cohomology classes of arithmetic subgroups of

GL_n lift to characteristic zero after passing to a finite index subgroup. However, Conjecture 1.2.3 of [15] combined with Theorem 3.8 implies that any irreducible representation $\bar{\rho}$ satisfying the hypotheses of the theorem is the reduction of the Galois representation associated to the cohomology of an arithmetic subgroup of GL_n with coefficients in a suitable infinite dimensional module (see [16, §5]) which is a $\mathbb{Q}_p$-vector space.

3E. *Trianguline lifts for representations into connected reductive groups.* Using Lemma 2.7 (and the results of [13] and [14]) we now prove a generalisation of Theorem 3.8 for representations valued in general split connected reductive groups G satisfying the assumptions of the lemma. Before we can do this, we need an analogue of the definition of GL_n-odd used for GL_n. The definition below, motivated by the numerics of the trianguline local condition, is weaker than GL_n-odd when applied to GL_n, but it still suffices to prove our lifting result.

Definition 3.15. (1) Let K be any field and G a split reductive group (not necessarily connected) over K. An involution $c \in G(K)$ is said to be *t-odd* if $\dim(\mathfrak{g}^{\mathrm{der}})^c \leq \dim(B^{\mathrm{der}})$, where B^{der} is a Borel subgroup of G^{der}.

(2) Let F be a number field, K any topological field, and G as above. We say that a continuous representation $\bar{\rho} : \Gamma_F \to G(K)$ is *t-odd* if for every real place v of F and $c_v \in \Gamma_{F_v}$ the corresponding complex conjugation, the involution $\bar{\rho}(c_v) \in G(K)$ is t-odd as in (1).

We then have the following generalisation of Theorem 3.8.

Theorem 3.16. *Let F be a number field, G a connected and split reductive group, and $\bar{\rho} : \Gamma_{F,\mathcal{S}} \to G(k)$ a t-odd continuous representation. Assume that $p \gg_G 0$, $\bar{\rho}$ satisfies the conditions of Assumption A and G has no factor of type* G_2, F_4 *or* E_8. *Fix a lift μ of $\bar{\mu}$ as in Section 3C and also assume that after possibly replacing E by a finite extension, for each finite prime v of F, $\bar{\rho}|_{\Gamma_{F_v}}$ has a lift to $G(\mathcal{O})$ with determinant $\mu|_{\Gamma_{F_v}}$ which is trianguline and very regular if $v \mid p$. Then there exists a finite set of places $\mathcal{S}' \supset \mathcal{S}$ and an extension E' of E with ring of integers $\mathcal{O}'$ such that $\bar{\rho}$ lifts to a t-odd representation $\rho : \Gamma_{F,\mathcal{S}'} \to G(\mathcal{O}')$ with determinant μ which is very regular trianguline at all primes of F above p. Furthermore, one can also ensure that $\rho(\Gamma_{F,\mathcal{S}'})$ contains an open subgroup of $G^{\mathrm{der}}(\mathcal{O}')$.*

Proof. The proof is essentially the same as that of Theorem 3.8 after noting that the definition of t-odd is designed precisely to ensure that the cocycles obtained by applying Proposition 2.2 and Lemma 2.7 for primes $v \mid p$ and [13, Proposition 4.7] for other primes are adequate in the sense of Definition 3.3. This ensures, by Lemma 3.4, that the (relative) Selmer and dual Selmer groups are semibalanced, which is the key condition needed for the arguments of [13] to go through as in the proof of Theorem 3.8. □

It will be clear to the reader that using this theorem one may also prove an analogue of Corollary 3.12, but we do not formulate it explicitly here. The t-odd assumption in the theorem can also be replaced by the slightly weaker condition $\sum_{v|\infty} \dim(\mathfrak{g}^{\mathrm{der}})^{\Gamma_{F_v}} \leq [F:\mathbb{Q}] \dim(B^{\mathrm{der}})$.

Remark 3.17. The existence of lifts for primes $v \nmid p$ has been proved for some groups other than GL_n by Booher [4] and very regular trianguline (in fact, crystalline) lifts for primes $v \mid p$ are known to exist when $\bar{\rho}|_{\Gamma_{F_v}}$ is absolutely irreducible by recent work of Lin [18].

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