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The Expected Embedding Dimension, Type and Weight of a Numerical Semigroup

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ABSTRACT: We study statistical properties of numerical semigroups of genus g as g goes to infinity. More specifically, we answer a question of Delgado and Eliahou by showing that as g goes to infinity, the proportion of numerical semigroups of genus g with embedding dimension close to $g/\sqrt{5}$ approaches 1. We prove similar results for the type and weight of a numerical semigroup of genus g.

Keywords: Embedding dimension of a numerical semigroup; Genus of a numerical semigroup; Numerical semigroup; Type of a numerical semigroup; Weight of a numerical semigroup
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1. Introduction

A *numerical semigroup* S is an additive submonoid of $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ for which $|\mathbb{N}_0 \setminus S| < \infty$. The *set of gaps* of S is $\mathcal{H}(S) = \mathbb{N}_0 \setminus S$. The *Frobenius number* of S, denoted by $F(S)$, is the largest element of $\mathcal{H}(S)$. By convention we define $F(\mathbb{N}_0) = -1$. The *genus* of S, denoted by $g(S)$, is the number of elements of $\mathcal{H}(S)$. The *multiplicity* of S, denoted $m(S)$, is the smallest nonzero element of S. For a general reference on numerical semigroups, see [6].

There has been extensive recent interest in counting numerical semigroups ordered by genus and in studying invariants of 'typical' numerical semigroups of a given genus. Our goal is to prove several statistical results about these numerical semigroups. Every numerical semigroup has a unique minimal generating set, which we denote by $\mathcal{A}(S)$. This set consists of positive elements of S that are not the sum of two positive elements of S. That is, $\mathcal{A}(S) = (S \setminus \{0\}) \setminus ((S \setminus \{0\}) + (S \setminus \{0\}))$. The size of the minimal generating set is called the *embedding dimension* of S, and is denoted by $e(S) = |\mathcal{A}(S)|$. The *pseudo-Frobenius* numbers of S are defined as follows:

$$PF(S) = \{P \in \mathcal{H}(S) \colon \text{for every } s \in S \setminus \{0\} \text{ we have } P + s \in S\}.$$

The number of pseudo-Frobenius numbers is the *type* of S, denoted by $t(S) = |PF(S)|$. The *weight* of S is defined as

$$w(S) = \left(\sum_{x \in \mathcal{H}(S)} x \right) - \frac{g(S)(g(S)+1)}{2}.$$

The motivation for studying the weight of a numerical semigroup comes from the theory of Weierstrass semigroups of algebraic curves. For a reference, see [1, Chapter 1, Appendix E].

There are infinitely many numerical semigroups, so in order to prove statistical statements about their invariants we must order them in some way. Let $\mathcal{S}_g$ denote the set of numerical semigroups of genus g. It is not difficult to show that $\mathcal{S}_g$ is finite. There is extensive literature about how the size of this set varies with g. Let $N(g) = |\mathcal{S}_g|$ be the number of numerical semigroups of genus g. Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio.

Theorem 1. *[16, Theorem 1] There exists a constant $c > 3.78$ such that*

$$\lim_{g \to \infty} \frac{N(g)}{\varphi^g} = c.$$

We denote the uniform probability distribution on $\mathcal{S}_g$ by $\mathbb{P}_g$. If X is a random variable on $\mathcal{S}_g$, we denote its expectation by $\mathbb{E}_g[X]$ and its variance by $\mathrm{Var}_g[X]$.

Question 1. *How are the quantities $m(S), F(S), e(S), t(S)$, and $w(S)$ distributed as we vary through the semigroups in $\mathcal{S}_g$?*

Let $\gamma = \frac{5+\sqrt{5}}{10} = \frac{1}{\sqrt{5}}\varphi$. Kaplan and Ye show that most numerical semigroups $S \in \mathcal{S}_g$ have multiplicity close to γg and Frobenius number close to twice the multiplicity [9]. The main goal of this paper is to prove analogous statements for $e(S), t(S)$, and $w(S)$.

Theorem 2. *[9, Proposition 16 and Theorem 4] For fixed $\epsilon > 0$, we have*

1.
$$\lim_{g\to\infty} \mathbb{P}_g[|m(S) - \gamma g| < \epsilon g] = 1, \text{ and}$$

2.
$$\lim_{g\to\infty} \mathbb{P}_g[|F(S) - 2m(S)| < \epsilon g] = 1.$$

Theorem 2 implies that for fixed $\epsilon > 0$,

$$\lim_{g\to\infty} \mathbb{P}_g[|F(S) - 2\gamma g| < \epsilon g] = 1.$$

Singhal strengthens Theorem 2(2) in [13].

Proposition 3. *[13, Theorem 8] Given $\epsilon > 0$, there is an $M(\epsilon) > 0$ such that for all $g > 0$ we have*

$$\mathbb{P}_g[|F(S) - 2m(S)| > M(\epsilon)] < \epsilon.$$

Recently, Zhu has proven a stronger result of this kind but we will not need it for the applications in this paper [18, Theorem 6.1]. We say that *a typical numerical semigroup has property $\mathcal{P}$* if $\lim_{g\to\infty} \mathbb{P}_g[S \text{ has property } \mathcal{P}] = 1$. For example, Theorem 2(2) says that for any $\epsilon > 0$, a typical numerical semigroup has $|F(S) - 2m(S)| < \epsilon g$.

1.1 The Distribution of Invariants of Numerical Semigroups in $\mathcal{S}_g$

We show that most numerical semigroups of genus g have embedding dimension close to $\frac{1}{\sqrt{5}}g$, type close to $(1-\gamma)g$, and weight close to $\frac{1}{10\varphi}g^2$.

Theorem 4. *Fix $\epsilon > 0$. We have*

1.
$$\lim_{g\to\infty} \mathbb{P}_g\left[\left|e(S) - \frac{1}{\sqrt{5}}g\right| < \epsilon g\right] = 1, \text{ and}$$

2.
$$\lim_{g\to\infty} \mathbb{P}_g\left[|t(S) - (1-\gamma)g| < \epsilon g\right] = 1.$$

As a direct consequence, we can compute the expected values of $e(S)$ and $t(S)$ taken over semigroups in $\mathcal{S}_g$.

Corollary 5. *We have*

1.
$$\lim_{g\to\infty} \frac{1}{g}\mathbb{E}_g[e(S)] = \frac{1}{\sqrt{5}}, \text{ and}$$

2.
$$\lim_{g\to\infty} \frac{1}{g}\mathbb{E}_g[t(S)] = 1 - \gamma.$$

In [9, Theorem 22], Kaplan and Ye use the Hardy-Ramanujan asymptotic formula for the number of partitions of n to prove that with high probability, a random $S \in \mathcal{S}_g$ satisfies

$$0.03519g^2 < w(S) < 0.0885g^2.$$

They state that it would be interesting to try to improve the constants in these inequalities. We achieve this goal by proving a result for the distribution of $w(S)$ taken over semigroups in $\mathcal{S}_g$ analogous to Theorem 4. The proof strategy for this result is different than the strategy for Theorem 4. We first compute the expected value and variance of $w(S)$, and then use these results to deduce our result about the distribution.

Theorem 6. *Fix $\epsilon > 0$. We have*

1.
$$\lim_{g\to\infty} \frac{1}{g^2}\mathbb{E}_g[w(S)] = \frac{1}{10\varphi}, \text{ and}$$

2.
$$\lim_{g\to\infty} \mathbb{P}_g\left[\left|w(S) - \frac{1}{10\varphi}g^2\right| < \epsilon g^2\right] = 1.$$

Note that $\frac{1}{10\varphi} \approx 0.0618$.

The main tool to prove this theorem is a result about the independence of the probability that a set of elements is contained in a semigroup in $\mathcal{S}_g$. We expect that this result is of independent interest. In joint work with Bras-Amorós we have applied it to a different statistical problem about semigroups in $\mathcal{S}_g$ [3]. We return to this result in Section 6.

Wilf asked in [15] whether every numerical semigroup S satisfies

$$\frac{1}{e(S)} \leq \frac{F(S)+1-g(S)}{F(S)+1}.$$

This question is now commonly known as *Wilf's conjecture* and has been the subject of extensive work in the numerical semigroups community. Sammartano proved that if S if a numerical semigroup with $e(S) \geq m(S)/2$, then S satisfies Wilf's conjecture [12, Theorem 18]. This result was improved by Eliahou, who showed that if S satisfies $e(S) \geq m(S)/3$ then S satisfies Wilf's conjecture [5, Theorem 1]. Eliahou explains that Delgado has observed that more than 99.999% of the numerical semigroups with genus at most 45 have $e(S) \geq m(S)/3$. Delgado and Eliahou ask whether the proportion of such semigroups goes to 1 as g goes to infinity [4, Section 4]. A direct consequence of Theorem 4(1) is that not only do most semigroups satisfy the condition in Eliahou's result but most also satisfy the stronger condition in Sammartano's result.

Corollary 7. *We have*
$$\lim_{g\to\infty} \mathbb{P}_g[e(S) \geq m(S)/2] = 1.$$

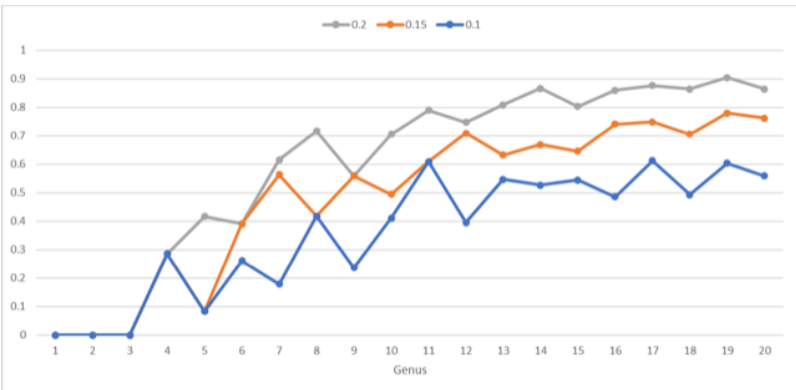

Figure 1: Proportion of $S \in \mathcal{S}_g$ with $|e(S) - \frac{1}{\sqrt{5}}g| < \epsilon g$. Plotted for $\epsilon = 0.2$, $\epsilon = 0.15$ and $\epsilon = 0.1$.

In order to prove Theorem 4(1) we partition the minimal generating set $\mathcal{A}(S)$ into two parts. For every numerical semigroup S we have
$$[m(S), 2m(S)-1] \cap S \subseteq \mathcal{A}(S).$$
Let $e_1(S) = \#\left([m(S), 2m(S)-1] \cap S\right)$ and $e_2(S) = e(S) - e_1(S)$. We prove separate results about the typical size of $e_1(S)$ and $e_2(S)$ for a semigroup in $\mathcal{S}_g$. Combining these estimates proves Theorem 4(1).

Proposition 8. *For fixed $\epsilon > 0$, we have*

1.
$$\lim_{g\to\infty} \mathbb{P}_g\left[\left|e_1(S) - \frac{1}{\sqrt{5}}g\right| < \epsilon g\right] = 1, \text{ and}$$

2.
$$\lim_{g\to\infty} \mathbb{P}_g\left[e_2(S) < \epsilon g\right] = 1.$$

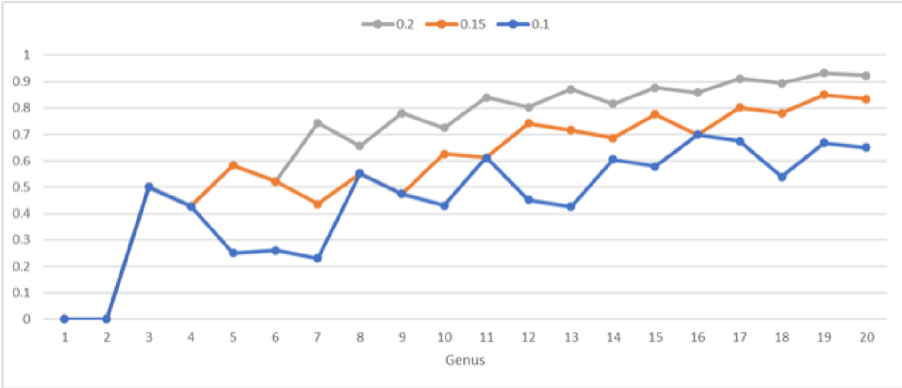

Figure 2: Proportion of $S \in \mathcal{S}_g$ with $|t(S) - (1-\gamma)g| < \epsilon g$. Plotted for $\epsilon = 0.2$, $\epsilon = 0.15$ and $\epsilon = 0.1$.

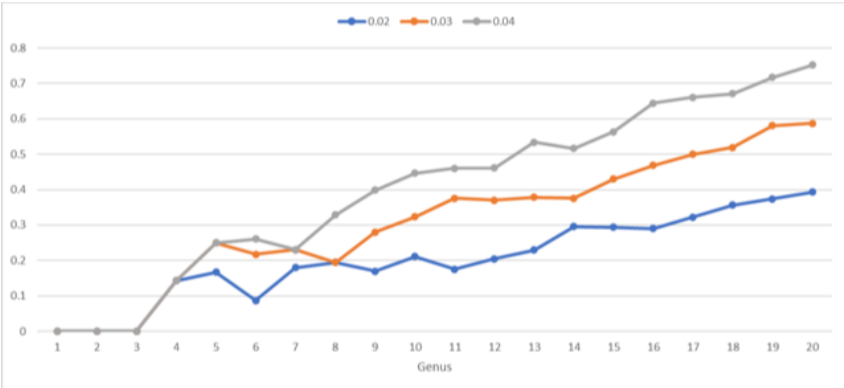

Figure 3: Proportion of $S \in \mathcal{S}_g$ with $\left|w(S) - \frac{1}{10\varphi}g^2\right| < \epsilon g^2$. Plotted for $\epsilon = 0.02$, $\epsilon = 0.03$ and $\epsilon = 0.04$.

Similarly, in order to prove Theorem 4(2) we partition $PF(S)$ into two parts. For any numerical semigroup S with Frobenius number F and multiplicity m, we have

$$\mathcal{H}(S) \cap [F - m + 1, F] \subseteq PF(S).$$

Let $t_1(S) = \#(\mathcal{H}(S) \cap [F - m + 1, F])$ and $t_2(S) = t(S) - t_1(S)$. We separately estimate $t_1(S)$ and $t_2(S)$ for a typical numerical semigroup in $\mathcal{S}_g$. Combining these estimates proves Theorem 4(2).

Proposition 9. *For fixed $\epsilon > 0$, we have*

1. $$\lim_{g \to \infty} \mathbb{P}_g \left[|t_1(S) - (1-\gamma)g| < \epsilon g\right] = 1, \text{ and}$$

2. $$\lim_{g \to \infty} \mathbb{P}_g \left[t_2(S) < \epsilon g\right] = 1.$$

1.2 Counting Numerical Semigroups with Large Invariants

Among numerical semigroups in $\mathcal{S}_g$, we have seen that most have

- multiplicity close to γg,
- Frobenius number close to $2\gamma g$,
- embedding dimension close to $\frac{1}{\sqrt{5}}g$,
- type close to $(1-\gamma)g$, and
- weight close to $\frac{1}{10\varphi}g^2$.

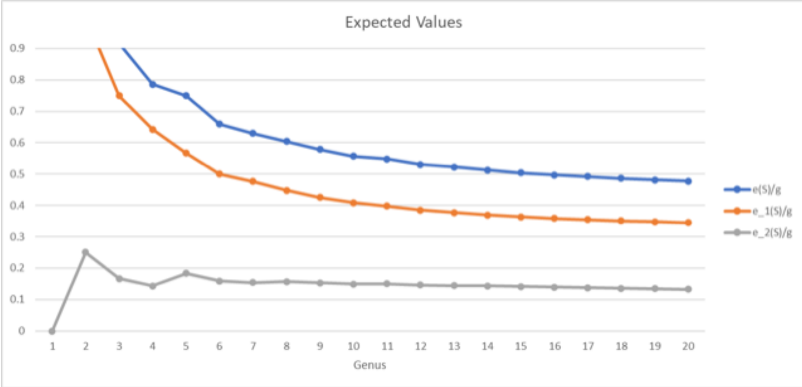

Figure 4: Expected values of $\frac{e(S)}{g}$, $\frac{e_1(S)}{g}$ and $\frac{e_2(S)}{g}$ taken over $S \in \mathcal{S}_g$.

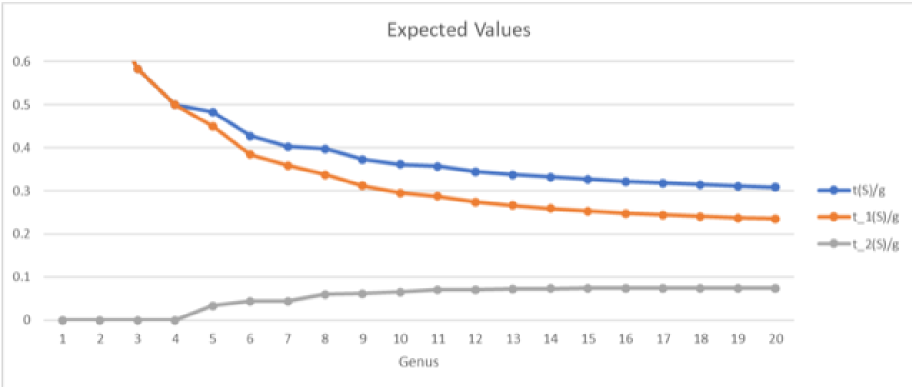

Figure 5: Expected values of $\frac{t(S)}{g}$, $\frac{t_1(S)}{g}$ and $\frac{t_2(S)}{g}$ taken over $S \in \mathcal{S}_g$.

One could also ask about the extreme values of these invariants, and try to count numerical semigroups in $\mathcal{S}_g$ with invariants close to these maximum or minimum values. Basic properties of numerical semigroups imply that for $S \in \mathcal{S}_g$, $m(S) \leq g+1$, $F(S) \leq 2g-1$, $e(S) \leq g+1$, and $t(S) \leq g$.

Numerical semigroups for which $F(S) = 2g(S) - 1$ are called *symmetric*. Backelin has studied the problem of counting symmetric numerical semigroups [2].

Theorem 10. *[2, Proposition 1] For $i \in \{0, 1, 2\}$, the following limit exists and is positive:*

$$\lim_{\substack{g \to \infty \\ g \equiv i \pmod 3}} \frac{\#\{S \in \mathcal{S}_g \mid F(S) = 2g - 1\}}{\sqrt[3]{2}^g}.$$

Kaplan has studied the problem of counting $S \in \mathcal{S}_g$ with $m(S) = g - k$ for fixed k [7].

Theorem 11. *[7, Proposition 13] For each $k \geq 0$, there is a monic polynomial $f_k(x) \in \mathbb{Q}[x]$ of degree $k+1$ such that for $g > 3k$ we have*

$$\#\{S \in \mathcal{S}_g \mid m(S) = g - k\} = \frac{1}{(k+1)!} f_k(g).$$

The following fact is not stated in [7], so we provide a proof here.

Corollary 12. *The polynomials $f_k(x)$ have integer coefficients.*

Proof. Define polynomials $F_k(x) = \frac{1}{(k+1)!} f_k(x+3k+1)$. Therefore $F_k(x) \in \mathbb{Q}[x]$ has degree $k+1$ and $F_k(n) \in \mathbb{Z}$ for all $n \in \mathbb{N}_0$. Fix k and recursively define a_i for $0 \leq i \leq k+1$ as follows. Let $a_0 = F_k(0)$ and $a_i = F_k(i) - \sum_{j=0}^{i-1} a_j \binom{i}{j}$. It is clear that each $a_i \in \mathbb{Z}$. Now $F_k(x)$ and $\sum_{i=0}^{k+1} a_i \binom{x}{i}$ are two polynomials of degree $k+1$ whose values match at $k+2$ points. It follows that

$$F_k(x) = \sum_{i=0}^{k+1} a_i \binom{x}{i},$$

and hence $(k+1)! F_k(x) \in \mathbb{Z}[x]$. Therefore, $f_k(x) \in \mathbb{Z}[x]$. □

Singhal has studied the problem of counting $S \in \mathcal{S}_g$ with $t(S) = g - k$ for fixed k [14].

Theorem 13. *[14, Theorem 1.7] For each $k \geq 0$ there is a positive integer c_k such that for $g \geq 3k - 1$,*

$$\#\{S \in \mathcal{S}_g \mid t(S) = g - k\} = c_k.$$

We prove an analogous result for numerical semigroups in $\mathcal{S}_g$ with embedding dimension close to g.

Theorem 14. *For each $k \geq -1$, there is a polynomial $h_k(t) \in \mathbb{Q}[t]$ of degree $\lceil \frac{k}{2} \rceil$ such that for all $g \geq \frac{9k+7}{2}$ we have*

$$\#\{S \in \mathcal{S}_g \mid e(S) = g - k\} = h_k(g).$$

Moreover, $\lceil \frac{k}{2} \rceil ! h_k(t)$ is a monic polynomial with integer coefficients.

1.3 Outline of the Paper

In Section 2, we review results of Zhao that characterize numerical semigroups in $\mathcal{S}_g$ with $F(S) < 3m(S)$. In Section 3 we prove several results about random variables on $\mathcal{S}_g$ and show how to deduce Corollary 5 from Theorem 4. In Section 4 we prove Proposition 8 and in Section 5 we prove Proposition 9. In Section 6 we prove a result about the probability that a subset of elements is contained in a random element of $\mathcal{S}_g$. We use this result in Section 7 to prove Theorem 6. In Section 8 we prove Theorem 14.

2. Numerical semigroups with $F(S) < 3m(S)$

A major step in Zhai's proof of Theorem 1 is to prove a conjecture of Zhao [17, Conjecture 4.1], which states that

$$\lim_{g \to \infty} \mathbb{P}_g[F(S) < 3m(S)] = 1.$$

We define the following two subsets of $\mathcal{S}_g$:

$$\begin{aligned} \mathcal{B}(g) &= \{S \in \mathcal{S}_g \mid F(S) < 2m(S)\}, \\ \mathcal{C}(g) &= \{S \in \mathcal{S}_g \mid 2m(S) < F(S) < 3m(S)\}. \end{aligned}$$

We further divide up the elements of $\mathcal{B}(g)$ by multiplicity. Let

$$\mathcal{B}(g, m) = \{S \in \mathcal{S}_g \mid m(S) = m, F(S) < 2m\}.$$

Throughout this paper when describing a numerical semigroup by listing its elements, we use the symbol $\rightarrow$ to indicate that it contains all larger elements. For example, the numerical semigroup of genus g containing all positive integers larger than g is $S = \{0, g + 1 \rightarrow\}$.

Proposition 15. *[17, Corollary 2.2] Numerical semigroups in $\mathcal{B}(g, m)$ are in bijection with subsets $B \subseteq \{1, 2, \ldots, m - 1\}$ of size $2m - g - 2$. The bijection is as follows. Given such a subset B, let*

$$S_{m,B} = (m + B) \cup \{0, m, 2m \rightarrow\} \in \mathcal{B}(g, m).$$

Note that $\mathcal{B}(g, m) \neq \emptyset$ if and only if $\frac{g}{2} + 1 \leq m \leq g + 1$. We further divide up the elements of $\mathcal{C}(g)$, first by $F(S) - 2m(S)$ and then by multiplicity. For a fixed positive integer k, we define the following two sets:

$$\begin{aligned} \mathcal{C}(k, g) &= \{S \in \mathcal{C}(g) \mid F(S) = 2m(S) + k\}, \\ \mathcal{C}(m, k, g) &= \{S \in \mathcal{C}(k, g) \mid m(S) = m\}. \end{aligned}$$

Zhao counts numerical semigroups with $2m(S) < F(S) < 3m(S)$ by dividing them up by *type* [17, Section 3.1]. (Note that this use of type is unrelated to how we have used it earlier in the paper.) Let

$$A_k = \{A \subseteq [0, k - 1] \mid 0 \in A, k \notin A + A\}.$$

For $A \in A_k$, we define the following two sets:

$$\begin{aligned} \mathcal{C}(k, A, g) &= \{S \in \mathcal{C}(k, g) \mid m(S) + A = S \cap [m(S), m(S) + k]\}, \\ \mathcal{C}(m, k, A, g) &= \{S \in \mathcal{C}(k, A, g) \mid m(S) = m\}. \end{aligned}$$

Proposition 16. *[17, Proposition 3.3, Corollary 3.4] For $g \geq 3k$, numerical semigroups in $\mathcal{C}(m, k, A, g)$ are in bijection with subsets $B \subseteq [m + k + 1, 2m + k - 1] \setminus (2m + A + A)$ of size $2m - g + k - |A| - |(A + A) \cap [0, k]|$. The bijection is as follows. Given such a subset B, let*

$$S_{m,A,B} = \{0\} \cup (m + A) \cup \big(2m + ((A + A) \cap [0, k])\big) \cup B \cup \{2m + k + 1 \rightarrow\} \in \mathcal{C}(m, k, A, g).$$

3. Random Variables on $\mathcal{S}_g$

In this section, we prove several results about nonnegative random variables on $\mathcal{S}_g$ and show how to deduce Corollary 5 from Theorem 4.

Lemma 17. *Suppose we are given $n \geq 1$ and a sequence of nonnegative random variables X_g on $\mathcal{S}_g$ such that*

$$\lim_{g\to\infty} \frac{1}{g^n}\mathbb{E}_g[X_g] = 0.$$

Then for every $\epsilon > 0$

$$\lim_{g\to\infty} \mathbb{P}_g[X_g(S) < \epsilon g^n] = 1.$$

Proof. Assume for the sake of contradiction that

$$\liminf_{g\to\infty} \mathbb{P}_g[X_g(S) < \epsilon g^n] < 1.$$

Pick $0 < \delta < 1$ such that

$$\liminf_{g\to\infty} \mathbb{P}_g[X_g(S) < \epsilon g^n] < 1 - \delta.$$

This implies that we have a sequence g_i such that $\lim_{i\to\infty} g_i = \infty$ and for all i we have

$$\mathbb{P}_{g_i}[X_{g_i}(S) \geq \epsilon g_i^n] \geq \delta.$$

Therefore, we see that for all i, we have

$$\mathbb{E}_{g_i}[X_{g_i}] \geq \delta\epsilon g_i^n.$$

This contradicts the fact that

$$\lim_{g\to\infty} \frac{1}{g^n}\mathbb{E}_g[X_g] = 0.$$

□

Lemma 18. *Let X_g be a sequence of nonnegative random variables on $\mathcal{S}_g$. Suppose that there is a positive integer n and constant M such that for every g and every $S \in \mathcal{S}_g$, we have $X_g(S) \leq Mg^n$. Suppose further that there is a β such that for every $\epsilon > 0$, we have*

$$\lim_{g\to\infty} \mathbb{P}_g[|X_g(S) - \beta g^n| < \epsilon g^n] = 1.$$

Then,

$$\lim_{g\to\infty} \frac{1}{g^n}\mathbb{E}_g[X_g] = \beta.$$

Proof. Fix $\epsilon_1, \epsilon_2 > 0$. We know that

$$\lim_{g\to\infty} \mathbb{P}_g[|X_g(S) - \beta g^n| < \epsilon_1 g^n] = 1.$$

This means there is $M_1 > 0$ such that for $g > M_1$ we have

$$\mathbb{P}_g[|X_g(S) - \beta g^n| < \epsilon_1 g^n] > 1 - \epsilon_2.$$

This implies that for $g > M_1$, we have

$$\frac{1}{g^n}\mathbb{E}_g[X_g] = \frac{1}{g^n}\frac{1}{N(g)}\sum_{S\in\mathcal{S}_g} X_g(S) \geq \frac{1}{g^n}\frac{1}{N(g)}(1-\epsilon_2)N(g)\left(\beta - \epsilon_1\right)g^n = (1-\epsilon_2)\left(\beta-\epsilon_1\right).$$

For $g > M_1$, we also have

$$\frac{1}{g^n}\mathbb{E}_g[X_g] = \frac{1}{g^n}\frac{1}{N(g)}\sum_{S\in\mathcal{S}_g} X_g(S) \leq \frac{1}{g^n}\frac{1}{N(g)}N(g)\left(\beta + \epsilon_1\right)g^n + \frac{1}{g^n}\frac{1}{N(g)}\epsilon_2 N(g)Mg^n = (\beta+\epsilon_1) + \epsilon_2 M.$$

Since ϵ_1 and ϵ_2 were arbitrary we see that

$$\lim_{g\to\infty} \frac{1}{g^n}\mathbb{E}_g[X_g] = \beta.$$

□

We now show how to apply this result to determine the expected value of certain invariants of numerical semigroups.

Proof that Theorem 4 implies Corollary 5. Since $e(S) \leq g(S)+1$, we see that Theorem 4(1) and Lemma 18 imply Corollary 5(1). Similarly, since $t(S) \leq g(S)$, we see that Theorem 4(2) and Lemma 18 imply Corollary 5(2). □

Proof that Theorem 4 implies Corollary 7. Pick $\epsilon > 0$ such that $\frac{3}{2}\epsilon < \frac{1}{\sqrt{5}} - \frac{\gamma}{2}$ (numerically check that $\gamma < \frac{2}{\sqrt{5}}$). Note that if $|e(S) - \frac{1}{\sqrt{5}}g| < \epsilon g$ and $|\frac{m(S)}{2} - \frac{\gamma}{2}g| < \frac{\epsilon}{2}g$ then $\frac{m(S)}{2} < e(S)$. Therefore, Theorem 2(1) and Theorem 4(1) imply that $\lim_{g\to\infty} \mathbb{P}_g[e(S) \geq m(S)/2] = 1$. □

We apply the following result in Section 7 about the distribution of weights of $S \in \mathcal{S}_g$.

Lemma 19. *Suppose we have a sequence of random variables X_g on $\mathcal{S}_g$. Suppose further that there is a positive integer n and constant β such that*

$$\lim_{g\to\infty} \frac{1}{g^n}\mathbb{E}_g[X_g] = \beta, \quad \lim_{g\to\infty} \frac{1}{g^{2n}}\mathbb{E}_g[X_g^2] = \beta^2.$$

Then for every $\epsilon > 0$, we have

$$\lim_{g\to\infty} \mathbb{P}_g[|X_g(S) - \beta g^n| < \epsilon g^n] = 1.$$

Proof. Fix $\epsilon > 0$. We have $\mathbb{E}_g[X_g] = \beta g^n + o(g^n)$ and $\mathbb{E}_g[X_g^2] = \beta^2 g^{2n} + o(g^{2n})$. Therefore, $\mathrm{Var}_g[X_g] = o(g^{2n})$. This means given $\epsilon_1 > 0$, there is an $M > 0$ such that for all $g > M$, we have

$$\begin{aligned}\mathrm{Var}_g[X_g] &< \epsilon_1 g^{2n}, \text{ and}\\ |\mathbb{E}_g[X_g] - \beta g^n| &< \frac{\epsilon}{2} g^n.\end{aligned}$$

By Chebychev's inequality, we see that for $g > M$,

$$\mathbb{P}_g[|X_g(S) - \beta g^n| > \epsilon g^n] \leq \mathbb{P}_g\left[|X_g(S) - \mathbb{E}[X_g(S)]| > \frac{\epsilon}{2}g^n\right] \leq \frac{4\mathrm{Var}_g[X_g]}{\epsilon^2 g^{2n}} \leq \frac{4}{\epsilon^2}\epsilon_1.$$

We conclude that

$$\lim_{g\to\infty} \mathbb{P}_g[|X_g(S) - \beta g^n| < \epsilon g^n] = 1.$$ □

4. Embedding dimension of a typical numerical semigroup

The goal of this section is to prove Proposition 8. We first prove the part about the typical size of $e_1(S)$.

Proof of Proposition 8(1). For $S \in \mathcal{S}_g$ we have $\#\left(\mathcal{H}(S) \cap [1, m(S)-1]\right) = m(S) - 1$ and

$$\#\left(\mathcal{H}(S) \cap [2m(S)+1, F(S)]\right) \leq \max(0, F(S) - 2m(S)) \leq |F(S) - 2m(S)|.$$

It follows that

$$0 \leq g - (m(S)-1) - \#(\mathcal{H}(S) \cap [m(S), 2m(S)-1]) \leq |F(S) - 2m(S)|.$$

Note that $\#(\mathcal{H}(S) \cap [m(S), 2m(S)-1]) = m(S) - e_1(S)$. This implies that

$$2m(S) - g - 1 \leq e_1(S) \leq 2m(S) - g - 1 + |F(S) - 2m(S)|.$$

Therefore,

$$\left|2m(S) - g - e_1(S)\right| \leq 1 + |F(S) - 2m(S)|.$$

We note that $2\gamma - 1 = \frac{1}{\sqrt{5}}$ and conclude that

$$\begin{aligned}\left|e_1(S) - \frac{1}{\sqrt{5}}g\right| &\leq \left|e_1(S) - (2m(S) - g)\right| + \left|(2m(S) - g) - (2\gamma - 1)g\right|\\ &\leq 1 + |F(S) - 2m(S)| + 2|m(S) - \gamma g|.\end{aligned}$$

We see that for $g > \frac{3}{\epsilon}$, we have

$$\mathbb{P}_g\left[\left|e_1(S) - \frac{1}{\sqrt{5}}g\right| \geq \epsilon g\right] \leq \mathbb{P}_g\left[|F(S) - 2m(S)| \geq \frac{\epsilon}{3}g\right] + \mathbb{P}_g\left[|m(S) - \gamma g| \geq \frac{\epsilon}{6}g\right].$$

The result now follows from Theorem 2. □

Next we bound $e_2(S)$ for numerical semigroups with $F(S) < 2m(S)$. Let F_n denote the n^{th} Fibonacci number, where $F_1 = F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. Recall that

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - (1-\varphi)^n\right).$$

Lemma 20. *For any $g \geq 0$ we have*

$$\sum_{S \in \mathcal{B}(g)} e_2(S) \leq 2F_{g+1}.$$

Proof. In this proof, we see the first instance of a style of argument that will appear several times later in this paper, so we give an outline of the strategy. If $F(x)$ is a polynomial we write $[x^m]\left(F(x)\right)$ for its x^m-coefficient. We show that the following estimate holds.
Claim:

$$\sum_{S \in \mathcal{B}(g,m)} e_2(S) \leq 2 \cdot [x^{g-m}] \left((1+x)^m - x^{\lfloor \frac{m}{2} \rfloor}(x+2)^{\lfloor \frac{m}{2} \rfloor}(1+x)^{m-2\lfloor \frac{m}{2} \rfloor}\right). \tag{1}$$

Assuming this for now, we complete the proof of the lemma. Since $x^{\lfloor \frac{m}{2} \rfloor}(x+2)^{\lfloor \frac{m}{2} \rfloor}(1+x)^{m-2\lfloor \frac{m}{2} \rfloor}$ has nonnegative coefficients, we see that

$$\sum_{S \in \mathcal{B}(g,m)} e_2(S) \leq 2 \cdot [x^{g-m}]\left((1+x)^m\right) = 2\binom{m}{g-m}.$$

For $S \in \mathcal{B}(g)$, it is clear that $m(S) \in [\lceil \frac{g}{2} \rceil + 1, g+1]$. Taking a sum over $m(S)$ gives

$$\sum_{S \in \mathcal{B}(g)} e_2(S) = \sum_{m=\lceil \frac{g}{2} \rceil+1}^{g+1} \sum_{S \in \mathcal{B}(g,m)} e_2(S) \leq 2 \sum_{m=\lceil \frac{g}{2} \rceil+1}^{g+1} \binom{m}{g-m} \leq 2F_{g+1}.$$

We now prove the inequality (1). We have

$$\mathcal{A}(S_{m,B}) = \{m\} \cup \{m+i \mid i \in B\} \cup \{2m+j \mid 1 \leq j \leq m-1, j \notin B, j \notin B+B\},$$

and so

$$e_2(S_{m,B}) = \#\{2m+j \mid 1 \leq j \leq m-1, j \notin B, j \notin B+B\}.$$

For $j \in [1, m-1]$, $2m+j \in \mathcal{A}(S_{m,B})$ if and only if $j \notin B \cup (B+B)$. Let $j_1 = \lfloor \frac{j-1}{2} \rfloor$. A necessary condition for $2m+j \in \mathcal{A}(S_{m,B})$ is that $j \notin B$ and none of $\{1, j-1\}, \{2, j-2\}, \ldots, \{j_1, j-j_1\}$ is a subset of B. By inclusion-exclusion, we see that

$$\#\{S \in \mathcal{B}(g,m) \mid 2m+j \in \mathcal{A}(S)\} \leq \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} \binom{m-2-2l}{2(m-1)-g-2l} = \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} \binom{m-2-2l}{g-m}.$$

Taking a sum over these terms gives

$$\sum_{S \in \mathcal{B}(g,m)} e_2(S) \leq 2 \sum_{j_1=0}^{\lfloor \frac{m}{2} \rfloor - 1} \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} \binom{m-2-2l}{g-m}.$$

Now, notice that

$$\sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} \binom{m-2-2l}{g-m} = [x^{g-m}] \left((1+x)^{m-2} \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} (1+x)^{-2l} \right).$$

Since

$$(1+x)^{m-2} \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} (1+x)^{-2l} = (1+x)^{m-2} \left(1 - \frac{1}{(1+x)^2}\right)^{j_1} = (1+x)^{m-2} \frac{x^{j_1}(x+2)^{j_1}}{(x+1)^{2j_1}},$$

we see that

$$\sum_{j_1=0}^{\lfloor \frac{m}{2} \rfloor - 1} \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} \binom{m-2-2l}{g-m} = [x^{g-m}] \left(\sum_{j_1=0}^{\lfloor \frac{m}{2} \rfloor - 1} (1+x)^{m-2} \frac{x^{j_1}(x+2)^{j_1}}{(x+1)^{2j_1}} \right).$$

Noting that

$$\sum_{j_1=0}^{\lfloor \frac{m}{2} \rfloor - 1} (1+x)^{m-2} \frac{x^{j_1}(x+2)^{j_1}}{(x+1)^{2j_1}} = \frac{(1+x)^m}{(1+x)^2} \frac{\left(1 - \left(\frac{x(x+2)}{(1+x)^2}\right)^{\lfloor \frac{m}{2} \rfloor}\right)}{\left(1 - \frac{x(x+2)}{(1+x)^2}\right)} = (1+x)^m - x^{\lfloor \frac{m}{2} \rfloor}(x+2)^{\lfloor \frac{m}{2} \rfloor}(1+x)^{m-2\lfloor \frac{m}{2} \rfloor}$$

completes the proof. □

We now give a similar, but more complicated, argument to bound $e_2(S)$ for numerical semigroups with $2m(S) < F(S) < 3m(S)$.

Lemma 21. *For any positive integers g and k, we have*

$$\sum_{S \in \mathcal{C}(k,g)} e_2(S) \leq 2F_{g+k}.$$

Proof. A major step in the proof is to prove the following inequality.
Claim:

$$\sum_{S \in \mathcal{C}(m,k,g)} e_2(S) \leq 2 \cdot [x^{g-m}] \left((1+x)^{m+k+1} - x^{\lfloor \frac{m+k+1}{2} \rfloor}(x+2)^{\lfloor \frac{m+k+1}{2} \rfloor}(1+x)^{m+k+1-2\lfloor \frac{m+k+1}{2} \rfloor}\right). \quad (2)$$

Assuming this for now, we complete the proof of the lemma. Since $x^{\lfloor \frac{m+k+1}{2} \rfloor}(x+2)^{\lfloor \frac{m+k+1}{2} \rfloor}(1+x)^{m+k+1-2\lfloor \frac{m+k+1}{2} \rfloor}$ has nonnegative coefficients, we see that

$$\sum_{S \in \mathcal{C}(m,k,g)} e_2(S) \leq 2 \cdot [x^{g-m}] \left((1+x)^{m+k+1}\right) = \binom{m+k-1}{g-m}.$$

Taking a sum over m shows that

$$\sum_{S \in \mathcal{C}(k,g)} e_2(S) \leq 2 \sum_m \binom{m+k-1}{g-m} = 2F_{g+k},$$

completing the proof.

We now prove the inequality (2). Numerical semigroups S in $\mathcal{C}(m,k,g)$ are determined by a subset $B = S \cap [m+1, 2m+k-1]$ of size $2m-g+k-1$. For $j \in [1, m+k]$, let $j_1 = \lfloor \frac{j-1}{2} \rfloor$. A necessary condition for $2m+j \in \mathcal{A}(S)$ is that none of $\{m+1, m+j-1\}, \ldots, \{m+j_1, m+j-j_1\}$ is a subset of B. This means that we can bound the number of $S \in \mathcal{C}(m,k,g)$ with $2m+j \in \mathcal{A}(S)$ by the number of subsets $B \subseteq [m+1, 2m+k-1]$ of size $2m-g+k-1$ for which none of $\{m+1, m+j-1\}, \ldots, \{m+j_1, m+j-j_1\}$ is a subset of B. By inclusion-exclusion, we see that

$$\#\{S \in \mathcal{C}(m,k,g) \mid 2m+j \in \mathcal{A}(S)\} \leq \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} \binom{m+k-1-2l}{2m-g+k-1-2l} = \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} \binom{m+k-1-2l}{g-m}.$$

Taking a sum over these terms gives

$$\sum_{S \in \mathcal{C}(m,k,g)} e_2(S) \leq 2 \sum_{j_1=0}^{\lfloor \frac{m+k-1}{2} \rfloor} \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} \binom{m+k-1-2l}{g-m}.$$

Now, notice that

$$\sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} \binom{m+k-1-2l}{g-m} = [x^{g-m}] \left((1+x)^{m+k-1} \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} (1+x)^{-2l}\right).$$

Since,

$$(1+x)^{m+k-1} \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} (1+x)^{-2l} = (1+x)^{m+k-1} \left(1 - \frac{1}{(1+x)^2}\right)^{j_1} = (1+x)^{m+k-1} \frac{x^{j_1}(x+2)^{j_1}}{(x+1)^{2j_1}},$$

we see that

$$\sum_{j_1=0}^{\lfloor \frac{m+k-1}{2} \rfloor} \sum_{l=0}^{j_1} (-1)^l \binom{j_1}{l} \binom{m+k-1-2l}{g-m} = [x^{g-m}] \left(\sum_{j_1=0}^{\lfloor \frac{m+k-1}{2} \rfloor} (1+x)^{m+k-1} \frac{x^{j_1}(x+2)^{j_1}}{(x+1)^{2j_1}}\right).$$

Noting that

$$\sum_{j_1=0}^{\lfloor\frac{m+k-1}{2}\rfloor}(1+x)^{m+k-1}\frac{x^{j_1}(x+2)^{j_1}}{(x+1)^{2j_1}} = \frac{(1+x)^{m+k+1}}{(1+x)^2}\frac{\left(1-\left(\frac{x(x+2)}{(1+x)^2}\right)^{\lfloor\frac{m+k-1}{2}\rfloor+1}\right)}{\left(1-\frac{x(x+2)}{(1+x)^2}\right)}$$
$$= (1+x)^{m+k+1} - x^{\lfloor\frac{m+k+1}{2}\rfloor}(x+2)^{\lfloor\frac{m+k+1}{2}\rfloor}(1+x)^{m+k+1-2\lfloor\frac{m+k+1}{2}\rfloor}$$

completes the proof of (2). □

Lemma 22. *We have*

$$\lim_{g\to\infty}\frac{1}{g}\mathbb{E}_g[e_2] = 0.$$

Proposition 8(2) follows directly from this result together with Lemma 17. Therefore, proving this result completes the proof of Theorem 4(1).

Proof. Choose $\epsilon > 0$ and consider the $M(\epsilon)$ given by Proposition 3. For any g, applying Lemma 20 and Lemma 21 gives

$$\sum_{S\in\mathcal{S}_g} e_2(S) = \sum_{S\in\mathcal{B}(g)} e_2(S) + \sum_{k=1}^{M(\epsilon)}\sum_{S\in\mathcal{C}(k,g)} e_2(S) + \sum_{\substack{S\in\mathcal{S}_g\\ F(S)>2m(S)+M(\epsilon)}} e_2(S)$$
$$\leq 2F_{g+1} + \sum_{k=1}^{M(\epsilon)} 2F_{g+k} + \sum_{\substack{S\in\mathcal{S}_g\\ F(S)>2m(S)+M(\epsilon)}} (g+1).$$

Noting that $F_n < \frac{\varphi^n+1}{\sqrt{5}}$ and applying Proposition 3 to the last term in this expression gives

$$\sum_{S\in\mathcal{S}_g} e_2(S) < 2\frac{\varphi^{g+1}+1}{\sqrt{5}} + 2\sum_{k=1}^{M(\epsilon)}\frac{\varphi^{g+k}+1}{\sqrt{5}} + (g+1)\epsilon N(g)$$
$$= \varphi^{g+1}\frac{2}{\sqrt{5}}\left(1+\frac{\varphi^{M(\epsilon)}-1}{\varphi-1}\right) + \frac{2}{\sqrt{5}}(M(\epsilon)+1) + (g+1)\epsilon N(g).$$

Therefore,

$$\limsup_{g\to\infty}\frac{1}{g}\frac{1}{N(g)}\sum_{S\in\mathcal{S}_g} e_2(S) \leq \epsilon.$$

Since ϵ was arbitrary, we see that

$$\lim_{g\to\infty}\frac{1}{g}\frac{1}{N(g)}\sum_{S\in\mathcal{S}_g} e_2(S) = 0.$$ □

5. Type of a typical numerical semigroup

The goal of this section is to prove Proposition 9. We first prove the part about the typical size of $t_1(S)$. The arguments in this section are quite similar to the arguments in Section 4.

Proof of Proposition 9(1). Let $S \in \mathcal{S}_g$. We know that

$$\#(\mathcal{H}(S)\cap[1,m(S)-1]) = m(S)-1,$$
$$\#(\mathcal{H}(S)\cap[F(S)-m(S)+1,F(S)]) = t_1(S),$$

and

$$\#(\mathcal{H}(S)\cap[m(S),F(S)-m(S)]|) \leq \max(0,F(S)-2m(S)) \leq |F(S)-2m(S)|.$$

Therefore,

$$|g-(m(S)-1)-t_1(S)| \leq |F(S)-2m(S)|,$$

and

$$|t_1(S)-(1-\gamma)g| \leq |t_1(S)-g+m(S)-1|+1+|m(S)-\gamma g| \leq 1+|F(S)-2m(S)|+|m(S)-\gamma g|.$$

Therefore, for $g > \frac{3}{\epsilon}$, we have

$$\mathbb{P}_g\left[|t_1(S) - (1-\gamma)g| \geq \epsilon g\right] \leq \mathbb{P}_g\left[|F(S) - 2m(S)| \geq \frac{\epsilon}{3}g\right] + \mathbb{P}_g\left[|m(S) - \gamma g| \geq \frac{\epsilon}{3}g\right].$$

The result now follows from Theorem 2. □

Next we bound $t_2(S)$ for numerical semigroups with $F(S) < 2m(S)$.

Lemma 23. *For any $g \geq 0$, we have*

$$\sum_{S \in \mathcal{B}(g)} t_2(S) \leq F_{g+4}.$$

Proof. Suppose $S \in \mathcal{B}(g)$. Since $F(S) - m(S) \leq m(S) - 1$, we see that

$$t_2(S) \leq \#\left(PF(S) \cap [1, m(S) - 1]\right),$$

so it is enough to prove that

$$\sum_{S \in \mathcal{B}(g)} \#\left(PF(S) \cap [1, m(S) - 1]\right) \leq F_{g+4}.$$

We divide the set $PF(S) \cap [1, m(S) - 1]$ into two pieces and bound the size of each. Note that $\lceil \frac{m}{2} \rceil - 1 = \lfloor \frac{m-1}{2} \rfloor$.
Claim:

$$\sum_{S \in \mathcal{B}(g,m)} \#\left(PF(S) \cap \left[\left\lceil \frac{m}{2} \right\rceil, m-1\right]\right) = [x^{2m-g-3}]\left(x^{-1}\left((1+x)^m - (1+x+x^2)^{m-\lceil \frac{m}{2} \rceil}(1+x)^{2\lceil \frac{m}{2} \rceil - m}\right)\right) \tag{3}$$

and

$$\sum_{S \in \mathcal{B}(g,m)} \#\left(PF(S) \cap \left[1, \left\lfloor \frac{m-1}{2} \right\rfloor\right]\right) \leq [x^{2m-g-4}]\left(x^{-1}\left((1+x)^{m-1} - (1+x+x^2)^{\lfloor \frac{m-1}{2} \rfloor}(1+x)^{(m-1)-2\lfloor \frac{m-1}{2} \rfloor}\right)\right). \tag{4}$$

Assuming these results for now, we complete the proof of the lemma. Since both $(1+x+x^2)^{m-\lceil \frac{m}{2} \rceil}(1+x)^{2\lceil \frac{m}{2} \rceil - m}$ and $(1+x+x^2)^{\lfloor \frac{m-1}{2} \rfloor}(1+x)^{(m-1)-2\lfloor \frac{m-1}{2} \rfloor}$ have nonnegative coefficients, we see that the following inequalities hold:

$$\begin{aligned}
\sum_{S \in \mathcal{B}(g,m)} \#\left(PF(S) \cap \left[\left\lceil \frac{m}{2} \right\rceil, m-1\right]\right) &\leq [x^{2m-g-2}]\left((1+x)^m\right) = \binom{m}{2m-g-2} \\
\sum_{S \in \mathcal{B}(g,m)} \#\left(PF(S) \cap \left[1, \left\lfloor \frac{m-1}{2} \right\rfloor\right]\right) &\leq [x^{2m-g-3}]\left((1+x)^{m-1}\right) = \binom{m-1}{2m-g-3}.
\end{aligned}$$

Taking a sum over m shows that

$$\begin{aligned}
\sum_{S \in \mathcal{B}(g)} \#\left(PF(S) \cap [1, m(S) - 1]\right) &\leq \sum_m \binom{m}{2m-g-2} + \sum_m \binom{m-1}{2m-g-3} \\
&= \sum_m \binom{m}{g-m+2} + \sum_m \binom{m-1}{g-m+2} = F_{g+3} + F_{g+2} = F_{g+4},
\end{aligned}$$

completing the proof.

Now we only need to prove (3) and (4). We recall the definition of $S_{m,B}$ from Proposition 15 and note that

$$PF(S_{m,B}) = \{m + j \mid j \in [1, m-1] \setminus B\} \cup \{j \in B \mid \forall i \in [1, m-1-j] \cap B \colon i + j \in B\}.$$

We see that

$$PF(S_{m,B}) \cap [1, m-1] = \{j \in B \mid \forall i \in [1, m-1-j] \cap B \colon i + j \in B\}.$$

For $j \in [\lceil \frac{m}{2} \rceil, m-1]$, we have $[1, m-1-j] \cap (j + [1, m-1-j]) = \emptyset$. For $j \in [\lceil \frac{m}{2} \rceil, m-1]$, by inclusion-exclusion we have

$$\#\{S \in \mathcal{B}(g,m) \mid j \in PF(S)\} = \sum_{l=0}^{m-1-j} (-1)^l \binom{m-1-j}{l} \binom{(m-1)-1-2l}{2(m-1)-g-1-l}$$

$$
\begin{aligned}
&= [x^{2m-g-3}]\left(\sum_{l=0}^{m-1-j}(-1)^l\binom{m-1-j}{l}x^l(1+x)^{m-2-2l}\right)\\
&= [x^{2m-g-3}]\left((1+x)^{m-2}\left(1-\frac{x}{(1+x)^2}\right)^{m-1-j}\right)\\
&= [x^{2m-g-3}]\left((1+x)^{m-2}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{m-1-j}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{S\in\mathcal{B}(g,m)}\#\left(PF(S)\cap\left[\left\lceil\frac{m}{2}\right\rceil,m-1\right]\right) &= \sum_{j=\lceil\frac{m}{2}\rceil}^{m-1}\sum_{l=0}^{m-1-j}(-1)^l\binom{m-1-j}{l}\binom{(m-1)-1-2l}{2(m-1)-g-1-l}\\
&= [x^{2m-g-3}]\left(\sum_{j=\lceil\frac{m}{2}\rceil}^{m-1}(1+x)^{m-2}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{m-1-j}\right).
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\sum_{j=\lceil\frac{m}{2}\rceil}^{m-1}(1+x)^{m-2}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{m-1-j} &= (1+x)^{m-2}\sum_{k=0}^{m-1-\lceil\frac{m}{2}\rceil}\left(\frac{x^2+x+1}{(1+x)^2}\right)^k\\
&= \frac{(1+x)^m}{(1+x)^2}\frac{\left(1-\left(\frac{x^2+x+1}{(1+x)^2}\right)^{m-\lceil\frac{m}{2}\rceil}\right)}{\left(1-\left(\frac{x^2+x+1}{(1+x)^2}\right)\right)}\\
&= \frac{(1+x)^m-(1+x+x^2)^{m-\lceil\frac{m}{2}\rceil}(1+x)^{2\lceil\frac{m}{2}\rceil-m}}{x}
\end{aligned}
$$

completes the proof of (3).

Consider $j\in[1,\lfloor\frac{m-1}{2}\rfloor]$, so $2j\le m-1$. A necessary condition for $j\in PF(S_{m,B})$ is that $j,2j\in B$ and for every $i\in[1,j-1]$, if $i\in B$ then $j+i\in B$. For $j\in[1,\lfloor\frac{m-1}{2}\rfloor]$, by inclusion-exclusion we have

$$
\begin{aligned}
\#\{S\in\mathcal{B}(g,m)\mid j\in PF(S)\} &\le \sum_{l=0}^{j-1}(-1)^l\binom{j-1}{l}\binom{(m-1)-2-2l}{2(m-1)-g-2-l}\\
&= [x^{2m-g-4}]\left(\sum_{l=0}^{j-1}(-1)^l\binom{j-1}{l}x^l(1+x)^{m-3-2l}\right)\\
&= [x^{2m-g-4}]\left((1+x)^{m-3}\left(1-\frac{x}{(1+x)^2}\right)^{j-1}\right)\\
&= [x^{2m-g-4}]\left((1+x)^{m-3}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{j-1}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{S\in\mathcal{B}(g,m)}\#\left(PF(S)\cap\left[1,\left\lfloor\frac{m-1}{2}\right\rfloor\right]\right) &\le \sum_{j=1}^{\lfloor\frac{m-1}{2}\rfloor}\sum_{l=0}^{j-1}(-1)^l\binom{j-1}{l}\binom{(m-1)-2-2l}{2(m-1)-g-2-l}\\
&= [x^{2m-g-4}]\left(\sum_{j=1}^{\lfloor\frac{m-1}{2}\rfloor}(1+x)^{m-3}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{j-1}\right).
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\sum_{j=1}^{\lfloor\frac{m-1}{2}\rfloor}(1+x)^{m-3}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{j-1} &= \frac{(1+x)^{m-1}}{(1+x)^2}\frac{\left(1-\left(\frac{x^2+x+1}{(1+x)^2}\right)^{\lfloor\frac{m-1}{2}\rfloor}\right)}{\left(1-\left(\frac{x^2+x+1}{(1+x)^2}\right)\right)}\\
&= \frac{(1+x)^{m-1}-(1+x+x^2)^{\lfloor\frac{m-1}{2}\rfloor}(1+x)^{(m-1)-2\lfloor\frac{m-1}{2}\rfloor}}{x}
\end{aligned}
$$

completes the proof of (4). □

Next we bound $t_2(S)$ for numerical semigroups with $2m(S) < F(S) < 3m(S)$.

Lemma 24. *For any positive integers g and k, we have*

$$\sum_{S\in\mathcal{C}(k,g)} t_2(S) \le F_{g+k+3}$$

Proof. First recall from Section 2 that for positive integers k and m with $k < m$, we have

$$\mathcal{C}(m,k,g) = \{S \in \mathcal{S}_g \mid m(S) = m,\ F(S) = 2m(S) + k\}.$$

An $S \in \mathcal{C}(m,k,g)$ is determined by $B = S \cap [m+1, 2m+k-1]$ where $|B| = 2m-g+k-1$. Let $j \in [1, m+k-1]$. If $j \in PF(S)$ then:

- $j + m \in B$,
- for every $i \in [1, m+k-1-j]$, if $m+i \in B$ then $m+i+j \in B$.

We bound $\#\{S \in \mathcal{C}(m,k,g) \mid j \in PF(S)\}$ by counting subsets B satisfying these conditions. Our argument is similar to the proof of Lemma 23.

For $S \in \mathcal{C}(m,k,g)$ we have $t_2(S) = \#\left(PF(S) \cap [1, m+k-1]\right)$. Note that $\lceil \frac{m+k}{2} \rceil - 1 = \lfloor \frac{m+k-1}{2} \rfloor$. We divide the elements of $PF(S) \cap [1, m+k-1]$ into two sets and consider each separately.
Claim:

$$\begin{aligned}&\sum_{S\in\mathcal{C}(m,k,g)} \#\left(PF(S) \cap \left[\left\lceil \frac{m+k}{2} \right\rceil, m+k-1\right]\right)\\ &\le [x^{2m-g+k-2}]\left(x^{-1}\left((1+x)^{m+k} - (1+x+x^2)^{m+k-\lceil \frac{m+k}{2}\rceil}(1+x)^{2\lceil \frac{m+k}{2}\rceil - m-k}\right)\right)\end{aligned} \tag{5}$$

and

$$\begin{aligned}&\sum_{S\in\mathcal{C}(m,k,g)} \#\left(PF(S) \cap \left[1, \left\lfloor \frac{m+k-1}{2} \right\rfloor\right]\right)\\ &\le [x^{2m-g+k-3}]\left(x^{-1}\left((1+x)^{m+k-1} - (1+x+x^2)^{\lfloor \frac{m+k-1}{2}\rfloor}(1+x)^{(m+k-1)-2\lfloor \frac{m+k-1}{2}\rfloor}\right)\right).\end{aligned} \tag{6}$$

Assuming these results for now, we complete the proof of the lemma. Since $(1+x+x^2)^{m+k-\lceil \frac{m+k}{2}\rceil}(1+x)^{2\lceil \frac{m+k}{2}\rceil - m-k}$ and $(1+x+x^2)^{\lfloor \frac{m+k-1}{2}\rfloor}(1+x)^{(m+k-1)-2\lfloor \frac{m+k-1}{2}\rfloor}$ have nonnegative coefficients, we see that the following inequalities hold:

$$\begin{aligned}\sum_{S\in\mathcal{C}(m,k,g)} \#\left(PF(S) \cap \left[\left\lceil \frac{m+k}{2} \right\rceil, m+k-1\right]\right) &\le [x^{2m-g+k-1}]\left((1+x)^{m+k}\right) = \binom{m+k}{2m-g+k-1},\\ \sum_{S\in\mathcal{C}(m,k,g)} \#\left(PF(S) \cap \left[1, \left\lfloor \frac{m+k-1}{2} \right\rfloor\right]\right) &\le [x^{2m-g+k-2}]\left((1+x)^{m+k-1}\right) = \binom{m+k-1}{2m-g+k-2}.\end{aligned}$$

Therefore,

$$\sum_{S\in\mathcal{C}(m,k,g)} t_2(S) = \sum_{S\in\mathcal{C}(m,k,g)} \#\left(PF(S) \cap [1, m+k-1]\right) \le \binom{m+k}{2m-g+k-1} + \binom{m+k-1}{2m-g+k-2}.$$

Taking a sum over m shows that

$$\begin{aligned}\sum_{S\in\mathcal{C}(k,g)} t_2(S) &\le \sum_m \binom{m+k}{2m-g+k-1} + \sum_m \binom{m+k-1}{2m-g+k-2}\\ &= \sum_m \binom{m+k}{g-m+1} + \sum_m \binom{m+k-1}{g-m+1} = F_{g+k+2} + F_{g+k+1} = F_{g+k+3}.\end{aligned}$$

This completes the proof.

We now need only prove (5) and (6). For $j \in [\lceil \frac{m+k}{2} \rceil, m+k-1]$, we have

$$[1, m+k-1-j] \cap (j + [1, m+k-1-j]) = \emptyset.$$

For $j \in [\lceil \frac{m+k}{2} \rceil, m+k-1]$, by inclusion-exclusion we have that $\#\{S \in \mathcal{C}(m,k,g) \mid j \in PF(S)\}$ is at most

$$\sum_{l=0}^{m+k-1-j} (-1)^l \binom{m+k-1-j}{l}\binom{(m+k-1)-1-2l}{(2m-g+k-1)-1-l}$$

$$
\begin{aligned}
&= [x^{2m-g+k-2}]\left(\sum_{l=0}^{m+k-1-j}(-1)^l\binom{m+k-1-j}{l}x^l(1+x)^{m+k-2-2l}\right)\\
&= [x^{2m-g+k-2}]\left((1+x)^{m+k-2}\left(1-\frac{x}{(1+x)^2}\right)^{m+k-1-j}\right)\\
&= [x^{2m-g+k-2}]\left((1+x)^{m+k-2}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{m+k-1-j}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{j=\lceil\frac{m+k}{2}\rceil}^{m+k-1}\sum_{l=0}^{m+k-1-j}(-1)^l\binom{m+k-1-j}{l}\binom{(m+k-1)-1-2l}{(2m-g+k-1)-1-l}\\
&= [x^{2m-g+k-2}]\left(\sum_{j=\lceil\frac{m+k}{2}\rceil}^{m+k-1}(1+x)^{m+k-2}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{m+k-1-j}\right).
\end{aligned}
$$

Noting that

$$
\begin{aligned}
&\sum_{j=\lceil\frac{m+k}{2}\rceil}^{m+k-1}(1+x)^{m+k-2}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{m+k-1-j}\\
&= (1+x)^{m+k-2}\sum_{s=0}^{m+k-1-\lceil\frac{m+k}{2}\rceil}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{s} = \frac{(1+x)^{m+k}}{(1+x)^2}\frac{\left(1-\left(\frac{x^2+x+1}{(1+x)^2}\right)^{m+k-\lceil\frac{m+k}{2}\rceil}\right)}{\left(1-\left(\frac{x^2+x+1}{(1+x)^2}\right)\right)}\\
&= \frac{(1+x)^{m+k}-(1+x+x^2)^{m+k-\lceil\frac{m+k}{2}\rceil}(1+x)^{2\lceil\frac{m+k}{2}\rceil-m-k}}{x},
\end{aligned}
$$

completes the proof of (5).

Consider $j\in[1,\lfloor\frac{m+k-1}{2}\rfloor]$, so $2j\le m+k-1$. A necessary condition for $j\in PF(S)$ is that $j+m, 2j+m\in B$ and for every $i\in[1,j-1]$, if $m+i\in B$ then $m+j+i\in B$. For $j\in[1,\lfloor\frac{m+k-1}{2}\rfloor]$, by inclusion-exclusion we have

$$
\begin{aligned}
\#\{S\in\mathcal{C}(m,k,g)\mid j\in PF(S)\} &\le \sum_{l=0}^{j-1}(-1)^l\binom{j-1}{l}\binom{(m+k-1)-2-2l}{(2m-g+k-1)-2-l}\\
&= [x^{2m-g+k-3}]\left(\sum_{l=0}^{j-1}(-1)^l\binom{j-1}{l}x^l(1+x)^{m+k-3-2l}\right)\\
&= [x^{2m-g+k-3}]\left((1+x)^{m+k-3}\left(1-\frac{x}{(1+x)^2}\right)^{j-1}\right)\\
&= [x^{2m-g+k-3}]\left((1+x)^{m+k-3}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{j-1}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{S\in\mathcal{C}(m,k,g)}\#\left(PF(S)\cap\left[1,\left\lfloor\frac{m+k-1}{2}\right\rfloor\right]\right) &\le \sum_{j=1}^{\lfloor\frac{m+k-1}{2}\rfloor}\sum_{l=0}^{j-1}(-1)^l\binom{j-1}{l}\binom{(m+k-1)-2-2l}{(2m-g+k-1)-2-l}\\
&= [x^{2m-g+k-3}]\left(\sum_{j=1}^{\lfloor\frac{m+k-1}{2}\rfloor}(1+x)^{m+k-3}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{j-1}\right).
\end{aligned}
$$

Noting that

$$
\sum_{j=1}^{\lfloor\frac{m+k-1}{2}\rfloor}(1+x)^{m+k-3}\left(\frac{x^2+x+1}{(1+x)^2}\right)^{j-1} = \frac{(1+x)^{m+k-1}}{(1+x)^2}\frac{\left(1-\left(\frac{x^2+x+1}{(1+x)^2}\right)^{\lfloor\frac{m+k-1}{2}\rfloor}\right)}{\left(1-\left(\frac{x^2+x+1}{(1+x)^2}\right)\right)}
$$

$$= \frac{(1+x)^{m+k-1} - (1+x+x^2)^{\lfloor \frac{m+k-1}{2} \rfloor}(1+x)^{(m+k-1)-2\lfloor \frac{m+k-1}{2} \rfloor}}{x}$$

completes the proof of (6). □

Lemma 25. *We have*

$$\lim_{g\to\infty} \frac{1}{g}\mathbb{E}_g[t_2] = 0.$$

Proposition 9(2) follows directly from this result together with Lemma 17. Therefore, proving this result completes the proof of Theorem 4(2).

Proof. Choose $\epsilon > 0$ and consider the $M(\epsilon)$ given by Proposition 3. For any g, applying Lemma 23 and Lemma 24 gives

$$\begin{aligned} \sum_{S\in\mathcal{S}_g} t_2(S) &= \sum_{S\in\mathcal{B}(g)} t_2(S) + \sum_{k=1}^{M(\epsilon)} \sum_{S\in\mathcal{C}(k,g)} t_2(S) + \sum_{\substack{S\in\mathcal{S}_g \\ F(S)>2m(S)+M(\epsilon)}} t_2(S) \\ &\leq 2F_{g+4} + \sum_{k=1}^{M(\epsilon)} 2F_{g+k+3} + \sum_{\substack{S\in\mathcal{S}_g \\ F(S)>2m(S)+M(\epsilon)}} (g+1). \end{aligned}$$

Noting that $F_n < \frac{\varphi^n+1}{\sqrt{5}}$ and applying Proposition 3 shows that this is less than

$$2\frac{\varphi^{g+4}+1}{\sqrt{5}} + 2\sum_{k=1}^{M(\epsilon)} \frac{\varphi^{g+k+3}+1}{\sqrt{5}} + (g+1)\epsilon N(g) = \varphi^{g+4}\frac{2}{\sqrt{5}}\left(1+\frac{\varphi^{M(\epsilon)}-1}{\varphi-1}\right) + \frac{2}{\sqrt{5}}(M(\epsilon)+1) + (g+1)\epsilon N(g).$$

Therefore,

$$\limsup_{g\to\infty} \frac{1}{g}\frac{1}{N(g)} \sum_{S\in\mathcal{S}_g} t_2(S) \leq \epsilon.$$

Since ϵ was arbitrary, we see that

$$\lim_{g\to\infty} \frac{1}{g}\frac{1}{N(g)} \sum_{S\in\mathcal{S}_g} t_2(S) = 0.$$ □

6. The probability that a subset is contained in a semigroup of genus g

In this section, we consider a set of positive integers and study the proportion of semigroups in $\mathcal{S}_g$ that contain some of them but not others. Intuitively, integers that are large relative to g should be contained in most, or all, semigroups in $\mathcal{S}_g$, and integers that are small relative to g should be contained in very few semigroups in $\mathcal{S}_g$. It is the integers in 'the middle' where the statistical behavior is not so obvious. Our goal is to make this kind of reasoning precise.

Define a step function $f_1 : [0,2] \setminus \{\gamma, 2\gamma\} \to [0,1]$ by

$$f_1(x) = \begin{cases} 0 & \text{if } 0 \leq x < \gamma \\ \frac{\sqrt{5}-1}{2} & \text{if } \gamma < x < 2\gamma \\ 1 & \text{if } 2\gamma < x \leq 2. \end{cases}$$

We show that $f_1\left(\frac{n}{g}\right)$ is a good approximation for the probability that n is contained in a random element of $\mathcal{S}_g$, and also that these probabilities for various n are independent.

Theorem 26. *Fix $l_1, l_2 \geq 0$ and $\epsilon_1, \epsilon_2 > 0$. There exists an $M(\epsilon_1, \epsilon_2) > 0$ such that for all $g > M(\epsilon_1, \epsilon_2)$ and all pairs of subsets*

$$C, C' \subseteq [1, (\gamma-\epsilon_1)g) \cup ((\gamma+\epsilon_1)g, (2\gamma-\epsilon_1)g) \cup ((2\gamma+\epsilon_1)g, 2g]$$

with $|C| = l_1$, $|C'| = l_2$ and $C \cap C' = \emptyset$, we have

$$\left|\mathbb{P}_g[C \subseteq S \text{ and } C' \cap S = \emptyset] - \prod_{n\in C} f_1\left(\frac{n}{g}\right) \prod_{n\in C'} \left(1 - f_1\left(\frac{n}{g}\right)\right)\right| < \epsilon_2.$$

In the next section, we apply two special cases of this result.

Corollary 27. *Fix* $\epsilon_1, \epsilon_2 > 0$.

1. *There exists an* $M(\epsilon_1, \epsilon_2) > 0$ *such that for all* $g > M(\epsilon_1, \epsilon_2)$ *and*

$$n \in \big[1, (\gamma - \epsilon_1)g\big) \cup \big((\gamma + \epsilon_1)g, (2\gamma - \epsilon_1)g\big) \cup \big((2\gamma + \epsilon_1)g, 2g\big]$$

we have

$$\left|\mathbb{P}_g[n \in S] - f_1\left(\frac{n}{g}\right)\right| < \epsilon_2.$$

2. *There exists an* $M(\epsilon_1, \epsilon_2) > 0$ *such that for all* $g > M(\epsilon_1, \epsilon_2)$ *and*

$$i, j \in \big[1, (\gamma - \epsilon_1)g\big) \cup \big((\gamma + \epsilon_1)g, (2\gamma - \epsilon_1)g\big) \cup \big((2\gamma + \epsilon_1)g, 2g\big]$$

with $i \neq j$*, we have*

$$\left|\mathbb{P}_g[\{i, j\} \cap S = \emptyset] - \left(1 - f_1\left(\frac{i}{g}\right)\right)\left(1 - f_1\left(\frac{j}{g}\right)\right)\right| < \epsilon_2.$$

The second statement may be surprising because in general whether two positive integers i, j are contained in a numerical semigroup S are *not* independent events. For example, if $i \mid j$ then $i \in S$ implies $j \in S$. The point here is that if i is 'too small', $\mathbb{P}_g[i \in S]$ is very small, and if neither of i, j is small, then either they are in S with very high probability, or they are too close together for the condition $i \in S$ to influence $\mathbb{P}_g[j \in S]$ in a meaningful way.

We start the proof of Theorem 26 by recalling a result of Zhai that was conjectured by Zhao [17, Conjecture 2]. Recall that the set A_k is defined in Section 2.

Theorem 28. *[16, Theorem 3.11] The sum*

$$c = \frac{\varphi}{\sqrt{5}} + \frac{1}{\sqrt{5}} \sum_{k=1}^{\infty} \sum_{A \in A_k} \varphi^{|A| - |(A+A) \cap [0,k]| - k - 1}$$

converges. Moreover,

$$N(g) = c\varphi^g + o(\varphi^g).$$

We next need a technical result about sums of binomial coefficients. The proof is similar to the proof of [9, Proposition 7], so we do not include it here.

Lemma 29. *Fix* $l_1, l_2 \in \mathbb{Z}_{\geq 0}$ *and* $\epsilon > 0$*. As* $g \to \infty$*, we have*

$$\sum_{m=\lceil(\gamma-\epsilon)g\rceil}^{\lfloor(\gamma+\epsilon)g\rfloor} \binom{m - 1 - l_1}{g - m + 1 - l_2} = \frac{1}{\sqrt{5}} \varphi^{g - l_1 - l_2 + 1} + o(\varphi^g).$$

A main step in the proof of Theorem 26 is to study elements between the expected size of the multiplicity and the expected size of the Frobenius number of a random $S \in \mathcal{S}_g$.

Lemma 30. *Fix* $l \geq 0$ *and* $\epsilon_1, \epsilon_2 > 0$*. There is an* $M(\epsilon_1, \epsilon_2) > 0$ *such that for all* $g > M(\epsilon_1, \epsilon_2)$ *and all subsets* $C \subseteq \big((\gamma + \epsilon_1)g, (2\gamma - \epsilon_1)g\big)$ *with* $|C| = l$*, we have*

$$\left|\mathbb{P}_g[C \subseteq S] - \varphi^{-l}\right| < \epsilon_2.$$

Proof. We separately prove that for sufficiently large g, we have $\mathbb{P}_g[C \subseteq S] < \varphi^{-l} + \epsilon_2$ and $\mathbb{P}_g[C \subseteq S] > \varphi^{-l} - \epsilon_2$. We first phrase this in terms of the sets $\mathcal{B}(g, m)$ and A_k from Section 2.

Fix $m \in \big((\gamma - \frac{\epsilon_1}{2})g, (\gamma + \frac{\epsilon_1}{2})g\big)$, so $m < \min(C) \leq \max(C) < 2m$. Proposition 15 implies that

$$\#\{S \in \mathcal{B}(g, m) \mid C \subseteq S\} = \binom{m - 1 - l}{2m - g - 2 - l} = \binom{m - 1 - l}{g - m + 1}.$$

Next, for $k \leq \min(\frac{g}{3}, \frac{\epsilon_1}{2}g)$ and $A \in A_k$ we have $m + k < \min(C) \leq \max(C) < 2m$ and

$$\#\{S \in \mathcal{C}(m, k, A, g) \mid C \subseteq S\} = \binom{m - 1 - |(A+A) \cap [0,k]| - l}{2m - g + k - |A| - |(A+A) \cap [0,k]| - l} = \binom{m - 1 - |(A+A) \cap [0,k]| - l}{g - m + |A| - k - 1}.$$

We start with the lower bound for $\mathbb{P}_g[C \subseteq S]$. By Theorem 28, we can choose M sufficiently large so that

$$\frac{\varphi}{\sqrt{5}} + \frac{1}{\sqrt{5}} \sum_{k=1}^{M} \sum_{A \in A_k} \varphi^{|A|-|(A+A)\cap[0,k]|-k-1} > c\left(1 - \frac{\epsilon_2}{2}\varphi^{-l}\right).$$

For $g > \max(3M, \frac{2}{\epsilon_1}M)$, we have

$$\mathbb{P}_g[C \subseteq S] \geq \frac{1}{N(g)} \sum_{m=\lceil(\gamma-\frac{\epsilon_1}{2})g\rceil}^{\lfloor(\gamma+\frac{\epsilon_1}{2})g\rfloor} \binom{m-1-l}{g-m+1} + \frac{1}{N(g)} \sum_{k=1}^{M} \sum_{A \in A_k} \sum_{m=\lceil(\gamma-\frac{\epsilon_1}{2})g\rceil}^{\lfloor(\gamma+\frac{\epsilon_1}{2})g\rfloor} \binom{m-1-|(A+A)\cap[0,k]|-l}{g-m+|A|-k-1}.$$

Applying Lemma 29 with $l_1 = l$ and $l_2 = 0$ shows that

$$\sum_{m=\lceil(\gamma-\frac{\epsilon_1}{2})g\rceil}^{\lfloor(\gamma+\frac{\epsilon_1}{2})g\rfloor} \binom{m-1-l}{g-m+1} = \frac{1}{\sqrt{5}}\varphi^{g+1-l} + o(\varphi^g).$$

Applying Lemma 29 with $l_1 = |(A+A)\cap[0,k]| + l$ and $l_2 = k+2-|A|$ shows that

$$\sum_{m=\lceil(\gamma-\frac{\epsilon_1}{2})g\rceil}^{\lfloor(\gamma+\frac{\epsilon_1}{2})g\rfloor} \binom{m-1-|(A+A)\cap[0,k]|-l}{g-m+|A|-k-1} = \frac{1}{\sqrt{5}}\varphi^{g-l+|A|-|(A+A)\cap[0,k]|-k-1} + o(\varphi^g).$$

We now have

$$\begin{aligned}\mathbb{P}_g[C \subseteq S] &\geq \varphi^{-l}\frac{\varphi^g}{N(g)}\left(\frac{\varphi}{\sqrt{5}} + \frac{1}{\sqrt{5}} \sum_{k=1}^{M} \sum_{A \in A_k} \varphi^{|A|-|(A+A)\cap[0,k]|-k-1}\right) + o(1) \\ &> \varphi^{-l}\left(\frac{1}{c} + o(1)\right) c\left(1 - \frac{\epsilon_2}{2}\varphi^{-l}\right) + o(1) = \varphi^{-l} - \frac{\epsilon_2}{2} + o(1).\end{aligned}$$

Therefore, for sufficiently large g, all subsets $C \subseteq \big((\gamma+\epsilon_1)g, (2\gamma-\epsilon_1)g\big)$ with $|C| = l$ satisfy $\mathbb{P}_g[C \subseteq S] > \varphi^{-l} - \epsilon_2$.

We now turn to the upper bound for $\mathbb{P}_g[C \subseteq S]$. For $g > \max(3M, \frac{2}{\epsilon_1}M)$, we have

$$\begin{aligned}\mathbb{P}_g[C \subseteq S] \leq\ & \frac{1}{N(g)} \sum_{m=\lceil(\gamma-\frac{\epsilon_1}{2})g\rceil}^{\lfloor(\gamma+\frac{\epsilon_1}{2})g\rfloor} \binom{m-1-l}{g-m+1} + \frac{1}{N(g)} \sum_{k=1}^{M} \sum_{A \in A_k} \sum_{m=\lceil(\gamma-\frac{\epsilon_1}{2})g\rceil}^{\lfloor(\gamma+\frac{\epsilon_1}{2})g\rfloor} \binom{m-1-|(A+A)\cap[0,k]|-l}{g-m+|A|-k-1} \\ &+ \mathbb{P}_g\left[|m(S) - \gamma g| > \frac{\epsilon_1}{2}g\right] + \mathbb{P}_g\left[|F(S) - 2m(S)| > M\right].\end{aligned}$$

As we analyzed above, the sum of the first two terms is

$$\varphi^{-l}\frac{\varphi^g}{N(g)}\left(\frac{\varphi}{\sqrt{5}} + \frac{1}{\sqrt{5}} \sum_{k=1}^{M} \sum_{A \in A_k} \varphi^{|A|-|(A+A)\cap[0,k]|-k-1}\right) + o(1) < \varphi^{-l}\left(\frac{1}{c} + o(1)\right) c + o(1) = \varphi^{-l} + o(1).$$

Applying Theorem 2 and Proposition 3 shows that for all sufficiently large g, we have

$$\mathbb{P}_g\left[|m(S) - \gamma g| > \frac{\epsilon_1}{2}g\right] + \mathbb{P}_g\left[|F(S) - 2m(S)| > M\right] \leq \frac{\epsilon_2}{2}.$$

We conclude that for sufficiently large g, all subsets $C \subseteq \big((\gamma+\epsilon_1)g, (2\gamma-\epsilon_1)g\big)$ with $|C| = l$ satisfy $\mathbb{P}_g[C \subseteq S] < \varphi^{-l} + \epsilon_2$. □

We now prove a result about the probability that a set is contained in a random $S \in \mathcal{S}_g$.

Theorem 31. *Fix $l \geq 0$ and $\epsilon_1, \epsilon_2 > 0$. There exists an $M(\epsilon_1, \epsilon_2) > 0$ such that for all $g > M(\epsilon_1, \epsilon_2)$ and all subsets*

$$C \subseteq \big[1, (\gamma-\epsilon_1)g\big) \cup \big((\gamma+\epsilon_1)g, (2\gamma-\epsilon_1)g\big) \cup \big((2\gamma+\epsilon_1)g, 2g\big]$$

of size $|C| = l$, we have

$$\left|\mathbb{P}_g[C \subseteq S] - \prod_{n \in C} f_1\left(\frac{n}{g}\right)\right| < \epsilon_2.$$

Proof. By Theorem 2, there is $M_1(\epsilon_1,\epsilon_2)$ such that $g > M_1(\epsilon_1,\epsilon_2)$ implies that

$$\begin{aligned} \mathbb{P}_g[m(S) \le (\gamma-\epsilon_1)g] &< \epsilon_2, \\ \mathbb{P}_g[F(S) \ge (2\gamma+\epsilon_1)g] &< \frac{\epsilon_2}{2}. \end{aligned}$$

Lemma 30 implies that there is $M_2(\epsilon_1,\epsilon_2) > 0$ such that for all $g > M_2(\epsilon_1,\epsilon_2)$ and all subsets

$$C' \subseteq \big((\gamma+\epsilon_1)g, (2\gamma-\epsilon_1)g\big),$$

of size $|C'| \le l$, we have

$$\left|\mathbb{P}_g[C' \subseteq S] - \varphi^{-|C'|}\right| < \frac{\epsilon_2}{2}.$$

Let $M = \max(M_1(\epsilon_1,\epsilon_2), M_2(\epsilon_1,\epsilon_2))$. Pick $g > M$ and a subset

$$C \subseteq \big[1, (\gamma-\epsilon_1)g\big) \cup \big((\gamma+\epsilon_1)g, (2\gamma-\epsilon_1)g\big) \cup \big((2\gamma+\epsilon_1)g, 2g\big]$$

of size $|C| = l$. We split C into three parts:

$$\begin{aligned} C_1 &= C \cap \big[1, (\gamma-\epsilon_1)g\big), \\ C_2 &= C \cap \big((\gamma+\epsilon_1)g, (2\gamma-\epsilon_1)g\big), \\ C_3 &= C \cap \big((2\gamma+\epsilon_1)g, 2g\big]. \end{aligned}$$

We consider two cases.

- Case 1: $C_1 \ne \emptyset$. Then $\prod_{n\in C} f_1\left(\frac{n}{g}\right) = 0$. Moreover,

  $$\mathbb{P}_g[C \subseteq S] \le \mathbb{P}_g[m(S) < (\gamma-\epsilon_1)g] < \epsilon_2.$$

- Case 2: $C_1 = \emptyset$. Suppose $|C_2| = l_1 \le l$, so $\prod_{n\in C} f_1\left(\frac{n}{g}\right) = \varphi^{-l_1}$. Now

  $$\mathbb{P}_g[C \subseteq S] \le \mathbb{P}_g[C_2 \subseteq S] < \varphi^{-l_1} + \frac{\epsilon_2}{2}.$$

  We have

  $$\mathbb{P}_g[C \subseteq S] \ge \mathbb{P}_g[C_2 \subseteq S] - \mathbb{P}_g[C_3 \not\subseteq S].$$

  Now if $C_3 = \emptyset$, then $\mathbb{P}_g[C_3 \not\subseteq S] = 0$ and if $C_3 \ne \emptyset$, then

  $$\mathbb{P}_g[C_3 \not\subseteq S] \le P_g\Big[F(S) \ge (2\gamma+\epsilon_1)g\Big] < \frac{\epsilon_2}{2}.$$

  Therefore, we have

  $$\mathbb{P}_g[C \subseteq S] > \varphi^{-l_1} - \frac{\epsilon_2}{2} - \frac{\epsilon_2}{2}.$$

  We conclude that

  $$\left|\mathbb{P}_g[C \subseteq S] - \varphi^{-l_1}\right| < \epsilon_2.$$ □

We now have all the tools to prove the main result of this section.

Proof of Theorem 26. By Theorem 31, there exists an $M(\epsilon_1,\epsilon_2) > 0$ such that for all $g > M(\epsilon_1,\epsilon_2)$ and all subsets

$$D \subseteq \big[1, (\gamma-\epsilon_1)g\big) \cup \big((\gamma+\epsilon_1)g, (2\gamma-\epsilon_1)g\big) \cup \big((2\gamma+\epsilon_1)g, 2g\big]$$

of size $|D| \le l_1 + l_2$, we have

$$\left|\mathbb{P}_g[D \subseteq S] - \prod_{n\in D} f_1\left(\frac{n}{g}\right)\right| < \frac{\epsilon_2}{2^{l_2}}.$$

Consider $g > M(\epsilon_1,\epsilon_2)$ and a pair of subsets

$$C, C' \subseteq \big[1, (\gamma-\epsilon_1)g\big) \cup \big((\gamma+\epsilon_1)g, (2\gamma-\epsilon_1)g\big) \cup \big((2\gamma+\epsilon_1)g, 2g\big]$$

with $|C| = l_1,\ |C'| = l_2$, and $C \cap C' = \emptyset$. By inclusion-exclusion, we know that

$$\mathbb{P}_g[C \subseteq S \text{ and } C' \cap S = \emptyset] = \sum_{B\subseteq C'} (-1)^{|B|} \mathbb{P}_g[(C \cup B) \subseteq S].$$

We see that

$$\begin{aligned}
&\left|\mathbb{P}_g[C \subseteq S \text{ and } C' \cap S = \emptyset] - \prod_{n\in C} f_1\left(\frac{n}{g}\right)\prod_{n\in C'}\left(1 - f_1\left(\frac{n}{g}\right)\right)\right| \\
= &\left|\sum_{B\subseteq C'}(-1)^{|B|}\mathbb{P}_g[(C\cup B)\subseteq S] - \sum_{B\subseteq C'}(-1)^{|B|}\prod_{n\in C\cup B} f_1\left(\frac{n}{g}\right)\right| \\
\leq &\sum_{B\subseteq C'}\left|\mathbb{P}_g[(C\cup B)\subseteq S] - \prod_{n\in C\cup B} f_1\left(\frac{n}{g}\right)\right| \\
< &\sum_{B\subseteq C'}\frac{\epsilon_2}{2^{l_2}} = \epsilon_2,
\end{aligned}$$

where in the last step we applied Theorem 31 with $D = C \cup B$. □

7. The weight of a typical numerical semigroup

We first determine the expected value of the weight of a numerical semigroup of genus g.

Proof of Theorem 6(1). Let $\alpha(S) = \sum_{x\in\mathcal{H}(S)} x$. So for $S \in S_g$ we have $w(S) = \alpha(S) - \frac{g(g+1)}{2}$. We will prove that

$$\lim_{g\to\infty}\frac{1}{g^2}\mathbb{E}_g[\alpha(S)] = \frac{9+\sqrt{5}}{20}. \tag{7}$$

Assuming this for now, noting that $\frac{1}{10\varphi} = \frac{9+\sqrt{5}}{20} - \frac{1}{2}$ completes the proof.

Our goal is now to prove (7). Every numerical semigroup S satisfies $F(S) \leq 2g(S) - 1$. By linearity of expectation, we know that

$$\mathbb{E}_g[\alpha(S)] = \sum_{n=1}^{2g-1} n\mathbb{P}_g[n \notin S] = \sum_{n=1}^{2g-1} n\left(1 - f_1\left(\frac{n}{g}\right)\right) + \sum_{n=1}^{2g-1} n\left(f_1\left(\frac{n}{g}\right) - \mathbb{P}_g[n \in S]\right).$$

We have

$$\sum_{n=1}^{2g-1} n\left(1 - f_1\left(\frac{n}{g}\right)\right) = \frac{(\gamma g)^2}{2} + (1-\varphi^{-1})\left(\frac{(2\gamma g)^2}{2} - \frac{(\gamma g)^2}{2}\right) + O(g).$$

Applying Corollary 27(1) gives an $M(\epsilon_1, \epsilon_2)$ such that for all $g > M(\epsilon_1, \epsilon_2)$, we have

$$\left|\sum_{n=1}^{2g-1} n\left(f_1\left(\frac{n}{g}\right) - \mathbb{P}_g[n \in S]\right)\right| \leq 2g\sum_{n=1}^{2g-1}\left|f_1\left(\frac{n}{g}\right) - \mathbb{P}_g[n\in S]\right| \leq (2g)(4\epsilon_1 g + (2-4\epsilon_1)g\epsilon_2).$$

Note that $\frac{\gamma^2}{2} + (1-\varphi^{-1})\frac{3\gamma^2}{2} = \frac{9+\sqrt{5}}{20}$. Since ϵ_1, ϵ_2 were arbitrary, we conclude that

$$\mathbb{E}_g[\alpha(S)] = \frac{9+\sqrt{5}}{20}g^2 + o(g^2),$$

completing the proof of (7). □

Theorem 32. *We have*

$$\lim_{g\to\infty}\frac{1}{g^4}\mathbb{E}_g[w(S)^2] = \left(\frac{1}{10\varphi}\right)^2.$$

Before giving the proof we use this result to complete the proof of Theorem 6(2).

Proof of Theorem 6(2). Apply Lemma 19 together with Theorem 6(1) and Theorem 32. □

Proof of Theorem 32. We will show that

$$\lim_{g\to\infty}\frac{1}{g^4}\mathbb{E}_g[\alpha(S)^2] = \left(\frac{9+\sqrt{5}}{20}\right)^2. \tag{8}$$

Assuming this, for now, we complete the proof. Since $w(S) = \alpha(S) - \frac{g(g+1)}{2}$, we see that

$$\begin{aligned}\lim_{g\to\infty} \frac{1}{g^4}\mathbb{E}_g[w(S)^2] &= \lim_{g\to\infty} \frac{1}{g^4}\mathbb{E}_g[\alpha(S)^2] - \lim_{g\to\infty} \frac{g(g+1)}{g^4}\mathbb{E}_g[\alpha(S)] + \lim_{g\to\infty} \frac{1}{g^4}\frac{g^2(g+1)^2}{4} \\ &= \left(\frac{9+\sqrt{5}}{20}\right)^2 - \frac{9+\sqrt{5}}{20} + \frac{1}{4} = \left(\frac{1}{10\varphi}\right)^2,\end{aligned}$$

where we used the expressions in (7) and (8). Therefore, we only need to prove (8).

For $1 \le i \le 2g-1$, consider the following random variables on $\mathcal{S}_g$:

$$\psi_i(S) = \begin{cases} 1 & \text{if } i \notin S \\ 0 & \text{if } i \in S \end{cases}.$$

Therefore, $\alpha(S) = \sum_{i=1}^{2g-1} i\psi_i(S)$. By linearity of expectation we have

$$\begin{aligned}\mathbb{E}_g[\alpha(S)^2] &= \sum_{i=1}^{2g-1}\sum_{j=1}^{2g-1} ij\mathbb{E}_g[\psi_i\psi_j] = \sum_{i=1}^{2g-1}\sum_{j=1}^{2g-1} ij\mathbb{P}_g[\{i,j\}\cap S = \emptyset] \\ &= \sum_{i=1}^{2g-1}\sum_{j=1}^{2g-1} ij\left(1 - f_1\left(\frac{i}{g}\right)\right)\left(1 - f_1\left(\frac{j}{g}\right)\right) \\ &\quad - \sum_{i=1}^{2g-1}\sum_{j=1}^{2g-1} ij\left(\left(1 - f_1\left(\frac{i}{g}\right)\right)\left(1 - f_1\left(\frac{j}{g}\right)\right) - \mathbb{P}_g[\{i,j\}\cap S = \emptyset]\right).\end{aligned}$$

We estimate the size of each term separately, starting with the first. We have

$$\sum_{i=1}^{2g-1}\sum_{j=1}^{2g-1} ij\left(1 - f_1\left(\frac{i}{g}\right)\right)\left(1 - f_1\left(\frac{j}{g}\right)\right) = \left(\sum_{i=1}^{2g-1} i\left(1 - f_1\left(\frac{i}{g}\right)\right)\right)^2.$$

Note that,

$$\sum_{i=1}^{2g-1} i\left(1 - f_1\left(\frac{i}{g}\right)\right) = \frac{\gamma^2}{2}g^2 + (1-\varphi^{-1})\left(\frac{4\gamma^2}{2} - \frac{\gamma^2}{2}\right)g^2 + O(g).$$

Therefore,

$$\sum_{i=1}^{2g-1}\sum_{j=1}^{2g-1} ij\left(1 - f_1\left(\frac{i}{g}\right)\right)\left(1 - f_1\left(\frac{j}{g}\right)\right) = \left(\frac{9+\sqrt{5}}{20}\right)^2 g^4 + O(g^3).$$

Now we estimate the second term. It is clear that

$$\begin{aligned}&\left|\sum_{i=1}^{2g-1}\sum_{j=1}^{2g-1} ij\left(\left(1 - f_1\left(\frac{i}{g}\right)\right)\left(1 - f_1\left(\frac{j}{g}\right)\right) - \mathbb{P}_g[\{i,j\}\cap S = \emptyset]\right)\right| \\ \le\ &\sum_{i=1}^{2g-1}\sum_{j=1}^{2g-1} ij\left|\left(1 - f_1\left(\frac{i}{g}\right)\right)\left(1 - f_1\left(\frac{j}{g}\right)\right) - \mathbb{P}_g[\{i,j\}\cap S = \emptyset]\right|.\end{aligned}$$

We first consider the terms with $i = j$, and see that

$$\sum_{i=1}^{2g-1} i^2\left|\left(1 - f_1\left(\frac{i}{g}\right)\right)^2 - \mathbb{P}_g[\{i\}\cap S = \emptyset]\right| \le \sum_{i=1}^{2g-1} i^2 = O(g^3).$$

Next, we consider the terms with $i \neq j$. Fix $\epsilon_1, \epsilon_2 > 0$. Corollary 27(2) gives an $M_2(\epsilon_1, \epsilon_2)$ such that for all $g > M_2(\epsilon_1, \epsilon_2)$ and all $i \neq j$ with

$$\{i,j\} \subseteq \big[1, (\gamma - \epsilon_1)g\big) \cup \big((\gamma+\epsilon_1)g, (2\gamma - \epsilon_1)g\big) \cup \big((2\gamma+\epsilon_1)g, 2g\big]$$

we have

$$\left|\mathbb{P}_g[\{i,j\}\cap S = \emptyset] - \left(1 - f_1\left(\frac{i}{g}\right)\right)\left(1 - f_1\left(\frac{j}{g}\right)\right)\right| < \epsilon_2.$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{2g-1}\sum_{\substack{j=1\\ j\neq i}}^{2g-1} ij\left|\left(1-f_1\left(\frac{i}{g}\right)\right)\left(1-f_1\left(\frac{j}{g}\right)\right)-\mathbb{P}_g[\{i,j\}\cap S=\emptyset]\right|\\
\leq\ & \sum_{i=1}^{2g-1}\sum_{j=1}^{2g-1} ij\epsilon_2+\sum_{i\in\left((\gamma-\epsilon_1)g,(\gamma+\epsilon_1)g\right)}\sum_{j=1}^{2g-1} ij+\sum_{i\in\left((2\gamma-\epsilon_1)g,(2\gamma+\epsilon_1)g\right)}\sum_{j=1}^{2g-1} ij\\
&+\sum_{i=1}^{2g-1}\sum_{j\in\left((\gamma-\epsilon_1)g,(\gamma+\epsilon_1)g\right)} ij+\sum_{i=1}^{2g-1}\sum_{j\in\left((2\gamma-\epsilon_1)g,(2\gamma+\epsilon_1)g\right)} ij\\
\leq\ & \epsilon_2\frac{(2g)^2}{2}\frac{(2g)^2}{2}+4(2\epsilon_1 g)(2g)\frac{(2g)^2}{2}=\epsilon_2 4g^4+\epsilon_1 32g^4.
\end{aligned}$$

Combining everything, since ϵ_1, ϵ_2 were arbitrary, we get

$$\mathbb{E}_g[\alpha(S)^2]=\left(\frac{9+\sqrt{5}}{20}\right)^2 g^4+o(g^4).$$

□

8. Counting Numerical Semigroups with Large Embedding Dimension

A main idea of this section is to construct a bijection between numerical semigroups with fixed multiplicity, genus, and embedding dimension and certain finite sequences of positive integers. These sequences are the initial segments of Kunz coordinate vectors of numerical semigroups. We recall some notation and basic facts about these objects.

Definition. *Let S be a numerical semigroup. The Apéry set of S with respect to an element $m \in S$ is*

$$\mathrm{Ap}(S;m)=\{s\in S\colon s-m\notin S\}.$$

It is easy to see that there is one element of $\mathrm{Ap}(S;m)$ in each residue class modulo m. We can write $\mathrm{Ap}(S;m)=\{0,a_1,a_2,\ldots,a_{m-1}\}$ where each $a_i \equiv i \pmod{m}$. For each $i\in\{1,2,\ldots,m-1\}$, we define the nonnegative integer k_i by $a_i = k_i m + i$. Note that if $m = m(S)$, then each $k_i \geq 1$.

Definition. *The Kunz coordinate vector of S with respect to m is $(k_1,\ldots,k_{m-1})$. Let KV_m denote the function that takes a numerical semigroup containing m to its Kunz coordinate vector with respect to m.*

We collect some results about Kunz coordinate vectors of numerical semigroups.

Theorem 33. *[10, 11] The map KV_m gives a bijection between $S\in\mathcal{S}_g$ with $m(S)=m$ and $(x_1,\ldots,x_{m-1})\in\mathbb{Z}_{\geq 1}^{m-1}$ satisfying:*

$$\begin{aligned}
x_i+x_j &\geq x_{i+j}, && \text{for all}\quad 1\leq i\leq j\leq m-1 \text{ with } i+j<m,\\
x_i+x_j+1 &\geq x_{i+j-m}, && \text{for all}\quad 1\leq i\leq j\leq m-1 \text{ with } i+j>m,\\
\sum_{i=1}^{m-1} x_i &= g.
\end{aligned}$$

Proposition 34. *If S is a numerical semigroup with $m=m(S)$, then $\mathcal{A}(S)\setminus\{m\}\subseteq \mathrm{Ap}(S;m)$. More precisely, if $\mathrm{KV}_m(S)=(k_1,\ldots,k_{m-1})$ we have*

$$\begin{aligned}
\mathcal{A}(S)=\{m\}\cup\{mk_i+i \mid &\nexists j_1,j_2\in[1,m-1]\colon j_1+j_2=i,\ k_{j_1}+k_{j_2}=k_i,\\
&\text{and}\ \ \nexists j_1,j_2\in[1,m-1]\colon j_1+j_2=m+i,\ k_{j_1}+k_{j_2}+1=k_i\}.
\end{aligned}$$

For a more detailed discussion of this material, see [8, Section 4].

In order to state the first main result of this section, we introduce some notation. Suppose $\overline{x}=(x_1,\ldots,x_t)\in\{1,2,3\}^t$. We define

1. $a(\overline{x}) = \#\{i \in [1,t] \mid x_i = 2\}$,
2. $b(\overline{x}) = \#\{i \in [1,t] \mid x_i = 3\}$,
3. $c(\overline{x}) = \#\{i \in [1,t] \mid \exists j_1, j_2 \in [1,t] :\ j_1 + j_2 = i, (x_{j_1}, x_{j_2}, x_i) = (1,1,2)\}$.

Theorem 33 gives a bijection between numerical semigroups S with $m(S) = m$ and $g(S) = g$ and a certain set of integer tuples of length $m-1$. We consider a refined version of this result that applies in the case where $g(S)$ and $e(S)$ are not too far away from $m(S)$.

Theorem 35. *Fix integers k_1 and k_2 satisfying $-1 \le k_1 \le k_2$ and $m \ge 2k_1 + 2$. For $(x_1, \ldots, x_{m-1}) \in \mathbb{Z}_{\ge 0}^{m-1}$, let $\overline{x} = (x_1, \ldots, x_{2k_1+1})$.*

There is a bijection between the set of numerical semigroups S satisfying $m(S) = m$, $g(S) = m + k_1$, and $e(S) = g(S) - k_2$, and sequences $(x_1, \ldots, x_{m-1})$ satisfying:

1. *$x_1, \ldots, x_{m-1} \in \{1,2,3\}$.*
2. *If $i \ge 2k_1$, then $x_i \in \{1,2\}$.*
3. *Whenever $i_1, i_2, i_3 \in [1, 2k_1+1]$ satisfy $i_1 + i_2 = i_3$, we have $(x_{i_1}, x_{i_2}, x_{i_3}) \ne (1,1,3)$.*
4. *$\#\{i \in [2k_1+2, m-1] \mid x_i = 2\} = k_1 + 1 - a(\overline{x}) - 2b(\overline{x})$.*
5. *$a(\overline{x}) + b(\overline{x}) - c(\overline{x}) = 2k_1 + 1 - k_2$.*

Note that conditions (4) and (5) imply that if such an S exists, then

- $a(\overline{x}) + 2b(\overline{x}) \le k_1 + 1$.
- $k_2 \le 2k_1 + 1$.

In the course of proving Theorem 35, we will express $e(S)$ in terms of $(x_1, \ldots, x_{m-1})$. We highlight this result because we will apply it in the discussion that follows.

Proposition 36. *Suppose $(x_1, \ldots, x_{m-1}) = \mathrm{KV}_m(S)$ where S is a numerical semigroup satisfying $m(S) = m$, $g(S) = m + k_1$, and $m \ge 2k_1 + 2$. Let $\overline{x} = (x_1, \ldots, x_{2k_1+1})$. Then*

$$e(S) = g - 2k_1 - 1 + a(\overline{x}) + b(\overline{x}) - c(\overline{x}).$$

Before proving Theorem 35, we show how to use it to prove Theorem 14.

Proposition 37. *Fix integers k_1, k_2 satisfying $-1 \le k_1 \le k_2$ and $g \ge 4k_1 + 3$. Suppose $\overline{x} = (x_1, \ldots, x_{2k_1+1}) \in \{1,2,3\}^{2k_1+1}$ satisfies the following conditions:*

1. *Whenever $i_1, i_2, i_3 \in [1, 2k_1+1]$ satisfy $i_1 + i_2 = i_3$, we have $(x_{i_1}, x_{i_2}, x_{i_3}) \ne (1,1,3)$.*
2. *$a(\overline{x}) + b(\overline{x}) - c(\overline{x}) = 2k_1 + 1 - k_2$.*
3. *$a(\overline{x}) + 2b(\overline{x}) \le k_1 + 1$.*

The number of numerical semigroups S for which $g(S) = g$, $m(S) = g - k_1$, $e(S) = g - k_2$, and the first $2k_1 + 1$ coordinates of $\mathrm{KV}_m(S)$ are given by $\overline{x}$ is $\binom{g-3k_1-2}{k_1+1-a(\overline{x})-2b(\overline{x})}$.

Proof. Suppose S satisfies $g(S) = g$, $m(S) = m = g - k_1$, $e(S) = g - k_2$, and $\mathrm{KV}_m(S) = (x_1, \ldots, x_{2k_1+1}, k_{2k_1+2}, \ldots, k_{m-1})$. Since $g(S) = g$, Theorem 35 implies that the number of $k_{2k_1+2}, \ldots, k_{m-1}$ equal to 2 must be $k_1 + 1 - a(\overline{x}) - 2b(\overline{x})$, and the rest of the elements must be equal to 1. Note that $(m-1) - (2k_1+1) = g - 3k_1 - 2$. Therefore, we have $\binom{g-3k_1-2}{k_1+1-a(\overline{x})-2b(\overline{x})}$ choices for $k_{2k_1+2}, \ldots, k_{m-1}$. Theorem 35 says that each choice gives a semigroup satisfying the properties we are looking for and that these are all such semigroups. Condition (3) ensures that $k + 1 - a(\overline{x}) - 2b(\overline{x}) \ge 0$ and $g \ge 4k_1 + 3$ ensures that $k_1 + 1 - a(\overline{x}) - 2b(\overline{x}) \le g - 3k_1 - 2$. ☐

Consider the collection of all sequences $\overline{x}$ satisfying the conditions of Theorem 35, but now where we allow k_2 to vary. This leads to the following definition.

Definition. *Fix $k \in \mathbb{Z}_{\ge 0}$. Let $\mathcal{Y}(k)$ be the collection of all tuples $\overline{x} = (x_1, \ldots, x_{2k+1}) \in \{1,2,3\}^{2k+1}$ satisfying the following conditions:*

1. *Whenever $i_1, i_2, i_3 \in [1, 2k+1]$ satisfy $i_1 + i_2 = i_3$, we have $(x_{i_1}, x_{i_2}, x_{i_3}) \ne (1,1,3)$.*
2. *$a(\overline{x}) + 2b(\overline{x}) \le k + 1$.*

Theorem 38. *Fix an integer $k \geq -1$. For $g \geq 4k+3$ we have*

$$\#\{S \in \mathcal{S}_g \mid m(S) = g-k\} = \sum_{\overline{x} \in \mathcal{Y}(k)} \binom{g-3k-2}{k+1-a(\overline{x})-2b(\overline{x})}.$$

Proof. For $\overline{x} \in \mathcal{Y}(k)$, let $k_2 = 2k+1-a(\overline{x})-b(\overline{x})+c(\overline{x})$. We have

$$k_2 = k - a(\overline{x}) - b(\overline{x}) + c(\overline{x}) + k + 1 \geq k + b(\overline{x}) + c(\overline{x}) \geq k.$$

We apply Proposition 37 for each $\overline{x} \in \mathcal{Y}(k)$ with the corresponding k_2 and add the results. □

In the notation of Theorem 11, this means that for each $k \geq -1$ and $g \geq 4k+3$, we have

$$\frac{1}{(k+1)!} f_k(x) = \sum_{\overline{x} \in \mathcal{Y}(k)} \binom{x-3k-2}{k+1-a(\overline{x})-2b(\overline{x})}.$$

At the end of this paper, we list the sets $\mathcal{Y}(k)$ for $-1 \leq k \leq 2$. A simple computation gives $f_{-1}(x), \ldots, f_2(x)$. We see that they match the formulas in [7, Corollary 14].

We return to the problem of counting semigroups $S \in \mathcal{S}_g$ with a large embedding dimension.

Theorem 39. *Fix an integer $l \geq -1$. For $g \geq 4l+3$ we have*

$$\#\{S \in \mathcal{S}_g \mid e(S) = g-l\} = \sum_{k=-1}^{l} \sum_{\substack{\overline{x} \in \mathcal{Y}(k) \\ a(\overline{x})+b(\overline{x})-c(\overline{x})=2k+1-l}} \binom{g-3k-2}{k+1-a(\overline{x})-2b(\overline{x})}.$$

Proof. We divide up the semigroups in $\mathcal{S}_g$ with $e(S) = g-l$ by multiplicity and see that

$$\{S \in \mathcal{S}_g \mid e(S) = g-l\} = \bigcup_{k=-1}^{l} \{S \in \mathcal{S}_g \mid e(S) = g-l, m(S) = g-k\}.$$

By Theorem 35, if $S \in S_g$ satisfies $e(S) = g-l$ and $m(S) = g-k$, then $\overline{x} \in \mathcal{Y}(k)$ and $a(\overline{x})+b(\overline{x})-c(\overline{x}) = 2k+1-l$. By Proposition 37, the number of numerical semigroups corresponding to a given $\overline{x}$ is $\binom{g-3k-2}{k+1-a(\overline{x})-2b(\overline{x})}$. The result follows. □

The only remaining thing needed to complete the proof of Theorem 14 is to establish some basic properties of the polynomials on the right-hand side of Theorem 39. Define

$$H_l(x) = \sum_{k=-1}^{l} \sum_{\substack{\overline{x} \in \mathcal{Y}(k) \\ a(\overline{x})+b(\overline{x})-c(\overline{x})=2k+1-l}} \binom{x-3k-2}{k+1-a(\overline{x})-2b(\overline{x})}.$$

Proposition 40. *Fix an integer $l \geq -1$. Let $l_1 = \lfloor \frac{l+1}{2} \rfloor$. Then $H_l(x)$ is a polynomial of degree l_1 and $l_1! H_l(x)$ is a monic polynomial with integer coefficients.*

Proving this statement completes the proof of Theorem 14.

Proof. The degree of $H_l(x)$ is

$$\max\{k+1-a(\overline{x})-2b(\overline{x}) \mid -1 \leq k \leq l,\ \overline{x} \in \mathcal{Y}(k), 2k+1+c(\overline{x})-a(\overline{x})-b(\overline{x}) = l\}.$$

Suppose that $-1 \leq k \leq l$ and $\overline{x} \in \mathcal{Y}(k)$ satisfies $2k+1+c(\overline{x})-a(\overline{x})-b(\overline{x}) = l$. We have

$$2k+1 = l + (a(\overline{x}) - c(\overline{x})) + b(\overline{x}) \geq l.$$

This means that $k \geq \frac{l-1}{2}$. Next,

$$k+1-a(\overline{x})-2b(\overline{x}) = l-k-c(\overline{x})-b(\overline{x}) \leq l-k \leq \frac{l+1}{2}.$$

This implies $\deg(H_l(x)) \leq \frac{l+1}{2}$. Since $\deg(H_l(x))$ is an integer, we see that $\deg(H_l(x)) \leq l_1$. Note that $k+1-a(\overline{x})-2b(\overline{x}) = l_1$ if and only if $b(\overline{x}) = c(\overline{x}) = 0$ and $k = l - l_1$.

- If l is odd, then $l = 2l_1 - 1$. Take $k = l - l_1 = l_1 - 1$. If $\overline{x} \in \mathcal{Y}(k)$ satisfies $b(\overline{x}) = c(\overline{x}) = 0$, then
$$l = 2k + 1 - a(\overline{x}) = 2l_1 - 1 + a(\overline{x}) = l - a(\overline{x}).$$
This implies $a(\overline{x}) = 0$. There is a unique such $\overline{x}$, which is $\overline{x} = (1, 1, \dots, 1) \in \mathcal{Y}(k)$.
- If l is even, then $l = 2l_1$. Take $k = l - l_1 = l_1$. If $\overline{x} \in \mathcal{Y}(k)$ satisfies $b(\overline{x}) = c(\overline{x}) = 0$, then
$$l = 2k + 1 - a(\overline{x}) = 2l_1 + 1 + a(\overline{x}) = l + 1 - a(\overline{x}).$$
This implies $a(\overline{x}) = 1$. There is a unique such $\overline{x}$, which is $\overline{x} = (2, 1, \dots, 1) \in \mathcal{Y}(k)$.

This completes the proof. □

8.1 The proof of Theorem 35.

The goal of the rest of this section is to prove Theorem 35.

We need several facts about the Kunz coordinate vector of a numerical semigroup with multiplicity m and genus g.

Lemma 41. *[7, Lemma 11] Suppose S is a numerical semigroup with $g(S) = g$, $m(S) = m$, and $\mathrm{KV}_m(S) = (x_1, \dots, x_{m-1})$. If $2g < 3m + 2$, then $\{x_1, \dots, x_{m-1}\} \subseteq \{1, 2, 3\}$.*

Lemma 42. *Suppose S is a numerical semigroup with $g(S) = g$ and $m(S) = m$. Let $\mathrm{KV}_m(S) = (x_1, \dots, x_{m-1})$. If $i \in [1, m-1]$ satisfies $x_i = 3$, then*
$$g \geq m + 1 + \left\lceil \frac{i-1}{2} \right\rceil.$$

Proof. The set $[1, i-1]$ can be partitioned as a union of $\lceil \frac{i-1}{2} \rceil$ subsets of the form $\{j_1, j_2\}$ with $j_1 + j_2 = i$. For each such $\{j_1, j_2\}$, at least one of x_{j_1}, x_{j_2} must be at least 2. Therefore,
$$g - (m-1) = \sum_{j=1}^{m-1} (x_j - 1) \geq 2 + \left\lceil \frac{i-1}{2} \right\rceil. \qquad \square$$

Lemma 43. *Suppose S is a numerical semigroup with $g(S) = g$ and $m(S) = m$. Let $\mathrm{KV}_m(S) = (x_1, \dots, x_{m-1})$. If $i \in [1, m-1]$ satisfies $x_i = 2$ and $mx_i + i \in \mathcal{A}(S)$, then*
$$g \geq m + \left\lceil \frac{i-1}{2} \right\rceil.$$

Proof. The set $[1, i-1]$ can be partitioned as a union of $\lceil \frac{i-1}{2} \rceil$ subsets of the form $\{j_1, j_2\}$ with $j_1 + j_2 = i$. For each such $\{j_1, j_2\}$, at least one of x_{j_1}, x_{j_2} must be at least 2. Therefore,
$$g - (m-1) = \sum_{j=1}^{m-1} (x_j - 1) \geq 1 + \left\lceil \frac{i-1}{2} \right\rceil. \qquad \square$$

Lemma 44. *Suppose S is a numerical semigroup with $g(S) = g$ and $m(S) = m$. Let $\mathrm{KV}_m(S) = (x_1, \dots, x_{m-1})$. If $3m + i \in \mathcal{A}(S)$ for some $i \in [1, m-1]$, then $g \geq \frac{3m}{2}$.*

Proof. The set $[1, m-1] \setminus \{i\}$ can be partitioned as a union of subsets of the form $\{j_1, j_2\}$ with $j_1 + j_2 \equiv i \pmod{m}$. Since $3m + i \in \mathcal{A}(S)$ we know that for each subset $\{j_1, j_2\}$ in the partition $x_{j_1} + x_{j_2} + 1 > x_i = 3$. Hence at least one of x_{j_1}, x_{j_2} is at least 2. The number of subsets $\{j_1, j_2\}$ in the partition is at least $\lceil \frac{m-2}{2} \rceil$. Therefore,
$$g - (m-1) = \sum_{j=1}^{m-1} (x_j - 1) \geq 2 + \left\lceil \frac{m-2}{2} \right\rceil \geq 2 + \frac{m-2}{2}. \qquad \square$$

We are now ready to prove Theorem 35. We prove it in two parts.

Proposition 45. *Fix integers $k_2 \geq k_1 \geq -1$. Suppose $g \geq 3k_1 + 2$, and $S \in \mathcal{S}_g$ satisfies $m(S) = g - k_1$ and $e(S) = g - k_2$. Let $m = m(S)$. Suppose $\mathrm{KV}_m(S) = (x_1, \dots, x_{m-1})$ and let $\overline{x} = (x_1, \dots, x_{2k_1+1})$. Then the following hold:*

1. *$\{x_1, \dots, x_{m-1}\} \subseteq \{1, 2, 3\}$.*
2. *If $i \geq 2k_1$, then $x_i \in \{1, 2\}$.*

3. *Whenever* $i_1, i_2, i_3 \in [1, 2k_1+1]$ *satisfy* $i_1 + i_2 = i_3$*, we have* $(x_{i_1}, x_{i_2}, x_{i_3}) \neq (1,1,3)$.
4. $\#\{i \in [2k_1+2, m-1] \mid x_i = 2\} = k_1 + 1 - a(\overline{x}) - 2b(\overline{x})$.
5. $a(\overline{x}) + b(\overline{x}) - c(\overline{x}) = 2k_1 + 1 - k_2$.
6. $e(S) = g - 2k_1 - 1 + a(\overline{x}) + b(\overline{x}) - c(\overline{x})$.
7. $k_2 \leq 2k_1 + 1$.

We note that verifying property (6) proves Proposition 36.

Proof. We know that $m = g - k_1 \geq 2k_1 + 2$. Next,

$$3m + 2 - 2g = 3(g - k_1) + 2 - 2g = g - (3k_1 + 2) + 4 \geq 4.$$

This means that $2g < 3m + 2$. Lemma 41 implies $\{x_1, \ldots, x_{m-1}\} \subseteq \{1, 2, 3\}$.

Suppose i satisfies $2k_1 \leq i \leq m - 1$. Assume for the sake of contradiction that $x_i = 3$. By Lemma 42 we have $g \geq m + 1 + \lceil \frac{i-1}{2} \rceil$. However,

$$g \geq m + 1 + \left\lceil \frac{i-1}{2} \right\rceil \geq m + 1 + \left\lceil \frac{2k_1 - 1}{2} \right\rceil = m + 1 + k_1 = g + 1.$$

This is a contradiction and we conclude that $x_i \in \{1, 2\}$.

Suppose $i_1, i_2, i_3 \in [1, 2k_1+1]$ satisfy $i_1 + i_2 = i_3$. Suppose $x_{i_1} = x_{i_2} = 1$. Since $(x_1, \ldots, x_{m-1})$ is the Kunz coordinate vector of a numerical semigroup, we know that $x_{i_3} \leq x_{i_1} + x_{i_2} = 2$.

Next, note that

$$k_1 + 1 = g - (m-1) = \sum_{i=1}^{m-1} (x_i - 1) = a(\overline{x}) + 2b(\overline{x}) + \#\{i \in [2k_1+2, m-1] \mid x_i = 2\}.$$

Next, we claim that $\mathcal{A}(S)$ is given by m together with the elements $mx_i + i$ satisfying either

1. $x_i = 1$, or
2. $x_i = 2$ where $i \in [1, 2k_1+1]$ and there does not exist a j satisfying $1 \leq j < i$ with $x_j = x_{i-j} = 1$.

It is clear that all these elements are elements of $\mathcal{A}(S)$. Suppose $a = mx_i + i$ is some other element of $\mathrm{Ap}(S; m)$. Then one of the following must hold:

- Case 1: $x_i = 2$ where $i \in [1, 2k_1+1]$ and there does exist a j satisfying $1 \leq j < i$ with $x_j = x_{i-j} = 1$. Then $x_i = x_j + x_{i-j}$, so $a \notin \mathcal{A}(S)$.
- Case 2: $x_i = 2$ where $i \in [2k_1+2, m-1]$. Assume for the sake of contradiction that $a \in \mathcal{A}(S)$. Lemma 43 implies that $g \geq m + \lceil \frac{i-1}{2} \rceil$. This implies that

  $$g \geq m + \left\lceil \frac{2k_1+1}{2} \right\rceil = m + k_1 + 1 = g + 1.$$

  This is a contradiction. Therefore $a \notin \mathcal{A}(S)$.
- Case 3: $x_i = 3$. Assume for the sake of contradiction that $a \in \mathcal{A}(S)$. By Lemma 44, $g \geq \frac{3m}{2}$. This implies that

  $$k_1 = g - m \geq \frac{m}{2} = \frac{g - k_1}{2} \geq \frac{2k_1 + 2}{2} = k_1 + 1,$$

  which is a contradiction. Therefore $a \notin \mathcal{A}(S)$.

This characterization of $\mathcal{A}(S)$ implies that

$$\begin{aligned} g - k_2 &= e(S) \\ &= 1 + \big(2k_1 + 1 - a(\overline{x}) - b(\overline{x})\big) + \#\{i \in [2k_1+2, m-1] \mid x_i = 1\} + \big(a(\overline{x}) - c(\overline{x})\big) \\ &= 2k_1 + 2 - b(\overline{x}) - c(\overline{x}) + \big(m - 1 - (2k_1+1) - \#\{i \in [2k_1+2, m-1] \mid x_i = 2\}\big) \\ &= -b(\overline{x}) - c(\overline{x}) + (g - k_1) - \big(k_1 + 1 - a(\overline{x}) - 2b(\overline{x})\big) \\ &= g - 2k_1 - 1 + a(\overline{x}) + b(\overline{x}) - c(\overline{x}). \end{aligned}$$

We conclude that $a(\overline{x}) + b(\overline{x}) - c(\overline{x}) = 2k_1 + 1 - k_2$.

Finally since $c(\overline{x}) \leq a(\overline{x})$, we see that $a(\overline{x}) + b(\overline{x}) - c(\overline{x}) \geq 0$ so $k_2 \leq 2k_1 + 1$. □

We now prove the other direction in Theorem 35.

Proposition 46. *Fix integers* $-1 \le k_1 \le k_2 \le 2k_1+1$. *Also fix* $m \ge 2k_1+2$ *and a tuple of positive integers* $(x_1, x_2, \dots, x_{m-1})$. *Let* $\overline{x} = (x_1, \dots, x_{2k_1+1})$. *Suppose we have the following:*

1. $\{x_1, \dots, x_{m-1}\} \subseteq \{1,2,3\}$.
2. *If* $i \ge 2k_1+2$ *then* $x_i \in \{1,2\}$.
3. *Whenever* $i_1, i_2, i_3 \in [1, 2k_1+1]$ *satisfy* $i_1+i_2=i_3$, *we have* $(x_{i_1}, x_{i_2}, x_{i_3}) \neq (1,1,3)$.
4. $\#\{i \in [2k_1+2, m-1] \mid x_i = 2\} = k_1+1-a(\overline{x})-2b(\overline{x})$.
5. $a(\overline{x})+b(\overline{x})-c(\overline{x}) = 2k_1+1-k_2$.

Let

$$S = m\mathbb{N}_0 \cup \bigcup_{i=1}^{m-1} (i+mx_i+m\mathbb{N}_0).$$

Then S *is a numerical semigroup satisfying* $m(S)=m$, $g(S)=m+k_1$, *and* $e(S)=g(S)-k_2$.

Proof. Theorem 33 implies that if $(x_1, \dots, x_{m-1})$ satisfies the first three conditions, then it is the Kunz coordinate vector of numerical semigroups of multiplicity m. Notice that

$$\begin{aligned} g(S)-(m-1) &= \sum_{i=1}^{m-1}(x_i-1) = \#\{i \in [1,m-1] \mid x_i=2\} + 2\#\{i \in [1,m-1] \mid x_i = 3\} \\ &= a(\overline{x}) + \#\{i \in [2k_1+2, m-1] \mid x_i = 2\} + 2b(\overline{x}) \\ &= a(\overline{x}) + 2b(\overline{x}) + \big(k_1+1-a(\overline{x})-2b(\overline{x})\big) \\ &= k_1+1. \end{aligned}$$

This means that $g(S)=m+k_1$. Suppose $e(S)=g(S)-k$. It is clear that $k_1 \le k$, and also that $g(S) \ge 3k_1+2$. Therefore, by Proposition 45(5) we see that $a(\overline{x})+b(\overline{x})-c(\overline{x}) = 2k_1+1-k$. This implies $k=k_1$. ☐

We end this paper by including some data related to the sets $\mathcal{Y}(k)$ and the polynomials $H_l(x)$ and $f_k(x)$ of this section. The initial $\mathcal{Y}(k)$ are as follows:

$$\mathcal{Y}(-1)=\{\emptyset\},\quad \mathcal{Y}(0)=\{(1),(2)\},\quad \mathcal{Y}(1)=\{(1,1,1),(1,1,2),(1,2,1),(2,1,1),(2,2,1),(2,1,2),(1,2,2),(3,1,1)\},$$

$$\begin{aligned} \mathcal{Y}(2) = \{&(1,1,1,1,1),(2,1,1,1,1),(1,2,1,1,1),(1,1,2,1,1),(1,1,1,2,1),(1,1,1,1,2),(2,2,1,1,1),(2,1,2,1,1),\\ &(2,1,1,2,1),(2,1,1,1,2),(1,2,2,1,1),(1,2,1,2,1),(1,2,1,1,2),(1,1,2,2,1),(1,1,2,1,2),(1,1,1,2,2),\\ &(2,2,2,1,1),(2,2,1,2,1),(2,2,1,1,2),(2,1,2,2,1),(2,1,2,1,2),(2,1,1,2,2),(1,2,2,2,1),(1,2,2,1,2),\\ &(1,2,1,2,2),(1,1,2,2,2),(3,1,1,1,1),(3,2,1,1,1),(3,1,2,1,1),(3,1,1,2,1),(3,1,1,1,2),(2,3,1,1,1),\\ &(2,1,3,1,1),(1,2,3,1,1)\}. \end{aligned}$$

The first few polynomials are as follows:

$$H_{-1}(t)=1,\quad H_0(t)=1,\quad H_1(t)=t,\quad H_2(t)=t+1,\quad H_3(t)=\frac{t^2}{2}-\frac{3t}{2}+2,\quad H_4(t)=\frac{t^2}{2}-\frac{t}{2}-2.$$

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