

# The cotype zeta function of $\mathbb{Z}^d$

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## Abstract

We give an asymptotic formula for the number of sublattices  $\Lambda \subseteq \mathbb{Z}^d$  of index at most  $X$  for which  $\mathbb{Z}^d/\Lambda$  has rank at most  $m$ , answering a question of Nguyen and Shparlinski. We compare this result to work of Stanley and Wang on Smith normal forms of random integral matrices and discuss connections to the Cohen–Lenstra heuristics. Our arguments are based on Petrogradsky’s formulas for the cotype zeta function of  $\mathbb{Z}^d$ , a multivariable generalization of the subgroup growth zeta function of  $\mathbb{Z}^d$ .

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## 1. Introduction

A fundamental problem in the field of subgroup growth is understanding the number of subgroups of finite index  $n$  in a fixed group  $G$ . In many cases, analytic properties of the *subgroup growth zeta function*  $\zeta_G(s)$  provide useful information. This is the Dirichlet series

$$\zeta_G(s) = \sum_{H \subseteq G} \frac{1}{[G : H]^s} \quad (1.1)$$

where  $H$  ranges over all finite index subgroups of  $G$ . If the number of subgroups in  $G$  of index  $n$  grows at most polynomially, then the Dirichlet series defining  $\zeta_G(s)$  converges absolutely for

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$\operatorname{Re}(s)$  sufficiently large. An analytic continuation of the series and knowledge of the locations and orders of its poles would provide information on asymptotics for the number of subgroups of index less than  $X$  as  $X \rightarrow \infty$ .

One of the most basic examples is the subgroup growth zeta function of the integer lattice  $\mathbb{Z}^d$  which turns out to have a simple expression as a product of Riemann zeta functions:

$$\zeta_{\mathbb{Z}^d}(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-(d-1)). \quad (1.2)$$

See the book of Lubotzky and Segal for five proofs of this fact [21]. Since  $\zeta(s)$  has a simple pole at  $s = 1$ , standard Tauberian techniques immediately give the asymptotic

$$\begin{aligned} N_d(X) &:= \#\{\text{sublattices of } \mathbb{Z}^d \text{ of index } < X\} \\ &= \frac{\zeta(d)\zeta(d-1)\cdots\zeta(2)}{d} X^d + O(X^{d-1} \log(X)) \\ &\text{as } X \rightarrow \infty. \end{aligned} \quad (1.3)$$

### 1.1. The proportion of lattices with given corank

A number of more refined questions about the distribution of sublattices of  $\mathbb{Z}^d$  can be asked. Motivated by work of Nguyen and Shparlinski [23], we investigate the distribution of sublattices of  $\mathbb{Z}^d$  whose *cotype* has a certain form. The cotype of a sublattice  $\Lambda \subset \mathbb{Z}^d$  is defined as follows. By elementary divisor theory, there is a unique  $d$ -tuple of positive integers  $(\alpha_1, \dots, \alpha_d) = (\alpha_1(\Lambda), \dots, \alpha_d(\Lambda))$  such that the finite abelian group  $\mathbb{Z}^d/\Lambda$  is isomorphic to the sum of cyclic groups

$$(\mathbb{Z}/\alpha_1\mathbb{Z}) \oplus (\mathbb{Z}/\alpha_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/\alpha_d\mathbb{Z}) \quad (1.4)$$

where  $\alpha_{i+1} \mid \alpha_i$  for  $1 \leq i \leq d-1$ . We call the  $d$ -tuple  $\alpha(\Lambda) := (\alpha_1(\Lambda), \dots, \alpha_d(\Lambda))$  the *cotype* of  $\Lambda$ . The largest index  $i$  for which  $\alpha_i(\Lambda) \neq 1$  is called the *rank* of  $\mathbb{Z}^d/\Lambda$  and the *corank* of  $\Lambda$ . By convention,  $\mathbb{Z}^d$  has corank 0. A sublattice  $\Lambda$  of corank 0 or 1 is called *cocyclic*, i.e.,  $\mathbb{Z}^d/\Lambda$  is cyclic, or equivalently,  $\Lambda$  has cotype  $([\mathbb{Z}^d : \Lambda], 1, \dots, 1)$ .

Nguyen and Shparlinski study the distribution of cocyclic sublattices of  $\mathbb{Z}^d$  and pose several related questions. Let  $N_d^{(m)}(X)$  be the number of sublattices  $\Lambda$  of  $\mathbb{Z}^d$  of index less than  $X$  such that  $\Lambda$  has corank at most  $m$ . In particular,  $N_d^{(1)}(X)$  is the number of cocyclic sublattices of  $\mathbb{Z}^d$  of index less than  $X$ . Throughout this paper we use  $\prod_p$  to denote a product over all primes. Rediscovering a result of Petrogradsky [24] by more elementary means, they show that

$$N_d^{(1)}(X) \sim \frac{\theta_d}{d} X^d, \text{ where } \theta_d = \prod_p \left(1 + \frac{p^{d-1} - 1}{p^{d+1} - p^d}\right) \quad (1.5)$$

as  $X \rightarrow \infty$ . Comparing this to the asymptotic (1.3) for all sublattices, Nguyen–Shparlinski and Petrogradsky both observe that the probability that a “random” sublattice of  $\mathbb{Z}^d$  is cocyclic is about 85% for  $d$  large.

Nguyen and Shparlinski conclude their paper by stating that it would be of interest to obtain similar asymptotic formulas for  $N_d^{(m)}(X)$  for  $m > 1$  and to show that the sublattices of corank  $m$  form a negligible proportion of all sublattices of  $\mathbb{Z}^d$  when  $m$  is sufficiently large.

In this paper we show the following theorem.

**Theorem 1.1.** *Let  $1 \leq m \leq d$ . As  $X \rightarrow \infty$ ,*

$$N_d^{(m)}(X) \sim \frac{X^d}{d} \cdot \prod_p \left( (1 - p^{-1}) \sum_{i=0}^m \begin{bmatrix} d \\ i \end{bmatrix}_{p^{-1}} \frac{p^{-i^2}}{\prod_{j=1}^i (1 - p^{-j})} \right). \quad (1.6)$$

We recall in Section 2 the definition of the  $q$ -binomial coefficient  $\begin{bmatrix} d \\ i \end{bmatrix}_{p^{-1}}$ .

Dividing by the number of all sublattices of index less than  $X$  as given in (1.3) gives the proportion of sublattices with corank at most  $m$ .

**Corollary 1.2.** As  $X \rightarrow \infty$ ,

$$\frac{N_d^{(m)}(X)}{N_d(X)} \sim \prod_p \left( \prod_{j=1}^d (1 - p^{-j}) \sum_{i=0}^m \begin{bmatrix} d \\ i \end{bmatrix}_{p^{-1}} \frac{p^{-i^2}}{\prod_{j=1}^i (1 - p^{-j})} \right). \quad (1.7)$$

For example, the proportion of sublattices of  $\mathbb{Z}^d$  of corank at most 2 converges to approximately 99.4% as  $d \rightarrow \infty$ , and the proportion of sublattices of  $\mathbb{Z}^d$  of corank at most 3 converges to approximately 99.995%. Therefore, while sublattices of any fixed corank have positive density among all sublattices of  $\mathbb{Z}^d$ , they become sparser as the corank grows. This confirms an expectation of Nguyen–Shparlinski.

Also of interest in our work is the method of proof. Nguyen and Shparlinski prove their results by counting solutions of linear congruence equations. Our proofs extend Petrogradsky’s methods and make systematic use of the *cotype zeta function* of  $\mathbb{Z}^d$ , which he introduced in [24]. This is a multivariate generalization of the subgroup growth zeta function  $\zeta_{\mathbb{Z}^d}(s)$  from (1.2). Petrogradsky computes it explicitly in terms of permutation descent polynomials.

**Remark 1.** We have not attempted to bound the error term in the asymptotic of Theorem 1.1. The expression (1.2) for  $\zeta_{\mathbb{Z}^d}(s)$  as a product of Riemann zeta functions implies that the number of subgroups of  $\mathbb{Z}^d$  of index  $n$  can be expressed as a divisor sum. Restricting to subgroups of index  $n$  of corank less than  $m$  imposes a local condition at each prime  $p$ . We therefore see the problem of obtaining error terms in the asymptotic for  $N_d^{(m)}(X)$  as an analogue of bounding the error in asymptotics for divisor sums in arithmetic progressions, see e.g. Friedlander–Iwaniec [12] for early work on this problem and more recently, the paper of Kowalski–Ricotta [19]. Such error bounds typically come from a Voronoi formula and Weil’s estimate for Kloosterman sums. Lacking an automorphic interpretation in our setting, any investigation of error terms will probably require a deeper understanding of the combinatorial properties of the polynomials arising in the Euler products (3.3).

**Remark 2.** We make use of Petrogradsky’s multivariate cotype zeta function to derive a formula for a single variable zeta function counting subgroups of corank less than or equal to  $m$ . The asymptotic of Theorem 1.1 comes from analytic properties of this single variable series. Using recent results on Tauberian theorems for multivariate Dirichlet series [4, 11, 39], one could also investigate asymptotics for the number of subgroups of  $\mathbb{Z}^d$  with cotype satisfying the inequalities:

$$1 \leq \alpha_1 \leq c_1 X, \quad 1 \leq \alpha_1 \alpha_2 \leq c_2 X, \dots, \quad 1 \leq \alpha_1 \cdots \alpha_{d-1} \leq c_{d-1} X, \quad 1 \leq \alpha_1 \cdots \alpha_d \leq X$$

for fixed constants  $0 < c_1 \leq \cdots \leq c_{d-1} \leq 1$  with  $X \rightarrow \infty$ . We have not attempted to pursue such questions.

## 1.2. Coranks of lattices and cokernels of matrices in Hermite normal form

Throughout this paper, for a ring  $R$  we let  $M_d(R)$  denote the  $d \times d$  matrices with entries in  $R$ . For a finite abelian group  $G$ , we write  $(G)_p$  for its Sylow  $p$ -subgroup. Consider the

distribution on finite abelian  $p$ -groups of rank at most  $d$  that chooses a group  $G$  of rank  $r$  where  $r \leq d$  with probability

$$P_d^p(G) = |\mathrm{Aut}(G)|^{-1} \left( \prod_{j=1}^d (1 - p^{-j}) \right) \left( \prod_{j=d-r+1}^d (1 - p^{-j}) \right). \quad (1.8)$$

It follows from a result of Cohen and Lenstra [9, Theorem 6.1] that the right-hand side of (1.7) is equal to the product over all primes  $p$  of the probability that a group chosen from  $P_d^p$  has rank at most  $m$ .

Motivated by famous conjectures of Cohen and Lenstra on distributions of Sylow  $p$ -subgroups of class groups of number fields [9], Friedman and Washington prove that the distribution of cokernels of  $d \times d$  random matrices with entries in the  $p$ -adic integers  $\mathbb{Z}_p$ , drawn from Haar measure on the space of all such matrices, is the distribution of (1.8) [13, Proposition 1]. Stanley and Wang show that this distribution arises in the study of Smith normal forms of random  $d \times d$  integer matrices with entries chosen uniformly from  $[-k, k]$ , as  $k \rightarrow \infty$ . The Smith normal form of an integer matrix carries the same information as its cokernel. As  $k \rightarrow \infty$ , each entry is uniformly distributed modulo  $p^r$  for each prime power, so this distribution of cokernels matches the one studied by Friedman and Washington, and therefore is equal to the one defined by (1.8). Going from a result for a single prime to a result involving infinitely many primes is often challenging. Stanley and Wang prove that the probability that the cokernel of a random integer matrix chosen from the model described above has rank at most  $m$  is given by the right-hand side of (1.7) [34, Theorem 4.13]. The proof uses nontrivial results from number theory of Ekedahl and Poonen on greatest common divisors of outputs of multivariable polynomials [10, 25].

We now interpret of Corollary 1.2 in terms of cokernels of special classes of random integer matrices. A nonsingular  $M \in M_d(\mathbb{Z})$  with entries  $a_{ij}$  is in *Hermite normal form* if:

1.  $M$  is upper triangular, and
2.  $0 \leq a_{ij} < a_{jj}$  for  $1 \leq i < j \leq d$ .

We recall a basic fact about lattices and matrices in Hermite normal form.

**Proposition 1.3.** *Every sublattice  $\Lambda \subseteq \mathbb{Z}^d$  is the row span of a unique matrix  $H(\Lambda)$  in Hermite normal form. Moreover,  $\mathbb{Z}^d / \Lambda \cong \mathrm{cok}(H(\Lambda))$ , which implies  $[\mathbb{Z}^d : \Lambda] = \det(H(\Lambda))$ .*

*This gives a bijection between the set of sublattices of  $\mathbb{Z}^d$  of index less than  $X$  and nonsingular  $d \times d$  matrices in Hermite normal form with determinant less than  $X$ .*

Let  $\mathcal{H}_d(\mathbb{Z}) \subset M_d(\mathbb{Z})$  denote the subset of nonsingular matrices in Hermite normal form and  $\mathcal{H}_d(X) \subset \mathcal{H}_d(\mathbb{Z})$  denote the subset of these matrices with determinant less than  $X$ . By Proposition 1.3, Corollary 1.2 is equivalent to the following statement.

**Corollary 1.4.** *We have*

$$\begin{aligned} & \lim_{X \rightarrow \infty} \frac{\#\{M \in \mathcal{H}_d(X) : \mathrm{rank}(\mathrm{cok}(M)) \leq m\}}{\#\mathcal{H}_d(X)} \\ &= \prod_p \left( \prod_{j=1}^d (1 - p^{-j}) \sum_{i=0}^m \begin{bmatrix} d \\ i \end{bmatrix}_{p^{-1}} \frac{p^{-i^2}}{\prod_{j=1}^i (1 - p^{-j})} \right). \end{aligned}$$

In Section 4 we consider the distribution of Sylow  $p$ -subgroups of cokernels of matrices in Hermite normal form, giving an explanation for this result.

**Theorem 1.5.** *Let  $G$  be a finite abelian  $p$ -group of rank  $r \leq d$ . Then*

$$\lim_{X \rightarrow \infty} \frac{\#\{\Lambda \subseteq \mathbb{Z}^d : [\mathbb{Z}^d : \Lambda] < X \text{ and } (\mathbb{Z}^d / \Lambda)_p \cong G\}}{\#\{\Lambda \subseteq \mathbb{Z}^d : [\mathbb{Z}^d : \Lambda] < X\}} = P_d^p(G).$$

Equivalently,

$$\lim_{X \rightarrow \infty} \frac{\#\{M \in \mathcal{H}_d(X) : \text{cok}(M)_p \cong G\}}{\#\mathcal{H}_d(X)} = P_d^p(G).$$

We note that this result does not directly imply Corollary 1.4 because of subtleties involved in going from a single prime to a product over all primes.

The main point of Theorem 1.5 is that Sylow  $p$ -subgroups of cokernels of matrices in Hermite normal form follow the same distribution as cokernels of Haar random matrices in  $M_d(\mathbb{Z}_p)$ . This fits in with universality results for cokernels of families of random integer and  $p$ -adic matrices due to Wood [42]. However, Wood’s results do not directly imply Theorem 1.5 because, in the language of [42, Definition 1], matrices in Hermite normal form are not  $\epsilon$ -balanced, as many entries are fixed to be 0.

### 1.3. Related work

More general functions of the type considered in this paper have their origin in Igusa’s study of zeta functions of algebraic groups [16]. This work has become an essential tool in the theory of zeta functions of groups and rings and has been extended in various directions. See for example the paper of du Sautoy and Lubotzky [28] and the further references listed in Section 5.2. In this context, Petrogradsky’s local zeta function and generalizations thereof arise naturally as multivariate  $p$ -adic integrals. This point of view is developed in Voll [40], in both the context of counting subgroups of a finitely generated torsion-free nilpotent group and in the study of their representation zeta functions. Further generalizations of Igusa’s local zeta functions are introduced in Klopsch–Voll [17] and Schein–Voll [30], leading to numerous applications in subgroup, subring and representation growth; see e.g. [6, 7, 20, 31, 36, 37].

In the theory of automorphic forms related functions appear in the computation of Fourier coefficients of Eisenstein series on  $GL_{2n}$  and the local singular series of an  $n$ -by- $n$  square matrix, as noted in the papers of F. Sato [27] and Beineke–Bump [1]. Sato raises the interesting open question of finding a corresponding relation between local singular series and the enumeration of subgroups of an abelian group in the symplectic case.

### Outline of the paper

We review Petrogradsky’s work in Section 2. In Section 3 we prove our main results on the distribution of the corank. In Section 4 we prove Theorem 1.5. The utility of the cotype zeta function in the resolution of these corank problems suggests that it may be fruitful to introduce multivariate Dirichlet series to address analogous subgroup and subring growth problems in a broader context. We elaborate on this and present some further concluding remarks in Section 5.

## 2. The cotype zeta function of $\mathbb{Z}^d$

We recall Petrogradsky's definition of the *cotype zeta function* of  $\mathbb{Z}^d$ , which he calls the *multiple zeta function* of  $\mathbb{Z}^d$ .

**Definition 2.1** ([24]). Let  $d$  be a positive integer and let  $a_\alpha(\mathbb{Z}^d)$  be the number of subgroups  $\Lambda \subseteq \mathbb{Z}^d$  of cotype  $\alpha$ . We define the *cotype zeta function* of  $\mathbb{Z}^d$ :

$$\zeta_{\mathbb{Z}^d}(s_1, \dots, s_d) = \sum_{H \subseteq \mathbb{Z}^d} \alpha_1(H)^{-s_1} \cdots \alpha_d(H)^{-s_d} = \sum_{\alpha=(\alpha_1, \dots, \alpha_d)} a_\alpha(\mathbb{Z}^d) \cdot \alpha_1^{-s_1} \cdots \alpha_d^{-s_d}.$$

Note that  $\zeta_{\mathbb{Z}^d}(s, \dots, s) = \zeta_{\mathbb{Z}^d}(s)$ , so this multivariable function generalizes the subgroup growth zeta function of  $\mathbb{Z}^d$ .

The subgroup growth zeta function of  $\mathbb{Z}^d$  has an Euler product, and Petrogradsky shows that this multivariable generalization has one as well.

**Lemma 2.2** ([24, Lemma 1.1]). For each  $d \geq 1$  we have

$$\zeta_{\mathbb{Z}^d}(s_1, \dots, s_d) = \prod_p \zeta_{\mathbb{Z}^d, p}(s_1, \dots, s_d),$$

where the local factor for each prime  $p$  is defined as

$$\zeta_{\mathbb{Z}^d, p}(s_1, \dots, s_d) = \sum_{m=0}^{\infty} \sum_{\substack{H \subseteq \mathbb{Z}^d \\ [\mathbb{Z}^d : H] = p^m}} \alpha_1(H)^{-s_1} \cdots \alpha_d(H)^{-s_d}.$$

One of the main results of [24] is the computation of the local factors of the cotype zeta function of  $\mathbb{Z}^d$  in terms of permutation descents and  $q$ -binomial coefficients. We fix some notation and recall basic properties of these combinatorial objects following [24, Section 3]:

$$\begin{aligned} [n]_q &= \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}; \\ [n]_q! &= [n]_q [n-1]_q \cdots [2]_q; \\ \begin{bmatrix} n \\ j \end{bmatrix}_q &= \frac{[n]_q!}{[j]_q! [n-j]_q!}; \\ \begin{bmatrix} m_1 + m_2 + \cdots + m_k \\ m_1, m_2, \dots, m_k \end{bmatrix}_q &= \frac{[m_1 + m_2 + \cdots + m_k]_q!}{[m_1]_q! [m_2]_q! \cdots [m_k]_q!} \\ &= \begin{bmatrix} m_1 + m_2 + \cdots + m_k \\ m_1 \end{bmatrix}_q \begin{bmatrix} m_2 + \cdots + m_k \\ m_2 \end{bmatrix}_q \cdots \begin{bmatrix} m_{k-1} + m_k \\ m_{k-1} \end{bmatrix}_q. \end{aligned}$$

Let  $\mathbb{N}_d = \{1, 2, \dots, d\}$  and suppose  $\lambda \subseteq \mathbb{N}_d$ . If  $\lambda = \emptyset$  then we set  $\begin{bmatrix} d \\ \lambda \end{bmatrix}_q = 1$ . Otherwise, if  $\lambda = \{\lambda_1, \dots, \lambda_k\}$ , where  $d \geq \lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 1$ , let  $|\lambda| = k$  and  $m_i = \lambda_i - \lambda_{i+1}$  for  $0 \leq i \leq k$ , where we set  $\lambda_{k+1} = 0$  and  $\lambda_0 = d$ . Note that  $d = m_0 + m_1 + \cdots + m_k$ . We define the following polynomials in  $q$ :

$$\begin{bmatrix} d \\ \lambda \end{bmatrix}_q = \begin{bmatrix} d \\ m_0, m_1, \dots, m_k \end{bmatrix}_q, \quad \lambda \subseteq \mathbb{N}_d; \tag{2.1}$$

$$w_{d, \lambda}(q) = \sum_{\mu \subseteq \lambda} (-1)^{|\lambda| - |\mu|} \begin{bmatrix} d \\ \mu \end{bmatrix}_q, \quad \lambda \subseteq \mathbb{N}_d. \tag{2.2}$$

**Theorem 2.3** ([24, Theorem 3.1]). Consider the cotype zeta function of  $\mathbb{Z}^d$ . For each  $j$  satisfying  $1 \leq j \leq d$ , we introduce the new variable  $z_j = s_1 + \cdots + s_j - j(d - j)$ . Then

1.

$$\zeta_{\mathbb{Z}^d}(s_1, \dots, s_d) = \zeta(z_1)\zeta(z_2) \cdots \zeta(z_d) \cdot f(s_1, \dots, s_{d-1}), \quad \text{where} \quad (2.3)$$

$$f(s_1, \dots, s_{d-1}) = \prod_p \left( 1 + \sum_{\emptyset \neq \lambda \subseteq \mathbb{N}_{d-1}} w_{d,\lambda}(p^{-1}) \prod_{j \in \lambda} p^{-z_j} \right), \quad (2.4)$$

with the sum taken over all nonempty subsets  $\lambda \subseteq \mathbb{N}_{d-1}$ .

2. The region of absolute convergence of the cotype zeta function is  $\operatorname{Re}(z_j) > 1$ ,  $j = 1, \dots, d$ .
3. The region of absolute convergence of the product over primes is  $\operatorname{Re}(z_j) > 0$ ,  $j = 1, \dots, d - 1$ ; and (2.3) is the analytic continuation to this region.
4. The local factors are rational functions in the variables  $t_i = p^{-z_i}$ ,  $i = 1, \dots, d$ :

$$\zeta_{\mathbb{Z}^d,p}(s_1, \dots, s_d) = \frac{\sum_{\lambda \subseteq \mathbb{N}_{d-1}} w_{d,\lambda}(p^{-1}) \prod_{j \in \lambda} t_j}{(1 - t_1)(1 - t_2) \cdots (1 - t_d)}. \quad (2.5)$$

The polynomials  $w_{d,\lambda}(q)$  that arise have been studied extensively in the combinatorial literature. The first part of Theorem 2.5 is stated in [24, Theorem 3.1 (2)], while the other two parts are due to Stanley [32, Corollary 3.2] and [33, Section 2.2.5].

**Definition 2.4.** Let  $\pi \in S_d$  be a permutation. We call  $i \in \{1, 2, \dots, d - 1\}$  a *descent* of  $\pi$ , provided that  $\pi(i) > \pi(i + 1)$ . For  $\pi \in S_d$ , let  $D'(\pi)$  denote its set of descents.

A pair  $(i, j)$  is called an *inversion* of  $\pi$  if and only if  $i < j$  and  $\pi(i) > \pi(j)$ . Let  $\operatorname{inv}(\pi)$  denote the number of inversions of  $\pi$ .

Note that  $d$  cannot be a descent of a permutation in  $S_d$ .

**Theorem 2.5.** Let  $\lambda \subseteq \mathbb{N}_{d-1}$  be fixed.

1. There exists a number  $N \geq |\lambda|$  such that  $w_{d,\lambda}(q)$  is a polynomial in  $q$  with nonnegative integer coefficients of the form

$$w_{d,\lambda}(q) = q^N + \sum_{i > N} \tau_i q^i.$$

2. We have that

$$w_{d,\lambda}(q) = \sum_{\substack{\pi \in S_d \\ D'(\pi) = \lambda}} q^{\operatorname{inv}(\pi)}.$$

3. We have that

$$w_{d,\lambda}(q) = \det \left( \begin{bmatrix} d - \lambda_{i+1} \\ \lambda_j - \lambda_{i+1} \end{bmatrix}_q \mid 0 \leq i, j \leq k \right).$$

To conclude this section, we compare the results of Petrogradsky described here to the work of du Sautoy and Lubotzky [28], which builds on earlier work of Igusa [16]. Theorem 5.9 of [28], specialized to  $G = \operatorname{GL}_d$  and  $\rho$  the standard representation gives (2.5). (The result of [28] is specialized to a single variable, but the multivariate extension is obvious.) Petrogradsky's

proof uses a cotype-preserving bijective correspondence between finite index subgroups  $\Lambda$  of  $\mathbb{Z}^d$  and subgroups of the finite group  $\mathbb{Z}^d/N\mathbb{Z}^d$ , where  $N = \alpha_1(\Lambda)$ . The number of the latter can be expressed in terms of  $q$ -binomial coefficients [5]. On the other hand, du Sautoy and Lubotzky interpret the  $p$ -part of the zeta function as a  $p$ -adic integral over  $\mathrm{GL}_d(\mathbb{Z}_p)$ , which they compute using the Iwahori decomposition. This leads to a sum over the (affine) Weyl group equivalent to (2.5). These ideas have been further developed to prove local functional equations for zeta functions of nilpotent groups and other Igusa-type zeta functions; see, e.g., [17, 40].

### 3. Density results for the corank

We begin by introducing the Dirichlet series counting sublattices of  $\mathbb{Z}^d$  of corank less than or equal to  $m$ . This is given by

$$\zeta_{\mathbb{Z}^d}^{(m)}(s) = \sum_{\substack{\Lambda \subset \mathbb{Z}^d \\ \mathrm{corank}(\Lambda) \leq m}} \frac{1}{[\mathbb{Z}^d : \Lambda]^s}. \quad (3.1)$$

Recall that a sublattice of corank at most  $m$  will have cotype  $(\alpha_1, \alpha_2, \dots, \alpha_d)$  with  $\alpha_{m+1} = \dots = \alpha_d = 1$ . Therefore, in terms of Petrogradsky's expression for the cotype zeta function given in Theorem 2.3, we have

$$\begin{aligned} \zeta_{\mathbb{Z}^d}^{(m)}(s) &= \lim_{s_{m+1} \rightarrow \infty} \cdots \lim_{s_d \rightarrow \infty} \zeta_{\mathbb{Z}^d}(s, \dots, s, s_{m+1}, \dots, s_d) \\ &= \zeta(s - (d-1))\zeta(2s - 2(d-2)) \cdots \zeta(ms - m(d-m))f_d^{(m)}(s), \end{aligned} \quad (3.2)$$

where

$$f_d^{(m)}(s) = \prod_p f_{d,p}^{(m)}(s) = \prod_p \left( \sum_{\lambda \subset \{1, \dots, m\}} \omega_{d,\lambda}(p^{-1}) \prod_{j \in \lambda} p^{-js + j(d-j)} \right). \quad (3.3)$$

The analytic properties of  $\zeta_{\mathbb{Z}^d}^{(m)}(s)$  will lead to our desired density results.

**Proposition 3.1.** *The corank at most  $m$  zeta function  $\zeta_{\mathbb{Z}^d}^{(m)}(s)$  has a simple pole at  $s = d$  with residue*

$$\prod_p \left( (1 - p^{-1}) \sum_{i=0}^m \begin{bmatrix} d \\ i \end{bmatrix}_q \frac{p^{-i^2}}{\prod_{j=1}^i (1 - p^{-j})} \right). \quad (3.4)$$

The simple pole comes from the simple pole of the Riemann zeta function  $\zeta(s - (d-1))$  at  $s = d$ . The other zeta factors in (3.2) are holomorphic at  $s = d$  and collectively contribute a factor of  $\prod_{2 \leq j \leq m} \zeta(j^2)$  at  $s = d$  to the residue. Thus

$$\mathrm{Res}_{s=d} \zeta_{\mathbb{Z}^d}^{(m)}(s) = \left( \prod_{2 \leq j \leq m} \zeta(j^2) \right) f_d^{(m)}(d). \quad (3.5)$$

To complete the proof of Proposition 3.1, it remains to evaluate

$$f_{d,p}^{(m)}(d) = \sum_{\lambda \subset \{1, \dots, m\}} \omega_{d,\lambda}(p^{-1}) \prod_{j \in \lambda} p^{-j^2} \quad (3.6)$$



and take the product over all primes  $p$ . Setting  $q = p^{-1}$ , we compute

$$\begin{aligned} \sum_{\lambda \subset \{1, \dots, m\}} \omega_{d, \lambda}(q) \prod_{j \in \lambda} q^{j^2} &= \sum_{\lambda \subset \{1, \dots, m\}} \left( \sum_{\mu \subset \lambda} (-1)^{|\lambda| - |\mu|} \begin{bmatrix} d \\ \mu \end{bmatrix}_q \right) \prod_{j \in \lambda} q^{j^2} \\ &= \sum_{\mu \subset \{1, \dots, m\}} \begin{bmatrix} d \\ \mu \end{bmatrix}_q \left( \sum_{\lambda \supset \mu} (-1)^{|\lambda| - |\mu|} \prod_{j \in \lambda} q^{j^2} \right) \\ &= \sum_{\mu \subset \{1, \dots, m\}} \begin{bmatrix} d \\ \mu \end{bmatrix}_q \prod_{j \in \mu} q^{j^2} \prod_{j \notin \mu} (1 - q^{j^2}). \end{aligned} \quad (3.7)$$

In order to go further we need the intermediate result of [Lemma 3.3](#).

### 3.1. A $q$ -multinomial identity

The lemma below follows from the  $q$ -binomial theorem.

**Lemma 3.2.** *Let  $e, n$  be nonnegative integers. We have*

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2 + ek} \prod_{j=k+1+e}^{n+e} (1 - q^j) = 1. \quad (3.8)$$

This lemma will be used in the proof of [Lemma 3.3](#). We note in passing that setting  $e = 0$  and letting  $n \rightarrow \infty$  yields the generating series for partitions in terms of the Durfee number generating series.

**Lemma 3.3.** *Let  $1 \leq i \leq d$  be integers. We have*

$$\sum_{\mu \subset \{1, \dots, i-1\}} \begin{bmatrix} d \\ \mu \cup \{i\} \end{bmatrix}_q \prod_{j \in \mu} q^{j^2} \prod_{j \in \{1, \dots, i-1\} \setminus \mu} (1 - q^{j^2}) = \begin{bmatrix} d \\ i \end{bmatrix}_q \prod_{j=1}^i \frac{1 - q^{j^2}}{1 - q^j}. \quad (3.9)$$

The argument below is similar to that given in Section 4.1 of Stasinski–Voll [35]. We give a proof for completeness.

**Proof of Lemma 3.3.** We argue by induction on  $i$ . The base case  $i = 1$  is immediate. Assume the identity is true for all  $i_0$  satisfying  $1 \leq i_0 < i$ . We remove the contribution of  $\mu = \emptyset$  from the left-hand side of (3.9) and write it as

$$\begin{aligned} &\sum_{\mu \subset \{1, \dots, i-1\}} \begin{bmatrix} d \\ \mu \cup \{i\} \end{bmatrix}_q \prod_{j \in \mu} q^{j^2} \prod_{j \in \{1, \dots, i-1\} \setminus \mu} (1 - q^{j^2}) - \begin{bmatrix} d \\ i \end{bmatrix}_q \prod_{j=1}^{i-1} (1 - q^{j^2}) \\ &= \sum_{i_0=1}^{i-1} \sum_{\mu \subset \{1, \dots, i_0-1\}} \begin{bmatrix} d \\ \mu \cup \{i_0, i\} \end{bmatrix}_q \prod_{j \in \mu \cup \{i_0\}} q^{j^2} \prod_{j \in \{1, \dots, i-1\} \setminus \mu \cup \{i_0\}} (1 - q^{j^2}) \\ &= \sum_{i_0=1}^{i-1} \begin{bmatrix} d - i_0 \\ i - i_0 \end{bmatrix}_q q^{i_0^2} \prod_{j=i_0+1}^{i-1} (1 - q^{j^2}) \\ &\quad \times \sum_{\mu \subset \{1, \dots, i_0-1\}} \begin{bmatrix} d \\ \mu \cup \{i_0\} \end{bmatrix}_q \prod_{j \in \mu} q^{j^2} \prod_{j \in \{1, \dots, i_0-1\} \setminus \mu} (1 - q^{j^2}), \end{aligned} \quad (3.10)$$

where we have used the identity

$$\left[ \begin{matrix} d \\ \mu \cup \{i_0, i\} \end{matrix} \right]_q = \left[ \begin{matrix} d \\ \mu \cup \{i_0\} \end{matrix} \right]_q \left[ \begin{matrix} d - i_0 \\ i - i_0 \end{matrix} \right]_q$$

in the final step. We continue by using the inductive hypothesis on the inner sum, i.e., the expression in (3.10), and see that the left-hand side of (3.9) is equal to:

$$\begin{aligned} \left[ \begin{matrix} d \\ i \end{matrix} \right]_q \prod_{j=1}^{i-1} (1 - q^{j^2}) + \sum_{i_0=1}^{i-1} \left[ \begin{matrix} d - i_0 \\ i - i_0 \end{matrix} \right]_q q^{i_0^2} \prod_{j=i_0+1}^{i-1} (1 - q^{j^2}) \left[ \begin{matrix} d \\ i_0 \end{matrix} \right]_q \prod_{j=1}^{i_0} \frac{1 - q^{j^2}}{1 - q^j} \\ = \prod_{j=1}^{i-1} (1 - q^{j^2}) \sum_{i_0=0}^{i-1} \left[ \begin{matrix} d \\ i \end{matrix} \right]_q \left[ \begin{matrix} i \\ i_0 \end{matrix} \right]_q q^{i_0^2} \prod_{j=1}^{i_0} (1 - q^j)^{-1} \end{aligned}$$

where we have used the subset-of-a-subset identity. Comparing with the right-hand side of (3.9), we are reduced to proving

$$\sum_{i_0=0}^{i-1} \left[ \begin{matrix} i \\ i_0 \end{matrix} \right]_q q^{i_0^2} \prod_{j=1}^{i_0} (1 - q^j)^{-1} = \frac{1 - q^{i^2}}{\prod_{j=1}^i (1 - q^j)}$$

or equivalently,

$$\sum_{i_0=0}^{i-1} \left[ \begin{matrix} i \\ i_0 \end{matrix} \right]_q q^{i_0^2} \prod_{j=i_0+1}^i (1 - q^j) = 1 - q^{i^2}.$$

We can write this a little more nicely:

$$\sum_{i_0=0}^i \left[ \begin{matrix} i \\ i_0 \end{matrix} \right]_q q^{i_0^2} \prod_{j=i_0+1}^i (1 - q^j) = 1.$$

This is the case  $e = 0$  of Lemma 3.2.  $\square$

### 3.2. Conclusion of the proof of Proposition 3.1

We return to the evaluation of  $f_{d,p}^{(m)}(d)$  using the expression of (3.7):

$$f_{d,p}^{(m)}(d) = \sum_{\mu \subset \{1, \dots, m\}} \left[ \begin{matrix} d \\ \mu \end{matrix} \right]_q \prod_{j \in \mu} q^{j^2} \prod_{j \notin \mu} (1 - q^{j^2}).$$

By Lemma 3.3, the above sum restricted to subsets with largest element  $i$  yields

$$\begin{aligned} \left[ q^{i^2} \prod_{j=i+1}^m (1 - q^{j^2}) \right] \sum_{\mu \subset \{1, \dots, i-1\}} \left[ \begin{matrix} d \\ \mu \cup \{i\} \end{matrix} \right]_q \prod_{j \in \mu} q^{j^2} \prod_{j \in \{1, \dots, i-1\} \setminus \mu} (1 - q^{j^2}) \\ = \left[ \begin{matrix} d \\ i \end{matrix} \right]_q \frac{q^{i^2} \prod_{j=1}^m (1 - q^{j^2})}{\prod_{j=1}^i (1 - q^j)}. \end{aligned}$$

Noting that  $i = 0$  corresponds to the contribution of  $\mu = \emptyset$ , we sum over all  $i$  to obtain

$$f_{d,p}^{(m)}(d) = \left[ \prod_{j=1}^m (1 - q^{j^2}) \right] \sum_{i=0}^m \left[ \begin{matrix} d \\ i \end{matrix} \right]_q \frac{q^{i^2}}{\prod_{j=1}^i (1 - q^j)}.$$

Now taking the product over  $p$  cancels the zeta factors in (3.5) and we are left with

$$\text{Res}_{s=d} \zeta_{\mathbb{Z}^d}^{(m)}(s) = \prod_p \left( (1 - p^{-1}) \sum_{i=0}^m \begin{bmatrix} d \\ i \end{bmatrix}_{p^{-1}} \frac{p^{-i^2}}{\prod_{j=1}^i (1 - p^{-j})} \right).$$

This concludes the proof of Proposition 3.1.

### 3.3. The density of sublattices of corank at most $m$

Theorem 1.1, the asymptotic expression for the number of sublattices with corank at most  $m$ , follows immediately from Proposition 3.1 and the analytic continuation statements from Theorem 2.3.

We note that the constant term in the expression (1.3) is

$$\frac{\zeta(d)\zeta(d-1)\cdots\zeta(2)}{d} = \frac{1}{d} \prod_p \prod_{j=0}^{d-2} (1 - p^{-d+j})^{-1}.$$

Taking the quotient of this term with the constant term in Theorem 1.1 completes the proof of Corollary 1.2.

## 4. Sylow $p$ -subgroups of cokernels of matrices in Hermite normal form

The goal of this section is to prove the second of the equivalent statements in Theorem 1.5. Our strategy for determining the distribution of Sylow  $p$ -subgroups of cokernels of matrices in Hermite normal form is to relate this distribution to the distribution of cokernels of Haar random  $p$ -adic matrices. Haar measure on the  $p$ -adic integers  $\mathbb{Z}_p$  gives rise to Haar measure on  $M_d(\mathbb{Z}_p)$ , normalized so that the total volume is 1. More concretely, each matrix entry can be chosen independently with respect to Haar measure on  $\mathbb{Z}_p$ . Throughout the rest of this section, we use  $\text{Prob}_{M \in M_d(\mathbb{Z}_p)}(\cdot)$  to denote the probability that a Haar random matrix  $M \in M_d(\mathbb{Z}_p)$  has some property. This is equal to the volume of the subset of  $M_d(\mathbb{Z}_p)$  consisting of matrices with this property. We give an example from the introduction. Recall the distribution  $P_d^p$  on finite abelian  $p$ -groups of rank at most  $d$  defined in (1.8).

**Proposition 4.1** ([13, Proposition 1]). *Let  $G$  be a finite abelian  $p$ -group of rank at most  $d$ . Let  $M \in M_d(\mathbb{Z}_p)$  be a random matrix. Then*

$$\text{Prob}_{M \in M_d(\mathbb{Z}_p)}(\text{cok}(M) \cong G) = P_d^p(G).$$

We use the following fact, which follows from Proposition 4.1 and [9, Corollary 3.8].

**Proposition 4.2.** *Let  $e \geq 0$  be an integer and  $M \in M_d(\mathbb{Z}_p)$  be a random matrix. Then*

$$\text{Prob}_{M \in M_d(\mathbb{Z}_p)}(|\text{cok}(M)| = p^e) = \frac{\prod_{j=1}^d (1 - p^{-j})}{p^e} \begin{bmatrix} d + e - 1 \\ e \end{bmatrix}_{p^{-1}}.$$

Any  $\alpha \in \mathbb{Z}_p$  can be written uniquely as  $\alpha = p^e u$  where  $e \in \mathbb{Z}_{\geq 0}$  and  $u \in \mathbb{Z}_p$  is a unit. In this case, we write  $v_p(\alpha) = e$ . Note that  $|\text{cok}(M)| = p^e$  if and only if  $v_p(\det(M)) = e$ .

We will use the following analogue of Proposition 1.3 for matrices with entries in  $\mathbb{Z}_p$ .

**Proposition 4.3** ([8, Theorem 3.1.7]). *Any invertible  $M \in M_d(\mathbb{Z}_p)$  can be written uniquely as a product  $M = UH$  where  $U \in \text{GL}_d(\mathbb{Z}_p)$  and  $H$  is an upper triangular matrix with entries*

$a_{ij}$  such that  $a_{jj} = p^{n_j}$  where  $n_j \geq 0$  and each  $a_{ij}$  with  $i < j$  is an integer satisfying  $0 \leq a_{ij} < p^{n_j}$ .

The matrix  $H$  in Proposition 4.3 is the  $p$ -adic version of an integer matrix in Hermite normal form. We call  $H$  the *Hermite normal form* of  $M$  and write  $\text{HNF}(M) = H$ . We say that  $H \in M_d(\mathbb{Z}_p)$  of this type is in *Hermite normal form*.

The left multiplication by the matrix  $U$  corresponds to operations on the rows of  $M$  that are operations on the standard basis vectors  $e_i$ . The invertibility of  $U$  over  $\mathbb{Z}_p$  ensures that the rows of  $M$  and the rows of  $H$  generate the same sublattice of  $\mathbb{Z}_p^n$ . Therefore, if  $M = UH$  are as in Proposition 4.3, then  $\text{cok}(M) \cong \text{cok}(H)$ .

**Proposition 4.4.** Suppose  $H \in M_d(\mathbb{Z}_p)$  is in Hermite normal form and  $\det(H) = p^e$ . Then,

$$\begin{aligned} \text{Prob}_{M \in M_d(\mathbb{Z}_p)}(\text{HNF}(M) = H) &= \frac{\text{Prob}_{M \in M_d(\mathbb{Z}_p)}(v_p(\det(M)) = v_p(\det(H)))}{\#\{\Lambda \subseteq \mathbb{Z}^d : [\mathbb{Z}^d : \Lambda] = \det(H)\}} \\ &= \frac{\prod_{j=1}^d (1 - p^{-j})}{p^{ed}}. \end{aligned}$$

**Proof.** Since  $M = U \text{HNF}(M)$  for some  $U \in \text{GL}_d(\mathbb{Z}_p)$ , and  $\det(U)$  is a unit in  $\mathbb{Z}_p$ , we see that  $v_p(\det(M)) = v_p(\det(\text{HNF}(M)))$ . The set of  $M \in M_d(\mathbb{Z}_p)$  with  $v_p(\det(M)) = p^e$  is a disjoint union of orbits  $\text{GL}_d(\mathbb{Z}_p)H_1, \dots, \text{GL}_d(\mathbb{Z}_p)H_k$ , where  $H_1, \dots, H_k$  are the finitely many distinct matrices in Hermite normal form of determinant  $p^e$ . These matrices are in bijection with the sublattices of  $\mathbb{Z}_p^d$  of index  $p^e$ , which are in bijection with the sublattices of  $\mathbb{Z}^d$  of index  $p^e$ .

The volume of the orbit  $\text{GL}_d(\mathbb{Z}_p)H_i$  is equal to the probability that  $d$  randomly chosen vectors of  $\mathbb{Z}_p^d$  each lie in the lattice spanned by the rows of  $H_i$  and are linearly independent. This probability depends on  $\det(H_i)$ , but not on the particular choice of  $H_i$ . This completes the proof of the first equality in Proposition 4.4.

Proposition 4.2 gives  $\text{Prob}_{M \in M_d(\mathbb{Z}_p)}(v_p(\det(M)) = v_p(\det(H)))$ . We know that

$$\#\{\Lambda \subseteq \mathbb{Z}^d : [\mathbb{Z}^d : \Lambda] = \det(H)\}$$

is the  $p^{-es}$  coefficient of the power series expansion of  $\zeta_{\mathbb{Z}^d}(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-(d-1))$ , which is  $\left[ \begin{smallmatrix} d+e-1 \\ e \end{smallmatrix} \right]_p$ .

For the final equality in Proposition 4.4, recall that

$$\left[ \begin{smallmatrix} d+e-1 \\ e \end{smallmatrix} \right]_p = \left[ \begin{smallmatrix} d+e-1 \\ e \end{smallmatrix} \right]_{p^{-1}} p^{e(d-1)},$$

and therefore,

$$\frac{\prod_{j=1}^d (1 - p^{-j})}{\frac{\left[ \begin{smallmatrix} d+e-1 \\ e \end{smallmatrix} \right]_{p^{-1}}}{p^e}} = \frac{\prod_{j=1}^d (1 - p^{-j})}{p^{ed}}. \quad \square$$

By Proposition 1.3, a sublattice  $\Lambda \subseteq \mathbb{Z}^d$  gives rise to a  $d \times d$  integer matrix in Hermite normal form  $H(\Lambda)$  with  $[\mathbb{Z}^d : \Lambda] = \det(H(\Lambda))$ . An application of the Chinese remainder theorem shows that for each prime  $p$ ,  $H(\Lambda)$  determines a matrix in Hermite normal form with determinant equal to a power of  $p$ .

**Definition 4.5.** Suppose  $H \in \mathcal{H}_d(\mathbb{Z})$  has entries  $a_{ij}$  and  $\det(H) = p_1^{e_1} \cdots p_r^{e_r}$  where  $p_1, \dots, p_r$  are distinct primes and each  $e_i$  is a positive integer. For each prime  $p_i$ , define  $H_{p_i} \in \mathcal{H}_d(\mathbb{Z})$

with  $\det(H_{p_i}) = p_i^{e_i}$  as follows. Write each  $a_{jj} = p_i^{b_j} u_j$  where  $p_i \nmid u_j$ . Let  $H_{p_i}$  be the upper-triangular matrix with entries  $b_{jk}$  defined so that:

- $b_{jj} = p_i^{b_j}$ , and
- for  $1 \leq j < k \leq d$ ,  $b_{jk}$  is the unique integer  $0 \leq b_{jk} < p_i^{b_k}$  with  $b_{jk} \equiv \frac{a_{jk}}{u_j} \pmod{p_i^{b_k}}$ .

If  $p$  is a prime for which  $p \nmid \det(H)$ , define  $H_p = I_d$ .

With this definition, it is clear that if  $H \in \mathcal{H}_d(\mathbb{Z})$ , then each  $H_p \in \mathcal{H}_d(\mathbb{Z})$  as well. The following proposition follows from an application of the Chinese remainder theorem.

**Proposition 4.6.** *Definition 4.5 gives a bijection between matrices  $H \in \mathcal{H}_d(\mathbb{Z})$  and collections  $\{H_p\}$  consisting of one  $H_p \in \mathcal{H}_d(\mathbb{Z})$  for each prime  $p$  and where all but finitely many  $H_p = I_d$ . Moreover,  $\det(H) = \prod_p \det(H_p)$ .*

Note that for any  $H \in \mathcal{H}_d(\mathbb{Z})$ ,  $\text{cok}(H)_p = \text{cok}(H_p)$ . That is, the Sylow  $p$ -subgroup of the cokernel of  $H$  depends only on  $H_p$ . Suppose  $Q \in \mathcal{H}_d(\mathbb{Z})$  has  $\det(Q) = p^e$ . Theorem 1.5 follows from showing that the probability that a random  $H \in \mathcal{H}_d(\mathbb{Z})$  has  $H_p = Q$  is equal to the probability that a random  $M \in M_d(\mathbb{Z}_p)$  has  $\text{HNF}(M) = Q$ .

**Proposition 4.7.** *Suppose  $Q \in \mathcal{H}_d(\mathbb{Z})$  satisfies  $\det(Q) = p^e$  for some prime  $p$  and positive integer  $e$ . Then*

$$\lim_{X \rightarrow \infty} \frac{\#\{H \in \mathcal{H}_d(X) : H_p = Q\}}{\#\mathcal{H}_d(X)} = \frac{\prod_{j=1}^d (1 - p^{-j})}{p^{ed}}.$$

**Proof.** There is a bijection between  $\{H \in \mathcal{H}_d(X) : H_p = Q\}$  and  $\{H \in \mathcal{H}_d(\frac{X}{p^e}) : H_p = I_d\}$ . Note that

$$\lim_{X \rightarrow \infty} \frac{\#\{H \in \mathcal{H}_d(\frac{X}{p^e}) : H_p = I_d\}}{\#\mathcal{H}_d(X)} = \lim_{X \rightarrow \infty} \frac{\#\{H \in \mathcal{H}_d(\frac{X}{p^e}) : H_p = I_d\}}{\#\mathcal{H}_d(\frac{X}{p^e})} \frac{\#\mathcal{H}_d(\frac{X}{p^e})}{\#\mathcal{H}_d(X)}.$$

By (1.3), we have that

$$\lim_{X \rightarrow \infty} \frac{\#\mathcal{H}_d(\frac{X}{p^e})}{\#\mathcal{H}_d(X)} = \frac{1}{(p^e)^d}. \quad (4.1)$$

Let  $g_d(k)$  be the number of sublattices  $\Lambda \subseteq \mathbb{Z}^d$  of index  $k$  for which  $H(\Lambda)_p = I_d$ . We have

$$\sum_{k=1}^{\infty} g_d(k) k^{-s} = ((1 - p^{-s})(1 - p^{-s+1}) \cdots (1 - p^{-s+d-1})) \zeta_{\mathbb{Z}^d}(s). \quad (4.2)$$

This expression equals  $\zeta_{\mathbb{Z}^d}(s)$  except that the local factor at  $p$  in its Euler product is replaced with 1. It is still clear that the right-most pole of this function is a simple pole at  $s = d$ , so applying a Tauberian theorem as we did to derive expression (1.3) shows that

$$\lim_{X \rightarrow \infty} \frac{\#\{H \in \mathcal{H}_d(\frac{X}{p^e}) : H_p = I_d\}}{\#\mathcal{H}_d(\frac{X}{p^e})} = (1 - p^{-d})(1 - p^{-d+1}) \cdots (1 - p^{-1}). \quad (4.3)$$

Combining Eqs. (4.1) and (4.3) completes the proof.  $\square$

Combining [Propositions 4.1, 4.4, and 4.7](#) completes the proof of [Theorem 1.5](#).

## 5. Conclusion

The results and methods of this paper suggest several natural directions for further study.

### 5.1. Subgroup and subring growth zeta functions

We may also try to construct multivariate Dirichlet series to study subgroup growth for other groups. A first case of potential interest is the discrete Heisenberg group

$$H_3 = \langle a, b, c \mid c = [a, b], [a, c] = [b, c] = 1 \rangle.$$

The normal subgroup zeta function of  $H_3$  is

$$\zeta_{H_3}(s) = \sum_{N \trianglelefteq_f H_3} [H_3 : N]^{-s} \quad (5.1)$$

where the sum is over all finite index normal subgroups of  $H_3$ . It has been shown [\[14\]](#) that

$$\zeta_{H_3}(s) = \zeta(s)\zeta(s-1)\zeta(3s-2). \quad (5.2)$$

A multivariate generalization of this series might give more refined information on the distribution of the finite groups which arise as quotients of  $H_3$ .

Similar questions can be asked for subring growth. For example, we expect that the cotype subring zeta function of  $\mathbb{Z}^3$  can be used to show that in contrast to the case studied here, most of the subrings of  $\mathbb{Z}^3$  (ordered by index) are not cocyclic. In a nonabelian setting, the Lie ring  $\mathfrak{sl}_2(\mathbb{Z})$  has an explicitly computed zeta function

$$\zeta_{\mathfrak{sl}_2(\mathbb{Z})}(s) = \sum_L [\mathfrak{sl}_2(\mathbb{Z}) : L]^{-s} = P(2^{-s}) \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-1)}{\zeta(3s-1)}, \quad (5.3)$$

where the sum is over all finite index Lie subrings of  $\mathfrak{sl}_2(\mathbb{Z})$  and  $P(x) = (1 + 6x^2 - 8x^3)/(1 - x^3)$  [\[29\]](#). It would be interesting to compute the cotype subring zeta function of  $\mathfrak{sl}_2(\mathbb{Z})$  and use it to find the density of Lie subrings with cyclic quotient.

Klopsch and Voll compute the subring zeta functions of all 3-dimensional Lie algebras over  $\mathbb{Z}_p$  in a uniform manner [\[18\]](#). Their techniques, in particular, should allow one to compute the cotype zeta function for both  $H_3$  and  $\mathfrak{sl}_2(\mathbb{Z})$ .

### 5.2. Zeta functions of classical groups

The subgroup growth zeta function  $\zeta_{\mathbb{Z}^d}(s)$  of  $\mathbb{Z}^d$  also arises in the more general context of the zeta functions associated to algebraic groups studied by Hey, Weil, Tamagawa, Satake, Macdonald and Igusa [\[15,16,22,26,38,41\]](#). For  $G$  a linear algebraic group over  $\mathbb{Q}_p$  and a rational representation  $\rho : G \rightarrow \mathrm{GL}_n$  they define

$$Z_{G,\rho}(s) = \int_{G^+} |\det \rho(g)|^s dg \quad (5.4)$$

where  $G^+ = \rho^{-1}(\rho(G(k) \cap M_n(\mathbb{Z}_p)))$ , where  $\mathbb{Z}_p$  is the ring of integers of  $\mathbb{Q}_p$ . When  $G = \mathrm{GL}_n$  and  $\rho$  is the natural representation,  $Z_{G,\rho}(s)$  is just the  $p$ -part of the subgroup growth zeta function  $\zeta_{\mathbb{Z}^d}(s)$ . In more recent work, du Sautoy and Lubotzky [\[28\]](#) show that  $Z_{G,\rho}(s)$  for

more general  $G$  and  $\rho$  continues to have an interpretation as a generating series counting substructures of algebras.

We take an explicit example from Bhowmik–Grunewald [3], see also [2, Theorem 12]. Let  $\beta$  be the alternating bilinear form on a  $2n$  dimensional space associated to the matrix

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

A sublattice  $\Lambda$  of  $\mathbb{Z}^{2n}$  is  $\beta$ -polarized if  $\hat{\Lambda} = c\Lambda$  for some constant  $c \in \mathbb{Q}^\times$ , where

$$\hat{\Lambda} = \{v \in \mathbb{Z}^{2n} : \beta(u, v) \in \mathbb{Z} \text{ for all } u \in \mathbb{Z}^{2n}\}.$$

Define the group  $\mathrm{GSp}_{2n}(\mathbb{Q})$  of symplectic similitudes by

$$\begin{aligned} \mathrm{GSp}_{2n}(\mathbb{Q}) \\ = \{g \in \mathrm{GL}_{2n}(\mathbb{Q}) : \beta(gx, gy) = \mu_g \beta(x, y) \text{ for some } \mu_g \in \mathbb{Q}^\times \text{ and all } x, y \in \mathbb{Q}^n\}. \end{aligned}$$

Following computations of Satake [26] and Macdonald [22], the zeta function of the group  $\mathrm{GSp}_6(\mathbb{Q})$  is written down explicitly in [28]. Bhowmik and Grunewald use this to show that the number of  $\beta$ -polarized sublattices of  $\mathbb{Z}^6$  of index less than  $X$  is asymptotic to  $cX^{7/3}$  for an explicit constant  $c$ . The results of [28] indicate a way to extend these computations, both to higher rank and to include the distribution of cotype.

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