

Article

Algebraic Properties of Quantum Reference Frames: Does Time Fluctuate?

Martin Bojowald ^{1,*}  and Artur Tsobanjan ²

¹ Institute for Gravitation and the Cosmos, The Pennsylvania State University, 104 Davey Lab, University Park, PA 16802, USA

² College of Arts and Sciences, King's College, 133 North River Street, Wilkes-Barre, PA 18711, USA

* Correspondence: bojowald@psu.edu

Abstract: Quantum reference frames are expected to differ from classical reference frames because they have to implement typical quantum features such as fluctuations and correlations. Here, we show that fluctuations and correlations of reference variables, in particular of time, are restricted by their very nature of being used for reference. Mathematically, this property is implemented by imposing constraints on the system to make sure that reference variables are not physical degrees of freedom. These constraints not only relate physical degrees of freedom to reference variables in order to describe their behavior, they also restrict quantum fluctuations of reference variables and their correlations with system degrees of freedom. We introduce the notion of “almost-positive” states as a suitable mathematical method. An explicit application of their properties to examples of recent interest in quantum reference frames reveals previously unrecognized restrictions on possible frame–system interactions. While currently discussed clock models rely on assumptions that, as shown here, make them consistent as quantum reference frames, relaxing these assumptions will expose the models to new restrictions that appear to be rather strong. Almost-positive states also shed some light on a recent debate about the consistency of relational quantum mechanics.

Keywords: quantum reference frames; time; Hamiltonian constraint; relational quantum mechanics



Citation: Bojowald, M.; Tsobanjan, A. Algebraic Properties of Quantum Reference Frames: Does Time Fluctuate?. *Quantum Rep.* **2022**, *1*, 22–37. <https://doi.org/10.3390/quantum5010003>

Academic Editor: J. Gonzalo Muga

Received: 8 November 2022

Revised: 17 December 2022

Accepted: 24 December 2022

Published: 30 December 2022



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1. Introduction

The unavoidable fact that measurements are made with measurement devices implies that a complete description of the underlying process must use degrees of freedom for both the measured and the measuring systems. In quantum mechanics, both are represented by states or, rather, a single combined state, and they may be expected to be subject to fluctuations, correlations, and entanglement. The accessibility and consistency of relationships between a measured system and a frame set up to describe the measuring device has given rise to much recent work [1–8]. A common method is the application of quantized constraints to impose the reference relations between parts of the combined system. The purpose of the present paper is to point out that constrained reference states produced by imposing quantum constraints are distinct from ordinary entangled states in a precise mathematical sense. The essential feature of constrained states is that they contain redundant information, which is used in computations of probabilities for outcomes of physical measurements, but is not itself directly measurable or subject to dynamics. While we will focus here on temporal reference frames, much of the discussion is general and applies whenever quantum constraints are used.

Examples of redundant information abound in space-time physics (and are therefore prominent in the endeavor of quantum gravity), the most important one being the coordinates used to describe motion in space-time. In our discussion, it will be important to distinguish between the values of coordinates (time or space) and the physical devices used to measure them (clocks or rulers, or their generalized versions). The former amount to

redundant information because they may well be chosen differently. It is important to determine how one could transform between different coordinate choices in order to compare different descriptions of the same physics, but this task is distinct from the determination of physical properties of space and time. A clock or a ruler or some other spatial measurement device, by contrast, is a physical instrument that, in a sensitive quantum measurement, may well interact with the system in some way and influence the measurement outcome. Quantum fluctuations or entanglement between clock and system may then be relevant. However, a clock is not the same as time, and a ruler is not the same as space. Therefore, their mathematical treatments and physical roles may well be different from each other. In the usual formalized treatment, a clock or a ruler is represented by a degree of freedom that directly appears in a Hamiltonian or in a constraint (We discuss clocks within our construction in Section 6.). On the other hand, time and space coordinates are numbers extracted from certain properties of solutions, for instance when one computes the gauge flow generated by a dynamical constraint. Time and space are therefore effective or emergent parameters rather than fundamental quantum degrees of freedom. A dedicated analysis is required in order to determine whether time and space (as opposed to clocks and rulers) should be subject to quantum properties such as fluctuations and entanglement.

In ordinary quantum mechanics, the state of a system is described relative to some external classical reference frame using “coordinates” (or analogous references) that have no inherent physical significance to the system being described. Changing the coordinate frame (for example by rotating Cartesian axes) therefore leads to a physically equivalent state of the system seen from a different coordinate perspective. Schematically, we can represent this transformation as

$$\sum_c \psi_a(c) |c\rangle_a \rightarrow \sum_c \psi_b(c) |c\rangle_b.$$

Here a and b label classical reference systems which are not part of the quantum degrees of freedom, $\psi_i(c)$ are numerical coefficients, and e.g., $|c\rangle_a$ denotes the basis in which c is described in frame a (which may or may not coincide with $|c\rangle_b$). Ultimately, however, measurement references are made with respect to physical objects (clocks, rulers, etc.) which are fundamentally quantum mechanical, and are therefore subject to the usual quantum mechanical effects such as fluctuation, discreteness, and quantum superposition. One can explore the quantum effects of reference frames by incorporating them into the quantum mechanical description. A measurement device a will start in some pre-measurement state $|\phi_a\rangle$ and the measurement of c relative to a can be modeled by some interaction Hamiltonian which will induce a transformation on the combined state of a and c to an entangled state $|\phi_a\rangle \otimes (\sum_c \psi_a(c) |c\rangle_a) \rightarrow \sum_c \psi_a(c) |a(c)\rangle \otimes |c\rangle_a$, where state $|a(c)\rangle$ of the device correlates with state $|c\rangle_a$ of the measured system. Frame transformation therefore corresponds to linking a pair of entangled states

$$\left(\sum_c \psi_a(c) |a(c)\rangle \otimes |c\rangle_a \right) \otimes |\phi_b\rangle \rightarrow \left(\sum_c \psi_b(c) |b(c)\rangle \otimes |c\rangle_b \right) \otimes |\phi_a\rangle.$$

The ongoing work on quantum reference frames deals with the quantum nature of measurement references in a structurally different way (see for example [2]). Here, in any given reference frame, one constructs a more or less ordinary quantum mechanical description of the observed system of interest as well as of all available reference frames other than the one being used. Schematically, given quantum reference frames a and b , and using c to denote “the rest” of the quantum system being modeled, the transformation from frame a to frame b has the form

$$\sum_{b,c} \psi_a(b,c) |b\rangle \otimes |c\rangle \rightarrow \sum_{a,c} \psi_b(a,c) |a\rangle \otimes |c\rangle.$$

In a given reference frame, the degrees of freedom corresponding to the frame itself are not part of the quantum representation, however they appear in the quantum description relative to a different reference frame.

Let us compare the schematic workings of a quantum reference frame to the two uses of reference systems in ordinary quantum mechanics described earlier. It is immediately obvious that quantum reference frames are in general inequivalent to classical reference frames. Classical reference systems are not dynamical within the quantum description, while, at least from the perspective of quantum reference frame a , quantum reference frame b can be quantum mechanically coupled to the observed system. At first, incorporating quantum references as subsystems within an ordinary quantum mechanical description appears as a promising way to simultaneously treat quantum degrees of freedom of a , b , and c , and then to somehow project to individual quantum reference frame descriptions. At the very least, this would require additional couplings, since otherwise subsystem a will not have a description of subsystem b , but only of subsystem c , etc. The viability of embedding a quantum reference frame within a larger ordinary quantum state description of all quantum degrees of freedom can be further restricted if we use some of the details of how quantum reference frames are constructed. Here we turn to some of the work on quantum reference frames that implements quantum frame covariance using Dirac constraint quantization [4,5], especially in the case of temporal references [6,9,10]. The advantage of using quantum constraints is that the process of establishing a quantum reference frame proceeds via a perspective neutral framework of constrained states defined on the algebra of operators that represent the degrees of freedom of all of the reference frames at once.

Here we use a very general algebraic analysis of constrained quantum systems developed in [11] to explore the properties of temporal quantum reference frames. In particular, we assume that such a frame is constructed by solving a quantum Hamiltonian constraint and mapping solutions to the quantum description of a reduced system associated with selecting some observable as a reference time. This approach does not constitute a new definition of a quantum temporal reference frame; it explicitly agrees with the approach taken in [6,9,10]. However, it does provide a novel vantage point: Our algebraic analysis allows us to embed reference states, which are typically defined only on a reduced system, within the states of the entire “unreduced” constrained system. In Section 2, we describe a simple model of a Hamiltonian constraint and briefly review its classical properties as the regulator of temporal reference frames. In Section 3, we review some general properties of constrained quantum states and in Section 4 we introduce a class of constrained states that have a relational interpretation. We argue that constrained states that have an interpretation as (temporal) reference states are quite different from correlated states of quantum mechanics in that the former encode redundancy rather than correlation. We further argue that, in light of this redundancy, a temporal quantum reference frame is not the same as a physical clock. In Section 5, we discuss the restrictions that the form of the Hamiltonian constraint places on the viability of a given temporal quantum reference frame. In Section 6, we discuss the implication of these restrictions on the viability of interacting physical clocks (such as the models considered in [12–14]) as temporal reference variables.

2. Hamiltonian Constraint and Temporal Reference Frames

In canonical descriptions of space-time physics, redundancy of information is mathematically incorporated through the use of constraints. One first describes the properties of a system using degrees of freedom that contain redundancies, and then imposes constraints to make sure that only physical degrees of freedom remain. In quantum gravity, suitable constraints arise automatically within the canonical formulation of general relativity, in which general covariance is realized not manifestly by transformations on space-time tensors but rather by restrictions on allowed configurations.

We will build our present discussion around the technically much simpler example of time-reparameterization-invariant formulation of an ordinary classical mechanical system.

Starting with an N -component system with degrees of freedom $q_i, p_i, i = 1, 2 \dots N$, and a Hamiltonian $H(q_i, p_i; t)$, we extend the phase-space to include a canonical pair given by time q_0 and its canonically conjugate momentum p_0 (often referred to as the energy of the “clock” q_0). Instead of a Hamiltonian, the redundant dynamics is captured by the Hamiltonian constraint

$$C = p_0 + H(q_i, p_i; q_0), \quad (1)$$

where $i \neq 0$ and we have replaced parameter time t in the Hamiltonian by the new phase-space time variable q_0 . Physically allowed states of the system lie on the constraint surface $C = 0$ where $p_0 = -H(q_i, p_i; q_0)$. Hamilton’s equations generated by C through the canonical Poisson bracket are

$$\frac{dq_0}{d\tau} = \{q_0, C\} = \frac{\partial C}{\partial p_0} = 1, \quad \frac{dp_0}{d\tau} = \{p_0, C\} = -\frac{\partial C}{\partial q_0} = 0, \quad (2)$$

and similarly for $i \neq 0$

$$\frac{dq_i}{d\tau} = \{q_i, C\} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{d\tau} = \{p_i, C\} = -\frac{\partial H}{\partial q_i}. \quad (3)$$

In this view, the time and mechanical degrees of freedom are restricted to the constraint surface and evolve along orbits generated by C relative to an external parameter τ . However, this parameter keeping track of evolution along the Hamiltonian flow of C is arbitrary. Rescaling the constraint by any non-vanishing phase-space function $\mathcal{N}$, gives us an entirely equivalent constraint $\tilde{C} = \mathcal{N}C$: it defines the same constraint surface as $\tilde{C} = 0$ implies $C = 0$, and the flow that it produces on the constraint surface reads

$$\frac{df}{d\tilde{\tau}} = \{f, \tilde{C}\} = \{f, \mathcal{N}\}C + \{f, C\}\mathcal{N} \approx \{f, C\}\mathcal{N} = \mathcal{N}\frac{df}{d\tau},$$

where “ $\approx$ ” denotes equality on the constraint surface and f is any phase space function. Evidently, $d/d\tilde{\tau} = \mathcal{N}d/d\tau$ and the new flow is simply a reparameterized version of the original one. Because of this freedom of parameterization, it may be more fruitful to think of constrained evolution as phase-space degrees of freedom evolving relative to each other rather than relative to the arbitrary flow parameter τ .

While our language will be adapted to questions about time and evolution, we note that analogous considerations apply to spatial or other reference systems. The nature of ever-progressing time means that we are forced to consider time translations, generated by a Hamiltonian. One may assume that for a spatial reference frame we do not need to worry about translations: we could stay at a constant position or choose our direction of motion. However, the simple case of a ruler as a spatial reference system shows that we must translate it whenever its finite size does not cover the entire range of a measured system. Spatial translations are therefore essential to spatial measurements, just as evolution or time translations are essential for the nature of time. Since spatial translations are generated by spatial momentum components, the corresponding reference systems are of a form similar to (1), but usually with a simpler expression, such as the sum of system momenta, replacing the interaction term H . For example, the system studied in [4] possesses a true Hamiltonian (rather than a constraint) as well as a constraint $C = \sum_i p_i$ enforcing spatial translation invariance, which structurally is a special case of (1) and can be reduced in the same way. If we aim to allow for non-linear transformations of spatial coordinates, we also have an analog of reparameterizations where C is replaced by $\mathcal{N}C$ with a position-dependent $\mathcal{N}$. From now on, we will focus on the more challenging task of formulating quantum evolution with respect to a temporal quantum reference frame.

The constrained system can be easily reduced to its unextended form in two steps. First we eliminate p_0 using the constraint $p_0 = -H(q_i, p_i; q_0)$. Second, we note that, according to (2), along the orbits generated by the constraint function $q_0(\tau) = \text{const.} + \tau$. We can therefore simply replace q_0 in the Hamiltonian with a parameter time $t = \text{const.} + \tau$ and

$d/d\tau = d/dt$. This reduces the system to just the non-time degrees of freedom q_i, p_i , $i = 1, 2, \dots, N$, which evolve relative to an external parameter time t along orbits of the Hamiltonian $H(q_i, p_i; t)$.

An important feature of the constrained formulation of such a system is that reduction can be performed relative to any function on the extended phase space that has a non-vanishing Poisson bracket with C and therefore uniformly increases or decreases along its flow. Relative to a general time function and its conjugate momentum, the constraint does not have the simple form of Equation (1) and the analysis is geometrically more subtle. Nevertheless, in a classical treatment, for each suitable time, one gets a reduced unconstrained mechanical system with a Hamiltonian generating the dynamics relative to a time parameter. Each such distinct reduction can be interpreted as selection of the particular temporal reference frame associated with the corresponding time function. While the end results of reductions relative to different time functions will, in general, look quite different, they will nevertheless be linked via the “timeless” constrained framework: on the extended phase space different parameterizations of the flow of C on the constraint surface can be linked by following the flow itself with a suitable local rescaling $\mathcal{M}C$, where $\mathcal{M}$ is some (possibly vanishing) function of q_0, p_0, q_i, p_i .

As in this simple example, constraints in canonical systems control redundancy in two ways: they constrain the extended phase space ($C = 0$) and they generate gauge transformations (here along the parameter τ) that relate variables on the constraint surface (after solving $C = 0$) to flow parameters that transform some of the redundant information. In the given example, and in related but technically more involved examples from general relativity, the reduced system, where the leftover variables evolve relative to an external parameter, and the constrained system, where phase space degrees of freedom linked by redundancy evolve relative to each other, are clearly just two equivalent viewpoints of a given system. However, the equivalence works only if we apply a suitable treatment to the constraint on the extended phase space. A relational interpretation of dynamics does not inherently necessitate a Hamiltonian constraint. For example, the Hamiltonian of the original N -component mechanical system generates time-evolution trajectories on the original phase space which (at least locally) can be interpreted as $(2N - 1)$ canonical variables evolving relative to the one remaining variable. In this situation, however, there is no redundancy in the description: all points of the original phase space represent allowed configurations (there is no constraint surface) that are physically distinct (there is no gauge flow).

At the quantum level, a constraint $C = 0$ is implemented by the requirement that a corresponding constraint operator, $\hat{C}$, annihilate all admissible states. If a Hilbert-space representation is used for the quantum system, the condition reads

$$\hat{C}|\psi\rangle = 0 \quad (4)$$

for all admissible states $|\psi\rangle$. In the following section, we use generalized algebraic states to highlight important differences between states that satisfy the quantum constraint (4) and ordinary quantum states that contain correlations between subsystems.

3. Redundancy versus Correlation in the Quantum Setting

Algebraically, a state is defined as a linear functional ω from the operator algebra $\mathcal{A}$ of the system to the complex numbers. The functional is required to be positive if it refers to observable information, where positivity is defined by $\omega(\hat{A}^*\hat{A}) \geq 0$ for all $\hat{A} \in \mathcal{A}$, using a $*$ -relation on the algebra that corresponds to adjointness when represented on a Hilbert space. Such a state can be thought of as an expectation-value functional that assigns the expectation value $\omega(\hat{A})$ to an operator $\hat{A}$, just like $\langle\psi|\hat{A}|\psi\rangle$ or $\text{tr}(\rho\hat{A})$ in a representation. It can be shown, see for instance [15], that every positive linear functional obeys the Cauchy–Schwarz inequality

$$\omega(\hat{A}^*\hat{A})\omega(\hat{B}^*\hat{B}) \geq |\omega(\hat{A}^*\hat{B})|^2 \quad (5)$$

for all $\hat{A}, \hat{B} \in \mathcal{A}$, from which uncertainty relations follow in the textbook manner. If the algebra contains a unit element, $\mathbb{I} \in \mathcal{A}$, a positive state is real: $\omega(\hat{A}^*) = \overline{\omega(\hat{A})}$ and therefore $\omega(\hat{A}) \in \mathbb{R}$ if $\hat{A} = \hat{A}^*$. A state is a minimal requirement for a meaningful description of observable information, given by numbers rather than operators, but it need not be constructed via a Hilbert-space representation or wave functions.

Suppose $\mathcal{A}$ is the operator algebra corresponding to an unconstrained N -component quantum system, generated by $\hat{q}_i, \hat{p}_i$, with $i = 0, 1, 2, \dots$, subject to the usual canonical commutation relations $[\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij} \mathbb{I}$, and suppose ω is an algebraic state in which, say, configurations of subsystems labeled 0 and 1 are entangled. Then we make the following observations.

- Such a state makes probabilistic predictions for measurements of all observables of the system. Therefore it must be positive on the entire algebra $\mathcal{A}$ as discussed above.
- Entanglement requires that correlations between the observables $\hat{q}_0$ and $\hat{q}_1$ are non zero: $\Delta_{q_0 q_1} = \frac{1}{2}\omega(\hat{q}_0 \hat{q}_1 + \hat{q}_1 \hat{q}_0) - \omega(\hat{q}_0)\omega(\hat{q}_1) \neq 0$, and similarly for other correlations involving powers of $\hat{q}_0$ and $\hat{q}_1$.
- Such a correlated state is special: since $[\hat{q}_0, \hat{q}_1] = 0$, positivity of ω places no bounds on the correlations between the configurations of the two subsystems. While there may be dynamical or symmetry considerations that strongly select for correlated states, the system possesses physically valid states in which the two subsystems are completely uncorrelated.
- Even though they are correlated in an entangled state, both $\hat{q}_0$ and $\hat{q}_1$ are individually observable. The state makes probabilistic predictions for measuring either of them individually without the other.

Now let us contrast this with an algebraic state on a constrained system. For simplicity, we will use the classical constraint of Equation (1) so that the quantized constraint is

$$\hat{C} = \hat{p}_0 + H(\hat{q}_i, \hat{p}_i; \hat{q}_0), \quad (6)$$

with $i \neq 0$, where some suitable operator ordering has been chosen for the operator-valued function $H(\hat{q}_i, \hat{p}_i; \hat{q}_0) = H(\hat{q}_i, \hat{p}_i; \hat{q}_0)^*$. We enforce the quantum constraint (4) on an algebraic state by demanding

$$\omega(\hat{A}\hat{C}) = 0 \quad \text{for all } \hat{A} \in \mathcal{A}. \quad (7)$$

It is clear that the above conditions are satisfied by an ordinary null-eigenstate of the constraint operator, where expectation values can be constructed via the Hilbert-space inner product as usual $\langle \psi | \hat{A} \hat{C} | \psi \rangle$. However, a constraint operator of the form considered here will generically have zero in the continuous part of the spectrum. In a representation of algebra $\mathcal{A}$ as acting on a Hilbert space $\mathcal{H}$ the null eigenstates of $\hat{C}$ will belong to the space of linear functionals on $\mathcal{H}$ (the dual space $\mathcal{H}^*$). There will then be no default prescription for taking expectation values of operators relative to such states. Our generalized algebraic states, however, are not tethered to any particular Hilbert space representation of the algebra $\mathcal{A}$ and there is no immediate obstruction for assigning numerical values to all operators in $\mathcal{A}$.

In contrast to an ordinary quantum state (entangled or not), an algebraic state that solves constraint (6) cannot be positive on the entire algebra $\mathcal{A}$: we have $[\hat{q}_0, \hat{C}] = i\hbar \mathbb{I}$, and therefore

$$\frac{1}{2}\omega(\hat{C}\hat{q}_0 + \hat{q}_0\hat{C}) = \omega(\hat{q}_0\hat{C}) + \frac{1}{2}\omega([\hat{C}, \hat{q}_0]) = -\frac{1}{2}i\hbar \mathbb{I} \quad (8)$$

using (7). The result is imaginary, even though $(\hat{C}\hat{q}_0 + \hat{q}_0\hat{C})^* = (\hat{C}\hat{q}_0 + \hat{q}_0\hat{C})$ as long as $\hat{C}^* = \hat{C}$ and $\hat{q}_0^* = \hat{q}_0$, making the corresponding state non-positive.

While constrained quantum states cannot be positive on the entire algebra, they are still subject to certain positivity conditions. In a quantum system subject to a constraint $\hat{C}$, a notion of observables is given by Dirac observables, defined as operators $\hat{O} \in \mathcal{A}$ such that $[\hat{O}, \hat{C}] = 0$. Dirac observables therefore form a subalgebra $\mathcal{A}_{\text{obs}}$ of $\mathcal{A}$ given by the commutant of $\hat{C}$. Their action preserves the space of solutions of $\hat{C}$ in any Hilbert

space representation since $\hat{C}|\psi\rangle = 0$ and $[\hat{O}, \hat{C}] = 0$ implies that $\hat{C}\hat{O}|\psi\rangle = \hat{O}\hat{C}|\psi\rangle = 0$. Furthermore, Dirac observables are invariant under the infinitesimal gauge flow generated by $\hat{C}$ (which exponentiates to the unitary flow $\exp(i\tau\hat{C})$)—they are the quantum counterparts of extended phase space functions that are invariant along the flow generated by the constraint. Classically, such invariant functions are mapped to constants of motion during reduction of the constrained system relative to any valid time function. It therefore makes sense to insist that, in addition to condition (7), physical algebraic states of a constrained system are positive on $\mathcal{A}_{\text{obs}}$. Indeed, since Dirac observables commute with $\hat{C}$, condition (7) does not in any way restrict positivity of the state on $\mathcal{A}_{\text{obs}}$. Which brings us to a second contrasting point.

Constrained quantum states are not inherently entangled. Suppose $\hat{O}_1, \hat{O}_2 \in \mathcal{A}_{\text{obs}}$ are real observables, $\hat{O}_i^* = \hat{O}_i$, correspond to different Dirac observable subsystems $[\hat{O}_1, \hat{O}_2] = 0$, and are not proportional to the constraint (so that condition (7) does not demand $\omega(\hat{A}\hat{O}_i) = 0$ for all $\hat{A} \in \mathcal{A}$). In this case neither the constraint condition nor positivity of states on $\mathcal{A}_{\text{obs}}$ place any restriction on the value of subsystem correlations such as $\Delta_{O_1 O_2} = \frac{1}{2}\omega(\hat{O}_1\hat{O}_2 + \hat{O}_2\hat{O}_1) - \omega(\hat{O}_1)\omega(\hat{O}_2)$ which may well be small or zero.

Finally, constrained quantum states are not special states of the constrained system. Without additional constructions, they are in fact the only states on the full algebra $\mathcal{A}$ that have a meaningful physical interpretation consistent with enforcing the quantum constraint.

4. Almost-Positive States

Much of the immediately preceding discussion references Dirac observables. However, except for simple examples, it is usually hard to construct a complete set of Dirac observables, and, in fact, in a general situation, such a complete set may not exist [16,17]. While a constrained quantum state has to be positive on $\mathcal{A}_{\text{obs}}$ and cannot be positive on the entire algebra $\mathcal{A}$, there is a way to consistently extend positivity to additional subalgebras of $\mathcal{A}$ that are more readily available. In [11] we define a subclass of constrained states, which we christened almost-positive states, that are positive on a subalgebra associated with a reference observable and possess a relational interpretation. In this section we review some of the properties of these states and discuss additional ways in which they differ from ordinary quantum states that carry subsystem entanglement.

Any operator $\hat{Z} \in \mathcal{A}$ that would correspond to a measurement in the absence of a constraint, so that $\hat{Z}^* = \hat{Z}$, can potentially serve as reference for constrained states. In a straightforward analogy with ordinary quantum mechanics, an observable $\hat{A} \in \mathcal{A}$ can be determined simultaneously with $\hat{Z}$ if $[\hat{A}, \hat{Z}] = 0$, and so without modifying the algebra structure, $\hat{Z}$ can serve as reference for measuring all such observables. Analogously to $\mathcal{A}_{\text{obs}}$, the commutant of $\hat{Z}$ is a subalgebra, which we will denote by $\mathcal{A}_Z \subset \mathcal{A}$. The new definitions of [11] are based on the observation that the commutant of a reference operator $\mathcal{A}_Z$ can, with some additional constructions, replace the hard-to-obtain algebra of Dirac observables $\mathcal{A}_{\text{obs}}$. Since $\hat{Z}$ is usually a simpler operator than $\hat{C}$ because it does not refer to the interacting dynamics, it is much easier to construct the commutant of $\hat{Z}$ than the commutant of $\hat{C}$. In many cases, $\hat{Z}$ may be one of the basic canonical operators such as $\hat{q}_0$ in our example, in which case $\mathcal{A}_Z$ is simply the span of all basic operators other than the conjugate momentum of $\hat{Z}$.

Just like in the classical example discussed in Section 2, the reference variable needs to characterize the flow generated by the constraint. It must therefore vary along this flow, so that $[\hat{Z}, \hat{C}] \neq 0$. Here we will require that $[\hat{Z}, \hat{C}] = i\hbar\mathbb{I}$ in order to mimic the energy-time relationship of our example, though this condition can be made somewhat more general. We immediately obtain the result that the infinitesimal gauge flow of $\hat{C}$ preserves the commutant of $\hat{Z}$, mimicking the corresponding property of Dirac observables: According to the Jacobi identity, we have

$$[[\hat{A}, \hat{C}], \hat{Z}] = [[\hat{Z}, \hat{C}], \hat{A}] + [[\hat{A}, \hat{Z}], \hat{C}] = 0 \quad (9)$$

because $[\hat{Z}, \hat{C}]$ is proportional to the identity operator, and $[\hat{A}, \hat{Z}] = 0$ for $\hat{A}$ in the commutant of $\hat{Z}$. For any $\hat{A}$ in $\mathcal{A}_Z$, therefore, $[\hat{A}, \hat{C}]$ is also in $\mathcal{A}_Z$.

Through the condition in Equation (7) and positivity on $\mathcal{A}_{\text{obs}}$, we are already demanding that constrained states give numerical probabilistic predictions for measurements of all Dirac observables such that the constraint is identically zero. We now want to find a subset of constrained states that, in addition, can also be used to assign probabilistic predictions to all observables in $\mathcal{A}_Z$ and correspond to a fixed configuration of the reference observable $\hat{Z}$. We summarize these conditions in our definition of almost-positive states, which are states ω such that

1. they solve the constraint, $\omega(\hat{A}\hat{C}) = 0$ for all $\hat{A} \in \mathcal{A}$;
2. they parameterize $\hat{Z}$ as a reference variable, $\omega(\hat{Z}\hat{A}) = \omega(\hat{Z})\omega(\hat{A})$ for all $\hat{A} \in \mathcal{A}$; and
3. they are positive on the commutant of $\hat{Z}$, $\omega(\hat{A}^*\hat{A}) \geq 0$ for all $\hat{A} \in \mathcal{A}_Z$.

Note that parameterization of $\hat{Z}$ in condition 2 above is required only in the specified ordering where $\hat{Z}$ appears on the left. The ordering is important for consistency with the commutation relations involving $\hat{Z}$. For example, when computing the value of the commutator $\omega([\hat{Z}, \hat{E}]) = \omega(\hat{Z}\hat{E}) - \omega(\hat{E}\hat{Z})$, where $\hat{E}$ denotes the conjugate momentum of $\hat{Z}$, we have to re-order the product in the second term before we can apply the parameterization condition; otherwise we would obtain the inconsistent result $0 = \omega(\hat{Z})\omega(\hat{E}) - \omega(\hat{E})\omega(\hat{Z}) = i\hbar$. For $\hat{A}$ in the commutant of $\hat{Z}$, of course, re-ordering does not change the expression.

One should be worried that none of the above conditions explicitly mention positivity on $\mathcal{A}_{\text{obs}}$. However, as we show in [11], under some additional algebraic conditions on $\hat{C}$, $\hat{Z}$ and $\mathcal{A}$, which are satisfied by constraints of the form (6), the above three conditions automatically imply positivity of ω on $\mathcal{A}_{\text{obs}}$, without the need to explicitly construct this algebra. The theorems in [11] further show that the flow generated by the constraint preserves almost-positivity and linearly evolves $\omega(\hat{Z})$. (This last statement is somewhat obvious given that $[\hat{Z}, \hat{C}] = i\hbar\mathbb{I}$.) An almost-positive state ω can therefore be interpreted as a quantum state on $\mathcal{A}_Z$ at a fixed value of time $t_Z = \omega(\hat{Z})$ and it can be evolved to a different time using $\hat{C}$, such that reality and the Cauchy–Schwarz inequality are preserved. (This is the algebraic analogue of unitarity.)

In fact, almost-positive states provide a consistent embedding of the states of a reduced quantum system within the states of its parent constrained quantum system (Almost-positive states are therefore more powerful than the usual distinction between kinematical and physical Hilbert spaces.). Let us illustrate this with the concrete example of the constraint of Equation (6) with $\hat{Z} = \hat{q}_0$. Here the degrees of freedom of the reduced system are generated by $\hat{q}_i$ and $\hat{p}_i$ with $i \neq 0$. A solution of the reduced dynamics assigns a positive state on this algebra for each value of parameter time t . This solution can be linearly extended to a state on the commutant of $\hat{q}_0$, which is generated by $\hat{q}_0$ in addition to $\hat{q}_i$ and $\hat{p}_i$ with $i \neq 0$, by setting $\omega(\hat{q}_0) = t$ and using condition 2: since t is real the extension will remain positive on the commutant of $\hat{q}_0$. (Conversely, enforcing positivity on the whole commutant of $\hat{Z}$, including $\hat{Z}$ itself, guarantees reality of the reference variable $\omega(\hat{Z})$.) The constraint Equation (7) can then be used to further extend the state to all of $\mathcal{A}$ by setting $\omega(\hat{A}\hat{p}_0) = -\omega(\hat{A}H(\hat{q}_i, \hat{p}_i; \hat{q}_0))$. For each positive state of the reduced system there is therefore a functional on the full algebra of the parent constrained quantum system, which as we already discussed, cannot be completely positive. Almost-positivity gives the specific ways in which positivity must be relaxed on the constraint operator and the chosen reference variable.

The parameterization condition of almost-positive states has an immediate implication: the reference variable $\hat{Z}$ is not a physical degree of freedom but a parameter. If we apply the condition to $\hat{A} = \hat{Z}$, we obtain $\omega(\hat{Z}^2) = \omega(\hat{Z})^2$ and therefore $\Delta Z = 0$ for quantum fluctuations of the reference variable. For any $\hat{A}$ in the commutant of $\hat{Z}$ we obtain that the quantum correlations

$$\Delta_{ZA} = \frac{1}{2}\omega(\hat{Z}\hat{A} + \hat{A}\hat{Z}) - \omega(\hat{Z})\omega(\hat{A}) = 0 \quad (10)$$

vanish. A reference variable therefore does not fluctuate, and it cannot be entangled with system degrees of freedom. It is correlated with its own momentum, but not in a real way, since positivity does not extend to $\hat{E}$:

$$\Delta_{ZE} = \frac{1}{2}\omega(\hat{Z}\hat{E} + \hat{E}\hat{Z}) - \omega(\hat{Z})\omega(\hat{E}) = \frac{1}{2}\omega([\hat{E}, \hat{Z}]) + \omega(\hat{Z}\hat{E}) - \omega(\hat{Z})\omega(\hat{E}) = -\frac{1}{2}i\hbar. \quad (11)$$

This result underlines the non-physical nature of the reference degrees of freedom. (Interestingly, even though Z does not fluctuate, the uncertainty relation of the pair $(\hat{Z}, \hat{E})$ is formally satisfied (and always saturated): $(\Delta Z)^2(\Delta E)^2 - \Delta_{ZE}^2 = \hbar^2/4$.) If $\hat{Z}$ is the quantum analog of a relational time, it is not a physical degree of freedom but rather a parameter used to characterize redundant information. In our classical example of Section 2, for instance, q_0 was simply replaced by the evolution parameter t once the constraint has been solved and its gauge flow eliminated. Similarly, we expect a quantum reference variable to be replaced by a parameter (a number rather than an operator) once the constraints are solved.

Viewed from this algebraic perspective, the reduced theory is not a description of co-evolution of a physical clock $\hat{Z}$ in relation to observables compatible with $\hat{Z}$. Instead, it is a theory of reduced degrees of freedom unitarily evolving relative to a parameter time t_Z that, a priori does not possess a physical clock.

5. Switching Temporal Reference Frames

The preceding section introduced almost-positive states as a way to embed a reduced quantum system within its parent constrained system with a particular focus on the situation where the constraint governs dynamics. Similar to the classical situation, the quantum constraint can, in general be reduced relative to multiple internal times (though the situation is quite restrictive as we shall see below). The simplest example of such a situation would be the special case of constraint in Equation (6) where one other component subsystem (or more) behaves in the same way as $\hat{q}_0$

$$\hat{C} = \hat{p}_0 + \hat{p}_1 + H(\hat{q}_i, \hat{p}_i; \hat{q}_0, \hat{q}_1), \quad i \geq 2. \quad (12)$$

Here both $\hat{q}_0$ and $\hat{q}_1$ can be used as references and define almost-positive states. It is easy to see that almost-positive states with respect to $\hat{q}_0$ will not be almost-positive with respect to $\hat{q}_1$. From Equation (11) we see that for any almost-positive state of $\hat{q}_0$ we have $\Delta_{q_0 p_0} = -i\hbar/2$. On the other hand, since $[\hat{q}_0, \hat{q}_1] = [\hat{p}_0, \hat{q}_1] = 0$, both $\hat{q}_0$ and $\hat{p}_0$ are in $\mathcal{A}_{q_1}$, so that an almost-positive state of $\hat{q}_1$ is positive on their products, which requires $\Delta_{q_0 p_0} \in \mathbb{R}$.

In general, mapping the description of physics relative to one time variable $\hat{Z}_a$ to the description of the same physics relative to another time variable $\hat{Z}_b$ involves creating an identification between the two corresponding sets of almost-positive states. Structurally, this is in good agreement with the general form of quantum reference frame transformations discussed in the introduction. However, one might wonder why there would be any such equivalence between algebraic constrained states considered here. In the case where a complete set of Dirac observables is available (for an N -component constrained canonical system that would mean that $2(N-1)$ independent invariant operators have been constructed) the answer is straightforward: a pair of constrained states are physically equivalent if they assign the same values to $\mathcal{A}_{\text{obs}}$. In the more general situation considered in [11] one looks at infinitesimal properties of the flows on algebraic states that are generated by placing the constraint operator on the left (as in $\omega(\hat{C}\hat{A})$), which are guaranteed to preserve the value of Dirac observables. Our analysis demonstrates that, once almost-positivity is imposed, the only equivalence relation left is time evolution generated by $\hat{C}$, so at a fixed time $t_Z = \omega_1(\hat{Z}) = \omega_2(\hat{Z})$, different almost-positive states $\omega_1 \neq \omega_2$ are physically distinct.

Since the algebraic analysis of [11] focuses on infinitesimal properties of physical equivalence relations, it does not provide a straightforward method for constructing finite transformations between temporal reference frames. Instead, it can be used to shed light

on which frames are viable. For a time variable that is part of a canonical subsystem, like $\hat{q}_0$ in our main example, our results show that (locally) complete reduction requires the constraint to have the form given in Equation (6) or to factorize $\hat{C} = \hat{\mathcal{N}}\hat{C}_H$, so that the right factor $\hat{C}_H$ is of this form. In addition, adjointness relations $\hat{C}^* = \hat{C}$ and $\hat{C}_H^* = \hat{C}_H$ severely restrict possible forms of the left factor $\hat{\mathcal{N}}$. An example relevant to models considered in [1,12–14] is a $\hat{C}$ linear in $\hat{p}_0$. In this case $\hat{\mathcal{N}} = \frac{1}{i\hbar}[\hat{q}_0, \hat{C}]$, and our results require $\hat{\mathcal{N}} = \hat{\mathcal{N}}^*$, $[\hat{\mathcal{N}}, \hat{C}_H] = 0 = [\hat{\mathcal{N}}, \hat{q}_0]$. For example, this precludes the exact reduction of

$$\hat{C} = \hat{q}_1\hat{p}_0 + \frac{1}{2}(\hat{q}_1\hat{p}_1 + \hat{p}_1\hat{q}_1).$$

Even though $\hat{C}$ is hermitian, when we attempt to factorize we get

$$\hat{C} = \hat{q}_1 \left(\hat{p}_0 + \hat{p}_1 - \frac{1}{2}i\hbar \frac{\widehat{1}}{q_1} \right).$$

where $\hat{\mathcal{N}} = \hat{q}_1 = \hat{\mathcal{N}}^*$ and $\hat{C}_H = \hat{p}_0 + \hat{p}_1 - \frac{1}{2}i\hbar \frac{\widehat{1}}{q_1} \neq \hat{C}_H^*$. In such a situation we conclude that $\hat{q}_0$ cannot be used to exactly reduce the constraint by providing a temporal reference. This still leaves the door open for approximate reduction under additional conditions satisfied by states. Examples of current interest are shown in the following section and in Appendix A.

6. Modeling Time-Keeping Devices

We have so far not addressed an important question that naturally arises in our analysis: Within the reduced theory, if the reference variable $\hat{Z}$ is not an observable, then how could one measure time? This situation is not that different from ordinary quantum mechanics where the evolution equation for a system is specified relative to a parameter time. The studied system itself does not by default possess a clock, but one can always add a subsystem the measurement of which will correlate with (changes in) the value of time. In the case of a constraint, where its right factor $\hat{C}_H$ is reduced by using $\hat{Z}$ as the time reference (as in the previous section), a physical clock must first and foremost be some observable of the reduced system $\hat{U} = \hat{U}^* \in \mathcal{A}_Z$ that is independent of $\hat{Z}$. Furthermore, the clock must evolve along in time $[\hat{U}, \hat{C}_H] \neq 0$, with an ideal clock evolving at a constant rate so that $[\hat{U}, \hat{C}_H] = i\hbar\mathbb{I}$ up to a real constant factor. A simple model for this situation is provided by the constraint in Equation (12), where $\hat{q}_0$ can play the role of reference time $\hat{Z}$ and $\hat{q}_1$ the role of the ideal physical clock $\hat{U}$. Indeed, the roles can also be reversed: an ideal clock is also a valid time reference variable. However a valid time reference variable will not always define an ideal clock relative to another time reference variable—for an example look at the discussion of constraint $\hat{C}_1$ below.

A useful physical clock need not be ideal: for example $\hat{U}$ could represent a degree of freedom that oscillates with a stable frequency like an atomic clock. However, unless the clock is also a valid reference time, there is, in general, no exact temporal reference frame associated with it. If we select an algebraic reference state that satisfies the conditions laid out in Section 4 relative to a clock $\hat{U}$ that does not satisfy the conditions described in Section 5, such a state will not retain positivity during time-evolution and, therefore, would not result in future probabilistic predictions for measuring observables that commute with $\hat{U}$. In this light it is interesting to consider the interacting clock models used in [12] and models for clocks experiencing an external gravitational field of different strengths employed in [13,14]. Here we are not attempting to analyze the claims about real physical behavior of clocks subject to non-uniform gravitational interactions, but merely point out that, according to our algebraic analysis, whether such clocks define exact temporal quantum reference frames sensitively depends on the interactions one introduces.

For a concrete example, we will take a closer look at one of the models in [12], the same algebraic analysis can be applied to other models considered there and in [13,14].

The Hamiltonian constraint in Equation (12) in [12] models two gravitationally interacting clocks, A and B , a third distant clock, labeled C , and an event localized in time relative to clock A . In our notation, the constraint has the form

$$\hat{C}_1 = \hat{p}_A + \hat{p}_B + \hat{p}_C + \lambda_1 \hat{p}_A \hat{p}_B + f_A(\hat{q}_A)(\mathbb{I} + \lambda_2 \hat{p}_B). \quad (13)$$

Here $f_A(\hat{q}_A)$ is a function of $\hat{q}_A$ sharply localized around a (real) reading t_A of clock A and multiplied by an additional operator that couples this clock to a record-keeping subsystem, while λ_i are numerical coefficients that characterize the gravitational interaction of clocks A and B . (Larger λ_i correspond to clocks being closer to each other and interacting more strongly.) The purpose of $f_A(\hat{q}_A)$ is to set up a “recorded event” that is temporally localized relative to clock A . Here we will ignore its action on the record-keeping subsystem and use the fact that $f_A(\hat{q}_A)$ has vanishing commutators with $\hat{p}_B$, $\hat{p}_C$, and all three configuration observables $\hat{q}_I$, $I = A, B, C$, while $[\hat{p}_A, f_A(\hat{q}_A)] \neq 0$.

In [12], the two coupling constants are equal $\lambda_1 = \lambda_2 =: \lambda$. With this choice one can re-define the momentum of clock A , $\hat{p}'_A := \hat{p}_A + f_A(\hat{q}_A)$ which retains canonical commutators with $\hat{q}_A$ and observables of the other two clocks. The constraint can then be re-written

$$\hat{C}'_1 = \hat{p}'_A + \hat{p}_B + \hat{p}_C + \lambda \hat{p}'_A \hat{p}_B. \quad (14)$$

Because $[\hat{q}_C, \hat{C}'_1] = i\hbar\mathbb{I}$, the non-interacting distant clock C is a valid time reference variable with trivial factorization $\hat{C}_{H;C} = \hat{C}'_1$ and $\hat{N}_C = \mathbb{I}$. Furthermore, the interaction term in this model on its own does not destroy the ideal nature of clocks A and B , which can be used as time reference variables with factorization, for example for A

$$\hat{C}'_1 = (\mathbb{I} + \lambda \hat{p}_B) \left(\hat{p}'_A + \frac{\hat{p}_B + \hat{p}_C}{\mathbb{I} + \lambda \hat{p}_B} \right) := \hat{N}_A \hat{C}_{H;A},$$

valid on the domain where $(\mathbb{I} + \lambda \hat{p}_B)$ is invertible. Since this operator commutes with $\hat{C}_{H;A}$, its invertibility is preserved by time evolution. Factorization for clock B is entirely symmetric. Note that, in this model, if e.g., $\hat{q}_C$ defines the temporal reference frame, the clock $\hat{q}_A$ is no longer ideal, since its rate

$$[\hat{q}_A, \hat{C}_{H;C}] = [\hat{q}_A, \hat{C}'_1] = i\hbar(\mathbb{I} + \lambda \hat{p}_B),$$

which is not of the form $\alpha i\hbar\mathbb{I}$, for $\alpha \in \mathbb{R}$. Nevertheless, it is still a “good” clock since its rate is a constant of motion $[[\hat{q}_A, \hat{C}_{H;C}], \hat{C}_{H;C}] = 0$. The same scenario plays out if we pick another pair as reference time and a physical clock.

If $\lambda_1 \neq \lambda_2$, the constraint can no longer be written in the simple form (14). Clocks A and C remain valid time reference variables, but clock B does not: the combination of interaction with clock A , and the event-recording process spoil its behavior in this model. The quickest way to see that the algebraic conditions for factorization are violated is to note that $\hat{C}'_1$ is linear in $\hat{p}_B$, so that

$$\hat{N}_B = \frac{1}{i\hbar} [\hat{q}_B, \hat{C}'_1] = \mathbb{I} + \lambda_1 \hat{p}_A + \lambda_2 f_A(\hat{q}_A),$$

from which it follows that

$$[\hat{N}_B, \hat{C}'_1] = i\hbar(\lambda_1 - \lambda_2)[\hat{p}_A, f_A(\hat{q}_A)] \neq 0,$$

violating the requirements of Section 5. If we attempt to factorize this constraint

$$\hat{C}_1 = (\mathbb{I} + \lambda_1 \hat{p}_A + \lambda_2 f_A(\hat{q}_A)) \left[\hat{p}_B + \frac{1}{\mathbb{I} + \lambda_1 \hat{p}_A + \lambda_2 f_A(\hat{q}_A)} (\hat{p}_A + \hat{p}_C + f_A(\hat{q}_A)) \right],$$

we end up with a non-hermitian factor on the right because $(\mathbb{I} + \lambda_1 \hat{p}_A + \lambda_2 f_A(\hat{q}_A))^{-1}$ does not commute with $(\hat{p}_A + \hat{p}_C + f_A(\hat{q}_A))$. In other words, with this minor change in the model of coupling, the ticks of clock B no longer map out a unitary evolution history of the rest of the system.

7. Implications for Relational Quantum Mechanics

Heuristically, based on thinking about the establishment of a reference frame as similar to measurement of the properties of a subsystem of interest by a reference device (ruler, clock), one would expect a quantum state that simultaneously describes the entire system—that is the reference subsystem and the subsystem of interest—to generically exhibit a high degree of entanglement between the two. The Hilbert space based constructions of quantum reference frames cited here do not construct such states: expectation values are only taken once a quantum reference frame has been selected and the reference variable is no longer one of the quantum degrees of freedom. In contrast, our approach explicitly relies on states that are defined on the entire “unreduced” kinematical algebra (see Section 4). Our conclusion is that the “relationality” of quantum data—that is, the fact that a reference variable needs to be chosen before quantum probability computations on non-Dirac-observables can be performed—is encoded as a redundancy rather than as an entanglement-type correlation. This may have additional implications in the context of the attempts to define (or rule out) a technical description of interacting quantum systems consistent with the stated principles of relational quantum mechanics (RQM) of [18,19]. The distinction between ordinary (positive) quantum states that freely carry correlations and constrained (almost-positive) reference states with partially restricted correlations may be relevant to a recent debate in this context [20–24].

Relational approaches to quantum mechanics including RQM were inspired by the older attempts [25,26] to define meaningful observables in quantum gravity, a theory in which the geometry of space and time is subject to the rule of quantum mechanics. However, since space and time coordinates are not part of the underlying phase space because they represent redundant information, it is impossible to describe observables in a way similar to the classical method of, say, geodesics on a space-time manifold used to set up the local inertial frame of an observer who measures properties of a moving object. Instead of using a time coordinate, the gist of those older proposals was to describe evolution relationally, for instance by specifying the position of one particle relative to the position of another particle. In this example the second particle is used as a clock, and the first particle can be said to evolve with respect to time as determined by the second particle.

The more recent program of RQM [18] takes a relational view of all measurements because they are always performed relative to some system describing the observer, even if gravity is not quantized. (The review [19] emphasizes the connection with quantum gravity.) Both sides of the ongoing debate about the consistency of RQM make some use of ordinary entangled quantum states when describing one subsystem from the perspective of another (although the proponents of RQM prefer to de-emphasize the role of states in favor of interactions). Such entangled states provide no obvious relation to the description of that same subsystem from a third perspective, as, for example, given by $\hat{q}_I$, $I = A, B, M$, in Section 6. Since the choice of a reference degree of freedom for relational statements is not unique, however, it is necessary to consider multiple perspectives and to demonstrate some degree of invariance of physical statements with respect to changes of relational dependencies.

The discussion therefore requires a formulation with redundant degrees of freedom and constraints, as considered in the main part of this paper: If consistent transformations of reference choices exist, the overall description of all the relevant interacting subsystems together in one consistent setting necessarily carries a large degree of redundancy. Different choices of reference within the same redundant setting are related by gauge transformations, enforced by some form of quantum constraints. As a result, and as per our discussion, any relational setting necessitates certain restrictions on which subsystems can

be simultaneously assigned positive quantum states (i.e., given a probabilistic quantum description), and a full state for the entire system including reference degrees of freedom can only be almost-positive. The set of available states that can be used to construct suitable versions of relational quantum mechanics (or counter-examples to their consistency) is therefore limited.

8. Discussion

Our new notion of almost-positive algebraic states provides a consistent embedding of the states associated with all valid time reference frames of a system with a Hamiltonian constraint as states on the full kinematical algebra. We were therefore able to analyze properties of reference variables within a quantum setting, going beyond previous studies in which the reference system completely disappeared from the quantum realm after the transition to a physical Hilbert space. The systems considered here are algebraically different from unconstrained systems in that the presence of the constraint and corresponding reference variables requires a weakened form of positivity of states. Almost-positivity ensures that the resulting evolution picture relative to the reference variable is consistent, but results in a state that does not give probabilistic predictions for the measurement of some kinematical observables. As we have seen, in almost-positive states reference variables themselves do not behave in a quantum manner: They do not fluctuate, $\Delta Z = 0$, and they are uncorrelated with system degrees of freedom, $C_{ZA} = 0$. They are, in fact, not observables, but generally require additional subsystems in order to be measured. In a consistent treatment of quantum constrained systems, a quantum reference frame therefore remains largely classical. In particular, time and space in quantum gravity would be represented by reference variables that characterize the action of constraints. If exact time and space reference variables are found, the resulting space-time that they will map out should retain many of its classical properties: time does not fluctuate and is not correlated with other subsystems.

It may certainly be possible that physical clocks, defined as quantum devices that measure time but do not represent time at a fundamental level, exhibit fluctuations or correlations that limit our observational access to time. On the purely mathematical level, the reference variable of one temporal reference frame can serve as a clock in another temporal reference frame and possess quantum fluctuations and correlations when viewed from the latter perspective. However, our general discussion shows that correlations or entanglement between a physical clock and some system are not necessarily implied by the mere fact that the combined state is described with respect to a quantum reference frame: Since any positive state on the commutant of $\hat{Z}$ is uniquely extended to an almost-positive state of the full algebra, the presence of a quantum reference frame does not impose any new conditions on correlations and entanglement, other than those features that may be implied by interactions between clock and system in the standard way. Given the same interactions, the same correlations and entanglement would evolve if one were to use a classical-type background time. There is then no fundamental limitation on the accessibility of time measurements because interactions between clock and system depend on how the clock is constructed and can in principle be reduced by judicious choices or placements of clocks. The question of explicit transformations between different quantum reference frames is more complicated and still has to be addressed in our setting. It is, however, clear that the common realization of almost-positive states for different quantum reference frames on a single object, given by the kinematical algebra, will lend more clarity to this task compared with the traditional treatment of mappings between different physical Hilbert spaces.

At the current stage of developments, mathematical consistency requirements, rather than practical questions, seem to play a more decisive role in the admissibility of clock-system interactions. As we saw from the example in Section 6, and a related one in Appendix A, the viability of a kinematical observable as a time reference variable is very sensitive to the interactions one includes. A general quantum system subject to a sensible

Hamiltonian constraint may possess one or more reference times, where there may or may not be ideal physical clocks corresponding to each choice of time; it may also possess no valid reference times at all (and therefore also no corresponding ideal clocks). The latter case calls for state-based approximations, where in some states a given reference time can be used to characterize a portion of evolution (as was done within the semiclassical approximation in [27–29]). Here too, the ability to embed all (approximate) reference time perspectives under one roof becomes important and the algebraic construction provides a helpful framework (see [30]). If admissible interactions can be classified completely for a given set of reference, clock, and system variables, they may lead to a restricted set of possible outcomes for correlations and entanglement between clock and system. At present, however, such a classification is unavailable.

Author Contributions: Both authors contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Science Foundation (NSF) of United States grant number PHY-2206591.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank Philipp Höhn for discussions.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. A Clock Model with Time Dilation

As a specific application of our results, we here outline how recent constructions of clocks in space-time [13,14] fit within the algebraic framework and the new restrictions it imposes on deparameterizability as relational evolution. We use the slightly simpler constraint system of [13], focusing on the example of Equation (A1) there. The system consists of two light particles A and B and a massive particle M on a $1 + 1$ -dimensional space-time, with corresponding basic canonical observables $[\hat{q}_J^i, \hat{p}_I^K] = i\hbar \mathbb{I} \delta_J^K \delta_I^i$. Here, $i, l = 0, 1$ are space-time indices and $J, K = A, B, M$ particle indices. The light particles also possess internal clocks $[\hat{Z}_I, \hat{E}_J] = i\hbar \mathbb{I} \delta_{IJ}$, with $I, J = A, B$. (The model in [14] includes an internal clock for the massive particle and an additional constraint.)

There are four constraints. The first two impose the energy-momentum relations for the light particles,

$$\hat{C}_I = \hat{g}_I \hat{p}_0^I - \hat{\omega}_I \quad (\text{A1})$$

where $\hat{g}_I$ depends on the metric at the location of the light particle $I = A, B$, distorted from Minkowski metric by the presence of the massive particle M ,

$$\hat{g}_I = \sqrt{g^{00}(\hat{q}_I^1 - \hat{q}_M^1)},$$

and $\hat{\omega}_I$ only depends on the spatial momentum of particle I

$$\hat{\omega}_I = \sqrt{m_I^2 c^2 \mathbb{I} + (\hat{p}_1^I)^2}.$$

The third constraint enforces spatial translational invariance

$$\hat{f}^1 = \hat{p}_1^A + \hat{p}_1^B + \hat{p}_1^M. \quad (\text{A2})$$

The last is the overall Hamiltonian constraint

$$\hat{f}^0 = \hat{p}_0^A + \hat{p}_0^B + \hat{p}_0^M + m_A (\hat{g}_A)^{-1} (\hat{\omega}_A)^{-1} \hat{E}_A + m_B (\hat{g}_B)^{-1} (\hat{\omega}_B)^{-1} (\hat{E}_B + \hat{\theta}_B), \quad (\text{A3})$$

where $\hat{\theta}_B$ is a function of the internal configuration of the clock $\hat{Z}_B$ that is sharply localized about a specific value t_B (in fact [13] uses a delta function), multiplied by an operator that commutes with all four constraints. (One way to construct such an operator is to add a “record-keeping” subsystem that does not interact with the rest of the system otherwise.)

While the first three constraints are manifestly hermitian and all explicitly commute with each other, $\hat{f}^0$ is not because $[\hat{g}_I, \hat{\omega}_I] \neq 0$. In [13] it is assumed that the value of this commutator is small when applied to states, and that

$$[(\hat{g}_I)^{-1}, (\hat{\omega}_I)^{-1}] = (\hat{\omega}_I)^{-1}(\hat{g}_I)^{-1}[\hat{g}_I, \hat{\omega}_I](\hat{g}_I)^{-1}(\hat{\omega}_I)^{-1} \quad (\text{A4})$$

enjoys the same property. Time dilation is therefore required to be position-independent.

Our aim here is to show that this condition is not only an approximation that may simplify some calculations, but is in fact necessary in order to deparameterize the constraints (A1) with respect to $\hat{q}_I^0$ as the clock. In this model, deparameterization is achieved by factorization

$$\hat{C}_I = \hat{g}_I \left(\hat{p}_0^I - (\hat{g}_I)^{-1} \hat{\omega}_I \right) \quad (\text{A5})$$

with $\hat{N} = \hat{g}_I$ in our previous notation. If $[\hat{g}_I, \hat{\omega}_I] \neq 0$, the right factor in (A5) is not hermitian and cannot play the role of a consistent evolution generator $\hat{C}_H$ (See our discussion in Section 5.).

If this commutator as well as (A4) can be dropped, then all constraints commute with each other and the constrained system can be solved by removing each of them one at a time.

- First, we eliminate $\hat{f}^1$ by removing $\hat{p}_1^M = -\hat{p}_1^A - \hat{p}_1^B$ and switching to relative positions $\hat{Q}_I^1 = \hat{q}_I^1 - \hat{q}_M^1$, $I = A, B$.
- Second, we deparameterize the constraints (A1) by factorization with respect to $\hat{q}_I^0$ as clocks. Individually, space-time degrees of freedom of each light particle are reduced to the observables $\hat{q}_I^1$ and $\hat{p}_1^I$ in the commutant of the clocks. They evolve in parameter time τ_I with respect to the Hamiltonian $\hat{h}_I = -(\hat{g}_I)^{-1} \hat{\omega}_I$. The constraint (A3) reduces to

$$\hat{f}^0 = -\hat{h}_A - \hat{h}_B + \hat{p}_0^M + m_A (\hat{g}_A)^{-1} (\hat{\omega}_A)^{-1} \hat{E}_A + m_B (\hat{g}_B)^{-1} (\hat{\omega}_B)^{-1} (\hat{E}_B + \hat{\theta}_B) .$$

- Finally, the above constraint can be deparameterized directly by using $\hat{q}_C^0$ as time reference, or by factorization if we instead use $\hat{Z}_A$ or $\hat{Z}_B$. For example, in the case of $\hat{Z}_A$ we factorize

$$\hat{f}^0 = m_A (\hat{g}_A)^{-1} (\hat{\omega}_A)^{-1} \left[\hat{E}_A - \frac{(\hat{\omega}_A)^2}{m_A} + \frac{\hat{\omega}_A \hat{g}_A}{m_A} \left(\hat{p}_0^M - \hat{h}_B + m_B (\hat{g}_B)^{-1} (\hat{\omega}_B)^{-1} (\hat{E}_B + \hat{\theta}_B) \right) \right] .$$

This latter deparameterization, once again, requires that we ignore the commutator $[\hat{g}_I, \hat{\omega}_I]$, to make the third term inside the square parentheses hermitian, and to make the left factor $m_A (\hat{g}_A)^{-1} (\hat{\omega}_A)^{-1}$ a constant of motion.

These constructions and our general discussion indicate that an extension of time dilation in clock models to more relativistic settings, in which commutators such as $[\hat{g}_I, \hat{\omega}_I]$ can no longer be ignored, will be more challenging.

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