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Reconceptualizing a mathematical domain on the basis of student reasoning: Considering teachers' perspectives about integers

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ABSTRACT

Integers have historically been approached as a system of rules. However, to teach any mathematical domain for understanding, teachers must conceptualize it as comprised of more than procedures. Using as a lens the four types of integer reasoning identified by Bishop et al. (2014a, 2014b), we interviewed 7th-grade teachers to investigate their own integer reasoning and how this corresponds to their approaches to teaching integers and to their interpretations of students' reasoning. The teachers not only correctly solved integer tasks but also most reasoned using more than rules, demonstrating a flexibility of strategies. Additionally, although they attempted to introduce integers in meaningful ways, most teachers viewed teaching integers as helping their students apply procedures, an orientation that constrained their understanding of students' integer reasoning. Results indicate that teachers possess productive conceptual resources but need a structure to leverage their understandings to teach integers as more than a set of rules.

1. Introduction

The vision of effective mathematics instruction has become one in which teachers move beyond simply demonstrating techniques to correctly calculate answers and instead strive to teach for understanding (e.g., Hiebert, 2013). To support teachers to see mathematics as more than simply computational, educators have developed explicit descriptions of this broader view of mathematics, identifying multiple strands of mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics (NCTM), 2000) and various habits of mind and mathematical practices associated with engaging in mathematics (Cuoco et al., 1996; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Moreover, researchers have elaborated on what a meaningful understanding of mathematics entails in different content areas through the development of frameworks by outlining the nature of the reasoning and conceptual benchmarks inherent to the topic. Typically, these frameworks are organized as a hierarchy of more sophisticated ways of thinking and strategies in *learning trajectories* or *progressions* (Lobato & Walters, 2017). Rather than basing the hierarchies on disciplinary logic from the expert perspective, researchers capture the mathematical thinking and understanding from the students' perspectives formulated through extensive student interviews. Such work has led researchers to reconceptualize instruction in multidigit arithmetic (e.g., Carpenter et al., 2014), rational number,

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(Maloney & Confrey, 2010), and algebra (e.g., Kaput et al., 2008) from domains dominated by algorithms and rules for symbolic manipulation to developing ways of reasoning about patterns and operations. Furthermore, although researchers have only begun to investigate how teachers interpret and use these frameworks, evidence indicates that they can have significant positive effects on both student and teacher learning. Mathematics teachers experiencing repeated professional development around such frameworks were able to restructure their understandings of the mathematics around student thinking, enabling them to develop models of students' thinking and consequently to more readily anticipate and interpret student reasoning (Wilson et al., 2013). They began to modify their instruction to focus less on teaching explicit procedures and more on developing students' reasoning (Carpenter et al., 1989; Jacobs et al., 2007).

We propose that to support the development of high-quality teaching in other mathematical domains, we must educate prospective and practicing teachers about similar frameworks so that they have organized understandings of the types of reasoning associated with rich understandings of the domains. To address this issue, we select one domain—integers—and consider aspects of teachers' current understandings from a conceptual perspective in an effort to identify conceptual resources and constraints that can be leveraged to support them in (re)conceptualizing integers as a set of interrelated concepts and procedures rather than memorized algorithms and rules.

2. Research on integer instruction

Students' struggles with integer arithmetic are well documented and stem from a range of inherent challenges (Gallardo, 1995, 2002; Reck & Mora, 2004; Vlassis, 2002). Certain properties of integers contradict many of the intuitions developed around natural numbers, such as "adding makes bigger" (Bishop et al., 2011; Gallardo, 2002; Linchevski & Williams, 1999). Furthermore, the meanings of integer operations do not follow directly from concrete objects but have traditionally been achieved through formal mathematical reasoning (Hefendehl-Hebeker, 1991; Linchevski & Williams, 1999). Such intrinsically counterintuitive qualities present difficulties not only for today's students in the classroom but historically for mathematicians as well (Bishop et al., 2014a, 2014b; Hefendehl-Hebeker, 1991). Additionally, the meaning of the minus sign, the symbol most fundamental to integers, is ambiguous. Common meanings include the meaning of an operation (take away), a value (negative), and "the opposite of," and learners often apply multiple meanings during manipulation (Lamb et al., 2012).

2.1. Design experiments

Finding instructional approaches to overcome these inherent challenges for students has been a focus of research in the domain of integers. A literature review revealed that most research has consisted of design experiments to investigate the effectiveness of different models and representations. Although the primary concentration of this work has been on the affordances and challenges of various models, one consensus from these studies is that no single representation for integer operations accurately or completely embodies the underlying mathematical ideas (Linchevski & Williams, 1999; Stephan & Akyuz, 2012). Moreover, Peled and Carraher (2007) concluded that "didactical models for signed numbers have not been very successful" (p. 305), arguing that signed numbers entail more than computational skills and that these models do not facilitate conceptual growth. They contended that teachers used these instructional models on the basis of the belief that it is "easier to remember a set of strange rules in context … than a set of number rules" (p. 305). Peled and Carraher challenged this belief, citing Mukhopadhyay et al. (1990), who found that students more successfully solved integer problems based in context than naked-number problems but did so by using only positive numbers rather than invoking signed numbers (see also, Whitacre et al., 2014, 2016).

2.2. Teachers' perspectives

Although these design experiments address the teaching of integers, they are predominantly focused on instruction from the students' perspective, providing limited insight into teachers' understandings of integers and their corresponding goals. In contrast, a few researchers—Kinach (2002), Kumar et al. (2017), Mitchell et al. (2013), and Wessman-Enzinger and Tobias (2020)—have specifically targeted integer instruction from the perspective of teachers, with their research primarily centered around characterizing the mathematics knowledge for teaching teachers possess (MKT) (Hill et al., 2008). In particular, they have focused on teachers' *specialized content knowledge*, which refers to ways of knowing mathematics that are particularly useful to teaching and that go beyond the professional knowledge held by laypeople. For example, in the domain of integers, teachers need to know how a context or representation could be used to illustrate why subtracting a negative number results in a greater quantity.

Results of these studies highlight two main conclusions. The first is that teachers often do not possess the specialized content knowledge to fully understand the nuances of different representations and models associated with integers. Wessman-Enzinger and Tobias (2020) found that teachers struggled to conceive of different contextual problem types that correctly and realistically aligned with different integer operations and instead consistently posed a single structure, even though it was often problematic. Whereas Mitchell et al. (2013) described how one teacher successfully used varied models to help students draw connections to the underlying algorithms, they also provided details of another teacher who confused the conventions associated with different models. Furthermore, the authors found that the latter teacher was more representative of the other teachers in their study. Consequently, they argued that teachers need increased knowledge of the conventions as well as the affordances and misconceptions associated with different integer representations. Similarly, Kumar et al. (2017) noted that the teachers in their study, while highly experienced, lacked understanding of different interpretations of the minus sign. Likewise, Kinach (2002) found that the preservice teachers (PSTs) in her study switched among rules for interpreting the minus sign and were not explicit about when and why these inconsistencies existed.

A second finding from these studies is that teachers do not understand how various models embody the underlying mathematical ideas. For example, Kumar et al. (2017) found that teachers believed that contexts were useful to motivate the need for integers but did not draw on them to develop meanings of the various operations. Although they were aware of typical student errors, they attributed these mistakes to students' forgetting rules rather than to conceptual difficulties associated with reasoning about the underlying meanings. In addition, Mitchell et al. (2013) found that teachers struggled to gradually abstract the mathematical rules and operations from the use of concrete representations. As a consequence of this disconnect between contexts/models and the abstract mathematical ideas, teachers preferred rules and symbolic representations to using contexts and models for teaching integers. Finally, Kinach (2002) noted that PSTs did not attend to the conceptual meanings of the rules they explained. For example, although the PSTs recognized the mathematically formal equivalence of $5 + (-3)$ and $5 - 3$, they did not appreciate how these expressions might differ for students. Ultimately, although they professed dedication to teaching for understanding, being unable to provide rationales for the rules of integer operations, they provided students with procedures to calculate answers.

A consistent theme through these studies is that although teachers express desire to teach integers in more meaningful ways, they lack the understandings necessary to do so. Without the specialized content knowledge to fully understand the nuances of various representations and models associated with integers, teachers are unable to support students in developing understandings that go beyond instrumental understanding (Skemp, 2006). Moreover, developing integer reasoning does not seem to be a goal. Given no alternatives, teachers equate understanding integers as knowing the rules.

Such results underscore the difficulties associated with teachers transitioning their integer instruction beyond rules. However, to explore teachers' mathematical knowledge for teaching, we took a different approach in this study. Rather than analyze teachers' understandings according to expectations set by the research community that has served to identify teachers' struggles, we chose to explore the conceptual resources that teachers possess and might leverage to teach integers with understanding. Specifically, we were curious about the ways of reasoning they brought to bear when solving problems as well as their understandings in recognizing nonprocedural reasoning in students. In addition, we wanted to characterize the teachers' goals that framed their instruction of integers and look for connections with their own thinking.

2.3. Students' integer reasoning

To help us understand the reasoning that teachers use, we turned to the work of a group of researchers (Bishop et al., 2014b; Lamb et al., 2018) who analyzed the thinking of 160 students—40 in each of Grades 2, 4, 7, and 11—when they reasoned through a variety of integer tasks consisting of open number sentences with the missing unknown in different locations. This cross-sectional work led to a variety of noteworthy findings. First, investigating the thinking and solution methods of students across a broad range of ages enabled the authors to identify the types of understanding that younger students possess prior to formal instruction. In fact, not only did Grade 2 and Grade 4 students in their study engage in a variety of informal, non-rule-based ways of thinking, these younger students were also able to more accurately solve problems of certain structure types using their informal ways of reasoning than the older students who had been taught integer-operation rules (Lamb et al., 2018). Moreover, these younger students generally approached these problems from a meaning-making stance and, in doing so, treated mathematics as a discipline devoted less to learning how to apply procedures and more to finding ways of reasoning that would enable them to extend their understanding of number, serving as the foundation to both understand and competently operate with these new types of numbers. Such a finding shows that teachers who are aware of such ways of reasoning, could look for them among their students, identify them, and encourage students to share them.

2.4. Integer reasoning framework

In addition to establishing that young children bring a variety of rich resources to integer instruction, Bishop et al. (2014b) compared the integer thinking that students in their study exhibited across the grade levels with an historical analysis of mathematicians' integer reasoning. From these side-by-side analyses, the authors found striking similarities that led them to identify five types of integer reasoning: Order-based, Analogy-based (previously referred to as Magnitude-based [Bishop et al., 2014b]), Formal, Computational, and Emergent (previously referred to as Limited [Bishop et al., 2014b]). Although the authors do not claim to have identified the only ways of reasoning about integers, they argue that because these appear commonly in historical texts and in children's thinking, they are worthy of consideration. Below we briefly describe four of these ways of reasoning, and in the Appendix we provide descriptions and associated examples from children's engagement in each type of reasoning (Note, we chose not to discuss Emergent reasoning here because it pertains to children who are coming to understand integer operations and would not be pertinent for our study of 7th-grade teachers.)

First, *Order-based Reasoning* draws upon the ordered nature of integers. Invoking the number line is an example of order-based reasoning, but one can engage in order-based reasoning without using a number line. For example, one may solve $3 - 5 = \square$ by counting back by ones from 3, in which case the ordered nature of integers is invoked.

Second, *Analogy-based Reasoning* involves relating (negative) numbers to an analogous concept and reasoning about the numbers on the basis of characteristics observed in this other concept. Such reasoning is often characterized by separately considering the sign and magnitude of a number and interpreting the sign as taking on the meaning based in the analogy. An example of Analogy-based Reasoning is when a student solves $-5 - -2$ by reasoning about the corresponding positive numbers, "I know that 5 minus 2 is 3, so negative 5 minus negative 2 must be negative 3." Other examples of Analogy-based Reasoning include using colored chips to refer to positive and negative numbers or drawing upon a context such as gaining or losing yards in football or increasing or losing altitude.

Third, *Formal Reasoning* is characterized by treating negative numbers as formal objects that exist in a system and are subject to fundamental mathematical principles. In reasoning formally, one often draws upon underlying properties and structures to reason about two similar tasks. Frequently, students reasoning formally create a comparison to a familiar problem by holding all but one feature constant, leading to the logical necessity that the two must have different, and often exact opposite, outcomes. For example, when students reason that the effect of subtraction in $5 - -3$ must be opposite of the effect of addition in $5 + -3$, they are invoking formal reasoning.

Finally, *Computational Reasoning* refers to using a procedure, rule, property, or calculation to compute the answer to an integer problem. For example, when students employ "the *double-stick trick"* to change $6 - -2$ to $6 + +2$, or they apply the rule that to add two negative integers, one adds the absolute values of the numbers and appends the negative sign to the sum, they are using Computational Reasoning. Computational Reasoning can be powerful because of its efficiency and broad applicability, and as such, rules and procedures for operating on integers are widely taught.

A notable relationship identified by Lamb et al. (2018) is that success with integer tasks was correlated with the flexibility in reasoning that students demonstrated. Even older college-tracked high school students continued to invoke a range of reasoning types when they solved integer-arithmetic problems. Consequently, Lamb et al. suggested that flexibility with these ways of reasoning, in addition to supporting the development of individual ways of reasoning about integers, should be another instructional goal, so that students are positioned to effectively choose among solution approaches on the bases of the features of the tasks (including locations, signs, and magnitudes of the numbers).

2.5. Reconceptualizing integer instruction around reasoning

The identification of multiple ways of reasoning is a powerful result, serving to establish that the domain of integers, like all mathematics, involves more than merely the skills to correctly calculate. As such, this organized framework of integer reasoning provides a foundation for a new conceptualization of integer instruction with mathematical goals that extend beyond students' learning to operate. To illustrate this shift, we look at the Analogy-based Reasoning, which is related to, but different from, the use of models. In focusing on Analogy-based Reasoning, one's goal is not to support students in understanding the conventions of the model. Instead, the model serves as a tool to develop a more general understanding that can be leveraged and applied in other contexts. In particular, Analogy-based Reasoning encompasses the idea that numbers can have magnitude and direction. For example, although we know that -6 is less than -4, we also know that whereas -6 is comprised of six negative ones, -4 is comprises of only 4. As such $-6 - (-4)$ can be interpreted as taking 4 negatives from 6 negatives, resulting in 2 negatives. Focusing on such reasoning, students grapple with the relationship between magnitude and direction, a critical idea that paves the way for conceptualizations applied in other mathematical topics, for example, vectors and polar coordinates. In contrast, when the emphasis is on the model, students often engage in contrived approaches that satisfy the rules associated with the model but may not feel intuitive. For example, a student taught only to use the conventions associated with the chip model must solve $1 - 4$ by adding 3 zero pairs to get $1 + (3 + -3) - 4$, rather than using a more meaningful number-line interpretation.

Given the richness of mathematics associated with these ways of reasoning, we argue that these ways of thinking should not be viewed simply as stepping stones to more efficient recall of facts but rather should frame the overarching instructional goals themselves. Shifting the instructional focus to ways of reasoning will not only support students in developing deeper understandings of integers but will also foster valuable mathematical habits of mind and ways of knowing mathematics (Cuoco et al., 1996; Harel, 2008).

No doubt every decision a teacher makes communicates to students what mathematics is. As Bishop et al. (2014a) found, students exposed to integer rules without justification often develop an alternative view of the discipline. Almost 40% of the students they interviewed who employed rules such as *Keep Change Change* to transform $6 - -2$ to $6 + +2$, changed their answers when pressed to solve the problem without using the rule. Notably, when asked to clarify this discrepancy, they were not bothered by the resulting inconsistency and often attributed the difference to the arbitrariness of mathematics.

We see correlations with other domains that have been previously examined through a lens of student thinking and ultimately organized around critical mathematical understandings. For example, the CGI framework led to the identification of children's use of derived facts that significantly shifted the instructional focus of whole numbers. When students, faced with 19×8, reconceptualize the 19 as $(20 - 1)$ to mentally multiply $20 \times 8 - 1 \times 8$, they are learning the distributive property, developing a deeper understanding of our base-10 number system, and engaging in the mathematical practice of looking for and using structure (CCSS, 2010). Because of the richness in such reasoning, supporting students to strategically decompose numbers has now become a substantial goal of whole number operations.

Whereas integer operations have historically been approached as a system of rules, we contend that by conceptualizing integers as a domain rich in mathematical reasoning, students are more likely to approach integers more meaningfully and effectively. Overall, we want to broaden the field's understanding of teachers' perspectives about integers by exploring the conceptual resources teachers possess to support their students in developing deeper understandings of integers as well as limitations that might hinder this endeavor. Such productive and robust ways of reasoning, used by students and mirroring those of mathematicians, seemed like a constructive and logical starting point for considering teachers' ways of integer reasoning. Therefore, using this Integer Reasoning Framework as a guide, we investigated the following research questions:

1) What instructional goals did the teachers hold for teaching integers, and how flexible were they in their instructional approaches?

2) What understanding of integers did the teachers hold, and what ways of reasoning did they invoke?
3) In what ways (or to what extent) did the teachers understand students' integer reasoning, and what pedagogical implications did they draw, if any, from the students' comments?

3. Methods

3.1. Setting and participants

To identify teachers to interview about their teaching and thinking about integers, we created an online survey about integer teaching and invited approximately 45 Grade 7 teachers to complete it. Of the 45, 27 participants accepted the invitation and were paid $20 to complete the survey. Questions on the computerized survey targeted teachers' specialized and pedagogical content knowledge, specifically asking participants to write story problems relevant to various integer number sentences, describe their approaches to teaching integers, and identify the ways and extents to which they use various models associated with integer instruction (*Number Line*, *Manipulatives*, *Computer Applications*, *Other,* and *None*). In the final question, we asked whether the respondent was willing to be interviewed. Although the survey was completed by teachers from several states, we invited only those within a 30-mile radius from our location for in-person interviews so we could audiotape the interviews and have participants, during the interviews, complete pencil-and-paper integer items and react to video clips of children solving integer problems. We report on 9 teachers for whom we have complete data. These teachers range in experience from 5 to 25 years of teaching, and their majors varied, including 2 in mathematics, 2 in biology, and 2 in liberal studies, with the majority holding a K – 8 multiple-subject teaching credential. Teachers were paid $50 for participating in the interviews.

3.2. Interview

To provide a comprehensive view of teachers' orientations and mathematical knowledge for teaching associated with integers, we conducted semistructured interviews comprised of a broad range of questions. The first section targeted the participating teachers' instructional goals. Wanting to elicit their broader views of teaching integers, we began by asking teachers to describe everything they would want students to learn about integers under idealized circumstances. To steer away from a description of their current practices, we elaborated as follows:

> I would like you to imagine that your students learned *everything* about integers that you wanted them to learn. Everything! … Talk about what they would have learned about integers. Just to be clear, I'm not referring here to *how* they learned or *how* you taught. That's important, but that is not what this question is about. It is just about what they would end up *knowing*.

If a teacher's initial response focused more on how students would operate on integers, we followed up probing more directly any conceptual understanding they might possess, distinguishing integer reasoning from what students would be able to do:

> You described what you would like students to be able to do—to calculate or compute with integers correctly. Are there ways that you would like them to be able to *reason* about integers that might not be captured by what you have already said? If so, can you talk about those ways of reasoning?

We then shifted focus, asking about their current practices, including clarifying questions about their integer-instruction descriptions provided in the survey.

The second section was focused on teachers' personal integer reasoning and involved a range of questions in which teachers were asked to solve several open number sentences and explain their thinking (see Fig. 1). Throughout, we asked each interviewee to "take off your teacher hat" and apply approaches comfortable to "you as a problem solver."

In addition, this section included tasks involving the comparison of quantities such as -5 and -6; – (-4) and -4; x and $x + 1$; $-x$ and x. Teachers were asked to *circle the larger, write "=" if they have the same value, or indicate using a question mark if there is insufficient information.*

Finally, to explore teachers' interpretations of student thinking, we showed videos of three elementary school students solving open number sentences and asked the teacher to describe and reflect upon each student's thinking. In the first video, a student sings a song created for integer addition, sung to the tune of "*Row, Row, Row Your Boat,*" with the rule, "Same sign add and keep, different signs subtract. Keep the sign of the bigger number, then you'll be exact." The child then misapplies the rule she provided, illustrating a student carrying out a procedure without meaning, a method that often characterizes integer instruction. In the second video, a fourth grader, prior to formal instruction on negative numbers, reasons about the problems $-5 + -1$ and $-5 - -3$ by comparing them to

- **(a)** □ + -2 = -10
- **(b)** -9 + □ = -4
- **(c)** -3 + 6 = □
- **(d)** 5 – □ = 8
- **(e)** -3 – □ = 2
- **(f)** 6 + □ = 4

Fig. 1. The open number sentences completed by the teachers.

Table 1
The Construct Names, Descriptions, and Emergent Categories.

Construct name	Construct description	Emergent categories
Instructional goals	The teaching objectives, including procedures, concepts, and ways of reasoning, that the teacher held for students	• Rules-only • Procedures with understanding • Explicit reasoning goals
Commitment to instructional method	Whether the teacher was committed to one particular instructional method	• Single model • No commitment to approach
Teachers' ways of reasoning about integers	The way(s) of reasoning a teacher used on each task	• Ordered-based • Analogy-based • Formal • Computational
Teachers' interpretations of students' integer thinking	Alignment of the teacher's interpretation of student thinking with research-based frameworks	• Aligned • Partially aligned • Not aligned

operations involving similar positive numbers. The third video shows a first grader solving a sequence of four problems: $5 + \square = 3$, $-2 + 5 = \square$, $5 + -2 = \square$, and $6 + \square = 4$. Initially, he argues that addition cannot result in a smaller number but then eventually reasons through the problems to understand how it is possible with negative numbers.

3.3. Analysis

The second and last authors together conducted the teacher interviews, which were audiotaped and transcribed. Participants' responses were analyzed using a grounded theory approach (Strauss & Corbin, 1994). To answer the first research question, we analyzed the data related to teachers' espoused goals for integer instruction and the teachers' responses about their instructional methods. Specifically, we coded the nature of their goals, determining whether they focused solely on rules, attempted to develop the rules along with understanding of the underlying mathematics, or were able to articulate some reasoning goals. We also categorized teachers' descriptions of their instruction on the basis of whether or not a teacher relied exclusively on a single method or integer model.

To answer the second research question, we analyzed the teachers' integer reasoning by applying a coding scheme developed from Bishop et al.'s (2014b) study of students' integer reasoning. Each teacher's solution and explanation for every open number sentence was coded for the ways of reasoning exhibited (see Appendix for the Ways of Reasoning codes). Trained, experienced coders of the student data double coded all teacher responses with an interrater agreement of 86%. Coding disagreements were resolved through discussion and resulted in the final set of codes reported.

To answer the third research question, we analyzed teachers' interpretations of student thinking evident in the videos to determine the degree to which their interpretations aligned with research-based frameworks of student thinking as well as the pedagogical ramifications they attributed to the students' reasoning. We determined three categories: (a) aligned, which indicated an interpretation that captured nuances of student thinking elevated by research and included meaningful pedagogical ramifications; (b) partially aligned, which was assigned to vague interpretation often stemming from an assumption that the method had been taught or that the students had uncommon, innate ability; (c) not aligned, which was characterized by over- or under-generalizations based on personal pedagogical views rather than reasoning identified by research. These categories are further explained and exemplified in the results section. To be clear, these categories emerged when we compared teachers' responses to our understanding of student thinking interpreted through the Ways of Reasoning framework, and, as such, have a degree of subjectivity.

4. Results

We organize the results into three sections: (a) teachers' instructional goals and methods, (b) teachers' ways of thinking about integers, and (c) teachers' understanding of students' integer reasoning. Additionally, we explore the interrelationships among these areas. Table 1 contains these constructs, their descriptions, and the emergent categories.

4.1. Research Question 1: teachers' instructional goals and methods

4.1.1. Instructional goals

To understand teachers' instructional goals, we asked the participants to describe their overall instructional objectives within this content domain, their efforts to achieve these objectives, and the characteristics they take as evidence of students' success. Although teachers varied in pedagogical strategies for teaching integers, these teachers' foci were primarily procedural. They repeatedly stated that they wanted students to develop the ability to correctly compute with the four operations. Three teacher categories emerged from the data: one characterized as teachers with a *rules-only* orientation, a second comprised of teachers who indicated an appreciation for *procedures with understanding*, and a third marked by the ability to articulate *explicit reasoning goals*. Although teachers in the second category emphasized the desire for students to be able to reason through integer tasks, their responses indicated that they did not view integer concepts or ways of reasoning as topics to be developed in their own right, but only in the service of computational proficiency. We were not surprised by the teachers' struggles to articulate ways of reasoning because such ideas have emerged only after extensive research involving a detailed literature review and the analysis of many student interviews.

4.1.1.1. Rules-only. Two of the 9 participants, Bret and Ann (see Table 2), were classified as holding instructional goals identified as rules–only. These teachers, when asked to describe the full extent of what they hoped students would learn within the context of integers, expressed in detail the rules they expected students to learn. The precision with which they articulated these rules indicates that for these two teachers, the procedures, exactly as they expressed them, are essential for the success of their students. Moreover, both teachers specified that difficulties with integers stem from students' lack of attention to the explicitly taught rules and failure to sufficiently practice them as such.

For example, Bret highlighted that students must "do repetitions over and over and over again … until they have it embedded in their memory—until they get 100% … and not make a mistake." Bret revealed more of his rules-only thinking when asked to watch the video clip and analyze the thinking of a fifth-grader, Jane, attempting to apply a rule by reciting a song. She rewrote the problem, changing the initial expression of $-12 + 7$ to $-12 - -7$ and then misapplied the rule of "same sign add and keep, different sign subtract" to the new subtraction problem. She obtained an answer of -19, not recognizing that the "different signs, subtract" part of the rule applied to the original addition problem. Responding to this video clip, Bret stated that the student should "leave the original problem always and rewrite whatever the rule is. … No matter what order the numbers are in, you have to subtract the smaller number from the larger number—their absolute values." Bret's interpretation was that Jane's error was due to her failure to methodically apply the rule, reflecting his belief that clarity in articulating and practicing rules is essential to learning integer arithmetic.

Similarly, Ann explained that students' difficulties result from a lack of "drill and kill." She further emphasized this approach by arguing that alternative approaches only contribute to confusion and mistakes. She explained that once, when she had allowed students to solve problems using their own methods, the results were "disastrous." "They all got it wrong. … It was amazing to me. It's almost like they needed the steps, to feel comfortable. So maybe that is my manipulative—the steps." Ann now instructs her students to use only her prescribed methods. For example, she mandates that students, given a subtraction problem, rewrite the problem as addition and change the sign of the subtrahend. When asked how many of her students would make use of this technique and change $4 - -12$ to $4 + 12$, she replied, "They would all do that or they would get it wrong. I call my classroom "Thompsonville" (note, her last name is Thompson), and you do it the Thompson way." Furthermore, she said that she has her students practice the procedures "85,000 times."

Later in the interview, when Bret and Ann were asked the ways in which they like students to *reason about integers,* their answers indicated no clear conception of what reasoning entails. Both replied by repeating the strict adherence to a step-by-step procedure in applying the rules. Ann responded with a particularly illustrative example of this phenomenon. When initially asked about *reasoning*, Ann, who taught sixth and seventh grades, appeared confused:

> Reason … I think that that comes more in seventh grade, when we take it to the next level, because we will add variables to it, and we add the distributive property to that. In sixth grade, we are not doing that. We are just going straight forward with it.

Not seeing what role reasoning could play in solving integer tasks, Ann attempted to make sense of such an idea by associating it with topics that arise in the subsequent formal study of algebra. As these comments illustrate, for Bret and Ann, the ability to reason about and solve integer problems on one's own is not simply absent from their instructional goals; it is actively discouraged.

Table 2
Categorization of Teachers' Instructional Goals and Methods.

Participants	Instructional goals	Instructional Methods
Ann	Rules only	Single model
Bret	Rules only	Single model
Kalani	Procedural with understanding	Single model
Kate	Procedural with understanding	Single model
Jill	Procedural with understanding	No commitment
Ron	Procedural with understanding	No commitment
Tom	Procedural with understanding	No commitment
Gabriel	Procedural with understanding	No commitment
Cheryl	Explicit reasoning goals	No commitment

4.1.1.2. Procedural with understanding. Six of the other seven teachers fell into the second category characterized by a focus on procedural fluency accompanied with some appreciation for instilling an understanding or justification of the rules. Although these teachers reported the ability to accurately carry out the four operations as their main instructional goal with integers, they also voiced the desire for students to reason through their calculations. The teachers' descriptions of what such reasoning might entail were vague, but they appeared to possess views of procedural fluency that incorporated an associated understanding of the rationales behind the rules. The following quotes from Gabriel are indicative of the coexistence of these two elements. He began by asserting that he hoped students "would know that integers are whole numbers, positives, negatives, and that they would be able to do the operations on those numbers," but then added that he tries to "stay away from gimmicks [referring to rules without reasoning] that … didn't really explain what was going on." He continued, "I want them to reason through it as much as they can. They need to know that there is some other reason [beyond the rule]." Gabriel valued the ability to correctly carry out the operations but, unlike Bret and Ann, also wanted students to develop a sense of the rationale for these procedures.

One of the more consistent results was that the teachers were unable to explain what integer reasoning entails, even though they had multiple opportunities and were asked in several ways to distinguish between ways of *operating on* integers and underlying ways of *reasoning about* integers. Each time the teachers were asked to explain what they interpreted integer reasoning to be, they responded in vague, nondescriptive terms. For example, Kate expressed a desire for students to be able to "just kind of picture it in their head." Gabriel described this capacity as being "able to play with it a little bit in their head." Others spoke of an intuition with numbers. For example, Tom emphasized that he wanted students "to be able to look at a simple integer problem and be able to tell, off the top of their head, if it is going to be a positive answer or a negative answer." Jill expressed a desire for students to "have a good understanding of what positives and negatives mean." Although these teachers seemed to value some sort of reasoning, they were unable to articulate what this might involve.

4.1.1.3. Explicit reasoning goals. Only one teacher, Cheryl, articulated details about what *integer reasoning* entails. She stated specifically that she wanted students to develop understanding of "value and quantity" associated with integers. She distinguished the two by explaining that -3 has a larger value than -7 but less quantity in terms of the visual context of swimming-pool depth (implicitly referring to magnitude). She also spoke of the need to make clear different interpretations of the minus sign as "negatives and opposites." She emphasized understanding the minus sign as meaning the *opposite* of whatever follows and not simply the sign of the number. Of the 7 teachers who valued reasoning, only Cheryl spoke of specific understandings of integers she hoped to develop in students. For the other participants, their nebulous descriptions seem to indicate that a *way of thinking about integers* is not tangible for them. Although the teachers value and (as we will show later) possess a variety of ways of reasoning, without an explicit understanding of what integer reasoning is, they lack a clear picture of what understanding they could hold as goals for their students.

4.1.2. Instructional methods

When asked how they taught integers, all the participants mentioned the use of various models and manipulatives but struggled to cite any purpose for their inclusion except as a way to demonstrate the rules. In analyzing the teachers' descriptions of the pedagogical approaches they used, we coded their responses into one of two categories: a firm commitment to a *single model* for integer instruction or *no commitment to an approach* (see Table 2).

4.1.2.1. Single model. Four of the teachers, Ann, Bret, Kate, and Kalani, fell into the first category. What distinguished these teachers was not simply their exclusive use of a single method but also their strong beliefs that their chosen approaches were the best (and often *only*) way to master and understand integer operations. As highlighted previously, a trait of the rules-based-instruction orientation is the conviction that only through the strict adherence to rules can students learn integer operations. Consequently, both Ann and Bret fell into the single-model category. In addition, two of the teachers who included understanding as part of their instructional goals also fell into this category. Kate used exclusively the balloons and weights model (Lamb & Thanheiser, 2006), which situates integers on a number line and uses balloons to represent positive quantities and weights to represent negative quantities. Removing (a balloon or weight) represents the operation of subtraction, whereas adding (a balloon or weight) represents the operation of addition. Kalani related all integer problems to various contextual situations, for example highlighting how filling in holes explains the subtraction of a negative. All four of these teachers were so dedicated to their specific approaches that they repeatedly explained student difficulty as a lack of exposure to their prescribed methods. In addition, they interpreted student reasoning that did not incorporate their approach as lacking understanding. Consider Kalani, who was confused by the thinking of a 4th grader who correctly leveraged the inverse relationship between addition and subtraction to solve $-5 - (-3)$ by reasoning that the subtraction of a negative must have the "opposite" effect of addition. In response, Kalani said, "I think he got confused. There's no context involved. … I don't quite understand him when he used the *opposite*. Opposite of what? … I don't see a clear understanding on his part at all." As a follow-up question to clarify the 4th grader's thinking, Kalani said that he would ask if he could explain what -5 represents in context. In this exchange, Kalani seemed unable to follow the student's approach and appeared to assume that because the student did not mention context, he possessed no meaningful conceptualization. Kalani's reaction surprised us because in solving integer tasks, he had invoked reasoning almost identical to this child's.

4.1.2.2. No commitment to an approach. The other five teachers fell into the second category, defined by their lack of commitment to any particular approach. To be clear, what characterized the teachers in this category is not simply that they used multiple approaches but that they expressed indifference about which models they used when teaching. They each described how they selected instructional aides on the basis of convenience and time constraints without having a particular commitment to those aides, often changing their approaches from year to year. For example, Tom told of using number lines he had found in a mathematics locker one year, but because the number lines had been destroyed as a result of students' taping them to their desks, he moved on to the chips model the subsequent year. In addition, these teachers pointed out that the various models they had tried were inadequate, explaining that each inevitably led to student confusion and in the end required the teacher to tell the students the rules. Although the exact approach was unimportant, these teachers described the need to use some instructional aides to *explain* the rules or develop some sort of understanding of the rules they eventually taught. Such a belief aligns with Kumar et al.'s (2017) finding that teachers see contexts as useful to introduce students to provide a "need for integers,'' but do not see them as effective in teaching operations with integers. Although these teachers wanted students to develop understandings that went beyond simply rules, lacking clear understandings of what this entails, they ultimately focused on providing their students procedures to correctly calculate with integers. See Table 2 for a summary of results on Research Question 1.

4.2. Research Question 2: teachers' ways of thinking about integers

In the previous section, we shared data supporting the findings that the teachers' goals for instruction were primarily focused on the mastery of operations with some appreciation for understanding (although not well-defined). Although one might infer from these findings that the teachers would approach solving integer tasks in a procedural manner, the results indicate otherwise: The teachers consistently invoked various ways of reasoning while solving integer tasks and even invoked methods that did not align with their endorsed instructional methods! Even when they applied rules, the teachers usually did so strategically, knowing when a particular rule or property would be of service, and often combining rules with other noncomputational forms of reasoning. (Note that 100% of the tasks were answered correctly by the teachers.)

4.2.1. Teachers' ways of reasoning

To answer the second research question regarding teachers' ways of reasoning, we turn to the teachers' responses to the number sentences (see Fig. 1). In Table 3 we present the various ways of reasoning teachers used across all six problems. The Total column indicates the total number of times each way of reasoning was used across all the open number sentences. Note that for a single problem, a teacher might have invoked multiple ways of reasoning, as Bret did for Problem (b). Collectively, teachers' solutions were coded as 54% Computational, 15% Formal, 15% Order-based, and 37% Analogy-based (more than 100%, because one solution may be coded for more than one way of reasoning). As such, the number of instances of noncomputational reasoning was larger than the number of instances of Computational reasoning. In fact, 2 teachers (Kalani and Gabriel) used no Computational reasoning at all (see the frequency counts in the Total column), and the responses of 4 of the 9 teachers indicate that it was not their preferred method. Not only did they use noncomputational ways of reasoning, they also demonstrated flexibility by using various ways of reasoning. Only one teacher, Ann, used a single way of reasoning. In the other cases, although teachers might favor a particular way of reasoning, they combined this preferred approach with other ways of reasoning.

To illustrate the range of strategies used by the teachers, we describe three responses by Tom, chosen because of the clarity of his responses and because he favored a noncomputational way of reasoning, in particular, Order-based Reasoning. As such, his responses serve as examples of the variability and flexibility of the reasoning demonstrated by the teachers as a group, contrasting starkly with their instructional approaches.

4.2.2. Tom's reasoning

Tom's solution to $-9 + \square = -4$ exemplifies his preferred Order-based Reasoning. Here he located -9 and -4 using the sequential and ordered nature of numbers, stating, "If I start at negative 9 and have to get to negative 4, I have to add 5. I don't know if I actually pictured a number line myself, … but I know I have to go upwards towards negative 4 and upwards is obviously a positive number."

Tom's response to $5 - \square = 8$ illustrates the combination of three ways of reasoning, Order-based, Formal, and Computational. He first compared 5 and 8 in an ordered way: "I know I am starting with 5 and my answer is higher than 5, so I know I have to go upwards, … and the difference between the two is 3." He then used formal reasoning to infer the sign of the unknown on the basis of the

Table 3
Ways of Reasoning Invoked by Teachers on Interview Tasks.

Teacher	$\square + -2 = -10$	$-9 + \square = -4$	$-3 + 6 = \square$	$5 - \square = 8$	$-3 - \square = 2$	$6 + \square = 4$	Total
Ann	C	C	C	C	C	C	C = 6
Bret	C	F, C	C	C	C	F	C = 5, F = 2
Kalani	A	A	A	O	O	A	A = 4, O = 2
Kate	C	C	C	F	C	C	C = 5, F = 1
Jill	A	A	A	A, C	O, A	F, A	A = 6, C = 1, F = 1, O = 1
Ron	C	C	A	C	C	C, F	A = 1, C = 5, F = 1
Tom	A	O	O	O, C, F	O, C	O, F	A = 1, C = 2, F = 2, O = 5
Gabriel	A	A	A	F	F	A	A = 4, F = 2
Cheryl	A, C	A, C	A	C	C	A, C	A = 4, C = 5

Note. A = Analogy-based, C = Computational, F = Formal, and O = Order-based.

operation of subtraction and the fact that the ending value of 8 has a greater magnitude than the starting value of 5. He said, "But the subtraction sign leads me to think, 'How can we go upwards to eight?' So I know it has to be a negative number, negative 3." He then invoked a rule (Computational reasoning) saying, "Two negatives turn into a positive. Explaining to the kids, '5 minus negative,' they can't visually see that minus a negative means adding. But if you just tell them, 'Two negatives in a row make a positive,' they can understand that a little bit better." Using a mixture of Ordered-based, Formal, and Computational reasoning, Tom correctly answered -3.

Finally, to solve $\square + -2 = -10$, Tom reasoned quite differently, using an analogy-based approach. He stated, "I am thinking there are 10 negatives, so you have to add up to 10 negatives, so if I have 8 negatives and 2 negatives it makes 10 negatives." Notice the focus here on magnitude, that is, the absolute values of the numbers. He separated the negative sign from the magnitude, viewing -10 as representing a collection of 10 objects that are "negatives." If he already has 2 of these objects, he needs 8 more.

4.2.3. Relationships between teachers' reasoning and their pedagogical goals

After investigating teachers' personal reasoning and their instructional goals, we wondered whether a relationship existed between these two areas. Overall, we found a limited relationship. All participants did indicate, to varying degrees, that integer rules were part of their instruction, and this did emerge as the preferred method. Nonetheless, all except Ann used other noncomputational approaches to solve the six open number sentences, and most tended to flexibly modify their thinking depending on the problem at hand. The one strong correlation that emerged was with the two teachers who reported a rules-only orientation. As mentioned above, Ann, whose instruction was characterized by strict adherence to rules, exclusively invoked Computational reasoning. Similarly, Bret tended to invoke Computational reasoning, but he also engaged in formal reasoning for two tasks to justify which sign the answer should have.

For the others, these data highlight that teachers possess a wide range of resources that they invoke when reasoning but appear to not fully leverage in their instruction. In fact, the thinking of the other seven participants differed quite noticeably from their stated instructional goals and strategies. This contrast was particularly visible with the two teachers who were dedicated exclusively to a single instructional model. Kate, who reported using a model that combined an analogy-based and ordered-based approach, used neither the model nor either of these two ways of reasoning to answer any of the questions! Kalani, who viewed integer instruction almost exclusively through a lens of context, solved the integer tasks using a variety of approaches. Although he did show some preference for Analogy-based Reasoning, he explicitly stated that applying a context would be unsuitable for two of the problems. Similarly, none of the other teachers, whose instruction did not align with a single model, favored a single approach.

4.3. Research Question 3: teachers' understanding of students' integer reasoning

During the interview, each teacher viewed videos of three elementary school students solving open number sentences and were asked to describe and reflect upon each student's thinking. Even though the teachers were not asked to address possible pedagogical ramifications associated with these approaches, most offered explanations for students' thinking, cited possible instructional responses, or reflected on the students' thinking in terms of their own pedagogy. Therefore, although we focused on teachers' interpretations of student thinking, we also analyzed various pedagogical implications and interpretations teachers chose to incorporate in their answers.

The data for our analysis come primarily from teachers' reflections on the second and third videos. The second video was of a fourth grader, Raymond, solving two problems. In solving the first problem, $-5 + -1$, he reasoned, "If you add these two together … it would make it farther from the positive numbers, and it's like mainly just doing 5 plus 1 equals 6." By comparing $-5 + -1$ to the sum of $5 + 1$, he used analogy-based reasoning to conclude that adding negatives depends only on the magnitudes of the numbers, resulting in a number six away from the positives just as $5 + 1$ results in a larger positive number. In addition, Raymond's statement that the result was "farther from the positive numbers" indicates that he also invoked Order-based Reasoning.

Raymond used a formal approach to solve $-5 - -3$. He held the two values of -5 and -3 constant and compared the problem he was asked to solve to the related addition problem $-5 + -3$. Through this comparison, he conjectured that because "addition [of the two negative numbers] … would be farther from positive numbers, if you do the opposite, it should be closer [to the positive numbers]." Using formal reasoning and invoking the idea that subtraction is the inverse of addition, he concluded that the effect of subtracting a number is opposite the effect of adding that number, resulting in this case a final value three units closer to the positive numbers than the first term. Again, Raymond's description of being closer to and farther from the positive numbers indicates that he also invoked Order-Based Reasoning.

The third video showed a first-grader, Ryan, solving a sequence of four problems: $5 + \square = 3$, $-2 + 5 = \square$, $5 + -2 = \square$, and $6 + \square = 4$. In solving the first problem, he stated that $5 + \square = 3$ has no solution because "it's [the sum is] always past 3 if it's plus," alluding to the idea that *addition makes bigger.* He then solved $-2 + 5$ by using ordered-based reasoning and counting on five units from -2. Across these two problems, we conclude that Ryan was able to conceptualize negatives as a starting position on the number line (as in the 2nd problem) but not as a decreasing change action requiring movement to the left (as in the first). Ryan then solved $5 + -2$ (with a negative change action) by applying the commutative property (using formal reasoning) to the previous problem, stating "because it's pretty much the same thing. Five plus negative 2, [and] negative 2 plus 5. 'Cause you add the same things." Finally, Ryan reproduced this sequence of reasoning to solve the problem $6 + \square = 4$ (a negative change as the unknown instead of given), stating, "It was kind of like that (pointing to $5 + -2 = \square$). Plus a negative." When asked to explain, Ryan summarized his method, arguing that "negative is like minus" and negative 2 means "go back two." Ryan began to make sense of adding a negative as change and connecting this understanding to that of the operation of subtraction. Remember, Ryan was in *first* grade.

4.3.1. *Teachers' interpretations of students' thinking*

In our analysis we coded the teachers' interpretations of these students' thinking into three categories: aligned, partially aligned, and not aligned (see Table 4). *Aligned* indicates responses showing that the teacher understood and could articulate nuances of student thinking in a way that aligns with research-based frameworks. Furthermore, the teacher made meaningful connections to instruction, explaining how students' thinking informed her about changes she should make to her own teaching. *Partially aligned* means that the teacher's explanations of student thinking, though correct, were vague, without attention to details, often incorporating an overgeneralization of the student's understanding and generally attributing student thinking to a learned method or innate intelligence. *Not aligned* denotes inaccurate explanations of the students' reasoning with interpretations of the students' reasoning made almost solely through the lens of the teacher's preferred instructional approach. Unable to detach their understanding of the students' ideas from their views of instruction, teachers whose responses were categorized as *not aligned* overestimated the student's understanding when it corresponded with their views of instruction or described the approach as unproductive when it did not, offering pedagogical suggestions to improve or correct the student's understanding. To illustrate, we provide an example of teacher responses to either Raymond's or Ryan's thinking from each classification. To help the reader contextualize the data, we also present the categorization of each teacher's interpretation of student thinking in a table along with the classification of goals and methods and the ways of reasoning (see Table 4).

4.3.1.1. Aligned. Cheryl's analysis of Ryan's thinking illustrates the nuanced characteristic of interpretations that align with research-based frameworks along with the associated pedagogical reflections. Cheryl began by clearly explaining Ryan's reasoning: "He has a number-line sense of where the numbers are placed. So if he started from a negative, he can move up, and he didn't quite have that understanding of a negative movement." She then continued to reflect on this observation, considering implications for instruction:

> It made me really think of—especially the children that are struggling with integers—is to definitely start it with the negative because that was really apparent to him. And the commutative property, changing the order of them around, wasn't a big deal for him.

Cheryl indicated that she saw how the order of the addends in integer addition (i.e., $5 + -2$ vs. $-2 + 5$) could make a significant difference for students while they come to understand integers. She concluded that starting with problems in which the first addend is negative provides a more accessible approach for students and that such sequencing may allow students ways to reason about problems in which the second addend is negative.

4.3.1.2. Partially aligned. Tom's response to Raymond's video is an example of a partially aligned interpretation. When asked about Raymond's solution for $-5 + -1$, Tom expressed satisfaction with Raymond's thinking, explaining that he would like all his students to be able to identify the answer as negative before actually calculating the exact value. Tom offered no details about Raymond's reasoning, stating, "He [Raymond] just sat there and thought about it." He attributed Raymond's method to innate ability, describing him as "a bright mathematical student," clarifying that he believed that some students "have these math instincts that give them a push towards something or another, and some just don't have those kind of math—they don't understand it as well." When asked to comment on Raymond's solution for $-5 - -3$, Tom was initially unable to follow Raymond's reasoning but assumed that the method was "something he was taught." When shown the video a second time, Tom was able to explain the crux of Raymond's thinking, stating, "I think because the problem beforehand was plus, it was more negative. So he says, it has to be the other way then." Tom followed this explanation with a comment clarifying how he saw this answer as another example of Raymond's "math instincts." Again, Tom offered some articulation of Raymond's thinking, but he interpreted his reasoning as evidence of some natural mathematics aptitude or a specifically instructed method.

4.3.1.3. Not aligned. Finally, we present responses from Ann, whose analysis and comparison of both students' mathematical thinking provides an example of the overgeneralized nature and strong pedagogical evaluation of interpretations in this category. After being shown Ryan working on the first two problems in which he classified $5 - \square = 3$ as an impossible problem and accurately reasoned through $-2 + 5$, Ann concluded that Ryan possessed no understanding. She explained, "Obviously it [his understanding] didn't

Table 4
Teachers' Interpretations of Student Thinking With Summary of Integer Instruction and Personal Ways of Reasoning Across Teachers.

Teacher	Instructional goals	Instructional methods	Teachers' ways of reasoning about integers	Interpretation of students' integer thinking
Ann	Rules only	Single model	C = 6	Not aligned
Bret	Rules only	Single model	C = 5, F = 2	Not aligned
Kalani	Procedural with understanding	Single model	A = 4, O = 2	Not aligned
Kate	Procedural with understanding	Single model	C = 5, F = 1	Partially aligned
Jill	Procedural with understanding	No commitment	A = 6, C = 1, F = 1, O = 1	Partially aligned
Ron	Procedural with understanding	No commitment	A = 1, C = 5, F = 1	Partially aligned
Tom	Procedural with understanding	No commitment	A = 1, C = 2, F = 2, O = 5	Partially aligned
Gabriel	Procedural with understanding	No commitment	A = 4, F = 2	Aligned
Cheryl	Explicit reasoning goals	No commitment	A = 4, C = 5	Aligned

Note. A = Analogy-based, C = Computational, F = Formal, and O = Order-based.

translate to the other problem. So, does he have a full working knowledge of it? If he did, he would have been able to transfer it there." Her assessment of Ryan changed though after she saw him correctly solve the final two problems. Much like Tom, Ann attributed Ryan's success to an innate ability, describing him as possessing "a very promising understanding of numbers." When asked what evidence she had, she commented that Ryan used a method that corresponded to her instructional method, what she called "double-duty" to describe how a negative could be an operation or the sign of a number, stating, "Because he did *my* double-duty. He said, 'Subtraction is like.' … He saw negative as taking away and he called it 'negative.' He said, 'The negative sign is like the subtraction sign. You are taking it away.'" It seems that because Ryan's approach was consistent with a rule and vocabulary that Ann teaches in her classroom, she attributed this to strong understanding. She then contrasted Ryan's thinking to Raymond's, which she did not value:

> So, there is a better number sense there than I think Raymond had. … Raymond was calling it *minus*. He didn't have a concept of negative and positive. He was looking at it like a subtraction sign. I think Raymond can get the right answers, but his thinking would have to be adjusted.

Ann demonstrated strong feelings about not only the procedures that ought to be taught but also the language that ought to be used, regarding Raymond's use of the term *minus* to be problematic. Furthermore, when Ann was asked whether she would like someone to come into her class thinking like Raymond or if would it make it worse, she responded without hesitation, "Worse! I absolutely do not want them learning positive and negative numbers with the operations in fourth grade." We conclude that Ann not only failed to recognize the rich and valid reasoning embedded in Raymond's responses but that she also did not consider that an approach different from her own could be valid. Finally, she clearly wished to avoid Raymond's reasoning, and, we suspect, *any* reasoning that was not consistent with meeting her instructional goals.

4.3.2. Relationship between teachers' interpretations of student thinking and their instructional goals

Although the teachers' instructional goals and their own personal reasoning seemed disconnected, we did find a correspondence between teachers' pedagogical orientations and their interpretations of student thinking (see Table 4 above). Those teachers, like Ann and Bret, who possessed a rules-only perspective of integer instruction seemed to have a binary view of student understanding and were unable to follow students' ways of reasoning unless they mapped onto their own instructional methods for teaching. Similarly, Kalani, who valued students' developing some understanding of the rules but was rigid in his pedagogical approaches, struggled to follow students' reasoning. These three teachers were critical of methods not corresponding to their own instructional goals and overgeneralized students' strategies and understandings related to their preferred instructional approaches. We conjecture that without an appreciation of integer reasoning, these teachers failed to understand how a student could solve one problem, such as $-2 + 5 = \square$, but not another, such as $5 + \square = 3$. Furthermore, they incorrectly concluded that one correct solution by a student implied that the student had a robust understanding of integers. In contrast, Cheryl, who articulated some nonprocedural goals, provided more detail in her interpretations of student reasoning and was able to formulate pedagogical conclusions on the bases of the nuances of student thinking.

5. Discussion

Our goal of this study was to investigate seventh-grade teachers' perspectives about integers, including their goals and methods for instruction, their ways of reasoning when solving integer tasks, and their understanding of students' integer reasoning. Results indicate that the seventh-grade teachers in this study did not explicitly articulate reasoning about integers as an instructional goal and instead focused primarily on the mastery of the four operations, as evidenced by the first column of Table 1, "Instructional goals." Furthermore, although most of the teachers shared that they would like for this mastery to be associated with understanding, only one teacher, Cheryl, was able to provide specificity as to what an understanding of integers might entail. Additionally, all the teachers correctly completed integer tasks, and all but one invoked more than one way of reasoning across the tasks (see Table 3). Notably, whereas the teachers used a range of noncomputational strategies associated with a variety of ways of reasoning in their own thinking, such reasoning was not evident in their instructional goals. Overall, we found limited relationships among teachers' personal reasoning, their instructional goals, and their interpretations of student thinking. However, we did find that teachers who strongly preferred a single model for instruction or held a singular focus on rules as an instructional goal seemed to be constrained by these relatively narrow instructional models and goals when interpreting student thinking. Thus, a flexible approach to integer instruction appeared to be a necessary, but not sufficient, attribute for appreciating student reasoning.

Such results point to a disconnect between the types of reasoning that teachers strive for as part of their instructional goals for integers and their own personal reasoning. Even though the participants in our study drew on a range of ways of reasoning themselves, they were, for the most part, unable to explicitly identify such reasoning as instructional goals and struggled to recognize the corresponding integer concepts in students' strategies. The lack of conceptual specificity in the teachers' responses highlights an important dimension of Mathematical Knowledge for Teaching. Although seven of the nine teachers in this study were explicitly committed to developing integer understanding, they lacked an organized structure for their knowledge that would enable them to leverage the conceptual resources they possessed. Without a framework to guide them, they ultimately focused their instruction on integer models and rules.

Previous research designed to explore teacher knowledge in the realm of integers has been focused primarily on the lack of Specialized Content Knowledge (SCK) involved in understanding of different representations, models, and contexts associated with integers (Kinach, 2002; Kumar et al., 2017; Mitchell et al., 2013; Wessman-Enzinger & Tobias, 2020). We build on this research, but take a different approach. The inability of teachers in this study to connect the rich integer understanding they demonstrated to their classroom instruction suggests an additional component of SCK, the ability to coherently organize an instructional domain (like integers) around key concepts and reasoning. Within the domain of integers such a structure would incorporate the various meanings of the minus sign, the underlying meanings and perspectives of number (ordinal, cardinal) and operations (motion, movement, acquisition, etc), the use of fundamental properties to justify procedures and rules, and various ways of reasoning students can use to solve problems. We believe that the Ways of Reasoning framework provides such an organizational piece, connecting and giving purpose to these different aspects of specialized knowledge.

In addition, we extend previous research by describing not only the importance of a structure for organizing SCK but also how SCK for integers may interact with the ways teachers interpret and respond to students' mathematical thinking within a particular content domain. Those teachers whose instructional goals included understanding in addition to computational proficiency and indicated more flexibility in their instructional approaches were able to appreciate and recognize student reasoning when observing videos, with some even articulating clear pedagogical implications.

Shifting teachers' current pedagogical methods so that their instructional practices encompass teaching for understanding and responding to student thinking entails not only knowing how to solve problems but also possessing an organized understanding of the associated types of reasoning. We believe that the mathematics education community needs to reconceptualize mathematical domains around the types of student reasoning associated with a rich understanding of the content, as has been done with whole number operations through the CGI–based research (Carpenter et al., 2014). By identifying and organizing such knowledge into frameworks, researchers elevate ways of reasoning for teachers, enabling them to more productively attend to and learn from their students' thinking. To effectively listen and respond to students' thinking, teachers must not only want to know what their students are thinking, they must also hold a structured understanding of the underlying ways of reasoning that enables them to meaningfully interpret their students' reasoning and use that reasoning to make effective in-the-moment pedagogical decisions (Jacobs et al., 2010). As CGI research has shown (e.g., Carpenter et al., 1989), a restructuring of teachers' thinking can have powerful effects on teachers' understanding of the goals related to a mathematical domain and, as a consequence, on their instruction.

A profound principle about learning that applies not just to children but to all people is that we must start with what one already knows when trying to move that person to a richer understanding. Our study shows that although many teachers did not recognize the various ways of reasoning in student's responses, they were able to use them in thoughtful and productive ways. Clearly, in the domain of integers we have a productive starting point that can be leveraged once we provide teachers the right support.

Author statement

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Appendix A. Integer Reasoning Framework

Ways of Integer Reasoning	Descriptions	Examples
Order-based Reasoning	Using the sequential and ordered nature of numbers to reason about a problem (e.g., counting strategies or a number line with motion).	Maria (3 – ☐ = −3) Referring to a number line, Maria puts her finger on 3, and while she moves her finger to the left, she counts her touches, starting at 1, "One (touches 2), 2 (touches 1), 3 (touches 0), 4 (touches −1), 5 (touches −2), 6 (touches −3). Six." Maria's approach is common in that the number line is a powerful representation that supports ordered-based reasoning. Ryan (−2 + 5 = ☐) "It was negative 2, negative 1 (puts up one finger), 0 (puts up a second finger), 1 (puts up a third finger), 2 (puts up a fourth finger), 3 (puts up a fifth finger)." Ryan uses the ordering of numbers to interpret −2 as a starting place and counts on five numbers to 3. Ryan's counting strategy shows that ordered-based reasoning does not require the number line.
Analogy-based Reasoning	Relating negative numbers to another idea and reasoning about negative numbers on the basis of characteristics observed in this other concept. This way of reasoning may be characterized by	Felisha (4 + −5 = ☐) Using colored chips, Felisha counts four blue chips and five red chips, matches each blue chip to one red chip, touches the remaining unmatched red chip, and says, "Negative one." Note that

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(*continued*)

Ways of Integer Reasoning	Descriptions	Examples
	thinking of the sign and magnitude of the number separately and interpreting the sign as taking on the meaning based in the analogy.	the chips model is associated with analogical reasoning, but as Julio demonstrates in the following example, analogical reasoning need not involve physical models. Julio (−9 + ☐ = −4) "The idea here is you have a negative quantity or a negative nine holes in the ground. How do you get to having only four holes in the ground? You fill up five of those." Julio used analogy-based reasoning to relate negative numbers to holes in the ground and interpreted addition of positive numbers as the act of filling holes.
Formal Reasoning	Negative numbers are treated as formal objects that exist in a system and are subject to fundamental mathematical principles. Formal strategies often involve comparisons to other, known problems so that the logic of the approach remains consistent and underlying structural principles are not violated.	Francisco (−8 – 3 = ☐) When describing his answer of −11, Francisco explained that he used the previous result. "Well I looked back up at this problem [pointing to −3 + 6 = ☐]. I got 3 because I counted up; negative 3, negative 2… And then I thought, 'Well, minus [as indicated by the subtraction sign in −8 – 3] must be going down. And I got 11… Negative 8, negative 9 (raises one finger), negative 10 (raises second finger), negative 11 (raises third finger)." Francisco used what he knew about the sum of −3 and 6 to help him make a reasonable conjecture about how subtraction might function when the starting value is a negative number. Francisco reasoned that if addition means to count up, then subtraction must do the opposite and be going down.
Computational Reasoning	Using a procedure, rule, property, or calculation to arrive at an answer.	Omar (6 – −2 = ☐) Omar changes the minus sign to addition and the negative sign to positive to rewrite the left-hand side of the equation as 6 + +2 and says, "The answer would be 8." When asked to explain, he refers to this practice as using the *double-stick trick*. Katie (−3 + 6 = ☐) "I know that negative 3 plus 6 is the same as six plus negative three, and adding a negative is the same as subtracting a positive, and 6 minus 3 is 3."

References

Bishop, J. P., Lamb, L. C., Philipp, R. A., Schappelle, B. P., & Whitacre, I. (2011). First graders outwit a famous mathematician. *Teaching Children Mathematics, 17*(6), 350–358.

Bishop, J. P., Lamb, L. L., Philipp, R. A., & Schappelle, B. P. (2014a). Opportunities for algebraic reasoning in the context of integers. Epistemic algebraic students: Emerging models of students' algebraic knowing. *WISDOMe Monographs, 4*, 303–316.

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. L. (2014b). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education, 45*(1), 19–61. https://doi.org/10.5951/jresematheduc.45.1.0019

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2014). *Children's mathematics: Cognitively guided instruction* (2nd ed.). Portsmouth, NH: Heinemann.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C.-P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499–531. https://doi.org/10.3102/00028312026004499

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The Journal of Mathematical Behavior, 15*(4), 375–402. https://doi.org/10.1016/S0732-3123(96)90023-1

Gallardo, A. (1995). Negative numbers in the teaching of arithmetic: Repercussions in elementary algebra. In D. Owens, M. Reeds, & G. Millsaps (Eds.), *Proceedings of the Seventeenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 158–163). Columbus, OH: ERIC.

Gallardo, A. (2002). The extension of the natural-number domain to the integers in the transition from arithmetic to algebra. *Educational Studies in Mathematics, 49*, 171–192. https://doi.org/10.1023/A:1016210906658

Harel, G. (2008). What is mathematics? A pedagogical answer to a philosophical question. In R. B. Gold, & R. Simons (Eds.), *Proof and other dilemmas: Mathematics and philosophy* (pp. 265–290). Mathematical Association of America.

Hefendehl-Hebeker, L. (1991). Negative numbers: Obstacles in their evolution from intuitive to intellectual constructs. *For the Learning of Mathematics, 11*(1), 26–32. https://www.jstor.org/stable/40248003.

Hiebert, J. (Ed.). (2013). *Conceptual and procedural knowledge: The case of mathematics*. Routledge.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking "pedagogical content knowledge. *Journal for Research in Mathematics Education, 39*, 372–400. https://doi.org/10.5951/jresematheduc.39.4.0372

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38*(3), 258–288. https://doi.org/10.2307/30034868

Jacobs, V., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169–202. https://doi.org/10.5951/jresematheduc.41.2.0169

Kaput, J. J., Carraher, D. W., & Blanton, M. L. (Eds.). (2008). *Algebra in the early grades*. New York: Erlbaum/National Council of Teachers of Mathematics.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Kinach, B. M. (2002). A cognitive strategy for developing pedagogical content knowledge in the secondary mathematics methods course: Toward a model of effective practice. *Teaching and Teacher Education, 18*, 51–71. https://doi.org/10.1016/S0742-051X(01)00050-6

Kumar, R. S., Subramaniam, K., & Naik, S. S. (2017). Teachers' construction of meanings of signed quantities and integer operation. *Journal of Mathematics Teacher Education, 20*(6), 557–590. https://doi.org/10.1007/s10857-015-9340-9

Lamb, L. C., & Thanheiser, E. (2006). Understanding integers: Using balloons and weights software. *Algebraic Thinking, 2*, 163–164.

Lamb, L. L., Bishop, J. P., Philipp, R. A., Schappelle, B. P., Whitacre, I., & Lewis, M. (2012). Developing symbol sense for the minus sign. *Mathematics Teaching in the Middle School, 18*(1), 5–9.

Lamb, L. L., Bishop, J. P., Philipp, R. A., Whitacre, I., & Schappelle, B. P. (2018). A cross-sectional investigation of students' reasoning about integer addition and subtraction: Ways of reasoning, problem types, and flexibility. *Journal for Research in Mathematics Education, 49*(5), 575–613. https://doi.org/10.5951/jresematheduc.49.5.0575

Linchevski, L., & Williams, J. (1999). Using intuition from everyday life in 'filling' the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics, 39*, 131–147. https://doi.org/10.1023/A:1003726317920

Lobato, J., & Walters, C. D. (2017). A taxonomy of approaches to learning trajectories and progressions. In J. Cai (Ed.), *Compendium for research in mathematics education*. Reston, VA: National Council of Teachers of Mathematics.

Maloney, A., & Confrey, J. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory. In *Proceedings of the 9th International Conference of the Learning Sciences-Volume 1* (pp. 968–975).

Mitchell, R., Charalambous, C., & Hill, H. (2013). Examining the task and knowledge demands needed to teach with representations. *Journal of Mathematics Teacher Education, 17*(1), 37–60. https://doi.org/10.1007/s10857-013-9253-4

Mukhopadhyay, S., Resnick, L. B., & Schauble, L. (1990). Social sense-making in mathematics: Children's ideas of negative numbers. In , *Proceedings of the 14th Conference of the International Group for the Psychology in Mathematics Education (PME) with the North American Chapter 12th PME-NA Conference* (Vol. 3, pp. 281–288).

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards initiative*. Retrieved, September 10, 2015, from http://www.corestandards.org/the-standards/mathematics.

Peled, I., & Carraher, D. W. (2007). Signed numbers and algebraic thinking. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 303–327). Mahwah, NJ: Erlbaum.

Reck, E., & Mora, M. (2004). *Before and after algebra, how students solve problems*. Unpublished Master of Arts Final Project. San Diego State University.

Skemp, R. R. (2006). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School, 12*(2), 88–95.

Stephan, M., & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education, 43*(4), 428–464. https://doi.org/10.5951/jresematheduc.43.4.0428

Strauss, A., & Corbin, J. (1994). Grounded theory methodology: An overview. In N. K. Denzin, & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 273–285). Thousand Oaks, CA: Sage.

Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics, 49*(3), 341–359.

Wessman-Enzinger, N. M., & Tobias, J. M. (2020). The dimensions of prospective elementary and middle school teachers' problem posing for integer addition and subtraction. *Journal of Mathematics Teacher Education*, 1–33. https://doi.org/10.1007/s10857-020-09477-x

Whitacre, I., Bouhjar, K., Bishop, J. P., Philipp, R. A., Schappelle, B. P., & Lamb, L. L. (2016). Regular numbers and mathematical worlds. *For the Learning of Mathematics, 36*(2), 20–25.

Whitacre, I., Bishop, J. P., Lamb, L. L. C., Philipp, R. A., & Schappelle, B. P. (2014). Dollars and sense: Students' integer perspectives. *Mathematics Teaching in the Middle School, 20*(2), 84–89.

Wilson, P. H., Mojica, G. F., & Confrey, J. (2013). Learning trajectories in teacher education: Supporting teachers' understandings of students' mathematical thinking. *The Journal of Mathematical Behavior, 32*(2), 103–121. https://doi.org/10.1016/j.jmathb.2012.12.003