

AN EXPONENTIAL BOUND ON THE NUMBER OF NON-ISOTOPIC COMMUTATIVE SEMIFIELDS

FARUK GÖLOĞLU AND LUKAS KÖLSCH

Abstract. We show that the number of non-isotopic commutative semifields of odd order p^n is exponential in n when $n = 4t$ and t is not a power of 2. We introduce a new family of commutative semifields and a method for proving isotopy results on commutative semifields that we use to deduce the aforementioned bound. The previous best bound on the number of non-isotopic commutative semifields of odd order was quadratic in n and given by Zhou and Pott [Adv. Math. 234 (2013), pp. 43–60]. Similar bounds in the case of even order were given in Kantor [J. Algebra 270 (2003), pp. 96–114] and Kantor and Williams [Trans. Amer. Math. Soc. 356 (2004), pp. 895–938].

1. Introduction

In this paper, we show that the number N_{p^n} of non-isotopic commutative semifields of odd order p^n is exponential in n when $n = 4t$. To be precise, we prove for every odd prime p ,

$$N_{p^n} \geq \frac{(\sigma(n) - 1)(p^{n/4} - 1)}{2n},$$

when $v_2(n) \geq 2$, where we denote by $\sigma(n)$ the odd part of an integer n (i.e., $\sigma(n) = n/2^{v_2(n)}$), and by $v_2(n)$ the 2-adic valuation of n (i.e., $2^{v_2(n)} \mid n$ and $2^{v_2(n)+1} \nmid n$). For odd p , the previous best bound on N_{p^n} was quadratic in n and was proved in [35, Corollary 1]:

$$N_{p^n} \geq \frac{n(\sigma(n) - 1)}{8} + cn,$$

when $v_2(n) \geq 1$ and c a constant. When p and n are odd, the known number for N_{p^n} is linear in n . The problem of determining whether the number N_{p^n} can be bounded by a polynomial in n has been described [29, p. 180] as “the main problem in connection with commutative semifields of [odd] order p^n .” Note that it is impossible to find families with exponentially many non-isotopic commutative semifields of order p^n for arbitrary p, n . Indeed, by a result of Menichetti [27, Corollary 33], all commutative semifields of order p^n with n prime and p large enough are isotopic to the finite field or a twisted field (see Section 3.2). It is thus impossible to give an exponential count for all p, n . The problem in the characteristic 2 case was solved almost two decades ago. Kantor [19, Theorem 1.1] showed that the number of non-isotopic commutative semifields of order 2^{km} is at

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least

$$N_{2^k m} \geq \frac{2^{km(\rho(m)-1)}}{k^2 m^4},$$

when $m > 1$ is odd and m is not a power of 3 (where we denote by $\rho(m)$ the number of prime factors of m counting multiplicities), using a construction by Kantor and Williams [20, Theorem 1.7]. In these papers, finding a large number of (commutative) semifields in odd characteristic and finding a general approach to proving non-isotopy were posed as important open problems [20, p. 936], [19, p. 112].

To prove the bound we introduce a new family of commutative semifields. These semifields satisfy a property that we call *biprojectivity*, which also applies to many known semifields of square order. The biprojective structure allows us to develop a technique of determining isotopy between semifields. This technique is key to proving the exponential bound on non-isotopic commutative semifields.

In Section 2, we give the preliminaries. In Section 3, we define biprojective semifields and give a quick survey on known commutative semifields and their counts. Section 4 is devoted to proving the semifield property of our family (Theorem 4.4). Section 5 introduces our technique of proving isotopy between semifields (Theorem 5.10). In Section 6, we give the number of non-isotopic semifields arising from our family (Theorem 6.2). Section 6 contains our main result that the number of non-isotopic commutative semifields of odd order p^n is exponential in n (Corollaries 6.4 and 6.5). In Section 7, we compute the nuclei associated to our semifields (Theorem 7.2). Finally, in Section 8, we show that our semifields are indeed new and not isotopic to most known semifields (Theorem 8.1).

2. Preliminaries

A finite semifield $S = (S, +, \circ)$ is a set S equipped with two operations $(+, \circ)$ satisfying the following axioms.

(S1) $(S, +)$ is a group.

(S2) For all $x, y, z \in S$,

- $x \circ (y + z) = x \circ y + x \circ z$,
- $(x + y) \circ z = x \circ z + y \circ z$.

(S3) For all $x, y \in S$, $x \circ y = 0$ implies $x = 0$ or $y = 0$.

(S4) There exists $2 \in S$ such that $x \circ 2 = x = 2 \circ x$.

In this paper, we will be interested only in finite semifields. Henceforth, when we say a semifield we will mean a finite semifield. An algebraic object satisfying the first three of the above axioms is called a pre-semifield. If $P = (P, +, \circ)$ is a pre-semifield, then $(P, +)$ is an elementary abelian p -group [22, p. 185], and $(P, +)$ can be viewed as an n -dimensional F_p -vector space F^n . If \circ is associative then S is the finite field F_{p^n} by Wedderburn's theorem which states that a finite division ring is a field. By a result of Menichetti (known as Kaplansky's conjecture [26]) when $n > 2$, there exist *proper* semifields of odd order p^n where \circ is non-associative. There are no proper semifields of order 2^3 . For $n > 3$, there exist proper semifields of order 2^n [22]. A pre-semifield $P = (F^n, +, \circ)$ can be converted to a semifield $S = (F^n, +, \cdot)$ using *Kaplansky's trick* by defining the new multiplication as

$$(x \circ e) \cdot (e \circ y) = (x \circ y),$$

for any nonzero element $e \in F_p^n$ making $(e \circ e)$ the multiplicative identity of S . A pre-semifield is an F_p -algebra, thus the multiplication is bilinear. Therefore we have F_p -bilinear $B : F_p \times F_p^n \rightarrow F_p^n$, satisfying

$$B(x, y) = x \circ y,$$

and F_p -linear left and right multiplications $L_x, R_y : F_p^n \rightarrow F_p^n$, with

$$L_x(y) := B(x, y) =: R_y(x).$$

The mapping L_x (resp. R_y) is a bijection whenever $x \neq 0$ (resp. $y \neq 0$) by (S3). Thus,

$$R_e(x) \in L_e(y) = x \circ y.$$

Two pre-semifields $P_1 = (F_p^n, +, \circ_1)$ and $P_2 = (F_p^n, +, \circ_2)$ are said to be isotopic if there exist F_p -linear bijections L, M and N of F_p^n satisfying

$$N(x \circ_1 y) = L(x) \circ_2 M(y).$$

Such a triple $\gamma = (N, L, M)$ is called an isotopism between P_1 and P_2 . If additionally $L = M$ holds, we call γ a strong isotopism and P_1 and P_2 strongly isotopic. Isotopisms between a pre-semifield P and itself are called autotopisms. Thus the pre-semifield P and the corresponding semifield S constructed by Kaplansky's trick are isotopic and even strongly isotopic if P is commutative. Isotopy of pre-semifields is an equivalence relation and the isotopism class of a pre-semifield P is denoted by $[P]$. Semifields coordinatize projective planes and different semifields coordinatize isomorphic planes if and only if they are isotopic ([1], see [22, Section 3] for a detailed treatment). Semifields are further equivalent to maximum rank distance codes with certain parameters (see e.g. [31]) and can be used to construct relative difference sets (see [30]). Associative substructures of a semifield $S = (F_p^n, +, \cdot)$, namely the left, middle and right nuclei, are defined as follows:

$$N_l(S) := \{x \in F_p^n : (x \cdot y) \cdot z = x \cdot (y \cdot z), \ y, z \in F_p^n\},$$

$$N_m(S) := \{y \in F_p^n : (x \cdot y) \cdot z = x \cdot (y \cdot z), \ x, z \in F_p^n\},$$

$$N_r(S) := \{z \in F_p^n : (x \cdot y) \cdot z = x \cdot (y \cdot z), \ x, y \in F_p^n\}.$$

It is easy to check that $N_l(S), N_m(S), N_r(S) \subseteq F_p^n$ are finite fields and if S is commutative then $N_l(S) = N_r(S)$. Nuclei are isotopy invariants for semifields. Since every pre-semifield $P \in [S]$ for some semifield S , the nuclei can be thought to extend to pre-semifields. Thus, when we speak of the nuclei of a pre-semifield P we mean the nuclei of the isotopic semifield S .

Let $\text{End}(F_p)$ denote the F_p -linear endomorphisms of the vector space F_p . Every F_p -linear mapping $L \in \text{End}(F_p)$ can be written uniquely as a linearized polynomial

$$L(x) = \sum_{i=0}^{p-1} b_i x^{p^i},$$

in the polynomial ring $F_{p^n}[x]$. We will not make distinction between mappings and the polynomials. Let p be an odd prime and consider the polynomials from $F_{p^n}[x]$ of the form

$$F(x) = \sum_{0 \leq i, j < n} a_{ij} x^{p^i + p^j}.$$

These polynomials are called Dembowski-Ostrom (DO) polynomials. The polarization of a DO polynomial F is defined as

$$\Delta_F(x, y) = F(x + y) - F(x) - F(y).$$

The mapping $\Delta_F : \mathbb{F}_p^n \times \mathbb{F}_p^n \rightarrow \mathbb{F}_p^n$ is symmetric and \mathbb{F}_p -bilinear, thus if $\Delta_F(x, a) = 0$ implies $x = 0$ for all $a \in \mathbb{F}_{p^n} \setminus \{0\}$, then $\Delta_F(x, y)$ describes a commutative pre-semifield multiplication [10]. Conversely, by a counting argument, every commutative pre-semifield multiplication can be written as $\Delta_F(x, y)$ for some DO polynomial F [9]. In that case we will call F a planar DO polynomial/mapping. Strong isotopy between pre-semifields can be recognized also in the corresponding planar DO polynomials:

Theorem 2.1 ([8, Theorem 3.5.]). *Let $F, G \in \mathbb{F}_{p^n}[x]$ be planar DO polynomials and P_1, P_2 be the corresponding pre-semifields. Then P_1 and P_2 are strongly isotopic via an isotopism $\gamma = (N, L, L)$ if and only if $F = NGL^{-1}$.*

Consequently, we say that two planar DO polynomials F, G are equivalent if bijective linear mappings L_1, L_2 exist such that $F = L_1GL_2$. Note that this type of equivalence is the most general equivalence known to preserve the planarity of a DO polynomial, see [23].

3. Bijective planar mappings and commutative semifields

In this paper we are interested in planar DO polynomials of a specific form. Let $F \in \mathbb{F}_{p^n}$ be a finite field of square odd order and $M = \mathbb{F}_{p^m}$ with $n = 2m$. Let

$$F(x, y) = [f(x, y), g(x, y)],$$

where

$$\begin{aligned} f(x, y) &= a_0x^{q+1} + b_0x^qy + c_0xy^q + d_0y^{q+1}, \\ g(x, y) &= a_1x^{r+1} + b_1x^ry + c_1xy^r + d_1y^{r+1}, \end{aligned}$$

with $q = p^k, r = p^l, 1 \leq k, l \leq m$. We will call $f(x, y)$ a q -bijective polynomial and $(x, y) \mapsto F(x, y)$ a (q, r) -bijective mapping (of $M \times M$). Note that $F(x, y)$ is a (q, r) -bijective polynomial pair. We will not make any distinction between the polynomials and the mappings defined by them. We also let $\bar{q} = p^{m-k}$ and $\bar{r} = p^{m-l}$, so that $q\bar{q} \equiv r\bar{r} \equiv 1 \pmod{p^m - 1}$. We are going to use the shorthand notation

$$\begin{aligned} f(x, y) &= (a_0, b_0, c_0, d_0)_q, \\ g(x, y) &= (a_1, b_1, c_1, d_1)_r. \end{aligned}$$

We are going to refer to f and g as (left and right) components of F . We refer the reader to [5] for projective polynomials over finite fields.

The polarization of a planar (q, r) -bijective mapping defines a (q, r) -bijective (commutative) pre-semifield $P = (M \times M, +, \cdot)$. It is easy to see that both components correspond to homogeneous operations due to bijectivity:

$$\begin{aligned} (3.1) \quad (x, y) \cdot (y, v) &= ((a_0u + b_0v)x^q + (a_0u^q + c_0v^q)x + (c_0u + d_0v)y^q + (b_0u^q + d_0v^q)y, \\ &\quad (a_1u + b_1v)x^r + (a_1u^r + c_1v^r)x + (c_1u + d_1v)y^r + (b_1u^r + d_1v^r)y). \end{aligned}$$

Define

$$\begin{aligned} D_f^0(x, y) &= b_0x^q + c_0x + d_0y^q + d_0y, & D_f^\infty(x, y) &= a_0x^q + a_0x + c_0y^q + b_0y, \\ D_g^0(x, y) &= b_1x^r + c_1x + d_1y^r + d_1y, & D_g^\infty(x, y) &= a_1x^r + a_1x + c_1y^r + b_1y, \end{aligned}$$

and for $u \in P^1(M) \setminus \{0, \infty\}$,

$$\begin{aligned} D_f^u(x, y) &= (a_0u + b_0)x^q + (a_0u^q + c_0)x + (c_0u + d_0)y^q + (b_0u^q + d_0)y, \\ D_g^u(x, y) &= (a_1u + b_1)x^r + (a_1u^r + c_1)x + (c_1u + d_1)y^r + (b_1u^r + d_1)y. \end{aligned}$$

Lemma 3.1 is straightforward.

Lemma 3.1. *Let $(x, y) \rightarrow F(x, y) = [f(x, y), g(x, y)]$ be a (q, r) -biprojective mapping of $M \times M$. Then F is planar if and only if the pair of equations*

$$D_f^u(x, y) = 0 = D_g^u(x, y)$$

has exactly one solution for each $u \in P^1(M)$.

Proof. We need to show that the polarization $\Delta_F((x, y), (u, v)) = (x, y) \boxtimes (u, v) = 0$ has a unique zero for each $(u, v) \in M \times M \setminus (0, 0)$ if and only if $D_f^w(x, y) = 0 = D_g^w(x, y)$ has a unique solution for each $w \in P^1(M)$. Inspecting Eq. (3.1), one immediately sees that the case $v = 0$ corresponds to $D_f^\infty(x, y) = 0 = D_g^\infty(x, y)$ after applying $x \rightarrow xu$ and $y \rightarrow yu$. For $v \in M^\times$, apply $x \rightarrow xv$, $y \rightarrow yv$ and $u \rightarrow uv$ to get the remaining cases $D^w(x, y) = 0 = D^w(x, y)$ for $w \in M$. \square

In the following we will show that many known semifields fall into the (q, r) -biprojective setting.

3.1. Dickson semifields D . Dickson introduced the commutative semifields $S = (M \times M, +, \circ)$ with

$$(x, y) \circ (u, v) = (xu + ay^q v^q, xv + yu),$$

where $q = p^k$ with $0 < k < l$ and $a \in M^\times \setminus (M^\times)^2$. Note that the isotopic multiplication

$$(x, y) \boxtimes (u, v) = (xu + ayv, xv^{\bar{q}} + y^{\bar{q}}u)$$

is $(1, \bar{q})$ -biprojective and isotopic to the polarization of the $(1, q)$ -biprojective planar mapping

$$F_D = [(1, 0, 0, a)_1, (0, 1, 0, 0)_q].$$

Different choices for $a \in M^\times \setminus (M^\times)^2$ produce isotopic semifields and there are a total of $b \frac{n}{4} c$ non-isotopic Dickson semifields [19, p.107].

3.2. Albert's generalized twisted fields A . Albert introduced [2] a family of commutative and non-commutative semifields. The commutative ones may be given as $S = (F, +, \circ)$ with

$$X \circ U = X^q U + U^q X,$$

where $q = p^k$ with $0 < k < n$ satisfying $n/\gcd(k, n)$ odd. When $F = M(\xi)$ with $[F : M] = 2$, one can write $X = x\xi + y$ with $x, y \in M$. One can choose $\xi \in F \setminus M$

satisfying $\xi^2 = a \boxtimes M^\times \setminus (M^\times)^2$.

$$\begin{aligned} (x\xi + y) \circ (u\xi + v) &= (x\xi + y)^q(u\xi + v) + (u\xi + v)^q(x\xi + y) \\ &= \xi^{q+1}(x^q u + u^q x) + \xi^q(x^q v + u^q y) + \xi(y^q u + v^q x) + (y^q v + v^q y) \\ &= a^{(q+1)/2}(x^q u + u^q x) + a^{(q-1)/2}\xi(x^q v + u^q y) + \xi(y^q u + v^q x) + (y^q v + v^q y). \end{aligned}$$

Identifying $\xi M + M$ with $M \times M$, and

$$\begin{aligned} (x, y) \boxtimes (u, v) &= a^{(q-1)/2}(x^q v + u^q y) + (y^q u + v^q x), \quad a^{(q+1)/2}(x^q u + u^q x) + (y^q v + v^q y) \end{aligned}$$

is (q, q) -biprojective and isotopic to the polarization of the (q, q) -biprojective planar mapping

$$F_A = [(0, a^{(q-1)/2}, 1, 0)_q, (a^{(q+1)/2}, 0, 0, 1)_q].$$

Different choices for $a \boxtimes M^\times \setminus (M^\times)^2$ produce isotopic semifields and there are a total of $\frac{\sigma(n)-1}{2}$ non-isotopic generalized twisted fields [2, 19].

3.3. Zhou-Pott semifields ZP . Zhou and Pott [35] gave a family of pre-semifields $S = (M \times M, +, \circ)$ given by

$$(x, y) \circ (u, v) = (x^q u + u^q x + a(y^q v + y v^q))^r, x v + y u,$$

where $a \boxtimes M \setminus (M^\times)^2$, $q = p^k$ and $r = p^j$ with $0 \leq j, k \leq m$ where $m/\gcd(k, m)$ is odd. The isotopic multiplication

$$(x, y) \boxtimes (u, v) = (x^q u + u^q x + a(y^q v + y v^q), x^r v + y u^r),$$

is (q, r) -biprojective and isotopic to the polarization of the (q, r) -biprojective planar mapping

$$F_{ZP} = [(1, 0, 0, a)_q, (0, 1, 0, 0)_r].$$

Different choices for $a \boxtimes M^\times \setminus (M^\times)^2$ produce isotopic semifields and there are a total of $\frac{\sigma(n)}{2} \cdot \frac{n}{4}$ non-isotopic ZP semifields [35].

3.4. Budaghyan-Helleseth semifields $(BH, ZW, LMPTB)$. These semifields were found in [6] and independently in [34]. The commutative semifields given later in [24] and [4] were shown to be isotopic to the previous ones [25]. We note that Bierbrauer's construction in [4] gives also non-commutative semifields. We will use the definition from [4]. Let $S = (M \times M, +, \circ)$ be the pre-semifield given by

$$\begin{aligned} (x, y) \circ (u, v) &= \begin{cases} (xv + yu, x^q u + xu^q + a(y^q v + yv^q)) & \text{if } m/\gcd(k, m) \text{ is odd,} \\ (xu + ayv, x^q v + yu^q + a^{(q-1)/2}(xv^q + y^q u)) & \text{if } m/\gcd(k, m) \text{ is even,} \end{cases} \end{aligned}$$

where $a \boxtimes M \setminus (M^\times)^2$ and $q = p^k$ with $0 < k < m$. The pre-semifield multiplication is $(1, q)$ -biprojective. Similarly, the corresponding $(1, q)$ -biprojective planar mapping whose polarization is isotopic to S is given by

$$F_{BH} = \begin{cases} [(0, 0, 1, 0)_1, (1, 0, 0, a)_q] & \text{if } m/\gcd(k, m) \text{ is odd,} \\ (1, 0, 0, a)_1, (0, 1, a^{(q-1)/2}, 0)_q] & \text{if } m/\gcd(k, m) \text{ is even.} \end{cases}$$

The number of non-isotopic semifields in this family is $b \frac{n}{4} c$ which is proved in [14].

Table 1. Known infinite families of biprojective commutative semifields of order p^n

Family	Planar Mapping	#S	Notes	$(\#N_i, \#N_m)$ (p^n, p^n)	Count	Proved in
F	X^2	p^n			1	
	$[(0, 1, 0, 0)_1, (1, 0, 0, a)_1]$		$a \in M \setminus (M^*)^2, n = 2m$			
A	X^{q+1}	p^n	$q = p^k, 0 < k < m,$ $\gcd(k, n) = d, n/d \text{ odd.}$	(p^d, p^d)	$j \sim \frac{k}{2} \dots k$	[2]
	$[(0, a^{\frac{q-1}{2}}, 1, 0)_{q^*}, (a^{\frac{q+1}{2}}, 0, 0, 1)_{q^*}]$		$a \in M \setminus (M^*)^2, n = 2m$			
D	$[(1, 0, 0, a)_1, (0, 1, 0, 0)_{q^*}]$	p^{2m}	$q = p^k, 0 < k < m,$ $\gcd(k, m) = d,$ $a \in M \setminus (M^*)^2.$	(p^d, p^m)	$\frac{m}{4}!$	[12]
ZP	$[(1, 0, 0, a)_{q^*}, (0, 1, 0, 0)_r]$	p^{2m}	$q = p^k, r = p^l, 0 < j, k < m,$ $\gcd(k, m) = d, \gcd(j, k, m) = d^0,$ $m/d \text{ odd, } a \in M \setminus (M^*)^2.$	(p^{d^0}, p^d)	$j \sim \frac{k}{2} \dots k \frac{m}{4}!$	[35]
BH	$[(0, 1, 0, 0)_1, (1, 0, 0, a)_{q^*}]$	p^{2m}	$q = p^k, 0 < k < m,$ $\gcd(k, m) = d, m/d \text{ odd,}$ $a \in M \setminus (M^*)^2.$	(p^d, p^{2d})	$\frac{m}{4}!$	[6, 14, 34]
	$[(1, 0, 0, a)_1, (0, 1, a^{\frac{q-1}{2}}, 0)_{q^*}]$		$q = p^k, 0 < k < m,$ $\gcd(k, m) = d, m/d \text{ even,}$ $a \in M \setminus (M^*)^2.$			
S	$[(1, 0, 0, B)_{q^*}, (0, 1, \frac{a}{B}, 0)_r]$	p^{4l}	$q = p^k, r = p^{k+1}, 0 < k < l,$ $m = 2l, \gcd(k, m) = e, m/e \text{ odd,}$ $a \in L^x, B \in M \setminus (M^*)^2.$	$(p^{e/2}, p^e)$	$\geq j \sim \frac{k}{2} \dots k! \frac{m}{4}!$	Theorem 4.4

The known infinite families of biprojective and other commutative semifields and their planar representations are summarized in Tables 1 and 2. Families A, D, BH

reduce to F when $k \notin \{0, m\}$. Family ZP reduces to D when $k = 0$, to BH when $j = 0$, and to F when $j = k = 0$. Family S reduces to ZP when $a = 0$, and to D when $k \notin \{0, l\}$. We excluded those cases in the Notes and also in the Counts columns of Table 1.

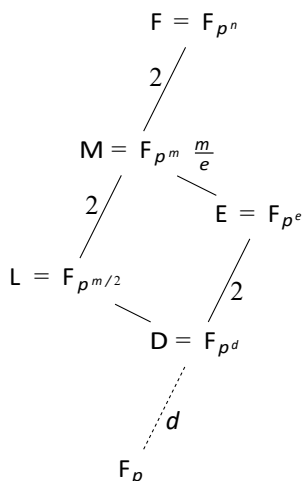
3.5. The Family S . The main result of this paper is to prove that

- Family S gives new commutative semifields, and
- Family S contains an exponential number of non-isotopic commutative semifields (in n).

An informal way to explain why Family S gives such a large number of commutative semifields is that their polarizations admit only a few M -linear isotopisms (within the family) due to their complexity—they contain two non-zero entries in either component of their planar representations $(1, 0, 0, a)_q$ and $(0, 1, b, 0)_r$; and the underlying field automorphisms q and r are nontrivial and are not simply related to each other. Indeed, our method in Section 5 will show that M -linear isotopisms are essentially the only ones for biprojective semifields whose autotopism groups satisfy a simply defined condition. The polarization of $(0, 1, 0, 0)_q$, which is a component polynomial of many other biprojective semifields (except A and BH_{even}), admits more such isotopisms and that is the main reason why all $a \in M$ allowed in these constructions lead to isotopic semifields. For F, D, BH_{even} and A the reasons for admitting only a small number of non-isotopic semifields include the simplicity of the defining field automorphisms, e.g., $q \in \{1, p^{m/2}\}$; or having the same (q, q) or conjugate (q, q) automorphisms. We will explain these in detail in Section 5 (see Theorem 5.10). We start by proving that Family S indeed gives commutative semifields.

4. The commutative semifield family S

The following diagram and its annotations describe our setting.



Notation 4.1.

- p is an odd prime.
- $n = 2m$, m is even.
- $Q = p^{m/2}$, $Q^2 = p^m$.
- $q = p^k$, $r = p^{k+m/2} = Qq$ with $1 \leq k \leq m-1$.
- $e = \gcd(k, m)$ with m/e odd.
- $d = \gcd(k + m/2, m)$.
- $e = 2d$ by Lemma 4.3.
- $(M^\times)^2$ — the subgroup of non-zero squares in M^\times .
- $L^\times = (M^\times)^{Q+1} \leq (M^\times)^2 \leq M^\times$.
- $(M^\times)^{Q-1} \leq (M^\times)^2 \leq M^\times$ — the subgroup of $(Q+1)^{\text{st}}$ roots of unity in M^\times .
- $E = F_q \cap M = F_{q^2} \cap M = F_{r^2} \cap M$ by Lemma 4.3.
- $D = F_r \cap M$.

We need the following well known result on the greatest common divisor of $p^i \pm 1$ and $p^m - 1$. A proof can be found in [28].

Lemma 4.2. Let $i, m \in \mathbb{N}$ and p be a prime. Then

- $\gcd(p^i - 1, p^m - 1) = \begin{cases} p^{\gcd(i, m)} - 1 & \text{if } m/\gcd(i, m) \text{ odd, and } p = 2, \\ 2 & \text{if } m/\gcd(i, m) \text{ odd, and } p > 2, \\ p^{\gcd(i, m)} + 1 & \text{if } m/\gcd(i, m) \text{ even.} \end{cases}$

First, we will prove a lemma.

Lemma 4.3. We have,

- (i) $(-1) \in (M^\times)^{q-1}$.
- (ii) Any $x \in (M^\times)^2$ can be written (twice) as $x = cg$ where $c \in L^\times$ and $g \in (M^\times)^{q-1}$.
- (iii) $\gcd(k + m/2, m) = \gcd(k, m)/2$.
- (iv) $E = F_q \cap M = F_{q^2} \cap M = F_{r^2} \cap M$.

Proof. (i) Recall that $v_2(x)$ denotes the index of 2 in x , that is, $v_2(x) = h$ if $2^h \mid x$ and $2^{h+1} \nmid x$. We have $v_2(p^e - 1) = v_2(Q^2 - 1)$ since $Q^2 - 1 = p^m - 1 = (p^e - 1) \prod_{i=0}^{m/e-1} p^i$ and the fact that an odd number of odd integers add up to an odd integer. Thus $v_2((p^m - 1)/2) = v_2(p^e - 1) - 1$. The result follows from Lemma 4.2 which shows $\gcd(q - 1, Q^2 - 1) = p^e - 1$.

(ii) It is easy to see that $(M^\times)^{q-1} \cap L^\times = \{\pm 1\}$ since $\gcd(Q - 1, Q + 1) = 2$ and $xg = yh$ if and only if $(x, g) = (y, h)$ or $(x, g) = (-y, -h)$.

(iii) Let $k = 2dk^0$ and $m = 2dm^0$ with $\gcd(k^0, m^0) = 1$ and m^0 odd. Then $\gcd(2dk^0 + dm^0, 2dm^0) = d \cdot \gcd(2k^0 + m^0, 2m^0) = d \cdot \gcd(2k^0 + m^0, m^0) = d \cdot \gcd(2k^0, m^0) = d$, since $m^0 + 2k^0$ is odd.

(iv) Obvious since m/e is odd and $(qQ)^2 = q^2 \pmod{Q^2 - 1}$.

□

Now we present the family of planar mappings.

Theorem 4.4. Let $a \in L^\times$ and $B \in M^\times \setminus (M^\times)^2$ and let

$$F : M \times M \rightarrow M \times M$$

be defined as

$$F : (x, y) \rightarrow F(x, y) = [(1, 0, 0, B)_q, (0, 1, a/B, 0)_r].$$

Then F is planar.

Proof. We are going to use Lemma 3.1. First,

$$D_f^0(x, y) = B(y^q + y) = 0, \quad \text{and}$$

$$D_g^0(x, y) = x^r + \frac{ax}{B} = 0$$

imply $(x, y) = (0, 0)$, since otherwise either of

$$y^{q-1} = -1, \quad \text{and} \quad x^{r-1} = -\frac{a}{B}$$

leads to a contradiction, since $-1 \notin (M^\times)^{q-1}$ by Lemma 4.3, and $2/\gcd(r - 1, Q^2 - 1)$ and $-a/B \notin (M^\times)^2$. Similarly,

$$D_f^\infty(x, y) = x^q + x = 0, \quad \text{and}$$

$$D_g^\infty(x, y) = y + \frac{ay^r}{B} = 0$$

have the unique common solution $(x, y) = (0, 0)$ with the same argument after changing variables. Now for $u \in M^\times$,

$$D_f^u(x, y) = ux^q + u^q x + B(y^q + y) = 0, \quad \text{and}$$

$$D_g^u(x, y) = x^r + \frac{a}{B}x + \frac{a}{B}uy^r + u^r y = 0.$$

Or,

$$D_f^u(ux, y) = u^{q+1}(x^q + x) + B(y^q + y) = 0, \quad \text{and}$$

$$D_g^u(ux, y) = u^r(x^r + y) + \frac{a}{B}u(x + y^r) = 0.$$

We will proceed to show that $(x, y) = (0, 0)$ is the only common solution of these equations for $x, y \in M$. Now we can assume $x, y \in M^\times$, since $x = 0$ implies $y = 0$ and vice versa for $D^u(ux, y) = 0$. Furthermore, $x = -y^r$ implies $y - y^{r^2} = 0$ and $x, y \in F_{r^2} \cap M = F_{q^2} \cap M = F_q \cap M = E$ by Lemma 4.3. Thus $x^q + x = 2x = -2y^r$ and $y + y = q2y$, in turn

$$2(-u^{q+1}y^r + By) = 0,$$

or

$$y^{r-1} = \frac{B}{u^{q+1}}.$$

This is impossible since $B \in (M^\times)^2$. The same argument shows $x^r = -y$ and we can concentrate on

$$(4.1) \quad u^{q+1} = -\frac{B(y^q + y)}{x^q + x}, \quad \text{and}$$

$$(4.2) \quad u^{r-1} = -\frac{a(x + y^r)}{B(x^r + y)},$$

for $x, y \in M^\times$ with $x^r = -y$ and $x = -y^r$. Now assume (4.1) and (4.2) hold for such $x, y \in M^\times$, and let

$$(4.3) \quad \varphi_q(x, y) = \frac{y^q + y}{x^q + x} = \frac{\gamma}{cg}, \quad \text{and}$$

$$(4.4) \quad \varphi_r(x, y) = \frac{x + y^r}{x^r + y} = \gamma^Q dh,$$

for some $c, d \in L^\times$, $g, h \in (M^\times)^{Q-1}$ and a fixed $\gamma \in M^\times \setminus (M^\times)^2$, since cg and dh both run through $(M^\times)^2$ independently by Lemma 4.3. Note that (4.1) and (4.2) guarantee that $\varphi_q(x, y)$ and $\varphi_r(x, y)$ are non-squares. Multiplying (4.3) and (4.4) (and also (4.1) and (4.2)) we get

$$\gamma^{Q+1} \frac{dh}{cg} = \varphi_q(x, y)\varphi_r(x, y) = \frac{u^{q+r}}{a} = \frac{u^{q(Q+1)}}{a} \in L^\times,$$

and therefore $^2g = h$ with $^2 \in \{\pm 1\}$ since $\gamma^{Q+1} \in L^\times$. Now let $z = y - x^Q$ and consider

$$cg(z^q + z + (x^q + x)^Q) = \gamma(x^q + x), \quad \text{and}$$

$$(x^q + x) + z^r = ^2\gamma^Q dg((x + x^q)^Q + z),$$

for $x \in M^\times$, which is a rewriting of (4.3) and (4.4). Or, equivalently,

$$(4.5) \quad cg(x^q + x)^Q - \gamma(x^q + x) = -cg(z^q + z), \quad \text{and}$$

$$(4.6) \quad -^2\gamma^Q dg(x^q + x)^Q + (x^q + x) = -z^r + ^2\gamma^Q dgz.$$

The two new equations generated by

- Eq. (4.5) plus γ times Eq. (4.6), and
- $d^2\gamma^Q$ times Eq. (4.5) plus c times Eq. (4.6),

are as follows:

$$(4.7) \quad (c - {}^2\gamma^{Q+1}d)(x^q + x)^Q g = -\gamma z^r - c g z^q - (c - {}^2\gamma^{Q+1}d) g z, \text{ and}$$

$$(4.8) \quad (c - {}^2\gamma^{Q+1}d)(x^q + x) = -c(z^r - d^2\gamma^Q g z^q).$$

We will show that the common solutions of these equations for $x \in M^\times$ lead to a contradiction to our assumption that (4.1) and (4.2) hold. Note that $x^q + x = 0$ since $x \in M^\times$ and $-1 \in (M^\times)^{q-1}$. Now if $z = \gamma - x^Q = 0$, then (4.3) becomes

$$(x^q + x)^{Q-1} = \frac{\gamma}{c g},$$

which is a contradiction since the left hand side is a square and the right hand side is not. If $c = {}^2\gamma^{Q+1}d$ then $z = 0$ by Eq. (4.8), which we have just handled, or $z^{r-q} = z^{q(Q-1)} = d^2\gamma^Q g$ by Eq. (4.8), which is again a contradiction since the right hand side is not a square. Then $c = {}^2\gamma^{Q+1}d$. Observe that $c - {}^2\gamma^{Q+1}d \in L$. Comparing g times Eq. (4.8) to the power Q with Eq. (4.7) yields

$$(-c(z^r - d^2\gamma^Q g z^q))^Q g = -\gamma z^r - c g z^q - (c - {}^2\gamma^{Q+1}d) g z,$$

$$-c g z^q + c d^2 \gamma g^{Q+1} z^r = -\gamma z^r - c g z^q - (c - {}^2\gamma^{Q+1}d) g z,$$

$$c d^2 \gamma z^r = -\gamma z^r - (c - {}^2\gamma^{Q+1}d) g z,$$

$$\gamma(c d^2 + 1) z^r = -(c - {}^2\gamma^{Q+1}d) g z.$$

Now $c d^2 + 1 = 0$ implies $z = 0$ or $c = {}^2\gamma^{Q+1}d$, which were handled before. Thus, we have

$$z^{r-1} = -\frac{g}{\gamma} \frac{c - {}^2\gamma^{Q+1}d}{c d^2 + 1},$$

and noting again that $\gamma^{Q+1} \in L^\times$, we reach another contradiction since the right hand side is not a square. \square

5. A method to determine isotopy of biprojective pre-semifields

There are two usual ways to determine whether two semifields are isotopic. The first one is to use isotopy invariants like the nuclei. Since there are less than n^2 possible configurations for the left/right and central nucleus for a commutative pre-semifield on F_p^n , this method is not enough to determine whether the number of non-isotopic pre-semifields grows exponentially in n or not. The second method works by directly considering all possible isotopisms (N, L, M) . This is, however, in many cases not feasible unless the pre-semifield has a very simple structure. Note that, for commutative semifields, some general results were obtained in [8] that make this approach slightly easier and allowed for instance to settle the isotopy question inside the family of Zhou-Pott semifields [35]. However, the semifields in our Family 5 are more delicate and such a direct approach does not seem possible. In this section, we develop a new general technique to determine whether two biprojective pre-semifields are isotopic or not.

5.1. Group theoretic preliminaries. We denote the set of all autotopisms of a pre-semifield \mathbf{P} by $\text{Aut}(\mathbf{P})$. It is easy to check that $\text{Aut}(\mathbf{P})$ is a group under component-wise composition, i.e., $(N_1, L_1, M_1)(N_2, L_2, M_2) = (N_1 N_2, L_1 L_2, M_1 M_2)$. We view $\text{Aut}(\mathbf{P})$ as a subgroup of $\text{GL}(\mathbb{F})^3 \cong \text{GL}(\mathbb{M} \times \mathbb{M})^3 \cong \text{GL}(n, \mathbb{F}_p)^3$. Our approach is based on the following simple and well-known result.

Lemma 5.1. *Let $\mathbf{P}_1 = (\mathbb{F}_p, +, \boxplus_1)$, $\mathbf{P}_2 = (\mathbb{F}_p, +, \boxplus_2)$ be isotopic pre-semifields via the isotopism $\gamma \in \text{GL}(\mathbb{F})^3$. Then $\gamma^{-1} \text{Aut}(\mathbf{P}_2) \gamma = \text{Aut}(\mathbf{P}_1)$.*

Proof. Let $\gamma = (N_1, L_1, M_1) \in \text{GL}(\mathbb{F})^3$ be an isotopism between \mathbf{P}_1 and \mathbf{P}_2 and $\delta = (N_2, L_2, M_2) \in \text{Aut}(\mathbf{P}_2)$. Then $\gamma^{-1} \delta \gamma \in \text{Aut}(\mathbf{P}_1)$. Indeed

$$\begin{aligned} (N_1^{-1} N_2 N_1)(x \boxplus_1 y) &= (N_1^{-1} N_2)(L_1(x) \boxplus_2 M_1(y)) \\ &= N_1^{-1}((L_2 L_1(x)) \boxplus_2 (M_2 M_1(y))) \\ &= (L_1^{-1} L_2 L_1(x)) \boxplus_1 (M_1^{-1} M_2 M_1(y)), \end{aligned}$$

so $\gamma^{-1} \text{Aut}(\mathbf{P}_2) \gamma \subseteq \text{Aut}(\mathbf{P}_1)$. The other inclusion follows by symmetry. \square

The central idea of the technique we are going to develop is to identify large abelian subgroups (in particular certain Sylow subgroups), in the autotopism group of biprojective semifields. We then use tools from group theory to obtain strong constraints on when the autotopism groups of two pre-semifields are conjugate. This approach is inspired by a similar technique for inequivalences of power functions on finite fields developed by Dempwolff [11] and Yoshiara [32].

First we recall the well-known Sylow Theorems (see for instance [17, Chapter 4]).

Theorem 5.2 (Sylow Theorems). *Let G be a group with order $p^m s$, with p prime, $m > 0$ and $p \nmid s$. Then,*

- (i) *G has a subgroup of order p^m , called a Sylow p -subgroup of G .*
- (ii) *Every p -subgroup of G is contained in a Sylow p -subgroup of G .*
- (iii) *The Sylow p -subgroups of G are conjugate in G .*

We will identify Sylow p^0 -subgroups of $\text{Aut}(\mathbf{P})$ when \mathbf{P} is a biprojective pre-semifield. In order to do that, we need to find a suitable prime p^0 , for which we will employ Zsigmondy's Theorem (see for instance [18, Chapter IX., Theorem 8.3]).

Theorem 5.3 (Zsigmondy's Theorem). *For every prime p and $m > 2$ except when $(p, m) = (2, 6)$, there exists a p -primitive divisor p^0 of $p^m - 1$, that is, p^0 prime, $p^0 | p^m - 1$ and $p^0 \nmid p^i - 1$ for all $1 \leq i \leq m - 1$.*

We write \mathbb{F}_p -linear mappings $L \in \text{End}(\mathbb{F})$ from \mathbb{F} to itself as 2×2 matrices of \mathbb{F}_p -linear mappings in $\text{End}(\mathbb{M})$. That is,

$$L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}, \text{ for } L_i \in \text{End}(\mathbb{M}).$$

Set

$$\gamma_a = (N_a, L_a, M_a) \in \text{GL}(\mathbb{F})^3$$

with

$$N^a = \begin{pmatrix} m_{a+1} & 0 \\ 0 & m_{a+r+1} \end{pmatrix}, \quad L_a = M_a = \begin{pmatrix} m_a & 0 \\ 0 & m_a \end{pmatrix},$$

where m_a denotes multiplication with the finite field element $a \in \mathbb{M}^\times$. For simplicity, we write these diagonal matrices also in the form $\text{diag}(m_a, m_a)$, so

$$\gamma_a = (\text{diag}(m_{a^{q+1}}, m_{a^{r+1}}), \text{diag}(m_a, m_a), \text{diag}(m_a, m_a)).$$

An important fact which follows immediately from bijectivity is that $\gamma_a \in \text{Aut}(\mathbb{P})$ for all $a \in \mathbb{M}^\times$ when \mathbb{P} is a (q, r) -bijective pre-semifield, which can be readily verified using Eq. (3.1). We fix some further notation that we will use from now on:

Notation 5.4.

- Set $q = p^k$ and $r = p^l$.
- Set $\bar{q} = p^{m-k}$ and $\bar{r} = p^{m-l}$, that is, $q\bar{q} \equiv r\bar{r} \equiv 1 \pmod{p^m - 1}$.
- Define the cyclic group

$$Z^{(q,r)} = \{\gamma_a : a \in \mathbb{M}^\times\},$$

of order $p^m - 1$.

- Let p^0 be a p -primitive divisor of $p^m - 1$. Such a prime p^0 always exists if $m > 2$ and $(p, m) = (2, 6)$ by Zsigmondy's Theorem. In our case, we have $p > 2$. We will also stipulate $m > 2$. Note that $p^0 = 2$ since $p^0 - p - 1$.
- Let R be the unique Sylow p^0 -subgroup of \mathbb{M}^\times .
- Define

$$Z_R^{(q,r)} = \{\gamma_a : a \in R\},$$

which is the unique Sylow p^0 -subgroup of $Z^{(q,r)}$ with $|R|$ elements.

- For a (q, r) -bijective pre-semifield \mathbb{P} , denote by

$$C_{\mathbb{P}} = C_{\text{Aut}(\mathbb{P})}(Z_R^{(q,r)})$$

the centralizer of $Z_R^{(q,r)}$ in $\text{Aut}(\mathbb{P})$.

- Define

$$S = \{\text{diag}(m_a, m_a) : a \in \mathbb{M}^\times\},$$

and

$$S_R = \{\text{diag}(m_a, m_a) : a \in R\}.$$

We start by identifying a subgroup of the autotopism group of any (q, r) -bijective pre-semifield. Lemma 5.5 is straightforward, but very important for our paper.

Lemma 5.5. Let \mathbb{P} be any (q, r) -bijective pre-semifield. Then

$$Z_R^{(q,r)} \leq Z^{(q,r)} \leq \text{Aut}(\mathbb{P}).$$

Proof. Follows directly from Eq. (3.1). □

We continue with a simple observation on R .

Lemma 5.6. Let $a \in R$, $a \neq 1$. Then a is not contained in a proper subfield of \mathbb{M} .

Proof. Clearly, a is contained in the subfield \mathbb{F}_{p^i} if and only if $a^{p^i-1} = 1$, i.e. the multiplicative order of a is a divisor of $p^i - 1$. The order of a is a power of p^0 and p^0 does not divide $p^i - 1$ for any $i < m$, so a is not contained in a proper subfield of \mathbb{F}_{p^m} . □

Now we observe that the normalizer and the centralizer of certain subgroups of $\text{GL}(F)$ must have certain shape.

Lemma 5.7. *Let $N_{\text{GL}(\mathbb{F})}(S_R)$, $N_{\text{GL}(\mathbb{F})}(S)$ and $C_{\text{GL}(\mathbb{F})}(S_R)$, $C_{\text{GL}(\mathbb{F})}(S)$ be the normalizers and the centralizers of S_R and S in $\text{GL}(\mathbb{F})$. Then*

(a)

$$N_{\text{GL}(\mathbb{F})}(S_R) = N_{\text{GL}(\mathbb{F})}(S) = \left\{ \begin{pmatrix} m_{c_1} \tau & m_{c_2} \tau \\ m_{c_3} \tau & m_{c_4} \tau \end{pmatrix} : c_1, c_2, c_3, c_4 \in \mathbb{M}, \tau \in \text{Gal}(\mathbb{M}/\mathbb{F}_p) \right\} \cap \text{GL}(\mathbb{F}),$$

$$(b) \quad C_{\text{GL}(\mathbb{F})}(S_R) = C_{\text{GL}(\mathbb{F})}(S) = \left\{ \begin{pmatrix} m_{c_1} & m_{c_2} \\ m_{c_3} & m_{c_4} \end{pmatrix} : c_1, c_2, c_3, c_4 \in \mathbb{M} \right\} \cap \text{GL}(\mathbb{F}).$$

Proof. We present the proof only for the more delicate case S_R . The proof for S is identical with \mathbb{M} substituting R throughout.

Let $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in N_{\text{GL}(\mathbb{F})}(S_R)$, where $A_1, A_2, A_3, A_4 \in \text{End}(\mathbb{M})$. Then

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \text{diag}(m_a, m_a) = \text{diag}(m_b, m_b) \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

for all $a \in R$ and some $b = \pi(a)$ where $\pi: R \rightarrow R$ is a bijection. Simple matrix multiplication implies $A_i(ax) = bA_i(x)$ for $i \in \{1, 2, 3, 4\}$.

We now write the mappings as linearized polynomials, i.e. $A_i = \sum_{j=0}^{p-1} c_{j,i} x^{p^j}$ for $i \in \{1, 2, 3, 4\}$. The equations above then immediately yield

$$\sum_{j=0}^{p-1} c_{j,i} a^{p^j} x^{p^j} = b \sum_{j=0}^{p-1} c_{j,i} x^{p^j}$$

for $i \in \{1, 2, 3, 4\}$ and all $a \in R$. We now compare the coefficients of these polynomials. If $a = 1$, it is not contained in any proper subfields of \mathbb{M} by Lemma 5.6, so we have $b = a^{p^j}$ only for at most one $j \in \{0, 1, \dots, p-1\}$. So A_1, A_2, A_3, A_4 are zero or monomials of the same degree p^j , which proves the statement for the normalizer. In the case of the centralizer, we have $b = a$, so we get the same possible mappings except with $j = 0$ forced. \square

We have shown in Lemma 5.5 that $Z_R^{(q,r)}$ is a subgroup of $\text{Aut}(\mathbb{P})$. Now we show that, under a certain condition which is key to our proofs, it is not just a Sylow p^0 -subgroup of $Z^{(q,r)}$, but even a Sylow p^0 -subgroup of $\text{Aut}(\mathbb{P})$.

Lemma 5.8. *Let \mathbb{P} be a (q, r) -biprojective pre-semifield. Assume that $C_{\mathbb{P}}$ contains $Z^{(q,r)}$ as an index l subgroup such that p^0 does not divide l . Then $Z_R^{(q,r)}$ is a Sylow p^0 -subgroup of $\text{Aut}(\mathbb{P})$.*

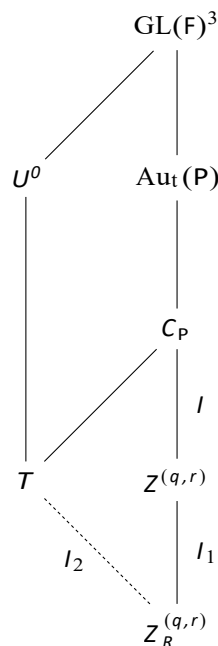
Proof. First define

$$U = \{(\text{diag}(m_a, m_b), \text{diag}(m_c, m_d), \text{diag}(m_e, m_f)) : a, b, c, d, e, f \in R\}.$$

Clearly, $|U| = |R|^6$. We will now show that U is a Sylow p^0 -subgroup of $\text{GL}(F)^3$. We have

$$|\text{GL}(F)| = |\text{GL}(2m, F_p)| = p^{m(2m-1)} \prod_{i=1}^{2m} (p^i - 1).$$

Clearly, $p^{m+i} - 1 \equiv p^i - 1 \pmod{p^m - 1}$. As p^0 is a p -primitive divisor of $p^m - 1$, all integers $p^j - 1$ with j in $[1, 2m]$ are coprime to p^0 , except for $j \in \{m, 2m\}$. Furthermore, the p^0 -part of $p^{2m} - 1 = (p^m - 1)(p^m + 1)$ is $|R|$ since $p^0 = 2$. Thus the p^0 -part of $|\text{GL}(F)|$ is $|R|^2$ and U is a Sylow p^0 -subgroup of $\text{GL}(F)^3$ as claimed. All Sylow p^0 -subgroups of $\text{GL}(F)^3$ are abelian since U is abelian, by Sylow Theorem (iii). Note that any p^0 -subgroup of a group G is contained in a Sylow p^0 -subgroup of G by Sylow Theorem (ii). Let T be a Sylow p^0 -subgroup of $\text{Aut}(P)$ that contains the p^0 -group $Z_R^{(q,r)}$. Then T itself is (again by Sylow Theorem (ii)) contained in a Sylow p^0 -subgroup of $\text{GL}(F)^3$, say U^0 . In particular, T is abelian. This implies that T is a subgroup of the centralizer C_P of $Z_R^{(q,r)}$ in $\text{Aut}(P)$. By assumption, $Z^{(q,r)}$ is an index l subgroup of C_P and p^0 does not divide l . Moreover $Z_R^{(q,r)}$ is a Sylow p^0 -subgroup of $Z^{(q,r)}$ and therefore $p^0 \cdot [Z^{(q,r)} : Z_R^{(q,r)}] = l_1$. Let $[T : Z_R^{(q,r)}] = l_2 = p^{0h}$ for $h \geq 0$, since both are p^0 -groups. Since $l_2 | l_1 l$, and $p^0 \nmid l_1 l$, we must have $p^0 \nmid l_2$ and $l_2 = 1$. Thus, $Z_R^{(q,r)} = T$ and $Z_R^{(q,r)}$ is a Sylow p^0 -subgroup of $\text{Aut}(P)$. \square



5.2. A theorem on isotopisms between biprojective semifields. Now we are going to show that if two biprojective pre-semifields are isotopic, an isotopism (N, L, M) that satisfies strong requirements on the shape of the linearized polynomials N , L and M has to exist, whenever the condition appearing in the assumption of Lemma 5.8 is satisfied. We will name it Condition (C). First we need a simple lemma.

Lemma 5.9. *Let*

$$(x, y) \rightarrow F(x, y) = [f(x, y), g(x, y)] = [(a_0, b_0, c_0, d_0)_q, (a_1, b_1, c_1, d_1)_r]$$

be a (q, r) -biprojective mapping for arbitrary values of q, r . If F is planar, then $(a_0, a_1) = (0, 0)$ and $(d_0, d_1) = (0, 0)$. That is to say, all biprojective pre-semifield polarizations $\Delta_F((x, y), (u, v))$ have a component that contains both monomials $x^\sigma u$ and xu^σ and a component (not necessarily different) that contains both $y^\tau v$ and yv^τ for $\sigma, \tau \in \{q, r\}$ depending on the component.

Proof. If $a_0 = a_1 = 0$, then $D_f^\infty(x, y) = 0 = D_g^\infty(x, y)$ for $y = 0$ and arbitrary x , contradicting Lemma 3.1 for $u = \infty$. The contradiction for d_0, d_1 is obtained

similarly by considering Lemma 3.1 for $u = 0$. The statement on the corresponding pre-semifield follows immediately by Eq. (3.1) and the fact that $(a_0, a_1) = (0, 0)$. \square

We are now ready to prove the main result of this section: If two biprojective semifields are isotopic, then there exists an isotopism between them of a very specific form.

Theorem 5.10. *Let $P_1 = (M \times M, +, \boxtimes_1)$ and $P_2 = (M \times M, +, \boxtimes_2)$ be (q_1, r_1) - and (q_2, r_2) -biprojective pre-semifields, respectively, such that $q_1 \boxtimes \{r_1, r_1^{-1}\}$, $1 \notin \{q_1, r_1\}$ and $Q \boxtimes \{q_1, r_1\}$, where $q_i = p^{k_i}$ and $r_i = p^{l_i}$ for $i \in \{1, 2\}$. Assume that*

(C) C_{P_1} contains $Z^{(q_1, r_1)}$ as an index l subgroup such that p^0 does not divide l .

If P_1, P_2 are isotopic, then there exists an isotopism $\gamma = (N, L, M) \in \text{GL}(F)^3$, with the following properties:

- All non-zero subfunctions of N, L and M are monomials.
- All non-zero subfunctions of L and M have the same degree p^t .
- We have,
 - either $k_1 \equiv \pm k_2 \pmod{m}$ and $l_1 \equiv \pm l_2 \pmod{m}$,
 - or $k_1 \equiv \pm l_2 \pmod{m}$ and $l_1 \equiv \pm k_2 \pmod{m}$.

More precisely, we have either,

- $N_2 = N_3 = 0$ and $N_1, N_4 = 0$,
- $k_1 \equiv \pm k_2 \pmod{m}$ and $l_1 \equiv \pm l_2 \pmod{m}$,
- if $k_1 \equiv k_2 \pmod{m}$ (resp. $l_1 \equiv l_2 \pmod{m}$) then N_1 (resp. N_4) is a monomial of degree p^t ,
- if $k_1 \equiv -k_2 \pmod{m}$ (resp. $l_1 \equiv -l_2 \pmod{m}$) then N_1 (resp. N_4) is a monomial of degree p^{t+k_2} (resp. p^{t+l_2}),

or,

- $N_1 = N_4 = 0$ and $N_2, N_3 = 0$,
- $k_1 \equiv \pm l_2 \pmod{m}$ and $l_1 \equiv \pm k_2 \pmod{m}$,
- if $k_1 \equiv l_2 \pmod{m}$ (resp. $l_1 \equiv k_2 \pmod{m}$) then N_3 (resp. N_2) is a monomial of degree p^t ,
- if $k_1 \equiv -l_2 \pmod{m}$ (resp. $l_1 \equiv -k_2 \pmod{m}$) then N_3 (resp. N_2) is a monomial of degree p^{t+l_2} (resp. p^{t+k_2}).

Proof. Set

$$(x, y) \boxtimes_1 (u, v) = (f_1(x, y, u, v), g_1(x, y, u, v)), \text{ and}$$

$$(x, y) \boxtimes_2 (u, v) = (f_2(x, y, u, v), g_2(x, y, u, v)).$$

By Lemma 5.5, we have $Z_R^{(q_1, r_1)} \leq \text{Aut}(P_1)$ and $Z_R^{(q_2, r_2)} \leq \text{Aut}(P_2)$. Assume P_1 and P_2 are isotopic via the isotopism $\delta \in \text{GL}(F)^3$ that maps P_1 to P_2 . Then $\delta^{-1} \text{Aut}(P_2) \delta = \text{Aut}(P_1)$ by Lemma 5.1. Observe that $|\delta^{-1} Z_R^{(q_2, r_2)} \delta| = |R| = |Z_R^{(q_1, r_1)}|$, so $Z_R^{(q_1, r_1)}$ and $\delta^{-1} Z_R^{(q_2, r_2)} \delta$ are Sylow p^0 -subgroups of $\text{Aut}(P_1)$ by Lemma 5.8 as long as Condition (C) holds. In particular, these two subgroups are conjugate in $\text{Aut}(P_1)$ by Sylow Theorem (iii), i.e., there exists a $\lambda \in \text{Aut}(P_1)$ such that

$$(5.1) \quad \lambda^{-1} \delta^{-1} Z_R^{(q_2, r_2)} \delta \lambda = (\delta \lambda)^{-1} Z_R^{(q_2, r_2)} (\delta \lambda) = Z_R^{(q_1, r_1)}.$$

Set $\gamma = (N, L, M) = \delta\lambda$. Note that γ is an isotopism between P_1 and P_2 since $\lambda \in \text{Aut}(P_1)$. Eq. (5.1) then immediately implies that

$$\begin{aligned}\text{diag}(m_{a^{q_2+1}}, m_{a^{r_2+1}})N &= N \text{diag}(m_{b^{q_1+1}}, m_{b^{r_1+1}}), \\ \text{diag}(m_a, m_a)L &= L \text{diag}(m_b, m_b), \\ \text{diag}(m_a, m_a)M &= M \text{diag}(m_b, m_b),\end{aligned}$$

for all $a \in R$ and $b = \pi(a)$ where $\pi: R \rightarrow R$ is a permutation. In particular, L and M are in the normalizer of $S_R = \{\text{diag}(m_a, m_a) : a \in R\}$. By Lemma 5.7, all of the four subfunctions of L and M are zero or monomials of the same degree, say p^{t_2} and p^{t_3} , respectively. Then, for all $(x, y), (u, v) \in M^2$,

$$\begin{aligned}L(x, y) \in_2 M(u, v) \\ &= (a_2 x^{p^{t_2}} + b_2 y^{p^{t_2}}, c_2 x^{p^{t_2}} + d_2 y^{p^{t_2}}) \in_2 (a_3 u^{p^{t_3}} + b_3 v^{p^{t_3}}, c_3 u^{p^{t_3}} + d_3 v^{p^{t_3}}), \\ &= (h_1(x, y, u, v), h_2(x, y, u, v))\end{aligned}$$

for some $a_2, b_2, c_2, d_2, a_3, b_3, c_3, d_3 \in M$. We also have

$$\begin{aligned}N((x, y) \in_1 (u, v)) &= (N_1(f_1(x, y, u, v)) + N_2(g_1(x, y, u, v)), \\ &\quad N_3(f_1(x, y, u, v)) + N_4(g_1(x, y, u, v))).\end{aligned}$$

Let us now assume that $N((x, y) \in_1 (u, v)) = L(x, y) \in_2 M(u, v)$. We consider the first component, i.e. $h_1(x, y, u, v) = N_1(f_1(x, y, u, v)) + N_2(g_1(x, y, u, v))$. Lemma 5.9 implies both monomials $x^q u$ and xu^q occur in one of the two components of P_1 . Since switching the components clearly preserves isotopy, we can assume without loss of generality that they occur in the first component f_1 . Let us for now assume $N_1 = 0$. We know that $N_1(f_1(x, y, u, v))$ has then terms of the form

$$(5.2) \quad x^{p^{k_1+t}} u^{p^t} \text{ and } x^{p^t} u^{p^{k_1+t}}$$

for at least one $0 \leq t \leq m-1$. Observe that the differences of p -adic valuations of exponents in the x - and u -degrees of the monomials are $k_1 + t - t = k_1$ and $t - k_1 - t = -k_1$, respectively. In particular, if $q_1 = r_1$, $q_1 = r_1$, then these terms cannot be canceled out by $N_2(g_1(x, y, u, v))$. Since P_2 is a (q_2, r_2) -biprojective pre-semifield, all possible terms in h_1 are of the form

$$(5.3) \quad w^{p^{k_2+t_2}} z^{p^{t_3}} \text{ or } w^{p^{t_2}} z^{p^{k_2+t_3}},$$

where $w \in \{x, y\}$, $z \in \{u, v\}$ and $0 \leq t_2, t_3 \leq m-1$. Comparing Eqs. (5.2) and (5.3) gives either

$$\begin{aligned}k_1 + t &\equiv k_2 + t_2 \pmod{m}, & k_1 + t &\equiv t_2 \pmod{m}, \\ t &\equiv t_3 \pmod{m}, & t &\equiv k_2 + t_3 \pmod{m}, \\ t &\equiv t_2 \pmod{m}, & t &\equiv k_2 + t_2 \pmod{m}, \\ k_1 + t &\equiv k_2 + t_3 \pmod{m}, \text{ or, } & k_1 + t &\equiv t_3 \pmod{m}.\end{aligned}$$

The first possibility is equivalent to $t = t_2 = t_3$, $k_1 \equiv k_2 \pmod{m}$ and the second implies $t_2 = t_3$, $t = t_2 + k_2$ and $k_1 \equiv -k_2 \pmod{m}$. Note in particular that in any case $t_2 = t_3$. Moreover, both cases cannot occur simultaneously since, by assumption, $k_1 \equiv m/2 \pmod{m}$. We conclude that N_1 is a monomial with the same degree p^{t_2} as the L_i, M_i if $k_1 \equiv k_2 \pmod{m}$, and a monomial with the degree $p^{t_2+k_2}$ if $k_1 \equiv -k_2 \pmod{m}$.

Now assume $N_2 = 0$ and observe that the terms of $N_2(g_1(x, y, u, v))$ are of the form

$$(5.4) \quad w^{p^{l_1+t}} z^{p^t} \text{ or } w^{p^t} z^{p^{l_1+t}},$$

where $w \in \{x, y\}$, $z \in \{u, v\}$ and some $0 \leq t \leq m-1$. In particular, the difference of p -adic valuations of exponents of the two monomials is l_1 or $-l_1$. This however yields a contradiction since the difference of p -adic valuations of exponents in Eq. (5.3) is k_2 or $-k_2$, that is by the considerations above, $-k_1$ or k_1 , which leads to $k_1 \equiv \pm l_1 \pmod{m}$ which is not possible since $q_1 = r_1$, $q_1 = \overline{r_1}$. We conclude that $N_2 = 0$.

Let us now consider the second component. Since $N_2 = 0$, we must have $N_4 = 0$ since N is bijective. The terms of $N_4(g_1(x, y, u, v))$ are then again of the same form as in Eq. (5.4). Similar to Eq. (5.3), the terms in h_2 are of the form (using $t_2 = t_3$)

$$(5.5) \quad w^{p^{l_2+t_2}} z^{p^{t_2}} \text{ or } w^{p^{t_2}} z^{p^{l_2+t_2}},$$

where $w \in \{x, y\}$, $z \in \{u, v\}$. A comparison between Eqs. (5.4) and (5.5) yields either $t = t_2$ and $l_1 \equiv l_2 \pmod{m}$ or $t = l_2 + t_2$ and $l_1 \equiv -l_2 \pmod{m}$. Again, this means that N_4 is a monomial of degree p^{t_2} or $p^{l_2+t_2}$ as both cases cannot occur simultaneously since $l_1 \equiv m/2 \pmod{m}$.

Since $q_1 = r_1$, $q_1 = \overline{r_1}$, we can again deduce $N_3 = 0$ with the same argument we used to prove $N_2 = 0$ before. This concludes the case $N_1 = 0$.

Now assume $N_1 = 0$. Since N is bijective, this implies $N_3 = 0$. We can then employ the entire argument, just starting with the second component and exchanging k_2 and l_2 , N_1 and N_3 , and N_2 and N_4 throughout. We conclude that in this case $N_3, N_2 = 0$ and $N_1 = N_4 = 0$. \square

Remark 5.11.

- (i) We exclude the cases $q_1 = r_1$, $q_1 = \overline{r_1}$, $1 \in \{q_1, r_1\}$, and $Q \in \{q_1, r_1\}$. It is possible to give (slightly weaker) versions of Theorem 5.10 also in the excluded cases. We avoided these cases to simplify the exposition. For instance, when we allow $Q \in \{q_1, r_1\}$, then the subfunctions of N may be binomials of the form $N_i = ax^{p^t} + bx^{p^{t+m/2}}$. We will showcase an isotopy of this kind in Remark 8.3 in Section 8. Observe that in the version we have given, all non-zero subfunctions of N, L, M are monomials. We chose this presentation of the theorem to avoid listing unnecessary special cases that we do not need in this paper.
- (ii) We will mainly use Theorem 5.10 to determine the number of isotopy classes in Family 5. Of course, it can also be used to give alternative (in most cases simpler) proofs of the number of isotopy classes of the known commutative biprojective pre-semifields.

Theorem 5.10 enables us to settle the isotopy question of biprojective pre-semifields with relative ease as long as Condition (C) is satisfied. In the next section we will first show that Condition (C) is satisfied for our family and then use Theorem 5.10 to determine the number of non-isotopic semifields in the family. Note that the condition $m > 2$ we stipulate in this section is not restrictive when considering S since it does not yield semifields when m is a power of 2.

6. Isotopisms within the Family S

In this section we will show that the number of non-isotopic semifields within the Family S is exponential in n (when $n = 4t$, where t is not a power of 2). We first need to check Condition (C) in Theorem 5.10 for the pre-semifields in the Family S . Lemma 6.1 does that in a straightforward manner.

Lemma 6.1. *Let $n = 2m$ and $P = (M \times M, +, \cdot)$ be a (q, r) -biprojective pre-semifield in the Family S . Then*

$$|C_P| = (p^m - 1)(p^{\gcd(k, m)} - 1), \text{ or} \\ |C_P| = 2(p^m - 1)(p^{\gcd(k, m)} - 1).$$

In particular, Condition (C) is always satisfied.

Proof. If $(N, L, M) \in C_P$ then, by Lemma 5.7, the subfunctions L_i and M_i for $i \in \{1, 2, 3, 4\}$ are zero or monomials of degree 1, we write

$$L_1(x) = a_2x, \quad L_2(x) = b_2x, \quad L_3(x) = c_2x, \quad L_4(x) = d_2x, \\ M_1(x) = a_3x, \quad M_2(x) = b_3x, \quad M_3(x) = c_3x, \quad M_4(x) = d_3x.$$

We then have

$$\begin{aligned} L(x, y) \cdot M(u, v) &= (a_2x + b_2y, c_2x + d_2y) \cdot (a_3u + b_3v, c_3u + d_3v) \\ &= ((a_2x + b_2y)^q(a_3u + b_3v) + (a_2x + b_2y)(a_3u + b_3v)^q \\ &\quad + B((c_2x + d_2y)^q(c_3u + d_3v) + (c_2x + d_2y)(c_3u + d_3v)^q), \\ &\quad (a_2x + b_2y)^{qQ}(c_3u + d_3v) + (c_2x + d_2y)(a_3u + b_3v)^{qQ} \\ (6.1) \quad &\quad + \frac{a}{B}((a_2x + b_2y)(c_3u + d_3v)^{qQ} + (c_2x + d_2y)^{qQ}(a_3u + b_3v))). \end{aligned}$$

Similarly, we have

$$\begin{aligned} N((x, y) \cdot (u, v)) &= (N_1(x^q u + x u^q + B(y^q v + y v^q)) + N_2(x^{qQ} v + y u^{qQ} + \frac{a}{B}(x v^{qQ} + y^{qQ} u)), \\ &\quad N_3(x^q u + x u^q + B(y^q v + y v^q)) + N_4(x^{qQ} v + y u^{qQ} + \frac{a}{B}(x v^{qQ} + y^{qQ} u))). \end{aligned}$$

By comparing the degrees, it is then easy to see that $N((x, y) \cdot (u, v)) = L(x, y) \cdot M(u, v)$ implies $N_2 = N_3 = 0$ and $N_1 = a_1x$, $N_4 = d_1x$ for some $a_1, d_1 \in M^\times$. Then

$$(6.2) \quad N((x, y) \cdot (u, v)) = (a_1(x^q u + x u^q + B(y^q v + y v^q)), d_1(x^{qQ} v + y u^{qQ} + \frac{a}{B}(x v^{qQ} + y^{qQ} u))).$$

(6.18) and the bijectivity of L, M then only leave two possibilities: Either $a_2 = a_3 = d_2 = d_3 = 0$ and $b_2, b_3, c_2, c_3 = 0$ or, the other way round, $a_2, a_3, d_2, d_3 = 0$ and $b_2 = b_3 = c_2 = c_3 = 0$. We will deal with these two cases separately. Note that in both cases, Eqs. (6.5), (6.6), (6.7), (6.8), (6.11), (6.12), (6.17), (6.18) are always satisfied.

Case $b_2, b_3, c_2, c_3 = 0$. We set $a_3 = \omega_1 a_2$, $d_3 = \omega_2 d_2$. Then Eqs. (6.3), (6.4), (6.9), (6.10) become

$$a_1 = a_2^{q+1} \omega_1 = a_2^{q+1} \omega_1^q = d_2^{q+1} \omega_2 = d_2^{q+1} \omega_2^q,$$

which is satisfied if and only if $\omega_1, \omega_2 \in E$ and

$$(6.20) \quad (a_2/d_2)^{q+1} = \omega_2/\omega_1.$$

Similarly, from Eqs. (6.13), (6.14), (6.15), (6.16), we get immediately (using $\omega_1, \omega_2 \in E$),

$$d_1 = a_2^{qQ} d_2 \omega_2 = a_2 d_2^{qQ} \omega_2^Q = a_2 d_2^{qQ} \omega_1 = a_2^{qQ} d_2 \omega_1^Q.$$

This is equivalent to $\omega_1 = \omega_2^Q$, $(a_2/d_2)^{qQ-1} = \omega_2^{Q-1}$. Multiplying this with Eq. (6.20) gives

$$(a_2/d_2)^{q(Q+1)} = 1,$$

i.e., $a_2/d_2 \in (M^\times)^{Q-1}$, say $\zeta^{Q-1} = a_2/d_2$. Rewriting Eq. (6.20) gives

$$(a_2/d_2)^{q+1} = (\zeta^{q+1})^{Q-1} = \omega_1^{Q-1}.$$

The equation cannot be satisfied if ω_1 is a non-square. If ω_1 is a square, then d_2 is uniquely determined up to the sign from ω_1 and a_2 . Thus we have in total $p^m - 1$ choices for a_2 , $(p^{\gcd(k,m)} - 1)/2$ choices for ω_1 and 2 choices for d_2 , making in total $(p^m - 1)(p^{\gcd(k,m)} - 1)$ choices in this case.

Case $a_2, a_3, d_2, d_3 = 0$. Similarly to the previous case, we set $b_3 = \omega_1 b_2$, $c_3 = \omega_2 c_2$. We get from the first set of equations:

$$a_1 = B c_2^{q+1} \omega_2 = B c_2^{q+1} \omega_2^q = (1/B) b_2^{q+1} \omega_1 = (1/B) b_2^{q+1} \omega_1^q,$$

which is equivalent to $\omega_1, \omega_2 \in E$ and

$$(6.21) \quad (b_2/c_2)^{q+1} = B^2 \omega_2/\omega_1.$$

The second set of equations gives

$$d_1 = (a/B) c_2^{qQ} b_2 \omega_1 = (B/a) c_2 b_2^{qQ} \omega_1^Q = (B/a) c_2 b_2^{qQ} \omega_2 = (a/B) c_2^{qQ} b_2 \omega_2^Q.$$

This again implies $\omega_1^Q = \omega_2$ and $(b_2/c_2)^{qQ-1} = (1/\omega_1^{Q-1})(a/B)^2$. Multiplying this with Eq. (6.21) gives

$$(b_2/c_2)^{q(Q+1)} = \frac{\omega_2}{\omega_1^Q} a^2 = a^2 = a^{Q+1}.$$

Thus, b_2/c_2 is determined up to multiplication with a $(Q+1)^{\text{st}}$ root of unity, that is, a $(Q-1)^{\text{st}}$ power, say $(b_2/c_2)^q = \zeta^{Q-1} a$. Eq. (6.21) can be rewritten as

$$(b_2/c_2)^{q+1} = a^{q+1} (\zeta^{q+1})^{Q-1} = B^2 \omega_1^{Q-1}.$$

This equation has a solution if and only if $(a^{q+1})/B^2$ is a $(Q-1)^{\text{st}}$ power, say $(a^{q+1})/B^2 = \rho^{Q-1}$. In this case, there are $(p^{\gcd(k,m)} - 1)/2$ possible choices for ω_1 , either all squares or all non-squares, depending on if ρ is a square or not. Then ζ is determined up to the sign, so there are $p^m - 1$ possible choices for b_2 , 2 possible choices for c_2 and $(p^{\gcd(k,m)} - 1)/2$ possible choices for ω_1 . This case thus contributes

either 0 or $(p^m - 1)(p^{\gcd(k,m)} - 1)$ elements. Both cases together show that $|C_p|$ is either $(p^m - 1)(p^{\gcd(k,m)} - 1)$ or $2(p^m - 1)(p^{\gcd(k,m)} - 1)$.

Now it is clear that $p^0 \in [C_p : Z^{(q,r)}] \setminus \{p^{\gcd(k,m)} - 1, 2(p^{\gcd(k,m)} - 1)\}$ by our assumption that p^0 is p -primitive (recall $p^0 = 2$). \square

We can now apply Theorem 5.10 to the pre-semifields in the Family S .

Theorem 6.2. *Let $P_{q,B,a} = (M \times M, +, \boxplus_1)$ and $P_{q^0,B^0,a^0} = (M \times M, +, \boxplus_2)$ be pre-semifields from the Family S . Then*

- (i) $P_{q,B,a}$ and P_{q^0,B^0,a^0} are isotopic if and only if they are strongly isotopic.
- (ii) $P_{q,B,a}$ is isotopic to P_{q,B,a^0} for $a^0 = B^{Q+1}/a$ and arbitrary q .
- (iii) $P_{q,B,a}$ is isotopic to P_{q,B^0,a^0} for arbitrary q, B, B^0, a and a suitable choice for a^0 .
- (iv) If $P_{q,B,a}$ is isotopic to P_{q,B,a^0} , then it is also isotopic to $P_{q,B,-a^0}$.
- (v) There are at most $2m = n$ different a^0 such that $P_{q,B,a}$ is isotopic to P_{q,B,a^0} .
- (vi) No other isotopisms exist.

Proof. Let (N, L, M) be an isotopism between $P_{q,B,a}$ and $P_{q^0,B^0,a^0} = (M \times M, +, \boxplus_2)$. All subfunctions of N, L, M are zero or monomials by Theorem 5.10. Moreover, $P_{q,B,a}$ and P_{q^0,B^0,a^0} can only be isotopic if $q^0 = q$, $q^0 = \bar{q}$, $q^0 = qQ$, or $q^0 = \bar{q}Q$. Note that if $m/\gcd(k, m)$ is odd, then $m/\gcd(k + m/2, m)$ is even by Lemma 4.3(iii), so the cases $q^0 = qQ$, $q^0 = \bar{q}Q$ do not satisfy the conditions of Theorem 4.4 and need not be considered.

We first show the isotopy in the case $q^0 = \bar{q}$. We have

$$(x, y) \boxplus_1 (u, v) = (x^q u + x u^q + B(y^q v + y v^q), x^{qQ} v + y u^{qQ} + (a/B)(x v^{qQ} + y^{qQ} u)).$$

A transformation with

$$N_1 = x, N_4 = (B^Q/a)x^Q, N_2 = N_3 = 0$$

and raising x, y, u, v to the \bar{q} -th power yields

$$\begin{aligned} N((x^{\bar{q}}, y^{\bar{q}}) \boxplus_1 (u^{\bar{q}}, v^{\bar{q}})) \\ = (x u^{\bar{q}} + x^{\bar{q}} u + B(y v^{\bar{q}} + y^{\bar{q}} v), (B^Q/a)(x v^{\bar{q}Q} + y^{\bar{q}Q} u) + x^{\bar{q}Q} v + y u^{\bar{q}Q}). \end{aligned}$$

Observe that one can write $B^Q/a = a^0/B$ for some $a^0 \in L$ (indeed this is equivalent to $B^{Q+1} = aa^0$ which has always a solution since $B^{Q+1} \in L$). We conclude that there is always a strong isotopism between $P_{q,B,a}$ and $P_{q,B,B^{Q+1}/a}$. Thus we have proved Part (ii) of the theorem.

It thus only remains to deal with the case $q^0 = q$. By Theorem 5.10, we only need to consider isotopisms (N, L, M) with subfunctions

$$\begin{aligned} N_1 &= a_1 x^{p^t}, & N_4 &= d_1 x^{p^t}, \\ L_1 &= a_2 x^{p^t}, & L_2 &= b_2 x^{p^t}, & L_3 &= c_2 x^{p^t}, & L_4 &= d_2 x^{p^t}, \\ M_1 &= a_3 x^{p^t}, & M_2 &= b_3 x^{p^t}, & M_3 &= c_3 x^{p^t}, & M_4 &= d_3 x^{p^t}, \end{aligned}$$

for some $t \in \{0, \dots, m-1\}$. Then

$$\begin{aligned} L(x, y) \otimes_2 M(u, v) &= (((a_2^0 x + b_2^0 y)^q (a_3^0 u + b_3^0 v) + (a_2^0 x + b_2^0 y)(a_3^0 u + b_3^0 v)^q)^{p^t} \\ &\quad + B^0((c_2^0 x + d_2^0 y)^q (c_3^0 u + d_3^0 v) + (c_2^0 x + d_2^0 y)(c_3^0 u + d_3^0 v)^q)^{p^t}, \\ &\quad ((a_2^0 x + b_2^0 y)^{qQ} (c_3^0 u + d_3^0 v) + (c_2^0 x + d_2^0 y)(a_3^0 u + b_3^0 v)^{qQ})^{p^t} \\ &\quad + (a^0/B^0)((a_2^0 x + b_2^0 y)(c_3^0 u + d_3^0 v)^{qQ} + (c_2^0 x + d_2^0 y)^{qQ} (a_3^0 u + b_3^0 v)^{p^t}), \end{aligned}$$

where $a_i^c = a_i^{p^m - t}$ and similarly for the other coefficients b_i, c_i, d_i . We also obtain

$$\begin{aligned} N((x, y) \otimes_1 (u, v)) &= (a_1(x^q u + x u^q + B(y^q v + y v^q))^{p^t}, d_1(x^{qQ} v + y u^{qQ} + (a/B)(x v^{qQ} + y^{qQ} u))^{p^t}). \end{aligned}$$

We compare the coefficients $(x^q u)^{p^t}, (x u^q)^{p^t}, (x^q v)^{p^t}, (x v^q)^{p^t}, (y^q u)^{p^t}, (y u^q)^{p^t}, (y^q v)^{p^t}, (y v^q)^{p^t}$ in the first component to get the following 8 equations.

$$(6.22) \quad a_1 = a_2^q a_3 + B^0 c_2^q c_3,$$

$$(6.23) \quad a_1 = a_2 a_3^q + B^0 c_2 c_3^q,$$

$$(6.24) \quad 0 = a_2^q b_3 + B^0 c_2^q d_3,$$

$$(6.25) \quad 0 = a_2 b_3^q + B^0 c_2 d_3^q,$$

$$(6.26) \quad 0 = b_2^q a_3 + B^0 d_2^q c_3,$$

$$(6.27) \quad 0 = b_2 a_3^q + B^0 d_2 c_3^q,$$

$$(6.28) \quad B^{p^t} a_1 = b_2^q b_3 + B^0 d_2^q d_3,$$

$$(6.29) \quad B^{p^t} a_1 = b_2 b_3^q + B^0 d_2 d_3^q.$$

And similarly the 8 equations that come from comparing the coefficients in the second component:

$$(6.30) \quad 0 = a_2^{qQ} c_3 + (a^0/B^0) c_2^{qQ} a_3,$$

$$(6.31) \quad 0 = (a^0/B^0) a_2 c_3^{qQ} + c_2 a_3^{qQ},$$

$$(6.32) \quad d_1 = a_2^{qQ} d_3 + (a^0/B^0) c_2^{qQ} b_3,$$

$$(6.33) \quad d_1 (a/B)^{p^t} = (a^0/B^0) a_2 a_3^{qQ} + c_2 b_3^{qQ},$$

$$(6.34) \quad d_1 (a/B)^{p^t} = b_2^{qQ} c_3 + (a^0/B^0) a_2^{qQ} a_3,$$

$$(6.35) \quad d_1 = (a^0/B^0) b_2 c_3^{qQ} + d_2 a_3^{qQ},$$

$$(6.36) \quad 0 = b_2^{qQ} d_3 + (a^0/B^0) d_2^{qQ} b_3,$$

$$(6.37) \quad 0 = (a^0/B^0) b_2 d_3^{qQ} + d_2 b_3^{qQ}.$$

Note that Eqs. (6.30), (6.31), (6.36), (6.37) are identical to Eqs. (6.11), (6.12), (6.17), (6.18) in the proof of Lemma 6.1, just with a/B substituted by (a^0/B^0) . We can thus conclude with the same reasoning as in the proof of Lemma 6.1 that either $b_2 = b_3 = c_2 = c_3 = 0$ or $a_2 = a_3 = d_2 = d_3 = 0$.

Case $b_2 = b_3 = c_2 = c_3 = 0$. Here, we also proceed similarly to the proof of Lemma 6.1. We set $a_3 = \omega_1 a_2$, $d_3 = \omega_2 d_2$ and get from Eqs. (6.22), (6.23),

(6.28), (6.29)

$$a_1 = a_2^{q+1} \omega_1 = a_2^{q+1} \omega_1^q = (B^0/B^{p^t}) a_2^{q+1} \omega_2 = (B^0/B^{p^t}) a_2^{q+1} \omega_2^q,$$

which is satisfied if and only if $\omega_1, \omega_2 \in E$ and

$$(6.38) \quad \frac{\mu}{d_2} \mathfrak{I}_{q+1}^0 = \frac{\omega_2 B^0}{\omega_1 B^{p^t}}.$$

Similarly, from Eqs. (6.32), (6.33), (6.34), (6.35), we get immediately (using $\omega_1, \omega_2 \in E$)

$$d_1 = a_2^{qQ} d_2 \omega_2 = \frac{a^0 B^{p^t}}{a^{p^t} B^0} a_2 d_2^{qQ} \omega_2^Q = \frac{a^0 B^{p^t}}{a^{p^t} B^0} a_2 d_2^{qQ} \omega_1 = a_2^{qQ} d_2 \omega_1^Q.$$

This is equivalent to $\omega_1^Q = \omega_2$ and $(a_2/d_2)^{qQ-1} = \omega_2^{Q-1} (a^0 B^{p^t}) / (a^{p^t} B^0)$. Multiplying the second condition with Eq. (6.38) gives

$$(6.39) \quad \frac{\mu}{d_2} \mathfrak{I}_{q(Q+1)}^0 = \frac{\omega_2^Q a^0}{\omega_1 a^{p^t}} = \frac{a^0}{a^{p^t}}.$$

Using $\omega_1^Q = \omega_2$, we rewrite Eq. (6.38):

$$(6.40) \quad \frac{\mu}{d_2} \mathfrak{I}_{q+1}^0 = \frac{\omega_1^{Q-1} B^0}{B^{p^t}}.$$

Observe that B, B^0, t, ω_1 uniquely determine (a_2/d_2) up to the sign from Eq. (6.40). Since $(a_2/d_2)^{q(Q+1)} \in L$, there is thus for each B, B^0, t, ω_1, a precisely one a^0 that satisfies all conditions. For all ω_1 that are squares (i.e., all $(q+1)^{\text{st}}$ powers), this a^0 is the same since $\omega_1^{(Q-1)(Q+1)} = 1$. Similarly, for all ω_1 that are non-squares, we have $\omega_1^{(Q-1)(Q+1)/2} = -1$, so they also all yield the same a^0 , and in fact precisely the same a^0 as when ω_1 is a square, just with different sign. In particular, we conclude that a pre-semifield $P_{q,B,a}$ is always isotopic to P_{q,B^0,a^0} for arbitrary B^0 and a suitable choice of a^0 . Since we can choose $\omega_1 = \omega_2 = 1$, we can even choose a^0 such that the pre-semifields are strongly isotopic. Thus, we have proved Part (iii) of our theorem.

Consequently, it is enough to consider isotopisms in the case $B = B^0$ for an arbitrary non-square B . When $B = B^0$, every possible choice of t yields an a^0 such that a pre-semifield $P_{q,B,a}$ is strongly isotopic to P_{q,B,a^0} and isotopic to $P_{q,B,-a^0}$. Assume the choice of t in the previously described procedure leads to a strong isotopy between $P_{q,B,a}$ and P_{q,B,a^0} . We now show that choosing t^0 defined by $t^0 - t \equiv m/2 \pmod{m}$ in the same procedure gives a strong isotopy to $P_{q,B,-a^0}$, i.e. $P_{q,B,a}$ and $P_{q,B,-a^0}$ are not just isotopic but also strongly isotopic.

Let $(a_2/d_2)^{q+1}$ be determined by $\omega_1 = 1$ and fixed $B = B^0, t$ via Eq. (6.40), i.e.

$$\frac{\mu}{d_2} \mathfrak{I}_{q+1}^0 = \frac{1}{B^{p^t-1}}.$$

Similarly, let $(a_2^0/d_2^0)^{q+1}$ be determined by $\omega_1 = 1$, the same $B = B^0$ and t^0 :

$$\frac{\mu}{d_2^0} \mathfrak{I}_{q+1}^0 = \frac{1}{B^{p^{t^0}-1}}.$$

We then have

$$\frac{\mu}{d_2^0} \mathfrak{I}_{q+1}^0 = \frac{\mu}{d_2} \mathfrak{I}_{q+1}^0 \cdot \frac{1}{B^{p^{t^0}-p^t}}.$$

Since $B^{p^{t^0}-p^t} = (B^{Q-1})^{p^t} \not\equiv (\mathbf{M}^\times)^{Q-1}$, we have $(a_2^0/d_2^0)^{q+1} = (\zeta a_2/d_2)^{q+1}$ where $\zeta^{q+1} = 1/(B^{p^{t^0}-p^t}) \not\equiv (\mathbf{M}^\times)^{Q-1}$. Note that $\zeta \not\equiv (\mathbf{M}^\times)^{Q-1}$ since B is a non-square, so $B^{(Q-1)(Q+1)/2} = 1$. In particular, we have $\zeta^{Q+1} = -1$. Then by Eq. (6.39), $(a^0/d_2^0)^{q(Q+1)} = \zeta^{q(Q+1)}(a_2/d_2)^{q(Q+1)} = -(a_2/d_2)^{q(Q+1)} = -a^0/a^{p^t}$. We conclude that $P_{q,B,a}$ and P_{q,B,a^0} are strongly isotopic if and only if $P_{q,B,a}$ and $P_{q,B,-a^0}$ are strongly isotopic.

Case $a_2, a_3, d_2, d_3 = 0$. Similarly to the previous case, we set $b_3 = \omega_1 b_2$, $c_3 = \omega_2 c_2$. Since we know from the previous case that different B, B^0 always lead to strongly isotopic pre-semifields (for suitable choices of a, a^0), we only consider the case $B = B^0$ without loss of generality. We get from the first set of equations:

$$a_1 = Bc_2^{q+1}\omega_2 = Bc_2^{q+1}\omega_2^q = (1/B^{p^t})b_2^{q+1}\omega_1 = (1/B^{p^t})b_2^{q+1}\omega_1^q,$$

which is equivalent to $\omega_1, \omega_2 \not\equiv \mathbb{E}$ and

$$(6.41) \quad \mu \frac{b_2}{c_2} \not\equiv_{q+1} = B^{p^t+1} \cdot \frac{\omega_2}{\omega_1}.$$

The second set of equations gives

$$d_1 = (a^0/B)c_2^{qQ}b_2\omega_1 = (B/a)^{p^t}c_2b_2^{qQ}\omega_1^Q = (B/a)^{p^t}c_2b_2^{qQ}\omega_2 = (a^0/B)c_2^{qQ}b_2\omega_2^Q.$$

This again implies $\omega_1^Q = \omega_2$ and $(b_2/c_2)^{qQ-1} = (1/\omega_1^{Q-1})(a/B)^{p^t}(a^0/B)$. Multiply this with Eq. (6.41) gives

$$\mu \frac{b_2}{c_2} \not\equiv_{q(Q+1)} = \frac{\omega_2}{\omega_1^Q} a^{p^t} a^0 = a^{p^t} a^0.$$

Eq. (6.41) can be rewritten as

$$\mu \frac{b_2}{c_2} \not\equiv_{q+1} = \omega_1^{Q-1} B^{p^t+1}.$$

These two equations are structurally identical to Eqs. (6.39) and (6.40) from the previous case. With the same argumentation, we conclude that every possible choice of t yields an a^0 such that a pre-semifield $P_{q,B,a}$ is strongly isotopic to P_{q,B,a^0} and isotopic to $P_{q,B,-a^0}$. Again, choosing t^0 such that $t^0 - t \equiv m/2 \pmod{m}$ gives also strong isotopy between $P_{q,B,a}$ and $P_{q,B,-a^0}$. This proves Parts (iv) and (i) of our theorem. Now we can simply prove Parts (v) and (vi). Considering both cases together, there are thus at most $2m = n$ different a^0 such that $P_{q,B,a}$ is strongly isotopic to P_{q,B,a^0} . We have considered all cases thus there are no more isotopisms. \square

Remark 6.3 (Planar equivalence and strong isotopy). Instead of the exposition we chose based on isotopy, we could have developed an approach based on (in)equivalences of planar (q, r) -biprojective mappings. Recall that Theorem 2.1 states that strong isotopy of pre-semifields corresponds to equivalence of the corresponding planar mappings. We give a very brief sketch of such an approach: One can define the automorphism group $\text{Aut}(F)$ of a planar mapping F of $F = \mathbb{F} \times \mathbb{M}$ from a DO polynomial as the set of all $(N, L) \in \text{GL}(F)^2$ such that $NFL^{-1} = F$. Note that, by Theorem 2.1, $\text{Aut}(F) = \text{Aut}_5(\mathbb{F})$, where $\text{Aut}_5(\mathbb{F})$ is the group of all strong

autotopisms of the pre-semifield \mathbf{P} belonging to F . It is then clear (identically to Lemma 5.5) that the set

$$Z_F^{(q,r)} = \{(\text{diag}(m_{a^{q+1}}, m_{a^{r+1}}), \text{diag}(m_a, m_a)) : a \in \mathbb{M}^\times\} \subseteq Z^{(q,r)}$$

is a subgroup of $\text{Aut}(F)$. The same group theoretic machinery can then be applied, with $Z_F^{(q,r)}$ and $\text{Aut}(F)$ taking the role of $Z^{(q,r)}$ and $\text{Aut}(\mathbf{P})$ in the approach we presented, proving an analogue of Theorem 5.10, with the conclusion that all subfunctions of N and L are zeros or monomials.

Then, the planar mappings from Family S can be tested for equivalence similar to Theorem 6.2 by comparing the coefficients of the polynomial equation $NF = FL$. One obtains the same set of equations as Eqs. (6.22)-(6.37), just with the simplification that $M = L$. Then the same argumentation of the proof of Theorem 6.2 can be followed, with the result that the only possible equivalences that need to be considered are equivalences via N, L where the subfunctions are

$$N_1 = a_1 x^{p^t}, \quad N_2 = N_3 = 0, \quad N_4 = d_1 x^{p^t}$$

and either

$$L_1 = a_2 x^{p^t}, \quad L_2 = L_3 = 0, \quad L_4 = d_2 x^{p^t}$$

or

$$L_1 = 0, \quad L_2 = b_2 x^{p^t}, \quad L_3 = c_2 x^{p^t}, \quad L_4 = 0.$$

The conditions on the coefficients are then identical to the ones in the proof of Theorem 6.2 (e.g. Eqs. (6.39) and (6.40)), just with $\omega_1 = \omega_2 = 1$. This way, one obtains the same result as Theorem 6.2, except that one only gets information on *strong* isotopy and not regular isotopy.

Since an isotopy class of a commutative semifields contains at most 2 strong-isotopy classes [8, Theorem 2.6.], this approach would suffice to prove the exponential count. With some more effort, the planar mapping approach can also be used to find all isotopisms between commutative semifields (not just strong isotopisms). Indeed, by [8, Theorem 2.6.], if two commutative semifields with corresponding planar mapping $F : x \rightarrow x \boxtimes_1 x$ and $G : x \rightarrow x \boxtimes_2 x$ are isotopic, then either F and G are equivalent (the semifields are then strongly isotopic) or F is equivalent to $G^0 : x \rightarrow x \boxtimes_2 (a \boxtimes_2 x)$ where a is an arbitrary non-square element in the middle nucleus of the semifield. So in order to settle the isotopy question one could check equivalence between F and both G and G^0 , yielding an alternative proof of Theorem 5.10. The two approaches are essentially equivalent and require similar effort.

The isotopy approach we chose has the advantage that it can be extended naturally to non-commutative semifields where the connection to planar mappings does not exist.

The number of distinct isotopy classes can now be counted.

Corollary 6.4. *Let $N_S(p, n)$ be the number of non-isotopic pre-semifields in Family S on F_p^n . Then*

$$\frac{\sigma(n) - 1}{2} \cdot \frac{p^{n/4} - 1}{n} \leq N_S(p, n) \leq \frac{\sigma(n) - 1}{2} p^{n/4 - 1}.$$

Proof. This follows directly from Theorem 6.2(i), (ii), (iv) and (v): There are $\sigma(n) - 1$ admissible values for q , and only q, \bar{q} yield isotopic pre-semifields. Then

there are $p^{n/4} - 1$ admissible values for a , with at most n of them yielding isotopic pre-semifields. \square

In particular, S is the first known family of commutative (pre-)semifields that yields exponentially many non-isotopic (pre-)semifields. Since non-isotopic pre-semifields lead to inequivalent planar mappings (see Theorem 2.1), this also shows that the number of inequivalent planar mappings grows exponentially in n .

Corollary 6.5. *The number of non-isotopic commutative semifields of order p^n and the number of inequivalent planar DO mappings of F_{p^n} are exponential in n for a fixed odd prime p and n divisible by 4.*

7. The nuclei

In this section we will compute the nuclear parameters of Family S . As explained in Section 2, the nuclei are defined for semifields and not for pre-semifields. However, the nuclei of the isotopic semifield can be computed using the following theorem of Marino and Polverino [25, Theorem 2.2] (we give the commutative version of their general theorem) that allows computing the nuclei directly from the pre-semifield.

Let $P = (F_p, +, \cdot)$ be a commutative pre-semifield with right multiplication defined as

$$R_U : X \rightarrow X \cdot U, \quad \text{for } U \in F_p^n.$$

Then the spread set associated to P is defined as

$$L = \{R_U : U \in F_p^n\}.$$

In the following $N_j(P)$ denotes the corresponding nucleus of the semifield isotopic to P , for $j \in \{l, m, r\}$.

Theorem 7.1 ([25, Theorem 2.2]). *Let $N_0, N_1 \subseteq \text{End}(F_p^n)$ be the largest sets (and then necessarily fields) such that*

$$LN_0 \subseteq L \text{ and } N_1L \subseteq L.$$

Then

$$N_m(P) \subseteq N_0 \text{ and } N_l(P) = N_r(P) \subseteq N_1.$$

Now, let $P = (M \times M, +, \cdot)$ be a pre-semifield in Family S . Then $L = \{R_{u,v} : (u, v) \in M \times M\}$, where

$$R_{u,v} : (x, y) \rightarrow (R_{u,v}^{(1)}(x, y), R_{u,v}^{(2)}(x, y)),$$

with

$$R_{u,v}^{(1)}(x, y) = x^q u + x u^q + B(y^q v + y v^q) \text{ and } R_{u,v}^{(2)}(x, y) = x^r v + A x v^r + A y^r u + y u^r.$$

We write again $L \subseteq \text{End}(M \times M)$ as $L : (x, y) \rightarrow (\alpha(x) + \beta(y), \gamma(x) + \delta(y))$, where $\alpha, \beta, \gamma, \delta \in \text{End}(M)$.

Theorem 7.2. *The left, middle and right nuclei $N_l(P)$, $N_m(P)$, $N_r(P)$ satisfy $N_l(P) = N_r(P) \subseteq D$ and $N_m(P) \subseteq E$.*

Proof. We have nonzero $L \not\subseteq N_1$, if and only if, for every $(u, v) \in M \times M$ there exists $(w, t) \in M \times M$ such that

$$\begin{aligned}\alpha R_{u,v}^{(1)} + \beta R_{u,v}^{(2)} &= R_{w,t}^{(1)}, \text{ and} \\ \gamma R_{u,v}^{(1)} + \delta R_{u,v}^{(2)} &= R_{w,t}^{(2)},\end{aligned}$$

that is

$$\begin{aligned}\alpha(x^q u + x u^q + B(y^q v + y v^q)) + \beta(x^r v + A x v^r + A y^r u + y u^r) \\ = x^q w + x w^q + B(y^q t + y t^q),\end{aligned}$$

and

$$\begin{aligned}\gamma(x^q u + x u^q + B(y^q v + y v^q)) + \delta(x^r v + A x v^r + A y^r u + y u^r) \\ = x^r t + A x t^r + A y^r w + y w^r.\end{aligned}$$

This implies (after a routine comparison of degrees of x, y, u, v as in previous sections) that $\beta = \gamma = 0$ and $\alpha(x) = z_1 x$ and $\delta(x) = z_4 x$ for some $z_1, z_4 \in M^\times$. Now, the above equations become

$$\begin{aligned}z_1(x^q u + x u^q + B(y^q v + y v^q)) &= x^q w + x w^q + B(y^q t + y t^q), \text{ and} \\ z_4(x^r v + A x v^r + A y^r u + y u^r) &= x^r t + A x t^r + A y^r w + y w^r,\end{aligned}$$

or (the case $uv = 0$ is easy to see)

$$\begin{aligned}z_1 &= w/u = (w/u)^q = t/v = (t/v)^q, \text{ and} \\ z_4 &= w/u = (w/u)^r = t/v = (t/v)^r.\end{aligned}$$

Thus for every $(u, v) \in M \times M$ there exists $(w, t) \in M \times M$ if and only if $z_4 = z_1 = z = z_1^q$ if and only if $z_1 \in F_q \cap F_r \cap M = D$. That is to say $L \not\subseteq N_1$ if and only if $L(x, y) = (zx, zy)$ for $z \in D$. Now Theorem 7.1 implies $N_l(P) = N_r(P) = D$. \square

Similarly for the middle nucleus, non-zero $L \not\subseteq N_0$, if and only if,

$$\begin{aligned}R_{u,v}^{(1)}(\alpha(x) + \beta(y), \gamma(x) + \delta(y)) &= R_{w,t}^{(1)}(x, y), \text{ and} \\ R_{u,v}^{(2)}(\alpha(x) + \beta(y), \gamma(x) + \delta(y)) &= R_{w,t}^{(2)}(x, y),\end{aligned}$$

that is

$$\begin{aligned}(\alpha(x) + \beta(y))^q u + (\alpha(x) + \beta(y)) u^q + B(\gamma(x) + \delta(y))^q v + B(\gamma(x) + \delta(y)) v^q \\ = x^q w + x w^q + B(y^q t + y t^q),\end{aligned}$$

and

$$\begin{aligned}(\alpha(x) + \beta(y))^r v + A(\alpha(x) + \beta(y)) v^r + A(\gamma(x) + \delta(y))^r u + (\gamma(x) + \delta(y)) u^r \\ = x^r t + A x t^r + A y^r w + y w^r.\end{aligned}$$

This implies (after a routine comparison of degrees of x, y, u, v) that $\alpha(x) = z_1 x$, $\beta(y) = z_2 y$, $\gamma(x) = z_3 x$ and $\delta(y) = z_4 y$ for $z_1, z_2, z_3, z_4 \in M$. Now, the x -part of the first of the above equation implies

$$z_1^q x^q u + z_1 x u^q + B(z_3^q x^q v + z_3 x v^q) = x^q w + x w^q,$$

in other words,

$$z_1^q u + B z_3^q v = w \text{ and } z_1 u^q + B z_3 v^q = w^q,$$

which implies

$$(z_1^{q^2} - z_1)u^q + (B^q z_3^{q^2} - Bz_3)v^q = 0,$$

for all $u, v \in M$. That is to say $z_1 \in F_{q^2} \cap M = E$. The x -part of the second equation yields (after simple calculations)

$$A^r z_3^{r^2} - \frac{z_3}{A} = 0.$$

That is to say, if $z_3 = 0$ then

$$z_3^{q^2-1} = \frac{1}{B^{q-1}} \text{ and } z_3^{r^2-1} = \frac{1}{A^{r-1}}.$$

By definition of S , B is a non-square in $M = F_{Q^2}$ and $A = a/B$ where $a \in F_Q^\times$. Recalling that $q^2 \equiv r^2 \pmod{Q^2}$, we reach

$$B^{q(Q+1)} = B^{q+r} = a^{r+1}.$$

Note that since B is a non-square in F_{Q^2} we have $B^{(Q+1)} \binom{Q-1}{2} = -1$ and B^{Q+1} is a non-square in F_Q . But a^{r+1} is a square in F_Q and we get $z_3 = 0$. By the y -parts of the equations we similarly reach $z_2 = 0$ and $z_4 \in E$. Thus,

$$\begin{aligned} z_1^q x^q u + z_1 x u^q + B(z_4^q y^q v + z_4 y v^q) &= x^q w + x w^q + B(y^q t + y t^q), \text{ and} \\ z_1^r x^r v + A z_1 x v^r + A z_4^r y^r u + z_4 y u^r &= x^r t + A x t^r + A y^r w + y w^r, \end{aligned}$$

implying (the case $uv = 0$ is easy to see)

$$\begin{aligned} z_1^q &= w/u \text{ and } z_1 = (w/u)^q, \\ z_4^q &= t/v \text{ and } z_4 = (t/v)^q, \\ z_1^r &= t/v \text{ and } z_1 = (t/v)^r, \\ z_4^r &= w/u \text{ and } z_4 = (w/u)^r. \end{aligned}$$

Thus for every $(u, v) \in M \times M$ there exists $(w, t) \in M \times M$ if and only if $z_4 \stackrel{q}{=} z_1 = z^q \stackrel{r}{=} z^{r^2}$ if and only if $z_1 \in F_{q^2} \cap M = E$. That is to say $L \in N_0$ if and only if $L(x, y) = (zx, zy)$ for $z \in E$. Now Theorem 7.1 implies $N_m(P) = E$. \square

8. Comparison to other commutative semifields and concluding remarks

Table 2 lists known commutative semifields that are not biprojective. We should say here that these commutative semifields are not *obviously* represented as biprojective semifields. When the order is square, there might be isotopic semifields that can be biprojective, but we are not aware of such isotopisms.

We now consider isotopisms between the new Family S and other commutative pre-semifields.

Theorem 8.1. *Let $P_{q,B,a} = (M \times M, +, \cdot)$ be a pre-semifield in the Family S . $P_{q,B,a}$ is not isotopic to any other known commutative semifield, except possibly semifields from Family B_4 . Family S yields new examples of commutative semifields.*

Proof. The non-isotopy with the biprojective pre-semifields follows directly from Theorem 5.10, except for possible isotopisms between the families S and ZP when the coefficients q, r coincide. We exclude this case by again applying Theorem 5.10: Consider the Zhou-Pott pre-semifield $P_\alpha = (M \times M, +, \cdot)$ with multiplication

$$(x, y) \cdot (u, v) = (x^q u + u^q x + \alpha(y^q v + y v^q), x^{qQ} v + y u^{qQ})$$

Table 2. Known infinite families of (non-biprojective) commutative semifields of order p^n

Family	Planar Mapping	#S	Notes	$(\#N_L, \#N_M)$	Count	Proved in
ZKW	$X^{q+1} - a^{Q-1} X^q Q + Q^2$	p^{3s}	$Q = p^s, q = p^t,$ $d = \gcd(s, t), s^0 = s/d, t^0 = t/d,$ $s^0 \text{ odd}, s^0 + t^0 \equiv 0 \pmod{3},$ $hai = F_{p^{3s}}^x.$	(p^d, p^d) [25]	≥ 1	[33]
	$X^{q+1} - a^{Q-1} X^q Q + Q^2$	p^{3s}	$Q = p^s, q = p^t,$ $d = \gcd(s, t), s/d \text{ odd},$ $q \equiv Q \equiv 1 \pmod{3},$ $hai = F_{p^{3s}}^x.$	(p^d, p^d) [25]	$\leq 9\sigma(s)$	[3]
B_4	$X^{q+1} - a^{Q-1} X^q Q + Q^3$	p^{4s}	$Q = p^s, q = p^t,$ $d = \gcd(2s, t), 2s/d \text{ odd},$ $q \equiv Q \equiv 1 \pmod{4},$ $hai = F_{p^{4s}}^x.$	$(p^{d/2}, p^d)$ [25]	$\leq 8\sigma(s)$	[3]
	$(x^2 + ay^2 + a^3y^{18}, xy - ay^6)$	3^{2m}	$m \geq 3, a \in F_{3^m} \setminus (F_{3^m})^2$ $hai = F_{3^{4s}}^x.$	$(3, 3^m)$	1	[7]
G	$(x^2 + y^{10}, xy - y^6)$	3^{2m}	$m \geq 3 \text{ odd}$	$(3, 3)$	1	[15]
CM/DY	$X^{10} \pm X^6 - X$	3^m	$m \geq 5 \text{ odd}$	$(3, 3)$	2	[9, 13]

for some (arbitrary) non-square α . Note that it is not possible to use the parameter qQ in the first component and q in the second component since $\gcd(k + m/2, m) = \gcd(k, m)/2$ by Lemma 4.3 (iii), contradicting the necessary conditions of a Zhou-Pott pre-semifield. If P_α is isotopic to $P_{q,B,\alpha}$, then (using Theorem 5.10), there is

an isotopism (N, L, M) , where

$$\begin{aligned} N_1 &= a_1 x^{p^t}, & N_4 &= d_1 x^{p^t}, & N_2 &= N_3 = 0, \\ L_1 &= a_2^{p^t} x^{p^t}, & L_2 &= b_2^{p^t} x^{p^t}, & L_3 &= c_2^{p^t} x^{p^t}, & L_4 &= d_2^{p^t} x^{p^t}, \\ M_1 &= a_3^{p^t} x^{p^t}, & M_2 &= b_3^{p^t} x^{p^t}, & M_3 &= c_3^{p^t} x^{p^t}, & M_4 &= d_3^{p^t} x^{p^t}, \end{aligned}$$

where $a_1, d_1 = 0$. Then (only considering the second components), we have

$$\begin{aligned} L((x, y)) \circ M((u, v)) \\ = (\underline{\quad} (a_2 x + b_2 y)^{qQ+p^t} (c_3 u + d_3 v)^{p^t} + (c_2 x + d_2 y)^{p^t} (a_3 u + b_3 v)^{qQ+p^t}) \end{aligned}$$

and

$$N((x, y)) \circ (u, v) = (\underline{\quad} d_1 (x^{qQ} v + y u^{qQ} + (a/B)(x v^{qQ} + y^{qQ} u))^{p^t}).$$

Comparing the coefficients of $(x^{qQ} v)^{p^t}$, $(x v^{qQ})^{p^t}$, $(x^{qQ} u)^{p^t}$ and $(x u^{qQ})^{p^t}$ yields the following four equations:

$$\begin{aligned} a_2^{qQ} d_3^{p^t} &= d_1, \\ c_2 b_3^{qQ} &= d_1 (a/B)^{p^t}, \\ a_2^{qQ} c_3 &= 0, \\ c_2 a_3^{qQ} &= 0. \end{aligned}$$

The bijectivity of L and M induces the conditions $(a_2, c_2) = (0, 0)$ and $(a_3, c_3) = (0, 0)$. Thus, the last two equations only allow $a_2 = a_3 = 0$ or $c_2 = c_3 = 0$. Both cases contradict the first two equations. We conclude that $P_{q,B,a}$ is not isotopic to a Zhou-Pott pre-semifield.

The pre-semifields from S are also not isotopic to the ones from CG , G , CM/DY , ZKW , B_3 by considering the order of the semifields and their nuclei (see Table 2). Furthermore, the Family S is not contained in B_4 since we can choose p, m, q in a way that the conditions for B_4 in Table 2 are violated. \square

Although the parameters p, m, q for the pre-semifields from Family S are more general than that of Family B_4 , for suitable choices of p, m, q the parameters may coincide. Proposition 8.2 shows that even in that case Family S contains new semifields thanks to its exponential count. More precisely, we show that the number of non-isotopic semifields from Families B_3 and B_4 of order p^{3s} and p^{4s} , respectively, is linear in s .

Proposition 8.2. *The number of non-isotopic pre-semifields in Family B_3 (and B_4 resp.) of order p^{3s} (and p^{4s} resp.) is at most $9\sigma(s)$ (and $8\sigma(s)$ resp.).*

Proof. The B_4 planar mappings are of the form

$$f(X) = X^{q+1} - a^{Q-1} X^{qQ+Q^3},$$

where a generates $F_{p^{4s}}^\times$. We count the number of different a 's which give inequivalent planar mappings. Consider the change of variable $X \rightarrow BX$, and rescaling of f to get

$$X^{q+1} - B^{Q^3+qQ-Q-1} a^{Q-1} X^{qQ+Q^3}.$$

Note that $Q^3 + qQ - q - 1 = (Q - 1)(Q^2 + Q + q + 1)$. We have by [3, Lemma 6]

$$\gcd(Q^2 + Q + q + 1, Q^3 + Q^2 + Q + 1) = \gcd(Q^3 - q, Q^3 + Q^2 + Q + 1) = 4,$$

when the semifield conditions on q, Q appearing on Table 2 are satisfied. Thus the number of inequivalent planar mappings in the Family B_4 for a given q is at most 4. This means that (using Theorem 2.1) for a given q , the number of pre-semifields, that are not strongly isotopic, is also at most 4. Any isotopy class of a commutative semifield contains at most 2 strong-isotopy classes [8, Theorem 2.6.], so for a given q there are at most 8 non-isotopic pre-semifields. Thus the total number of non-isotopic pre-semifields in the Family B_4 of order p^{4s} is bounded by $8\sigma(s)$. The B_3 case is essentially the same using [3, Lemma 5]. In this case (again with [8, Theorem 2.6.]) strong isotopy and isotopy coincide. \square

For the Family ZKW we are not aware of any result on the exact value or a bound on the number of non-isotopic pre-semifields.

Remark 8.3. We remark that we could also allow $q = 1$ in 5. However, in that case the resulting pre-semifields are strongly isotopic to Dickson semifields. Indeed, consider the planar mapping $F = [(1, 0, 0, B)_1, (0, 1, A, 0)_Q]$ with $A = a/B$ where B is a non-square and $a \in \mathbb{L}^\times$. Note that $A \notin (\mathbb{M}^\times)^{Q-1}$ since it is a non-square. Define N via its subfunctions $N_1 = x$, $N_2 = N_3 = 0$, $N_4 = d_1x + d^0x^Q$ with $d_1 = 1/(1 - A^{Q+1})$ and $d^0 = -A/(1 - A^{Q+1})$. Note that $x \rightarrow \alpha x - \beta x^Q$ is bijective if and only if $\alpha/\beta \notin (\mathbb{M}^\times)^{Q-1}$. Therefore, N_4 is bijective since $((A^{Q+1} - 1)/(A(1 - A^{Q+1})))^{Q+1} = (-1/A)^{Q+1} = 1$ since $A \notin (\mathbb{M}^\times)^{Q-1}$. We conclude that N is bijective. The subfunction N_4 is chosen such that $d_1 + A^Q d^0 = 1$ and $Ad_1 + d^0 = 0$. Then

$$\begin{aligned} NF &= [(1, 0, 0, B)_1, d_1(0, 1, A, 0)_Q + d^0(0, A^Q, 1, 0)_Q] \\ &= [(1, 0, 0, B)_1, (0, d_1 + A^Q d^0, d_1 A + d^0, 0)_Q] = [(1, 0, 0, B)_1, (0, 1, 0, 0)_Q], \end{aligned}$$

so F is equivalent to a planar mapping belonging to a Dickson pre-semifield and the corresponding semifields are strongly isotopic by Theorem 2.1. It makes thus sense to exclude the case $q = 1$ so that the different families do not intersect (as proven in Theorem 8.1). Note that the same choice of N also yields equivalence between the Budaghyan-Helleseth planar mapping and the planar mappings associated with Dickson semifields for the parameter $q = Q$.

Remark 8.4. Recall that Kantor [19] gave a family that contains an exponential number of non-isotopic commutative semifields in characteristic two using a construction of Kantor and Williams [20]. We remark that Family S (and in general, a planar mapping) does not exist in characteristic two. However, a conceptual analogue of planar functions in characteristic two is possible. These are the so-called almost perfect non-linear (APN) functions (whose polarizations are 2-to-1) that parallel planar mappings (whose polarizations are 1-to-1) without the connection to semifields. In a follow-up work to this one, we give an analogous method for determining equivalence of bijective APN functions and an analogous family that contains an exponential number of inequivalent APN functions in [16]. The first result to show that an APN family contains an exponential number of inequivalent functions was given recently by Kaspers and Zhou [21] using a different method.

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Faculty of Mathematics and Physics, Charles University in Prague, Czechia
Email address: Faruk.Gologlu@mff.cuni.cz

University of Rostock, Germany; and Department of Mathematics and Statistics,
 University of South Florida
Email address: lukas.koelsch.math@gmail.com