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THE GENERALIZED DOUBLING METHOD: LOCAL THEORY

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Abstract. A fundamental difficulty in the study of automorphic representations, representations of p-adic groups and the Langlands program is to handle the non-generic case. In a recent collaboration with David Ginzburg, we presented a new integral representation for the tensor product L-functions of $G \times \mathrm{GL}_k$ where G is a classical group, that applies to all cuspidal automorphic representations, generic or otherwise. In this work we develop the local theory of these integrals, define the local γ-factors and provide a complete description of their properties. We can then define L- and ϵ-factors at all places, and as a consequence obtain the global completed L-function and its functional equation.

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Introduction

Let $\mathbb{A}$ be the ring of adeles of a number field. Let G be either a symplectic group or a split special orthogonal group, of rank n, or a split general spin group of rank $n+1$. The classical doubling method of Piatetski-Shapiro and Rallis [PSR87] produced an integral representation for the standard L-function of an irreducible cuspidal automorphic representation of a classical group twisted by a grössencharacter. In the recent work [CFGK19] their construction was extended to include twists by arbitrary cuspidal representations of $\mathrm{GL}_k(\mathbb{A})$, for all k. The purpose of this work is to develop the local theory of these integrals and characterize the local γ-factors. As a result, we can define local L- and ϵ-factors, then obtain the completed L-function and its functional equation.

Let F be a local field of characteristic 0 and ψ be a nontrivial additive character of F. Let π and τ be a pair of irreducible admissible representations, π of $G(F)$ and τ of $\mathrm{GL}_k(F)$, and assume τ is generic. Based on the recent uniqueness result of Gourevitch and the third named author [GK], the local doubling integral satisfies a functional equation with respect to an intertwining operator. Our main result concerns the γ-factor arising from this equation:

Theorem A. *There exists a γ-factor $\gamma(s,\pi\times\tau,\psi)$ which satisfies the fundamental list of properties of Shahidi [Sha90, Theorem 3.5].*

See Theorem 4.2. In the classical case $k=1$, the local theory was fully developed by Lapid and Rallis [LR05] (and Gan [Gan12] for the metaplectic group). We follow their formulation of the canonical properties of the γ-factor.

Using standard arguments we can now define local L- and ϵ-factors. In turn, in a global context let π and τ be cuspidal representations of $G(\mathbb{A})$ and $\mathrm{GL}_k(\mathbb{A})$, resp. (throughout, cuspidal representations are always automorphic and irreducible). We can define the completed L-function as the Euler product of the local L-functions. We summarize our global results Theorems 8.2, 8.3 and Corollary 8.5:

Theorem B. *The L-function $L(s,\pi\times\tau)$ admits meromorphic continuation to the plane and satisfies a standard functional equation $L(s,\pi\times\tau)=\epsilon(s,\pi\times\tau)L(1-s,\pi^\vee\times\tau^\vee)$. Moreover, if $L(s,\pi\times\tau)$ and $L(s,\pi^\vee\times\tau^\vee)$ are entire, they are bounded in vertical strips of finite width.*

Over the past few decades, local factors and γ-factors in particular have been a ubiquitous part of the Langlands Program. In the generic case the definitive theory was developed by Shahidi (e.g., [Sha90]) and the cornerstone of his theory was the existence and uniqueness of the Whittaker model. Because of this, it was considered difficult to envision similar results in the non-generic case. Among the few attempts to attack this problem, we mention the work of the second named author and Goldberg [FG99] and the doubling method itself, for $k=1$. While we can now define the local factors using the theory of Arthur, the trace formula does not provide us with any information on the poles. By contrast, the generalized doubling method can be used to study the poles of the local and global L-functions of $G \times \mathrm{GL}_k$, which are typically highly interesting. See e.g., the work of Yamana [Yam14] on the global theta lifting using the doubling integrals for $k = 1$.

To place our results in context we turn to the global setting and recall the global construction of the generalized doubling integral, following [CFGK19]. Let now F be a number field and $\mathbb{A} = \mathbb{A}_F$. Let G be the split group Sp_{2n}, SO_{2n} or SO_{2n+1} (minor modifications are needed for general spin groups; these are described below). Then $G(F)$ acts naturally on a c-dimensional F-vector space ($c = 2n$ or $2n + 1$). Denote the Borel subgroup of upper triangular invertible matrices in GL_{kc} by $B_{\mathrm{GL}_{kc}}$, let $P_{(k^c)} < \mathrm{GL}_{kc}$ be the standard parabolic subgroup corresponding to the partition $(k^c) = (k, \dots, k)$, and $K_{\mathrm{GL}_{kc}}$ be a maximal compact subgroup of $\mathrm{GL}_{kc}(\mathbb{A})$ (chosen as in, e.g., [MW95, § I.1.4]).

Let τ be a cuspidal representation of $\mathrm{GL}_k(\mathbb{A})$. Consider the generalized Speh representation $\mathcal{E}_\tau$ of Jacquet [Jac84], which is the residual representation attached to the Eisenstein series $E(g; \zeta, \xi)$ associated with a standard $K_{\mathrm{GL}_{kc}}$-finite section ξ of the induced representation $\mathrm{Ind}_{P_{(k^c)}(\mathbb{A})}^{\mathrm{GL}_{kc}(\mathbb{A})}(|\det|^{\zeta_1}\tau \otimes \ldots \otimes |\det|^{\zeta_c}\tau)$ at the point $((c-1)/2, (c-3)/2, \dots, (1-c)/2)$. The automorphic representation $\mathcal{E}_\tau$ is irreducible and when τ is unitary, belongs to the discrete spectrum of the space of square-integrable automorphic forms of $\mathrm{GL}_{kc}(\mathbb{A})$. Jiang and Liu [JL13] studied the Fourier coefficients of $\mathcal{E}_\tau$ (elaborating on [Gin06]). In particular, they proved that $\mathcal{E}_\tau$ admits a nonzero Fourier coefficient along the unipotent orbit attached to (k^c). Fix a nontrivial additive character ψ of $F\backslash\mathbb{A}$. Then they showed that for some automorphic form ϕ in the space of $\mathcal{E}_\tau$,

$$W_\psi(\phi) = \int\limits_{V_{(c^k)}(F)\backslash V_{(c^k)}(\mathbb{A})} \phi(v)\,\psi^{-1}\left(\mathrm{tr}\left(\sum_{i=1}^{k-1} v_{i,i+1}\right)\right) dv \neq 0. \tag{0.1}$$

Here $V_{(c^k)}$ is the unipotent radical of $P_{(c^k)}$ (note the interchange of c and k) and for $v \in V_{(c^k)}$, $v = (v_{i,j})_{1\le i,j\le k}$ where $v_{i,j}$ are $c \times c$ blocks. Call this Fourier coefficient a global (k, c) functional.

We define an auxiliary group H, on which we construct an Eisenstein series with inducing data $\mathcal{E}_\tau$. Let H be either Sp_{2kc} if G is symplectic or SO_{2kc} if G is orthogonal, and fix the Borel subgroup $B_H = H \cap B_{\mathrm{GL}_{2kc}}$. Take a standard maximal parabolic subgroup $P < H$ with a Levi part isomorphic to GL_{kc}. Define the Eisenstein series

$$E(h; s, f) = \sum_{\delta\in P(F)\backslash H(F)} f(s, \delta h), \qquad h \in H(\mathbb{A}), \tag{0.2}$$

where $s \in \mathbb{C}$ and f is a standard K_H-finite section of the representation $\mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(|\det|^{s-1/2}\mathcal{E}_\tau)$, regarded as a complex-valued function. This series converges absolutely for $\mathrm{Re}(s) \gg 0$ and has meromorphic continuation to $\mathbb{C}$.

We construct the following Fourier coefficient of $E(h; s, f)$. Let Q be a standard parabolic subgroup of H, whose Levi part M_Q is isomorphic to $\mathrm{GL}_c \times \ldots \times \mathrm{GL}_c \times H_0$, where GL_c appears $k-1$ times and $H_0 = \mathrm{Sp}_{2c}$ or SO_{2c}. Let $U = U_Q$ be the unipotent radical of Q. We define a character ψ_U of $U(\mathbb{A})$, which is trivial on $U(F)$, such that the direct product $G(\mathbb{A}) \times G(\mathbb{A})$ can be embedded in the stabilizer of ψ_U inside $M_Q(\mathbb{A})$.

Now let π be a unitary cuspidal representation of $G(\mathbb{A})$, and let φ_1 and φ_2 be two cusp forms in the space of π. The global integral is defined by

$$Z(s,\varphi_1,\varphi_2,f) = \int\limits_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} \varphi_1(g_1)\,\overline{{}^{\iota}\varphi_2(g_2)}\,E^{U,\psi_U}((g_1,g_2);s,f)\,dg_1\,dg_2, \tag{0.3}$$

where $g \mapsto {}^{\iota}g = \iota g \iota^{-1}$ is an involution of $G(\mathbb{A})$ and ${}^{\iota}\varphi_2(g_2) = \varphi_2({}^{\iota}g_2)$; (g_1, g_2) is the embedding of $G \times G$ in H; and

$$E^{U,\psi_U}(h;s,f) = \int\limits_{U(F)\backslash U(\mathbb{A})} E(uh;s,f)\psi_U(u)\,du \tag{0.4}$$

is the Fourier coefficient of E with respect to U and ψ_U. In particular for $k = 1$, $H_0 = H$ and U is trivial, and this recovers the doubling integral of Piatetski-Shapiro and Rallis [PSR87].

Integral (0.3) admits meromorphic continuation to $\mathbb{C}$, which is analytic except perhaps at the poles of the series. In a right half plane $Z(s,\varphi_1,\varphi_2,f)$ unfolds to an adelic integral:

$$\int\limits_{G(\mathbb{A})}\int\limits_{U_0(\mathbb{A})} \langle\varphi_1,\pi(g)\varphi_2\rangle f_{W_\psi(\mathcal{E}_\tau)}(s,\delta u_0(1,{}^{\iota}g))\,\psi_U(u_0)\,du_0\,dg. \tag{0.5}$$

Here U_0 is a subgroup of U; $\langle,\rangle$ is the standard inner product

$$\langle\varphi_1,\varphi_2\rangle = \int\limits_{G(F)\backslash G(\mathbb{A})} \varphi_1(g_0)\overline{\varphi_2(g_0)}\,dg_0; \tag{0.6}$$

$f_{W_\psi(\mathcal{E}_\tau)}$ is the composition of f with the Fourier coefficient (0.1); and $\delta \in G(F)$ is a representative of the open double coset $P\backslash H/(G \times G)U$. For additional details see § 2.

By [CFGoK, Theorem 4], for decomposable data (0.5) is Eulerian (in [CFGK19] we proved (0.5) is "almost Eulerian"). At almost all places ν of F, the local integral with unramified data equals $L(s,\pi_\nu \times \tau_\nu)/b(s,c,\tau_\nu)$, where $b(s,c,\tau_\nu)$ (a product of local L-functions) is the local component of the normalizing factor of (0.2). Consequently the integral (0.3) represents the partial L-function $L^S(s,\pi \times \tau)$, for a sufficiently large finite set S of places of F.

In [CFGK19] we treated Sp_{2n} and SO_{2n}. To extend the applicability of the doubling method, here we treat several other classes of groups, each of which follows the model of [CFGK19] but requires modifications. The first class is $G = \mathrm{SO}_{2n+1}$. Here the embedding of $G \times G$ in H is more involved, and several computations, most notably the calculation of the integrals with unramified data, are more difficult. The second class is $G = \mathrm{GL}_n$, which appeared briefly in [CFGK19] because it was needed for the induction step in the unramified calculation. In this case the global construction involves $\tau \otimes \tau^\vee$ instead of τ, and in (0.3) we divide the integration domain by the center of $H(\mathbb{A}) = \mathrm{GL}_{2kn}(\mathbb{A})$. The third class is the split general spin group $G = \mathrm{GSpin}_c$ (in hindsight, [PSR87, § 4.3] hinted at this). The group H is then GSpin_{2kc}. There are two main differences in the global construction. First, the inducing data of the series (0.2) is $\mathcal{E}_\tau \otimes \chi_\pi$, where χ_π is the restriction of the central character of π to the connected component $C_G^\circ(\mathbb{A})$ of the center of $G(\mathbb{A})$. Second, we divide the domain of integration of (0.3) by the two copies of $C_G^\circ(\mathbb{A})$. For more details see § 2.5.

We mention that the proof of the global unfolding which equates (0.3) and (0.5) in $\mathrm{Re}(s) \gg 0$ was only recently completed, in [GK, § 3.2] (a preliminary version was sketched in [CFGK19]).

Our main application of the local and global theory, which will appear in a follow-up to this work, is a new proof of global functoriality from $G(\mathbb{A})$ to the appropriate general linear group, using the Converse Theorem of Cogdell and Piatetski-Shapiro [CPS94, CPS99]. This result will extend the global result of [CKPSS01, CKPSS04, AS06] in the sense that it will be applicable to all cuspidal representations of $G(\mathbb{A})$, i.e., not only the globally generic ones. While global functoriality is now already included in the work of Arthur on the trace formula (e.g., [Art13]), our proof will be independent of the trace formula and its prerequisites.

The integrals described here have been recently used by Ginzburg and Soudry [GS21, GS22] in a global context, to construct the inverse image of the weak functorial lift from the classical group to the general linear group, via their method of global descent. A possible application of the local theory here would be to construct the local descent.

We expect the local and global theories developed here to have further applications, due to the role of the doubling method in a wide range of problems. We mention the studies of [KR94, HKS96, GS12, GI14, Yam14] on the theta correspondence, which is related to the doubling method by the Siegel–Weil formula; the works of [BS00, HLS05, HLS06, EHLS20] who used the doubling integrals for cohomological automorphic representations, in the context of p-adic L-functions; and also [Gar84, KR90, Tak97, Kim00].

The doubling method was originally developed for classical groups of symplectic, orthogonal or unitary type, including non-split cases [PSR87, LR05]. It was extended to the classical metaplectic group, i.e., the double cover of the symplectic group, by Gan [Gan12]. These cases, as well as unitary groups of hermitian or skew-hermitian forms over division algebras, were included in [Yam14]. In this work we deal with a subset of these groups, but also describe split general spin groups. We expect that our methods can be extended to the other cases studied, in particular quasi-split orthogonal groups, and to quasi-split general spin groups. As opposed to the aforementioned works, here we deal with connected groups. This is in line with the theories of Langlands and Shahidi, which were formulated for connected groups, and with several other works on Rankin–Selberg integrals.

For the extension of the generalized doubling method to arbitrary rank central extensions of the symplectic group see [Kapa, Kapb].

There are two appendices to this manuscript. Appendix A by Dmitry Gourevitch contains two results on families of representations depending on a complex parameter: an extension of the Dixmier–Malliavin Theorem ([DM78]), and a precise density result for smooth sections. Appendix B by the third named author contains the proof of a uniqueness result underlying the functional equation of the local intertwining operators.

1 Preliminaries

1.1 Groups and general notions. Let F be a local field of characteristic zero. If F is p-adic, $\mathcal{O}$ denotes its ring of integers, q is the cardinality of its residue field and ϖ is a uniformizer with $|\varpi| = q^{-1}$. When referring to unramified representations or data, we implicitly mean over p-adic fields. Throughout, linear algebraic groups will be defined and split over F, and for such a group H we usually identify $H = H(F)$. We fix a Borel subgroup $B_H = T_H \ltimes N_H$ where N_H is the unipotent radical, and for a standard parabolic subgroup P of H denote its Levi decomposition by $P = M_P \ltimes U_P$, with $U_P < N_H$. The modulus character of P is δ_P and the unipotent subgroup opposite to U_P is U_P^-. Also $W(H)$ denotes the Weyl group of H. When H is reductive, fix a maximal compact subgroup K_H in H which is the hyperspecial subgroup $H(\mathcal{O})$ for p-adic fields. The center of H is denoted C_H. For $x, y \in H$, ${}^x y = xyx^{-1}$, and if $Y < H$, ${}^x Y = \{{}^x y : y \in Y\}$.

Specifically for GL_l, B_{GL_l} is the subgroup of upper triangular invertible matrices, $P_\beta = M_\beta \ltimes V_\beta$ denotes the standard parabolic subgroup corresponding to a d parts composition $\beta = (\beta_1, \ldots, \beta_d)$ of l, and $V_\beta < N_{\mathrm{GL}_l}$. For $c \geq 0$, $\beta c = (\beta_1 c, \ldots, \beta_d c)$ is a composition of lc. Let $\mathrm{Mat}_{a\times b}$ be the space of $a \times b$ matrices and $\mathrm{Mat}_a = \mathrm{Mat}_{a\times a}$. Let w_β be the permutation matrix consisting of blocks of identity matrices $I_{\beta_1}, \ldots, I_{\beta_d}$, with $I_{\beta_i} \in \mathrm{Mat}_{\beta_i}$ on its anti-diagonal, beginning with I_{β_1} on the top right, then I_{β_2}, etc. In particular $J_l = w_{(1^l)}$, the permutation matrix with 1 on the anti-diagonal. We use τ_β to denote a representation of M_β, where $\tau_\beta = \otimes_{i=1}^d \tau_i$ (τ_i is then a representation of GL_{β_i}). The transpose of $g \in \mathrm{Mat}_{a\times b}$ is denoted ${}^t g$, and tr is the trace map. For $g \in \mathrm{GL}_l$, put $g^* = J_l {}^t g^{-1} J_l$. For a representation τ of GL_l which admits a central character, $\tau(a)$ denotes the value of the central character on aI_l.

Throughout, representations are assumed to be complex and smooth. Representations of reductive groups are in addition assumed to be admissible, and over archimedean fields they are also Fréchet of moderate growth. Induction is understood to be normalized and smooth.

For a representation π of a closed unipotent subgroup $U < H$ on a space $\mathcal{V}$, and a character ψ of U, the Jacquet module $J_{U,\psi}(\pi)$ is the quotient of $\mathcal{V}$ by the subspace spanned by $\{\pi(u)\xi - \psi(u)\xi : \xi \in \mathcal{V}, u \in U\}$ over non-archimedean fields, and by the closure of this subspace for archimedean fields. If $R < H$ is a closed subgroup containing U and π is a representation of R, the normalizer of U and stabilizer of ψ in R acts on $J_{U,\psi}(\pi)$, we normalize the action as in [BZ77, 1.8].

When the field is p-adic, an entire function $f(s) : \mathbb{C} \to \mathbb{C}$ will always be an element of $\mathbb{C}[q^{-s}, q^s]$, and a meromorphic function will belong to $\mathbb{C}(q^{-s})$ (so, meromorphic is actually rational). When a property holds outside a discrete subset of s, it means for all but finitely many values of q^{-s}. Similarly, $f(\zeta) : \mathbb{C}^k \to \mathbb{C}$ is entire (resp., meromorphic) if it belongs to $\mathbb{C}[q^{\mp\zeta_1}, \ldots, q^{\mp\zeta_k}]$ (resp., $\mathbb{C}(q^{-\zeta_1}, \ldots, q^{-\zeta_k})$), where $\zeta = (\zeta_1, \ldots, \zeta_k)$.

1.2 Representations of type (k, c). We briefly recall the results of [CFGoK] that will be needed throughout this work. Let k and c be positive integers. Fix a nontrivial additive character ψ of F and extend it to a generic character of $V_{(c^k)}$ by

$$\psi(v) = \psi\left(\sum_{i=1}^{k-1} \mathrm{tr}(v_{i,i+1})\right), \qquad v = (v_{i,j})_{1 \le i,j \le k}, \qquad v_{i,j} \in \mathrm{Mat}_c. \tag{1.1}$$

Let ρ be a finite length (and admissible) representation of GL_{kc}. We say that ρ is a (k, c) representation if (k^c) is the unique maximal orbit in its wave-front set and $\dim \mathrm{Hom}_{V_{(c^k)}}(\rho, \psi) = 1$. See [GK, § 1.4] and [CFGoK, § 2.1] for details and an equivalent definition in terms of orbits. E.g., ρ is $(k, 1)$ if it affords a unique Whittaker model, and $(1, c)$ representations are plainly characters of GL_c.

For a (k, c) representation ρ, its (k, c) model $W_\psi(\rho)$ is the space of functions $g \mapsto \lambda(\rho(g)\xi)$ where $g \in \mathrm{GL}_{kc}$ and ξ is a vector in the space of ρ, and $0 \ne \lambda \in \mathrm{Hom}_{V_{(c^k)}}(\rho, \psi)$ is fixed.

In [CFGoK, § 2.2], for an irreducible generic representation τ of GL_k we defined a (k, c) representation $\rho_c(\tau)$. If τ is unitary, $\rho_c(\tau)$ is the unique irreducible subrepresentation of

$$\mathrm{Ind}_{P_{(k^c)}}^{\mathrm{GL}_{kc}}((\tau \otimes \ldots \otimes \tau)\delta_{P_{(k^c)}}^{-1/(2k)}). \tag{1.2}$$

In general $\tau = \mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\otimes_{i=1}^d |\det|^{a_i}\tau_i)$ where β is a composition of d parts of k, $a_1 > \ldots > a_d$ and each τ_i is tempered, then $\rho_c(\tau) = \mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^d |\det|^{a_i}\rho_c(\tau_i))$. Note that $\rho_c(\tau)$ admits a central character.

For a representation ϱ of GL_l let $\varrho^*(g) = \varrho(g^*)$. If ϱ is irreducible, $\varrho^* \cong \varrho^\vee$. By [CFGoK, Claim 6], $\rho_c(\tau)^\vee = \rho_c(\tau^\vee)$ when τ is tempered, and $\rho_c(\tau)^* = \rho_c(\tau^\vee)$ in general. Let $g \mapsto g^\triangle$ denote the diagonal embedding of GL_c in GL_{kc}. By [CFGoK, Lemma 12]:

LEMMA 1.1. *Let $\lambda \in \mathrm{Hom}_{V_{(c^k)}}(\rho_c(\tau), \psi)$ and ξ be a vector in the space of $\rho_c(\tau)$. For any $g \in \mathrm{GL}_c$, $\lambda(\rho_c(\tau)(g^\triangle)\xi) = \tau(\det(g))\lambda(\xi)$.*

We will utilize the following realizations of (k,c) representations.

First assume $k>1$ and consider a representation $\mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\tau_\beta)$ where $\beta=(\beta_1,\ldots,\beta_l)$ is a nontrivial l parts composition and τ_β is an irreducible generic representation. Denote $\beta'=(\beta_l,\ldots,\beta_1)$. Consider the Jacquet integral

$$\int\limits_{V_{\beta' c}} \xi(w_{\beta c}v)\psi^{-1}(v)\,dv, \tag{1.3}$$

where ξ belongs to the space of $\mathrm{I}=\mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^{l}W_\psi(\rho_c(\tau_i)))$ and ψ is the restriction of (1.1) to $V_{\beta' c}$. Note that ξ can be regarded as a complex-valued function. As explained in [CFGoK, § 3.1] we can twist the inducing data of I by auxiliary complex parameters $\zeta\in\mathbb{C}^l$, then (1.3) becomes a meromorphic function which realizes the (k,c) model of each twisted representation I_ζ and in particular, if $\rho_c(\tau)$ is a quotient of I, of $\rho_c(\tau)$. Note that if F is archimedean, the analytic continuation and continuity of (1.3) are at present known only when $\beta=(1^k)$, but we can always assume this.

Second, assume $0<l<c$ and an unramified twist of τ is unitary. Fix $0<l<c$. Since now both $\rho_l(\tau)$ and $\rho_{c-l}(\tau)$ embed in the corresponding spaces (1.2),

$$\rho_c(\tau)\subset \mathrm{Ind}_{P_{(kl,k(c-l))}}^{\mathrm{GL}_{kc}}((W_\psi(\rho_l(\tau))\otimes W_\psi(\rho_{c-l}(\tau)))\delta_{P_{(kl,k(c-l))}}^{-1/(2k)}). \tag{1.4}$$

To construct the (k,c) functional we introduce the following notation. For $v\in V_{(c^k)}$ set $v_{i,j}=\left(\begin{smallmatrix} v_{i,j}^1 & v_{i,j}^2\\ v_{i,j}^3 & v_{i,j}^4\end{smallmatrix}\right)$, where $v_{i,j}^1\in\mathrm{Mat}_l$ and $v_{i,j}^4\in\mathrm{Mat}_{c-l}$. For $t\in\{1,\ldots,4\}$, let $V^t<V_{(c^k)}$ be the subgroup obtained by deleting the blocks $v_{i,j}^{t'}$ for all $i<j$ and $t'\neq t$, and $V=V^3$. Also define

$$\kappa=\kappa_{l,c-l}=\begin{pmatrix} I_l &&&&&&& \\ 0 & 0 & I_l &&&&& \\ 0 & 0 & 0 & 0 & I_l & \ddots && \\ &&&&&& I_l & 0 \\ 0 & I_{c-l} &&&&&& \\ 0 & 0 & 0 & I_{c-l} && \ddots && \\ &&&&&&& I_{c-l} \end{pmatrix}\in\mathrm{GL}_{kc}\,.$$

For ξ in the space of $\rho_c(\tau)$ under the embedding (1.4) (and regarded as a complex-valued function), consider the functional

$$\xi\mapsto\int\limits_{V}\xi(\kappa v)\,dv. \tag{1.5}$$

By [CFGoK, § 3.2, Lemma 9] this integral is absolutely convergent and realizes the (k,c) functional on $\rho_c(\tau)$.

2 The Integrals

We define the local integral with details for the different groups, starting with classical groups in § 2.1–2.4, then general spin groups in § 2.5.

2.1 Classical groups. Let G be either a split classical group of rank n, or GL_n. Fix a nontrivial additive character ψ of F. Given an integer k, introduce the following group H and auxiliary notation, used in the definition of the integral and the local factors below:

G	Sp_{2n}	SO_{2n}	SO_{2n+1}	GL_n
c	$2n$	$2n$	$2n+1$	n
H	Sp_{2kc}	SO_{2kc}	SO_{2kc}	GL_{2kc}
A	I_{2n}	$\begin{pmatrix} -I_n & \\ & I_n \end{pmatrix}$	$\begin{pmatrix} & -I_n & \\ & & I_n \\ 0 & & \end{pmatrix}$	I_n

Here Sp_{2n} is realized as the subgroup of $g \in \mathrm{GL}_{2n}$ such that ${}^tgJg = J$, where $J = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$ (${}^tg$ is the transpose of g and $J_n = w_{(1^n)}$); and SO_c consists of all $g \in \mathrm{SL}_c$ satisfying ${}^tgJ_cg = J_c$. Take $B_H = H \cap B_{\mathrm{GL}_{2kc}}$.

For an integer $l \geq 0$, put $\jmath_{2l} = I_2$ and $\jmath_{2l+1} = J_2$. For $m \geq 1$ and $h \in \mathrm{GL}_{2m}$, when we write ${}^{\jmath_l}h$ we identify $\jmath_l$ with $\mathrm{diag}(I_{m-1}, \jmath_l, I_{m-1})$. Also set $\epsilon_0 = -1$ for $G = \mathrm{Sp}_{2n}$ and $\epsilon_0 = 1$ otherwise.

2.2 The embedding: U,ψ_U and $G \times G$. Let $Q = M_Q \ltimes U_Q$ be the following standard parabolic subgroup of H: if $G \neq \mathrm{GL}_n$, $M_Q = \mathrm{GL}_c \times \ldots \times \mathrm{GL}_c \times H_0$ ($k-1$ copies of GL_c) and H_0 is of the type of H with rank c, and for $G = \mathrm{GL}_n$, $M_Q = M_{(c^{k-1},2c,c^{k-1})}$. Let $U = U_Q$. For $k > 1$, denote the middle $4c \times 4c$ block of an element in U by

$$\begin{pmatrix} I_c & u & v \\ & I_{2c} & u' \\ & & I_c \end{pmatrix}. \tag{2.1}$$

Denote by $u^{1,1} \in \mathrm{Mat}_n$ the top left block of u; let $u^{2,2} \in \mathrm{Mat}_n$ be the bottom right block of u if $G \neq \mathrm{GL}_n$, and for GL_n it denotes the top block of u'; for SO_{2n+1} also let $(u^3, u^4) \in \mathrm{Mat}_{1\times 2}$ be the middle two coordinates of row $n+1$ of u.

For $G \neq \mathrm{GL}_n$, regard $V_{(c^{k-1})}$ as a subgroup of U by embedding it in the top left block, and for $k > 1$, the character ψ_U restricts to (1.1) on $V_{(c^{k-1})}$. For $G = \mathrm{GL}_n$, there are two copies of $V_{(c^{k-1})}$, in the top left and bottom right blocks of U, and ψ_U restricts (for $k > 1$) to the inverse of (1.1) on each copy. The character ψ_U is given on (2.1) by

$$\begin{cases} \psi(\mathrm{tr}(-u^{1,1} + u^{2,2})) & G = \mathrm{GL}_n, \\ \psi(\mathrm{tr}(u^{1,1} + u^{2,2})) & G = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}, \\ \psi(\mathrm{tr}(u^{1,1} + u^{2,2}) + \epsilon_1 u^3 - \epsilon_2 u^4) & G = \mathrm{SO}_{2n+1}, \end{cases}$$

where $\epsilon_1 = 1$ if k is even and $\epsilon_1 = 1/2$ if k is odd, and $\epsilon_2 = \epsilon_1^{-1}/2$. For all $k \geq 1$ we describe the embedding (g_1, g_2) of $G \times G$ in M_Q, in the stabilizer of ψ_U:

$$(g_1, g_2) = \begin{cases} \mathrm{diag}(g_1, \ldots, g_1, \begin{pmatrix} g_{1,1} & & g_{1,2} \\ & g_2 & \\ g_{1,3} & & g_{1,4} \end{pmatrix}, g_1^*, \ldots, g_1^*) & G = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}, \\ \mathrm{diag}(g_1, \ldots, g_1, g_1, g_2, g_1, \ldots, g_1), & G = \mathrm{GL}_n, \end{cases}$$

where g_1^* appears $k-1$ times and is uniquely defined by g_1 and H; for $G \neq \mathrm{GL}_n$, $g_1 = \left(\begin{smallmatrix} g_{1,1} & g_{1,2} \\ g_{1,3} & g_{1,4} \end{smallmatrix}\right)$, $g_{1,i} \in \mathrm{Mat}_n$; and for GL_n, g_1 appears k times on the left of g_2 and $k-1$ on the right. When we write $(g_1, 1)$ or $(1, g_2)$, we use 1 to denote the identity element of G.

For SO_{2n+1} the embedding is defined as follows. Take column vectors $e_{\pm i}$, $1 \leq i \leq 2n+1$, whose Gram matrix is $J_{2(2n+1)}$ (i.e., ${}^t e_i e_{-j} = \delta_{i,j}$). Let

$$\begin{aligned} b &= (e_1, \ldots, e_{2n}, \epsilon_1 e_{2n+1} - \epsilon_2 e_{-2n-1}, \epsilon_1 e_{2n+1} + \epsilon_2 e_{-2n-1}, e_{-2n}, \ldots, e_{-1}), \\ b_1 &= (e_1, \ldots, e_n, \epsilon_1 e_{2n+1} - \epsilon_2 e_{-2n-1}, e_{-n}, \ldots, e_{-1}), \\ b_2 &= (e_{n+1}, \ldots, e_{2n}, \epsilon_1 e_{2n+1} + \epsilon_2 e_{-2n-1}, e_{-2n}, \ldots, e_{-n-1}), \\ m &= \mathrm{diag}(I_{c-1}, \left(\begin{smallmatrix} \epsilon_1 & \epsilon_1 \\ -\epsilon_2 & \epsilon_2 \end{smallmatrix}\right), I_{c-1}). \end{aligned}$$

The Gram matrices of (b, b_1, b_2) are $(J_{2(2n+1)}, \mathrm{diag}(I_n, -1, I_n) J_{2n+1}, J_{2n+1})$. Define the left copy of SO_{2n+1} using b_1, i.e., the group of matrices $g_1 \in \mathrm{SL}_{2n+1}$ such that

$${}^t g_1 \,\mathrm{diag}(I_n, -1, I_n) J_{2n+1} g_1 = \mathrm{diag}(I_n, -1, I_n) J_{2n+1},$$

and the right copy using b_2, which is our convention for SO_{2n+1}. For each i, extend g_i by letting it fix the vectors of b_{3-i}, then write this extension as a matrix $g_i' \in \mathrm{SO}_{2(2n+1)}$ with respect to b. Now ${}^m g_1'$ and ${}^m g_2'$ commute and

$$(g_1, g_2) = \mathrm{diag}(g_1, \ldots, g_1, {}^m g_1' \, {}^m g_2', g_1^*, \ldots, g_1^*).$$

We also mention that over archimedean fields, we can choose K_H such that $K_G \times K_G < K_H$ (under this embedding); over p-adic fields when c is even clearly $K_G \times K_G < K_H$ ($K_G = G(\mathcal{O})$, $K_H = H(\mathcal{O})$), and when c is odd this also holds assuming $|2| = 1$.

Example 2.1. Here are a few examples for the embedding in the odd orthogonal case. We assume $k = 2$ and n is arbitrary, but the only difference for other values of k would be in the number of copies of SO_{2n+1} above the middle $2c \times 2c$ block, because we keep ϵ_1 and ϵ_2 in the notation (we only assume $2\epsilon_1\epsilon_2 = 1$). We can write $u \in U$ in the form

$$\begin{aligned} u &= \begin{pmatrix} I_c & X & Y \\ & I_{2c} & vX' \\ & & I_c \end{pmatrix}, \\ X &= \begin{pmatrix} z_1 & b_1 & a_1 & a_2 & b_4 & z_4 \\ z_2 & b_2 & a_3 & a_4 & b_5 & z_5 \\ z_3 & b_3 & a_5 & a_6 & b_6 & z_6 \end{pmatrix}, \quad z_1, b_1, z_6 \in \mathrm{Mat}_n,\ a_3, a_4 \in \mathrm{Mat}_1, \end{aligned}$$

then $\psi_U(u) = \psi(\mathrm{tr}(z_1) + \epsilon_1 a_3 - \epsilon_2 a_4 + \mathrm{tr}(z_6))$. For the embedding of the left copy of G in H,

$$\left(\begin{pmatrix} a & & \\ & 1 & \\ & & a^* \end{pmatrix}, 1\right) = \mathrm{diag}\left(\begin{pmatrix} a & & \\ & 1 & \\ & & a^* \end{pmatrix}, \begin{pmatrix} a & & \\ & I_{2n+2} & \\ & & a^* \end{pmatrix}\right), \qquad a^* = J_n{}^t a^{-1} J_n,$$

$$\left(\begin{pmatrix} I_n & x & y \\ & 1 & x' \\ & & I_n \end{pmatrix}, 1\right)$$

$$= \mathrm{diag}\left(\begin{pmatrix} I_n & x & y \\ & 1 & x' \\ & & \\ & & I_n \end{pmatrix}, \begin{pmatrix} I_n & & \epsilon_2 x & -\epsilon_1 x & & y \\ & I_n & & & & \\ & & 1 & & & \epsilon_1 x' \\ & & & 1 & & -\epsilon_2 x' \\ & & & & I_n & \\ & & & & & I_n \end{pmatrix}\right),$$

$$\begin{matrix} x' = {}^t x J_n \\ {}^t y J_n + J_n y = J_n x {}^t x J_n \end{matrix}, \quad \left(\begin{pmatrix} I_{n-1} & & & & \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ & & & & I_{n-1} \end{pmatrix}, 1\right)$$

$$= \mathrm{diag}\left(\begin{pmatrix} I_{n-1} & & & & \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ & & & & I_{n-1} \end{pmatrix}, \right.$$

$$\left.\begin{pmatrix} I_{n-1} & & & & & & & \\ & & & & & & 1 & \\ & & I_n & & & & & \\ & & & & 2\epsilon_1^2 & & & \\ & & & 2\epsilon_2^2 & & & & \\ & & & & & I_n & & \\ & 1 & & & & & & \\ & & & & & & & I_{n-1} \end{pmatrix}\right).$$

Here and below we omitted the bottom right $c \times c$ block, because it is uniquely determined by the top left $c \times c$ block and H. For the right copy,

$$\left(1, \begin{pmatrix} a & & \\ & 1 & \\ & & a^* \end{pmatrix}\right) = \mathrm{diag}\left(I_c, \begin{pmatrix} I_n & & & & \\ & a & & & \\ & & I_2 & & \\ & & & a^* & \\ & & & & I_n \end{pmatrix}\right), \qquad a^* = J_n{}^t a^{-1} J_n,$$

$$\left(1, \begin{pmatrix} I_n & x & y \\ & 1 & x' \\ & & I_n \end{pmatrix}\right)$$

$$= \mathrm{diag}\left(I_c, \begin{pmatrix} I_n & & & & & \\ & I_n & \epsilon_2 x & \epsilon_1 x & y & \\ & & 1 & & \epsilon_1 x' & \\ & & & 1 & \epsilon_2 x' & \\ & & & & I_n & \\ & & & & & I_n \end{pmatrix}\right), \quad \begin{matrix} x' = -{}^t x J_n \\ {}^t y J_n + J_n y = -J_n x {}^t x J_n \end{matrix},$$

$$\left(1, \begin{pmatrix} I_{n-1} & & & & \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ & & & & I_{n-1} \end{pmatrix}\right) = \mathrm{diag}\left(I_c, \begin{pmatrix} I_{2n-1} & & & & & \\ & & & & 1 & \\ & & & -2\epsilon_1^2 & & \\ & & -2\epsilon_2^2 & & & \\ & 1 & & & & \\ & & & & & I_{2n-1} \end{pmatrix}\right).$$

2.3 Sections. We define the local spaces of sections that we use for the integral. These are the local analogs of the space on which we constructed the Eisenstein series (0.2). Let H be one of the groups given in § 2.1, and P be a standard maximal parabolic subgroup of H with $M_P = \mathrm{GL}_{kc}$, or $P = P_{((kc)^2)}$ when $H = \mathrm{GL}_{2kc}$. Let ρ be a finite length representation of M_P realized in a space of complex-valued functions. We assume $\rho = \rho_1 \otimes \rho_2$ if $P = P_{((kc)^2)}$. For a complex parameter s, let $V(s, \rho)$ be the space of $\mathrm{Ind}_P^H(|\det|^{s-1/2}\rho)$, or the space of $\mathrm{Ind}_P^H(|\det|^{s-1/2}\rho_1 \otimes |\det|^{-s+1/2}\rho_2)$ when $H = \mathrm{GL}_{2kc}$.

Extend the notation also to the case of $H = \mathrm{GL}_{kc}$ and $P = P_{\beta c}$ for an arbitrary composition $\beta = (\beta_1, \beta_2)$ of k, then $V(s,\rho)$ is still the space of the representation induced from $|\det|^{s-1/2}\rho_1 \otimes |\det|^{-s+1/2}\rho_2$. This space does not appear in the construction of the integral, but must be considered for multiplicativity arguments.

For $m \in M_P$, let a_m be the projection of m onto GL_{kc} if H is a classical group, otherwise $m = \mathrm{diag}(m_1, m_2)$ and $a_m = \mathrm{diag}(m_1, m_2^{-1})$. The elements of $V(s,\rho)$ are smooth functions ε on H, such that for all $h \in H$, $m \in M_P$ and $u \in U_P$, $\varepsilon(muh) = \delta_P^{1/2}(m)|\det a_m|^{s-1/2}\varepsilon(h)$, and the mapping $m \mapsto \delta_P^{-1/2}(m)|\det a_m|^{-s+1/2}\varepsilon(mh)$ belongs to the space of ρ. In particular $h \mapsto \varepsilon(h)$ is a complex-valued function, namely the evaluation of a function in the space of ρ at the identity. By virtue of the Iwasawa decomposition, the spaces $V(s,\rho)$ where s varies are all isomorphic as representations of K_H.

A function f on $\mathbb{C} \times H$ is called an entire section of $V(\rho)$ if for all $s \in \mathbb{C}$, $f(s,\cdot) \in V(s,\rho)$, and for each $h \in H$, the function $s \mapsto f(s,h)$ is entire. A standard section is then an entire section whose restriction to K_H is independent of s. A meromorphic section of $V(\rho)$ is a function f on $\mathbb{C} \times H$, such that for some entire function $\varphi : \mathbb{C} \to \mathbb{C}$ not identically zero, $\varphi(s)f(s,h)$ is an entire section (see e.g., [Yam14, § 3.1]). Away from the zeros of φ, $f(s,\cdot) \in V(s,\rho)$. The group H acts by right translations in the second parameter of sections, we denote this action by $h \cdot f$. Also if a group H' acts on H by conjugation, $f^{h'}$ is the section given by $f^{h'}(s,h) = f(s, {}^{h'}h)$.

Recall that when the field is p-adic, an entire section f satisfies, for all h, $s \mapsto f(s,h) \in \mathbb{C}[q^{-s}, q^s]$, and if f is meromorphic, $s \mapsto f(s,h) \in \mathbb{C}(q^{-s})$ (see § 1.1). If the representation ρ is unramified, the normalized unramified section of $V(\rho)$ is the unique element f such that $f(s,\cdot)$ is the normalized unramified vector for all s.

Over archimedean fields, f is called smooth if $f(s,\cdot)$ is smooth for all s (similarly for K_H-finite). If s is fixed, this is the usual notion of smooth or K_H-finite vectors of $V(s,\rho)$. For any smooth section f we can find a sequence of K_H-finite sections converging uniformly to f on each compact subset of $\mathbb{C}$. See [Cas89, Jac09] for the topological considerations, and also Appendix A. The Dixmier–Malliavin Theorem [DM78] can be applied separately to each $V(s,\rho)$, i.e., once s is fixed. Several arguments involving sections require us to treat s as a parameter (e.g., for the purpose of meromorphic continuation, or computations of integrals). The extension of [DM78] to this general setup is established in Appendix A, Theorem A.1 and Corollary A.3.

2.4 The integral. Let π be an irreducible representation of G. If $G \neq \mathrm{GL}_n$, let τ be an irreducible generic representation of GL_k, and P be the standard maximal parabolic subgroup of H such that $M_P = \{\left(\begin{smallmatrix} a & \\ & a^* \end{smallmatrix}\right) : a \in \mathrm{GL}_{kc}\}$. For GL_n, $\tau = \tau_0 \otimes \chi^{-1}\tau_0^\vee$ for an irreducible generic representation τ_0 of GL_k, a quasi-character χ of F^* (implicitly lifted to GL_k using det), and $P = P_{((kc)^2)}$.

Let ω be a matrix coefficient of $\pi^\vee$ and f be a meromorphic section of $V(W_\psi(\rho_c(\tau)))$, where for GL_n, $W_\psi(\rho_c(\tau)) = W_\psi(\rho_c(\tau_0)) \otimes \chi^{-1} W_\psi(\rho_c(\tau_0^\vee))$. The local integral takes the form

$$Z(s,\omega,f) = \int\limits_{G}\int\limits_{U_0} \omega(g) f(s, \delta u_0(1, {}^{\iota}g))\, \psi_U(u_0)\, du_0\, dg.$$

Here $\delta = \delta_0\delta_1$,

$$\delta_0 = \begin{pmatrix} & I_{kc} \\ \epsilon_0 I_{kc} & \end{pmatrix} \quad (G \neq \mathrm{SO}_{2n+1}), \qquad \delta_1 = \begin{pmatrix} I_{(k-1)c} & & & \\ & I_c & A & \\ & & I_c & \\ & & & I_{(k-1)c} \end{pmatrix},$$

$${}^{\jmath_{kc}}U_0 = \left\{ \begin{pmatrix} I_{(k-1)c} & & X & Z \\ & I_c & & Y \\ & & I_c & \\ & & & I_{(k-1)c} \end{pmatrix} \right\}$$

and for $G = \mathrm{SO}_{2n+1}$,

$$\delta_0 = \begin{pmatrix} & I_{kc} \\ I_{kc} & \end{pmatrix} \mathrm{diag}\left(I_{(k-1)c}, \begin{pmatrix} I_n & & \\ & & (-1)^k \\ & I_n & \end{pmatrix}, \begin{pmatrix} & I_n & \\ (-1)^k & & \\ & & I_n \end{pmatrix}, I_{(k-1)c}\right)\jmath_{kc}; \qquad (2.2)$$

$$\iota = \begin{cases} \begin{pmatrix} & I_n \\ -\epsilon_0 I_n & \end{pmatrix} & G = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}, \\ I_n & G = \mathrm{GL}_n, \\ \begin{pmatrix} & & I_n \\ & I_2 & \\ I_n & & \end{pmatrix} & G = \mathrm{SO}_{2n+1}, \text{odd } k, \\ \begin{pmatrix} & & & I_n \\ & & -2\epsilon_1^2 & \\ & -2\epsilon_2^2 & & \\ I_n & & & \end{pmatrix} & G = \mathrm{SO}_{2n+1}, \text{even } k; \end{cases}$$

for Sp_{2n} and SO_{2n}, $g \mapsto {}^{\iota}g (= \iota g \iota^{-1})$ is an involution; for SO_{2n+1}, note that $\iota = {}^{m}\iota_0{}'$ with

$$\iota_0 = \begin{pmatrix} & & I_n \\ & (-1)^{k+1} & \\ I_n & & \end{pmatrix} \in \mathrm{O}_{2n+1}$$

(written with respect to the basis b_2) and ${}^{\iota}g = {}^{m}({}^{\iota_0}g)'$ ($g \mapsto {}^{\iota_0}g$ is an inner or outer involution of G, depending on the parities of k and n); and when we write the middle block of $u_0 \in U_0$ as in (2.1),

$$\psi_U(u_0) = \begin{cases} \psi(\mathrm{tr}(u^{2,2})) & G \neq \mathrm{SO}_{2n+1}, \\ \psi(\mathrm{tr}(u^{2,2}) - \frac{1}{2}u^4) & G = \mathrm{SO}_{2n+1}, \text{even } k, \\ \psi(\mathrm{tr}(u^{2,2}) + \frac{1}{2}u^3) & G = \mathrm{SO}_{2n+1}, \text{odd } k. \end{cases}$$

When $G = \mathrm{SO}_{2n+1}$ and k is odd we need a similar version of the integral above, when the section belongs to the representation induced from ${}^{\jmath_{kc}}P$ and ${}^{\jmath_{kc}}W_\psi(\rho_c(\tau))$.

This is because when we apply an intertwining operator to the section, the Weyl element $\left(\begin{smallmatrix} & I_{kc} \\ I_{kc} & \end{smallmatrix}\right)$ is not in H. We still denote the integral by $Z(s,\omega,f)$, but the notation changes as follows:

$$\begin{aligned}
\delta_0 &= \jmath_{kc}\left(\begin{smallmatrix} & I_{kc} \\ I_{kc} & \end{smallmatrix}\right)\operatorname{diag}\left(I_{(k-1)c}, \begin{pmatrix} I_n & & \\ & & 2 \\ & I_n & \end{pmatrix}, \begin{pmatrix} & I_n & \\ 1/2 & & \\ & & I_n \end{pmatrix}, I_{(k-1)c}\right),\\
\delta_1 &= \begin{pmatrix} I_{(k-1)c} & & & \\ & I_c & A & \\ & & I_c & \\ & & & I_{(k-1)c} \end{pmatrix}\operatorname{diag}(I_{(k-1)c+n}, -I_n, I_2, -I_n, I_{(k-1)c+n}), \qquad (2.3)\\
U_0 &= \left\{\begin{pmatrix} I_{(k-1)c} & & X & Z \\ & I_c & & Y \\ & & I_c & \\ & & & I_{(k-1)c} \end{pmatrix}\right\},\\
\psi_U(u_0) &= \psi(\operatorname{tr}(u^{2,2}) - u^4) \qquad \text{(the middle block of } u_0 \text{ is given by (2.1))}.
\end{aligned}$$

The integrals are absolutely convergent in a right half plane, which for entire sections depends only on the representations (see Proposition 2.5 below). Over p-adic fields they can be made constant (Proposition 2.6), and over archimedean fields they can be made nonvanishing and finite in a neighborhood of a given s (Proposition 2.6, and Corollary 6.9 with a K_H-finite section). Furthermore, they admit meromorphic continuation: over p-adic fields this continuation belongs to $\mathbb{C}(q^{-s})$ (see § 4), over archimedean fields the continuation is continuous in the input data—more precisely continuous as a trilinear map (see § 6.10). For similar assertions in the literature see, e.g., [GJ72, JPSS83, GPSR87, JS90, KR90, BG92, Sou93, Sou95, GRS98, LR05, RS05, Jac09, Kap13b, Kap13c, FK19].

We explain how to obtain the form of the local integral from the global. The local integral is defined once we prove that (0.3) unfolds to (0.5). We describe this procedure briefly, complete details for all groups can be obtained from the comprehensive local treatment in [GK], see [GK, § 3.2] (see also [CFGK19, § 2.3] for Sp_{2n}).

Assume $G \neq \mathrm{GL}_n$. The global integral defined by (0.3) is

$$Z(s,\varphi_1,\varphi_2,f) = \int\limits_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} \varphi_1(g_1)\,\overline{{}^{\iota}\varphi_2(g_2)}\,E^{U,\psi_U}((g_1,g_2);s,f)\,dg_1\,dg_2,$$

with the notation of the introduction. For $\mathrm{Re}(s) \gg 0$, after unfolding the Eisenstein series (0.2), $Z(s,\varphi_1,\varphi_2,f) = \sum_\gamma \mathrm{I}(\gamma)$ where $\gamma \in H(F)$ varies over the representatives of $P\backslash H/L$, $L = (G\times G)U$, and

$$\mathrm{I}(\gamma) = \int\limits_{L_\gamma(F)\backslash L(\mathbb{A})} \varphi_1(g_1)\,\overline{{}^{\iota}\varphi_2(g_2)}\,f(s,\gamma u(g_1,g_2))\,\psi_U(u)\,du\,dg_1\,dg_2.$$

Here $L_\gamma = {}^{\gamma^{-1}}P \cap L$. All but one summand vanish. This is proved by arguments utilizing the character ψ_U, the cuspidality of π or the "smallness" of the representation $\mathcal{E}_\tau$ in the inducing data of the series, namely its Fourier coefficients attached to orbits greater than or not comparable with (k^c) vanish (it is globally (k,c)). For the remaining summand $\mathrm{I}(\gamma)$, we see that $L_\gamma \cap U_P$ is trivial, $V_{(c^k)} < {}^{\gamma}L_\gamma$ and if we factor through $L_\gamma(\mathbb{A})$, the integration over $V_{(c^k)}(F)\backslash V_{(c^k)}(\mathbb{A})$ together with ψ_U form a global (k,c) functional, namely a Fourier coefficient along $V_{(c^k)}$ and a character in the orbit of (1.1). We can modify γ using left multiplication by an element of $M_P(F)$, to obtain the character (1.1) and coefficient (0.1). Denote the new representative by δ. We then see that the reductive part of L_δ is $\{(g_0, {}^{\iota}g_0) : g_0 \in G\}$ and ${}^{\delta}L_\delta = G \ltimes V_{(c^k)} < M_P$, where G is embedded in the stabilizer of (1.1), in the diagonal embedding of SL_c. In [CFGK19, Claim 8] we proved that (0.1) is invariant with respect to the diagonal embedding of $\mathrm{SL}_c(\mathbb{A})$, which implies in particular invariance under $G(\mathbb{A})$. Therefore $\mathrm{I}(\delta)$ becomes

$$\int\limits_{G^{\triangle}(\mathbb{A})\backslash G(\mathbb{A})\times G(\mathbb{A})} \int\limits_{U_0(\mathbb{A})} \int\limits_{G(F)\backslash G(\mathbb{A})} \varphi_1(g_0g_1)\overline{{}^{\iota}\varphi_2({}^{\iota}g_0g_2)} f_{W_\psi(\mathcal{E}_\tau)}(s, \delta u_0(g_1,g_2)) \times \psi_U(u_0)\, dg_0\, du_0\, dg_1\, dg_2. \tag{2.4}$$

Here $G^{\triangle}$ is the diagonal embedding in $G \times G$, $U_0 = {}^{\jmath_{kc}}U_P \cap U$, and $f_{W(\mathcal{E}_\tau)}$ is the composition of the section with (0.1): for any $s \in \mathbb{C}$ and $h \in H(\mathbb{A})$,

$$f_{W_\psi(\mathcal{E}_\tau)}(s,h) = \int\limits_{V_{(c^k)}(F)\backslash V_{(c^k)}(\mathbb{A})} f(s, vh)\, \psi^{-1}\left(\mathrm{tr}\left(\sum_{i=1}^{k-1} v_{i,i+1}\right)\right) dv. \tag{2.5}$$

It remains to apply ${}^{\iota}$ to g_2 and use (0.6) to obtain (0.5).

Returning to the local context, at a place ν of F, $\rho_c(\tau_\nu)$ is the local component of $\mathcal{E}_\tau$ (τ is now global) and for a decomposable f, $f = \prod_\nu f_\nu$ where for all ν, f_ν is a section of $V(W_{\psi_\nu}(\rho_c(\tau_\nu)))$, which is normalized and unramified for almost all ν.

For $G = \mathrm{GL}_n$ the definition of (0.3) is modified to handle the center. We take φ_1 in the space of $\chi^k\pi$ where χ is a continuous character of $F^*\backslash\mathbb{A}^*$, φ_2 in the space of $\pi^\vee$ (if π is unitary, $\pi^\vee = \overline{\pi}$) and the representation of $M_P(\mathbb{A})$ is $|\det|^{s-1/2}\mathcal{E}_{\tau_0} \otimes |\det|^{-s+1/2}\chi^{-1}\mathcal{E}_{\tau_0^\vee}$, where τ_0 is a cuspidal representation of $\mathrm{GL}_k(\mathbb{A})$. The modified version of (0.3) is given by

$$Z(s,\varphi_1,\varphi_2,f,\varrho_l) = \int\limits_{(C_H(\mathbb{A})G(F)\times G(F))\backslash(G(\mathbb{A})\times G(\mathbb{A}))} \varphi_1(g_1)\varphi_2(g_2) E^{U,\psi_U}((g_1,g_2);s,f)\varrho_l(|\det(g_2g_1^{-1})|)\, dg_1\, dg_2,$$

where ϱ_l is a compactly supported Schwartz function on $\mathbb{R}^*_{>0}$, introduced to ensure convergence as in [PSR87, § 4.2] (they used this to extend their construction from PGL_n to GL_n). Note that the integrand is indeed invariant under $C_H(\mathbb{A})$.

In the unfolding of $Z(s,\varphi_1,\varphi_2,f,\varrho_l)$ we obtain two (k,c) functionals: one on $\mathcal{E}_{\tau_0}$, the other on $\chi^{-1}\mathcal{E}_{\tau_0^\vee}$. By Lemma 1.1, at any place ν, the local (k,c) functional on

$\rho_c((\tau_0)_\nu)$ transforms under $G^{\triangle}(F_\nu)$ with respect to $g^{\triangle} \mapsto (\tau_0)_\nu(\det(g))$. Using this we reach an integral similar to (2.4),

$$\int\limits_{G^{\triangle}(\mathbb{A})\backslash G(\mathbb{A})\times G(\mathbb{A})} \int\limits_{U_0(\mathbb{A})} \int\limits_{C_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} \chi^k(\det g_1)\varphi_1(g_0g_1)\varphi_2(g_0g_2)$$
$$f_{W_\psi(\mathcal{E}_\tau)}(s, \delta u_0(g_1,g_2))\varrho_l(|\det(g_2g_1^{-1})|)\,\psi_U(u_0)\,dg_0\,du_0\,dg_1\,dg_2$$
$$= \int\limits_{G(\mathbb{A})} \int\limits_{U_0(\mathbb{A})} \langle\varphi_1, \pi(g)\varphi_2\rangle f_{W_\psi(\mathcal{E}_\tau)}(s, \delta u_0(1,g))\varrho_l(|\det g|)\,\psi_U(u_0)\,du_0\,dg. \tag{2.6}$$

Note that $C_H < G^{\triangle}$, $W_\psi(\mathcal{E}_\tau)$ is defined to be $W_\psi(\mathcal{E}_{\tau_0}) \otimes \chi^{-1}W_\psi(\mathcal{E}_{\tau_0^\vee})$, and in the integral defining the inner product the domain is divided by $C_G(\mathbb{A})$. As explained in [PSR87, § 4.2], the convergence of (2.6) for $\mathrm{Re}(s) \gg 0$ is independent of ϱ_l, and an application of the Monotone Convergence Theorem implies we can define, for $\mathrm{Re}(s) \gg 0$,

$$Z(s,\varphi_1,\varphi_2,f) = \lim_{l\to\infty} Z(s,\varphi_1,\varphi_2,f,\varrho_l)$$
$$= \int\limits_{G(\mathbb{A})} \int\limits_{U_0(\mathbb{A})} \langle\varphi_1, \pi(g)\varphi_2\rangle f_{W_\psi(\mathcal{E}_\tau)}(s, \delta u_0(1,g))\,\psi_U(u_0)\,du\,dg.$$

Here $\{\varrho_l\}_l$ is an arbitrary monotonic increasing sequence such that $\varrho_l \to 1$ (the limit of the integrals is independent of the choice of sequence), but is not used for the local integral.

The reason for introducing the character χ is to study the GL_n integral arising from the integral for general spin groups, then χ will essentially be the central character of the representation of GSpin_c.

2.5 Split general spin groups. For any integer $c \geq 2$, the group Spin_c is the simple split simply connected algebraic group of type D_n if c is even, or B_n if it is odd, where $n = \lfloor c/2 \rfloor$. It is also the algebraic double cover of SO_c. We fix the Borel subgroup $B_{\mathrm{Spin}_c} < \mathrm{Spin}_c$ to be the preimage of B_{SO_c}. Denote the pullback of the i-th coordinate function of T_{SO_c} to T_{Spin_c} by ϵ_i, $0 \leq i \leq n-1$. Then define $\epsilon_j^\vee$ such that $\langle \epsilon_i, \epsilon_j^\vee \rangle = \delta_{i,j}$, where $\langle , \rangle$ is the standard pairing. The set of simple roots of Spin_c is $\Delta_c = \{\alpha_0, \ldots, \alpha_{n-1}\}$, where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ if $0 \leq i < n-1$, $\alpha_{n-1} = \epsilon_{n-2} + \epsilon_{n-1}$ for even c, $\alpha_{n-1} = \epsilon_{n-1}$ otherwise. For convenience, we include the cases $c = 0, 1$ in the notation, then $n = 0$ and Spin_c is the trivial group.

Identify the split general spin group $G = \mathrm{GSpin}_c$ with the Levi subgroup of Spin_{c+2} obtained by removing α_0 from Δ_{c+2}. In particular, this fixes a Borel subgroup B_G. Note that Spin_c is the derived group of GSpin_c, $\mathrm{GSpin}_0 = \mathrm{GSpin}_1 = \mathrm{GL}_1$ and $\mathrm{GSpin}_2 = \mathrm{GL}_1 \times \mathrm{GL}_1$. Define a "canonical" character Υ of GSpin_c as the lift of

$-\epsilon_0$ (see [Kap17, § 1.2]). Let

$$C_G^\circ=\{\mathfrak{r}_c^\vee(t):t\in F^*\},\qquad \mathfrak{r}_c=\begin{cases}\alpha_0 & c=0,\\ \alpha_0+\alpha_1 & c=2,\\ 2\sum_{i=0}^{n-2}\alpha_i+\alpha_{n-1}+\alpha_n & c=2n>2,\\ 2\sum_{i=0}^{n-1}\alpha_i+\alpha_n & c=2n+1, n\geq 0.\end{cases}$$

For odd c or $c=0$, $C_G^\circ=C_G$; for even $c>2$, C_G° is the connected component of C_G,

$$C_G=C_G^\circ\coprod \mathfrak{i}_G C_G^\circ,\qquad \mathfrak{i}_G=\prod_{i=0}^{n-2}\alpha_i^\vee((-1)^{n-2-i})\alpha_{n-1}^\vee(1)\alpha_n^\vee(-1).$$

We use this definition of $\mathfrak{i}_G$ also for $c=2$, and for $c=0$ put $\mathfrak{i}_G=-1$. For the computation of C_G and in particular $\mathfrak{i}_G$, note that a general element $t\in T_G$ can be written uniquely in the form $t=\prod_{i=0}^n\alpha_i^\vee(t_i)$, then $t\in C_G$ if and only if $\prod_{i=0}^n t^{\langle\alpha_j,\alpha_i^\vee\rangle}=1$ for all $1\leq j\leq n$ ($\mathfrak{i}_G$ is $e_0^*(-1)\zeta_0$ in the notation of [AS06, Proposition 2.3]; for even n, $\mathfrak{r}_c^\vee(-1)\mathfrak{i}_G$ is the image of z of [AS06, Remark 2.4] in Spin_{c+2}).

For a detailed definition of general spin groups using based root datum refer to [Asg02, AS06, HS16]. We work directly with the coroots of Spin_{c+2} to describe torus elements of GSpin_c (in those works the coroots of GSpin_c were used). See also [Mat09].

Let $R=R_{l,c}<G$ be a standard parabolic subgroup, obtained by removing one of the roots α_l, $1\leq l\leq n$. The Levi part M_R is isomorphic to $\mathrm{GL}_l\times\mathrm{GSpin}_{c-2l}$. We describe an isomorphism explicitly. First assume c is odd or $l<n-1$. The derived group of GL_l is the group generated by the root subgroups of $\alpha_1,\ldots,\alpha_{l-1}$ (if $l>1$, otherwise it is trivial) and if $\theta_i^\vee(t)=\mathrm{diag}(I_{i-1},t,I_{l-i})$ is the i-th standard coordinate of T_{GL_l}, $\theta_i^\vee\mapsto\epsilon_i^\vee-\epsilon_0^\vee$. It follows that $\Upsilon|_{\mathrm{GL}_l}=\det$. The copy of Spin_{c-2l+2} is identified with the roots $\sum_{i=0}^l\alpha_i,\alpha_{l+1},\ldots,\alpha_n$, then GSpin_{c-2l} is obtained by removing $\sum_{i=0}^l\alpha_i$. Under this embedding, the first coordinate map of $T_{\mathrm{Spin}_{c-2l+2}}$ is mapped to ϵ_0 and Υ restricts to the same character on GSpin_{c-2l}. In the remaining cases c is even, define GL_l as above and identify Spin_{c-2l+2} with $\sum_{i=0}^{n-1}\alpha_i,\sum_{i=0}^{n-2}\alpha_i+\alpha_n$ when $l=n-1$, and GSpin_0 with $\mathfrak{r}_c^\vee$ for $l=n$. Then $\Upsilon|_{\mathrm{GSpin}_{c-2l}}=\det^{-1}$ when $l=n-1$ or $l=n=1$, and $\det^{-2}$ if $l=n>1$. Under this identification, in all cases $C_G^\circ=C_{\mathrm{GSpin}_{c-2l}}^\circ$ and if c is even, $\mathfrak{i}_G=[-I_l,\mathfrak{i}_{\mathrm{GSpin}_{c-2l}}]\in\mathrm{GL}_l\times\mathrm{GSpin}_{c-2l}$. The image of $[\prod_{i=1}^l\theta_i^\vee(t_i),\prod_{i=0}^{n-l}\beta_i^\vee(x_i)]\in T_{\mathrm{GL}_l}\times T_{\mathrm{GSpin}_{c-2l}}$ ($\beta_i^\vee$ - the cocharacters of

Spin_{c-2l+2}) in M_R is

$$\begin{cases} \prod_{i=0}^{n-2}\alpha_i^\vee(\prod_{j=i+1}^{n-1}t_j^{-1})\prod_{i=0}^{n-2}\alpha_i^\vee(x_0x_1)\alpha_{n-1}^\vee(x_0)\alpha_n^\vee(x_1) & \text{even } c,\ l=n-1,\\ \prod_{i=0}^{n-1}\alpha_i^\vee(\prod_{j=i+1}^{n}t_j^{-1})\prod_{i=0}^{n-2}\alpha_i^\vee(x_0^2)\alpha_{n-1}^\vee(x_0)\alpha_n^\vee(x_0) & \text{even } c,\ l=n,\\ \prod_{i=0}^{n-1}\alpha_i^\vee(x_0^2\prod_{j=i+1}^{l}t_j^{-1})\alpha_n^\vee(x_0) & \text{odd } c,\ l=n,\\ \prod_{i=0}^{l-1}\alpha_i^\vee(\prod_{j=i+1}^{l}t_j^{-1})\prod_{i=0}^{l}\alpha_i^\vee(x_0)\prod_{i=l+1}^{n}\alpha_i^\vee(x_{i-l}) & \text{otherwise.} \end{cases}$$

When considering $a\in\mathrm{GL}_n$ as an element of G, we implicitly use the identification above of GL_n with a direct factor of $R_{n,c}$. The same applies to $t\in T_{\mathrm{SO}_c}$, since $t=\mathrm{diag}(a,a^*)$ or $\mathrm{diag}(a,1,a^*)$. The Weyl group $W(G)$ of G is canonically isomorphic to $W(\mathrm{SO}_c)$. Given a permutation matrix $w_0\in\mathrm{SO}_c$, the preimage of w_0 in Spin_c consists of 2 elements, which differ by an element in C_{Spin_c}. Choosing one representative w, we then regard it as an element in G. In this manner we identify each w_0 with $w\in H$ (this is not a homomorphism). To compute the action of $W(G)$ on T_G we appeal to the formulas from [HS16] (our choice of representatives eliminates the need for the implicit inner automorphisms in [HS16]).

Let k be given. Define $H=\mathrm{GSpin}_{2kc}$. If $kc\leq 1$, $G=\mathrm{GL}_1$ and we already constructed this integral, so assume $kc>1$. Since the unipotent subgroups of H are isomorphic to those of SO_{2kc} (as algebraic groups), we can define the data (A,U,ψ_U) exactly as we did above, for the corresponding orthogonal group.

We turn to the embedding of the two copies of G in the stabilizer of ψ_U in M_Q. This stabilizer contains two commuting copies of G, but they intersect in C_H° (it cannot contain the direct product $G\times G$, e.g., for $k=1$ and $c=2n$, the rank of H is $2n+1$ but the rank of $G\times G$ is $2n+2$). Adapting the convention (g_1,g_2), we describe the mapping $(,):G\times G\to H$, which is an embedding in each of the variables separately, and also injective on the product of derived groups. We have a left copy and a right copy.

Starting with the derived groups, the embedding described above for the orthogonal groups extends to an embedding of the direct product of derived groups, since it identifies each root subgroup of a copy of G with a unipotent subgroup in H. Also identify the first coordinate map of the left copy with $-\epsilon_0$, and the right copy with ϵ_0. This completes the definition for $c\neq 2$. If $c=2$, regard GSpin_2 as $M_{R_{1,2}}$, then since we already identified the first coordinate map of each copy (with $\mp\epsilon_0$), it remains to embed the GL_1 part of each copy, which is done using the embedding $\mathrm{SO}_2\times\mathrm{SO}_2<T_{\mathrm{SO}_{4k}}$. Observe that for even c, the right copy of G is the natural subgroup of $M_{R_{kc-c/2,2kc}}$.

To deduce that both copies of G are subgroups of M_Q which fix ψ_U, it remains to consider the image of C_G°. The definition implies that if $z \in C_G^{\circ}$, $(z,1) = (1, z^{-1}) \in C_H^{\circ}$. Hence $(G,1)$ and $(1,G)$ belong to the stabilizer of ψ_U in M_Q. Moreover,

$$(G,1) \cap (1,G) = (C_G^{\circ},1) \cap (1, C_G^{\circ}) = C_H^{\circ},$$

and (z,z) is the identity element.

For the global construction, π is cuspidal (unitary or not), φ_1 and φ_2 are cusp forms in the spaces of π and $\pi^{\vee}$, and in the integration domain of (0.3) we replace $G(F)\backslash G(\mathbb{A})$ with $C_G^{\circ}(\mathbb{A})G(F)\backslash G(\mathbb{A})$ (on both copies). Put $P = R_{kc,2kc} < H$. Since $C_G^{\circ} = \mathrm{GL}_1$, the restriction of π to C_G° is a continuous character χ_π of $F^*\backslash\mathbb{A}^*$. The inducing data for the Eisenstein series is the representation $|\det|^{s-1/2}\mathcal{E}_\tau \otimes \chi_\pi$ of $M_P(\mathbb{A})$. Since φ_1 (resp., φ_2) transforms under C_G° by χ_π (resp., χ_π^{-1}) and $\chi_\pi((z_1,z_2)) = \chi_\pi^{-1}(z_1)\chi_\pi(z_2)$ for all $z_1, z_2 \in C_G^{\circ}$,

$$\varphi_1(z_1g_1)\varphi_2(z_2g_2)E^{U,\psi_U}((z_1g_1, z_2g_2); s, f) = \varphi_1(g_1)\varphi_2(g_2)E^{U,\psi_U}((g_1,g_2); s, f).$$

Thus the global integral is well defined. There are no additional convergence issues, because we divided by the centers (analogous to the case of PGL_n in [PSR87]). The unfolding process is carried out as in the orthogonal cases: the choice of representatives of $P\backslash H/(G,G)U$ is similar because these are either Weyl group elements or unipotent elements. Then the arguments showing $\mathrm{I}(\gamma) = 0$ are the same, since they only involve unipotent subgroups. For details see [GK].

The remaining summand is $\mathrm{I}(\delta)$. First note that for $c \geq 3$, ι lifts uniquely to an involution of Spin_c, because Spin_c is the universal cover of SO_c. When $c = 2$ (ι was defined for $c \geq 2$), ι can be replaced with $\jmath_1$, then ι acts on Spin_c by conjugation, when regarded as an element of the algebraic double cover Pin_c of O_c. Since in all cases ι fixes C_{SO_c}, it also fixes C_{Spin_c}, hence can be extended to an involution of G which fixes C_G. Now we may compute the reductive part of L_δ, and it is again the group $\{(g_0, {}^{\iota}g_0) : g_0 \in G\}$: in terms of unipotent subgroups and the torus element $[t,1]$ with $t \in \mathrm{GL}_n$, the computation is similar to the computation for the orthogonal group, and C_G° is embedded in C_H°.

To compute ${}^{\iota}g$ we may regard ι (for odd c) or $\left(\begin{smallmatrix} & I_n \\ I_n & \end{smallmatrix}\right)$ (even c), as a Weyl element of a higher rank special orthogonal group (non-uniquely), thereby a Weyl element in the corresponding spin group. When c is even, $\iota = d\left(\begin{smallmatrix} & I_n \\ I_n & \end{smallmatrix}\right)$ with $d = \mathrm{diag}(I_n, -I_n)$. The element d acts trivially on the Levi subgroup $\{\left(\begin{smallmatrix} a & \\ & a^* \end{smallmatrix}\right) : a \in \mathrm{GL}_n\}$ of SO_c, hence on its preimage in Spin_c. Since d belongs to the similitude group GSO_c, it also acts trivially on T_G, and therefore can be ignored when computing ${}^{\iota}M_{R_{n,c}}$. If $g = [a,x] \in M_{R_{n,c}}$ with $a \in \mathrm{GL}_n$ and $x \in \mathrm{GL}_1$, for $c > 2$ we have ${}^{\iota}g = \mathfrak{r}_c^{\vee}(\det a^{-1})[a^*, x] = [a^*, x\det a^{-1}] \in M_{R_{n,c}}$ and for $c = 2$, ${}^{\iota}g = [a^*, xa^{-2}]$.

For $g_0 \in G$, denote $\mathrm{e}(g_0) = {}^{\delta}(g_0, {}^{\iota}g_0) \in M_P$. We show the (k,c) functional transforms under $\mathrm{e}(G(\mathbb{A}))$ with respect to the trivial character. We start with the torus. For $g_0 \in T_G$, write $g_0 = [t_0, x_0]$ with $t_0 \in T_{\mathrm{GL}_n}$ and $x_0 \in \mathrm{GL}_1$. The image of

x_0 in G is then $\mathfrak{r}_c^\vee(x_0)$. We see that

$$(g_0,1)=\mathfrak{r}_{2kc}^\vee(x_0^{-1}\det t_0)[\mathrm{diag}(\mathrm{diag}(t_0,t_0^*),\ldots,\mathrm{diag}(t_0,t_0^*)\,\mathrm{diag}(t_0,I_{\lceil c/2\rceil})),1] \quad (2.7)$$

($k-1$ copies of $\mathrm{diag}(t_0,t_0^*)$), and note that $\mathfrak{r}_{2kc}^\vee(\det t_0)$ appears because the leftmost coordinate map of the left copy of G is mapped to $-\epsilon_0$. Then

$$\mathrm{e}(g_0)={}^\delta\left([t_0,x_0],[t_0^*,x_0\det t_0^{-1}]\right)={}^\delta[\mathrm{diag}(\mathrm{diag}(t_0,t_0^*),\ldots,\mathrm{diag}(t_0,t_0^*)),1],$$

and since $\det(\mathrm{diag}(t_0,t_0^*))=1$, the last conjugation belongs to the diagonal embedding of SL_c in GL_{kc}. (For example, when c is even $\mathfrak{i}_G=[-I_n,-1]$, ${}^\iota\mathfrak{i}_G=[-I_n,(-1)^{n+1}]$ and ${}^\delta(\mathfrak{i}_G,{}^\iota\mathfrak{i}_G)={}^\delta[-I_{kc},1]=[-I_{kc},1]$, by a direct verification.) Thus for $g_0\in T_G(\mathbb{A})$ the (k,c) functional transforms under $\mathrm{e}(g_0)$ with respect to the trivial character. Regarding $\mathrm{Spin}_c(\mathbb{A})$, it suffices to check that $\mathrm{e}(\mathrm{Spin}_c)<\mathrm{SL}_{kc}$, i.e., the projection of $\mathrm{e}(\mathrm{Spin}_c)$ on the GL_1 part of M_P is trivial. This follows because otherwise we would obtain a nontrivial character of Spin_c, which is perfect. In more detail, put $\mathrm{e}(x)=[\ell(x),\zeta_x]\in M_P$, where $\ell(x)\in\mathrm{SL}_{kc}$. Since $x\in\mathrm{Spin}_c$, ζ_x belongs to the projection C' of $C_{\mathrm{Spin}_{2kc}}$ into the GL_1 part of M_P, and we claim ζ_x is identically 1. Suppose otherwise. The structure of $C_{\mathrm{Spin}_{2kc}}$ depends on the parity of kc, but C' is a nontrivial finite abelian group, namely $\mathrm{r}_{2kc}^\vee(-1)\in C'$ ($\mathrm{r}_{2kc}^\vee(-1)$ is c of [Asg02, Proposition 2.2]). Then since e is a homomorphism,

$$[\ell(xy),\zeta_{xy}]=\mathrm{e}(xy)=\mathrm{e}(x)\mathrm{e}(y)=[\ell(x),\zeta_x][\ell(y),\zeta_y]=[\ell(xy),\zeta_x\zeta_y].$$

Hence $x\mapsto\zeta_x$ is a homomorphism and composing it with a character of C' we obtain a nontrivial character of Spin_c, which is a contradiction.

We conclude that the du-integral in $\mathrm{I}(\delta)$ is invariant under the reductive part of $L_\delta(\mathbb{A})$. Factor $\mathrm{I}(\delta)$ through $G(\mathbb{A})$. The dg_0-integral in (2.4) becomes

$$\int\limits_{C_G^\circ(\mathbb{A})G(F)\backslash G(\mathbb{A})}\varphi_1(g_0g_1)\,{}^\iota\varphi_2({}^\iota g_0g_2)\,dg_0=\langle\pi(g_1)\varphi_1,\pi({}^\iota g_2)\varphi_2\rangle \quad (2.8)$$

and the global integral analogous to (0.5) is

$$\int\limits_{C_G^\circ(\mathbb{A})\backslash G(\mathbb{A})}\int\limits_{U_0(\mathbb{A})}\langle\varphi_1,\pi(g)\varphi_2\rangle f_{W_\psi(\mathcal{E}_\tau)\otimes\chi_\pi}(s,\delta u_0(1,{}^\iota g))\,\psi_U(u_0)\,du_0\,dg.$$

The definition of a local space $V(s,\rho)$ from § 2.3 changes, taking into account the fact that $M_P=\mathrm{GL}_{kc}\times\mathrm{GL}_1$. Now $\rho=\rho_1\otimes\rho_2$, where ρ_1 is a representation of GL_{kc}, and $V(s,\rho)$ is the space of $\mathrm{Ind}_P^H(|\det|^{s-1/2}\rho_1\otimes\rho_2)$. The only changes to the local integral are that f is a meromorphic section of $V(W_\psi(\rho_c(\tau))\otimes\chi_\pi)$, where χ_π is the restriction of the central character of π to C_G° regarded as a character of F^*, and the domain is $C_G^\circ\backslash G$ (δ and ι are defined as explained above, e.g., $\delta_0\in H$ is obtained from the matrix in SO_{2kc}).

2.6 Basic properties of the integrals. First we establish two formal properties of the integrals, which can be regarded as immediate consequences of the global construction, then turn to prove convergence and show that the integrals can be made nonzero.

Consider the space

$$\mathrm{Hom}_{(G,G)}(J_{U,\psi_U^{-1}}(V(s,W_\psi(\rho_c(\tau))\otimes\chi_\pi)),\pi^\vee\otimes\pi^\iota). \tag{2.9}$$

Here $J_{U,\psi_U^{-1}}(\cdots)$ is considered as a representation of (G,G); χ_π is omitted unless $G=\mathrm{GSpin}_c$; π^ι is the representation of G acting on the same space as π, where the action is defined by $\pi^\iota(g)=\pi({}^\iota g)$; and when $G=\mathrm{GL}_n$, $\pi^\vee\otimes\pi^\iota$ is replaced with $(\chi^k\pi)^\vee\otimes\pi$ (for GL_n, $\iota=I_c$).

Proposition 2.2. *The integral can be regarded, at least formally, as a morphism in* (2.9).

Proof. Given ω, by definition there are vectors φ and $\varphi^\vee$ in the spaces of π and $\pi^\vee$, such that $\omega(g)=\omega_{\varphi,\varphi^\vee}(g)=\varphi^\vee(\pi(g^{-1})\varphi)$ for $g\in G$. Regarding the integral as a trilinear form on

$$V(s,W_\psi(\rho_c(\tau))\otimes\chi_\pi)\times(\chi^k\pi)\times(\pi^\iota)^\vee,$$

where $\chi=1$ unless $G=\mathrm{GL}_n$, we can show the equivalent statement

$$Z(s,\omega_{\chi^k\pi(g_1)\varphi,(\pi^\iota)^\vee(g_2)\varphi^\vee},(g_1,g_2)u\cdot f)=\psi_U^{-1}(u)Z(s,\omega,f),\qquad\forall g_1,g_2\in G,\quad u\in U.$$

It is straightforward to show the equivariance property for u, using the definition of the embedding and $W_\psi(\rho_c(\tau))$. Regarding g_1 and g_2, since

$$\begin{aligned}\omega_{\chi^k\pi(g_1)\varphi,(\pi^\iota)^\vee(g_2)\varphi^\vee}(g)&=\chi^k(\det g_1)(\pi^\iota)^\vee(g_2)\varphi^\vee(\pi(g^{-1})\pi(g_1)\varphi)\\&=\chi^k(\det g_1)\varphi^\vee(\pi({}^\iota g_2^{-1}g^{-1}g_1)\varphi)=\chi^k(\det g_1)\omega(g_1^{-1}g({}^\iota g_2)),\end{aligned}$$

$$\begin{aligned}&Z(s,\omega_{\chi^k\pi(g_1)\varphi,(\pi^\iota)^\vee(g_2)\varphi^\vee},(g_1,g_2)\cdot f)\\&=\chi^k(\det g_1)\int\limits_G\int\limits_{U_0}\omega(g_1^{-1}g({}^\iota g_2))f(s,\delta u_0(1,{}^\iota g)(g_1,g_2))\,\psi_U(u_0)\,du_0\,dg.\end{aligned}$$

Changing variables $g\mapsto g_1g({}^\iota g_2)^{-1}$, we obtain

$$\chi^k(\det g_1)\int\limits_G\int\limits_{U_0}\omega(g)f(s,\delta u_0(g_1,{}^\iota g_1)(1,{}^\iota g))\,\psi_U(u_0)\,du_0\,dg.$$

It remains to conjugate $(g_1,{}^\iota g_1)$ to the left. Note that ${}^\delta(g_1,{}^\iota g_1)\in P$ and by Lemma 1.1, $f(s,{}^\delta(g_1,{}^\iota g_1)h)=\chi^{-k}(\det g_1)f(s,h)$ for any $h\in H$ (for $G=\mathrm{GL}_n$ see the definition of $W_\psi(\rho_c(\tau))$). Also ${}^{(g_1,{}^\iota g_1)^{-1}}u_0\in U$ and when we change variables in u_0 and use ψ_U and the equivariance properties of $W_\psi(\rho_c(\tau))$ on $U\cap M_P$, we obtain $Z(s,\omega,f)$. $\square$

As a corollary of the computation, we have the following result:

COROLLARY 2.3. *For any section f of $V(W_\psi(\rho_c(\tau)) \otimes \chi_\pi)$, $g_0 \in G$ and $h \in H$,*

$$\int\limits_{U_0} f(s, \delta u_0(g_0, {}^\iota g_0)h)\, \psi_U(u_0)\, du_0 = \chi^{-k}(\det g_0) \int\limits_{U_0} f(s, \delta u_0 h)\, \psi_U(u_0)\, du_0.$$

REMARK 2.4. Note that (2.9) was slightly different in the work of [PSR87] for $k = 1$ (see [LR05, (10)]). This difference is caused by a different choice of embedding for $G \times G$ in H. E.g., the local integral of [PSR87, LR05] does not contain δ; and the global and local invariance with respect to $(g, {}^\iota g)$ was with respect to (g, g) in *loc. cit.*

PROPOSITION 2.5. *The integrals with entire sections are absolutely convergent in a right half plane depending only on the representations. Over archimedean fields, in the domain of absolute convergence they are continuous in the input data (as trilinear forms, see (2.9)).*

Proof. For $k = 1$ this was already proved in [LR05, Theorem 3], albeit for O_c instead of SO_c, and GSpin_c was not included. The only ingredient in their proof which is not straightforward to extend to SO_c and GSpin_c is the multiplicative property [LR05, Proposition 2], but we prove this here in § 5.3.1–§ 5.3.4 (the proofs apply in particular when $k = 1$).

Assume $k \geq 1$ but if $G = \mathrm{GL}_n$, $k > 1$. We can prove the stronger statement,

$$\int\limits_{G} \int\limits_{U_P} |\omega(g) f(s, \delta_0 u(1, {}^\iota g))|\, du\, dg < \infty. \tag{2.10}$$

If $G = \mathrm{GSpin}_c$, the domain G is replaced by $C_G^\circ \backslash G$. (For $G = \mathrm{GL}_n$ and $k = 1$ (2.10) does not hold, the element δ_1 is used.) Assume F is p-adic. We may assume that ω is bi-K_G-invariant and f is right K_H-invariant, because we may introduce auxiliary integrations over K_G and K_H. Using Corollary 2.3, the integral (2.10) reduces to an integral over the cone T_G^-, which is the subset of $t \in T_G$ such that $|\alpha(t)| \leq 1$ for all the simple roots α of T_G. We need to bound

$$\sum_{t \in (C_G^\circ (T_G \cap K_G)) \backslash T_G^-} \int\limits_{U_P} |\omega(t) f(s, \delta_0 t u)| m(t)\, du.$$

Here and below C_G° is omitted unless $G = \mathrm{GSpin}_c$, and m is a modulus character multiplied by $|\chi^{-k}(\det t)|$ if $G = \mathrm{GL}_n$. If W is a function in $W_\psi(\rho_c(\tau))$ and $k > 1$, its restriction to torus elements of the form $\mathrm{diag}(t, I_{(k-1)c})$ can be bounded using a gauge ξ (see [Sou93, § 2] for the definition and method of proof, and also [Cas80b, § 6]). Here in particular ξ vanishes unless all coordinates of t are small, and also note that ξ is non-negative. If $k=1$, $\xi(t)$ is taken to be a power of $|\det t|$. Then as in [Sou93, § 4], for each t, the integral over U_P is bounded by

$$\xi(t)|\det t|^{\mathrm{Re}(s)d+d'}\int\limits_{U_P}|f(s,\delta_0 u)|\,du,$$

where d and d' are constants depending only on τ and H, and $d>0$. This integral is finite for $\mathrm{Re}(s)\gg 0$, as an integral defining an intertwining operator, and we are left with

$$\sum_{t\in(C_G^{\circ}(T_G\cap K_G))\backslash T_G^-}|\omega(t)|\ \xi(t)\,|\det t|^{\mathrm{Re}(s)d+d'}.$$

We can bound the matrix coefficient on T_G^- using the exponents of $\pi^\vee$, and since the coordinates of t are small, this integral is finite for $\mathrm{Re}(s)\gg 0$, depending only on $\pi^\vee$ and τ. Over archimedean fields the proof is similar, one uses the bound from [Wal92, Theorem 15.2.4] (see also [FK19, Theorem 1.1] for an asymptotic expansion of matrix coefficients), and [Sou93, § 3 and § 5]. Continuity in the domain of convergence can be shown as in [Sou95, § 6, Lemma 1]. □

Proposition 2.6. *Assume F is p-adic. There is a choice of data (ω,f) where f is an entire section, such that $Z(s,\omega,f)$ is absolutely convergent and equals 1, for all s. Over an archimedean field, for any given s, there is data (ω,f) where f is a smooth entire section (but not K_H-finite), such that the integral is absolutely convergent and nonzero.*

Proof. The proof is similar to [Sou93, § 6], [GRS98, Proposition 6.6], [Kap13a, Lemma 4.1]. For the doubling method with $k=1$ this was proved in [RS05, p. 298] (p-adic fields) and [KR90, Theorem 3.2.2] (archimedean fields), for Sp_{2n} and O_c; the arguments can be easily adapted to SO_c, and GSpin_c is proved along the same lines. At any rate assume $k\geq 1$.

Consider the p-adic case first. Briefly, let $\mathcal{N}$ be a small compact open neighborhood of the identity in H, which is normalized by ${}^{\delta_0}\delta_1$. Take an entire section f which is right-invariant by $\mathcal{N}$, and such that $\delta_0\cdot f$ is supported in $P({}^{\delta_0}\delta_1)\mathcal{N}$. Using Corollary 2.3 (or directly when $G=\mathrm{GL}_n$) we obtain

$$\int\limits_{U_0}f(s,\delta u_0(1,{}^{\iota}g))\psi_U(u_0)\,du_0=\int\limits_{U_0}\delta_0\cdot f(s,p(g)\ {}^{\delta_0}(u_0\delta_g))\psi_U(u_0)\,du_0,$$

where $p(g)\in P$ and δ_g is obtained from δ_1 by multiplying the block A in δ_1 by coordinates of g. Moreover, when ${}^{\delta_0}(u_0\delta_g)\in P({}^{\delta_0}\delta_1)\mathcal{N}$, g varies in a small compact open subgroup of G, f is left invariant by $p(g)$ (e.g., $\delta_P(p(g))=1$) and the coordinates of u_0 are small. For a sufficiently small $\mathcal{N}$, with respect to ω and ψ_U, the integral reduces to a nonzero measure constant multiplied by $\omega(1)$, and thus can be chosen to be nonzero, independently of s. The argument on the support also implies absolute convergence.

Over archimedean fields we can define an entire section f such that $\delta_0 \cdot f$ is supported in PU_P^- and $\delta_0 \cdot f(s, mu) = \phi(u)\delta_0 \cdot f(s, m)$ for $u \in U_P^-$ and $m \in M_P$, where ϕ is a Schwartz function on U_P^-. Choosing ϕ with compact support near ${}^{\delta_0}\delta_1$ we obtain

$$\int_{U_0} \phi'(u_0)\psi_U(u_0)\, du_0 \int_G \omega(g)\delta_0 \cdot f(s, p(g))\phi''(g)\, dg,$$

where ϕ' and ϕ'' are Schwartz functions on U_0 and G, obtained from ϕ. The support of ϕ' (resp., ϕ'') can be taken arbitrarily small (resp., near the identity of G). Thus the du_0-integral can be made nonzero, and for a given s, the dg-integral can also be made nonzero (even in a small neighborhood of s). □

3 The Normalized Intertwining Operator

We define the intertwining operators that we apply to the spaces of sections defined in § 2.3, and introduce their normalized versions, to be used for the definition of the γ-factor in § 4. Let k and c be integers. Let H, τ and P be given by § 2 (see § 2.1, § 2.4 and § 2.5), or $H = \mathrm{GL}_{kc}$, $P = P_{\beta c}$ for a 2 parts composition β of k, and $\tau = \tau_\beta$ is irreducible and generic. Consider the intertwining operators

$$\begin{aligned} &M(s, W_\psi(\rho_c(\tau)), w_P) : V(s, W_\psi(\rho_c(\tau))) \to V(1-s, W_\psi(\rho_c(\tau'))), \\ &M(1-s, W_\psi(\rho_c(\tau')), w_{P'}) : V(1-s, W_\psi(\rho_c(\tau'))) \to V(s, W_\psi(\rho_c(\tau))). \end{aligned} \tag{3.1}$$

Here w_P, τ', P', and $w_{P'}$ are given as follows.

(1) For a classical group H, $P' = {}^{\jmath_{kc}}P$, $w_P = \jmath_{kc} \left(\begin{smallmatrix} & \epsilon_0 I_{kc} \\ I_{kc} & \end{smallmatrix} \right) \mathrm{d}_{k,c}$ where $\mathrm{d}_{k,c} \in T_{\mathrm{GL}_{kc}}$ is the matrix $\mathrm{diag}(-I_c, I_c, \ldots, (-1)^k I_c)$ regarded as an element in M_P, $\tau' = \tau^\vee$, and $w_{P'} = {}^{\jmath_{kc}}w_P$. The representation on $V(1-s, W_\psi(\rho_c(\tau')))$ is induced from P' and ${}^{\jmath_{kc}}(|\det|^{1/2-s} W_\psi(\rho_c(\tau')))$.
The image of $M(s, W_\psi(\rho_c(\tau)), w_P)$ is a priori contained in $V(1-s, W_\psi(\rho_c(\tau))^*)$ ($g^* = J_{kc}{}^t g^{-1} J_{kc}$). Since the application of the intertwining operator commutes with the application of the (k, c) functional, we may assume the intertwining operator is into $V(1-s, W_\psi(\rho_c(\tau)^*))$, and then by [CFGoK, Claim 6], $\rho_c(\tau)^* = \rho_c(\tau^\vee)$.

(2) For $H = \mathrm{GSpin}_{2kc}$, the representation $W_\psi(\rho_c(\tau))$ is twisted by a quasi-character χ to form a representation of M_P. Then $M(s, W_\psi(\rho_c(\tau)) \otimes \chi, w_P)$ is into $V(1-s, W_\psi(\rho_c(\tau')) \otimes \chi)$ with $\tau' = \chi^{-1}\tau^\vee$. The remaining definitions are similar to SO_{2kc}, but w_P is a representative in H.

(3) For $H = \mathrm{GL}_{kc}$, $\beta' = (\beta_2, \beta_1)$, $P' = P_{\beta' c}$, $w_P = w_{\beta' c}$, $\tau' = \tau_2 \otimes \tau_1$ and $w_{P'} = w_{\beta c}$. Further denote $W_\psi(\rho_c(\tau_\beta)) = W_\psi(\rho_c(\tau_1)) \otimes W_\psi(\rho_c(\tau_2))$. In particular for $H = \mathrm{GL}_{2kc}$ and $\beta = (k^2)$, we usually take $\tau = \tau_0 \otimes \chi^{-1}\tau_0^\vee$ and then $\tau' = \chi^{-1}\tau^\vee$. If $\beta \neq (k^2)$, also set $\delta_0 = w_P^{-1}$.

Remark 3.1. The purpose of $\mathrm{d}_{k,c}$ is to preserve the character ψ in the model, i.e., for $\tau^\vee$ we still use the (k,c) model with respect to ψ (instead of ψ^{-1}). Globally, we then use (0.1) with the same character on both sides of the global functional equation. This is simpler because we may then keep the same representative δ in the unfolding argument. Note that this symmetry breaks down when kc is odd, as explained above; see (2.3).

To avoid burdensome notation, we exclude general spin groups until the end of the section. For a meromorphic section f of $V(W_\psi(\rho_c(\tau)))$ (induction from P), the operator $M(s,W_\psi(\rho_c(\tau)),w_P)$ is defined for $\mathrm{Re}(s)\gg 0$ by the absolutely convergent integral

$$M(s,W_\psi(\rho_c(\tau)),w_P)f(s,h)=\int\limits_{U_{P'}} f(s,w_P^{-1}uh)\,du, \tag{3.2}$$

then by meromorphic continuation to $\mathbb{C}$. By definition, when $M(s,W_\psi(\rho_c(\tau)),w_P)$ is holomorphic

$$M(s,W_\psi(\rho_c(\tau)),w_P):V(s,W_\psi(\rho_c(\tau)))\to V(1-s,W_\psi(\rho_c(\tau'))).$$

The picture is similar for $M(1-s,W_\psi(\rho_c(\tau')),w_{P'})$.

We further define

$$\lambda(s,c,\tau,\psi)f=\int_{U_{P'}} f(s,\delta_0 u)\psi^{-1}(u)\,du. \tag{3.3}$$

Here if $H=\mathrm{SO}_{2kc}$ and c is odd, δ_0 is given by (2.2). The character ψ is defined as follows. If H is a classical group, ψ is the character of $U_{P'}$ given by ${}^{\jmath_{kc}}\left(\begin{smallmatrix} I_{kc} & u\\ & I_{kc}\end{smallmatrix}\right)\mapsto \psi(\mathrm{tr}({}^tAx))$, where x is the bottom left $c\times c$ block of u (${}^tA=A$ unless kc is odd). In this case we also put $Y_{k,c}={}^{\jmath_{kc}}V_{(c^k)}\ltimes U_{P'}$ and define a character $\psi_{k,c}$ of $Y_{k,c}$ by taking the product of characters (1.1) (of ${}^{\jmath_{kc}}V_{(c^k)}$) and ψ of $U_{P'}$. For GL_{kc} ($k=\beta_1+\beta_2$), ψ is the character (1.1) of $V_{(c^k)}$ restricted to $U_{P'}$, $Y_{k,c}=V_{(c^k)}$ and $\psi_{k,c}$ is again (1.1).

The integral defining $\lambda(s,c,\tau,\psi)$ is absolutely convergent for $\mathrm{Re}(s)\gg 0$ (similarly to an intertwining operator), and can be made nonzero for a given s. Over p-adic fields, there is an entire section f such that for all s, $\lambda(s,c,\tau,\psi)$ is absolutely convergent and equals a constant (independent of s).

Theorem 3.2. *For all s, the space $\mathrm{Hom}_{Y_{k,c}}(V(s,\rho_c(\tau)),\psi_{k,c})$ is at most one dimensional.*

Proof. For $H=\mathrm{GL}_{kc}$ the dimension is precisely 1, this follows from [CFGoK, Proposition 2]. The proof for $H\neq\mathrm{GL}_{kc}$ for all $k\geq 1$ appears in Appendix B. We note that the case of $k=1$ and non-archimedean fields was already proved by Karel [Kar79] (see also [Wal88, LR05]). □

Corollary 3.3. *The functional $\lambda(s,c,\tau,\psi)$ admits meromorphic continuation, which is continuous in f over archimedean fields.*

Proof. For p-adic fields this follows from Bernstein's continuation principle (in [Ban98]), since we have uniqueness by Theorem 3.2 and the integral can be made constant. Over archimedean fields we prove this using a multiplicativity argument in § 6.10 below. □

By virtue of Theorem 3.2 and its corollary, there is a meromorphic function $C(s, c, \tau, \psi)$ satisfying the following functional equation for all f: if kc is even,

$$\lambda(s, c, \tau, \psi)f = C(s, c, \tau, \psi)\lambda(1 - s, c, \tau', \psi)M(s, W_\psi(\rho_c(\tau)), w_P)f. \tag{3.4}$$

For odd kc (i.e., $H = \mathrm{SO}_{2kc}$ and both k and c are odd), we modify this equation by replacing $M(s, W_\psi(\rho_c(\tau)), w_P)f$ with $(t_0 \cdot M(s, W_\psi(\rho_c(\tau)), w_P)f)^{\jmath_{kc}}$, $t_0 = \mathrm{diag}(I_{kc-1}, -2, -1/2, I_{kc-1})$, and note that on both sides of the equation λ is defined with δ_0 given by (2.2).

Equation (3.4) depends on the choice of measures on $U_{P'}$, but we may choose the measures for λ on both sides in the same way, and then $C(s, c, \tau, \psi)$ depends only on the measure chosen for the intertwining operator. Specifically, let $d_\psi x$ be the additive measure of F which is self-dual with respect to ψ. When H is a classical group, each root subgroup of $U_{P'}$ is identified with F by choosing the nontrivial coordinate above or on the anti-diagonal (the identification is clear when $H = \mathrm{GL}_{kc}$). The measure on $U_{P'}$ is then the product of measures $d_\psi x$ over each of these root subgroups. This measure is chosen for all integrations over subgroups of $U_{P'}$. Changing ψ affects the measure.

Following (3.4) we define the normalized version of the intertwining operator,

$$M^*(s, c, \tau, \psi) = C(s, c, \tau, \psi)M(s, W_\psi(\rho_c(\tau)), w_P). \tag{3.5}$$

Outside a discrete subset of s, the product

$$M(1 - s, W_\psi(\rho_c(\tau')), w_{P'})M(s, W_\psi(\rho_c(\tau)), w_P)$$

is a scalar, because by the construction of $\rho_c(\tau)$ (see § 1.2), we can use the multiplicative properties of intertwining operators to write $M(s, W_\psi(\rho_c(\tau)), w_P)$ as a product of operators on spaces $V(s, \rho)$ with irreducible generic representations ρ. Therefore

$$M^*(1 - s, c, \tau', \psi)M^*(s, c, \tau, \psi) = 1. \tag{3.6}$$

We fix notation for certain products of L-functions, which appear below in the normalizing factors of the intertwining operators (and globally, Eisenstein series).

Put

$$a_0(s,c,\tau) = \prod_{j=1}^{\lfloor c/2 \rfloor} L(2s-c+2j-1,\tau,\vee^2) \prod_{j=1}^{\lceil c/2 \rceil} L(2s-c+2j-2,\tau,\wedge^2),$$

$$b_0(s,c,\tau) = \begin{cases} \prod_{j=1}^{c/2} L(2s+2j-2,\tau,\vee^2)L(2s+2j-1,\tau,\wedge^2) & \text{even } c, \\ \prod_{j=1}^{\lfloor c/2 \rfloor} L(2s+2j-1,\tau,\vee^2) \prod_{j=1}^{\lceil c/2 \rceil} L(2s+2j-2,\tau,\wedge^2) & \text{odd } c. \end{cases}$$

Also for $H = \mathrm{GL}_{2kc}$ and $\beta = (k^2)$, set $\tau = \tau_1 \otimes \tau_2$. Define:

H	$a(s,c,\tau)$	$b(s,c,\tau)$
Sp_{2kc}	$L(s-c/2,\tau)a_0(s,c,\tau)$	$L(s+c/2,\tau)b_0(s,c,\tau)$
SO_{2kc}	$a_0(s,c,\tau)$	$b_0(s,c,\tau)$
GL_{2kc}	$\prod_{1\leq j\leq c} L(2s+j-c-1,\tau_1\times{\tau_2}^\vee)$	$\prod_{1\leq j\leq c} L(2s+j-1,\tau_1\times{\tau_2}^\vee)$

These L-functions were defined by Shahidi [Sha90] for any (generic) τ, although we only use the definition for unramified representations. Note that for $k=1$, $a(s,c,\tau)$ and $b(s,c,\tau)$ are the functions given in [Yam14, § 3.5] ((s,c) here corresponds to $(s-1/2,n')$ in *loc. cit.*).

The computations in [CFGK19, Lemmas 27 and 33] show that if τ and ψ are unramified and f_τ (resp., $f_{\tau'}$) is the normalized unramified section of $V(W_\psi(\rho_c(\tau)))$ (resp., $V(W_\psi(\rho_c(\tau'))))$,

$$M(s,W_\psi(\rho_c(\tau)),w_P)f_\tau = a(s,c,\tau)b(s,c,\tau)^{-1}f_{\tau'}. \quad (3.7)$$

(This also holds for GL_{kc} and any $\beta=(\beta_1,\beta_2)$, but will not be used.) Using this result and the usual multiplicative properties of the intertwining operators, it is possible to state the fundamental properties of the factors $C(s,c,\tau,\psi)$ which define them uniquely, e.g., multiplicativity and their values for unramified data (see (6.13) below), as we shall do for the γ-factors; see [Sha90, LR05]. Here we only prove the properties needed for the purpose of the γ-factors, in the process of establishing the properties of the latter.

For general spin groups the arguments are similar to the orthogonal cases. The notation can be adapted to incorporate twisting by χ, e.g., $\lambda(s,c,\tau\otimes\chi,\psi)$ and $C(s,c,\tau\otimes\chi,\psi)$. For odd c and k, one uses the modified version of (3.4) with $\jmath_{kc}$ and t_0, noting that conjugation by $\jmath_{kc}$ defines an involution of GSpin_{2kc}, and t_0 is regarded as an element in $T_{\mathrm{GL}_{kc}} < M_{R_{kc,2kc}}$ (see § 2.5). The functions $a(s,c,\tau\otimes\chi)$ and $b(s,c,\tau\otimes\chi)$ are defined as in the orthogonal cases, but the formulas for $a_0(\cdots)$ and $b_0(\cdots)$ are modified by replacing $(\wedge^2,\vee^2)$ with $(\wedge^2\otimes\chi,\vee^2\otimes\chi)$, thereby $a(s,c,\tau\otimes\chi)$ and $b(s,c,\tau\otimes\chi)$ are products of twisted L-functions.

4 The γ-factor

In this section we define the local γ-factors, following [GJ72, JPSS83, Sha90, Sou93, LR05, Kap15]. We proceed with the notation of § 2 and in particular consider the irreducible representations π of G and τ of GL_k, or if $G = \mathrm{GL}_n$, $\tau = \tau_0 \otimes \chi^{-1}\tau_0^\vee$ for an irreducible representation τ_0 of GL_k (τ is also generic). Also define $\chi_\pi = 1$ unless $G = \mathrm{GSpin}_c$, in which case it is the restriction of the central character of π to C_G°, regarded as a character of F^*.

With the notation of § 2.6, consider the space (2.9), i.e.,

$$\mathrm{Hom}_{(G,G)}(J_{U,\psi_U^{-1}}(V(s, W_\psi(\rho_c(\tau)) \otimes \chi_\pi)), \pi^\vee \otimes \pi^\iota).$$

(Recall that for $G = \mathrm{GL}_n$, $\pi^\vee \otimes \pi^\iota$ is replaced with $(\chi^k\pi)^\vee \otimes \pi$.) According to [GK, Theorem 2.1], outside a discrete subset of s the dimension of (2.9) is at most 1.

In its domain of absolute convergence, which for entire sections depends only on the representations, the integral $Z(s, \omega, f)$ belongs to (2.9), by Proposition 2.2. Over p-adic fields the uniqueness result combined with Proposition 2.6 readily imply the meromorphic continuation of the integral, by virtue of Bernstein's continuation principle (in [Ban98]). Over archimedean fields, in § 6.10 we deduce the meromorphic continuation of the integral along with the continuity of the continuation in the input data, using multiplicativity arguments and an idea of Soudry [Sou95] (see § 6.7.2). We proceed over any local field.

The meromorphic continuation of $Z(s, \omega, f)$, regarded as a bilinear form, belongs to (2.9) (for any meromorphic f). Therefore we may study a functional equation relating the integrals $Z(s, \omega, f)$ and

$$Z^*(s, \omega, f) = Z(1 - s, \omega, M^*(s, c, \tau \otimes \chi_\pi, \psi)f).$$

We can define the equation directly using the proportionality factor between $Z(s, \omega, f)$ and $Z^*(s, \omega, f)$. However, as in [LR05, Kap15], it is advantageous for some applications (e.g., [ILM17]) to introduce an additional normalization, which produces better behaved multiplicative factors. Let

$$\vartheta(s, c, \tau \otimes \chi_\pi, \psi) = \begin{cases} \gamma(s, \tau, \psi)\tau(-1)^n\tau(2)^{-2n}|2|^{-2kn(s-1/2)} & G = \mathrm{Sp}_{2n}, \\ \chi_\pi(2)^{-kn}\tau(-1)^n\tau(2)^{-2n}|2|^{-2kn(s-1/2)} & G = \mathrm{SO}_c, \mathrm{GSpin}_c, \\ \tau_0(-1)^n & G = \mathrm{GL}_n\,. \end{cases}$$

Here for SO_c and GSpin_c, $n = \lfloor c/2 \rfloor$ (as we use throughout). Recall that for GSpin_{2n} we defined the element $\mathfrak{i}_G$ in § 2.5. Set $\mathfrak{i}_G = -I_c$ for $G = \mathrm{Sp}_{2n}$, SO_{2n} and GL_n ($\mathrm{Sp}_{2n}, \mathrm{SO}_{2n} < \mathrm{GL}_{2n}$), and let $\mathfrak{i}_G$ be the identity element if $G = \mathrm{SO}_{2n+1}$ or GSpin_{2n+1}. Also denote $N = 2n$ for all groups except $G = \mathrm{Sp}_{2n}$, where $N = 2n + 1$.

Since (2.9) is at most one dimensional (outside a discrete subset of s), there is a function $\gamma(s, \pi \times \tau, \psi)$ such that for all data (ω, f),

$$\gamma(s, \pi \times \tau, \psi)Z(s, \omega, f) = \pi(\mathfrak{i}_G)^k\vartheta(s, c, \tau \otimes \chi_\pi, \psi)Z^*(s, \omega, f). \tag{4.1}$$

Note that $\gamma(s,\pi\times\tau,\psi)$ is well defined, meromorphic and not identically zero. Indeed, one can choose data for which $Z(s,\omega,f)$ is nonzero, the local integrals are meromorphic and $M^*(s,c,\tau\otimes\chi_\pi,\psi)$ is onto, outside a discrete subset of s. We also need to address the following minimal cases: for Sp_{2n} and $n=0$, define Sp_0 as the trivial group and take $\gamma(s,\pi\times\tau,\psi)=\gamma(s,\tau,\psi)$; for GSpin_c and $c\leq 1$ put $\gamma(s,\pi\times\tau,\psi)=1$ (the integral is over $C^{\circ}_{\mathrm{GSpin}_c}\backslash\mathrm{GSpin}_c$).

REMARK 4.1. Equation (4.1) agrees with [LR05, § 9] up to factors depending only on the groups, the central character of τ and a constant to the power s (cf. [Kap15, Remark 4.4]). In addition here A is fixed and we only consider split groups (hence we defined SO_{2n} using J_{2n}, implying $D=1$ and $\epsilon(1/2,\tau_D,\psi)=1$ in the notation of [LR05, § 9]). The sign $\tau_0(-1)$ for GL_n is compatible with [Gan12, p. 82] (see also [Kak20]).

Here is our main result regarding the local factors, formulated as in [LR05, Theorem 4]. To simplify the presentation, in the following theorem the case of GL_n is excluded except for (4.8), which defines this γ-factor uniquely. In (4.8), $\gamma^{\mathrm{RS}}(\cdots)$ denotes the γ-factor of [JPSS83], or [JS90] over archimedean fields. The local factors in [JPSS83, JS90] were mainly defined for representations affording a unique Whittaker model, but as explained in [JPSS83, § 9.4] since all irreducible tempered representations of general linear groups satisfy this property, one can define these local factors for all irreducible representations.

Theorem 4.2. *The γ-factor satisfies the following properties.*

- *Unramified twisting: $\gamma(s,\pi\times|\det|^{s_0}\tau,\psi)=\gamma(s+s_0,\pi\times\tau,\psi)$. For the group GSpin_c we can also twist π, then $\gamma(s,|\Upsilon|^{-s_0}\pi\times\tau,\psi)=\gamma(s+s_0,\pi\times\tau,\psi)$ (Υ was defined in § 2.5).*
- *Multiplicativity: Let π be a quotient of $\mathrm{Ind}_R^G(\sigma_{\beta'}\otimes\pi')$, where R is a standard parabolic subgroup of G, $\sigma_{\beta'}\otimes\pi'$ is an irreducible representation of $M_R=M_{\beta'}\times G'$, and β' is a d' parts composition of $l\leq n$. Let $\tau=\mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\tau_\beta)$ with $\tau_\beta=\otimes_{i=1}^d\tau_i$, $\tau_i=|\det|^{a_i}\tau_{0,i}$, $a_1\geq\ldots\geq a_d$ and each $\tau_{0,i}$ is square-integrable, or τ is the essentially square-integrable quotient of $\mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\tau_\beta)$ and τ_β is irreducible supercuspidal (including the case $\beta=(1^k)$ over any local field). Then*

$$\gamma(s,\pi\times\tau,\psi)=\prod_{i=1}^{d}\gamma(s,\pi\times\tau_i,\psi),\tag{4.2}$$

$$\gamma(s,\pi\times\tau,\psi)=\gamma(s,\pi'\times\tau,\psi)\prod_{i=1}^{d'}\gamma(s,\sigma_i\times(\tau\otimes\chi_\pi^{-1}\tau^\vee),\psi).\tag{4.3}$$

  Here if $G=\mathrm{Sp}_{2n}$ and $l=n$, $\gamma(s,\pi'\times\tau,\psi)=\gamma(s,\tau,\psi)$ as defined above.
- *Unramified factors: When all data are unramified,*

$$\gamma(s,\pi\times\tau,\psi)=\frac{L(1-s,\pi^\vee\times\tau^\vee)}{L(s,\pi\times\tau)}.\tag{4.4}$$

- *Duality:*

$$\gamma(s,\pi^\vee\times\tau,\psi)=\gamma(s,\pi\times\chi_\pi^{-1}\tau,\psi). \tag{4.5}$$

- *Functional equation:*

$$\gamma(s,\pi\times\tau,\psi)\gamma(1-s,\pi^\vee\times\tau^\vee,\psi^{-1})=1. \tag{4.6}$$

- *Dependence on* ψ*: Denote* $\psi_b(x)=\psi(bx)$*, for* $b\in F^*$*. Then*

$$\gamma(s,\pi\times\tau,\psi_b)=\chi_\pi^{kn}(b)\tau(b)^N|b|^{kN(s-1/2)}\gamma(s,\pi\times\tau,\psi). \tag{4.7}$$

- GL_n*-factors:*

$$\gamma(s,\pi\times(\tau_0\otimes\chi^{-1}\tau_0^\vee),\psi)=\gamma^{\mathrm{RS}}(s,\pi\times\chi\tau_0,\psi)\gamma^{\mathrm{RS}}(s,\pi^\vee\times\tau_0,\psi). \tag{4.8}$$

- *Archimedean property: Over* $F=\mathbb{R}$ *or* $\mathbb{C}$*, let* $\varphi: W_F\to {}^L(\mathrm{GL}_k\times G)$ *be the homomorphism attached to* $\tau\otimes\pi$*, and let* $\epsilon(s,r\circ\varphi,\psi)$ *and* $L(s,r\circ\varphi)$ *be Artin's local factors attached to* $r\circ\varphi$ *by Langlands' correspondence ([Bor79, Lan89]). Here* W_F *is the Weil group of* F*;* ${}^L(\mathrm{GL}_k\times G)$ *is the* L*-group; and* r *is the standard representation. Then*

$$\gamma(s,\pi\times\tau,\psi)=\epsilon(s,r\circ\varphi,\psi)\frac{L(1-s,r^\vee\circ\varphi)}{L(s,r\circ\varphi)}. \tag{4.9}$$

- *Crude functional equation: Let* F *be a number field with a ring of adeles* $\mathbb{A}$*,* ψ *be a nontrivial character of* $F\backslash\mathbb{A}$*, and assume* π *and* τ *are cuspidal representations of* $G(\mathbb{A})$ *and* $\mathrm{GL}_k(\mathbb{A})$*. Let* S *be a finite set of places of* F *such that for* $\nu\notin S$*, all data are unramified. Then*

$$L^S(s,\pi\times\tau)=\prod_{\nu\in S}\gamma(s,\pi_\nu\times\tau_\nu,\psi_\nu)L^S(1-s,\pi^\vee\times\tau^\vee). \tag{4.10}$$

  Here $L^S(s,\pi\times\tau)$ *is the partial* L*-function with respect to* S*.*

Furthermore, the γ*-factors are uniquely determined by the properties of multiplicativity, dependence on* ψ*,* GL_n*-factors and the crude functional equation.*

Remark 4.3. For $k=1$, by the uniqueness property our γ-factor coincides with the γ-factor of [LR05] for Sp_{2n}; for SO_c, Rallis and Soudry [RS05, § 5] showed how to use the γ-factor of [LR05] defined for O_c to obtain a γ-factor for SO_c, which is then identical with ours.

Remark 4.4. For GSpin_c, the choice of Υ is not canonical (as opposed to det, see [Kap17, § 1.2]). Also regarding (4.4), if π is a quotient of $\mathrm{Ind}_{R_{n,c}}^{\mathrm{GSpin}_c}(\mathrm{Ind}_{B_{\mathrm{GL}_n}}^{\mathrm{GL}_n}(\otimes_{i=1}^n\pi_i)\otimes\chi_\pi)$ ($R_{n,c}$ was defined in § 2.5) and $\tau=\mathrm{Ind}_{B_{\mathrm{GL}_k}}^{\mathrm{GL}_k}(\otimes_{j=1}^k\eta_j)$,

$$L(s,\pi\times\tau)=\prod_{i,j}(1-\chi_\pi\pi_i\eta_j(\varpi)q^{-s})^{-1}\prod_{i,j}(1-\pi_i^{-1}\eta_j(\varpi)q^{-s})^{-1}.$$

The Satake parameter of π regarded as an element of $\mathrm{GL}_N(\mathbb{C})$ is

$$\mathrm{diag}(\chi_\pi\pi_1(\varpi),\ldots,\chi_\pi\pi_n(\varpi),\pi_n^{-1}(\varpi),\ldots,\pi_1^{-1}(\varpi)). \tag{4.11}$$

This is compatible with Asgari and Shahidi [AS06, (64)]: they wrote the Satake parameter using the characters $\chi_1,\ldots,\chi_n,\chi_0\chi_n^{-1},\ldots,\chi_0\chi_1^{-1}$, χ_0 was the central character which identifies with χ_π, and since $\theta_i^\vee\mapsto\epsilon_i^\vee-\epsilon_0^\vee$, π_i corresponds to $\chi_0^{-1}\chi_i$ of *loc. cit.*

COROLLARY 4.5. *If π is a generic representation, our γ-factor is identical with the γ-factor of Shahidi.*

Proof. Shahidi's γ-factors satisfy the same list of properties ([Sha90, Theorem 3.5]). For GSpin_c, to compare the multiplicative formulas (4.3) and (4.8) to those of Shahidi, note that the standard intertwining operator takes the representation induced from a maximal parabolic subgroup and $\tau\otimes\pi'$, to the representation induced from $\chi_{\pi'}^{-1}\tau^\vee\otimes\pi'$ ($\chi_{\pi'}=\chi_\pi$). □

REMARK 4.6. The Rankin–Selberg γ-factors for classical groups and generic representations were defined in [Sou93, Sou95, Sou00, Kap13a, Kap13c, Kap15]. A refined definition which satisfies the above list of canonical properties was given in [Kap15], where the notation $\Gamma(s,\pi\times\tau,\psi)$ was used. With the minor corrections described in [AK19], the Rankin–Selberg γ-factors for Sp_{2n} and SO_c are identical with Shahidi's, thereby also with the γ-factors defined here (for generic representations).

5 Proof of Theorem 4.2: Part I: Multiplicativity

The proof that the γ-factors are uniquely determined by the properties of multiplicativity, dependence on ψ, GL_n-factors and the crude functional equation follows from a standard globalization argument as in [LR05, p. 339], we omit the details. The main part of the proof is devoted to multiplicativity, and since several similar proofs of this property have appeared in this generality, see [Sou93, Sou95, Sou00, Kap13a, Kap15], we settle for brief justifications here (they are similar and simpler). For clarity, we usually treat Sp_{2n} and SO_{2n} together, and for SO_{2n+1} explain only the modifications; the proofs for GSpin_c then follow by an almost "uniform modification" of the SO_c case (except the unramified twisting); the GL_n case is usually simpler. The proof of the remaining parts of Theorem 4.2 is deferred to § 6 below.

Several arguments are important for deducing additional results. We try to point them out at the end of each section, to minimize the number of cross references between separated sections.

We will repeatedly apply the following standard argument to integrals over *unipotent* subgroups. Let $\mathcal{V}$ denote a space of complex-valued functions on H. The group H acts on $\mathcal{V}$ by right-translations and we assume this action is admissible.

LEMMA 5.1. *Let X, Y be unipotent subgroups of H and let $\xi \in \mathcal{V}$. Consider an integral $\int_X \xi(hx)dx$ and assume for each $y \in Y$, $\int_X y \cdot \xi(hx)dx = \int_X \xi(hx)\psi(<x,y>)dx$, with a non-degenerate pairing $<,>$. Then $x \mapsto \xi(hx)$ is a Schwartz function of X, $\int_X |\xi(hx)|dx < \infty$ and we can choose $\xi' \in \mathcal{V}$ such that $\int_X \xi'(hx)dx = \xi(h)$.*

Proof. The proof technique is called "root elimination", see e.g., [Sou93, § 6.1, § 7.2] and [Jac09, § 6.1] (see also the proof of [CFGoK, Lemma 9]). □

REMARK 5.2. In the archimedean case the proof uses [DM78]. If there is an auxiliary dependence of $\mathcal{V}$ on a complex parameter s as in § 2.3, one can replace [DM78] by Corollary A.3.

5.1 Unramified twisting. For the twisting of τ one only needs to observe

$$\begin{aligned}
&\rho_c(|\det|^{s_0}\tau) = |\det|^{s_0}\rho_c(\tau),\\
&M^*(s, c, |\det|^{s_0}\tau \otimes \chi_\pi, \psi) = M^*(s + s_0, c, \tau \otimes \chi_\pi, \psi),\\
&\vartheta(s, c, |\det|^{s_0}\tau \otimes \chi_\pi, \psi) = \vartheta(s + s_0, c, \tau \otimes \chi_\pi, \psi).
\end{aligned}$$

For GSpin_c, changing π by $|\Upsilon|^{-s_0}$ implies that the integrand of $Z(s, \omega, f)$ is multiplied by $|\Upsilon|^{s_0}(g)$. Regarding Υ also as a character of H, the definition of the embedding implies $|\Upsilon|^{s_0}(g) = |\Upsilon|^{s_0}((1, g)) = |\Upsilon|^{s_0}((1, {}^{\iota}g))$. Then since $|\Upsilon|^{-s_0}(\mathfrak{r}_c^{\vee}(x_0)) = |x_0|^{2s_0}$, we obtain $\chi_{(|\Upsilon|^{-s_0}\pi)} = |\cdot|^{2s_0}\chi_\pi$ and the section $|\Upsilon|^{s_0} f$ belongs to

$$|\Upsilon|^{s_0} V(W_\psi(\rho_c(\tau)) \otimes |\ |^{2s_0}\chi_\pi) = V(W_\psi(|\det|^{s_0}\rho_c(\tau)) \otimes \chi_\pi).$$

Also note that

$$\begin{aligned}
&M(s, W_\psi(\rho_c(\tau)) \otimes |\cdot|^{2s_0}\chi_\pi, w_P)|\Upsilon|^{s_0} f = M(s, W_\psi(\rho_c(|\det|^{s_0}\tau)) \otimes \chi_\pi, w_P)f,\\
&M^*(s, c, \tau \otimes |\cdot|^{2s_0}\chi_\pi, \psi) = M^*(s, c, |\det|^{s_0}\tau \otimes \chi_\pi, \psi),
\end{aligned}$$

where the second equality follows also because $|\Upsilon|$ is trivial on the Weyl elements and unipotent matrices appearing in (3.4). Then a simple computation shows

$$\vartheta(s, c, \tau \otimes \chi_{(|\Upsilon|^{-s_0}\pi)}, \psi) = \vartheta(s, c, |\det|^{s_0}\tau \otimes \chi_\pi, \psi),$$

and we conclude $\gamma(s, |\Upsilon|^{-s_0}\pi \times \tau, \psi) = \gamma(s, \pi \times |\det|^{s_0}\tau, \psi) = \gamma(s + s_0, \pi \times \tau, \psi)$, as proved above.

5.2 Multiplicativity II: Identity (4.2). We proceed as in [Kap15, § 8.1]. Start with $G = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}$. By [CFGoK, Lemma 7], $\rho_c(\tau)$ is a quotient of

$$\mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^{d}\rho_c(\tau_i)). \tag{5.1}$$

For simplicity, throughout the proof we assume $d = 2$, i.e., $\beta = (\beta_1, \beta_2)$. If F is non-archimedean and τ is a full induced representation we can always assume this, by the definition of $\rho_c(\tau)$ and transitivity of induction; if τ is essentially square-integrable or if F is archimedean, we should really work with any $d \geq 2$ (in the archimedean case even if τ is a full induced representation, we have to apply [Cas80a] because the analytic properties of (1.3) are only known for degenerate principal series).

The representation τ_β of M_β is irreducible and generic. Let H', P', U_0', $\delta'=\delta_0'\delta_1'$ be the groups and elements defined in § 2 for the $G\times\mathrm{GL}_{\beta_2}$ integral involving $\pi\times\tau_2$. Let L be the standard parabolic subgroup of H with $M_L=\mathrm{GL}_{\beta_1 c}\times H'$. As explained in [CFGoK, § 3.1] we form the twisted version of (5.1) which is also (k,c). We then realize the (k,c) model using (1.3). Let $\zeta\in\mathbb{C}$. If f_ζ is a section corresponding to

$$\mathrm{Ind}_P^H(|\det|^{s-1/2}\,\mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^2|\det|^{\zeta_i}W_\psi(\rho_c(\tau_i)))),\qquad (\zeta_1,\zeta_2)=(\zeta,-\zeta),$$

the integral takes the form

$$Z(s,\omega,f_\zeta)=\int\limits_G\omega(g)\int\limits_{U_0}\int\limits_{V_{\beta'c}}f_\zeta(s,w_{\beta c}v\delta u_0(1,{}^\iota g))\psi^{-1}(v)\psi_U(u_0)\,dv\,du_0\,dg. \tag{5.2}$$

For $\mathrm{Re}(s)\gg\mathrm{Re}(\zeta)\gg0$, the integral (5.2) is absolutely convergent as a triple integral (see e.g., [Sou00, Lemma 3.1]). We will prove

$$\frac{Z^*(s,\omega,f_\zeta)}{Z(s,\omega,f_\zeta)}=\prod_{i=1}^2\pi(\mathrm{i}_G)^{-k_i}\vartheta(s,c,|\det|^{\zeta_i}\tau_i,\psi)^{-1}\gamma(s,\pi\times|\det|^{\zeta_i}\tau_i,\psi).$$

Since ϑ is holomorphic in ζ, and γ satisfies the unramified twisting property, we may take $\zeta=0$ on the r.h.s. (right-hand side). Furthermore, $Z(s,\omega,f_\zeta)$ is a meromorphic function of ζ and s which is well defined as a meromorphic function of s for any fixed ζ. This follows from the uniqueness result for (2.9) when we include the twists by ζ in the non-archimedean case and from § 6.10 when the field is archimedean. Moreover, for a fixed compact set $\mathcal{C}\subset\mathbb{C}$ we can choose $A>0$ such that $Z(s,\omega,f_\zeta)$ is absolutely convergent for all $\mathrm{Re}(s)>A$ and $\zeta\in\mathcal{C}$. Hence the Dominated Convergence Theorem implies $\lim_{\zeta\to0}Z(s,\omega,f_\zeta)=Z(s,\omega,\lim_{\zeta\to0}f_\zeta)$ and because (1.3) is entire, the last limit equals $f_0=f$ and then (1.3) realizes the (k,c) model of $\rho_c(\tau)$. Similarly for $Z^*(s,\omega,f_\zeta)$ (convergence will be in a left half plane). In addition, the denominator on the l.h.s. can be taken to be not identically zero for $\zeta=0$ by Proposition 2.6. Thus we can take $\zeta=0$ on both sides and conclude (4.2). (One can also justify taking $\zeta=0$ by arguing as in [Sou93, p. 66].) Henceforth we omit ζ from the notation.

Denote the triple integral (5.2) by $\mathcal{I}(f)$. Write $U_0=U_0'\ltimes(U_0\cap U_L)$ and observe the following:

(1) ${}^{\delta^{-1}}V_{\beta'c}$ normalizes U_0 and $U_L={}^{\delta^{-1}}V_{\beta'c}\ltimes(U_0\cap U_L)$,
(2) $w_{\beta c}\delta_0=\delta_0'w_L$, where ${}^{w_L}U_L=U_L^-$ ($w_L=\begin{pmatrix}&&I_{\beta_1c}\\&I_{2\beta_2c}&\\\epsilon_0I_{\beta_1c}&&\end{pmatrix}$),
(3) $\delta_1=\delta_1'$,
(4) w_L commutes with δ_1' and U_0',
(5) $(1,{}^\iota g)$ normalizes U_L,
(6) ${}^{w_L}(1,{}^\iota g)$ is the element $(1,{}^\iota g)$ appearing in the $G\times\mathrm{GL}_{\beta_2}$ integral.

Using these properties,

$$\mathcal{I}(f) = \int\limits_{U_L} Z'(s, \omega, (w_L u) \cdot f)\psi^{-1}(u)\, du. \tag{5.3}$$

Here Z' is the $G\times \mathrm{GL}_{\beta_2}$ integral for π and τ_2; $\psi(u)$ is defined by the trivial extension of the character of ${}^{\delta^{-1}}V_{\beta' c}$ (the conjugation of the character of $V_{\beta' c}$) to U_L, and $(w_L u)\cdot f$ is regarded as a meromorphic section of $V(W_\psi(\rho_c(\tau_2)))$. Therefore by (4.1),

$$\gamma(s, \pi\times\tau_2, \psi)\mathcal{I}(f) = \pi(\mathrm{i}_G)^{k_2}\vartheta(s, c, \tau_2, \psi) \int\limits_{U_L} Z'^{*}(s, \omega, (w_L u) \cdot f)\psi^{-1}(u)\, du.$$

(The justification of this formal step is actually given in the proof of Corollary 5.3 below.) Reversing the manipulations (5.2)–(5.3) we obtain

$$\gamma(s, \pi\times\tau_2, \psi)\mathcal{I}(f) = \pi(\mathrm{i}_G)^{k_2}\vartheta(s, c, \tau_2, \psi)\mathcal{I}(M^*(s, c, \tau_2, \psi) f). \tag{5.4}$$

Here on M_P, $M^*(s, c, \tau_2, \psi)f$ is a function in the space of

$$\mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(W_\psi(\rho_c(\tau_1)) \otimes W_\psi(\rho_c(\tau_2^\vee))).$$

Next, since the dv-integration of (1.3) comprises the l.h.s. of (3.4),

$$\mathcal{I}(M^*(s, c, \tau_2, \psi)f) = \mathcal{I}(M^*(s, c, \tau_1 \otimes \tau_2^\vee, \psi)M^*(s, c, \tau_2, \psi) f). \tag{5.5}$$

Now on the r.h.s. β is replaced by (β_2, β_1), and the section (restricted to GL_{kc}) is a function in the space of

$$\mathrm{Ind}_{P_{(\beta_2,\beta_1)c}}^{\mathrm{GL}_{kc}}(W_\psi(\rho_c(\tau_2^\vee)) \otimes W_\psi(\rho_c(\tau_1))).$$

To complete the proof we use the multiplicativity of the intertwining operators, namely

$$M^*(s, c, \tau, \psi) = M^*(s, c, \tau_1, \psi)M^*(s, c, \tau_1 \otimes \tau_2^\vee, \psi)M^*(s, c, \tau_2, \psi). \tag{5.6}$$

To see this note that the application of (3.3) to f (with the realization (1.3)) takes the form

$$\int\limits_{U_P}\int\limits_{V_{\beta' c}} f(s, w_{\beta c} v \mathrm{d}_{k,c}\delta_0 u)\psi^{-1}(v)\psi^{-1}(u)\, dv\, du,$$

with the characters and $\mathrm{d}_{k,c}$ defined in § 3. Applying (1), (2) and (4) to this integral we obtain the application of (3.3) to f as a section of $V(W_\psi(\rho_c(\tau_2)))$, as an inner integral. Note that ${}^{w_{\beta c}}\mathrm{d}_{k,c} = \mathrm{diag}(d', \mathrm{d}_{\beta_2,c})$. Applying (3.4) for H', the section changes to $M^*(s, c, \tau_2, \psi)f$. Then we apply the functional equation (3.4) for GL_{kc} (to the dv-integral) to produce the operator $M^*(s, c, \tau_1 \otimes \tau_2^\vee, \psi)$, and repeat (1), (2) and (4) again for $M^*(s, c, \tau_1, \psi)$.

Applying the steps (5.2)–(5.3) to the r.h.s. of (5.5), using (5.6) and the identity

$$\vartheta(s,c,\tau_1,\psi)\vartheta(s,c,\tau_2,\psi)=\vartheta(s,c,\tau,\psi),$$

we conclude $\gamma(s,\pi\times\tau,\psi)=\gamma(s,\pi\times\tau_1,\psi)\gamma(s,\pi\times\tau_2,\psi)$. The proof is complete.

Exactly the same manipulations apply to the GL_n integral. In this case $\tau=\tau_0\otimes\chi^{-1}\tau_0^\vee$; we assume $\tau_0=\mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\varrho_1\otimes\varrho_2)$ (or a quotient if τ_0 is square-integrable and the inducing data is supercuspidal); $\tau_i=\varrho_i\otimes\chi^{-1}\varrho_i^\vee$ and the intertwining operator applied in (5.5) is replaced by $M^*(s,c,\varrho_1\otimes\chi^{-1}\varrho_2^\vee,\psi)M^*(s,c,\varrho_2\otimes\chi^{-1}\varrho_1^\vee,\psi)$. The formula (4.2) for GL_n is again

$$\gamma(s,\pi\times\tau,\psi)=\gamma(s,\pi\times\tau_1,\psi)\gamma(s,\pi\times\tau_2,\psi).$$

Consider $G=\mathrm{SO}_{2n+1}$. The proof is similar, except for modifications related to the embedding of $G\times G$ in H and the parity of k. Equality (5.2) remains valid. Also while U_0 and U_0' do depend on the parities of k and β_2, we always have ${}^{\jmath_{\beta_1}}U_0'<U_0$. Hence we write $U_0={}^{\jmath_{\beta_1}}U_0'\ltimes(U_0\cap U_L)$. Looking at the list of properties above, item (1) still holds. For (2) use

$$w_L=t_{\beta_1}\begin{pmatrix}&&I_{\beta_1c}\\&I_{2\beta_2c}&\\I_{\beta_1c}&&\end{pmatrix},\qquad t_{\beta_1}=\mathrm{diag}(I_{kc-1},(-1)^{\beta_1}I_2,I_{kc-1})\jmath_{\beta_1}.$$

Equality (3) holds; for (4), w_L still commutes with δ_1', but now ${}^{w_L}({}^{\jmath_{\beta_1}}U_0')=U_0'$ and this conjugation changes the character $\psi_U|_{U_0}$ to be the proper character for the $G\times\mathrm{GL}_{\beta_2}$ integral, i.e., for $G\times\mathrm{GL}_k$ it depends on the parity of k, after the conjugation it depends on the parity of β_2; and (5) is valid. Finally for (6), in the previous cases w_L commutes with $(1,{}^\iota g)$, but here this is a bit more subtle: when k and β_2 do not have the same parity (equivalently t_{β_1} is nontrivial), the constants ϵ_1 and ϵ_2 used in the construction of the integral are swapped and the matrices $\iota=\iota_k$ and $m=m_k$ defined in § 2 change (m depends only on the parity of k, and for fixed n so does ι). We see that $t_{\beta_1}\iota_k m_k=\iota_{\beta_2}m_{\beta_2}$. This completes the verification of the properties leading to (5.3).

We apply the functional equation and reverse the manipulations (5.2)–(5.3), but if β_2 is odd, the resulting inner integral for $\pi\times\tau_2^\vee$ is slightly modified, since the section belongs to a space of a representation induced from ${}^{\jmath_{\beta_2}}P'$: δ_0', δ_1', U_0' and its character $\psi_{U'}$ are different, e.g., δ_0' is now given by (2.3) (see § 2.4). In both cases we see that (4) still holds, but (2) and (3) are modified. Let

$$z_{\beta_2}=\mathrm{diag}(I_{(k-1)c+n},-I_n,{}^{\jmath_k}\mathrm{diag}(-2,-1/2)\jmath_{\beta_2},-I_n,I_{(k-1)c+n})$$

if β_2 is odd, otherwise $z_{\beta_2}=I_{2kc}$. The integral before reversing (1) is

$$\int\limits_G\omega(g)\int\limits_{U_L}\int\limits_{U_0'}M^*(s,c,\tau_2,\psi)f(s,\jmath_{\beta_2}w_{\beta c}\delta z_{\beta_2}({}^{w_L^{-1}}u_0')u(1,{}^\iota g))\psi_{U'}(u_0')\psi^{-1}(u)\,du_0'\,du\,dg.$$

Here if β_2 is even, ${}^{w_L^{-1}}U_0' = {}^{\jmath_{\beta_1}}U_0' < U_0$, but for odd β_2, ${}^{w_L^{-1}}U_0' < {}^{\jmath_{\beta_2}}U_0$. Thus in all cases ${}^{z_{\beta_2}}({}^{w_L^{-1}}U_0') < U_0$ and when we change variables in u_0', the character $\psi_{U'}$ changes back to its definition when the representation is induced from P'. We can also take a subgroup of U_L of the form ${}^{\delta^{-1}z_{\beta_2}^{-1}}V_{\beta' c}$, then we can follow (1) in the opposite direction. Also z_{β_2} commutes with ι and (if β_2 is odd) $(1, {}^{\iota}g) \mapsto {}^{z_{\beta_2}}(1, {}^{\iota}g)$ is an outer involution of $(1, G)$, hence we can conjugate z_{β_2} to the right. We obtain (5.4), except that the section on the r.h.s. is

$$((z_{\beta_2}\jmath_{\beta_2}) \cdot M^*(s, c, \tau_2, \psi)f)^{\jmath_{\beta_2}} \tag{5.7}$$

$(\det z_{\beta_2}\jmath_{\beta_2} = 1)$. The section $M^*(s, c, \tau_2, \psi)f(s, h)$ belongs to a space of a representation induced from ${}^{\jmath_{\beta_2}}P$, but the additional conjugation by $\jmath_{\beta_2}$ takes it back to a section of a space induced from P. Then we can apply (3.4) and obtain (5.5), but with (5.7) instead of $M^*(s, c, \tau_2, \psi)f$ on both sides. We may then repeat the steps above for τ_1, and again consider odd β_1 separately. If k is even, $z_{\beta_1}z_{\beta_2} = I_{2kc}$, hence after applying the functional equation for τ_1 we obtain the correct form of the integral for $M^*(s, c, \tau, \psi)f$ (regardless of the parity of, say, β_2). When k is odd, either z_{β_1} or z_{β_2} is trivial, and we obtain

$$Z(1 - s, \omega, ((z_k\jmath_k) \cdot M^*(s, c, \tau, \psi)f)^{\jmath_k}).$$

At this point conjugating z_k to the left and $\jmath_k$ to the right, we reach

$$\int\limits_{G} \omega(g) \int\limits_{U_0} \int\limits_{V_{\beta' c}} M^*(s, c, \tau, \psi)f(s, {}^{\jmath_k}(w_{\beta c}v)\delta u_0(1, {}^{\iota}g))\psi^{-1}(v) \\ \times \psi_U(u_0)\, dv\, du_0\, dg = Z^*(s, \omega, f),$$

with δ, U_0 and ψ_U defined correctly (i.e., for a section of a space induced from ${}^{\jmath_k}P$).

The proof of the orthogonal cases extends to GSpin_c as follows. All conjugations of unipotent subgroups above remain valid. When we write $w_{\beta c}\delta_0 = \delta_0' w_L$, the elements δ_0 and δ_0' were fixed in the definition of the integral, and the choice of $w_{\beta c}$ is canonical by our identification of GL_{kc} with a subgroup of M_P. Then w_L is already defined uniquely, it is a representative for the Weyl element corresponding to the permutation matrix w_L in SO_{2kc}. When c is even, $\det w_{\beta c} = 1$ hence $w_{\beta c} \in \mathrm{SL}_{kc}$, and by the definition of the embedding of GL_{kc} in M_P (see § 2.5), $w_{\beta c} \in \mathrm{Spin}_{2kc}$. Therefore $w_L \in \mathrm{Spin}_{2kc}$ is one of the elements in the preimage of the matrix w_L. For odd c, when $\det w_{\beta c} = (-1)^{\beta_1\beta_2} = -1$, $[\mathrm{diag}(-1, I_{kc-1}), 1]w_L \in \mathrm{Spin}_{2kc}$ (the element $[\mathrm{diag}(-1, I_{kc-1}), 1]$ commutes with δ_0'). In both cases (6) holds. Finally, the intertwining operators are now $M^*(s, c, \tau_i \otimes \chi_\pi, \psi)$ and $M^*(s, c, \tau_1 \otimes \chi_\pi^{-1}\tau_2^\vee, \psi)$.

The proof has the following corollary, which can be used to reduce the proof of several properties of the integrals to the case of an essentially tempered τ, or even a character for archimedean fields. Assume τ is an irreducible generic representation of GL_k such that $\rho_c(\tau)$ is a quotient of (5.1), with any $d \geq 2$ ($d = k$ over archimedean fields) and where the representations τ_i appearing in (5.1) are irreducible generic.

Let $V'(s, W_\psi(\rho_c(\tau_d)) \otimes \chi_\pi)$ be the space corresponding to the representation induced from P' to H', where H' and P' are the groups and elements defined in § 2 for the $G \times \mathrm{GL}_{\beta_d}$ integral involving $\pi \times \tau_d$. Also recall that π is an irreducible representation of G and let ω be a matrix coefficient of $\pi^\vee$.

COROLLARY 5.3. *For every entire section $f' \in V'(W_\psi(\rho_c(\tau_d)) \otimes \chi_\pi)$ there is an entire section $f \in V(W_\psi(\rho_c(\tau)) \otimes \chi_\pi)$ such that $Z(s, \omega, f) = Z(s, \omega, f')$. Over archimedean fields f is smooth.*

Proof. The proof is a similar to [Sou00, Lemma 3.4]. Since $\rho_c(\tau)$ is a quotient of (5.1) and using transitivity of induction, we can regard functions in $V(s, W_\psi(\rho_c(\tau)) \otimes \chi_\pi)$ as complex-valued functions on $H \times \mathrm{GL}_{\beta_1 c} \times \ldots \times \mathrm{GL}_{\beta_{d-1} c} \times H'$ such that the mapping $h' \mapsto f(s, h, a, h')$ in particular, belongs to $V'(s, W_\psi(\rho_c(\tau_d)) \otimes \chi_\pi)$. Again, for simplicity only we set $d = 2$.

Assume F is p-adic. Given f', choose an entire section f such that $w_L \cdot f$ is supported in $L\mathcal{N}$, where $\mathcal{N}$ is a small neighborhood of the identity in H, and $w_L \cdot f(s, v, I_{\beta_1 c}, h') = f'(s, h')$ for all s and $v \in \mathcal{N}$. Since f' is entire, one can take $\mathcal{N}$ independently of s even though it depends on f' (because there is a neighborhood of the identity in H' fixing f' for all s). According to (5.3),

$$\mathcal{I}(f) = \int\limits_{U_L} Z'(s, \omega, ({}^{w_L} u w_L) \cdot f) \psi^{-1}(u)\, du.$$

Then we see that ${}^{w_L} u$ belongs to the support of $w_L \cdot f$ if and only if the coordinates of u are small, hence the integral reduces to a nonzero measure constant multiplied by $Z'(s, \omega, w_L \cdot f) = Z'(s, \omega, f')$. This computation is justified for $\mathrm{Re}(s) \gg 0$ and $\zeta = 0$, since U_L contains the conjugation of $V_{\beta' c}$ (see (1) in the proof), hence the inner dv-integral in (5.2) is over elements of $V_{\beta' c}$ which belong to a compact subgroup of GL_{kc}. The result now follows by meromorphic continuation.

Over archimedean fields, we can define an entire section $f \in V(W_\psi(\rho_c(\tau)) \otimes \chi_\pi)$ such that $w_L \cdot f$ is supported in LU_L^-, $w_L \cdot f(s, h'u) = \phi(u) f'(s, h')$ for $u \in U_L^-$ where ϕ is a compactly supported Schwartz function, and $\int_{U_L^-} \phi(u) du = 1$. Then we proceed as above. □

5.3 Multiplicativity I: Identity (4.3).

5.3.1 The groups Sp_{2n} *and* SO_{2n}. Let $G = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}$. The case $l = n$ essentially follows from [CFGK19, Lemma 27], but the general case is more involved. It is enough to consider a maximal parabolic subgroup R, so assume σ is a representation of GL_l, $l \leq n$. For SO_{2n} and $l = n$ there are two choices for R, in this case we assume $R = \{\left(\begin{smallmatrix} a & z \\ & a^* \end{smallmatrix}\right) : a \in \mathrm{GL}_n\}$ (the other case of ${}^{j_1}R$ can be dealt with similarly). Put $\varepsilon = \sigma \otimes \pi'$. We prove the (stronger) statement for $\pi = \mathrm{Ind}_R^G(\varepsilon)$. Then $\pi^\vee = \mathrm{Ind}_R^G(\varepsilon^\vee)$. If $\langle , \rangle$ is the canonical pairing on $\varepsilon \otimes \varepsilon^\vee$ and $\varphi \otimes \varphi^\vee$ belongs to the space of $\pi \otimes \pi^\vee$,

$$\langle \varphi(rg_1), \varphi^\vee(rg_2) \rangle = \delta_R(r) \langle \varphi(g_1), \varphi^\vee(g_2) \rangle, \qquad \forall g_1, g_2 \in G, r \in R.$$

Thus we can realize the matrix coefficient on $\pi^\vee$ using a semi-invariant measure dg_0 on $R\backslash G$ (see [BZ76, 1.21]), as in [LR05, § 4]. Take

$$\omega(g) = \int\limits_{R\backslash G} \langle \varphi(g_0), \varphi^\vee(g_0 g)\rangle \, dg_0. \tag{5.8}$$

Let $G^\triangle < G \times G$ be the diagonal embedding. Since for any $g \in G$,

$$\int\limits_{R\backslash G} \langle \varphi(g_0 g g_1), \varphi^\vee(g_0 g g_2)\rangle \, dg_0 = \int\limits_{R\backslash G} \langle \varphi(g_0 g_1), \varphi^\vee(g_0 g_2)\rangle \, dg_0,$$

and by Corollary 2.3 the integral of f over U_0 is invariant under $(g, {}^\iota g)$ ($(g_1, g_2) \in H$ was defined in § 2.2), we can write $Z(s, \omega, f)$ in the form

$$\int\limits_{G^\triangle\backslash G\times G}\ \int\limits_{R\backslash G}\ \int\limits_{U_0} \langle \varphi(g_0 g_1), \varphi^\vee(g_0 g_2)\rangle f(s, \delta u_0 (g_1, {}^\iota g_2)) \, \psi_U(u_0) \, du_0 \, dg_0 \, d(g_1, g_2).$$

Regard the dg_0-integral as an integral over $R^\triangle\backslash G^\triangle$, collapse it into the $d(g_1, g_2)$-integral, and domain using

$$\int\limits_{R^\triangle\backslash G\times G} d(g_1, g_2) = \int\limits_{R\times R\backslash G\times G}\ \int\limits_{R^\triangle\backslash R\times R} d(r_1, r_2) \, d(g_1, g_2) = \int\limits_{R\times R\backslash G\times G}\ \int\limits_{R} dr \, d(g_1, g_2).$$

We obtain, in a right half plane (ensuring absolute convergence)

$$\int\limits_{R\times R\backslash G\times G}\ \int\limits_{M_R}\ \int\limits_{U_R}\ \int\limits_{U_0} \delta_R^{-1/2}(m) \langle \varphi(g_1), \varepsilon^\vee(m) \varphi^\vee(g_2)\rangle \tag{5.9}$$
$$f(s, \delta u_0 (g_1, {}^\iota(z m g_2))) \, \psi_U(u_0) \, du_0 \, dz \, dm \, d(g_1, g_2).$$

Recall that $f(s, \cdot)$ belongs to a space induced from $W_\psi(\rho_c(\tau))$. Since we already proved (4.2), we can assume τ is essentially tempered, thus the results of [CFGoK, § 3.2] are applicable to $\rho_c(\tau)$ and we can realize the (k, c) model using (1.4) and (1.5). Applying this to $f(s, \cdot)$ we obtain a section of the space of the representation

$$\mathrm{Ind}_P^H(|\det|^{s-1/2} \, \mathrm{Ind}_{P_{(kl, k(c-l))}}^{\mathrm{GL}_{kc}}((W_\psi(\rho_l(\tau)) \otimes W_\psi(\rho_{c-l}(\tau)))\delta_{P_{(kl, k(c-l))}}^{-1/(2k)})).$$

This adds the Weyl element $\kappa_{l, c-l}$ and a unipotent integration over a subgroup, which we denote by V_1. Then we apply (1.4) and (1.5) again, this time to the bottom right $k(c-l) \times k(c-l)$ block to obtain a section of the space of

$$\mathrm{Ind}_P^H(|\det|^{s-1/2} \, \mathrm{Ind}_{P_{(kl, kc', kl)}}^{\mathrm{GL}_{kc}}((W_\psi(\rho_l(\tau)) \otimes W_\psi(\rho_{c'}(\tau)) \otimes W_\psi(\rho_l(\tau)))\delta_{P_{(kl, kc', kl)}}^{-1/(2k)})), \tag{5.10}$$

with $c' = c - 2l = 2(n - l)$. The additional Weyl element is $\mathrm{diag}(I_{kl}, \kappa_{c', l})$ and the unipotent integration is over a subgroup V_2. Note that if $c' = 0$, $\mathrm{diag}(I_{kl}, \kappa_{c', l}) = I_{kc}$

and V_2 is trivial. Both applications do not change the dependence on s, because we only change the realization of $W_\psi(\rho_c(\tau))$. Now for any $h \in H$, the du_0-integration of (5.9) takes the form

$$\int_{U_0}\int_{V_1}\int_{V_2} f(s, \mathrm{diag}(I_{kl}, \kappa_{c',l})v_2\kappa_{l,c-l}v_1\delta u_0 h)\,\psi_U(u_0)\,dv_2\,dv_1\,du_0. \tag{5.11}$$

By matrix multiplication we see that ${}^{\delta^{-1}}v_1u_0 = u_{v_1}{}^{\delta_0^{-1}}v_1$ and ${}^{(\kappa_{l,c-l}\delta)^{-1}}v_2u_0 = u_{v_2}{}^{(\kappa_{l,c-l}\delta_0)^{-1}}v_2$, where the elements $u_{v_i} \in U_0$ satisfy $\psi_U(u_{v_i}) = \psi_U(u_0)$. Thus we may shift v_1 and v_2 to the right of u_0. Also note that ${}^{(\kappa_{l,c-l}\delta_0)^{-1}}v_2$ normalizes ${}^{\delta_0^{-1}}V_1$, and for simplicity denote the resulting semi-direct product (where v_i varies in V_i) by V, and set $\kappa = \mathrm{diag}(I_{kl}, \kappa_{c',l})\kappa_{l,c-l}$. Note that V is the subgroup of $V_{(c^k)}$ with blocks $v_{i,j}$ (in the notation of (1.1)) of the form

$$\begin{pmatrix} 0_l & 0 & 0 \\ * & 0_{c'} & 0 \\ * & * & 0_l \end{pmatrix}, \tag{5.12}$$

where for any j, $0_j \in \mathrm{Mat}_j$ is the zero matrix. Then the last integral equals

$$\int_V\int_{U_0} f(s, \kappa\delta u_0 v h)\,\psi_U(u_0)\,du_0\,dv.$$

Plugging this back into (5.9), we obtain

$$\int_{R\times R\backslash G\times G}\int_{M_R}\int_{U_R}\int_V\int_{U_0} \delta_R^{-1/2}(m)\langle\varphi(g_1), \varepsilon^\vee(m)\varphi^\vee(g_2)\rangle \tag{5.13}$$
$$f(s, \kappa\delta u_0 v(g_1, {}^\iota(zmg_2)))\,\psi_U(u_0)\,du_0\,dv\,dz\,dm\,d(g_1, g_2).$$

As above, we proceed in $\mathrm{Re}(s) \gg 0$ so that the multiple integral is absolutely convergent.

For $z \in U_R$ we see that

$$\kappa\delta u_0 v(1, {}^\iota z) = {}^{\kappa\delta_0}(1, {}^\iota z)\,\kappa\,\delta_0\,x_z\,\delta_1\,u_z\,r_z\,a_{u_0,z}\,b_z\,v, \tag{5.14}$$

where $x_z \in V_{((k-1)c+c/2,c/2)}$; $u_z \in U_0$ depends on z; $r_z = \begin{pmatrix} I_{kc} & u \\ & I_{kc} \end{pmatrix} \in U_P$ is such that all coordinates of u are zero except the bottom left $c \times c$ block, which equals

$$\begin{pmatrix} 0_l & z_1 & z_2 \\ & 0_{c'} & z_1' \\ & & 0_l \end{pmatrix}, \tag{5.15}$$

$a_{u_0,z} \in V \cap V_{((k-1)c,c)}$, $b_z \in V_{(kc-l,l)} \cap V_{((k-1)c,c)}$ (in particular $a_{u_0,z}$ and b_z commute) and $a_{u_0,z}$ depends on both u_0 and z. Observe the following properties.

(1) Since ${}^{\kappa\delta_0}(1, {}^\iota z) \in V_{(kl,k(c-l))} \ltimes U_P$, $h \mapsto f(s,h)$ is left-invariant under ${}^{\kappa\delta_0}(1, {}^\iota z)$.

(2) ${}^{\kappa\delta_0}x_z$ belongs to $(V_{(l^k)} \times V_{(c'^k)}) \ltimes V_{(kl,k(c-l))}$, changing variables in u_z affects ψ_U, but this cancels with the character emitted when ${}^{\kappa\delta_0}x_z$ transforms on the left of f, i.e., $f(s, \kappa\delta_0 x_z\delta_1 u_z h)\psi_U(u_0) = f(s, \kappa\delta u_0 h)\psi_U(u_0)$.

(3) $\delta u_0 r_z b_z = \delta_0 b_z \delta_1 u_{b_z} r_z$, where $u_{b_z} \in U_0$, and as with ${}^{\kappa\delta_0}x_z$, $f(s, \kappa\delta_0 b_z \delta_1 u_{b_z} r_z h)$ $\psi_U(u_0) = f(s, \kappa\delta u_0 r_z h)\psi_U(u_0)$.

(4) Lastly, by a change of variables $a_{u_0,z}v \mapsto v$.

Now if U° is the subgroup of elements $u_0 r_z$ and we extend ψ_U trivially to U°, (5.13) becomes

$$\int\limits_{R\times R\backslash G\times G}\int\limits_{M_R}\int\limits_{V}\int\limits_{U^\circ} \delta_R^{-1/2}(m)\langle\varphi(g_1), \varepsilon^\vee(m)\varphi^\vee(g_2)\rangle \quad (5.16)$$

$$f(s, \kappa\delta uv(g_1, {}^\iota(mg_2)))\,\psi_U(u)\,du\,dv\,dm\,d(g_1, g_2).$$

Let

$$H^\sigma = \mathrm{GL}_{2kl}, \qquad P^\sigma, \qquad U_0^\sigma, \qquad \delta^\sigma = \delta_0^\sigma\delta_1^\sigma$$

be the groups and elements defined in § 2 for the $\mathrm{GL}_l \times \mathrm{GL}_k$ integral, with the exception that for δ_1^σ we actually take $\delta_1^{-\epsilon_0}$ instead of δ_1 defined there. Also let

$$H', \qquad P', \qquad U_0', \qquad \delta' = \delta_0'\delta_1'$$

be the notation for the $G' \times \mathrm{GL}_k$ integral. Fix the standard parabolic subgroup $L < H$ with $M_L = H^\sigma \times H'$, and regard the groups H^σ and H' as subgroups of M_L.

Put $\kappa^\bullet = {}^{\delta_0^{-1}}\kappa = \mathrm{diag}(\kappa_{l,c'}, I_{kl})\kappa_{c-l,l}$. Conjugating U° by $\kappa^\bullet$, we obtain

$$U^\bullet = {}^{\kappa^\bullet}U^\circ < U_P.$$

Denote the top right $kc \times kc$ block of elements of $U^\bullet$ by $(u^{i,j})_{1\le i,j\le 3}$. We see that $\begin{pmatrix} I_{kl} & u^{1,1} \\ & I_{kl}\end{pmatrix}$ is a general element of U_0^σ and similarly $\begin{pmatrix} I_{kc'} & u^{2,2} \\ & I_{kc'}\end{pmatrix}$ of U_0', $u^{2,1} \in \mathrm{Mat}_{kc'\times kl}$ (resp., $u^{3,1} \in \mathrm{Mat}_{kl}$) and its bottom left $c' \times l$ (resp., $l \times l$) block is 0. This determines the blocks $u^{3,2}$ and $u^{3,3}$ and the dimensions of all the blocks uniquely, and the remaining blocks take arbitrary coordinates such that $U^\bullet < H$. The restriction of ψ_U to $U^\bullet$ is given by the product of characters $\psi_{U_0^\sigma}^{-\epsilon_0}$ and $\psi_{U_0'}$ defined on the corresponding coordinates $u^{1,1}$ and $u^{2,2}$ ($\psi_{U_0^\sigma}^{-\epsilon_0} = \psi_{U_0^\sigma}$ for Sp_{2n}).

Write $\delta_0 = w^{-1}\delta_0'\delta_0^\sigma w_1$ ($\delta_0' \in H' < M_L$), where

$$w^{-1} = \mathrm{diag}(I_{kl}, \begin{pmatrix} & I_{kc'} & & \\ & & & I_{kl} \\ \epsilon_0 I_{kl} & & & \\ & & I_{kc'} & \end{pmatrix}, I_{kl}). \quad (5.17)$$

For Sp_{2n}, $w_1 = w$. Here the case of SO_{2n} requires additional treatment: if kl is odd, $\det w = -1$, whence this decomposition of δ_0 does not hold in H. To remedy this we let $\jmath = \jmath_{kl}$, then $\delta_0 = w^{-1}\jmath\,{}^\jmath\delta_0'\,{}^\jmath\delta_0^\sigma\,\jmath w_1$ and we re-denote $w^{-1} = w^{-1}\jmath$, $\delta_0' = {}^\jmath\delta_0'$, $\delta_0^\sigma = {}^\jmath\delta_0^\sigma$ and $w_1 = \jmath w_1$, and also re-denote $H' = {}^\jmath H'$ and similarly for H^σ (then U_0', U_0^σ and the characters are conjugated by $\jmath$ as well). Also set $\jmath = I_{2kc}$ for Sp_{2n}.

Then

$${}^{w_1}({}^{(\delta_0^{-1}\kappa\delta_0)}\delta_1) = \delta_1^\sigma\delta_1'$$

and if $[u^{i,j}]$ is the subgroup of $U^{\bullet}$ generated by elements whose coordinates $u^{t,t'}$ are zeroed out for $(t,t') \neq (i,j)$,

$$U_0^{\sigma} = {}^{w_1}[u^{1,1}, u^{3,3}], \quad U_0' = {}^{\delta^{\sigma} w_1}[u^{2,2}], \quad Z = {}^{\delta' \delta^{\sigma} w_1}[u^{1,2}, u^{1,3}, u^{2,3}],$$
$$O = [u^{2,1}, u^{3,1}, u^{3,2}].$$

In coordinates

$$ {}^{\jmath}Z = \left\{ \mathrm{diag}(I_{kl}, \begin{pmatrix} I_{kl} & z_1 & & z_2 \\ & I_{kc'} & & \\ & & I_{kc'} & z_1^* \\ & & & I_{kl} \end{pmatrix}, I_{kl}) \in H \right\}. \tag{5.18}$$

We write the integration du as an iterated integral according to these subgroups.

Returning to (5.16), we obtain

$$\int\limits_{R\times R\backslash G\times G} \int\limits_{M_R} \int\limits_{V} \int\limits_{O} \int\limits_{U_0^{\sigma}} \int\limits_{U_0'} \int\limits_{Z} \delta_R^{-1/2}(m) \langle \varphi(g_1), \varepsilon(m)\varphi^{\vee}(g_2)\rangle \tag{5.19}$$
$$f(s, w^{-1} z \delta' u' \delta^{\sigma} u^{\sigma} w_1 o\, \kappa^{\bullet} v(g_1, {}^{\iota}m^{\iota}g_2))\, \psi_{U'}(u') \psi_{U^{\sigma}}^{-\epsilon_0}(u^{\sigma})$$
$$dz\, du'\, du^{\sigma}\, do\, dv\, dm\, d(g_1, g_2).$$

Denote $m(s,\tau,w)f(s,h) = \int_Z f(s, w^{-1}zh)dz$. Let $Y < H$ be the standard parabolic subgroup with $M_Y = \mathrm{GL}_{kl} \times \mathrm{GL}_{kc'} \times \mathrm{GL}_{kl}$ and $Y < P$. When $c' = 0$, ${}^{w}M_Y = \mathrm{GL}_{kl} \times {}^{\jmath}\mathrm{GL}_{kl}$ and if $c' > 0$, ${}^{w}M_Y = \mathrm{GL}_{kl} \times \mathrm{GL}_{kl} \times {}^{\jmath}\mathrm{GL}_{kc'}$. Let $D < H$ denote the standard parabolic subgroup with $M_D = {}^{w}M_Y$ and $D < {}^{\jmath}P$. Then $m(s,\tau,w)$ is a standard intertwining operator taking representations $\mathrm{Ind}_Y^H(\cdots)$ to $\mathrm{Ind}_D^H(\cdots)$. Using (5.10) and transitivity of induction, $m(s,\tau,w)$ becomes an intertwining operator from the space of

$$\mathrm{Ind}_Y^H\left((|\det|^{s-1/2}W_{\psi}(\rho_l(\tau)) \otimes |\det|^{s-1/2}W_{\psi}(\rho_{c'}(\tau)) \otimes |\det|^{s-1/2}W_{\psi}(\rho_l(\tau)))\delta_{P_{(kl,kc',kl)}}^{-1/(2k)}\right)$$

to the space of

$$\mathrm{Ind}_{{}^{\jmath}L}^H\left(\delta_{{}^{\jmath}L}^{-1/2}\left(|\det|^{d} V(s, W_{\psi}(\rho_l(\tau)) \otimes W_{\psi}(\rho_l(\tau^{\vee}))) \otimes V(s, W_{\psi}(\rho_{c'}(\tau)))\right)\right). \tag{5.20}$$

Here d is a constant obtained from the modulus characters ($d = (k-1/2)(c-l) - \epsilon_0/2$).

Let $m = \mathrm{diag}(a, g, a^*) \in M_R$, where $a \in \mathrm{GL}_l$, $g \in G'$ and a^* is uniquely determined by a. Then $dm = dadg$. We see that $(1, {}^{\iota}\mathrm{diag}(a, I_{c'}, a^*))$ which we briefly denote by $(1, {}^{\iota}a)$ commutes with $\kappa^{\bullet}v$, normalizes O (with a change of measure $|\det a|^{(1-k)(c-l)}$) and

$${}^{w_1}(1, {}^{\iota}a) = \mathrm{diag}(I_{kl}, a, I_{2(kc-kl-l)}, a^*, I_{kl}) = (1, a)^{\sigma},$$

that is, the embedding of GL_l in the $\mathrm{GL}_l \times \mathrm{GL}_k$ integral ($\iota^{\sigma} = I_l$).

Now consider $(1, {}^{\iota}\mathrm{diag}(I_l, g, I_l)) = (1, {}^{\iota}g)$. The complication here is that $(1, {}^{\iota}g)$ does not normalize the subgroup ${}^{\delta_0^{-1}}V_1 < V$ nor O. To handle this, consider the subgroup $O^1 < O$ where all the coordinates of $u^{2,1}$ are zero except the bottom right $c' \times ((k-1)l)$ block which is arbitrary, and $u^{3,1}$ is also zero except on the anti-diagonal of $l \times l$ blocks ($u^{3,1} \in \mathrm{Mat}_{kl}$), which are arbitrary, except the bottom left $l \times l$ block which is zero.

Then ${}^{(\kappa^\bullet)^{-1}}O^1$ is normalized by V, denote $V^\bullet = V \ltimes {}^{(\kappa^\bullet)^{-1}}O^1$. Also write O as a direct product $O^\circ \times O^1$ for a suitable $O^\circ < O$, and put $\widetilde{g} = {}^{\kappa^\bullet}(1, {}^\iota g)$. The upshot is that $(1, {}^\iota g)$ normalizes $V^\bullet$, $\widetilde{g}$ commutes with the elements of O° and ${}^{w_1}\widetilde{g} = (1, {}^{\iota'} g)'$, the embedding in the $G' \times \mathrm{GL}_k$ integral. After pushing $(1, {}^\iota g)$ to the left, we may rewrite the integration over O and V_1 as before. The integral becomes

$$\int\limits_{R\times R\backslash G\times G} \int\limits_{V} \int\limits_{O} \int\limits_{\mathrm{GL}_l} \int\limits_{U_0^\sigma} \int\limits_{G'} \int\limits_{U_0'} \delta_R^{-1/2}(a)|\det a|^{(1-k)(c-l)} \langle \varphi(g_1), \sigma^\vee(a) \otimes {\pi'}^\vee(g)\varphi^\vee(g_2)\rangle$$
$$m(s,\tau,w) f(s, (\delta' u'(1, {}^{\iota'} g)')\,(\delta^\sigma u^\sigma (1,a)^\sigma)\, w_1 o\, \kappa^\bullet v(g_1, {}^\iota g_2))$$
$$\psi_{U'}(u')\psi_{U^\sigma}^{-\epsilon_0}(u^\sigma)\, du'\, dg\, du^\sigma\, da\, do\, dv\, d(g_1,g_2). \tag{5.21}$$

Note that $\delta_R^{-1/2}(a)|\det a|^{(1-k)(c-l)} = |\det a|^{-d}$. Considering this integral as a function of the section $m(s,\tau,w)f$, denote it by $\mathcal{I}(m(s,\tau,w)f)$. The $du^\sigma da$-integral is the $\mathrm{GL}_l \times \mathrm{GL}_k$ integral of $\sigma \times (\tau \otimes \tau^\vee)$; the $du' dg$-integral is the $G' \times \mathrm{GL}_k$ integral of $\pi' \times \tau$. Thus multiplying (5.21) by the appropriate γ-factors we obtain, formally at first,

$$\begin{aligned} &\gamma(s, \sigma \times (\tau \otimes \tau^\vee), \psi)\gamma(s, \pi' \times \tau, \psi) Z(s, \omega, f) \\ &= \sigma(-1)^k \tau(-1)^l \pi'(-I_{c'})^k \vartheta(s, c', \tau, \psi)\mathcal{I}(M^*(s, l, \tau \otimes \tau^\vee, \psi) \\ &\quad \times M^*(s, c', \tau, \psi) m(s,\tau,w) f). \end{aligned} \tag{5.22}$$

Note that for SO_{2n} the integral varies slightly from the definition in § 2 because δ_1^σ and ψ_{U^σ} are the inverses of those defined there (i.e., $-\epsilon_0 = -1$ for SO_{2n}). However, this does not change the γ-factor, to see this replace f in (4.1) with its right translate by $\mathrm{diag}(-I_{kl}, I_{kl})$.

We justify the formal application of the functional equations. First note that the $d(g_1, g_2)$-integration is over a compact group, by the Iwasawa decomposition. Hence over p-adic fields it is immediate that this integration can be ignored for this purpose. Over archimedean fields, one can apply Corollary A.3 to replace f with a sum of convolutions against Schwartz functions on $G \times {}^\iota G$. The computation of the integrals will then justify (5.22) once the inner integrals are shown to be proportional (with the correct factor). Alternatively, once we know the inner integral is meromorphic and continuous in the input data, we can use the Banach–Steinhaus Theorem as in [Sou95, § 5, Lemma 1] (see § 6.10). We proceed to handle the intertwining operator and $dodv$-integral. Since we are not confined to a prescribed s, we can assume $m(s,\tau,w)$ as a mapping from (5.10) to (5.20) is onto. While $V(s, W_\psi(\rho_c(\tau)))$ is only a subrepresentation of (5.10) ($\rho_c(\tau)$ is embedded in (1.4)), for the proof of (5.22) we can consider an arbitrary (meromorphic section) f of (5.10) (which is a stronger statement). Now one can take $m(s,\tau,w)f$ which is supported in ${}^{\jmath}LU^{-}_{\jmath L}$, such that its restriction to $U^{-}_{\jmath L}$ is given by a Schwartz function. Then the integrals over V and O reduce to a constant (see Corollary 5.3). This justifies the formal step. Alternatively, note that for fixed g_1 and g_2 the integrand is a Schwartz function of o and v (see the proof of [CFGK19, Lemma 27] and repeatedly use Lemma 5.1); this can also be used for a justification.

Next, applying the same manipulations (5.9)–(5.21) to $Z^*(s,\omega,f)$ yields

$$Z^*(s,\omega,f)=\mathcal{I}(m(1-s,\tau^\vee,w)M^*(s,c,\tau,\psi)f).$$

For any $b\in F^*$ set $C(b)=\tau(b)^{2l}|b|^{2kl(s-1/2)}$. To complete the proof we claim

$$M^*(s,l,\tau\otimes\tau^\vee,\psi)M^*(s,c',\tau,\psi)m(s,\tau,w)=C(1/2)m(1-s,\tau^\vee,w)M^*(s,c,\tau,\psi). \tag{5.23}$$

Granted this, since $c=c'+2l$,

$$C(1/2)\tau(-1)^l\vartheta(s,c',\tau,\psi)=\vartheta(s,c,\tau,\psi), \tag{5.24}$$

and also $\pi(-I_c)=\sigma(-1)\pi'(-I_{c'})$, we obtain the result:

$$\gamma(s,\sigma\times(\tau\otimes\tau^\vee),\psi)\gamma(s,\pi'\times\tau,\psi)Z(s,\omega,f)=\pi(-I_c)^k\vartheta(s,c,\tau,\psi)Z^*(s,\omega,f).$$

We mention that for Sp_{2n}, if $c'=0$, by definition $\gamma(s,\pi'\times\tau,\psi)=\gamma(s,\tau,\psi)$.

Set $d_0=-(c-l)/2$ and $s_0=s-1/2$, and consider the representation

$$\begin{aligned}&\mathrm{Ind}_{\jmath L}^{H}\Big(\delta_{\jmath L}^{-1/2}\Big(\Big(|\det|^{d}V(1-s,W_\psi(\rho_l(\tau^\vee))\otimes W_\psi(\rho_l(\tau))\Big)\Big)\\ &\quad\otimes V(1-s,W_\psi(\rho_{c'}(\tau^\vee)))\Big)\Big) &(5.25)\\ &\cong \mathrm{Ind}_D^H(|\det|^{d_0-s_0}\rho_l(\tau^\vee)\otimes|\det|^{d_0+s_0}\rho_l(\tau)\otimes|\det|^{-s_0}\rho_{c'}(\tau^\vee)). &(5.26)\end{aligned}$$

The space $\mathcal{H}$ of intertwining operators from $V(s,W_\psi(\rho_c(\tau)))$ to the space of (5.26) is, outside a discrete subset of s, at most one dimensional. This follows from the filtration argument in [LR05, Lemma 5], which extends to any $k\geq 1$. Briefly, write $H=\coprod_h PhD$ where h varies over the representatives of $W(M_P)\backslash W(H)/W(M_D)$, and for h and $\nu\geq 0$ define

$$\begin{aligned}\mathcal{H}_\nu(h)=\mathrm{Hom}_{D_h}(&|\det|^{s_0}\rho_c(\tau)\otimes{}^h\Big(|\det|^{-d_0+s_0}\rho_l(\tau)\otimes|\det|^{-d_0-s_0}\rho_l(\tau^\vee)\\ &\otimes|\det|^{s_0}\rho_{c'}(\tau)\Big)\otimes\Lambda_{h,\nu},\theta_h).\end{aligned}$$

Here $D_h={}^hD\cap P$; over archimedean fields $\Lambda_{h,\nu}$ is the algebraic dual of the symmetric ν-th power of $\mathbb{W}_h=\mathrm{Lie}(H)/(\mathrm{Lie}(P)+\mathrm{Ad}(h)\mathrm{Lie}(D))$, and over non-archimedean fields it is simply omitted; and $\theta_h(x)=\delta_{D_h}(x)\delta_D^{-1/2}({}^{h^{-1}}x)\delta_P^{-1/2}(x)$. Denote $h\sim h'$ if $PhD=Ph'D$. According to the Bruhat Theory ([Sil79, Theorems 1.9.4–5] over non-archimedean, [War72, Proposition 5.2.1.2, Theorem 5.3.2.3] over archimedean fields), $\dim \mathcal{H} \leq \sum_{h,\nu} \dim \mathcal{H}_\nu(h)$ (a finite sum in the p-adic case). But arguing as in [LR05, Lemma 5] using central characters (see also [GK, § 2.1.2 and (2.7)]), there is a discrete subset $\mathcal{B} \subset \mathbb{C}$ such that for all $s \notin \mathcal{B}$, $\mathcal{H}_\nu(h) = 0$ except when

$$h \sim h_0 = \begin{pmatrix} & I_{k(c-l)} & & \\ I_{kl} & & & \\ & & & I_{kl} \\ & & I_{k(c-l)} & \end{pmatrix} \jmath \begin{pmatrix} & & & & & I_{kl} \\ & I_{kl} & & & & \\ & & & I_{kc'} & & \\ & & \epsilon_0 I_{kc'} & & & \\ & & & & I_{kl} & \\ \epsilon_0 I_{kl} & & & & & \end{pmatrix}, \qquad \nu = 0.$$

Note that $\theta_{h_0} = \delta^{1/2}_{P_{(kl,kc',kl)}}$. Here since h_0 is not the longest Weyl element, in the archimedean case $\dim \Lambda_{h_0,\nu} > 1$ unless $\nu = 0$. To eliminate $\mathcal{H}_\nu(h_0)$ for $\nu > 0$ observe that each nonzero subspace of $\mathbb{W}_{h_0}$ is an eigenspace for the action of $tI_{kc} \in C_{M_P} < D_h$ corresponding to an eigenvalue $|t|^a$ for some integer $a > 0$ (direct computation). Then we can consider a second filtration, of $\Lambda_{h_0,\nu}$, such that the action of C_{M_P} on the i-th constituent is given by $|t|^{a_i}$ with $a_i < 0$, and since C_{M_P} acts trivially on

$$|\det|^{s_0}\rho_c(\tau) \otimes {}^{h_0}\left(|\det|^{-d_0+s_0}\rho_l(\tau) \otimes |\det|^{-d_0-s_0}\rho_l(\tau^\vee) \otimes |\det|^{s_0}\rho_{c'}(\tau)\right)$$

($\rho_c(\tau)$ admits a central character) and θ_{h_0} is trivial on C_{M_P}, $\mathcal{H}_\nu(h_0) = 0$ when $\nu > 0$.

It remains to consider $\mathcal{H}_0(h_0)$. Since $V_{(kl,kc',kl)} < {}^{h_0}U_D \cap P < D_{h_0}$, each morphism in $\mathcal{H}_0(h_0)$ factors through $\delta^{1/2}_{P_{(kl,kc',kl)}} J_{V_{(kl,kc',kl)}}(\rho_c(\tau))$ (see e.g., [GK, (2.5)]) whence $\mathcal{H}_0(h_0)$ becomes

$$\begin{aligned} &\mathrm{Hom}_{M_D}(|\det|^{s_0} J_{V_{(kl,kc',kl)}}(\rho_c(\tau)) \otimes \Big(|\det|^{-d_0-s_0}\rho_l(\tau^\vee) \otimes |\det|^{-s_0}\rho_{c'}(\tau^\vee) \\ &\quad \otimes |\det|^{d_0-s_0}\rho_l(\tau^\vee)\Big), 1) \\ &= \mathrm{Hom}_{M_D}(J_{V_{(kl,kc',kl)}}(\rho_c(\tau)), |\det|^{d_0}\rho_l(\tau) \otimes \rho_{c'}(\tau) \otimes |\det|^{-d_0}\rho_l(\tau)). \end{aligned}$$

When $k = 1$, $\dim \mathcal{H}_0(h_0) = 1$ immediately because $\rho_c(\tau) = \tau \circ \det$. In light of (1.4) (applied twice, see (5.10)), the proof of [CFGoK, Lemma 9] (where we considered an arbitrary summand) and the Frobenius reciprocity law ([Cas80a], [HS83, Theorem 4.9], [Cas89]) there are $\dim \mathcal{H}_0(h_0)$ constituents of $\rho_c(\tau)$ which afford (k, c) functionals. Since $\rho_c(\tau)$ is (k, c) and the generalized Whittaker functor is exact ([GGS17, Corollary G], over p-adic fields [BZ76]), $\dim \mathcal{H}_0(h_0) = 1$.

Since both sides of (5.23) take $V(s, W_\psi(\rho_c(\tau)))$ into the space of (5.25) which is isomorphic to (5.26), they are proportional. It remains to compute the proportionality factor. We argue as in [LR05, Lemma 9]. Denote

$$\lambda = \lambda_2(s, l, \tau \otimes \tau^\vee, \psi)\lambda(s, c', \tau, \psi),$$
$$\lambda^\vee = \lambda_2(1 - s, l, \tau^\vee \otimes \tau, \psi)\lambda(1 - s, c', \tau^\vee, \psi).$$

Here $\lambda_2(\cdots)$ are the functionals appearing in (3.4) except that the character ψ appearing in (3.3) is replaced with $\psi_{-2\epsilon_0}$ (but $\rho_l(\tau)$ is still realized in $W_\psi(\rho_l(\tau))$).

Define the following functionals: for $f_0(s,\cdot)$ in the space of (5.20) and $f_0^\vee(1-s,\cdot)$ in the space of (5.25),

$$\Lambda_\lambda(f_0)=\int_{O^\bullet}\lambda f_0(s,w_1o^\bullet\kappa^\bullet)\,do^\bullet,\qquad \Lambda_{\lambda^\vee}(f_0^\vee)=\int_{O^\bullet}\lambda^\vee f_0^\vee(1-s,w_1o^\bullet\kappa^\bullet)\,do^\bullet.$$

Here $O^\bullet$ is the subgroup ${}^{\kappa^\bullet}V\ltimes O^{'}$, where $O^{'}$ is obtained from O be replacing the zero blocks in $u^{2,1}$ and $u^{3,1}$ by arbitrary coordinates; $\lambda_2(s,l,\tau\otimes\tau^\vee,\psi)$ and $\lambda(s,c',\tau,\psi)$ are applied to the restriction of $f_0(s,\cdot)$ to $M_{\jmath L}$. The integrands are Schwartz functions on $O^\bullet$. This follows from Lemma 5.1 and the fact that using right translations of f by unipotent elements, we can eliminate the roots in $O^\bullet$. For a description of these elements see the proof of [CFGK19, Lemma 27] (U^3 in their notation corresponds to O, the additional blocks of $O^{'}$ can be handled similarly). See also [LR05, Lemma 8] and the example on [LR05, p. 325].

First we show

$$\lambda(s,c,\tau,\psi)f=\Lambda_\lambda(m(s,\tau,w)f). \tag{5.27}$$

This actually follows from the arguments above: repeat the steps (5.9)–(5.19) (excluding arguments regarding G and δ_1), in particular apply (1.5) twice, and (5.16) is modified by replacing (U°,ψ_U) with U_P and its character defined by $\lambda(s,c,\tau,\psi)$. Specifically,

$$\begin{aligned}\lambda(s,c,\tau,\psi)f&=\int\limits_V\int\limits_{U_P}f(s,\kappa\delta_0uv)\,\psi^{-1}(u)\,du\,dv\\&=\int\limits_{O^\bullet}\int\limits_{U_{P^\sigma}}\int\limits_{U_{P'}}\int\limits_{Z}f(s,w^{-1}z\delta_0'u'\delta_0^\sigma u^\sigma w_1o^\bullet\kappa^\bullet)\,\psi^{-1}(u')\psi_{-2\epsilon_0}^{-1}(u^\sigma)\,d(\cdots)\\&=\Lambda_\lambda(m(s,\tau,w)f).\end{aligned}$$

Here $(U_{P^\sigma},U_{P'})$ replaced (U_0^σ,U_0') in (5.19) (keeping the identification of H^σ or H' with their conjugations by $\jmath$ as above), and note that we obtain $\psi_{-2\epsilon_0}$ on U^σ.

Now on the one hand, using (5.27) and applying (3.4) twice implies

$$\lambda(s,c,\tau,\psi)f=\Lambda_{\lambda^\vee}(C(2)M^*(s,l,\tau\otimes\tau^\vee,\psi)M^*(s,c',\tau,\psi)m(s,w,\tau)f). \tag{5.28}$$

Here $C(2)$ is obtained when in (3.4), f is replaced with its right translate by $\mathrm{diag}(-2\epsilon_0I_{kl},I_{kl})$. On the other hand again by (5.27),

$$\lambda(1-s,c,\tau^\vee,\psi)M^*(s,c,\tau,\psi)f=\Lambda_{\lambda^\vee}(m(1-s,\tau^\vee,w)M^*(s,c,\tau,\psi)f). \tag{5.29}$$

Then (5.23) follows when we equate the left hand sides of (5.28) and (5.29) using (3.4).

5.3.2 The group SO_{2n+1}. Let $G = \mathrm{SO}_{2n+1}$. We can argue as above, and reach (5.9). Then apply (1.4)–(1.5) twice and obtain a section of the space induced from (5.10), with $c' = c - 2l = 2(n-l)+1$. We still obtain (5.13), except that V is slightly different: this is because the last c columns of v_1 are affected differently by δ_0 in the conjugation ${}^{\delta_0^{-1}}v_1$ (permuted and for odd k, one column is negated). Now ${}^{\jmath_k}V < V_{(c^k)}$ ($\jmath_k = \jmath_{kc}$ because c is odd), the blocks $v_{i,j}$ of ${}^{\jmath_k}V$ are given by (5.12) for $j < k$, and the blocks $v_{i,k}$ take the form

$$\begin{pmatrix} 0_l & 0 & 0 \\ * & 0_{c'} & 0 \\ * & a_2 & a_3 \end{pmatrix}, \qquad a_3 \in \mathrm{Mat}_l, \tag{5.30}$$

where the rightmost column of a_2 and first $l-1$ columns of a_3 are zero.

For $z \in U_R$, we see that (5.14) holds except the following modifications: ${}^{\jmath_k}x_z \in V_{((k-1)c+n,n+1)}$; ${}^{\jmath_k}r_z = \left(\begin{smallmatrix} I_{kc} & u \\ & I_{kc} \end{smallmatrix}\right)$ and instead of (5.15), the bottom left $c \times c$ block of u becomes

$$\begin{pmatrix} z_0 & 0_l & z_1 & z_2 \\ & & 0_{2(n-l)} & z_1' \\ & & & 0_l \\ 0 & & & z_0' \end{pmatrix}; \tag{5.31}$$

${}^{\jmath_k}a_{u_0,z} \in V \cap V_{((k-1)c,c)}$ and ${}^{\jmath_k}b_z = b_z \in V_{(kc-l-1,l+1)} \cap V_{((k-1)c,c)}$. Properties (1)–(4) hold and we reach the analog of (5.16).

Now we use the notation $H^\sigma, U_0^\sigma, H', U_0'$ etc., for the $\mathrm{GL}_l \times \mathrm{GL}_k$ and $G' \times \mathrm{GL}_k$ integrals. As with the SO_{2n} case, we take δ_1^σ to be the inverse of this element defined in § 2. Put

$$\delta_{k,n} = \mathrm{diag}\left(\begin{pmatrix} I_n & & \\ & & (-1)^k \\ & I_n & \end{pmatrix}, \begin{pmatrix} & I_n & \\ (-1)^k & & \\ & & I_n \end{pmatrix}\right)\jmath_k \in \mathrm{O}_{2c}$$

and $w_0 = \left(\begin{smallmatrix} & I_{kc} \\ I_{kc} & \end{smallmatrix}\right)$, then $\delta_0 = w_0 \,\mathrm{diag}\left(I_{(k-1)c}, \delta_{k,n}, I_{(k-1)c}\right)$. Set

$$\kappa^\bullet = ({}^{w^{-1}}\mathrm{diag}(I_{(k-1)c+2l}, \delta_{k,n-l}^{-1}, I_{(k-1)c+2l}))({}^{w_0^{-1}}\kappa)\,\mathrm{diag}(I_{(k-1)c}, \delta_{k,n}, I_{(k-1)c}), \tag{5.32}$$

where w^{-1} is the matrix given in (5.17) (here $w_1 = w$). Then

$$\kappa\delta_0 = w^{-1}\delta_0'\delta_0^\sigma w\kappa^\bullet, \qquad \kappa\delta = w^{-1}\delta'\delta^\sigma w\kappa^\bullet.$$

Note that $\det\kappa^\bullet = 1$. Then $U^\bullet = {}^{\kappa^\bullet}U^\circ$, and observe that ${}^{(w^{-1}\jmath_k w)}U^\bullet < U_P$. The subgroup $U^\bullet$ now plays the same role as in the previous cases.

For odd kl we re-denote $w^{-1} = w^{-1}\jmath_{kl}$ (because then the determinant of (5.17) is -1), $\delta' = {}^{\jmath_{kl}}\delta'$, $H' = {}^{\jmath_{kl}}H'$ and similarly for U_0' and $\psi_{U'}$ ($c' > 0$, always, hence ${}^{\jmath_{kl}}\delta^\sigma = \delta^\sigma$ and H^σ remains the same when kl is odd). Let

$$U_0^\sigma = {}^{w}[u^{1,1}, u^{3,3}], \quad U_0' = {}^{\delta^\sigma w}[u^{2,2}], \quad Z = {}^{\delta'\delta^\sigma w}[u^{1,2}, u^{1,3}, u^{2,3}],$$
$$O = [u^{2,1}, u^{3,1}, u^{3,2}].$$

Here ${}^{\jmath_k}U'_0 < U_{P'}$ (the form of P' also depends on the parity of kl) and ${}^{\jmath_{kl}}Z$ is given by the r.h.s. of (5.18) (here $\jmath = \jmath_{kl}$). The integral becomes the analog of (5.19). We denote $m(s,\tau,w)f$ as above, it belongs to (5.20) (with $\jmath = \jmath_{kl}$, $d = (k-1/2)(c-l)$).

Let $m = \mathrm{diag}(a, g, a^*) \in M_R$. We see that $(1, {}^{\iota}a)$ commutes with v (look at (5.30)), ${}^{\kappa^\bullet}(1, {}^{\iota}a)$ normalizes O (multiplying the measure by $|\det a|^{(1-k)(c-l)}$) and ${}^{w\kappa^\bullet}(1, {}^{\iota}a) = (1, a)^\sigma$. Regarding $(1, {}^{\iota}g)$, we define $O^\circ \times O^1$ and $V^\bullet$ exactly as above (except that c' is different). Then $(1, {}^{\iota}g)$ normalizes $V^\bullet$, ${}^{\kappa^\bullet}(1, {}^{\iota}g)$ commutes with the elements of O° and ${}^{w\kappa^\bullet}(1, {}^{\iota}g) = (1, {}^{\iota'}g)'$, the embedding in the $G' \times \mathrm{GL}_k$ integral. Finally we obtain the analog of (5.21), i.e.,

$$\int\limits_{R\times R\backslash G\times G} \int\limits_{V} \int\limits_{O} \int\limits_{\mathrm{GL}_l} \int\limits_{U_0^\sigma} \int\limits_{G'} \int\limits_{U_0'} |\det a|^{-d} \langle \varphi(g_1), \sigma^\vee(a) \otimes {\pi'}^\vee(g)\varphi^\vee(g_2)\rangle \tag{5.33}$$
$$m(s,\tau,w)f(s, (\delta' u'(1, {}^{\iota'}g)')\,(\delta^\sigma u^\sigma (1,a)^\sigma)\, wo\,\kappa^\bullet v(g_1, {}^{\iota}g_2))$$
$$\psi_{U'}(u')\psi^{-1}_{U^\sigma}(u^\sigma)\, du'\, dg\, du^\sigma\, da\, do\, dv\, d(g_1, g_2).$$

Note that for $l = n$, $G' = \{1\}$ but if $k > 1$, $U'_0 = {}^{\jmath_k}\{\left(\begin{smallmatrix} I_k & x \\ & I_k \end{smallmatrix}\right) \in \mathrm{SO}_{2k}\}$ is nontrivial,

$$\delta' = \left(\begin{smallmatrix} & I_k \\ I_k & \end{smallmatrix}\right) \mathrm{diag}(I_{k-1}, (-1)^k \jmath_k, I_{k-1}), \qquad \psi_{U'}(u') = \psi^{-1}((-1)^k \tfrac{1}{2}({}^{\jmath_k}u')_{k-1,k+1})$$

and we have a Whittaker functional on $V(s, W_\psi(\tau))$. For even k, the rest of the proof now follows as above.

For odd k, recall that the integral $Z^*(s,\omega,f)$ is slightly different from $Z(s,\omega,f)$ (see § 2.4). Since we already proved (4.2), and also (4.3) for even k, it is enough to assume $k = 1$. This is clear over archimedean fields. Over p-adic fields, let τ be an irreducible tempered representation of GL_{2k+1} and take a unitary character τ_0 of F^*, then $\widehat{\tau} = \mathrm{Ind}_{P_{(2k+1,1)}}^{\mathrm{GL}_{2k+2}}(\tau \otimes \tau_0)$ is irreducible tempered,

$$\begin{aligned}\gamma(s, \pi\times\tau, \psi)\gamma(s, \pi\times\tau_0, \psi) &= \gamma(s, \pi\times\widehat{\tau}, \psi) = \gamma(s, \pi'\times\widehat{\tau}, \psi)\gamma(s, \sigma\times(\widehat{\tau}\otimes\widehat{\tau}^\vee), \psi)\\ &= \gamma(s, \pi'\times\tau, \psi)\gamma(s, \pi'\times\tau_0, \psi)\gamma(s, \sigma\times(\tau\otimes\tau^\vee), \psi)\gamma(s, \sigma\times(\tau_0\otimes\tau_0^{-1}), \psi).\end{aligned}$$

Hence (4.3) for $\gamma(s, \pi\times\tau_0, \psi)$ implies (4.3) for $\gamma(s, \pi\times\tau, \psi)$.

Let $\delta_{0,\mathrm{odd}}$ and $\delta_{1,\mathrm{odd}}$ be the corresponding elements δ_i in the construction of $Z^*(s,\omega,f)$, $\delta'_{i,\mathrm{odd}}$ be these elements for $G' \times \mathrm{GL}_1$, $\delta_{\mathrm{odd}} = \delta_{0,\mathrm{odd}}\delta_{1,\mathrm{odd}}$ and $\delta'_{\mathrm{odd}} = \delta'_{0,\mathrm{odd}}\delta'_{1,\mathrm{odd}}$. Put

$$t_0 = \mathrm{diag}(I_{2n}, -2, -1/2, I_{2n}), \qquad t_1 = \mathrm{diag}(I_n, -I_n, I_2, -I_n, I_n).$$

Then $\delta_{0,\mathrm{odd}} = \jmath_1\delta_0\jmath_1 t_0$ and $\delta_{1,\mathrm{odd}} = \delta_1 t_1$. Note that κ and V are trivial now (since $k = 1$). Define $\kappa^\bullet$ by (5.32). Then $w\kappa^\bullet$ commutes with $\jmath_1$ and t_0, and $w\kappa^\bullet t_1 (w\kappa^\bullet)^{-1} = t'_1 t^\sigma_1$, where

$$t'_1 = \mathrm{diag}(I_{n-l}, -I_{n-l}, I_2, -I_{n-l}, I_{n-l}), \qquad t^\sigma_1 = \mathrm{diag}(I_l, -I_l)$$

(for the product $t'_1t^\sigma_1$, t'_1 is regarded as an element of H', and $t^\sigma_1 \in H^\sigma$). Thus

$$\delta_{0,\mathrm{odd}} = \jmath_1\delta_0\jmath_1 t_0 = \jmath_1(w^{-1}\delta'_0\delta^\sigma_0 w\kappa^\bullet)\jmath_1 t_0 = {}^{\jmath_1}w^{-1}\delta'_{0,\mathrm{odd}}\delta^\sigma_0 w\kappa^\bullet,$$
$$\delta_{\mathrm{odd}} = \jmath_1\delta_0\jmath_1 t_0\delta_1 t_1 = \jmath_1(w^{-1}\delta'\delta^\sigma w\kappa^\bullet)\jmath_1 t_0 t_1 = {}^{\jmath_1}w^{-1}\delta'_{\mathrm{odd}}\delta^\sigma t^\sigma_1 w\kappa^\bullet.$$

Assume l is even. Denote

$$m(1-s,\tau^\vee,w)f = \int\limits_Z f(1-s, {}^{\jmath_1}w_1^{-1}\,{}^{\jmath_1}\operatorname{diag}(I_l, \begin{pmatrix} I_l & z_1 & & z_2 \\ & I_{c'} & & \\ & & I_{c'} & z_1^* \\ & & & I_l \end{pmatrix}, I_l))\,dz.$$

Taking $z \in U_R$ and conjugating to the left, integral $Z^*(s,\omega,f)$ becomes

$$\int\limits_{R\times R\backslash G\times G}\int\limits_{\mathrm{GL}_l}\int\limits_{G'} |\det a|^{-d}\langle\varphi(g_1), \sigma^\vee(a)\otimes{\pi'}^\vee(g)\varphi^\vee(g_2)\rangle m(1-s,\tau^\vee,w)M^*(s,c,\tau,\psi)$$
$$f(1-s,(\delta'_{\mathrm{odd}}(1,{}^{\iota'}g)')\,(\delta^\sigma t^\sigma_1(1,a)^\sigma)\,w\kappa^\bullet(g_1,{}^\iota g_2))\,dg\,da\,d(g_1,g_2). \tag{5.34}$$

We change variables $a \mapsto -a$ to remove t^σ_1 from the integrand, thereby emitting $\sigma(-1)$ (!). To relate between (5.33) (with $k=1$) and (5.34), we need the analog of (5.23).

First assume $l < n$. Then we claim

$$M^*(s,l,\tau\otimes\tau^\vee,\psi)M^*(s,c',\tau,\psi)m(s,\tau,w) = C(1/2)m(1-s,\tau^\vee,w)M^*(s,c,\tau,\psi). \tag{5.35}$$

($C(b) = \tau(b)^{2l}|b|^{2l(s-1/2)}$.) Now we may proceed as in § 5.3.1: apply the functional equations of $\mathrm{GL}_l\times\mathrm{GL}_1$ and $G'\times\mathrm{GL}_1$ to (5.33), use (5.35) and (5.24), and deduce

$$\begin{aligned}&\gamma(s,\sigma\times(\tau\otimes\tau^\vee),\psi)\gamma(s,\pi'\times\tau,\psi)Z(s,\omega,f) \\ &= \tau(-1)^l\vartheta(s,c',\tau,\psi)\mathcal{I}(M^*(s,l,\tau\otimes\tau^\vee,\psi)M^*(s,c',\tau,\psi)m(s,\tau,w)f) \\ &= \vartheta(s,c,\tau,\psi)\mathcal{I}(m(1-s,\tau^\vee,w)M^*(s,c,\tau,\psi)f) = \vartheta(s,c,\tau,\psi)Z^*(s,\omega,f).\end{aligned} \tag{5.36}$$

This completes the proof for $k=1$ (under (5.35)), even l and $l<n$. When $l=n$, we claim

$$M^*(s,l,\tau\otimes\tau^\vee,\psi)m(s,\tau,w)f = C(1/2)(t_0\cdot m(1-s,\tau^\vee,w)M^*(s,c,\tau,\psi)f)^{\jmath_1}. \tag{5.37}$$

Granted that, since in this case $\delta'_{\mathrm{odd}} = \jmath_1\delta'\jmath_1 t_0$, we can conjugate $\jmath_1 t_0$ to the right in (5.34) ($w\kappa^\bullet$ commutes with $\jmath_1 t_0$). Moreover, $\jmath_1 t_0$ is the image of $(1,\operatorname{diag}(I_n,-1,I_n))$ (see § 2.2), therefore commutes with $(g_1,1)$, and the conjugation of $(1,{}^\iota g_2)$ by $\jmath t_0$ is an outer involution of G. We can therefore rewrite (5.34) in the form

$$\mathcal{I}((t_0\cdot m(1-s,\tau^\vee,w)M^*(s,c,\tau,\psi)f)^{\jmath_1}),$$

where $\mathcal{I}(\cdots)$ is given by (5.33). Using (5.37) we obtain an analog of (5.36),

$$\gamma(s,\sigma\times(\tau\otimes\tau^\vee),\psi)Z(s,\omega,f)=\tau(-1)^l\mathcal{I}(M^*(s,l,\tau\otimes\tau^\vee,\psi)m(s,\tau,w)f)$$
$$=\vartheta(s,c,\tau,\psi)\mathcal{I}((t_0\cdot m(1-s,\tau^\vee,w)M^*(s,c,\tau,\psi)f)^{\jmath_1})=\vartheta(s,c,\tau,\psi)Z^*(s,\omega,f).$$

To prove (5.35), first recall that the functional equation (3.4) reads

$$\lambda(s,c,\tau,\psi)f=\lambda(1-s,c,\tau^\vee,\psi)(t_0\cdot M^*(s,c,\tau,\psi)f)^{\jmath_1}.$$

(Here $\jmath_{kc}=\jmath_1$.) For brevity, put

$$f_1=t_0\cdot M^*(s,l,\tau\otimes\tau^\vee,\psi)M^*(s,c',\tau,\psi)m(s,\tau,w)f,$$
$$f_2=t_0\cdot m(1-s,\tau^\vee,w)M^*(s,c,\tau,\psi)f.$$

Since l is even, these sections belong to the same space. We claim $C(2)f_1=f_2$. Starting with the l.h.s. of (3.4) and applying the functional equations defining the normalized intertwining operators on the Levi components, we obtain

$$\int\limits_{{}^{\jmath_1}U_P} f(s,\delta_0u)\,\psi^{-1}(u)\,du=C(2)\int\limits_{O^\bullet}\int\limits_{U_{P^\sigma}}\int\limits_{{}^{\jmath_1}U_{P'}} f_1^{\jmath_1}(s,\delta_0'u'\delta_0^\sigma u^\sigma w_1o^\bullet\kappa^\bullet)$$
$$\times\,\psi^{-1}(u')\psi_{-2}^{-1}(u^\sigma)\,d(\cdots).$$

On the r.h.s. we similarly have

$$\int\limits_{{}^{\jmath_1}U_P}(t_0\cdot M^*(s,c,\tau,\psi)f)^{\jmath_1}(s,\delta_0u)\,\psi^{-1}(u)\,du$$
$$=\int\limits_{O^\bullet}\int\limits_{U_{P^\sigma}}\int\limits_{{}^{\jmath_1}U_{P'}}\int\limits_{Z} M^*(s,c,\tau,\psi)f(s,{}^{\jmath_1}w^{-1}\,{}^{\jmath_1}z\,\jmath_1\delta_0'u'\delta_0^\sigma u^\sigma w_1o^\bullet\kappa^\bullet\jmath_1t_0)$$
$$\times\,\psi^{-1}(u')\psi_{-2}^{-1}(u^\sigma)\,d(\cdots)$$
$$=\int\limits_{O^\bullet}\int\limits_{U_{P^\sigma}}\int\limits_{{}^{\jmath_1}U_{P'}} f_2^{\jmath_1}(s,\delta_0'u'\delta_0^\sigma u^\sigma w_1o^\bullet\kappa^\bullet)\psi^{-1}(u')\psi_{-2}^{-1}(u^\sigma)\,d(\cdots).$$

We proceed as in § 5.3.1 to deduce $C(2)f_1=f_2$, i.e., (5.35). The difference in the proof of (5.37) is that there is no functional equation for H' ($U_{P'}=\{1\}$ when $l=n$ and $k=1$). In turn, we have t_0 and $\jmath_1$ on the r.h.s. of (5.35) but not on the left, and instead of $f_1^{\jmath_1}$, we have $M^*(s,l,\tau\otimes\tau^\vee,\psi)m(s,\tau,w)f$. The proof of the case $k=1$, even l with $l\le n$, is complete.

The case of odd l is treated as before, by taking ${}^{\jmath_1}w^{-1}\jmath_1=\jmath_1w^{-1}$, ${}^{\jmath_1}\delta_{0,\mathrm{odd}}'$, etc. When comparing both sides of (5.35) (now with odd $l<n$), note that on the l.h.s. w^{-1} was replaced with $w^{-1}\jmath_1$, so again both sides belong to the same space. For $l=n$ we have (5.37). The proof is similar.

5.3.3 Example: $\mathrm{SO}_3 \times \mathrm{GL}_1$. We provide an example illustrating § 5.3.2 for $n = k = 1$ (the integral in [LR05] is different because it was defined for O_3). We follow the steps leading to (5.19) and see that $Z(s, \omega, f)$ equals

$$\int\limits_{R\times R\backslash G\times G} \langle \varphi(g_1), \varphi^\vee(g_2)\rangle \int\limits_{F^*}\int\limits_{F} |a|^{-1/2}\sigma^{-1}(a) f(s, \delta \begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \\ & a^{-1} \end{pmatrix} (g_1, {}^\iota g_2))\, dz\, da\, d(g_1, g_2). \tag{5.38}$$

Here matrices in GL_r, $r = 2, 3$, are identified with elements in H using the mapping $m \mapsto \mathrm{diag}(m, I_{6-2r}, m^*)$. Write $\delta_0 = w^{-1}\jmath_1\delta_0'\delta_0^\sigma\jmath_1 w\kappa^\bullet$ with

$$\delta_0' = \mathrm{diag}(I_2, -I_2, I_2), \qquad \delta_0^\sigma = \mathrm{diag}(J_2, I_2, J_2),$$
$$w^{-1} = \begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & & & & 1 & \\ & 1 & & & & \\ & & & 1 & & \\ & & & & & 1 \end{pmatrix}, \qquad \kappa^\bullet = \mathrm{diag}(1, J_2, J_2, 1).$$

(For odd l, $\delta_0 = w^{-1}\jmath_1({}^{\jmath_1}\delta_0')\delta_0^\sigma\jmath_1 w\kappa^\bullet$, but since $l = n$ and $k = 1$, $\delta_0' = {}^{\jmath_1}\delta_0'$.) Also set

$$\jmath = \jmath_1 = \mathrm{diag}(I_2, J_2, I_2), \qquad t_0 = \mathrm{diag}(I_2, -2, -1/2, I_2),$$
$$t_1 = \mathrm{diag}(1, -1, I_2, -1, 1),$$

and $m(s, \tau, w) f(s, h) = \int_F f(s, w^{-1}\jmath u(z)h) dz$ with

$$u(z) = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & z & & \\ & & & 1 & -z & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

Then (5.38) becomes (5.33), which in this case is

$$\int\limits_{R\times R\backslash G\times G} \langle \varphi(g_1), \varphi^\vee(g_2)\rangle \int\limits_{F^*} |a|^{-1/2}\sigma^{-1}(a) \times m(s, \tau, w) f(s, \delta'\delta_0^\sigma \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & a \end{pmatrix} \jmath w\kappa^\bullet(g_1, {}^\iota g_2))\, d(\cdots).$$

It follows that

$$\gamma(s, \sigma \times (\tau \otimes \tau^{-1}), \psi) Z(s, \omega, f) = \sigma(-1)\tau(-1)\mathcal{I}(M^*(s, 1, \tau \otimes \tau^{-1}, \psi) m(s, \tau, w) f). \tag{5.39}$$

The integral $Z^*(s, \omega, f)$ is slightly different from $Z(s, \omega, f)$, because k is odd. The element δ_{odd} (e.g., with $\delta_{0,\mathrm{odd}}$ given by (2.3)) equals $\jmath\delta\jmath t_0 t_1$, where δ is the element appearing in (5.38), and $Z^*(s, \omega, f)$ equals (compare to (5.38))

$$\int\limits_{R\times R\backslash G\times G} \langle \varphi(g_1), \varphi^\vee(g_2)\rangle \int\limits_{F^*}\int\limits_{F} |a|^{-1/2}\sigma^{-1}(a) M^*(s, c, \tau, \psi) f$$
$$(1 - s, \jmath\delta\jmath t_0 t_1 \jmath \begin{pmatrix} 1 & & z/2 \\ & 1 & \\ & & 1 \end{pmatrix} \jmath \begin{pmatrix} 1 & \\ & a^{-1} \end{pmatrix} (g_1, {}^\iota g_2))\, dz\, da\, d(g_1, g_2).$$

Conjugating by t_1 and $\jmath t_0$ we obtain

$$\int\limits_{R\times R\backslash G\times G} \langle\varphi(g_1),\varphi^\vee(g_2)\rangle \int\limits_{F^*}\int\limits_{F} |a|^{-1/2}\sigma^{-1}(a)M^*(s,c,\tau,\psi)f$$
$$(1-s,\jmath\delta\begin{pmatrix}1&&z\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&\\&-a^{-1}\end{pmatrix}\jmath t_0(g_1,{}^\iota g_2))\,dz\,da\,d(g_1,g_2).$$

Decomposing δ_0 as above now gives

$$\int\limits_{R\times R\backslash G\times G} \langle\varphi(g_1),\varphi^\vee(g_2)\rangle \int\limits_{F^*}\int\limits_{F} |a|^{-1/2}\sigma^{-1}(a)m(1-s,\tau^{-1},w)M^*(s,c,\tau,\psi)f$$
$$(1-s,\jmath\delta'\delta_0^\sigma\begin{pmatrix}1&-1\\&1\end{pmatrix}\begin{pmatrix}1&\\&-a\end{pmatrix}\jmath w\kappa^\bullet\jmath t_0(g_1,{}^\iota g_2))\,da\,d(g_1,g_2),$$

where $m(1-s,\tau^{-1},w)f(1-s,h)=\int_F f(1-s,\jmath w^{-1}\,{}^\jmath u(z)h)dz$. Then changing variables $a\mapsto -a$ emits $\sigma(-1)$. Moreover, $\jmath t_0$ is the image of $(1,\mathrm{diag}(1,-1,1))$, hence commutes with $(g_1,1)$ and the conjugation of $(1,{}^\iota g_2)$ by $\jmath t_0$ is an outer involution. The integral becomes

$$\sigma(-1)\int\limits_{R\times R\backslash G\times G} \langle\varphi(g_1),\varphi^\vee(g_2)\rangle \int\limits_{F^*}\int\limits_{F} |a|^{-1/2}\sigma^{-1}(a)$$
$$m(1-s,\tau^{-1},w)M^*(s,c,\tau,\psi)(t_0\cdot f)(1-s,\jmath\delta'\delta_0^\sigma\begin{pmatrix}1&-1\\&1\end{pmatrix}\begin{pmatrix}1&\\&a\end{pmatrix}\jmath w\kappa^\bullet(g_1,{}^\iota g_2)\jmath)\,d(\cdots).$$

Therefore

$$Z^*(s,\omega,f)=\sigma(-1)\mathcal{I}((t_0\cdot m(1-s,\tau^{-1},w)M^*(s,c,\tau,\psi)f)^\jmath). \tag{5.40}$$

The functional equation (3.4) will show

$$M^*(s,1,\tau\otimes\tau^{-1},\psi)m(s,\tau,w)f=C(1/2)(t_0\cdot m(1-s,\tau^{-1},w)M^*(s,c,\tau,\psi)f)^\jmath, \tag{5.41}$$

where $C(2)=\tau(2)^2|2|^{2(s-1/2)}$. Thus (5.39) and (5.40) imply

$$\gamma(s,\pi\times\tau,\psi)=\gamma(s,\sigma\times(\tau\otimes\tau^{-1}),\psi).$$

The functional equation (3.4) reads

$$\lambda(s,c,\tau,\psi)f=\lambda(1-s,c,\tau^\vee,\psi)(t_0\cdot M^*(s,c,\tau,\psi)f)^\jmath.$$

The l.h.s. equals

$$\int\limits_{F^3} f(s,\begin{pmatrix}&I_3\\I_3&\end{pmatrix}\begin{pmatrix}1&&\\&&-1\\&1&\end{pmatrix}\jmath\begin{pmatrix}1&&z&&y&\\&1&x&&&-y\\&&1&&&\\&&&1&-x&-z\\&&&&1&\\&&&&&1\end{pmatrix})\psi^{-1}(-2y)\,dz\,dy\,dx$$
$$=\int_{F^2} m(s,w,\tau)f(s,\begin{pmatrix}&1\\1&\end{pmatrix}\begin{pmatrix}1&y\\&1\end{pmatrix}\delta_0'\jmath w\kappa^\bullet\begin{pmatrix}1&&\\&1&x\\&&1\end{pmatrix})\psi^{-1}(-2y)\,dy\,dx.$$

The r.h.s. equals

$$\int\limits_{F^3} (t_0 \cdot M^*(s,c,\tau,\psi)f)^{\jmath} \left(s, \begin{pmatrix} & I_3 \\ I_3 & \end{pmatrix} \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix} \jmath \begin{pmatrix} 1 & & z & & y & \\ & 1 & x & & & -y \\ & & 1 & & & \\ & & & 1 & -x & -z \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right)$$

$$\psi^{-1}(-2y)\, dz\, dy\, dx = \int_{F^2} (t_0 \cdot m(1-s, w, \tau^{-1}) M^*(s,c,\tau,\psi)f)^{\jmath}$$

$$\times \left(s, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \delta_0' \jmath w \kappa^{\bullet} \begin{pmatrix} 1 & & \\ & 1 & x \\ & & 1 \end{pmatrix}\right) \psi^{-1}(-2y)\, dy\, dx.$$

The dy-integrations are again related via (3.4) and we deduce (5.41).

5.3.4 The group GSpin_c. Assume $G = \mathrm{GSpin}_c$. The integration over M_R changes to $C_G^{\circ}\backslash M_R$. The applications of (1.5) are carried out in GL_{kc}, hence remain valid here. When we decompose δ_0, the elements δ_0' and δ_0^{σ} are already fixed, and the representative for w^{-1} is fixed as explained in § 2.5. This determines w_1, which belongs to Spin_{2kc}. The decomposition of the conjugation of δ_1 into $\delta_1^{\sigma}\delta_1'$, e.g., ${}^{w_1}({}^{(\delta_0^{-1}\kappa\delta_0)}\delta_1) = \delta_1^{\sigma}\delta_1'$ for even c, is still valid in H, because $\delta_1, \delta_1^{\sigma}, \delta_1' \in N_H$.

The representation (5.20) is now

$$\begin{aligned} \mathrm{Ind}_{{}^{\jmath}L}^{H} \Big(\delta_{{}^{\jmath}L}^{-1/2} \Big(& (|\det|^{d} V(s, W_{\psi}(\rho_l(\tau)) \otimes (\chi_{\pi}^{-1} \circ \det) W_{\psi}(\rho_l(\tau^{\vee}))) \\ & \otimes V(s, W_{\psi}(\rho_{c'}(\tau)) \otimes \chi_{\pi}) \Big) \Big). \end{aligned} \tag{5.42}$$

We write $m = [a, g] \in M_R$, where $a \in \mathrm{GL}_l$ and $g \in G' (= \mathrm{GSpin}_{c-2l})$. Recall that under the embedding defined in § 2.5, $C_G^{\circ} = C_{G'}^{\circ}$. This implies that as m varies in $C_G^{\circ}\backslash M_R$, $a \in \mathrm{GL}_l$ and $g \in C_{G'}^{\circ}\backslash G'$.

Recall that for the $\mathrm{GL}_l \times \mathrm{GL}_k$ integral arising here we use the representation $\tau \otimes \chi_{\pi}^{-1}\tau^{\vee}$ (see (4.3)). For $t_0 \in T_{\mathrm{GL}_n}$, the embedding $(t_0, 1)$ is given by (2.7). It follows that for $a, b \in \mathrm{GL}_l$,

$$\begin{aligned} {}^{w_1\kappa^{\bullet}}(b, {}^{\iota}a) &= {}^{w_1}(\mathfrak{r}_{2kc}^{\vee}(\det b) \operatorname{diag}(b, \ldots, b, I_{kc'}, b^*, \ldots, b^*, a^*, a, b, \ldots, b, I_{kc'}, b^*, \ldots, b^*)) \\ &= \mathfrak{r}_{2kc}^{\vee}(\det b^k)(b, a)^{\sigma}. \end{aligned}$$

Here on the first line, b appears k times before the first block $I_{kc'}$, then b^* appears $k-1$ times (recall $2l + c' = c$). So the $\mathrm{GL}_l \times \mathrm{GL}_k$ integral we obtain is of the following form: for vectors ξ_{σ} and $\xi_{\sigma}^{\vee}$ in the spaces of σ and $\sigma^{\vee}$ (resp.), and a section f_0^{σ} of $V(W_{\psi}(\rho_l(\tau)) \otimes \chi_{\pi}^{-1} W_{\psi}(\rho_l(\tau^{\vee})))$,

$$\int\limits_{\mathrm{GL}_l^{\triangle} \backslash \mathrm{GL}_l \times \mathrm{GL}_l} \int\limits_{U_0^{\sigma}} \langle \sigma(b)\xi_{\sigma}, \sigma^{\vee}(a)\xi_{\sigma}^{\vee} \rangle_{\sigma} f_0^{\sigma}(s, \delta^{\sigma} u_0^{\sigma}(b,a)^{\sigma}) \psi_{U^{\sigma}}^{-1}(u^{\sigma}) \chi_{\pi}(\det b^k)\, du^{\sigma}\, da\, db.$$

Note that for $a = b$, since $f_0^{\sigma}(s, (b,b)^{\sigma}h) = \chi_{\pi}^{-1}(\det b^k) f_0^{\sigma}(s, h)$, the integrand is well defined on the quotient. Writing this integral on the right copy of GL_l (i.e., factoring out the db-integral) gives us the $\mathrm{GL}_l \times \mathrm{GL}_k$ integral for $\sigma \times (\tau \otimes \chi_\pi^{-1}\tau^\vee)$ (given in § 2.4).

To prove ${}^{w_1\kappa^\bullet}(1, {}^\iota[I_l, g]) = (1, {}^{\iota'}g)'$, note that for each of the root subgroups X of G', ${}^{w_1\kappa^\bullet}(1, {}^\iota[I_l, X])$ belongs to $(1, G')'$ and ${}^{\iota'}({}^{w_1\kappa^\bullet}(1, {}^\iota[I_l, X])) = (1, X)'$, hence ${}^{w_1\kappa^\bullet}(1, {}^\iota[I_l, g]) = (1, {}^{\iota'}g)'$ for all $g \in \mathrm{Spin}_{c'}$. This applies to any $g \in G'$ by a direct verification for $t \in T_{G'}$.

In conclusion, when we reach the formula equivalent to (5.21) or (5.33), we have the inner $\mathrm{GL}_l \times \mathrm{GL}_k$ integral for $\sigma \times (\tau \otimes \chi_\pi^{-1}\tau^\vee)$ and $G' \times \mathrm{GL}_l$ integral for $\pi' \times \tau$. As in the orthogonal cases, the $\mathrm{GL}_l \times \mathrm{GL}_k$ γ-factor is essentially $\gamma(s, \sigma \times (\tau \otimes \chi_\pi^{-1}\tau^\vee), \psi)$, but because δ_1^σ and ψ_{U^σ} are the inverses of those defined in § 2, this factor is further multiplied by $\chi_\pi(-1)^{kl}$ (replace f in (4.1) with $\mathrm{diag}(-I_{kl}, I_{kl}) \cdot f$). The constant $C(2)$ becomes $\chi_\pi(-2)^{kl}\tau(2)^{2l}|2|^{2kl(s-1/2)}$. Also for even c, $\pi(\mathfrak{i}_G) = \sigma(-1)\pi'(\mathfrak{i}_{G'})$, because the definition of the embedding of $\mathrm{GL}_l \times \mathrm{GSpin}_{c-2l}$ in M_R implies $\mathfrak{i}_G = [\mathfrak{i}_{\mathrm{GL}_l}, \mathfrak{i}_{G'}]$ (see § 2.5). Now (5.24) reads

$$C(1/2)\chi_\pi(-1)^{kl}\tau(-1)^l\vartheta(s, c', \tau \otimes \chi_\pi, \psi) = \vartheta(s, c, \tau \otimes \chi_\pi, \psi),$$

and note that $\chi_\pi(-1)^{kl}$ cancels on the l.h.s. because it also appears in $C(1/2)$.

5.3.5 The group GL_n. The proof for any $l \le n$ is similar to the case $l = n$ for Sp_{2n}, and so is considerably simpler than the general case proved in § 5.3.1. This is mainly because even though here we also apply (1.5) twice, we apply it on commuting copies of GL_l and GL_{c-l}, hence these applications may be treated almost independently. In fact, most of the manipulations for GL_n were already described in [CFGK19, Lemma 33], where we handled the (unramified) case with π induced from $P_{(l,c-l)}$ for any $0 < l < c$. We provide a brief description and when applicable, use notation from § 5.3.1.

Assume $\varepsilon = \sigma \otimes \pi'$ is a representation of $\mathrm{GL}_l \times \mathrm{GL}_{c-l}$ and $\pi^\vee = \mathrm{Ind}_R^G(\varepsilon^\vee)$. The formula (4.3) takes the form

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \sigma \times \tau, \psi)\gamma(s, \pi' \times \tau, \psi)$$

(recall $\tau = \tau_0 \otimes \chi^{-1}\tau_0^\vee$). We obtain (5.9), except that the integrand is further multiplied by $\chi^k(\det(g_1))$. Assuming τ_0 is essentially tempered or unitary, we apply (1.4)–(1.5) to each of the (k, c) functionals $W_\psi(\rho_c(\tau_0))$ and $W_\psi(\rho_c(\tau_0^\vee))$ in the inducing data of $f(s, \cdot)$. We obtain a section in the space of

$$\begin{aligned}\mathrm{Ind}_{P_{(kl,k(c-l),kl,k(c-l))}}^H&(|\det|^{-(c-l)/2+s}W_\psi(\rho_l(\tau_0)) \otimes |\det|^{l/2+s}W_\psi(\rho_{c-l}(\tau_0))\\ &\otimes |\det|^{-(c-l)/2-s}\chi^{-1}W_\psi(\rho_l(\tau_0^\vee)) \quad \otimes |\det|^{l/2-s}\chi^{-1}W_\psi(\rho_{c-l}(\tau_0^\vee))).\end{aligned} \tag{5.43}$$

Let V_1 and V_2 be the additional unipotent subgroups introduced by the lemma ($V_1 = V_2$). The du_0-integration of (5.9) becomes, for fixed $g_1, g_2 \in G$,

$$\int\limits_{U_0}\int\limits_{V_1}\int\limits_{V_2} f(s, \mathrm{diag}(\kappa_{l,c-l}, \kappa_{l,c-l})\,\mathrm{diag}(v_1, v_2)\delta u_0(g_1, g_2))\,\psi_U(u_0)\,dv_2\,dv_1\,du_0.$$

We then observe the following properties, which simplify the passage to the analog of (5.16):

(1) ${}^{\delta_0^{-1}}\operatorname{diag}(v_1,v_2)=\operatorname{diag}(v_2,v_1)$.
(2) If $v_i\in V_i$, ${}^{\operatorname{diag}(v_2,v_1)}\delta_1=\delta_1u'$ where $u'\in U_0$ and $\psi_U(u')=1$.
(3) If $v_i\in V_i$, $\operatorname{diag}(v_2,v_1)$ normalizes U_0 and fixes $\psi_U|_{U_0}$.
(4) The subgroup $\operatorname{diag}(V_2,I_{kc})$ commutes with $(1,g_2)$.
(5) δ_0 commutes with $\operatorname{diag}(\kappa_{l,c-l},\kappa_{l,c-l})$.
(6) $\operatorname{diag}(\kappa_{l,c-l},I_{kc})$ commutes with $(1,g_2)$.

Applying these properties to the last integral gives

$$\int\limits_{U_0}\int\limits_{V_1}\int\limits_{V_2} f(s,\delta_0({}^{\kappa^\bullet}\delta_1)({}^{\kappa^\bullet}u_0)$$
$$\times\operatorname{diag}(I_{kc},\kappa_{l,c-l}v_1)(1,g_2)\operatorname{diag}(\kappa_{l,c-l}v_2,I_{kc})(g_1,1))\psi_U(u_0)\,dv_2\,dv_1\,du_0,$$

where $\kappa^\bullet=\operatorname{diag}(\kappa_{l,c-l},\kappa_{l,c-l})$. When we factor (5.9) through U_R we use the invariance properties of the top left (k,l) model in the inducing data of f (see (5.43)), and ψ_U. We then form the subgroup U° generated by U_0 and the additional coordinates obtained from the conjugation of U_0 by $z\in U_R$.

We use the notation H^σ,P^σ etc., for the data corresponding to the $\mathrm{GL}_l\times\mathrm{GL}_k$ integral, and H',P' etc., for the $\mathrm{GL}_{c-l}\times\mathrm{GL}_k$ integral. Let $L=P_{(2kl,2k(c-l))}$; then $\operatorname{diag}(H^\sigma,H')=M_L$. Define

$$U^\bullet={}^{\kappa^\bullet}U^\circ=\left\{\begin{pmatrix}I_{kl}&&u^{1,1}&u^{1,2}\\&I_{k(c-l)}&u^{2,1}&u^{2,2}\\&&I_{kl}&\\&&&I_{k(c-l)}\end{pmatrix}\right\}. \tag{5.44}$$

The bottom left $(c-l)\times l$ block of $u^{2,1}$ is zero, $U_0^\sigma=\{\begin{pmatrix}I_{kl}&u^{1,1}\\&I_{kl}\end{pmatrix}\}$ and $U_0'=\{\begin{pmatrix}I_{k(c-l)}&u^{2,2}\\&I_{k(c-l)}\end{pmatrix}\}$. Write $\delta_0=w^{-1}\delta_0'\delta_0^\sigma w_1$ with $w^{-1}=\operatorname{diag}(I_{kl},w_{(k(c-l),kl)},I_{k(c-l)})$ and $w_1=w$. Then

$$Z={}^{\delta'\delta^\sigma w_1}[u^{1,2}]=\operatorname{diag}(I_{kl},V_{(kl,k(c-l))},I_{k(c-l)}),\qquad O=[u^{2,1}].$$

(This notation was introduced before (5.18).) We obtain (5.19), where the character on U^σ is $\psi_{U^\sigma}(u^\sigma)$, and as above the integrand is twisted by $\chi^k(\det g_1)$.

The intertwining operator $m(s,\tau,w)$ takes f to the space of

$$\operatorname{Ind}_L^H(\delta_L^{-1/2}(|\det|^{d_{c-l}}V(s,W_\psi(\rho_l(\tau_0))\otimes\chi^{-1}W_\psi(\rho_l(\tau_0^\vee)))$$
$$\otimes|\det|^{-d_l}V(s,W_\psi(\rho_{c-l}(\tau_0))\otimes\chi^{-1}W_\psi(\rho_{c-l}(\tau_0^\vee))))),$$

where for an integer r, $d_r=(k-1/2)r$. Write $m=\operatorname{diag}(a,g)\in M_{(l,c-l)}$ and conjugate a and g to the left. We see that

$${}^{w_1}({}^{\operatorname{diag}(I_{kc},\kappa_{l,c-l})}(1,m))=\operatorname{diag}(I_{kl},a,I_{(k-1)l},I_{k(c-l)},g,I_{(k-1)(c-l)})=(1,a)^\sigma(1,g)'.$$

We obtain a formula similar to (5.21), except we have $\psi_{U^\sigma}(u^\sigma)$ instead of $\psi_{U^\sigma}^{-\epsilon_0}(u^\sigma)$ (and $\chi^k(\det g_1)$). Equality (5.22) takes the form

$$\gamma(s,\sigma\times\tau,\psi)\gamma(s,\pi'\times\tau,\psi)Z(s,\omega,f)$$
$$=\pi(-1)^k\vartheta(s,c,\tau,\psi)\mathcal{I}(M^*(s,l,\tau,\psi)M^*(s,c-l,\tau,\psi)m(s,\tau,w)f).$$

The proof is then complete once we prove

$$M^*(s,l,\tau,\psi)M^*(s,c-l,\tau,\psi)m(s,\tau,w)=m(1-s,\tau^\vee,w)M^*(s,c,\tau,\psi).$$

The argument is similar to the proof of (5.23), and simpler because there are no twists to the characters (e.g., in § 5.3.1 we used $\psi_{-2\epsilon_0}$ in the definition of λ_2).

6 Proof of Theorem 4.2: Part II

Here the exposition is ordered so that the flow of the proof is "linear" (not according to the order of properties in the statement of the theorem). For example in § 6.2 we prove the minimal case for the GL_n factors, then use it to compute the γ-factors for unramified data. This is needed for the proof of the crude functional equation in § 6.6, which is then used in § 6.7 to complete the computation of the GL_n factors in general.

6.1 Dependence on ψ. Consider Sp_{2n} and SO_{2n} first. Changing the character ψ entails changing the (k,c) model of $\rho_c(\tau)$ and the normalization of the intertwining operator. Fix a (k,c) functional λ on $\rho_c(\tau)$, with respect to ψ, and consider

$$t_b=\mathrm{diag}(b^{k-1}I_c,\ldots,bI_c,I_{2c},b^{-1}I_c,\ldots b^{1-k}I_c)\in T_H.$$

Then t_b commutes with the image of $G\times G$ in H; normalizes U_0; ${}^{t_b^{-1}}\psi_U=(\psi_b)_U$ on U_0 (${}^{x^{-1}}\psi_U(y)=\psi_U({}^xy)$); t_b commutes with δ_1; if $y_b={}^{\delta_0}t_b$ where the r.h.s. remove is regarded as an element of GL_{kc}, the mapping $\xi\mapsto\lambda(y_b\cdot\xi)$ is a (k,c) functional on $\rho_c(\tau)$ with respect to ψ_b. Therefore if f is a meromorphic section of $V(W_\psi(\rho_c(\tau)))$, $Z(s,\omega,t_b\cdot f)$ is equal to the similar integral when ψ is replaced by ψ_b, multiplied by a measure constant c_b. This constant appears because of the conjugation of U_0 by t_b and the changes to the measures of G and U_0, when the character ψ is changed to ψ_b. Also since

$${}^{w_P^{-1}\delta_0}t_b=(b^{k-1}I_{kc})y_b, \tag{6.1}$$

we see that

$$Z(s,\omega,M(s,W_\psi(\rho_c(\tau)),w_P)t_b\cdot f) \tag{6.2}$$
$$=c_b|b|^{-d/2}\rho_c(\tau)(b^{k-1})|b|^{(k-1)kc(s-1/2)}Z(s,\omega,M(s,W_{\psi_b}(\rho_c(\tau)),w_P)f).$$

Here d is the number of roots in U_P, $|b|^{-d/2}$ appears because the measure for the intertwining operator on the r.h.s. is defined with respect to $d_{\psi_b}x=|b|^{1/2}d_\psi x$ (see after (3.4)), and we used $\delta_P^{1/2}(b^{k-1}I_{kc})\delta_P({}^{\delta_0}t_b) = 1$. Also recall that $\rho_c(\tau)(b^{k-1})$ is shorthand for $\rho_c(\tau)(b^{k-1}I_{kc})$ (see § 1.1).

Next, we relate the normalizing factor $C(s, c, \tau, \psi_b)$ to $C(s, c, \tau, \psi)$. Take

$$h_b = \mathrm{diag}(b^k I_{c/2}, b^{k-1}I_c \ldots, bI_c, I_c, b^{-1}I_c, \ldots, b^{-k+1}I_c, b^{-k}I_{c/2}) \in T_H$$

(for $k = 1$, $h_b = \mathrm{diag}(bI_{c/2}, I_c, b^{-1}I_{c/2})$) and put $z_b = {}^{\delta_0}h_b$. The mapping $\xi \mapsto \lambda(z_b\xi)$ realizes $W_{\psi_b}(\rho_c(\tau))$. Again take a section f of $V(W_\psi(\rho_c(\tau)))$. Then

$$\lambda(s, c, \tau, \psi)h_b \cdot f = \delta_P(h_b)\int_{U_P} f(s, z_b\delta_0 u)\psi_b(u)\,du = |b|^{-d/2}\delta_P(h_b)\lambda(s, c, \tau, \psi_b)f,$$

and since ${}^{w_P^{-1}}z_b = (b^k I_{kc})z_b$,

$$\begin{aligned}
&\lambda(1 - s, c, \tau^\vee, \psi)M(s, W_\psi(\rho_c(\tau)), w_P)h_b \cdot f\\
&= |b|^{-d}\rho_c(\tau)(b^k)|b|^{k^2c(s-1/2)}\delta_P^{1/2}(b^k I_{kc})\delta_{P'}({}^{w_P^{-1}}h_b)\delta_P(h_b)\\
&\quad\times \lambda(1 - s, c, \tau^\vee, \psi_b)M(s, W_{\psi_b}(\rho_c(\tau)), w_P)f.
\end{aligned}$$

Note that $\delta_P^{1/2}(b^k I_{kc})\delta_{P'}({}^{w_P^{-1}}h_b) = 1$. Therefore by (3.4),

$$C(s, c, \tau, \psi) = \rho_c(\tau)(b)^{-k}|b|^{d/2-k^2c(s-1/2)}C(s, c, \tau, \psi_b). \tag{6.3}$$

Combining this with (6.2) and the definitions, and since $\rho_c(\tau)(b) = \tau^c(b)$,

$$\gamma(s, \pi \times \tau, \psi_b) = \tau^c(b)|b|^{kc(s-1/2)}\gamma(s, \pi \times \tau, \psi)\frac{\vartheta(s, c, \tau, \psi_b)}{\vartheta(s, c, \tau, \psi)}.$$

This proves the result for $G = \mathrm{SO}_{2n}$. For Sp_{2n} the result follows from the last equality using $\gamma(s, \tau, \psi_b) = |b|^{k(s-1/2)}\tau(b)\gamma(s, \tau, \psi)$ (see [JPSS83], or [FLO12, § 9]).

For SO_{2n+1} we proceed as above. The elements t_b and y_b are the same (${}^{\delta_0}t_b$ does not depend on the parity of k). The integral $Z^*(s, \omega, f)$ is defined differently when k is odd, and when we use the correct version of δ_0 (for even k (2.2), otherwise (2.3)), (6.1) still holds, leading to (6.2). To compute $C(s, c, \tau, \psi_b)$ take

$$h_b = \mathrm{diag}(b^k I_n, b^{k-1}I_c \ldots, bI_c, I_{c+1}, b^{-1}I_c, \ldots, b^{-k+1}I_c, b^{-k}I_n)$$

($c = 2n+1$) and put $z_b = {}^{\delta_0}h_b$, where δ_0 is given by (2.2). When we compute the r.h.s. of (3.4), we use the fact that $\jmath_{kc}$ commutes with h_b, and ${}^{w_P^{-1}\jmath_{kc}}z_b = (b^k I_{kc})m_b z_b$ where m_b is the diagonal embedding of $\mathrm{diag}(I_n, b^{-1}, I_n)$ in GL_{kc} ($\jmath_{kc}$ appears because on this side the section is $(t_0 \cdot M(s, W_\psi(\rho_c(\tau)), w_P)h_b \cdot f)^{\jmath_{kc}}$). The functional $\xi \mapsto \lambda(m_b z_b\xi)$ still realizes $W_{\psi_b}(\rho_c(\tau))$, and thus is proportional to $\lambda(z_b\xi)$ and by Lemma 1.1,

$$\lambda(m_b z_b\xi) = \tau(\det \mathrm{diag}(I_n, b^{-1}, I_n))\lambda(z_b\xi) = \tau(b)^{-1}\lambda(z_b\xi).$$

Then the r.h.s. of (6.3) is multiplied by $\tau(b)|b|^{k(s-1/2)}$. This explains the change from $\tau^c(b)|b|^{kc(s-1/2)}$ to $\tau^N(b)|b|^{kN(s-1/2)}$ (now $N = c - 1$) in (4.7).

For GSpin_c, $\mathfrak{r}^{\vee}_{2kc}(b^r)y_b = {}^{\delta_0}t_b$ with $r = -k(k-1)c/2$, so that the constant c_b emitted from $Z(s,\omega,t_b \cdot f)$ is multiplied by $\chi_\pi(b^r)$. Equality (6.1) (with δ_0 depending on the parity of k for odd c) still holds, whence (6.2) is unchanged. Similarly $\mathfrak{r}^{\vee}_{2kc}(b^{r-kn})z_b = {}^{\delta_0}h_b$. Thus (the odd or even version of) (6.3) is modified by multiplying the r.h.s. by $\chi_\pi(b^{r-kn})$, leading to the factor $\chi_\pi^{kn}(b)$ appearing in (4.7).

For the GL_n integral the argument is similar. We explain the modifications. The element t_b remains the same; ${}^{w_P^{-1}\delta_0}t_b = \mathrm{diag}(b^{k-1}I_{kc}, b^{1-k}I_{kc})y_b$;

$$h_b = \mathrm{diag}(b^{k-1}I_c,\ldots,bI_c,I_c,b^{-1}I_c,\ldots,b^{-k}I_c),$$

${}^{w_P^{-1}}z_b = \mathrm{diag}(b^kI_{kc}, b^{-k}I_{kc})z_b$; and $\rho_c(\tau) = \rho_c(\tau_0)\otimes\chi^{-1}\rho_c(\tau_0^{\vee})$. Altogether we obtain

$$\gamma(s,\pi\times(\tau_0\otimes\chi^{-1}\tau_0^{\vee}),\psi_b) = \chi(b)^{kc}\tau_0^{2c}(b)|b|^{2kc(s-1/2)}\gamma(s,\pi\times(\tau_0\otimes\chi^{-1}\tau_0^{\vee}),\psi).$$

6.2 GL_n factors: the minimal case. Next we establish (4.8) for $n = k = 1$, over any local field (simplifying [LR05, § 9.1] to some extent). The general cases of (4.4) (for all G) and (4.8) will follow from this.

For any $r \geq 1$, let $\mathcal{S}(F^r)$ be the space of Schwartz–Bruhat functions on the row space F^r. The Fourier transform of $\phi\in\mathcal{S}(F^r)$ with respect to ψ is given by $\widehat{\phi}(y) = \int_{F^r}\phi(z)\psi(z\,{}^ty)dz$. For a quasi-character η of F^*, $\phi\in\mathcal{S}(F)$ and $s\in\mathbb{C}$, Tate's integral [Tat67] is given by

$$\zeta(s,\phi,\eta) = \int_{F^*}\phi(x)\eta(x)|x|^s d^*x.$$

It is absolutely convergent in a right half plane, admits meromorphic continuation and satisfies the functional equation

$$\gamma^{\mathrm{Tate}}(s,\eta,\psi)\zeta(s,\phi,\eta) = \zeta(1-s,\widehat{\phi},\eta^{-1}). \tag{6.4}$$

Define the following meromorphic section $f_{\tau_0,\chi,\phi}$ of $V(\tau) = V(\tau_0\otimes\chi^{-1}\tau_0^{-1})$. For $\phi\in\mathcal{S}(F^2)$,

$$f_{\tau_0,\chi,\phi}(s,g) = \int\limits_{F^*}\phi(e_2\left(\begin{smallmatrix} z & \\ & z\end{smallmatrix}\right)g)\tau_0(\det(\left(\begin{smallmatrix} z & \\ & z\end{smallmatrix}\right)g))\chi(z)|\det\left(\begin{smallmatrix} z & \\ & z\end{smallmatrix}\right)g|^s\,d^*z.$$

Here $e_2 = (0,1)$. Since $n = 1$, we can take the matrix coefficient $\omega = \pi^{-1}$. Then

$$Z(s,\omega,f_{\tau_0,\chi,\phi}) = \int\limits_{F^*}\pi^{-1}(g)\int\limits_{F^*}\phi(e_2\left(\begin{smallmatrix} z & \\ & z\end{smallmatrix}\right)\delta\left(\begin{smallmatrix} 1 & \\ & g\end{smallmatrix}\right))\tau_0(-z^2g)\chi(z)|\det\left(\begin{smallmatrix} z & \\ & z\end{smallmatrix}\right)g|^s\,d^*z\,dg.$$

It is absolutely convergent for $\mathrm{Re}(s)\gg 0$, as a double integral. Consider $\phi = \phi_1\otimes\phi_2$ with $\phi_1,\phi_2\in\mathcal{S}(F)$. Using a change of variables $g\mapsto z^{-1}g$ we see that

$$Z(s,\omega,f_{\tau_0,\chi,\phi}) = \tau_0(-1)\zeta(s,\phi_1,\pi\tau_0\chi)\zeta(s,\phi_2,\pi^{-1}\tau_0). \tag{6.5}$$

Next we compute $M^*(s,1,\tau,\psi)f_{\tau_0,\chi,\phi}$. The l.h.s. of (3.4) is seen to be

$$\tau_0(-1)\int\limits_F\int\limits_{F^*}\phi(z,u)\psi^{-1}(z^{-1}u)|z|^{2s-1}\tau_0^2(z)\chi(z)\,d^*z\,du. \tag{6.6}$$

For any $\phi'\in\mathcal{S}(F^2)$, define $\mathcal{F}(\phi')\in\mathcal{S}(F)$ by $\mathcal{F}(\phi')(z)=\int\limits_F\phi(z,u)\,du$. Then

$$M(s,1,\tau,\psi)f_{\tau_0,\chi,\phi}(s,g)=\tau_0(-1)\tau_0(\det g)|\det g|^s\zeta(2s-1,\mathcal{F}(g\phi),\chi\tau_0^2).$$

Thus by (6.4), and using $\widehat{\mathcal{F}(g\phi)}(z)=\widehat{g\phi}(z,0)$ and $\widehat{g\phi}=|\det g|^{-1}({}^tg^{-1})\cdot\widehat{\phi}$,

$$\gamma^{\mathrm{Tate}}(2s-1,\chi\tau_0^2,\psi)M(s,1,\tau,\psi)f_{\tau_0,\chi,\phi}(s,g)=f_{\tau_0^{-1},\chi^{-1},\widehat{\phi}}(1-s,w_{1,1}{}^tg^{-1}).$$

Using this and a partial Fourier inversion,

$$\begin{aligned}&\lambda(1-s,1,\chi^{-1}\tau^{-1},\psi)M(s,1,\tau,\psi)f_{\tau_0,\chi,\phi}\\&=\chi(-1)\gamma^{\mathrm{Tate}}(2s-1,\chi\tau_0^2,\psi)^{-1}\int\limits_F\int\limits_{F^*}\phi(z,u)\psi^{-1}(z^{-1}u)|z|^{2s-1}\tau_0^2(z)\chi(z)\,d^*z\,du.\end{aligned} \tag{6.7}$$

Then from (6.6) and (6.7) we deduce

$$C(s,1,\tau,\psi)=\chi(-1)\tau_0(-1)\gamma^{\mathrm{Tate}}(2s-1,\chi\tau_0^2,\psi).$$

Returning to $Z^*(s,\omega,f_{\tau_0,\chi,\phi})$ and since $\widehat{\phi}=\widehat{\phi_1}\otimes\widehat{\phi_2}$,

$$Z^*(s,\omega,f_{\tau_0,\chi,\phi})=\pi(-1)\zeta(1-s,\widehat{\phi_1},\pi^{-1}\tau_0^{-1}\chi^{-1})\zeta(1-s,\widehat{\phi_2},\pi\tau_0^{-1}). \tag{6.8}$$

Now dividing (6.8) by (6.5) and using (6.4) we conclude

$$\gamma(s,\pi\times\tau,\psi)=\gamma^{\mathrm{Tate}}(s,\pi\tau_0\chi,\psi)\gamma^{\mathrm{Tate}}(s,\pi^{-1}\tau_0,\psi). \tag{6.9}$$

Of course, in this case the Rankin–Selberg γ-factors are identical with Tate's.

Remark 6.1. A similar choice of $f_{\tau_0,\chi,\phi}$ was used in [PSR87, § 6.1] (with $\chi=1$) for any n, for computing the integrals with unramified data by reducing to the integrals of Godement and Jacquet [GJ72]. Specifically, define $f_{\tau_0,\chi,\phi}$ as above with a Schwartz function ϕ on $\mathrm{Mat}_{n\times 2n}(F)$, e_2 replaced by $(\,0\ I_n\,)$, $z\in\mathrm{GL}_n$ and $|\cdots|^s$ replaced by $|\cdots|^{s+(n-1)/2}$. When the representations are unramified, take $\phi=\phi_1\otimes\phi_2$ where ϕ_1,ϕ_2 are the characteristic functions of $\mathrm{Mat}_n(\mathcal{O})$. Then $f_{\tau_0,\chi,\phi}$ is unramified and $f_{\tau_0,\chi,\phi}(s,I_{2n})=b(s,1,\tau_0\otimes\chi^{-1}\tau_0^{-1})$. For an unramified ω we obtain (6.5) for all n, with the integrals of [GJ72] on the r.h.s.

6.3 The minimal case of GSpin_2. We explain this case, where $G = \mathrm{GSpin}_2$ and $k = 1$, because of the unique structure of G. We identify G with $M_{R_{1,2}}$, then $\pi = \sigma \otimes \chi_\pi$ is a character. Since we divide by C_G°, the integral is written over the coordinate of GL_1, denoted $\theta_1^{\vee,G}(x) = \theta_1^\vee(x)$. The image of $\theta_1^{\vee,G}(x)$ in H is $\alpha_0^\vee(x^{-1})\alpha_1^\vee(x^{-1})$, which is the coordinate $\theta_2^{\vee,H}(x)$ of T_H when we identify T_H with $T_{\mathrm{GL}_2} \times T_{\mathrm{GSpin}_0}$. Thus

$$Z(s,\omega,f) = \int\limits_{F^*} \sigma^{-1}(x) f(s, \delta \operatorname{diag}(1, x^{-1}, x, 1))\, dx,$$

which is similar to the integral for SO_2. The same manipulations now lead to the $\mathrm{GL}_1 \times \mathrm{GL}_1$ integral for $\sigma \times (\tau \otimes \chi_\pi^{-1}\tau^\vee)$ and the γ-factor is hence $\gamma(s, \pi \times \tau, \psi) = \gamma(s, \sigma \times (\tau \otimes \chi_\pi^{-1}\tau^\vee))$.

6.4 The computation of the integral with unramified data. Although we will deduce (4.4) directly from (4.2)–(4.3) and (6.9), the value of the integral with unramified data can be used to determine $C(s, c, \tau, \psi)$ with unramified data, and is crucial for the crude functional equation. This computation was carried out for Sp_{2n}, SO_{2n} and GL_n in [CFGK19, Theorems 28, 29]: using (5.21) (proved in [CFGK19] for $l = n$ and when data are unramified) we reduced the integral to the $\mathrm{GL}_n \times \mathrm{GL}_k$ integral, which was computed using induction on n. To compute the $\mathrm{GL}_1 \times \mathrm{GL}_k$ integral, we reduced it to the Rankin–Selberg integrals of [JPSS83], by employing an idea of Soudry [Sou93, Sou95] (see § 6.7.2 below for more details). The result proved was that when all data are unramified,

$$Z(s,\omega,f) = \frac{L(s, \pi \times \tau)}{b(s, c, \tau)}. \tag{6.10}$$

Here if $G = \mathrm{GL}_n$, $\tau = \tau_0 \otimes \chi^{-1}\tau_0^\vee$ and $L(s, \pi \times \tau) = L(s, \pi \times \chi\tau_0)L(s, \pi^\vee \times \tau_0)$.

We now complete the cases of SO_{2n+1} and GSpin_c. Assume τ is unitary (see Remark 6.2 below). Let $G = \mathrm{SO}_{2n+1}$. According to the proof of (4.3) with $l = n$, $Z(s,\omega,f)$ is equal to (5.33). Since in this case we can assume that g_1 and g_2 belong to K_G, the integration $d(g_1, g_2)$ can be ignored (each integral dg_i reduces to the volume of K_G, which is 1). Since $c' = 1$, G' is trivial. Because f is unramified, using Lemma 5.1 the integrations over V and O can also be ignored (see [CFGK19, Lemma 27] for details in the case of Sp_{2n}). Thus (5.33) becomes an integral for $\mathrm{GL}_n \times \mathrm{GL}_k$, multiplied by a du'-integration in H', where the section is obtained by restricting $m(s, \tau, w)f$.

The section $m(s, \tau, w)f$ is a scalar multiple of the normalized unramified function in the space of (5.20) (with $(H, L, \jmath, d)$ as defined for SO_{2n+1}). To compute the scalar, we appeal to the Gindikin–Karpelevich formula ([Cas80b, Theorem 3.1]). Write $w^{-1} = w_0^{-1}({}^{\jmath_{kn}}w_1^{-1})$ with

$$w_0 = \jmath_{kn} \operatorname{diag}(I_{k(n+1)}, \begin{pmatrix} & I_{kn} \\ I_{kn} & \end{pmatrix}, I_{k(n+1)}), \qquad w_1 = \operatorname{diag}(I_{kn}, w_{(k,kn)}, w_{(kn,k)}, I_{kn}).$$

Then we use multiplicativity to compute $m(s,\tau,w)f$. To compute the contribution of the operator corresponding to w_0, note that the action of the L-group of the Levi part on the Lie algebra of the L-group of the unipotent subgroup (the subgroup corresponding to z_2 in (5.18), conjugated by ${}^{J_{kn}}w_1^{-1}$) is $\wedge^2$. If $\tau = \mathrm{Ind}_{B_{\mathrm{GL}_k}}^{\mathrm{GL}_k}(\tau_1 \otimes \ldots \otimes \tau_k)$, the unramified representation of SO_{2kn} is $\mathrm{Ind}_{B_{\mathrm{SO}_{2kn}}}^{\mathrm{SO}_{2kn}}(\otimes_{1\leq i\leq k, 1\leq j\leq n}\tau_i|\ |^{s-1/2+j})$ (use (5.10)). From this operator we obtain

$$\prod_{1\leq j\leq \lfloor n/2\rfloor}\frac{L(2s+2j,\tau,\vee^2)}{L(2s+2j+2\lceil n/2\rceil-1,\tau,\vee^2)}\prod_{1\leq j\leq \lceil n/2\rceil}\frac{L(2s+2j-1,\tau,\wedge^2)}{L(2s+2j+2\lfloor n/2\rfloor,\tau,\wedge^2)}. \tag{6.11}$$

For the second operator, the action on the Lie algebra of the L-group of the unipotent subgroup corresponding to z_1 in (5.18) is $\mathrm{st}\otimes\widetilde{\mathrm{st}}$, the unramified representation is

$$\mathrm{Ind}_{B_{\mathrm{GL}_{k(n+1)}}}^{\mathrm{GL}_{k(n+1)}}((\otimes_{1\leq i\leq k}\tau_i|\ |^{s-1+k(n+1)/2})\otimes(\otimes_{1\leq i\leq k,1\leq j\leq n}\tau_i^{-1}|\ |^{-s-j+k(n+1)/2}))$$

and the contribution is

$$\prod_{1\leq j\leq n}\frac{L(2s+j-1,\tau\times\tau)}{L(2s+j,\tau\times\tau)}=\frac{L(2s,\tau\times\tau)}{L(2s+n,\tau\times\tau)}. \tag{6.12}$$

Finally when $k > 1$, the du'-integral constitutes the Whittaker functional on $\mathrm{Ind}_{P'}^{H'}(|\det|^{s-1/2}\tau)$ given by the Jacquet integral, applied to the normalized unramified vector. According to the Casselman–Shalika formula [CS80] (or see [Sou93, p. 97]), the du'-integral equals $L(2s,\tau,\wedge^2)^{-1}$. Multiplying (6.11), (6.12), $L(2s,\tau,\wedge^2)^{-1}$ and (6.10) for $\mathrm{GL}_n\times\mathrm{GL}_k$, and since $\mathrm{Sp}_{2n}(\mathbb{C})$ is the L-group of SO_{2n+1}, we obtain (6.10) for SO_{2n+1}.

For GSpin_c one uses § 5.3.4 and follows the computation of SO_c. The contribution of the intertwining operator, which is given for odd c by (6.11) and (6.12), is now modified to the twisted versions. Specifically $\vee^2$ and $\wedge^2$ change to $\vee^2\chi_\pi$ and $\wedge^2\chi_\pi$; $\tau\otimes\tau$ changes to $\tau\otimes\chi_\pi\tau$; and $b(s,c,\tau)$ in (6.10) is replaced with $b(s,c,\tau\otimes\chi_\pi)$. The $\mathrm{GL}_n\times\mathrm{GL}_k$ integral becomes

$$\frac{L(s,\sigma\times\chi_\pi\tau)L(s,\sigma^\vee\times\tau)}{b(s,c,\tau\otimes\chi_\pi^{-1}\tau^\vee)}.$$

Remark 6.2. The assumption that τ is unitary is needed in order to apply (1.5), which is used for the proof of (4.3). One may study the case of unramified τ separately, and replace this assumption by taking an inducing character for τ in "general position".

6.5 Unramified factors. We handle all groups simultaneously. First use multiplicativity to reduce to the case $n = k = 1$, which is further reduced to the $\mathrm{GL}_1 \times \mathrm{GL}_1$ integral using (4.3). Then (4.4) follows from (6.9) and the computation of Tate's integrals with unramified data [Tat67]. Note that for GL_n the r.h.s. of (4.4) is replaced by

$$\frac{L(1-s,\pi^\vee \times \chi^{-1}\tau_0^\vee)L(1-s,\pi \times \tau_0^\vee)}{L(s,\pi \times \chi\tau_0)L(s,\pi^\vee \times \tau_0)}.$$

Now we may also deduce the value of $C(s,c,\tau \otimes \chi_\pi,\psi)$ for unramified data (assuming τ is unitary, see Remark 6.2). Indeed, combining (6.10) with (3.7), (4.1) and (4.4) we see that

$$C(s,c,\tau \otimes \chi_\pi,\psi) = \frac{b(1-s,c,\chi_\pi^{-1}\tau^\vee \otimes \chi_\pi)}{a(s,c,\tau \otimes \chi_\pi)} \left[\frac{L(s,\tau)}{L(1-s,\tau^\vee)}\right]. \tag{6.13}$$

Here the factor in square brackets appears only when $H = \mathrm{Sp}_{2kc}$ (because $\vartheta(s,c,\tau \otimes \chi_\pi,\psi)$ contains $\gamma(s,\tau,\psi)$ in this case).

6.6 The crude functional equation. We treat all groups $G \neq \mathrm{GL}_n$ together, the proof for GL_n is obtained by minor modifications to the notation. Also, to lighten the formulas we omit χ_π from the notation, it is easily recovered by looking at (6.13). The global construction was described in the introduction and in § 2. For the proof we may assume τ is unitary. Let $\mathcal{E}_\tau$ be the generalized Speh representation and $V(\mathcal{E}_\tau)$ be the global analog of the representation defined in § 2.3 (i.e., we induce from $P(\mathbb{A})$ and $|\det|^{s-1/2}\mathcal{E}_\tau$ to $H(\mathbb{A})$). Let

$$M(s,\mathcal{E}_\tau,w_P)f(s,h) = \int\limits_{U_{P'}(\mathbb{A})} f(s,w_P^{-1}uh)\,du$$

be the global intertwining operator ($U_{P'}$ was defined in § 3).

Take a standard K_H-finite section f of $V(\mathcal{E}_\tau)$ which is a pure tensor, and a large finite set S of places of F. According to the functional equation of the Eisenstein series and (3.7),

$$E(\cdot;s,f) = E(\cdot;1-s,M(s,\mathcal{E}_\tau,w_P)f) = \frac{a^S(s,c,\tau)}{b^S(s,c,\tau)}E(\cdot;1-s,f'), \tag{6.14}$$

where the superscript S denotes the infinite product of local factors over the places outside S. Since $\mathcal{E}_\tau$ is irreducible ([Jac84, § 2]), $f' \in V(\mathcal{E}_\tau^\vee)$, and because the local components of $\mathcal{E}_\tau$ are unitary, the representations $\rho_c(\tau_\nu)$ are irreducible and [CFGoK, Claim 6] implies $\rho_c(\tau_\nu)^\vee = \rho_c(\tau_\nu^\vee)$, thus $\mathcal{E}_\tau^\vee = \mathcal{E}_{\tau^\vee}$. Then $f' \in V(\mathcal{E}_{\tau^\vee})$ and for $\nu \in S$, $f'_\nu(s,h) = M_\nu(s,\rho_c(\tau_\nu),w_P)f_\nu(s,h)$ (see (3.2), $\rho_c(\tau_\nu)$ and $W_{\psi_\nu}(\rho_c(\tau_\nu))$ are isomorphic here).

The global integral (0.3) is Eulerian ([CFGK19, Theorem 1] and [CFGoK, Theorem 4]), and according to (6.10) we have

$$Z(s,\varphi_1,\varphi_2,f)=\frac{L^S(s,\pi\times\tau)}{b^S(s,c,\tau)}\prod_{\nu\in S}Z(s,\omega_\nu,f_\nu). \tag{6.15}$$

Combining (6.14) and (6.15) for the section $b^S(s,c,\tau)f$ we obtain

$$L^S(s,\pi\times\tau)\prod_{\nu\in S}Z(s,\omega_\nu,f_\nu)=\frac{a^S(s,c,\tau)}{b^S(1-s,c,\tau^\vee)}L^S(1-s,\pi^\vee\times\tau^\vee)\times\prod_{\nu\in S}Z(1-s,\omega_\nu,f'_\nu). \tag{6.16}$$

By the definition (4.1) for all $\nu\in S$,

$$\gamma(s,\pi_\nu\times\tau_\nu,\psi_\nu)=\pi_\nu(\mathrm{i}_G)^k\vartheta(s,c,\tau_\nu,\psi_\nu)C(s,c,\tau_\nu,\psi_\nu)\frac{Z(1-s,\omega_\nu,f'_\nu)}{Z(s,\omega_\nu,f_\nu)}.$$

For any ν let $\vartheta^\circ(s,c,\tau_\nu,\psi_\nu)=\vartheta(s,c,\tau_\nu,\psi_\nu)[\gamma(s,\tau_\nu,\psi_\nu)^{-1}]$, where $[\ldots]$ appears only for Sp_{2n}. Let $\vartheta^\circ(s,c,\tau,\psi)$ and $\gamma(s,\tau,\psi)$ be the products of the corresponding local factors over all places of F. Then $\vartheta^\circ(s,c,\tau,\psi)=(\vartheta^\circ)^S(s,c,\tau,\psi)=1$. Since $\gamma(s,\tau,\psi)=1$, for Sp_{2n} we have

$$\gamma^S(s,\tau,\psi)^{-1}\prod_{\nu\in S}\vartheta^\circ(s,c,\tau_\nu,\psi_\nu)=\prod_{\nu\in S}\vartheta(s,c,\tau_\nu,\psi_\nu).$$

Then by (6.13) and using $\pi(\mathrm{i}_G)=\pi^S(\mathrm{i}_G)=1$,

$$\begin{aligned}C^S(s,c,\tau,\psi)&=\frac{b^S(1-s,c,\tau^\vee)}{a^S(s,c,\tau)}[\gamma^S(s,\tau,\psi)^{-1}]\\&=\prod_{\nu\in S}\pi_\nu(\mathrm{i}_G)^k\vartheta_\nu(s,c,\tau_\nu,\psi_\nu)\frac{b^S(1-s,c,\tau^\vee)}{a^S(s,c,\tau)}.\end{aligned} \tag{6.17}$$

Let $C(s,c,\tau,\psi)=\prod_\nu C(s,c,\tau_\nu,\psi_\nu)$. Below we show $C(s,c,\tau,\psi)=1$. Then when we multiply (6.16) by $C(s,c,\tau,\psi)$ and use (6.17) and (4.1), we obtain (4.10), i.e.,

$$L^S(s,\pi\times\tau)=\prod_{\nu\in S}\gamma(s,\pi_\nu\times\tau_\nu,\psi_\nu)L^S(1-s,\pi^\vee\times\tau^\vee).$$

It remains to prove $C(s,c,\tau,\psi)=1$. To this end consider the Fourier coefficient

$$\int_{Y_{k,c}(F)\backslash Y_{k,c}(\mathbb{A})}E(u;s,f)\psi_{k,c}^{-1}(u)\,du, \tag{6.18}$$

with $Y_{k,c}$ and $\psi_{k,c}$ as defined after (3.3). We unfold the Eisenstein series in $\mathrm{Re}(s)\gg 0$ and analyze the contribution from each representative of $P\backslash H/Y_{k,c}$. The contributions from all but one representative vanish, this follows using the character $\psi_{k,c}$

and the "smallness" of $\mathcal{E}_\tau$ (the global version of the proof of Theorem 3.2). For the representative δ_0, ${}^{\delta_0^{-1}}P \cap Y_{k,c} = {}^{\jmath_{kc}}V_{(c^k)}$ whence (6.18) equals

$$\int_{U_{P'}(\mathbb{A})} \int_{V_{(c^k)}(F)\backslash V_{(c^k)}(\mathbb{A})} f(s, v\delta_0 u)\psi_{k,c}^{-1}(vu)\ dv\ du.$$

Denote the inner integration by $W_\psi(\delta_0 u \cdot f)$. It is factorizable because $(\mathcal{E}_\tau)_\nu$ supports a unique (k, c) functional for all ν. Thus for a factorizable f, $W_\psi(\delta_0 u \cdot f) = \prod_\nu W_{\psi_\nu}(\delta_0 u_\nu \cdot f_\nu)$ and we obtain

$$\prod_\nu \lambda_\nu(s, c, \tau_\nu, \psi_\nu) W_{\psi_\nu}(f_\nu). \tag{6.19}$$

On the other hand, applying the functional equation (6.14) then recomputing (6.18) we have

$$\prod_\nu \lambda_\nu(1 - s, c, \tau'_\nu, \psi_\nu) M_\nu(s, W_{\psi_\nu}(\rho_c(\tau_\nu)), w_P) W_{\psi_\nu}(f_\nu), \tag{6.20}$$

or the modified version for the groups $H = \mathrm{SO}_c, \mathrm{GSpin}_c$ when c and k are odd (with $\jmath_{kc}$ and t_0, see after (3.4)). Equating (6.19) and (6.20) and looking at (3.4) we conclude $C(s, c, \tau, \psi) = 1$.

6.7 The GL_n-factors.

6.7.1 Proof of (4.8). The multiplicativity property was proved above for GL_n as well. Hence over archimedean fields, by Casselman's subrepresentation theorem [Cas80a] the proof reduces to the minimal case (6.9) already proved in § 6.2. Over p-adic fields, by (4.2) and (4.3) to (irreducible) supercuspidal representations.

Now assume π is a supercuspidal representation of GL_n. Hence π is also generic (and τ is always generic). Then we can use the global argument in [Sha90, § 5]: take a number field F and embed π and τ as the components of two cuspidal representations at a place ν_0 of F, and similarly globalize ψ (implicitly using (4.7)). We can further assume that at all places $\nu \neq \nu_0$ the local representations are quotients of principal series. The p-adic case then follows from (4.10), (4.2), (4.3), (6.9) and because the same global property is satisfied by the product of Rankin–Selberg γ-factors appearing on the r.h.s. of (4.8).

6.7.2 The case $n = 1$. The results of this section will be used below to deduce the archimedean meromorphic continuation (§ 6.10). Along the way, although we already deduced (4.8), we provide a direct proof of this for $n = 1$ and $k > 1$. Assume $k > 1$, up to Corollary 6.7.

The argument was adapted from [CFGK19], where it was used to complete the computation of the integrals for $G \times \mathrm{GL}_k$ with unramified data (see § 6.4). We follow (and elaborate on) the proof of [CFGK19, Proposition 34], which was given for unramified data, but the relevant manipulations are valid in general and over any local field.

Let π and χ be quasi-characters of F^* and τ_0 be an irreducible generic representation of GL_k. Since $n=1$, $\omega(a)$ is a scalar multiple of $\pi^{-1}(a)$, so that we can re-denote the integral $Z(s,\omega,f)$ by $Z(s,f)$, where f is a section of $V(W_\psi(\tau_0)\otimes\chi^{-1}W_\psi(\tau_0^\vee))$. Then

$$Z(s,f)=\int\limits_{F^*}\int\limits_{U_0} f(s,\delta u_0\,\mathrm{diag}(I_k,a,I_{k-1}))\psi_U(u_0)\pi^{-1}(a)\,du_0\,d^*a. \tag{6.21}$$

It is absolutely convergent as a multiple integral, in a right half plane depending only on π and τ_0, and over archimedean fields it is continuous in the input data (in its domain of convergence). In this domain it belongs to (2.9), which here becomes

$$\mathrm{Hom}_{(\mathrm{GL}_1,\mathrm{GL}_1)}(J_{U,\psi_U^{-1}}(V(s,W_\psi(\tau_0)\otimes\chi^{-1}W_\psi(\tau_0^\vee))),(\chi^k\pi)^{-1}\otimes\pi). \tag{6.22}$$

Specifically,

$$Z(s,(b,a)u\cdot f)=\psi_U^{-1}(u)\chi^{-1}(b^k)\pi(a)\pi^{-1}(b)Z(s,f),\qquad \forall a,b\in\mathrm{GL}_1,u\in U. \tag{6.23}$$

We study $Z(s,f)$ by relating it to the integral

$$\int\limits_{V^-_{(k-1,1)}}\int\limits_{F^*}\lambda_{-1}((\mathrm{diag}(I_{2k-1},a)[v]w'_{(k-1,1)})\cdot f)\pi^{-1}(a)|a|^{-\eta+k-1}\,d^*a\,dv, \tag{6.24}$$

where $\lambda_{-1}=\lambda_{-1}(s,1,\tau_0\otimes\chi^{-1}\tau_0^\vee,\psi)$ is the functional from (3.4) except that ψ appearing in (3.3) is replaced with ψ^{-1} (cf. $\lambda_2(\cdots)$ in § 5.3.1); $[v]=\mathrm{diag}(I_k,v)$ for $v\in V^-_{(k-1,1)}$ and we also identify v with a row vector in F^{k-1}; $w'_{(k-1,1)}=\mathrm{diag}(I_k,w_{(k-1,1)})$; and η is an additional complex parameter. The proofs of the following two claims appear below.

Claim 6.3. *Integral (6.24) is absolutely convergent for* $\mathrm{Re}(\eta)\gg 0$ *and admits meromorphic continuation in η and s. It is a meromorphic function of s when $\eta=0$. Over archimedean fields the continuation in η and s, and only in s when $\eta=0$, is continuous in the input data.*

To relate $Z(s,f)$ to (6.24) we follow the idea of Soudry [Sou93, p. 70] (also used in [Sou00, Kap13a], the particular variant we use appeared in [Sou95] for archimedean fields). Since in its domain of convergence (6.24) belongs to (6.22) with π replaced by $|\ |^\eta\pi$ (direct verification of (6.23)), so does its meromorphic continuation. Taking $\eta=0$, the meromorphic continuation of (6.24) belongs to (6.22) itself. By [GK, Theorem 2.1] this space is at most one dimensional outside a discrete subset of s. In fact, this particular uniqueness result was already proved in [CFGK19, Lemma 35] over p-adic fields, and since there are only finitely many orbits to consider in the proof, the argument readily extends to the archimedean case. Thus comparing $Z(s,f)$ in its domain of convergence to the meromorphic continuation of (6.24), they are proportional.

We will compute the proportionality factor using a direct substitution. This factor will turn out to be the meromorphic function $\gamma^{\mathrm{RS}}(s,\pi^{-1}\times\tau_0,\psi)\pi(-1)^{k-1}$. Therefore we obtain a functional equation relating $Z(s,f)$ to (6.24) as meromorphic continuations. The bonus is over archimedean fields: we can deduce the meromorphic continuation and continuity of the continuation for $Z(s,f)$, from that of (6.24) (over p-adic fields we already know $Z(s,f)$ is meromorphic, though this is another method for proving it).

Claim 6.4. *As meromorphic continuations $\gamma^{\mathrm{RS}}(s,\pi^{-1}\times\tau_0,\psi)\pi(-1)^{k-1}Z(s,f)$ is equal to integral (6.24) with $\eta=0$.*

Next we apply (3.4) to (6.24) and obtain

$$\begin{aligned}&\gamma^{\mathrm{RS}}(s,\pi^{-1}\times\tau_0,\psi)\chi(-1)^k\pi(-1)^{k-1}Z(s,f)\\&=\int\limits_{V^-_{(k-1,1)}}\int\limits_{F^*}\lambda_{-1}((\mathrm{diag}(I_{2k-1},a)[v]w'_{(k-1,1)})\cdot\\&\quad\times M^*(s,1,\tau_0\otimes\chi^{-1}\tau_0^\vee,\psi)f)\\&\quad\times\pi^{-1}(a)|a|^{k-1}\,d^*a\,dv.\end{aligned}$$

Here the l.h.s. was multiplied by $\chi(-1)^k$, because we used (3.4) with λ_{-1}, and the r.h.s. The contributions from is regarded as the meromorphic continuation with $\eta=0$. Applying Claim 6.4 again, to the last integral, we obtain

$$\gamma^{\mathrm{RS}}(1-s,\pi^{-1}\times\chi^{-1}\tau_0^\vee,\psi)^{-1}\gamma^{\mathrm{RS}}(s,\pi^{-1}\times\tau_0,\psi)\chi(-1)^kZ(s,f)=Z^*(s,f).$$

($Z^*(s,f)=Z(1-s,M^*(s,1,\tau_0\otimes\chi^{-1}\tau_0^\vee,\psi)f)$.) Since by [JPSS83],

$$\gamma^{\mathrm{RS}}(s,\pi\times\chi\tau_0,\psi)\gamma^{\mathrm{RS}}(1-s,\pi^{-1}\times\chi^{-1}\tau_0^\vee,\psi)=\pi(-1)^k\chi(-1)^k\tau_0(-1)$$

(direct verification using [JPSS83, § 2], see also [FLO12, § 9]), we deduce

$$\gamma(s,\pi\times(\tau_0\otimes\chi^{-1}\tau_0^\vee),\psi)=\gamma^{\mathrm{RS}}(s,\pi\times\chi\tau_0,\psi)\gamma^{\mathrm{RS}}(s,\pi^{-1}\times\tau_0,\psi).$$

This completes the verification of (4.8) for $n=1$ and $k>1$, once we justify the formal application of (3.4).

Indeed write a general element of $V^-_{(k-1,1)}$ as $\left(\begin{smallmatrix}I_{k-1}&\\v&1\end{smallmatrix}\right)$. Let Y_i be the subgroup of elements of $V^-_{(k-1,1)}$ where all coordinates of v other than the i-th are zero, X be the subgroup of matrices

$$\mathrm{diag}(I_{k-1},\begin{pmatrix}I_{k-1}&&x\\&1&\\&&1\end{pmatrix}),$$

and $X_i<X$, $1\le i\le k-1$, be the subgroup of elements where all coordinates of x other than x_i are zero. Then for any $a\in F^*$, $v\in Y_i$ and $x\in X_i$,

$$\begin{aligned}&\lambda_{-1}((\mathrm{diag}(I_{2k-1},a)[v]w'_{(k-1,1)}{}^{{w'_{(k-1,1)}}^{-1}}x)\cdot f)\\&\quad=\psi(x_iv_i)\lambda_{-1}((\mathrm{diag}(I_{2k-1},a)[v]w'_{(k-1,1)})\cdot f).\end{aligned}$$

Thus by Lemma 5.1 we can "hide" the dv-integral in (6.24) by replacing f with a convolution against Schwartz functions, independently of a (see [CFGK19, pp. 1051–1052]).

Proof of Claim 6.3. First change $a \mapsto a^{-1}$ in (6.24). Recall that if σ is a finitely generated (admissible) representation of GL_{2k}, which admits a unique Whittaker model with respect to ψ, and W is a Whittaker function in this model, $\widetilde{W}(h) = W(J_{2k}{}^t h^{-1})$ is a Whittaker function in the Whittaker model of σ^* with respect to ψ^{-1} (see e.g., [JPSS83, § 2.1]; $\sigma^*(h) = \sigma(h^*)$). Using this we see that (6.24) is a Rankin–Selberg integral for $\mathrm{GL}_1 \times \mathrm{GL}_{2k}$ and

$$\pi \times \mathrm{Ind}_{P_{(k^2)}}^{\mathrm{GL}_{2k}}(|\det|^{s-1/2}\chi\tau_0 \otimes |\det|^{1/2-s}\tau_0^\vee), \tag{6.25}$$

of the type [JPSS83, § 2.4(3)] (with $j = k-1$ in the notation of *loc. cit.*). In particular it is absolutely convergent for $\eta \gg 0$ ([JPSS83, JS90]).

Over p-adic fields the integral is a meromorphic function of η and s, by Bernstein's continuation principle [Ban98], and by [JPSS83, Theorem 3.1] its poles are contained in

$$L(\eta + s, \pi \times \chi\tau_0)L(\eta + 1 - s, \pi \times \tau_0^\vee).$$

That is, when we divide the integral by this product of L-functions we obtain an entire function. Thus we may take $\eta = 0$ and still obtain a meromorphic function of s.

Over archimedean fields this integral admits meromorphic continuation in η and s by Jacquet [Jac09, Theorem 2.1(ii)] (see also [Jac09, Appendix]). In fact the continuation in η was already proved by Jacquet and Shalika [JS90, Theoerm 5.1]. By [Jac09, Theorem 2.3(i)] (see the proof of [Jac09, Proposition 12.5]) the continuation in η is continuous in the input data, namely the section from the induced representation. This applies to any fixed s. When we apply the Whittaker functional λ_{-1} to an entire section f, the result is a Whittaker function which is still entire in s, and we can bound it using a continuous semi-norm which is independent of s when s varies in a compact subset (see e.g., [Sou95, (4.16)]). Using this the proof of [Jac09, Proposition 12.5] implies that the continuation of (6.24) in both η and s is continuous in f. The poles are still located in the aforementioned product of L-factors, and again we can take $\eta = 0$. Hence (6.24) admits meromorphic continuation which is continuous in the input data. □

REMARK 6.5. As explained above, we can remove the dv-integral from (6.24) (now using Corollary A.3) and reduce to an integral over F^*, then obtain the continuity statement of Claim 6.3 directly using the asymptotic expansions of Whittaker functions from [Sou95, § 4].

Proof of Claim 6.4. We begin with a general observation. For $t, m \in F$, a section $\xi \in V(W_{\psi}(\tau_0) \otimes \chi^{-1} W_{\psi}(\tau_0^{\vee}))$ and a Schwartz function ϕ on F, define

$$\ell(m) = \mathrm{diag}(I_{k-1}, \left(\begin{smallmatrix} 1 & m \\ & 1 \end{smallmatrix}\right), I_{k-1}) \in U_P, \qquad [t] = \mathrm{diag}(I_k, \left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right), I_{k-2}),$$
$$\phi(\xi)(s, h) = \int\limits_F \xi(s, h\ell(m))\phi(m)\, dm.$$

For any ξ,

$$\int\limits_{U_0} \xi(s, \delta_0 \ell(b) u_0 [t]) \psi_U(u_0)\, du_0 = \psi((1-b)t) \int\limits_{U_0} \xi(s, \delta_0 \ell(b) u_0) \psi_U(u_0)\, du_0.$$

Since $\int_F \psi((1-b)t) dt = 0$ unless $b = 1$, and noting that $\delta_1 = \ell(1)$,

$$\int\limits_{U_0} \xi(s, \delta u_0) \psi_U(u_0)\, du_0 = \int\limits_F \int\limits_F \int\limits_{U_0} \xi(s, \delta_0 \ell(b) u_0 [t]) \psi_U(u_0)\, du_0\, dt\, db.$$

Applying this to $Z(s, f)$ we obtain

$$\int\limits_{F^*} \int\limits_F \int\limits_F \int\limits_{U_0} f(s, \delta_0 \ell(b) u_0 [t] e_a) \psi_U(u_0) \pi^{-1}(a)\, du_0\, dt\, db\, d^*a. \tag{6.26}$$

Here $e_a = \mathrm{diag}(I_k, a, I_{k-1})$, for brevity. This integral is defined in the domain of definition of $Z(s, f)$, but is not absolutely convergent as a multiple integral. Nonetheless, consider the integral formally obtained from (6.26) by changing the order of integration $dt\, db$ to $db\, dt$:

$$\int\limits_{F^*} \int\limits_F \int\limits_F \int\limits_{U_0} f(s, \delta_0 \ell(b) u_0 [t] e_a) \psi_U(u_0) \pi^{-1}(a)\, du_0\, db\, dt\, d^*a. \tag{6.27}$$

First we show that in a right half plane depending only on the representations,

$$\int\limits_{F^*} \int\limits_F \left| \int\limits_F \int\limits_{U_0} f(s, \delta_0 \ell(b) u_0 [t] e_a) \psi_U(u_0) \pi^{-1}(a)\, du_0\, db \right| dt\, d^*a < \infty. \tag{6.28}$$

We can shift e_a to the left of δ_0, multiplying the measure by $|a|^{1-k}$. Observe that, as in the proof of Proposition 2.5, in a domain of this form

$$\int\limits_{F^*} \int\limits_{U_P} \left| f(s, {}^{\delta_0} e_a \delta_0 u) \pi^{-1}(a) \right| |a|^{1-k}\, du\, d^*a < \infty. \tag{6.29}$$

It remains to show that t belongs to the support of a Schwartz function (which may depend on f). Consider a Schwartz function ϕ on F and the integral

$$\int\limits_{F^*}\int\limits_{F}\left|\int\limits_{F}\int\limits_{U_0}\int\limits_{F} f(s,{}^{\delta_0}e_a\delta_0\ell(b)u_0[t]\ell(m))\phi(m)\psi_U(u_0)\pi^{-1}(a)|a|^{1-k}\,dm\,du_0\,db\right|\,dt\,d^*a.$$

We see that ${}^{\ell(-m)}[t]=u_{m,t}[t]$, where $u_{m,t}\in U_P$ is such that $\psi(u_{m,t})=\psi^{-1}(mt)$. We can freely change the order of integration $dmdu_0db$ to du_0dbdm, because of (6.29) and the Schwartz function. Then we can also change variables $b\mapsto b-m$. We obtain

$$\int\limits_{F^*}\int\limits_{F}\left|\int\limits_{F}\int\limits_{U_0} f(s,{}^{\delta_0}e_a\delta_0\ell(b)u_0[t])\widehat{\phi}(t)\psi_U(u_0)\pi^{-1}(a)|a|^{1-k}\,du_0\,db\right|\,dt\,d^*a.$$

This proves (6.28) (see Lemma 5.1, here we can fix s and use [DM78]).

Now a direct verification shows that (6.27) also belongs to (6.22). Assume for the moment that (6.27) admits meromorphic continuation. Then we can compare (6.26) (or $Z(s,f)$) to (6.27) in the domain of definition of (6.26). We show the proportionality factor is 1, by proving that (6.27) for $\phi(f)$ is $Z(s,\phi(f))$. Indeed, after shifting e_a to the left, conjugating $[t]$ by $\ell(m)$ and changing $b\mapsto b-m$, the integral (6.27) becomes

$$\begin{aligned}&\int f(s,{}^{\delta_0}e_a\delta_0\ell(b)u_0[t])\psi(mt)\phi(m)\psi_U(u_0)\pi^{-1}(a)|a|^{1-k}\,dm\,du_0\,db\,dt\,d^*a\\&=\int f(s,\delta_0\ell(b)u_0[at]e_a)\widehat{\phi}(t)\psi_U(u_0)\pi^{-1}(a)|a|\,du_0\,db\,dt\,d^*a\\&=\int f(s,\delta_0\ell(b)u_0e_a)\psi(t(1-b))\widehat{\phi}(a^{-1}t)\psi_U(u_0)\pi^{-1}(a)\,du_0\,db\,dt\,d^*a.\end{aligned}$$

Changing $b\mapsto b+1$, then shifting $\ell(b)$ to the right (thereby changing $b\mapsto a^{-1}b$), we have

$$\int f(s,\delta u_0e_a\ell(b))\psi(-ba^{-1}t)\widehat{\phi}(a^{-1}t)\psi_U(u_0)\pi^{-1}(a)|a|^{-1}\,du_0\,db\,dt\,d^*a.$$

Now we change $t\mapsto at$ (eliminating $|a|^{-1}$), then we can change $db\,dt\mapsto dt\,db$, and since by the Fourier inversion formula $\int_F\widehat{\phi}(t)\psi(-bt)dt=\phi(b)$ we obtain $Z(s,\phi(f))$.

To proceed we describe a special choice of data, for which we can compute both (6.27) (in its domain of definition) and (6.24) (in $\mathrm{Re}(\eta)\gg 0$, then for $\eta=0$ in $\mathrm{Re}(s)\ll 0$, then in the domain of (6.27) by meromorphic continuation). The proportionality factor will be $\gamma^{\mathrm{RS}}(s,\pi^{-1}\times\tau_0,\psi)\pi(-1)^{k-1}$. The claim follows, because we deduce the meromorphic continuation of (6.27), then the argument above readily implies that the continuations of $Z(s,f)$ and (6.27) are identical.

Let $W\in W_\psi(\tau_0)$. Over a p-adic field, choose f' such that $\delta_0\cdot f'$ is right-invariant by a small neighborhood of the identity $\mathcal{N}$ in H, supported in $P\mathcal{N}$, and such that for all $a \in \mathrm{GL}_k$, $\delta_0 \cdot f'(s, \mathrm{diag}(a, I_k)) = |\det a|^{s-1/2+k/2} W(a)$ ($\mathcal{N}$ depends on W). Over archimedean fields take $\delta_0 \cdot f'$ supported in PU_P^- such that $\delta_0 \cdot f'(s, \mathrm{diag}(a, I_k)u) = |\det a|^{s-1/2+k/2} W(a)\phi'(u)$ for all $u \in U_P^-$, where ϕ' is a compactly supported Schwartz function on F^{k^2} and $\int_{F^{k^2}} \phi'(u)du = 1$.

Then over p-adic fields we take $f = \phi(f')$ where ϕ is such that for all s and h,

$$\int\limits_F f'(s, h[t])\widehat{\phi}(t)\, dt = f'(s, h)$$

(e.g., take $\widehat{\phi}$ supported near 0). Over archimedean fields, by Corollary A.3 we can take f_i and compactly supported Schwartz functions $\widehat{\phi}_i$ such that for all s and h,

$$\sum_{i=1}^{l} \int\limits_F f_i(s, h[t])\widehat{\phi}_i(t)\, dt = f'(s, h). \tag{6.30}$$

Then we take $f = \sum_{i=1}^{l} \phi_i(f_i)$. For convenience, we use this notation (with $l = 1$) also in the p-adic case.

Plugging f into (6.27), conjugating e_a to the left and using the definition of f, we obtain

$$\int\limits_{F^*}\int\limits_F\int\limits_F\int\limits_{U_0} \sum_{i=1}^{l} \int\limits_F f_i(s, {}^{\delta_0}e_a\delta_0\ell(b)u_0[t]\ell(m))\phi_i(m)\psi_U(u_0)\pi^{-1}(a)|a|^{1-k}\, dm\, du_0\, db\, dt\, d^*a$$

$$= \int\limits_{F^*}\int\limits_F\int\limits_F\int\limits_{U_0} \sum_{i=1}^{l} f_i(s, {}^{\delta_0}e_a\delta_0\ell(b)u_0[t])\widehat{\phi}_i(t)\psi_U(u_0)\pi^{-1}(a)|a|^{1-k}\, du_0\, db\, dt\, d^*a.$$

Here again we first changed $dmdu_0db$ to du_0dbdm, before changing variables $u_0 \mapsto u_0u_{m,t}^{-1}$ and $b \mapsto b - m$. As above, since $\widehat{\phi}_i$ is a Schwartz function and using (6.29), we can further change $du_0\, db\, dt$ to $dt\, du_0\, db$, then by the definition of the functions f_i and ϕ_i, we have

$$\int\limits_{F^*}\int\limits_F\int\limits_{U_0} f'(s, {}^{\delta_0}e_a\delta_0\ell(b)u_0)\psi_U(u_0)\pi^{-1}(a)|a|^{1-k}\, du_0\, db\, d^*a.$$

Again by (6.29) we may change the order of integration $du_0\, db\, d^*a$ to $d^*a\, du_0\, db$, and we also write $u = \ell(b)u_0$ and extend ψ_U to U_P trivially on $\ell(b)$. We reach

$$\int\limits_{U_P}\int\limits_{F^*} f'(s, {}^{\delta_0}e_a\delta_0u)\psi_U(u)\pi^{-1}(a)|a|^{1-k}\, d^*a\, du.$$

Then by our definition of f' (e.g., over p-adic fields the integrand vanishes unless the coordinates of u are small) we obtain

$$\int\limits_{F^*} W(\mathrm{diag}(a, I_{k-1}))\pi^{-1}(a)|a|^{s-(k-1)/2}\, d^*a. \tag{6.31}$$

This is the Rankin–Selberg integral for $\mathrm{GL}_1 \times \mathrm{GL}_k$ and $\pi^{-1} \times \tau_0$ ([JPSS83, § 2.4(3)] with $j = 0$).

To compute (6.24) for the same f, write $v = (t, v')$ where t is the leftmost coordinate of v, conjugate $\mathrm{diag}(I_{2k-1}, a)\,\mathrm{diag}(I_k, \left(\begin{smallmatrix} I_{k-1} & \\ (0,v') & 1 \end{smallmatrix}\right))w'_{(k-1,1)}$ to the left and arrive at

$$\int\limits_{F}\int\limits_{F^{k-2}}\int\limits_{F^*}\int\limits_{U_P} f(s, \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-2} \\ a & 0 & v' \end{smallmatrix}\right) \delta_0 u[t])\psi(u)\pi^{-1}(a)|a|^{-\eta+1-k}\, du\, d^*a\, dv'\, dt$$
$$= \int\limits_{F^{k-2}}\int\limits_{F^*}\int\limits_{U_P}\sum_{i=1}^{l}\int\limits_{F} f_i(s, \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-2} \\ a & 0 & v' \end{smallmatrix}\right) \delta_0 u[t])\widehat{\phi}_i(t)\psi(u)\pi^{-1}(a)|a|^{-\eta+1-k}\, dt\, du\, d^*a\, dv'.$$

The justification for the formal steps is similar to the above (but simpler) and again we use (6.29). Then exactly as above, we end up with the other side of the Rankin–Selberg functional equation (the version in [Sou93, p. 70])

$$\int\limits_{F^{k-2}}\int\limits_{F^*} W\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-2} \\ a & 0 & v' \end{pmatrix}\right)\pi^{-1}(a)|a|^{-\eta+s-(k-1)/2}\, d^*a\, dv'.$$

This integral is absolutely convergent for $\mathrm{Re}(\eta) \gg 0$, but moreover, for $\zeta = 0$ it is absolutely convergent for $\mathrm{Re}(s) \ll 0$ and admits meromorphic continuation given by

$$\int\limits_{F^{k-2}}\int\limits_{F^*} W\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{k-2} \\ a & 0 & v' \end{pmatrix}\right)\pi^{-1}(a)|a|^{s-(k-1)/2}\, d^*a\, dv'. \tag{6.32}$$

Since (6.31) and (6.32) are related by $\gamma^{\mathrm{RS}}(s, \pi^{-1} \times \tau_0, \psi)\pi(-1)^{k-1}$, the proof is complete. □

Corollary 6.6. *For each pole of the Rankin–Selberg $\mathrm{GL}_1 \times \mathrm{GL}_k$ L-function $L(s, \pi^{-1}\times\tau_0)$ with multiplicity m we can find an entire section f (smooth over archimedean fields) such that $Z(s, \omega, f)$ contains this pole with multiplicity m.*

Proof. Any such pole (with multiplicity) occurs in a $\mathrm{GL}_1 \times \mathrm{GL}_k$ integral for $\pi^{-1} \times \tau_0$ for some $W \in W_\psi(\tau_0)$ (over p-adic fields see [JPSS83], over archimedean fields [CPS04, § 1.3], one may even use K_{GL_k}-finite vectors). Taking the substitution f' from the proof of Claim 6.4 (e.g., $\delta_0 \cdot f'$ supported in PU_P^- and compactly supported in U_P^-, over archimedean fields) and computing $Z(s, \omega, f')$ directly, i.e., without using (6.27), we obtain (6.31) except the integrand is further multiplied by a compactly supported Schwartz function of a. By [Jac09, Proposition 6.1] (or more directly in the p-adic case) this is sufficient for the pole (we only need to produce the pole, as opposed to obtaining the precise Rankin–Selberg integral). □

Corollary 6.7. *If $k = 1$, Corollary 6.6 applies to any n, i.e., to $\mathrm{GL}_n \times \mathrm{GL}_1$.*

Proof. For a representation π of GL_n and $n \geq 1$, one uses f such that $\delta_0 \cdot f$ is supported in PU_P^- and restricts to a compactly supported Schwartz function on U_P^- (see also [Yam14, § 5.3, § 7]), to obtain a Godement–Jacquet integral ([GJ72]). The latter integral produces any pole with multiplicity by [GJ72, Jac79]. □

6.8 Duality. This is known for $G = \mathrm{Sp}_0$, clear for GSpin_c with $c < 2$ since the γ-factors are trivial, and also holds for SO_2 and GSpin_2 by (4.3) and (4.8). (Here $G \neq \mathrm{GL}_n$.)

For the general case we follow a local-global method similar to [Kap15, § 5]. According to the multiplicative properties, it is enough to show (4.5) for supercuspidal representations. These we can globalize as in § 6.7.1, using the globalization argument of Henniart [Hen84, Appendice 1] (π is in general not generic), and using (4.7) we simultaneously globalize ψ. We can assume that at all places except ν_0 (where we embed π and τ), the representations are quotients of principal series representations.

We can then write π_ν ($\nu \neq \nu_0$) as the quotient of $\mathrm{Ind}_{R(F_\nu)}^{G(F_\nu)}(\sigma_\nu \otimes \chi_{\pi_\nu})$, where $R < G$ is a maximal parabolic subgroup, σ_ν is a principal series representation of $\mathrm{GL}_n(F_\nu)$ and $\sigma_\nu \otimes \chi_{\pi_\nu}$ is a representation of $M_R(F_\nu)$. Then by (4.3) and (4.8),

$$\gamma(s, \pi_\nu \times \chi_{\pi_\nu}^{-1}\tau_\nu, \psi_\nu) = \gamma^{\mathrm{RS}}(s, \sigma_\nu \times \tau_\nu, \psi_\nu)\gamma^{\mathrm{RS}}(s, \sigma_\nu^\vee \times \chi_{\pi_\nu}^{-1}\tau_\nu, \psi_\nu). \tag{6.33}$$

Since σ_ν is a principal series, we can permute the inducing character of σ_ν to obtain a principal series representation σ'_ν such that $\pi_\nu^\vee$ is a quotient of $\mathrm{Ind}_{R(F_\nu)}^{G(F_\nu)}({\sigma'_\nu}^\vee \otimes \chi_{\pi_\nu}^{-1})$. Then

$$\gamma(s, \pi_\nu^\vee \times \tau_\nu, \psi_\nu) = \gamma^{\mathrm{RS}}(s, \sigma'_\nu \times \tau_\nu, \psi_\nu)\gamma^{\mathrm{RS}}(s, {\sigma'_\nu}^\vee \times \chi_{\pi_\nu}^{-1}\tau_\nu, \psi_\nu). \tag{6.34}$$

The Rankin–Selberg γ-factors appearing in (6.33) and (6.34) are equal, hence

$$\gamma(s, \pi_\nu^\vee \times \tau_\nu, \psi_\nu) = \gamma(s, \pi_\nu \times \chi_{\pi_\nu}^{-1}\tau_\nu, \psi_\nu).$$

This holds for all $\nu \neq \nu_0$, thus also at ν_0 by (4.10).

6.9 Functional equation. According to (3.6) and since

$$\vartheta(s, c, \tau \otimes \chi_\pi, \psi)\vartheta(1 - s, c, \chi_\pi^{-1}\tau^\vee \otimes \chi_\pi, \psi) = \chi_\pi(-1)^{kn}\tau(-1)^N,$$

$\gamma(s, \pi \times \tau, \psi)\gamma(1 - s, \pi \times \chi_\pi^{-1}\tau^\vee, \psi) = \chi_\pi(-1)^{kn}\tau(-1)^N$. Then (4.5) and (4.7) imply (4.6).

6.10 Archimedean meromorphic continuation. We deduce the meromorphic continuation of the integral $Z(s, \omega, f)$, and continuity of this continuation regarded as a trilinear form on $V(s, W_\psi(\rho_c(\tau)) \otimes \chi_\pi) \times \pi \times (\pi^\iota)^\vee$ or the similar space for GL_n. Recall that for Fréchet spaces, a separably continuous trilinear map extends to a continuous linear map on the inductive tensor. We will also prove the meromorphic continuation of $\lambda(s, c, \tau, \psi)$ and continuity of this continuation as a functional on $V(s, W_\psi(\rho_c(\tau)) \otimes \chi_\pi)$. Our proof is facilitated by the multiplicativity identities (5.21) and (5.27), which allow us to argue inductively.

We begin with the integral. Since the field is archimedean, we may assume (by [Cas80a]) that π is an irreducible quotient of a principal series representation induced from quasi-characters $\pi_1, \dots, \pi_n$ and χ_π (for GSpin_c) of F^*. Write $\tau = \mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\otimes_{i=1}^{d}|\det|^{a_i}\tau_i)$ where τ_i are tempered and $a_1 > \ldots > a_d$. For $\zeta \in \mathbb{C}^d$, let $\tau_\zeta = \mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\otimes_{i=1}^{d}|\det|^{\zeta_i + a_i}\tau_i)$. For ζ in general position, $\rho_c(\tau_\zeta)$ is a subrepresentation of (1.2) (for such ζ we may permute the representations in the inducing data to obtain this). Hence the realization of the (k, c) functional given by (1.5) is applicable to $\rho_c(\tau_\zeta)$ (see [CFGoK, § 3.2]), and we may then argue as in § 5.3 (e.g., § 5.3.1), to write the integral for $\pi \times \tau_\zeta$ in the form (5.21) (or similar, depending on G).

The realization (5.8) of ω is continuous on $\pi \otimes \pi^\vee$ (since it is a separably continuous bilinear form). The integral $d(g_1, g_2)$ in (5.21) is over a compact group, hence can be ignored for our purpose here, by virtue of the Banach–Steinhaus Theorem — see the proof of [Sou95, § 5, Lemma 1].

The outer integral over $V \times O$ can be handled as follows. Write the inner integral in the form $\Xi(s, \omega, f)$, then the iterated integral takes the form

$$\int\limits_{V \times O} \Xi(s, \omega, y \cdot f)\, dy.$$

Note that for $(v, o) \in V \times O$, $y = o\kappa^\bullet v$, and we can identify $V \times O$ with the subgroup ${}^{\kappa^\bullet}V \ltimes O$ of H. Assume $\Xi(s, \omega, f)$ is meromorphic for meromorphic sections f, and continuous as a trilinear form. The root subgroups of $V \times O$ are handled one after the other, with a predefined order (see [CFGK19, pp. 1037–1040] and the paragraph before (5.27)). Let $Y' < V \times O$ and

$$\Xi'(s, \omega, f) = \int\limits_{Y'} \Xi(s, \omega, y' \cdot f)\, dy'.$$

For the base case Y' is trivial and $\Xi'(s, \omega, f) = \Xi(s, \omega, f)$. Assume $\Xi'(s, \omega, f)$ is meromorphic and continuous, as above. At each step we take a subgroup $Y < H$ such that $Y \ltimes Y' < V \times O$, prove similar properties for

$$\int\limits_{Y} \Xi'(s, \omega, y \cdot f)\, dy = \int\limits_{Y}\int\limits_{Y'} \Xi(s, \omega, y'y \cdot f)\, dy'\, dy, \tag{6.35}$$

then re-denote $Y \ltimes Y'$ by Y', eventually obtaining the result for $V \times O$.

Regarding (6.35), there is a unipotent subgroup $X < H$ such that for all $x \in X$,

$$\int\limits_{Y} \Xi'(s, \omega, yx \cdot f)\, dy = \int\limits_{Y} \Xi'(s, \omega, y \cdot f)\psi(\langle x, y\rangle)\, dy, \tag{6.36}$$

where $\langle x, y\rangle$ is a non-degenerate pairing. For a compactly supported function ϕ in the space $\mathcal{S}(X)$ of Schwartz functions on X, denote

$$\phi(f)(s,h) = \int\limits_X x \cdot f(s,h)\phi(x)\,dx, \qquad \widehat{\phi}(f)(s,h) = \int\limits_Y y \cdot f(s,h)\widehat{\phi}(y)\,dy.$$

Here $\widehat{\phi} \in \mathcal{S}(Y)$. The sections $\phi(f)$ and $\widehat{\phi}(f)$ are meromorphic, because the representation of H on $V(s, W_\psi(\rho_c(\tau)) \otimes \chi_\pi)$ is of moderate growth uniformly when s varies in a compact set (see e.g., [Jac09, § 3.3]), and Schwartz functions are rapidly decreasing, so that we can differentiate $\widehat{\phi}(f)$ under the integral sign.

In $\mathrm{Re}(s) \gg 0$ the multiple integral (over $V \times O, X$ and the domains in the definition of Ξ) is absolutely convergent, whence by (6.36),

$$\int\limits_Y \Xi'(s, \omega, y \cdot \phi(f))\,dy = \Xi'(s, \omega, \widehat{\phi}(f))$$

is meromorphic for each (meromorphic) f. According to Corollary A.3 we can always write $f = \sum_i \phi_i(f_i)$ (a finite sum), then the l.h.s. of (6.35) becomes $\sum_i \Xi'(s, \omega, \widehat{\phi}_i(f_i))$ which is meromorphic (here it is crucial the functions ϕ_i are independent of s).

Moreover when we fix s, the bilinear map $(\phi, f) \mapsto \phi(f)$ extends to a continuous surjective and open map $\mathcal{S}(X) \otimes V(s, W_\psi(\rho_c(\tau)) \otimes \chi_\pi) \to V(s, W_\psi(\rho_c(\tau)) \otimes \chi_\pi)$ (see e.g., [Sou95, p. 199]). Thus the l.h.s. identify of (6.35) is continuous (as a trilinear form). This completes the reduction.

It remains to consider $\Xi(s, \omega, f)$, which is a $\mathrm{GL}_n \times \mathrm{GL}_k$ doubling integral for the matrix coefficient $a \mapsto \langle \varphi(1), \sigma^\vee(a)\varphi^\vee(1)\rangle$ of $\sigma^\vee$ and the section $m(s, \tau, w)f|_{M_{\jmath L}}$. Assuming the latter integral admits meromorphic continuation, so does the $G \times \mathrm{GL}_k$ integral. Moreover, if the continuation of the $\mathrm{GL}_n \times \mathrm{GL}_k$ integral is continuous in its data, the continuation of $\Xi(s, \omega, f)$ (and thereby, of the $G \times \mathrm{GL}_k$ integral) is continuous in ω and f (i.e., as a trilinear form), because evaluation at the identity is continuous in the topology on the smooth induced representations. Note that if c is odd, there is an additional inner integration du' which is a Whittaker functional (see after (5.33)), whose analytic properties are known ([Jac67, Sha80]).

Repeating the arguments of § 5.3.5 we reduce to the case of $n = 1$ and the representations $\pi_i \times (\tau_\zeta \otimes \chi_\pi^{-1}\tau_\zeta^\vee)$. We assume $k > 1$, since for $k = 1$ meromorphic continuation in s and continuity in the input data can be checked directly (when $n = k = 1$). Now as described in § 6.7.2, the analytic properties of the $\mathrm{GL}_1 \times \mathrm{GL}_k$ integral follow from those of (6.24), which here takes the form

$$\int\limits_{V^-_{(k-1,1)}} \int\limits_{F^*} \lambda_{-1}(s, 1, \tau_\zeta \otimes \chi_\pi^{-1}\tau_\zeta^\vee, \psi)((\mathrm{diag}(I_{2k-1}, a)[v]w'_{(k-1,1)}) \cdot f)$$
$$\pi_i^{-1}(a)|a|^{-\eta+k-1}\,d^*a\,dv.$$

By Claim 6.3 and its proof, this integral admits meromorphic continuation in η,ζ and s, which is continuous in the input data f ([JS90, Jac09]). The poles are contained in

$$L(\eta+s+\zeta,\pi_i\times\chi_\pi\tau)L(\eta+1-s-\zeta,\pi_i\times\tau^\vee). \tag{6.37}$$

Thus we may take $\eta=0$. We deduce that the original $G\times \mathrm{GL}_k$ integral for $\pi\times\tau_\zeta$ admits meromorphic continuation in ζ and s, which is continuous in the input data, and its poles are contained in the product of the above L-factors with $\eta=0$ over all $1\le i\le n$. Since $\lim_{\zeta\to0}Z(s,\omega,f_\zeta)=Z(s,\omega,\lim_{\zeta\to0}f_\zeta)$ (see the justification after (5.2)), we conclude the result for $\pi\times\tau$ by taking $\zeta=0$ (for $\zeta=0$, $\rho_c(\tau_\zeta)=\rho_c(\tau)$).

Regarding $\lambda(s,c,\tau,\psi)$, the meromorphicity and continuity properties are consequences of (5.27), which expresses the functional as the composition of an intertwining operator $m(s,\tau,w)$ with similar functionals on GL_{2kl} and on a lower rank group H' of the type of H, and with an additional outer integral which is handled similarly to the outer integral over $V\times O$ above (using Corollary A.3). Since the field is archimedean, we may already take $l=n$, then if c is odd we have one additional Whittaker functional ($\lambda(s,c',\tau\otimes\chi_\pi,\psi)$, $c'=1$). The intertwining operator satisfies the conditions we need (see e.g., [KS71, Sch71, KS80]). Applying the general linear groups analog of (5.27) to GL_{2kl} we reduce to products of such functionals on GL_{2k}, which are already Whittaker functionals. As above we first work with τ_ζ to utilize (1.5) (the proof of (5.27) also uses (1.5)), then take $\zeta=0$.

Remark 6.8. The twist by ζ is only needed in order to regard $\rho_c(\tau)$ as a summand of (1.2) and apply (1.5) (using [CFGoK, § 3.2]). If τ is unitary, no additional twist is needed.

Corollary 6.9. *For any given s, one can find ω and an entire section f of $V(W_\psi(\rho_c(\tau))\otimes\chi_\pi)$, which is also K_H-finite, such that $Z(s,\omega,f)\neq0$ and the integral is holomorphic in a neighborhood of s.*

Proof. A similar nonvanishing result was obtained in Proposition 2.6, albeit with a smooth section f, but one can find a sequence $\{f_m\}$ of entire K_H-finite sections converging to f. Since we proved the integral is continuous in the input data, we deduce $Z(s,\omega,f_m)$ is finite and nonzero at s for some (almost all) m. Then there is a neighborhood of s where the integral is also holomorphic (s is not a pole). □

Remark 6.10. Alternatively we may also prove Corollary 6.9 by applying Corollary 5.3 to reduce the proof to the case $k=1$, where it is known ([KR90, Theorem 3.2.2]).

We also have the following corollary.

Corollary 6.11. *Assume τ is unitary generic and f is a standard and K_H-finite section of $V(W_\psi(\rho_c(\tau))\otimes\chi_\pi)$. Let $\mathcal{D}$ be a vertical strip of finite width and $P(s)$ be a polynomial such that $P(s)Z(s,\omega,f)$ is holomorphic in $\mathcal{D}$. Then $P(s)Z(s,\omega,f)$ is of* *finite order in $\mathcal{D}$, i.e., $|P(s)Z(s,\omega,f)| \le ae^{|s|^r}$ for some constants $a>0$ and $r>0$, for all $s\in\mathcal{D}$.*

Proof. Since τ is unitary, we can carry out the reduction described above using τ directly (see Remark 6.8). However, instead of applying (5.21) once then using induction, we repeatedly apply the multiplicative identities ((5.21) is applied once, then we use § 5.3.5), each time introducing another intertwining operator $m(s,\tau,w)_i$, an integration over unipotent subgroups O_i and V_i, and an integration over maximal compact subgroups $K_{G_i}\times K_{G_i}$.

Following these reductions, the function $h\mapsto f(s,h)$ belongs to the space $V^{\otimes}(s,\tau)$ of

$$\mathrm{Ind}_L^H(\otimes_{i=1}^n V(s,W_\psi(\tau)\otimes\chi_\pi^{-1}W_\psi(\tau^\vee))\otimes V(s,W_\psi(\tau))\otimes\chi_\pi). \tag{6.38}$$

Here when $H\ne \mathrm{GL}_{2kc}$, $L<H$ is the standard parabolic subgroup with $M_L=M_{((2k)^n)}$ if $c=2n$, $M_L=M_{((2k)^n)}\times\mathrm{SO}_{2k}$ if $c=2n+1$, and in both cases we have the additional factor GL_1 for GSpin_c. For brevity, the minor modifications for GL_{2kc} are omitted, as well as the twist of M_L and the additional modulus characters which are independent of s (see (5.20)). The first n spaces in (6.38) correspond to the spaces of sections for the $\mathrm{GL}_1\times\mathrm{GL}_k$ doubling integrals, and $V(s,W_\psi(\tau))$ is included when c is odd.

By transitivity of induction, we can also identify (6.38) with the representation

$$\mathrm{Ind}_{L_0}^H(|\chi|^s(\otimes_{i=1}^c W_\psi(\tau^i))\otimes\chi_\pi),$$

where $L_0<L$ is a standard parabolic subgroup of H with $M_{L_0}=M_{(k^c)}(\times\,\mathrm{GL}_1)$, τ^i alternates between τ and $\chi_\pi^{-1}\tau^\vee$, and χ is a suitable algebraic character of M_{L_0} (e.g., $|\chi|^s=|\det|^{s-1/2}\otimes|\det|^{1/2-s}$ if $c=2$). One can then define entire sections of $V^{\otimes}(\tau)$, i.e., functions φ on $\mathbb{C}\times H$ such that for all s, $\varphi(s,\cdot)\in V^{\otimes}(s,\tau)$, and $s\mapsto\varphi(s,h)$ is entire, and also meromorphic sections, K_H-finite sections, etc. (see § 2.3).

Let $m(s,\tau)$ be the composition of the operators $m(s,\tau,w)_i$, it has finitely many poles in $\mathcal{D}$. Since $P(s)Z(s,\omega,f)$ is holomorphic in $\mathcal{D}$, we can assume for the proof that $|\mathrm{Im}(s)|\ge A\gg 0$, so that $m(s,\tau)$ is holomorphic for $s\in\mathcal{D}_A=\{s\in\mathcal{D}: |\mathrm{Im}(s)|\ge A\}$. Put $f'=m(s,\tau)f$.

We combine the integrations over the unipotent (resp., compact) subgroups into one subgroup Y (resp., $K_G\times K_G$). This is possible by reversing the passage from (5.19) to (5.21) once we apply the Iwasawa decomposition to the inner integral over the Levi subgroup of G. In fact each K_{G_i} is a maximal compact subgroup of a Levi subgroup M_{R_i}, where $R=R_1>R_2>\ldots>R_m=B_G$ is a finite decreasing chain of standard parabolic subgroups of G, hence we can simply integrate over $K_G\times K_G$. As for the unipotent subgroups, observe that the subgroups O_i are all subgroups of U_0 (on which ψ_U is trivial), and V_i all originate from the realization of $W_\psi(\rho_c(\tau))$. Combining them here amounts to writing the identity (4.3) with respect to induction from B_G, instead of going through maximal parabolic subgroups (as described in

§ 5.3). In this process we also shift the Weyl elements w_1 and $\kappa^\bullet$ from each reduction to the right, thereby conjugating the unipotent subgroups O_i and V_i and the images of $K_{G_i} \times {}^\iota K_{G_i}$ in H. The subgroup Y is taken to be the product of subgroups O_i and V_i, each conjugated by the appropriate Weyl elements. Then $Y < U_L^-$. Let w denote the product of Weyl elements. We obtain, first in $\mathrm{Re}(s) \gg 0$,

$$\int\limits_{K_G\times K_G}\int\limits_{Y}\int\limits_{\mathrm{GL}_1^n}\int\limits_{U_0^{\lceil c/2\rceil}} f'(s,\delta u_0(1,a)y({}^w(g_1,{}^\iota g_2))w)\psi_U(u_0)$$
$$\times\,\varepsilon^\vee(a)\langle\varphi(g_1),\varphi^\vee(g_2)\rangle\,du_0\,da\,dy\,d(g_1,g_2).$$

Here GL_1^n and U_0^n are the direct products of n copies of the groups GL_1 and U_0 for the $\mathrm{GL}_1 \times \mathrm{GL}_k$ doubling integrals; δ is the product of n elements δ from § 2.4 occurring in these integrals, and an element δ for the Whittaker functional if c is odd; $\varepsilon^\vee(a) = \prod_{i=1}^n \pi_i^{-1}(a_i)$; and if c is odd, the additional inner integral over U_0 is a Jacquet integral constituting the evaluation of a Whittaker functional at the identity (see after (5.33)).

We start "peeling off" the outer integrals, each time regarding an inner integral as a meromorphic function on $\mathbb{C}$ (as opposed to $\mathrm{Re}(s) \gg 0$); then the outer integral is defined for all s except at the poles of the inner integral.

Since $({}^w(g_1,{}^\iota g_2))w \in K_H$ and f is K_H-finite, it remains to bound the du_0dady-integral with $({}^w(g_1,{}^\iota g_2))\cdot f'$ replaced by $f_0' = m(s,\tau)f_0$ for an arbitrary standard K_H-finite section f_0. Arguing as explained above, the dy-integration can be traded for a sum of convolution sections, by a repeated application of Corollary A.3 (more details appear below). Note that the equivariance properties of the inner doubling integrals with respect to unipotent subgroups are preserved, even though the inner integrals are each further reduced to $\mathrm{GL}_1 \times \mathrm{GL}_k$ doubling integrals, and these equivariance properties are all that is needed in order to obtain (6.36). Eventually one obtains a finite sum of integrals of the form

$$\int\limits_{\mathrm{GL}_1^n}\int\limits_{U_0^{\lceil c/2\rceil}} \phi_j(f_j)(s,\delta u_0(1,a))\psi_U(u_0)\varepsilon^\vee(a)\,du_0\,da, \tag{6.39}$$

where f_j is a section and $\phi_j(f_j)(s,h) = \int_Y f_j(s,hy)\phi_j(y)dy$ for some Schwartz function ϕ_j (compactly supported or otherwise).

To describe f_j, first note that for any $h_0 \in K_H$, $f_0'(s,h_0)$ is a product of a rational function of s (depending on h_0) and fixed quotients of twisted classical Gamma functions due to the normalization factors of Langlands ([Art89, Theorem 2.1 and § 3]). Hence there is a constant $r > 0$ such that for all $h_0 \in K_H$, $f_0'(s,h_0)$ is holomorphic of order at most r in $\mathcal{D}_A$. We can then adapt the arguments of Appendix A to deduce that in the application of Corollary A.3, $f_0' = \sum_i \phi_i(f_i)$ where each f_i is also holomorphic of order at most r in $\mathcal{D}_A$ (i.e., $s \mapsto f(s,h_0)$ is of order $\leq r$ for all $h_0 \in K_H$). Specifically, with the notation of Appendix A, W consists of holomorphic functions from $\mathcal{D}_A$ into the space of $\mathrm{Ind}_{K_H\cap M_{L_0}}^{K_H}(((\otimes_{i=1}^c W_\psi(\tau^i))\otimes$

$\chi_\pi)|_{K_H\cap M_{L_0}})$, and the semi-norms $||f||^\nu_D$ are replaced with $||f||^\nu_m = \max_{s\in\mathcal{D}_A}\{\nu(f(s))e^{-|s|^{r+1/m}}\}$ where m varies over $\mathbb{N}-0$; then W is still a Fréchet space which is a continuous, smooth representation of moderate growth of H (argue as in [Jac09, § 3.3], but instead of compact subsets $\Omega\subset\mathbb{C}$ consider $\mathcal{D}_A$). We repeatedly apply Corollary A.3, each time obtaining new holomorphic sections f_j of order at most r in $\mathcal{D}_A$ (though no longer K_H-finite).

It remains to bound the integrals (6.39). Since the Rankin–Selberg integrals for $\mathrm{GL}_1\times\mathrm{GL}_{2k}$ are bounded at infinity in $\mathcal{D}$ ([Jac09, Theorem 2.1(ii)]), Claim 6.4 and the proof of Claim 6.3 imply that the product of n integrals over GL_1 and U_0 is bounded by a polynomial in $\mathcal{D}$ (the polynomial is needed because of the mediating γ-factor). When c is odd, the Whittaker functional is entire of finite order in $\mathcal{D}$ (in fact in the entire plane) by a result of McKee [McK13]. It remains to consider $|\phi_j(f_j)(s,1)|$, where 1 denotes the identity element of H. Note that we can assume $A\gg 0$ so that the Rankin–Selberg integrals are holomorphic in $\mathcal{D}_A$ (because the representations of GL_1 and GL_{2k} are already determined).

Write $y=vtb$ using the Iwasawa decomposition, with $v\in N_H$, $t\in T_H$ and $b\in K_H$. Each coordinate of t can be bounded by a polynomial in $||y||$, where $||\cdot||$ is a fixed norm on H. Specifically, there are $d_0>0$, $d_1>0$ and an integer $M>0$ such that $d_0||y||^{-M}\le|t_i|\le d_1||y||^M$ for all $1\le i\le kc$ (if $H=\mathrm{GSpin}_{2kc}$, we write the decomposition in Spin_{2kc}). See e.g., [Jac09, § 5.2] and [Sou93, § 7.3, Lemma 3]. If $t=\mathrm{diag}(b_1,\ldots,b_c)$ with $b_j\in T_{\mathrm{GL}_k}$,

$$|\chi|^s(t)=(\prod_{j=1}^{n}|\det b_{2j-1}|^{s-1/2}|\det b_{2j}|^{1/2-s})|\det b_{2n+1}|^{s-1/2}.$$

Since $||y||\ge 1$, we can bound this character from above by $||y||^{N_s}$ where $N_s>0$ is an integer depending on s. Moreover, since $\mathcal{D}$ is a vertical strip of finite width, we can take a uniform bound N for all $s\in\mathcal{D}$. In addition because the representation of M_{L_0} on $\otimes_{i=1}^c W_\psi(\tau^i)$ is of moderate growth, we can take a large N and a semi-norm on the space of $f_j(s)=f_j(s,\cdot)$ such that for all y, $\max_{h_0\in K_H}|f_j(s,yh_0)|\le||y||^N\nu(f_j(s))$. Then

$$|\phi_j(f_j)(s,1)|\le\nu(f_j(s))\int\limits_Y||y||^N|\phi_j(y)|\,dy.$$

The norm $\nu(f_j(s))$ is of finite order for $s\in\mathcal{D}_A$, completing the proof. □

Remark 6.12. Note that $P(s)$ exists and can be taken independently of ω and f. Indeed as we have seen in the discussion above (see (6.37)), the poles of the integrals together with their multiplicities are bounded independently of the data.

6.11 Archimedean property. Using (4.2)–(4.3) and the twisting property we reduce to the case where π and τ are square-integrable. Now we apply Casselman's subrepresentation theorem [Cas80a], to regard both π and τ as quotients of principal series representations. Then we can certainly use (4.2)–(4.3)and (4.8) to reduce to $\mathrm{GL}_1 \times \mathrm{GL}_1$ factors, which are Tate γ-factors, but it is a priori not clear how they relate to the Langlands parametrization.[1]

Recall that when the representations are generic, Shahidi [Sha85] computed the local coefficients and equated them with the corresponding Artin factors. The results of Knapp and Wallach [KW76] on extensions of roots, and Knapp and Zuckerman [KZ82] who related the inducing data from Casselman's result to the Harish-Chandra parameter (here — of $\tau \otimes \pi$), were crucial to his proof.

Shahidi expressed the local coefficient as a product of Tate factors involving Casselman's inducing data and simple reflections [Sha85, Lemma 1.4]. Then in a sequence of lemmas (*loc. cit.*, § 3),he used the results of [KW76, KZ82] to prove that this product is equal to the product of γ-factors defined using Artin's root number and L-factor for the homomorphism φ (attached to $\tau \otimes \pi$). The relation between these products is formal, and the lemmas from [Sha85, § 3] can be applied to the Tate factors we obtain (using multiplicativity). Note that in our setting, since we only treat split groups, only the SL_2 case of [Sha85, Lemma 1.4] appears in the computation, and we only obtain the factors for the standard representation r (the local coefficient consists of the finite list of representations r_i, see [Sha85]). This completes the proof of (4.9).

7 L- and ϵ-factors

Theorem 4.2 enables us to define the local L- and ϵ-factors, using the γ-factor. This was carried out in [LR05] for $k = 1$ (following [Sha90]), and we briefly recall the construction.

For $G = \mathrm{GL}_n$ define the L- and ϵ-factors as the products of Rankin–Selberg L- and ϵ- factors for $\pi \times \chi\tau_0$ and $\pi^\vee \times \tau_0$ defined in [JPSS83, JS90] (see (4.8)). Henceforth we assume $G \neq \mathrm{GL}_n$, until the end of this section.

Over p-adic fields we follow Shahidi [Sha90]. When π and τ are both tempered, define $L(s, \pi \times \tau) = P(q^{-s})^{-1}$, where $P(X) \in \mathbb{C}[X]$ is the polynomial such that the zeros of $P(q^{-s})$ are those of $\gamma(s, \pi \times \tau, \psi)$ and such that $P(0) = 1$. This does not depend on ψ, by (4.7). Then by (4.6),

$$\epsilon(s, \pi \times \tau, \psi) = \frac{\gamma(s, \pi \times \tau, \psi)L(s, \pi \times \tau)}{L(1 - s, \pi^\vee \times \tau^\vee)}$$

is invertible in $\mathbb{C}[q^{-s}, q^s]$. The final form of the functional equation is

$$\gamma(s, \pi \times \tau, \psi) = \epsilon(s, \pi \times \tau, \psi)\frac{L(1 - s, \pi^\vee \times \tau^\vee)}{L(s, \pi \times \tau)}. \tag{7.1}$$

[1] There was a gap in the proof in the first version of this manuscript; we would like to thank Freydoon Shahidi for pointing it out to us, and indicating the applicability of his results from [Sha85].

The general definition of the L- and ϵ- factors for an arbitrary irreducible representation π and irreducible generic τ is now given in terms of the Langlands' classification, using the unramified twisting and multiplicativity properties in Theorem 4.2, and using the GL_n case above. In more detail, assume π is the unique irreducible quotient of a representation parabolically induced from $\sigma_{\beta'} \otimes \pi'$, where β' is a d' parts composition of $l \leq n$, $\sigma_{\beta'} = \otimes_{i=1}^{d'} \sigma_i$, each σ_i is essentially tempered, π' is tempered unless $G = \mathrm{GSpin}_c$ then it is essentially tempered, and $\tau = \mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\otimes_{j=1}^{d} \tau_j)$ where each τ_j is essentially tempered. For each pair $\pi' \times \tau_j$, if $|\Upsilon|^{s_0}\pi' = \pi'_0$ and $|\det|^{-r_j}\tau_j = \tau_{0,j}$ are tempered, where $\pi' = \pi'_0$ and $s_0 = 0$ when $G \neq \mathrm{GSpin}_c$, $L(s, \pi'_0 \times \tau_{0,j})$ is defined using the zeros of $\gamma(s, \pi'_0 \times \tau_{0,j}, \psi)$ and

$$L(s, \pi' \times \tau_j) = L(s + s_0 + r_j, \pi'_0 \times \tau_{0,j}),$$
$$\epsilon(s, \pi' \times \tau_j, \psi) = \epsilon(s + s_0 + r_j, \pi'_0 \times \tau_{0,j}, \psi).$$

Then by definition

$$L(s, \pi \times \tau) = \prod_{i,j} L(s, \sigma_i \times \chi_\pi \tau_j) L(s, \sigma_i^\vee \times \tau_j) \prod_j L(s, \pi' \times \tau_j),$$
$$\epsilon(s, \pi \times \tau, \psi) = \prod_{i,j} \epsilon(s, \sigma_i \times \chi_\pi \tau_j, \psi) \epsilon(s, \sigma_i^\vee \times \tau_j, \psi) \prod_j \epsilon(s, \pi' \times \tau_j, \psi).$$

Now (7.1) holds in general. In addition by Corollary 4.5, for generic representations the local factors defined here agree with Shahidi's.

In particular when data are unramified, we obtain the L-function defined using the Satake isomorphism, and the ϵ-factor is trivial: for tempered representations this follows from (4.4) because for tempered unramified representations, the inducing data is unitary; the general case then follows from the definition and the tempered case.

Over archimedean fields we define the L- and ϵ-factors by the Langlands correspondence [Bor79, Lan89] (for details see [CKPSS04, § 5.1]). Specifically, if Π is the local functorial lift of π to GL_N, we define $L(s, \pi \times \tau) = L(s, \Pi \times \tau)$ and $\epsilon(s, \pi \times \tau, \psi) = \epsilon(s, \Pi \times \tau, \psi)$. Then (7.1) holds because of (4.9).

If π is unramified, we also consider its functorial lift Π to GL_N, defined by virtue of the Satake isomorphism [Sat63, Bor79, Hen00, HT01] (see [CKPSS04, § 5.2]). The local γ-, L- and ϵ-factors of $\Pi \times \tau$ are defined by [JPSS83].

Lemma 7.1. *If π is unramified, the γ-, L- and ϵ-factors of $\pi \times \tau$ and $\Pi \times \tau$ coincide.*

Proof. This follows as in [CKPSS04, Proposition 5.2], which was stated for generic representations, but extends to this setting using Theorem 4.2. We provide a proof. The multiplicativity properties (4.2)–(4.3) imply that the γ-factors coincide, then by (7.1) it suffices to prove $L(s, \pi \times \tau) = L(s, \Pi \times \tau)$. Both L-factors can be defined as the products of L-factors for essentially tempered representations, and by the unramified twisting property we may assume τ is tempered.

Assume that π is an irreducible quotient of an unramified principal series representation, induced from an unramified character $\otimes_{i=1}^{n}\pi_i$ of T_G if G is a classical group, or from $\otimes_{i=1}^{n}\pi_i \otimes \chi_\pi$ for GSpin_c as explained in Remark 4.4. Also write π as a quotient of a representation parabolically induced from $\sigma_{\beta'} \otimes \pi'$, where $\sigma_{\beta'} = \otimes_{i=1}^{d'}\sigma_i$, σ_i (resp., π') is an unramified essentially tempered representation of $\mathrm{GL}_{\beta'_i}$ (resp., G').

The representations σ_i and π' are also quotients of unramified principal series representations. If the unramified character corresponding to σ_i is $\otimes_j \sigma_{i,j}$, and the one corresponding to π' is $\otimes_l \pi'_l$ or $\otimes_l \pi'_l \otimes \chi_\pi$, up to reordering $\otimes_{i,j}\sigma_{i,j} \otimes_l \pi'_l$ is the character $\otimes_i \pi_i$.

By definition

$$L(s, \pi \times \tau) = L(s, \pi' \times \tau) \prod_{i=1}^{d'} L(s, \sigma_i \times (\tau \otimes \chi_\pi^{-1}\tau)).$$

(Recall $\chi_\pi = 1$ for $G \neq \mathrm{GSpin}_c$.) Again, by definition

$$L(s, \sigma_i \times (\tau \otimes \chi_\pi^{-1}\tau)) = L(s, \sigma_i \times \chi_\pi \tau) L(s, \sigma_i^\vee \times \tau),$$

where on the r.h.s. these are Rankin–Selberg L-functions for generic representations, and the description in [JPSS83, § 8.4] implies

$$L(s, \sigma_i \times (\tau \otimes \chi_\pi^{-1}\tau)) = \prod_j L(s, \sigma_{i,j} \times \chi_\pi \tau) L(s, \sigma_{i,j}^{-1} \times \tau).$$

Regarding π', by virtue of our definition we can already assume it is tempered. Then $L(s, \pi' \times \tau)$ is defined by the zeros of $\gamma(s, \pi' \times \tau, \psi)$, which by (4.3) are the zeros of

$$[\gamma(s, \tau, \psi)] \prod_l \gamma^{\mathrm{RS}}(s, \pi'_l \times \chi_\pi \tau, \psi) \gamma^{\mathrm{RS}}(s, {\pi'_l}^{-1} \times \tau, \psi). \tag{7.2}$$

Here $[\ldots]$ appears only for $G = \mathrm{Sp}_{2n}$. Since the inducing character of π' is unitary, when we write each γ-factor in (7.2) as a quotient of L-functions multiplied by the ϵ-factor, there are no cancellations, in each quotient as well as between pairs of quotients corresponding to pairs of γ-factors. Thus the zeros of (7.2) are precisely the poles of

$$[L(s, \tau)] \prod_l L(s, \pi'_l \times \chi_\pi \tau) L(s, {\pi'_l}^{-1} \times \tau).$$

We deduce

$$L(s, \pi \times \tau) = [L(s, \tau)] \prod_{i=1}^{n} L(s, \pi_i \times \chi_\pi \tau) L(s, \pi_i^{-1} \times \tau). \tag{7.3}$$

According to [JPSS83, § 8 and § 9.4], this identity is also satisfied by $L(s, \Pi \times \tau)$. □

Corollary 7.2. *Let π be tempered and τ be unitary generic, and if the field is p-adic also assume π is unramified. Then $L(s,\pi\times\tau)$ is holomorphic for* $\mathrm{Re}(s)\geq 1/2$. *If τ is tempered, $L(s,\pi\times\tau)$ is holomorphic for* $\mathrm{Re}(s)>0$.

Proof. Since τ is unitary (and generic), $\tau\cong\mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\otimes_{i=1}^d|\det|^{r_i}\tau_i)$ for square-integrable τ_i and $r_i>-1/2$ for all i ([Vog86, Tad86]). Over a p-adic field, since the inducing character of π is unitary, Π is tempered (it is a full induced representation). Thus $L(s,\pi\times\tau)=L(s,\Pi\times\tau)$ factors as the product of L-functions for unitary twists of $|\det|^{r_i}\tau_i$ ([JPSS83, § 8.4, § 9.4]), each holomorphic for $\mathrm{Re}(s)\geq 1/2$ because $L(s,\tau_i)$ is holomorphic for $\mathrm{Re}(s)>0$ ([GJ72]) and $r_i>-1/2$. Over archimedean fields we can directly deduce that $L(s,\pi\times\tau)$ is a product of L-functions $L(s+r_i,\pi\times\tau_j)$, each known to be holomorphic for $\mathrm{Re}(s)+r_i>0$. The case of a tempered τ follows at once since then $r_i=0$. □

Assume F is p-adic. We prove a stability result which essentially follows from the stability result of Rallis and Soudry [RS05] for the doubling method. Let π be an irreducible representation of G. Let Π be an irreducible generic representation of GL_N, where for $G\neq\mathrm{GSpin}_c$ we assume Π has a trivial central character, and for GSpin_c we assume Π is unramified with a central character $\chi_\pi^{N/2}$ (e.g., take Π whose Satake parameter takes the form (4.11)).

Lemma 7.3. *If η is a sufficiently highly ramified character of F^*, depending on π and Π, then for any $\tau=\eta\tau_0$ where τ_0 is an irreducible generic unramified representation of GL_k, the γ-, L- and ϵ-factors of $\pi\times\tau$ and $\Pi\times\tau$ coincide and moreover, $L(s,\pi\times\tau)=L(s,\Pi\times\tau)=1$. (Other than the condition on the central character for GSpin_c, π and Π are not related in any way.)*

Proof. Let π_{gen} be an irreducible generic representation of G. If $G=\mathrm{GSpin}_c$, π_{gen} must in addition have the same central character as π. This can be obtained, e.g., by taking π_{gen} to be an irreducible principal series with an inducing character $(\otimes_{l=1}^n\chi_l)\otimes\chi_\pi$, then it is automatically generic and if c is even, we can take $\otimes_{l=1}^n\chi_l$ such that $\prod_{l=1}^n\chi_l(-1)=\pi([-I_n,1])$ (in this case $C_G=C_G^\circ\coprod[-I_n,1]C_G^\circ$).

Assume $k=1$. According to the stability results [RS05, Wag] (see Remark 4.3), for a sufficiently highly ramified η (independent of τ_0), $\gamma(s,\pi\times\tau,\psi)$ is equal to the γ-factor $\gamma(s,\pi_{\mathrm{gen}}\times\tau,\psi)$ of Shahidi [Sha90]. The latter coincides with $\gamma(s,\Pi\times\tau,\psi)$ and moreover belongs to $\mathbb{C}[q^{-s},q^s]^*$, i.e., is invertible in $\mathbb{C}[q^{-s},q^s]$ (because it is equal to the ϵ-factor), by the stability results of [CKPSS04, § 4.5–4.6] for classical groups and [AS06, § 4] for general spin groups. By (4.2) we deduce that for all k,

$$\gamma(s,\pi\times\tau,\psi)=\gamma(s,\Pi\times\tau,\psi)\in\mathbb{C}[q^{-s},q^s]^*. \tag{7.4}$$

Now write π as the irreducible quotient of a representation parabolically induced from $\sigma_{\beta'}\otimes\pi'$, where $\sigma_{\beta'}=\otimes_i\sigma_i$, and all σ_i and π' are essentially tempered. For a sufficiently highly ramified η (still, depending only on π), by [JPSS83, 2.13],

$$L(s,\sigma_i\times\chi_\pi\tau)=L(s,\sigma_i^\vee\times\tau)=L(s,\sigma_i^\vee\times\chi_\pi^{-1}\tau^\vee)=L(s,\sigma_i\times\tau^\vee)=1$$

for all i, then $\gamma(\sigma_i\times(\tau\otimes\chi_\pi^{-1}\tau^\vee),\psi)\in\mathbb{C}[q^{-s},q^s]^*$. Therefore by (4.3), $\gamma(s,\pi'\times\tau,\psi)=\gamma(s,\pi\times\tau,\psi)$ up to factors in $\mathbb{C}[q^{-s},q^s]^*$. By (7.4), $\gamma(s,\pi'\times\tau,\psi)\in\mathbb{C}[q^{-s},q^s]^*$, thus by definition $L(s,\pi'\times\tau)=1$. Hence $L(s,\pi\times\tau)=1$ as a product of trivial L-factors. Also $L(s,\Pi\times\tau)=1$ ([JPSS83, 2.13]), whence (7.1) implies $\epsilon(s,\pi\times\tau,\psi)=\epsilon(s,\Pi\times\tau,\psi)$. □

Until the end of this section assume F is archimedean. The following lemma summarizes several basic properties of the L-function, which follow from known results on the classical Gamma functions.

Lemma 7.4. (1) *$L(s,\pi\times\tau)$ has finitely many poles in* $\mathrm{Re}(s)\geq 1/2$.

(2) *There is an $\epsilon>0$ (usually small) such that the poles of $L(s,\pi\times\tau)$ are contained in $\{s:|\mathrm{Im}(s)|<\epsilon\}$.*

(3) *$L(s,\pi\times\tau)$ decays exponentially in $|\mathrm{Im}(s)|$.*

(4) *Write $s=\sigma+it$ with $\sigma,t\in\mathbb{R}$. Fix σ, and let $\epsilon_0>0$. There are constants $A,B>0$ such that for all $|t|>\epsilon_0$, $|L(1-s,\pi^\vee\times\tau^\vee)/L(s,\pi\times\tau)|\leq A(1+|t|)^B$.*

Proof. By (4.9) and the definitions of the Artin factors,

$$L(s,\pi\times\tau)=C(s)\prod_{i=1}^{m}\Gamma(r_is+d_i),\qquad L(s,\pi^\vee\times\tau^\vee)=\widetilde{C}(s)\prod_{i=1}^{m}\Gamma(r_is+\widetilde{d}_i),$$

where $C(s)$ and $\widetilde{C}(s)$ are complex-valued functions such that $|C(s)|$ and $|\widetilde{C}(s)|$ are fixed when $\mathrm{Re}(s)$ is fixed; $r_i\in\{1,1/2\}$ and $d_i,\widetilde{d}_i\in\mathbb{C}$ (see e.g., [Sha85, § 3]). In particular $r_i>0$. The first two assertions follow immediately, and the third follows from Stirling's approximation for $\Gamma(s)$. For the last, observe that again by Stirling's approximation, under the assumption $|t|>\epsilon_0$, $|\Gamma(r_i(1-s)+\widetilde{d}_i)/\Gamma(r_is+d_i)|\leq A_i|t|^{B_i-2r_i\sigma}$, where $A_i>0$, $B_i\in\mathbb{R}$ and both depend on r_i,d_i and $\widetilde{d}_i$. □

Proposition 7.5. *Both $\vartheta(s,c,\tau\otimes\chi_\pi,\psi)$ and $C(s,c,\tau\otimes\chi_\pi,\psi)$ are bounded by a polynomial in vertical strips of finite width away from their poles.*

Proof. For any two meromorphic functions $q(s)$ and $q'(s)$, denote $q(s)\preceq q'(s)$ if $q(s)=\epsilon(s)q'(s)$ for an entire function $\epsilon(s)$, which is also invertible and bounded at infinity on vertical strips of finite width. E.g., $\vartheta(s,c,\tau\otimes\chi_\pi,\psi)\preceq 1$ unless $G=\mathrm{Sp}_{2n}$, in which case $\vartheta(s,c,\tau\otimes\chi_\pi,\psi)\preceq\gamma(s,\tau,\psi)$. The assertion on $\vartheta(s,c,\tau\otimes\chi_\pi,\psi)$ is now clear by Lemma 7.4 (4).

Next we claim

$$C(s,c,\tau\otimes\chi_\pi,\psi)\preceq[\gamma(s,\tau,\psi)^{-1}]\prod_{j=1}^{\lfloor c/2\rfloor}\gamma(2s-c+2j-1,\tau,\vee^2\otimes\chi_\pi,\psi)\qquad(7.5)$$

$$\times\prod_{j=1}^{\lceil c/2\rceil}\gamma(2s-c+2j-2,\tau,\wedge^2\otimes\chi_\pi,\psi),$$

where [...] appears only when $H = \mathrm{Sp}_{2kc}$. This will complete the proof by the aforementioned lemma. To prove (7.5), one reduces to the case $k = 1$ using Casselman's subrepresentation theorem [Cas80a] and (5.6), then it is essentially implied by the proof of [Yam14, Lemma B.1] (see also [Swe95, Yam11] and the conclusive [GI14, Appendix A.3]). In more detail, one argues by a globalization argument using (6.3), (6.13) and § 6.6. In our setup (as opposed to [Yam14]), at the p-adic places we can always compute $C(s, c, \tau \otimes \chi_\pi, \psi)$ using (6.10) (see § 6.5), unless c is odd then (6.10) is valid when $|2| = 1$ (see § 2.2). In the remaining case we can still determine $C(s, c, \tau \otimes \chi_\pi, \psi)$ using unramified sections and the multiplicative formulas for the functional λ on both sides of (3.4), given in § 5.3.2 (see also § 5.3.3). □

8 Global Theory: The Completed *L*-function

Assume $G \neq \mathrm{GL}_n$ (the results of this section are known for GL_n). Let F be a number field and $\mathbb{A}$ be its ring of adeles. Denote the set of infinite places of F by S_∞. Let π and τ be cuspidal representations of $G(\mathbb{A})$ and $\mathrm{GL}_k(\mathbb{A})$. If $G = \mathrm{GSpin}_c$ let $\chi_\pi = \pi|_{C_G^\circ(\mathbb{A})}$, otherwise χ_π is trivial. Also let $\chi_\tau : F^*\backslash\mathbb{A}^* \to \mathbb{C}$ be the central character of τ.

Theorem 8.1. *Let S be a finite set of places of F, such that outside S, all data are unramified. The partial L-function $L^S(s, \pi \times \tau)$ admits meromorphic continuation to the plane.*

Proof. The l.h.s. of (6.15) admits meromorphic continuation, and on the r.h.s. for any given s, by Proposition 2.6 and Corollary 6.9 we can choose data such that each integral at a place $\nu \in S$ is holomorphic and nonzero. □

In § 7 we defined local L- and ϵ-factors. Now we may define the global (completed) L-function $L(s, \pi \times \tau)$ and ϵ-factor $\epsilon(s, \pi \times \tau)$, as the Euler products of local factors over all places of F. Note that $\epsilon(s, \pi \times \tau)$ does not depend on ψ by (4.7).

Corollary 8.2. *The L-function $L(s, \pi \times \tau)$ admits meromorphic continuation to the plane.*

Proof. By Theorem 8.1, and since the local L-factors admit meromorphic continuation. □

Theorem 8.3. *The global functional equation holds:*

$$L(s, \pi \times \tau) = \epsilon(s, \pi \times \tau) L(1 - s, \pi^\vee \times \tau^\vee).$$

Proof. This follows from (4.10) with (7.1). □

We turn to proving boundedness in vertical strips.

Theorem 8.4. *Let S be a finite set of places of F, such that outside S, all data are unramified. Assume $L^S(s,\pi\times\tau)$ and $L^S(s,\pi^\vee\times\tau^\vee)$ have finitely many poles in* $\mathrm{Re}(s)\geq 1/2$, *and all of them are real. For any $\epsilon>0$, there are constants $A,B>0$ such that $|L^S(s,\pi\times\tau)|\leq A(1+|s|)^B$ for all s with* $\mathrm{Re}(s)\geq 1/2$ *and* $|\mathrm{Im}(s)|\geq\epsilon$.

Proof. We closely follow the arguments of [GL06, Proposition 1]. Let $S_0\subset S$ be the subset of finite places. Using (7.1), we may write (4.10) in the form

$$\frac{L^S(s,\pi\times\tau)}{L_{S_0}(1-s,\pi^\vee\times\tau^\vee)}=\epsilon_S(s,\pi\times\tau,\psi)\frac{L_{S_\infty}(1-s,\pi^\vee\times\tau^\vee)L^S(1-s,\pi^\vee\times\tau^\vee)}{L_S(s,\pi\times\tau)},$$

where the subscript S_0 (resp., S, S_∞) denotes the finite product of factors over the places in S_0 (resp., S, S_∞). According to our assumptions on the finiteness of poles, the l.h.s. has finitely many poles in $\mathrm{Re}(s)\geq 1/2$, and $L^S(1-s,\pi^\vee\times\tau^\vee)$ has finitely many poles in $\mathrm{Re}(s)\leq 1/2$. By Lemma 7.4 (1), $L_{S_\infty}(1-s,\pi^\vee\times\tau^\vee)$ also has finitely many poles in $\mathrm{Re}(s)\leq 1/2$. The remaining factors on the r.h.s. do not contribute (any) poles. Therefore, there is a polynomial $P(s)$ such that

$$L(s)=P(s)\frac{L^S(s,\pi\times\tau)}{L_{S_0}(1-s,\pi^\vee\times\tau^\vee)}$$

is entire.

Next, we can find $r_1\ll 0\ll r_2$ such that $L(s)$ is bounded on the boundary of the half-strip $\{s:r_1\leq\mathrm{Re}(s)\leq r_2,\mathrm{Im}(s)\geq\epsilon\}$ by $A(1+|s|)^B$ (by assumption the poles of $L^S(\cdots)$ are real, therefore any $\epsilon>0$ suffices).

Indeed, on the right boundary, this follows since $L^S(s,\pi\times\tau)$ is absolutely convergent for $\mathrm{Re}(s)\gg 0$ hence bounded there, and for $\nu<\infty$, $L_\nu(s,\pi_\nu\times\tau_\nu)^{-1}$ is bounded on any vertical line $\mathrm{Re}(s)=\sigma$ depending only on σ. On the left, this is because $L^S(1-s,\pi^\vee\times\tau^\vee)$ is absolutely convergent for $\mathrm{Re}(s)\ll 0$, $L_\nu(s,\pi_\nu\times\tau_\nu)^{-1}$ is again bounded on vertical lines for $\nu<\infty$, and $|L_{S_\infty}(1-s,\pi^\vee\times\tau^\vee)/L_{S_\infty}(s,\pi\times\tau)|\leq A(1+|s|)^B$ by Lemma 7.4 (4).

Moreover, by [GL06, Theorem 2] (stated also for non-generic representations) $L^S(s,\pi\times\tau)$ is a meromorphic function of finite order, whence so is $L(s)$. Now the Phragmén–Lindelöf principle implies $|L(s)|\leq A(1+|s|)^B$ on the half-strip, thereby on $\{s:r_1\leq\mathrm{Re}(s)\leq r_2\}$ (with possibly different constants A,B) because $L(s)$ is entire.

To obtain the bound for $L^S(s,\pi\times\tau)$ we apply the maximum modulus principle exactly as in [GL06, Proposition 1]. □

Corollary 8.5. *If $L(s,\pi\times\tau)$ and $L(1-s,\pi^\vee\times\tau^\vee)$ are entire, they are bounded in vertical strips of finite width.*

Proof. Let S and S_0 be as in Theorem 8.4. Since $L(s,\pi\times\tau)$ is entire, it is enough to prove boundedness in $\mathcal{D}=\{s:r_1\leq\mathrm{Re}(s)\leq r_2,|\mathrm{Im}(s)|\geq\epsilon\}$, where $r_1\ll 0\ll r_2$, and ϵ is sufficiently large such that $L_{S_\infty}(s, \pi \times \tau)$ is analytic in $\mathcal{D}$ (see Lemma 7.4 (2)). Put

$$L(s) = L_{S_0}(s, \pi \times \tau)^{-1} L(s, \pi \times \tau) = L_{S_\infty}(s, \pi \times \tau) L^S(s, \pi \times \tau).$$

Since $L(s, \pi \times \tau)$ and $L(1 - s, \pi^\vee \times \tau^\vee)$ are entire, so are $L^S(s, \pi \times \tau)$ and $L^S(1 - s, \pi^\vee \times \tau^\vee)$. Hence we may apply Theorem 8.4 without restricting to $\mathrm{Re}(s) \geq 1/2$ (see [GL06, Remark 2]), and deduce that $L^S(s, \pi \times \tau)$ is bounded by $A(1 + |s|)^B$ in $\mathcal{D}$. This bound is now polynomial in $|\operatorname{Im}(s)|$, while the L-functions appearing in $L_{S_\infty}(s, \pi \times \tau)$ decay exponentially in $|\operatorname{Im}(s)|$ (Lemma 7.4 (3)). Hence $L(s)$ is bounded in $\mathcal{D}$.

As in [GL06, Proposition 1], let $\mathcal{C}$ be the (discrete) union of discs of fixed radius $r > 0$ around the poles of $L_{S_0}(s, \pi \times \tau)$. Since $L_{S_0}(s, \pi \times \tau)$ is bounded in $\mathcal{D} - \mathcal{C}$, so is $L(s, \pi \times \tau)$, hence by the maximum modulus principle $L(s, \pi \times \tau)$ is bounded in $\mathcal{D}$, completing the proof. □

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Appendix A. Technical results on analytic families of representations (Dmitry Gourevitch)

Let [2] H be a real reductive group. Fix a maximal compact subgroup K_H of H. Let P be a parabolic subgroup of H, and M_P be its Levi quotient. Let ρ be a (complex) smooth Fréchet representation of M_P, of moderate growth. For an algebraic

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character χ of M_P and $s \in \mathbb{C}^l$, let $V(s,\rho)$ be the space of the smooth induced representation $\mathrm{Ind}_P^H(|\chi|^s\rho)$ (l is determined by M_P). For example H is a classical group, P is a Siegel parabolic subgroup of H, M_P is isomorphic to $\mathrm{GL}_r(\mathbb{R})$ or $\mathrm{GL}_r(\mathbb{C})$, ρ is in addition admissible of finite length, χ is the determinant character and $s \in \mathbb{C}$.
By virtue of the Iwasawa decomposition, the spaces $V(s,\rho)$ where s varies are all isomorphic as representations of K_H to the smooth induction $V{:=}\mathrm{Ind}_{M_P\cap K_H}^{K_H}(\rho|_{M_P\cap K_H})$. Let W denote the space of functions from $\mathbb{C}^l$ to V that are holomorphic in the sense that their composition with every continuous functional on V is a holomorphic function. This notion was discussed by Grothendieck [Gro53, § 2]. Since V is a Fréchet space, by [Gro53, § 2, Remarque 1 and footnote 4] a function $f : \mathbb{C}^l \to V$ is holomorphic if and only if it is continuous, and in addition $\psi \circ f$ is a holomorphic function $\mathbb{C}^l \to \mathbb{C}$ for every ψ in a separating set $\mathfrak{X}$ of functionals on V. Separating here means that they have no common zeros on V. For example, we can take $\mathfrak{X}$ to be the set of all functionals of the form $v \mapsto \langle w, v(k)\rangle$, where $k \in K_H$ (thus $v(k)$ belongs to the space of ρ), and w is a K_M-finite vector in the space of the continuous dual representation of ρ. Here, K_M is a maximal compact subgroup of M.
Define a topology on W by the system of semi-norms $||f||_D^\nu := \max_{s\in D}\nu(f(s))$, where D runs over all closed balls in $\mathbb{C}^l$, and ν over all the semi-norms on V. Note that this family of semi-norms defines a Fréchet topology on W. Indeed, the topology stays equivalent if we keep only balls with rational centers and radii, and thus can be given by a countable family of semi-norms. Furthermore, the topology is complete since for any Cauchy sequence f_n and any $s \in \mathbb{C}^l$, the sequence of vectors $f_n(s)$ converges, and the limit $f(s)$ is holomorphic in s by the Cauchy formula, since for every continuous functional ψ on V, the holomorphic functions $\psi(f_n(s))$ converge to $\psi(f(s))$ uniformly on compact sets.
Note that W is naturally a continuous representation of H of moderate growth (see e.g., [Jac09, § 3.3]). Furthermore, W is a smooth representation of H. Indeed, $V(s,\rho)$ is smooth for every s, and for every X in the Lie algebra of H and $f \in W$, the functions $t^{-1}(\exp(tX)f(s) - f(s))$ converge when $t \to 0$ to the derivative $X(f(s))$ uniformly on compact sets. The latter follows from the definition of the topology on the smooth induction (see e.g., [Cas89] for this definition).
Let R be a Lie subgroup of H. Let $C^\infty(R)$ denote the space of smooth functions on R, and let $C_c^\infty(R)$ be the subspace of compactly supported functions. Fix a (non-zero) left-invariant measure dx on R. For any $\phi \in C_c^\infty(R)$ and any $f \in W$, define $\phi(f) \in W$ by

$$\phi(f)(s) = \int\limits_R x \cdot f(s)\phi(x)\,dx.$$

Equivalently, we can define $\phi(f)$ using the action of ϕ on the representation W, rather than separately on $V(s,\rho)$ for each s. The Dixmier–Malliavin Theorem [DM78] (see also [Cas] for a modern exposition and [Dor] for an extension to bornological spaces) applied to W implies the following statement.

Theorem A.1. *For any $f \in W$ there exist $m \in \mathbb{N}$, $\phi_1, \dots \phi_m \in C_c^\infty(R)$ and $f_1, \dots, f_m \in W$ such that $f = \sum_{i=1}^m \phi_i(f_i)$, i.e., $f(s) = \sum_{i=1}^m \phi_i(f_i)(s)$ for all s.*

Remark A.2. As a rule, even if f does not depend on s, the sections f_i will still depend on s, unless $R < K_H$.

In the discussion above, and in the theorem, one can restrict the domain of the functions to any open subset U of $\mathbb{C}^l$. One can also define meromorphic sections of W as functions f from $U \setminus S$ to V for some discrete set S such that for some holomorphic function $\alpha : U \to \mathbb{C}$, the product αf extends to an element of W. Multiplying by α, Theorem A.1 implies the following corollary.

Corollary A.3. *For any meromorphic section $f \in W$ there exist $m \in \mathbb{N}$, $\phi_1, \dots, \phi_m \in C_c^\infty(R)$ and meromorphic sections $f_1, \dots, f_m \in W$ such that for all s for which $f(s)$ is defined, each $f_i(s)$ is also defined and we have $f(s) = \sum_{i=1}^m \phi_i(f_i)(s)$.*

Consider $f \in W$ (a holomorphic section), and let $\mathcal{D} \subset \mathbb{C}^l$ be a domain (in the paper $l = 1$ and the domains are vertical strips of finite width). We say that f is of finite order in $\mathcal{D}$ if for every continuous functional ψ on V, the holomorphic $\mathbb{C}^l \to \mathbb{C}$ function $\psi \circ f$ has a finite order in $\mathcal{D}$.

Theorem A.4. *For any $f \in W$ there exists a sequence $f_n \in W$ that converges to f, and for every n, f_n is a finite sum of the form $f_n = \sum_{i=1}^{m_n} \vartheta_{n,i} f_{n,i}$ with the following properties:*

1. *Each $f_{n,i} \in W$ is a standard section, in the sense that $f_{n,i}(s)$ is independent of s.*
2. *Each $f_{n,i}$ is K_H-finite.*
3. *Each $\vartheta_{n,i} : \mathbb{C}^l \to \mathbb{C}$ is holomorphic.*
4. *If f is of finite order in $\mathcal{D}$, so are all the functions $\vartheta_{n,i}$.*

Proof. According to Bishop [Bis62, Theorem 1], there exists a sequence p_k of continuous mutually annihilating projections on V, whose ranges are one dimensional subspaces of V, such that $f = \sum_k p_k \circ f$. Choosing for each k a nonzero vector $v_k \in V$ in the image of p_k, we can write $f = \sum_k \alpha_k v_k$ where each $\alpha_k : \mathbb{C}^l \to \mathbb{C}$ is holomorphic.

The vectors v_k uniquely define standard sections h_k. We then approximate each h_k by a sequence of standard K_H-finite vectors h_k^i. Since $f = \sum_{k=1}^\infty \alpha_k h_k$, and the sequences h_k^i converge to h_k for every k, there exist sequences of indices k_n and i_n such that the sequence $f_n := \sum_{k=1}^{k_n} \alpha_k h_k^{i_n}$ converges to f.

Finally if f is of finite order (in $\mathcal{D}$), each $p_k \circ f$ is of finite order, then so are the functions α_k. □

Appendix B. Proof of Theorem 3.2 (Eyal Kaplan)

We prove the result by adapting the arguments from [GK] to the present setup. We use the notation of § 1.1 and § 3. For brevity and to simplify the comparison to [GK], we put $D = Y_{k,c}$ and $\psi_D = \psi_{k,c}$ (D of *loc. cit.* is a different subgroup but plays the same role). Let ρ be a (k, c) representation of finite length, not necessarily of the form $\rho_c(\tau)$. We prove $\dim \mathrm{Hom}_D(V(s,\rho), \psi_D) \le 1$ by analyzing distributions on the orbits of the right action of D on the homogeneous space $P\backslash H$. For $h, h' \in H$, write $h \sim h'$ if $PhD = Ph'D$, otherwise $h \not\sim h'$. Denote $P_h = {}^{h^{-1}}P \cap D$. By the Frobenius reciprocity law, the space of distributions on the orbit PhD is given by

$$\mathcal{H}(h) = \mathrm{Hom}_{P_h}({}^{h^{-1}}(|\det|^{s-1/2}\rho) \otimes \psi_D^{-1} \otimes \Lambda_{h,\nu}, \theta_h). \tag{B.1}$$

Here $\Lambda_{h,\nu}$ is the trivial one dimensional representation if F is p-adic or $h \sim \delta_0$ (δ_0 was defined in § 2.4), otherwise for each integer $\nu \ge 0$, $\Lambda_{h,\nu}$ is the algebraic dual of the symmetric ν-th power of the normal bundle to PhD, and $\theta_h(x) = \delta_{P_h}(x)\delta_D^{-1}(x)\delta_P^{-1/2}({}^h x)$ $(x \in P_h)$. We prove $\mathcal{H}(h) = 0$ when $h \not\sim \delta_0$, and $\dim \mathcal{H}(\delta_0) = 1$. The local analysis on the orbits implies the result: in the non-archimedean case this follows from the theory of Bernstein and Zelevinsky [BZ76] of distributions on l-sheafs, note that the action of D is constructive; in the archimedean case the analysis is far more involved, but now follows transparently from Kolk and Varadarajan [KV96] and Aizenbud and Gourevitch [GK, Appendix], exactly as explained in [GK, § 2.1.3]. Note that for the vanishing arguments we only use the equivariance properties with respect to unipotent subgroups of P_h, and for these the representations $\Lambda_{h,\nu}$ can be ignored (see [GK, § 2.1.1]).

Fix $H = \mathrm{Sp}_{2kc}$. At the end of the proof we explain how to adapt it to SO_{2kc} and GSpin_{2kc} (for GL_{kc} the result already follows from [CFGoK, Proposition 2]).

Since $V_{(c^k)} \ltimes U_P = D < P$, we have $P\backslash H/D = \coprod_h PhD$ with $h = wu$, where w is a representative from $W(M_P)\backslash W(H)$ and $u \in N_H \cap M_{(c^k)} < M_P$. Identify w with a kc-tuple of 0's and 1's, where the i-th coordinate corresponds to

$$\begin{pmatrix} I_{kc-i} &&&& \\ & 0 && 1 & \\ && I_{2(i-1)} && \\ & \epsilon_0 && 0 & \\ &&&& I_{kc-i} \end{pmatrix}.$$

E.g., $w = (1, 0^{kc-1}) = \mathrm{diag}(I_{kc-1}, \left(\begin{smallmatrix} & 1 \\ \epsilon_0 & \end{smallmatrix}\right), I_{kc-1})$. Writing $v \in D$ in the form $(v_{i,j})_{1\le i,j \le 2k}$ with $v_{i,j} \in \mathrm{Mat}_c$, let B_i be the i-th block $v_{i,i+1}$, $1 \le i \le k$, then $B_k \in D \cap U_P$. Note that B_i takes arbitrary coordinates in Mat_c for $i < k$, while $B_k \in \{X \in \mathrm{Mat}_c : J_c({}^tX)J_c = X\}$. Also $\psi_D|_{B_i} = \psi \circ \mathrm{tr}$ for each i.

As shown in [GK, § 2.1.2], the condition

$$\psi_D|_{D \cap {}^{h^{-1}}U_P} \ne 1 \tag{B.2}$$

implies $\mathcal{H}(h) = 0$ (in *loc. cit.* ψ_U was restricted to $U \cap {}^{h^{-1}}U_P$).

Let $h = wu$. We have the following analog of [GK, Lemma 2.6].

LEMMA B.1. *Condition (B.2) is implied by*

$$\psi_D|_{D\cap {}^{w^{-1}}U_P} \neq 1. \tag{B.3}$$

Proof. By (B.3), there exists a root in D such that for the subgroup $Y < D$ generated by this root, ${}^{w}Y < U_P$ and $\psi_D|_Y \neq 1$. Since u normalizes D, it remains to show $\psi_D|_{{}^{u^{-1}}Y} \neq 1$. If this root belongs to B_i for $i < k$, it is identified by a diagonal coordinate d of B_i, and if $i = k$, by two diagonal coordinates (d, d) and $(c-d+1, c-d+1)$ of B_i. In both cases, since $u \in N_H \cap M_{(c^k)}$, the conjugation by u only changes coordinates above or to the right of these diagonal coordinates, whence $\psi_D|_{{}^{u^{-1}}Y} \neq 1$ (cf. the proof of [GK, Lemma 2.6]). □

Recall the embedding $\mathrm{GL}_c^{\triangle}$ of GL_c in GL_{kc}, and further embed $\mathrm{GL}_c^{\triangle}$ in M_P by $g^{\triangle} \mapsto \mathrm{diag}(g^{\triangle}, (g^{\triangle})^*)$. We see that $\mathrm{GL}_c^{\triangle}$ stabilizes the restriction of ψ_D to $B_1, \ldots, B_{k-1}$. Since $({}^{g^{\triangle}}\psi_D)|_{B_k}(X) = \psi(\mathrm{tr}(J_c {}^{t}g^{-1} J_c g^{-1} X))$, the stabilizer of ψ_D in M_P is $\{g^{\triangle} : g \in \mathrm{GL}_c, {}^{t}g J_c g = J_c\}$. In particular, the stabilizer contains $W(\mathrm{O}_c)$ (the Weyl group of O_c) regarded as a subgroup of permutation matrices. The following result simplifies the structure of w, at the cost of slightly modifying u. See [GK, Propositions 2.7–2.8].

PROPOSITION B.2. *We have* $\mathcal{H}(h) = 0$, *unless* $h \sim \hat{w}\hat{u}\sigma$ *such that for an integer* $0 \leq l \leq n$,

$$\hat{w} = (1^n, 0^{n-l}, 1^l, w_2, \ldots, w_k), \quad \forall 1 < i \leq k,$$
$$w_i = (1^n, 0^{n-l-d_{i-1}}, 1^{l+d_{i-1}}), \; 0 \leq d_1 \leq \ldots \leq d_{k-1} \leq n - l,$$

$\sigma = \sigma_0^{\triangle}$ *for* $\sigma_0 \in W(\mathrm{O}_c)$ *and* ${}^{\sigma^{-1}}\hat{u} \in N_H \cap M_{(c^k)}$.

Proof. Put $w = (w_1, \ldots, w_k)$ with $w_i \in \{0, 1\}^c$ and denote the j-th coordinate of w_i by $w_i[j]$. For $1 \leq j \leq n$, if $w_1[j] = w_1[c-j+1] = 0$, (B.3) holds, then by Lemma B.1 (B.2) holds whence $\mathcal{H}(h) = 0$. This already describes the first c coordinates of $\hat{w}$ up to a permutation. E.g., l is the number of coordinates with $w_1[j] = w_1[c-j+1] = 1$. Assume $w_i[j] = 1$ for some $1 \leq i < k$ and $1 \leq j \leq c$. Hence the j-th column of B_{k-i} is permuted into U_P, and if $w_{i+1}[j] = 0$, the j-th row of B_{k-i} is not permuted. Thus the (j, j)-th coordinate of B_{k-i} is permuted into U_P, and as above (B.3) implies $\mathcal{H}(h) = 0$.

Now as in the proof of [GK, Proposition 2.8], we can choose a suitable permutation $\sigma = \sigma_0^{\triangle}$ with $\sigma_0 \in W(\mathrm{O}_c)$ such that $\hat{w} = {}^{\sigma}w$ satisfies the required properties, then clearly so does $\hat{u} = {}^{\sigma}u$, and $h \sim \sigma h = \hat{w}\hat{u}\sigma$. □

Re-denote $w = \hat{w}$ and $u = \hat{u}$ with the properties of the proposition, then $h = wu\sigma$. To compute ${}^{h}D \cap M_P$ note that ${}^{h}D = {}^{w}D$. We can further multiply h on the left by elements of M_P, to change the blocks J_a appearing in the matrix corresponding to w to blocks I_a, then conjugate ${}^{h}D \cap M_P$ by permutation matrices in M_P to obtain a subgroup of N_{M_P} (see [GK, (2.26)] and the discussion after [GK, Proposition 2.8]).

EXAMPLE B.3. For $k = 2$, we first multiply h on the left by elements in M_P to obtain

$$w = \begin{pmatrix} &&&&&&&&&&& I_{l+d_1} \\ & I_{n-l-d_1} &&&&&&&&&& \\ &&&&&&&&& I_n && \\ &&&&&&&& I_l &&& \\ &&&& I_{n-l} &&&&&&& \\ &&&&&& I_n &&&&& \\ &&&&& \epsilon_0 I_n &&&&&& \\ &&&&&&& I_{n-l} &&&& \\ &&& \epsilon_0 I_l &&&&&&&& \\ && \epsilon_0 I_n &&&&&&&&& \\ &&&&&&&&&& I_{n-l-d_1} & \\ \epsilon_0 I_{l+d_1} &&&&&&&&&&& \end{pmatrix},$$

then conjugate ${}^hD \cap M_P$ by

$$\begin{pmatrix} I_{n-l-d_1} &&& \\ && I_{c-l} & \\ & I_l && \\ &&& I_{n+l+d_1} \end{pmatrix} \begin{pmatrix} I_{n-l-d_1} &&& \\ && I_c & \\ & I_n && \\ &&& I_{l+d_1} \end{pmatrix} \begin{pmatrix} & I_{2c-l-d_1} \\ I_{l+d_1} & \end{pmatrix}.$$

We see that ${}^hD \cap M_P = V_\beta$ for the composition β of kc given by

$$\beta = (n-l-d_{k-1}, \ldots, n-l-d_1, n-l, n+l, n+l+d_1, \ldots, n+l+d_{k-1}). \quad \text{(B.4)}$$

(Cf. [GK, (2.27)].) Denote $\psi_{V_\beta} = {}^h\psi_D|_{V_\beta}$. First we describe ${}^w\psi_D|_{V_\beta}$, then handle $u\sigma$. For

$$v = \begin{pmatrix} I_{n-l-d_{k-1}} & b_1 & \cdots &&&&& \\ & \ddots & \ddots &&&&& \\ && I_{n-l-d_1} & b_{k-1} & \cdots &&& \\ &&& I_{n-l} & b_k & \cdots && \\ &&&& I_{n+l} & b_{k+1} & \cdots & \\ &&&&& \ddots & \ddots & \\ &&&&&& I_{n+l+d_{k-2}} & b_{2k-1} \\ &&&&&&& I_{n+l+d_{k-1}} \end{pmatrix} \in V_\beta,$$

$$\begin{aligned} {}^w\psi_D(v) = \psi \Bigg(& \sum_{j=k-1}^{2} \mathrm{tr}\left(b_{k-j} \begin{pmatrix} 0_{d_j - d_{j-1} \times n-l-d_j} \\ I_{n-l-d_j} \end{pmatrix}\right) + \mathrm{tr}\left(b_{k-1} \begin{pmatrix} 0_{d_1 \times n-l-d_1} \\ I_{n-l-d_1} \end{pmatrix}\right) \\ & + \mathrm{tr}\left(b_k \begin{pmatrix} 0_{l \times n-l} \\ I_{n-l} \\ 0_{l \times n-l} \end{pmatrix}\right) - \mathrm{tr}\left(b_{k+1} \begin{pmatrix} I_n & 0_{n \times l} \\ 0_{d_1 \times n} & 0_{d_1 \times l} \\ 0_{l \times n} & I_l \end{pmatrix}\right) \\ & - \sum_{j=2}^{k-1} \mathrm{tr}\left(b_{k+j} \begin{pmatrix} I_n & 0_{n \times d_{j-1}+l} \\ 0_{d_j - d_{j-1} \times n} & 0_{d_j - d_{j-1} \times d_{j-1}+l} \\ 0_{d_{j-1}+l \times n} & I_{d_{j-1}+l} \end{pmatrix}\right) \Bigg). \end{aligned} \quad \text{(B.5)}$$

(The sum $\sum_{j=k-1}^{2}$ is omitted if $k \le 2$.)

Proposition B.4. *Assume $k > 1$ and $l < n$. If $\mathcal{H}(h) \neq 0$, ψ_{V_β} belongs to the orbit of*

$$v \mapsto \psi\left(\sum_{j=k-1}^{2} \mathrm{tr}\left(b_{k-j}\begin{pmatrix} *_{d_j-d_{j-1}\times n-l-d_j} \\ *_{n-l-d_j}\end{pmatrix}\right) + \mathrm{tr}\left(b_{k-1}\begin{pmatrix} *_{d_1\times n-l-d_1} \\ *_{n-l-d_1}\end{pmatrix}\right) + \mathrm{tr}\left(b_k\begin{pmatrix} *_{l\times n-l} \\ I_{n-l} \\ *_{l\times n-l}\end{pmatrix}\right)\right. \tag{B.6}$$

$$\left. - \mathrm{tr}\left(b_{k+1}\begin{pmatrix} I_n & 0_{n\times l} \\ *_{d_1\times n} & *_{d_1\times l} \\ *_{l\times n} & *_l\end{pmatrix}\right) - \sum_{j=2}^{k-1}\mathrm{tr}\left(b_{k+j}\begin{pmatrix} I_n & 0_{n\times d_{j-1}+l} \\ *_{d_j-d_{j-1}\times n} & *_{d_j-d_{j-1}\times d_{j-1}+l} \\ *_{d_{j-1}+l\times n} & *_{d_{j-1}+l}\end{pmatrix}\right)\right).$$

Here $*$ means undetermined block entries. When $u\sigma$ is the identity element, all coordinates were computed above and (B.6) coincides with (B.5).

Proof. The proof is a simplified version of [GK, Proposition 2.11]. We need some notation. Set $d_0 = 0$ and $d_k = d_{k-1}$. For each $1 \leq i \leq k-1$, write B_i as the upper right block of

$$\begin{pmatrix} I_{l+d_{k-i-1}} &&&& B_i^{1,1} & B_i^{1,2} & B_i^{1,3} & B_i^{1,4} \\ & I_{d_{k-i}-d_{k-i-1}} &&& B_i^{2,1} & B_i^{2,2} & B_i^{2,3} & B_i^{2,4} \\ && I_{n-l-d_{k-i}} && B_i^{3,1} & B_i^{3,2} & B_i^{3,3} & B_i^{3,4} \\ &&& I_n & B_i^{4,1} & B_i^{4,2} & B_i^{4,3} & B_i^{4,4} \\ &&&& I_{l+d_{k-i-1}} &&& \\ &&&&& I_{d_{k-i}-d_{k-i-1}} && \\ &&&&&& I_{n-l-d_{k-i}} & \\ &&&&&&& I_n \end{pmatrix}$$

and B_k as the upper right block of

$$\begin{pmatrix} I_l &&&& B_k^{1,1} & B_k^{1,2} & B_k^{1,3} & B_k^{1,4} \\ & I_{n-l} &&& B_k^{2,1} & B_k^{2,2} & B_k^{2,3} & B_k^{2,4} \\ && I_{n-l} && B_k^{3,1} & B_k^{3,2} & B_k^{3,3} & B_k^{3,4} \\ &&& I_l & B_k^{4,1} & B_k^{4,2} & B_k^{4,3} & B_k^{4,4} \\ &&&& I_l &&& \\ &&&&& I_{n-l} && \\ &&&&&& I_{n-l} & \\ &&&&&&& I_l \end{pmatrix}.$$

With this notation ψ_D is given by $\psi(\sum_{i=1}^{k}\sum_{j=1}^{4}\mathrm{tr}(B_i^{j,j}))$. Denote the lists of blocks $B_i^{t,t'}$ conjugated by w into M_P, U_P and U_P^- by $\mathscr{M}_P$, $\mathscr{U}_P$ and $\mathscr{U}_P^-$ (resp.). We have

$$\begin{aligned}\mathscr{M}_P =& \{B_i^{1,1}, B_i^{1,4}, B_i^{2,1}, B_i^{2,4}, B_i^{3,2}, B_i^{3,3}, B_i^{4,1}, B_i^{4,4} : 1 \leq i \leq k-1\} \\ & \coprod \{B_k^{1,3}, B_k^{2,1}, B_k^{2,2}, B_k^{2,4}, B_k^{3,3}, B_k^{4,3}\}, \\ \mathscr{U}_P =& \{B_i^{3,1}, B_i^{3,4} : 1 \leq i \leq k-1\} \coprod \{B_k^{2,3}\}\end{aligned}$$

and the remaining blocks belong to $\mathscr{U}_P^-$.

Recall $h = wu\sigma$. Since σ fixes ψ_D, ${}^h\psi_D = {}^{wu}\psi_D$, thus we can already assume $h = wu$ (but u is still given by Proposition B.2). Write $u = \mathrm{diag}(z_1, \ldots, z_k) \in M_{(c^k)}$ with $z_i = {}^{\sigma_0}v_i$ and $v_i \in N_{\mathrm{GL}_c}$ (recall ${}^{\sigma^{-1}}u \in N_H \cap M_{(c^k)}$). We can simplify the form of z_i

as follows. If $z_i = z_i' m_i$ such that ${}^w\operatorname{diag}(z_1', \ldots, z_k', z_k'^*, \ldots, z_1'^*) \in M_P$, then because $h \sim ph$ for any $p \in P$, we can already assume $z_i = m_i$. We take for $1 \le i \le k$,

$$m_i = \begin{pmatrix} I_{l+d_{k-i}}+M_i^1M_i^2 & M_i^1 & 0 \\ M_i^2 & I_{n-l-d_{k-i}}+M_i^3M_i^4 & M_i^3 \\ 0 & M_i^4 & I_n \end{pmatrix} \in \mathrm{GL}_c,$$
$$I_{l+d_{k-i}} + M_i^1 M_i^2 \in \mathrm{GL}_{l+d_{k-i}}, \qquad I_{n-l-d_{k-i}} + M_i^3 M_i^4 \in \mathrm{GL}_{n-l-d_{k-i}}.$$

These matrices are invertible because $m_i \in {}^{\sigma_0}N_{\mathrm{GL}_c}$, and so are the matrices $I_{n-l-d_{k-i}} + M_i^2 M_i^1$ (see the proof of [GK, Proposition 2.11]). Then

$$m_i^{-1} = \begin{pmatrix} I_{l+d_{k-i}} & -M_i^1 & M_i^1M_i^3 \\ -M_i^2 & I_{n-l-d_{k-i}}+M_i^2M_i^1 & -(I_{n-l-d_{k-i}}+M_i^2M_i^1)M_i^3 \\ M_i^4M_i^2 & -M_i^4(I_{n-l-d_{k-i}}+M_i^2M_i^1) & I_n+M_i^4(I_{n-l-d_{k-i}}+M_i^2M_i^1)M_i^3 \end{pmatrix}.$$

Also set for $X \in \mathrm{Mat}_{a\times b}$, $X' = -J_b{}^tXJ_a$.
To determine ψ_{V_β} we compute ${}^u\psi_D$ on the blocks of D conjugated by w into $b_k, b_{k+1}, \ldots, b_{2k-1}$. First, $b_k = \begin{pmatrix} B_k^{2,1} & B_k^{2,2} & B_k^{2,4} \end{pmatrix}$. To compute ${}^u\psi_D$ on b_k we consider $m_k^{-1}B_k(J_c{}^tm_k^{-1}J_c)$. Note that

$$J_c{}^tm_k^{-1}J_c = \begin{pmatrix} I_n+(M_k^3)'(I_{n-l}+(M_k^1)'(M_k^2)')(M_k^4)' & (M_k^3)'(I_{n-l}+(M_k^1)'(M_k^2)') & (M_k^3)'(M_k^1)' \\ (I_{n-l}+(M_k^1)'(M_k^2)')(M_k^4)' & I_{n-l}+(M_k^1)'(M_k^2)' & (M_k^1)' \\ (M_k^2)'(M_k^4)' & (M_k^2)' & I_l \end{pmatrix}.$$

Since $\psi_D|_{B_k} = \psi \circ \mathrm{tr}$, ${}^u\psi_D|_{B_k} = \psi(\mathrm{tr}(J_c{}^tm_k^{-1}J_cm_k^{-1}B_k))$. The restriction of ${}^u\psi_D$ to $B_k^{2,3}$ is given by the product of rows $n+1, \ldots, c-l$ of $J_c{}^tm_k^{-1}J_c$ and columns $l+1, \ldots, n$ of m_k^{-1}, and because $B_k^{2,3} \in \mathscr{U}_P$, we have

$$\begin{pmatrix} (I_{n-l}+(M_k^1)'(M_k^2)')(M_k^4)' & I_{n-l}+(M_k^1)'(M_k^2)' & (M_k^1)' \end{pmatrix} \begin{pmatrix} -M_k^1 \\ I_{n-l}+M_k^2M_k^1 \\ -M_k^4(I_{n-l}+M_k^2M_k^1) \end{pmatrix} = 0, \qquad \text{(B.7)}$$

otherwise $\mathcal{H}(h) = 0$ by (B.2). Since the restriction of ${}^u\psi_D$ to $\begin{pmatrix} B_k^{2,1}, B_k^{2,2} \end{pmatrix}$ is given by the product of rows $1, \ldots, n$ of $J_c{}^tm_k^{-1}J_c$ and columns $l+1, \ldots, n$ of m_k^{-1},

$$\begin{pmatrix} I_n+(M_k^3)'(I_{n-l}+(M_k^1)'(M_k^2)')(M_k^4)' & (M_k^3)'(I_{n-l}+(M_k^1)'(M_k^2)') \\ & (M_k^3)'(M_k^1)' \end{pmatrix}$$
$$\times \begin{pmatrix} -M_k^1 \\ I_{n-l}+M_k^2M_k^1 \\ -M_k^4(I_{n-l}+M_k^2M_k^1) \end{pmatrix} = \begin{pmatrix} -M_k^1 \\ a \end{pmatrix}, \qquad \text{(B.8)}$$

where $a = I_{n-l} + M_k^2M_k^1 \in \mathrm{GL}_{n-l}$. Set $d_a = \operatorname{diag}(I_{(k-1)c+l}, a, I_c, a^*, I_{(k-1)c+l}) \in M_P$. Since ${}^wd_a \in M_P$, $h \sim wd_au$ and when we repeat the computation above we obtain $\begin{pmatrix} -M_k^1a^{-1} \\ I_{n-l} \end{pmatrix}$, hence ${}^u\psi_D$ belongs to an orbit of a character which agrees with (B.6) on b_k.
For $1 \le i \le k-1$,

$$b_{k+i} = \begin{pmatrix} (B_{k-i}^{4,4})' & (B_{k-i}^{2,4})' & (B_{k-i}^{1,4})' \\ (B_{k-i}^{4,1})' & (B_{k-i}^{2,1})' & (B_{k-i}^{1,1})' \end{pmatrix}. \qquad \text{(B.9)}$$

To compute ${}^{u}\psi_D$ on b_{k+i} consider $m_{k-i}^{-1}B_{k-i}m_{k-i+1}$. Since $\psi_D|_{B_{k-i}} = \psi \circ \mathrm{tr}$,

$$ {}^{u}\psi_D|_{B_{k-i}} = \psi(\mathrm{tr}(m_{k-i+1}m_{k-i}^{-1}B_{k-i})). $$

This restriction must be trivial on $B_{k-i}^{3,4} \in \mathscr{U}_P$, otherwise $\mathcal{H}(h) = 0$ by (B.2). Thus we obtain, if $\mathcal{H}(h) \neq 0$,

$$ \begin{pmatrix} 0_{n\times l+d_{i-1}} & M_{k-i+1}^4 & I_n \end{pmatrix} \begin{pmatrix} -M_{k-i}^1 \\ I_{n-l-d_i}+M_{k-i}^2M_{k-i}^1 \\ -M_{k-i}^4(I_{n-l-d_i}+M_{k-i}^2M_{k-i}^1) \end{pmatrix} = 0. $$

Hence

$$ \begin{pmatrix} 0_{n\times l+d_{i-1}} & M_{k-i+1}^4 & I_n \end{pmatrix} \begin{pmatrix} M_{k-i}^1M_{k-i}^3 \\ -(I_{n-l-d_i}+M_{k-i}^2M_{k-i}^1)M_{k-i}^3 \\ I_n+M_{k-i}^4(I_{n-l-d_i}+M_{k-i}^2M_{k-i}^1)M_{k-i}^3 \end{pmatrix} = I_n. $$

Then the restriction of ${}^{u}\psi_D$ to $B_{k-i}^{4,4}$, which corresponds to the bottom right $n \times n$ block of $m_{k-i+1}m_{k-i}^{-1}$, is $\psi \circ \mathrm{tr} = \psi_D|_{B_{k-i}^{4,4}}$. Similarly, because $B_{k-i}^{3,1} \in \mathscr{U}_P$, $\mathcal{H}(h) = 0$ unless

$$ \begin{pmatrix} I_{l+d_{i-1}}+M_{k-i+1}^1M_{k-i+1}^2 & M_{k-i+1}^1 & 0_{l+d_{i-1}\times n} \end{pmatrix} \begin{pmatrix} -M_{k-i}^1 \\ I_{n-l-d_i}+M_{k-i}^2M_{k-i}^1 \\ -M_{k-i}^4(I_{n-l-d_i}+M_{k-i}^2M_{k-i}^1) \end{pmatrix} = 0. $$

Hence

$$ \begin{pmatrix} I_{l+d_{i-1}}+M_{k-i+1}^1M_{k-i+1}^2 & M_{k-i+1}^1 & 0_{l+d_{i-1}\times n} \end{pmatrix} \begin{pmatrix} M_{k-i}^1M_{k-i}^3 \\ -(I_{n-l-d_i}+M_{k-i}^2M_{k-i}^1)M_{k-i}^3 \\ I_n+M_{k-i}^4(I_{n-l-d_i}+M_{k-i}^2M_{k-i}^1)M_{k-i}^3 \end{pmatrix} = 0. $$

Therefore ${}^{u}\psi_D$ and ψ_D are both trivial on $B_{k-i}^{4,1}$. It then follows from (B.9) that ${}^{u}\psi_D$ is given on the blocks which w conjugates into b_{k+i} by

$$ \psi(\mathrm{tr}(\begin{pmatrix} (B_{k-i-1}^{4,4})' & (B_{k-i-1}^{2,4})' & (B_{k-i-1}^{1,4})' \\ (B_{k-i-1}^{4,1})' & (B_{k-i-1}^{2,1})' & (B_{k-i-1}^{1,1})' \end{pmatrix} \begin{pmatrix} I_n & 0_{n\times l+d_i} \\ *_{l+d_{i+1}\times n} & *_{l+d_{i+1}\times l+d_i} \end{pmatrix})). $$

We conclude ψ_{V_β} belongs to the orbit of (B.6). □

Proposition B.5. *If $l < n$, $\mathcal{H}(h) = 0$.*

Proof. The proof is a simplified version of [GK, Proposition 2.12]. The definitions imply any morphism in $\mathcal{H}(h)$ factors through $J_{V_\beta,\psi_{V_\beta}}(\rho)$ (see [GK, § 2.1.1]). The pair $(V_\beta, \psi_{V_\beta})$ defines a degenerate Whittaker model in the sense of [MW87]. Let φ be the transpose of the nilpotent element defined by ψ_{V_β}, which is an upper triangular nilpotent matrix in Mat_{kc}. We show φ is nilpotent of order at least $k+1$. Since ρ is (k, c), we deduce $J_{V_\beta,\psi_{V_\beta}}(\rho) = 0$ by [GGS17, Theorem E] (which over non-archimedean fields is based on [BZ76, 5.9–5.12]).

By Proposition B.4 we can assume ψ_{V_β} is given by (B.6), then the block b_i of φ is the transpose of the block appearing to the right of b_i in (B.6), up to the signs ± 1. Consider the blocks $b_k, \ldots, b_{2k-1}$ of φ: for $i > k$, the (n,n)-th coordinate of b_i is nonzero and is the only nonzero coordinate in its column, and the same applies to the $(n-l,n)$-th coordinate of b_k. These are k coordinates, and it follows that φ is nilpotent of order at least $k+1$. □

Remark B.6. The above reasoning in [GK] only implied $d_1 = n - l$; we had to use a third method to deduce vanishing (see [GK, Proposition 2.14]), and lose a discrete subset of s.

The remaining case to consider is $l = n$, which means $h \sim \delta_0$. Now since $P_{\delta_0} = V_{(c^k)}$ and $\psi_D|_{P_{\delta_0}}$ is the (k,c) character (1.1)(see(B.5)),

$$\mathcal{H}(\delta_0) = \mathrm{Hom}_{V_{(c^k)}}({}^{\delta_0^{-1}}\rho \otimes \psi_D^{-1}, 1) = \mathrm{Hom}_{V_{(c^k)}}(\rho \otimes \psi_D, 1) = \mathrm{Hom}_{V_{(c^k)}}(\rho, \psi_D^{-1}),$$

which is one dimensional (but the space in the theorem can still vanish) because ρ is (k,c) and ψ_D^{-1} belongs to the orbit of ψ_k, $\psi_D^{-1} = {}^{\mathrm{d}_{k,c}}\psi_k$. The proof is complete.

We now explain the case of $H = \mathrm{SO}_{2kc}$. The main difference is that here the restriction of ψ_D to the block B_k is given by $X \mapsto \psi(\mathrm{tr}({}^tAX))$ (A was defined in § 2.1, now $A \neq I_c$).

Assume momentarily that kc is even. First, for the kc-tuple representing the element w, the sum of coordinates must be even. Lemma B.1 remains valid, but now for the proof if the root belongs to B_k and c is odd, it is determined by a pair of coordinates $(d, d+1)$ and $(c-d, c-d+1)$ where $1 \le d \le n$.

The stabilizer of ψ_D in M_P does not contain $W(O_c)$, but $\mathrm{GL}_c^{\triangle}$ still fixes the restriction of ψ_D to the blocks $B_1, \ldots, B_{k-1}$. We argue as in Proposition B.2: Using conjugations by elements $\sigma = \sigma_0^{\triangle}$ for $\sigma_0 \in \mathrm{diag}(W(O_{2n}), I_{c-2n})$, we first deduce $\hat{w} = (a_1 I_{c-2n}, 1^n, 0^{n-l}, 1^l, w_2, \ldots, w_k)$ for some $0 \le l \le n$ and $w_i = (a_i I_{c-2n}, 1^n, 0^{n-l-d_{i-1}}, 1^{l+d_{i-1}})$ for $i > 1$. Here $a_1, \ldots, a_k \in \{0,1\}$ only appear when c is odd, and $a_1 \le \ldots \le a_k$. If c is odd we now conjugate $\hat{w}$ by $\mathrm{diag}(I_n, \left(\begin{smallmatrix} & 1 \\ I_n & \end{smallmatrix}\right))^{\triangle}$. Let $o \ge 1$ be minimal such that $a_o = 1$, where if $a_k = 0$ we set $o = k+1$. Then, for $j < o$ we have $w_j = (1^n, 0^{n+1-l-d_{j-1}}, 1^{l+d_{j-1}})$ ($d_0 = 0$) and for $j \ge o$, $w_j = (1^{n+1}, 0^{n-l-d_{j-1}}, 1^{l+d_{j-1}})$.

It follows that in the even case β is still given by (B.4). In the odd case the leftmost $k-o+1$ parts of β are $(n-l-d_{k-1}, \ldots, n-l-d_{o-1})$, the next $o-1$ parts are $(n+1-l-d_{o-2}, \ldots, n+1-l-d_0)$, the following $o-1$ parts are $(n+l+d_0, \ldots, n+l+d_{o-2})$, and the rightmost $k-o+1$ parts are $(n+1+l+d_{o-1}, \ldots, n+1+l+d_{k-1})$.

Now consider Proposition B.4. Besides minor modifications to the sizes of the parts of β in the odd case, the main difference concerns the restriction of (B.6) to b_k. This is because for $i \neq k$, $\psi_{V_\beta}|_{b_i}$ depends only on $\psi_D|_{B_j}$ for $j < k$ and then ${}^\sigma\psi_D|_{B_j} = \psi_D|_{B_j}$. However, σ does not fix $\psi_D|_{B_k}$ (which determines $\psi_{V_\beta}|_{b_k}$). We can write ${}^\sigma\psi_D|_{B_k}(X) = \psi(\mathrm{tr}(\varrho X))$ for $\varrho \in \mathrm{Mat}_c$, $\varrho = \mathrm{diag}(\varrho_1, \ldots, \varrho_c)$ where $\varrho_i = \pm 1$ for all i if c is even, and when c is odd $\varrho_i = \pm 1$ for all $i \neq n+1$ and $\varrho_{n+1} = 0$. The important observation is that ${}^\sigma\psi_D|_{B_k}$ will still be nonzero on n root subgroups. To determine

${}^{u\sigma}\psi_D$ on b_k we multiply the rows of $J_c{}^tm_k^{-1}J_c\varrho$ by columns of m_k^{-1}. On the l.h.s of both (B.7) and (B.8) we "inject" ϱ into the product. The r.h.s. of (B.7) still vanishes because $B_k^{2,3} \in \mathscr{U}_P$ (if $o > 1$, $B_k^{2,3}$ is taken to be an $n-l+1 \times n-l+1$ block), and the r.h.s. of (B.8) becomes $\begin{pmatrix} I_n & 0 \end{pmatrix} \varrho \begin{pmatrix} -M_k^1 \\ a \end{pmatrix}$ (if $o = 1$, I_n here is replaced by I_{n+1}). The only change to (B.6) (and in particular, to (B.5)) concerns the block I_{n-l} appearing in the restriction to b_k which is replaced by $\varrho^\circ = \mathrm{diag}(\varrho_{l+1}, \ldots, \varrho_n)$ when c is even, by $\begin{pmatrix} \varrho^\circ & 0 \end{pmatrix} \in \mathrm{Mat}_{n-l\times n+1-l}$ if $o > 1$ and by $\begin{pmatrix} \varrho^\circ \\ 0 \end{pmatrix} \in \mathrm{Mat}_{n+1-l\times n-l}$ for $o = 1$.
This change does not cause any new complications in the proof of Proposition B.5 and we conclude $l = n$. When c is even this implies $h \sim \delta_0$ and we complete the proof as above. When c is odd the remaining compositions β are uniquely determined by o, which varies over the numbers $1, \ldots, k+1$ such that $k-o+1$ is even. For each such β, the associated partition is $p_\beta = (k+o-1, k^{2n-1}, k-o+1)$ and the character ψ_{V_β} is generic. For $o > 1$ the partition p_β is greater than (k^c), thus $\mathcal{H}(h) = 0$ (because ρ is (k, c)). Since we are still considering the case where kc is even, $k - o + 1$ is even for $o = 1$. Then $h \sim \delta_0$ again, and the result holds. Lastly, when kc is odd we write $w = w'\jmath_1$ with $\det w' = -1$. Since now $D = {}^{\jmath_1}V_{(c^k)} \ltimes {}^{\jmath_1}U_P$ (see § 3), the same proof is applicable. In addition, since the proof only involves unipotent subgroups and the properties of (k, c) representations, the case of $H = \mathrm{GSpin}_{2kc}$ is now clear as well.

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