Adjusting for Unmeasured Confounding Variables in Dynamic Networks

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***Abstract*—The article presents a technique to identify a certain transfer function in a dynamic network when the input and the output of the transfer function are influenced by an unmeasured confounding variable. It is assumed that in an observational framework, only a subset of the variables of the network are measured and the topology of the interconnections between the variables is partially known. The focus of the paper is the challenging scenario where it is not possible to measure any variables on the directed paths from the confounding variable to either the input or the output of the transfer function of interest. Sufficient conditions are derived to determine a set of instrumental variables and a set of auxiliary variables that guarantee consistent identification of the transfer function using an algorithm based on prediction error method for the class of acyclic networks. It is also shown that similar ideas could be applied to cyclic networks. In particular, we show how consistent estimates of some transfer functions in a network with feedback loops could be used to identify some other transfer functions whose inputs and outputs are influenced by unmeasured confounding variables.**

***Index Terms*—Dynamic networks, Confounding Variables, Instrumental Variables, Identification**

I. Introduction

In recent years increasing attention has been given to the development of new tools for the identification of large-scale interconnected systems. While injecting a suitable input into a system and measuring the corresponding output is a standard strategy in system identification [1], [2], studying the relationship between two nodes of a network using only observational measurements is critically important for any large scale network fulfilling critical or uninterruptible functions (e.g., a power grid, a logistic system). or in situations where it is impractical (e.g. applications in medicine such as repeated drug testing, automatically assisted anesthesia, Deep Brain Stimulation for Parkinson disease) or too expensive to inject known probing signals into the system (e.g., a gene network, a financial network). This paper deals with the scenario where only observational data is available.

Statistical measures such as Cramer-Rao lower bound are often used to assess the accuracy of estimations. However, in a purely observational framework, guaranteeing the consistency of estimation is already a challenging task. A wealth of methodologies has been developed to deal with the problem of identifying a network of dynamic systems from observational data [3]–[12].

In several recent results, it has been shown that, by appropriately introducing additional measured variables to a set of predictor inputs, a consistent estimate of a certain transfer function can be obtained using prediction error method or substantially equivalent tools [13]–[16]. This idea has been explored in the extension of closed-loop identification techniques to network identification [13], [14], [17] and by applying graphical model tools [18]–[20].

Within a specific multi-input single-output prediction error framework , the results in [18] provide sufficient and necessary conditions, of purely graphical nature, to determine the set of auxiliary predictor inputs in order to guarantee a consistent identification of a single transfer function in a dynamic network. By consistent identification it is meant that as the number of data points used increases indefinitely, the estimated parameters converge in probability to their actual values [21].

In this paper, we deal with a special challenging scenario where the conditions required by [18] cannot be satisfied. Namely, we focus on the scenario where the input and the output of the transfer function of interest are influenced by an unmeasured confounding variable and it is not possible to measure any variables on the directed paths from the confounding variable to either the input or the output of the transfer function. We provide sufficient conditions to determine a set of instrumental variables [22]–[24] and a set of auxiliary variables that guarantee consistent identification of the transfer function using an algorithm based on the prediction error method for the class of acyclic networks. We also show that similar ideas could be applied to cyclic networks. In particular, we show how consistent estimates of some transfer functions in a network with feedback loops could be used to identify some other transfer functions whose inputs and outputs are influenced by unmeasured confounding variables.

The article is organized as follows. Section II reviews the concepts of dynamic networks, their graphical representations and some identification results. Section III presents the main results focusing on identification of a certain transfer function in an acyclic network in presence of an unmeasured confounding variable. Section IV shows that such results could be extended to networks with feedback loops. Concluding remarks are given in Section V.

II. Class of networked systems

In this section, we introduce the class of models that is going to be the object of our investigation along with some

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preliminary concepts and notions from the area of graphical models.

Definition 1. *A network $\mathcal{G}$ is a pair $(H(z),n)$ where $H(z)$ is a proper rational discrete-time $v\times v$ transfer matrix and n is a vector of v mutually independent stochastic processes with a full rank rational power spectral density matrix. The output signals of the network are defined by the relation*

$$x_j = n_j + \sum_{i\in V} H_{ji}(z)x_i, \qquad \text{for } j=1,\dots,v \tag{1}$$

Using a vector notation and defining $V=\{1,\dots,v\}$ we can represent the model in a more compact way as

$$x_V = n_V + H(z)x_V. \tag{2}$$

Models described by (2) lend themselves to be represented via graphs.

For a directed graph G, defined by the pair (V,E) where $V=\{1,2,\cdots,v\}$ is the set of nodes and $E\subseteq V\times V$ is the set of edges, we denote an edge $(i,j)\in E$ as $i\to j$ or $j\leftarrow i$ and say that the edge is oriented from i to j.

Definition 2. *Let $\mathcal{G}=(H,n)$ be a network with output processes x_V, where $V:=\{1,\dots,v\}$. We say that the graph $G=(V,E)$ is a graphical representation of the network if*
(a) *$i\to j\notin E$ implies $H_{ji}(z)=0$*

In other words, the absence of the edge $i\to j$ in a graphical representation implies that $H_{ji}(z)=0$.

Given a path π in a graph G we say that a node j is a collider, when there exist two consecutive edges in the path of the form $i\to j$ and $j\leftarrow k$. We say that node j is a descendant of node i if $j=i$ or if there is a directed path from i to j.

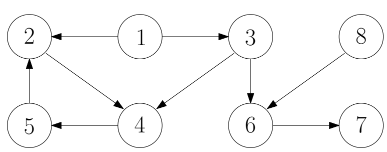

Fig. 1: Representation of a directed graph.

For example, in the graph of Figure 1, node 4 in the path $\{3\to 4\leftarrow 2\leftarrow 5\}$ is a collider. Also, node 7 is a descendant of node 1 because there is a directed path $\{1\to 3\to 6\to 7\}$ from node 1 to node 7.

Definition 3. *In a directed graph G, a path π between nodes i and j is blocked by a set of nodes Z if there is a non-collider on π that belongs to Z; or there is a collider c on π such that Z does not contain any descendants of c.*

In the theory of graphical models, a fundamental concept defined over the nodes of a directed graph is d-separation [25].

Definition 4. *In a directed graph $G=(V,E)$ let A, B, and C be disjoint subsets of V. A and B are d-separated by C if for all nodes $a\in A$ and $b\in B$, all paths between a and b are blocked by C.*

Example 1. *In the directed graph depicted in Fig. 1, by choosing $C=\{1,3,4\}$, we make $A=\{2\}$ and $B=\{6\}$ d-separated. Alternatively, $C=\{3\}$ would have been a smaller set making $A=\{2\}$ and $B=\{6\}$ d-separated.*

Furthermore we say that a path π is j-pointing if the last edge of π is of the form $k\to j$ for some node k [18]. For example, in the graph of Figure 1, the path $\{1\to 3\to 6\to 7\}$ is 7-pointing.

Definition 5. *Given a probability space, for a set of stochastic processes x_A where $A\subseteq V$, we denote the natural filtration generated by the processes x_A up to time t as $I_A(t)$.*

Denoting the set of real-rational causal transfer functions that are analytic and invertible on the unit circle by $\mathscr{F}^+$, we can interpret $I_A(t)$ as the rational causal transfer span:

$$I_A(t) := \Big\{ q = \sum_{i\in A} P_i(z)\ x_i \mid P_i(z)\in\mathscr{F}^+ \Big\}. \tag{3}$$

By the notation of Definition 5 the estimate $\hat{x}_j(t)$ of $x_j(t)$ in the least square sense based on the information of variables x_{D^+} up to time t and the information of variables x_{D^-} up to time $t-1$ could be written as

$$\hat{x}_j(t) = \mathbb{E}(x_j(t) \mid I_{D^+}(t), I_{D^-}(t-1)). \tag{4}$$

In the linear Gaussian case Equation (4) reduces to to

$$\hat{x}_j = \sum_{k\in D^+} W_{jk}(z)x_k + \sum_{k\in D^-} W_{jk}(z)x_k \tag{5}$$

where $W_{jk}(z)$ for $k\in D^+$ are proper modules and for $k\in D^-$ are strictly proper modules.

A. An identification result for the class of networked systems

In the following, we review some relatively recent results guaranteeing a consistent identification of a transfer function for the class of networked systems we have just introduced. These results rely on conditions that can be formulated directly on the graphical representation of the network. Here, we briefly summarize these identification results assuming that the reader is already familiar with standard notions of graph theory.

Within a specific multi-input single-output prediction error framework , Theorem III.2 and Theorem V.1 in [18] provide sufficient and necessary conditions, of purely graphical nature, to determine the set of auxiliary predictor inputs in order to guarantee a consistent identification of a single transfer function in a dynamic network. The following theorem combines Theorem III.2 and Theorem V.1 of [18].

Theorem II.1. *The application of Procedure 1 in [18] leads to a consistent estimate of $H_{ji}(z)$ for all networks with graphical representation G if and only if the predictor inputs set Z of observed nodes satisfies the following conditions.*

(i) Z blocks all the j-pointing paths between i and j with the exception of $i\to j$; and
(ii) $Z\cup\{i\}$ blocks all j-pointing paths from j to itself in G.

Proof. See the appendix. □

The above conditions (i) and (ii) of Theorem II.1 are applicable even in presence of feedback loops and/or confounding variables affecting the nodes i and j.

In our framework, we say that a node c is a confounding variable for the problem of identifying $H_{ji}(z)$, when there is a directed path from c to i and there is a directed path from c to j.

In this paper, however, we focus on a challenging scenario where Theorem II.1 is not directly applicable because its conditions cannot be satisfied due to the fact that some of the nodes are not measured.

III. Unmeasured confounders in acyclic networks

We start with a simplistic example to show that it is possible to leverage our knowledge of some transfer functions that we can straightforwardly identify to estimate some other transfer function. Consider a network with a graphical representation shown in Figure 2. The objective is the identification of the transfer function $H_{21}(z)$. If all the nodes were measured, we could apply Theorem II.1. Namely, the predictor inputs set $Z=\{4\}$ blocks the 2-pointing path $1 \leftarrow 4 \rightarrow 2$ and satisfies the conditions of Theorem II.1.

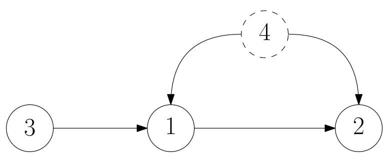

Fig. 2: Graphical representation of a network where the objective is the identification of $H_{21}(z)$. Node 4 which is a confounder influencing nodes 1 and 2 is not measured.

Suppose, however, that node 4 is not measured. Then, there is no way to block the 2-pointing path $1 \leftarrow 4 \rightarrow 2$. Since node 4 is a confounder influencing both nodes 1 and 2, the Wiener filter corresponding to x_1 when we estimate $x_2(t)$ using the information of x_1 is going to be, in general, a biased estimate of $H_{21}(z)$.

We show, however, that node 3 can be used as an instrumental variable to identify $H_{21}(z)$. Observe that $Z=\{\emptyset\}$ satisfies the conditions of Theorem II.1 for the identification of $H_{13}(z)$. Therefore, the transfer function $H_{13}(z)$ can be consistently identified by

$$\hat{H}_{13}(z) = W_{13}(z) \tag{6}$$

where $W_{13}(z)$ is the Wiener filter corresponding to x_3 when estimating $x_1(t)$ from the information of x_3 up to time t:

$$\mathbb{E}(x_1(t) \mid I_3(t)) = W_{13}(z)x_3(t). \tag{7}$$

Similarly, the product of $H_{21}(z)$ and $H_{13}(z)$, namely the transfer function $H_{21}(z)H_{13}(z)$, can be consistently identified by $\bar{W}_{23}(z)$ which is the Wiener filter corresponding to x_3 when estimating $x_2(t)$ from the information of x_3 up to time t

$$\mathbb{E}(x_2(t) \mid I_3(t)) = \bar{W}_{23}(z)x_3(t). \tag{8}$$

Therefore, $H_{21}(z)$ can be consistently identified by

$$\hat{H}_{21}(z) = \frac{\bar{W}_{23}(z)}{W_{13}(z)} \tag{9}$$

The fact that node 4 does not hinder the identification of $H_{23}(z)$ in the same way it does for $H_{21}(z)$ can also be seen from the algebraic equations governing the network. Indeed, for the process $x_2(t)$ we can write

$$\begin{aligned}
x_2(t) &= n_2(t) + H_{24}(z)x_4(t) + H_{21}(z)x_1(t) && (10)\\
&= n_2(t) + H_{21}(z)(n_1(t) + H_{13}(z)x_3(t) + && (11)\\
&\qquad H_{14}(z)x_4(t)) + H_{24}(z)x_4(t) && (12)\\
&= n_2(t) + \big(H_{24} + H_{21}(z)H_{14}\big)x_4(t) + && (13)\\
&\qquad H_{21}(z)n_1(t) + H_{21}(z)H_{13}(z)x_3(t) && (14)
\end{aligned}$$

From the first equality we can see that when we want to estimate $x_2(t)$ from the information of $x_1(t)$ the error term $n_2(t) + H_{24}(z)x_4(t)$ is not independent of $x_1(t)$ and will introduce bias in our estimation of $H_{21}(z)$. From the last equality, however, we can see that when we want to estimate $x_2(t)$ from the information of $x_3(t)$ the error term $n_2(t) + \big(H_{24}(z) + H_{21}(z)H_{14}(z)\big)x_4(t) + H_{21}(z)n_1(t)$ is independent of $x_3(t)$. Therefore our estimation of $H_{21}(z)H_{13}(z)$ will be unbiased.

Now consider a little more complicated network of Figure 3. The difference between the network of Figure 3 and the network of Figure 2 is that there are two new paths, $\{3 \rightarrow 5 \rightarrow 2\}$ and $\{3 \rightarrow 7 \leftarrow 2\}$, between nodes 2 and 3 in the network of Figure 3. In this case, the method above cannot be used to identify $H_{21}(z)$ because $\bar{W}_{23}$ in (8) is not going to be a consistent estimate of $H_{21}(z)H_{13}(z)$.

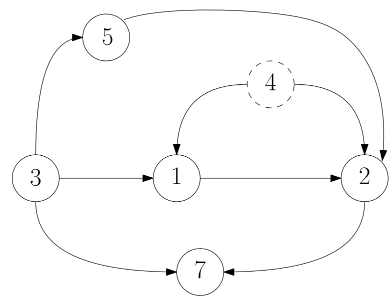

Fig. 3: Graphical representation of a more complicated network where the objective is the identification of $H_{21}(z)$ and there are multiple paths paths between nodes 2 and 3.

We will show, however, that using a set of auxiliary variables that satisfies certain conditions, it would be possible to determine an instrumental variable and consistently identify the transfer function of interest based on a prediction error method.

Theorem III.1. *Consider an acyclic network with a graphical representation G. Let G' be the mutilated graph obtained by removing the edge $i \rightarrow j$ from G. If a set $Z \cap de_G(j) = \emptyset$*

1) *d-separates w from j in G'; and*
2) *does not d-separate w from i in G',*

then, $H_{ji}(z)$ can be consistently identified by

$$\hat{H}_{ji}(z) = \frac{\bar{W}_{jw}(z)}{\bar{W}_{iw}(z)} \tag{15}$$

where $\bar{W}_{jw}(z)$ is computed from

$$\mathbb{E}(x_j(t) \mid I_w(t),\ I_Z(t)) = \sum_{r \in Z \cup \{w\}} \bar{W}_{jr}(z)x_r, \tag{16}$$

and $\bar{W}_{iw}(z)$ is computed from

$$\mathbb{E}(x_i(t) \mid I_w(t),\ I_Z(t)) = \sum_{r\in Z\cup\{w\}} \bar{W}_{ir}(z)x_r, \tag{17}$$

when the power spectral density matrix of (x_i, x_j, x_w, x_Z) is non-singular.

Proof. See the appendix. □

The condition $Z \cap \mathrm{de}_G(j) \neq 0$ in Theorem III.1 is a standard condition in graphical models to avoid creating new spurious paths. Indeed, Theorem III.1 can be seen as an extension of the instrumental variable technique [22] in the area of structural equation models which is only valid for the identification of simple proportional gains and not general transfer functions. Moreover, contrary to the the instrumental variable technique [22], Theorem III.1 can be extended to networks where the node j is involved in feedback loops.

Using Theorem III.1 we can reconsider the problem of identifying the transfer function $H_{21}(z)$ in the network of Figure 3.

Example 2. *Consider the graph G of the network of Figure 3. Suppose the objective is the identification of the transfer function $H_{21}(z)$. Figure 4 shows the mutilated graph G obtained by removing the edge $1 \to 2$ from G. We have that the set*

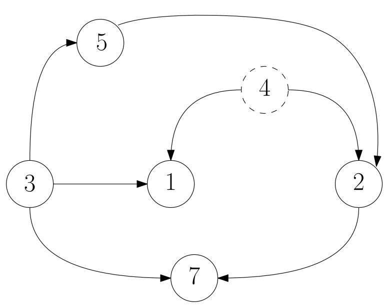

Fig. 4: The mutilated graph of Figure 3 obtained by removing the edge $1 \to 2$ where the objective is the identification of $H_{21}(z)$.

$Z = \{5\}$ d-separates nodes $w = \{3\}$ and 2 in G'. Therefore, by Theorem III.1 we have that

$$\hat{H}_{21}(z) = \frac{\bar{W}_{23}(z)}{\bar{W}_{13}(z)} \tag{18}$$

is a consistent estimate of $H_{21}(z)$ where $\bar{W}_{23}(z)$ is computed from

$$\mathbb{E}(x_2(t) \mid I_3(t),\ I_5(t)) = \bar{W}_{23}(z)x_3 + \bar{W}_{25}(z)x_5, \tag{19}$$

and $\bar{W}_{13}(z)$ is computed from

$$\mathbb{E}(x_1(t) \mid I_3(t),\ I_5(t)) = \bar{W}_{13}(z)x_3 + \bar{W}_{15}(z)x_5. \tag{20}$$

IV. APPICATION TO NETWORKS WITH FEEDBACK LOOPS

Consider a network with a graphical representation shown in Figure 5. It is assumed that the nodes 1,2, and 3 are measured while node 4 which influences both nodes 1 and 2 is not measured. Also, it is assumed that the forcing inputs n_j are mutually independent white Gaussian processes. The objective is to identify the transfer function $H_{21}(z)$.

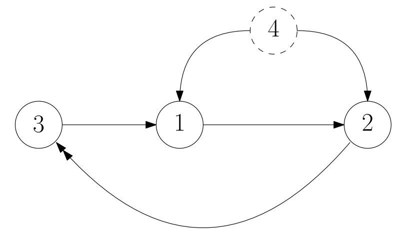

Fig. 5: Graphical representation of a network where the objective is the identification of $H_{21}(z)$ and nodes 1, 2, and 3 are in a feedback loop. Node 4 which is a confounder influencing nodes 1 and 2 is not measured.

Similar to the motivational example of Section III, the transfer function $H_{21}(z)$ cannot be identified using Theorem II.1 because the only way to block the 1-pointing path $\{2 \leftarrow 4 \to 1\}$ between nodes 1 and 2 (condition (i) of Theorem II.1) is to measure the confounder 4 which is hidden.

The difference between the network of Figure 5 and the acyclic network of Figure 2 is that the new edge $1 \to 3$ puts nodes 1, 2, and 3 in a feedback loop $\{3 \to 2 \to 1 \to 3\}$.

Because of this feedback loop, the results of Section III which were developed for acyclic networks cannot be applied and we briefly explain why. First, note that node 4 is now a confounder influencing both nodes 1 and 3 (this was not the case in the network of Figure 2). Indeed, node 4 influences node 1 through the path $\{4 \to 1\}$ and influences node 3 through the path $\{4 \to 2 \to 3\}$. One might think that it is possible to get a consistent estimate of $H_{13}(z)$ by including 2 in the predictor inputs, trying to block the path $\{4 \to 2 \to 3\}$ and adjusting for the confounder 4. This is, however, false and leads, in general, to a biased estimate of $H_{13}(z)$. Since this is a very delicate situation , we provide an example to show that including node 2 in the predictor inputs set will not necessarily result in a consistent estimate of $H_{13}(z)$.

Example 3. *Consider a network with a graphical representation as in Figure 5. Suppose that the external noises are jointly independent with power spectral density equal to identity and the transfer functions are given by $H_{21}(z) = \frac{1}{z}$, $H_{32}(z) = 0$, $H_{13}(z) = \frac{1}{z}$, $H_{14}(z) = \frac{1}{z^2}$ and $H_{24}(z) = \frac{1}{z}$.*

Then, $W_{13}(z)$ in

$$\mathbb{E}(x_1(t) \mid I_{2,3}(t-1)) = W_{12}(z)x_2(t) + W_{13}(z)x_3(t) \tag{21}$$

is a biased estimate of $H_{13}(z)$. This can also be seen by Theorem II.1. When the goal is the identification of the transfer function $H_{13}(z)$, the predictor inputs set Z needs to block all the 1-pointing paths between nodes 1 and 3; and $Z \cup \{3\}$ needs to block all the 1-pointing paths from 1 to itself. It is true that including node 2 in the predictor inputs set blocks the 1-pointing path $3 \leftarrow 2 \leftarrow 4 \to 1$ but it unblocks the 1-pointing path $1 \to 2 \leftarrow 4 \to 1$ from 1 to itself as an activated collider. Thus, the condition (ii) of Theorem II.1 is violated.

In what follows, however, we show that it is possible to use the information about a transfer function that we can consistently identify to identify another transfer function. Namely, we show that for the network of Figure 5 we can consistently identify the transfer function $H_{32}(z)$ and use the knowledge about $H_{32}(z)$ to identify $H_{13}(z)$ and use the knowledge about $H_{13}(z)$ to, eventually, identify $H_{12}(z)$.

Consider the transfer function $H_{32}(z)$ in the network of Figure 5. There is no 3-pointing path between the nodes 2 and 3 except the target edge $2 \to 3$ and the 3-pointing paths from node 3 to itself, $\{3 \to 1 \leftarrow 4 \to 2 \to 3\}$ and $\{3 \to 1 \to 2 \to 3\}$, are blocked by node 2. Therefore, by Theorem II.1 we can consistently identify $H_{32}(z)$ by

$$\hat{H}_{32}(z) = W_{32}(z) \tag{22}$$

where $W_{32}(z)$ comes from

$$\mathbb{E}(x_3(t) \mid I_2(t-1)) = W_{32}(z)x_2(t). \tag{23}$$

Now we use our knowledge about $H_{32}(z)$ to identify the transfer function $H_{13}(z)$. First, we define a new fictitious variable $e_{3|2}(t)$ as follows.

$$e_{3|2}(t) := x_3(t) - \hat{H}_{32}(z)x_2(t) \tag{24}$$

Note that $e_{3|2}(t)$ which will play the role of an instrumental variable is the residual of estimating $x_3(t)$ using the past information of $x_2(t)$ and in the case of the network of Figure 5 is equal to $n_3(t)$ (in more general and complex networks this is not the case). Therefore, we have that

$$I_{e_{3|2}}(t) \perp\!\!\!\perp I_4(t), \tag{25}$$

where $I_{e_{3|2}}(t)$ is the information of $e_{3|2}(t)$ up to time t. Thus, we can consistently identify $H_{13}(z)$ by

$$\hat{H}_{13}(z) = \tilde{W}_{13}(z) \tag{26}$$

where $\tilde{W}_{13}(z)$ comes from

$$\mathbb{E}(x_1(t) \mid I_{e_{3|2}}(t)) = \tilde{W}_{13}(z)e_{3|2}(t). \tag{27}$$

Now we can use our knowledge about $H_{13}(z)$ to identify the transfer function $H_{21}(z)$. Note that $\bar{W}_{23}(z)$ in

$$\mathbb{E}(x_2(t) \mid I_{e_{3|2}}(t)) = \bar{W}_{23}(z)e_{3|2}(t) \tag{28}$$

is a consistent estimate of $H_{21}(z)H_{13}(z)$. Therefore, $H_{21}(z)$ can be consistently identified by

$$\hat{H}_{21}(z) = \frac{\bar{W}_{23}(z)}{\tilde{W}_{13}(z)}. \tag{29}$$

Recalling the subtleties mentioned in the beginning of this section and in Example 3, the fact that we were able to identify a transfer function in a feedback loop in presence of an unmeasured confounder is noteworthy.

V. Conclusion

The article considered the problem of identifying a certain transfer function in a dynamic network when the input and the output of the transfer function are influenced by an unmeasured confounding variable. It was assumed that it was not possible to measure any variables on the directed paths from the confounding variable to either the input or the output of the transfer function of interest. Therefore, recent multi-input single-output identification results developed in the area of dynamic network identification were not applicable. A method based on determining a set of instrumental variables and a set of auxiliary variables that satisfied some sufficient conditions was presented to consistently identify the transfer function for the class of acyclic networks. Applying similar ideas to cyclic networks, it was shown that estimates of some transfer functions in a network could be used to identify some other transfer functions whose inputs and outputs are influenced by unmeasured confounding variables.

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Appendix

Proof of Theorem II.1

Proof. The sufficiency of conditions (i) and (ii) for consistent identification of $H_{ji}(z)$ follows from Theorem III.2 of [18]. The necessity of conditions (i) and (ii) for consistent identification of $H_{ji}(z)$ follows from Theorem V.1 of [18]. □

To prove Theorem III.1, we first need to provide a few lemmas.

Lemma V.1. *Consider a directed graph $G = (V,E)$. Suppose $Z \cap de_G(j) = \emptyset$. Let G' be be the mutilated graph obtained by removing the edge $i \to j$ from G. Define a new graph $\bar{G} = (\bar{V},\bar{E})$ from G such that*

$$\begin{aligned}\bar{V} &= V \cup q \\ \bar{E} &= E \cup \{k \to q \mid k \in K, q \to j\} \setminus \{k \to j \mid k \in K\},\end{aligned} \tag{30}$$

where $K := \{k \mid k \neq i \text{ and } k \to j \in E\}$. Then, the set Z d-separates w from j in G' if and only if Z d-separates w from q in $\bar{G}$.

Proof. First we prove that if Z d-separates w from j in G', then Z d-separates w from q in $\bar{G}$. Assume Z d-separates w from j in G'. By contradiction suppose there is an unblocked path $\bar{\pi}$ between w and q in $\bar{G}$. If $\bar{\pi}$ is of the form $\bar{\pi} = \{w \dots k \to q\}$, for some $k \in K$, then the path π' of the form $\pi' = \{w \dots k \to j\}$ is an unblocked path between w and j in G' which is a contradiction. Also, note that the paths of the form $\bar{\pi} = \{w \dots i \to j \leftarrow q\}$ in $\bar{G}$ are always blocked because j is a collider in $\bar{\pi}$ and $Z \cap \text{de}_G(j) = \emptyset$. If $\bar{\pi}$ is of the form $\bar{\pi} = \{w \dots \ell \leftarrow j \leftarrow q\}$, then the path π' of the form $\pi' = \{w \dots \ell \leftarrow j\}$ is an unblocked path between w from j in G' which is a contradiction. Now we prove that if Z d-separates w from q in $\bar{G}$, then, Z d-separates w from j in G'. Assume Z d-separates w from q in $\bar{G}$. By contradiction suppose there is an unblocked path π' between w and j in G'. If π' is of the form $\pi' = \{w \dots k \to j\}$, for some $k \in \text{pa}_{G'}$, then the path $\bar{\pi}$ of the form $\bar{\pi} = \{w \dots k \to q\}$ is an unblocked path between w and q in $\bar{G}$ which is a contradiction. If π' is of the form $\pi' = \{w \dots \ell \leftarrow j\}$, then the path $\bar{\pi}$ of the form $\bar{\pi} = \{w \dots \leftarrow j \leftarrow q\}$ is an unblocked path between w from q in $\bar{G}$ which is a contradiction. □

The following lemma relates the notion of estimating a certain node in a network and the notion of d-separation.

Lemma V.2. *Consider a network with a graphical representation G. Suppose $Z \cap de_G(j) = \emptyset$. If Z d-separates w and q in G, then we have*

$$\mathbb{E}(x_q(t) \mid I_w(t),\ I_Z(t)) = \mathbb{E}(x_q(t) \mid I_Z(t)), \tag{31}$$

and similarly

$$\mathbb{E}(x_w(t) \mid I_q(t),\ I_Z(t)) = \mathbb{E}(x_w(t) \mid I_Z(t)). \tag{32}$$

Proof. The results follow from lemmas 15 and 16 of [26]. □

Lemma V.2 states that when estimating a certain node from the information of a set of other nodes, the notion of d-separation in the graph translates to the notion of conditional independence.

We are now ready to present the proof of Theorem III.1.

Proof of Theorem III.1

Proof. Define a new variable $x_q(t) = x_j(t) - H_{ji}(z)x_i(t)$. Since Z d-separates w from j in G', it follows from Lemma V.1 that

$$x_q(t) \perp\!\!\!\perp I_w(t) \mid I_Z(t). \tag{33}$$

Therefore, we can write

$$\begin{aligned}
\mathbb{E}(x_j(t) \mid I_w(t),\ I_Z(t)) &= \quad (34)\\
\mathbb{E}(x_q(t) + H_{ji}(z)x_i(t) \mid I_w(t),\ I_Z(t)) &= \quad (35)\\
\mathbb{E}(x_q(t) \mid I_w(t),\ I_Z(t)) + \mathbb{E}(H_{ji}(z)x_i(t) \mid I_w(t),\ I_Z(t)) &= \quad (36)\\
\mathbb{E}(x_q(t) \mid I_Z(t)) + H_{ji}(z)\mathbb{E}(x_i(t) \mid I_w(t),\ I_Z(t)) &= \quad (37)\\
\sum_{r\in Z} \bar{W}_{qr}(z)x_r(t) + H_{ji}(z) \sum_{r\in Z\cup\{w\}} \bar{W}_{ir}(z)x_r(t) &= \quad (38)\\
H_{ji}(z)\bar{W}_{iw}(z)x_w(t) + \sum_{r\in Z} [\bar{W}_{qr}(z) + H_{ji}(z)\bar{W}_{ir}(z)]x_r(t), & \quad (39)
\end{aligned}$$

where the first equality follows from linearity of expectation and the second equality follows from (33) and Lemma V.2. Since the power spectral density matrix associated with (x_i, x_j, x_w, x_Z) is non-singular, comparing both expressions for $\mathbb{E}(x_j(t) \mid I_w(t),\ I_Z(t))$, namely (16) and the last equality of (34), we have that

$$\bar{W}_{jr} = \bar{W}_{qr}(z) + H_{ji}(z)\bar{W}_{ir}(z) \qquad \text{for } r \in Z \tag{40}$$

and

$$\bar{W}_{jw}(z) = H_{ji}(z)\bar{W}_{iw}(z), \tag{41}$$

which completes the proof. □