Arch. Rational Mech. Anal. 245 (2022) 819–861
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-022-01796-y

On the Regularity of Optimal Transports Between Degenerate Densities

Yash Jhaveri & Ovidiu Savin

Communicated by A. Figalli

Abstract

We study the most common image and informal description of the optimal transport problem for quadratic cost, also known as the second boundary value problem for the Monge–Ampère equation—what is the most efficient way to fill a hole with a given pile of sand?—by proving regularity results for optimal transports between degenerate densities. In particular, our work contains an analysis of the setting in which holes and sandpiles are represented by absolutely continuous measures concentrated on bounded convex domains whose densities behave like nonnegative powers of the distance functions to the boundaries of these domains.

1. Introduction

The optimal transport problem, formulated by Gaspard Monge in 1781, asks whether or not it is possible to find a map minimizing the total cost of moving a distribution of mass f to another g given that the cost of moving from x to y is measured by $c = c(x, y)$. Since its inception, optimal transportation has drawn together and impacted many areas of mathematics: fluid mechanics, functional analysis, geometry, general relativity, and probability, just to name a few (see, e.g., [1,15,23,25,30]). The most fundamental case is that of the quadratic cost on $\mathbb{R}^n$, when $c(x, y) = |x - y|^2$ for $x, y \in \mathbb{R}^n$. This is the model for all sufficiently smooth cost functions on all sufficiently smooth (Riemannian) geometries ([10]), and it is at the core of many applications ([30]). More precisely, it is

$$\min_T \left\{ \int_{\mathbb{R}^n} |x - T(x)|^2 \, \mathrm{d}f(x) : T_\# f = g \right\}.$$

Yash Jhaveri was supporting in part by NSF Grant DMS-1954363. Ovidiu Savin was supported in part by NSF Grants DMS-1800645 and DMS-2055617.

Under certain conditions on the nonnegative measures f and g, Brenier discovered that the optimal transport problem for the quadratic cost on $\mathbb{R}^n$ is uniquely solvable f-almost everywhere ([2]; see also [24]). Moreover, he characterized minimizing maps as gradient maps of convex potentials: $T_{min} = \nabla u$ for some convex function $u : \mathbb{R}^n \to \mathbb{R}$. When f and g are absolutely continuous with respect to Lebesgue measure, he also established that any convex potential u defining T_{min} satisfies a Monge–Ampère equation,

$$g(\nabla u) \det D^2 u = f \text{ and } \nabla u(\operatorname{spt} f) = \operatorname{spt} g,$$

in a suitable weak sense (the Brenier sense; see Lemma 3.1), where $f = f\,\mathrm{d}x$ and $g = g\,\mathrm{d}y$. (In this work, we equate absolutely continuous measures with their densities. It will either be clear from the context or explicitly stated when absolute continuity is assumed.) In turn, Brenier linked the optimal transport problem and the second boundary value problem for the Monge–Ampère equation: given two convex domains and a nonnegative function on their product, find a convex function whose gradient maps one domain onto the other with Jacobian determinant proportional to the given function.

Unfortunately, optimal transports can behave rather poorly. Indeed, Caffarelli observed that $T_{min} = \nabla u$ can be discontinuous under the seemingly ideal conditions that f and g are the characteristic functions of smooth, bounded domains of equal volume ([3]; see also [20]). In principle then, a convex potential of an optimal transport (on the support of the source measure) even between "nice" measures is no better than an arbitrary convex function. That said, in this same work, Caffarelli showed that the optimal transport must be locally Hölder continuous in $X := \operatorname{int}(\operatorname{spt}(f))$ under a geometric condition — $Y := \operatorname{int}(\operatorname{spt}(g))$ is convex — and a uniform ellipticity type condition — the Monge–Ampère measure associated to u is doubling in $\operatorname{spt}(f)$. In subsequent works, Caffarelli established the global Hölder continuity of ∇u assuming that both X and Y are convex ([4]), and the global Hölder continuity of $D^2 u$, the Hessian of u, additionally assuming X and Y are C^2 and uniformly convex and f and g are positive and Hölder continuous in $\overline{X}$ and $\overline{Y}$ respectively ([5]).

In this nondegenerate setting, Urbas also proved that $D^2 u$ is Hölder continuous up to ∂X when X and Y are uniformly convex, but under a C^3 regularity assumption on X and Y ([28]). More recently, Chen, Liu, and Wang demonstrated that these domain regularity assumptions can be weakened to $C^{1,1}$ in $n \geq 3$ dimensions and $C^{1,\alpha}$ in two dimensions ([7,8]). In two dimensions and at the same time as Caffarelli, Delanoë established the existence of globally smooth solutions to the second boundary value problem for the Monge–Ampère equation given smooth data ([9]).

In the degenerate setting of arbitrary open, bounded source and target domains, but still considering densities bounded away from zero and infinity, Figalli ([12]), Figalli–Kim ([14]), Goldman–Otto ([18]), and Goldman ([17]) showed that the closure of the discontinuity set of an optimal transport, also known as the singular set, has zero measure in $\overline{X}$.

In this paper, we consider a different degenerate setting, one in which f and g are permitted to vanish at times, e.g., continuously at the boundaries of X and

Y. This scenario encompasses a study of the most common image and informal description of the optimal transport problem:

What is the most efficient way to fill a hole with a given pile of sand?

Our first result is a global Hölder continuity regularity result for optimal transports between absolutely continuous, doubling measures (see Sect. 2 for the definition of a doubling measure in this context). This doubling assumption is different from Caffarelli's doubling assumption in that it is only on the data of the problem rather than on the data and the solution, as it is in Caffarelli's case.

Theorem 1.1. *Let X and Y be open, bounded convex sets in $\mathbb{R}^n$, and suppose that f and g are densities which define doubling measures concentrated on X and Y respectively. Let T_{min} be the optimal transport taking f to g. Then $T_{min} \in C^{\sigma}(\overline{X})$, for some $\sigma \in (0, 1)$, depending on n, the doubling constants of f and g, and the inner and outer diameters of X and Y.*

Our second result establishes (optimal) global regularity for the optimal transport in the plane when f and g are comparable to nonnegative powers of the distance functions to the boundaries of their supports, which we assume are convex:

$$f \sim d_{\partial X}^{\alpha} \text{ for some } \alpha \geq 0 \text{ and } g \sim d_{\partial Y}^{\beta} \text{ for some } \beta \geq 0.$$

In this work, $d_{\partial *}$ represents the distance function to the boundary of $*$; $d_{\partial *} > 0$ in $*$ and $d_{\partial *} = 0$ outside of $*$. Thus, we assume our sandpile and hole (turned upside down) have precise shapes at their boundaries.

Here we show that the optimal transport effectively splits along the tangential and normal directions to ∂X. Let $u' : [0, 1] \to [0, 1]$ be the optimal transport taking a density which behaves like x^{α} near 0 to another density which behaves like y^{β} near 0. Then, by the mass balance formula,

$$u'(t) \sim t^{\gamma} \text{ with } \gamma := \frac{1+\alpha}{1+\beta}.$$

In other words, informally, we find that T_{min} behaves like the identity map t moving along the boundary of X and the one dimensional transport t^{γ} moving orthogonally in from the boundary of X.

In order to precisely state our theorem and expansion, we must define three Hölder exponents, λ, μ, and ω, to formalize what we mean by $\sim$ above. We state our theorem assuming that $\alpha > 0$ and $\beta > 0$, and make a remark after to address the mild difference when either $\alpha = 0$ or $\beta = 0$. There are two cases to consider. When $\alpha \geq \beta$, let

$$\mu := \lambda \frac{1+\gamma}{2} \text{ and } \omega := \lambda, \text{ for any fixed } 0 < \lambda \leq \min\left\{\alpha \frac{2}{1+\gamma}, \frac{2}{1+\gamma}, \beta\right\}.$$

If $\alpha = \beta$, i.e., $\gamma = 1$, we additionally assume that $\lambda < 1$. On the other hand, when $\alpha < \beta$, set

$$\mu := \lambda \text{ and } \omega := \lambda \frac{1+\gamma}{2\gamma}, \text{ for any fixed } 0 < \lambda \leq \min\left\{\alpha, \frac{2\gamma}{1+\gamma}, \beta \frac{2\gamma}{1+\gamma}\right\}.$$

Theorem 1.2. *Let X and Y be open, bounded, and $C^{1,1}$ uniformly convex sets in $\mathbb{R}^2$. Suppose that α and β are two positive constants. Let $a \in C^{\mu}(\overline{X})$ and $b \in C^{\omega}(\overline{Y})$ be two positive functions. Suppose that T_{min} is the optimal transport taking $f = a d^{\alpha}_{\partial X}$ to $g = b d^{\beta}_{\partial Y}$. If $\gamma \geq 1$, then $T_{min} \in C^{1+\lambda}(\overline{X})$. On the other hand, if $\gamma < 1$, then $T_{min} \in C^{\gamma(1+\omega)}(\overline{X})$.*

At the heart of Theorem 1.2 is an expansion of a convex potential defining T_{min} at points on ∂X. In particular, up to a translation placing $0 \in \partial X$ and subtracting off the tangent plane to u at 0, we prove that

$$|u(Mx) - p_1 x_1^2 - p_2 x_2^{1+\gamma}| \leq C(|x_1|^2 + x_2^{1+\gamma})^{1+\frac{\lambda}{2}} \text{ in } \{x_2 \geq 0\} \cap M^{-1}\overline{X},$$

where M is some linear transformation and p_1, p_2, and C are three positive constants.

Remark 1.3. The case $\alpha = 0 = \beta$ is, by now, well-understood. When $\alpha = 0$ (respectively $\beta = 0$), the upper bound on λ becomes independent of any dependence on α (respectively β).

Our final main result is a Liouville theorem in the flat setting, $X = \{x_n > 0\}$ and $Y = \{y_n > 0\}$, with $a \equiv 1 \equiv b$. Here $u_n \geq 0$ in $\{x_n > 0\}$, and our Monge–Ampère equation is

$$\det D^2 u = \frac{x_n^{\alpha}}{u_n^{\beta}} \text{ in } \{x_n > 0\} \text{ and } u_n = 0 \text{ on } \{x_n = 0\}. \tag{1.1}$$

We remark that this equation is invariant under affine transformations that keep the tangential variables $x' = (x_1, \ldots, x_{n-1})$ separate from the normal variable x_n: $Ax = (A'x', a_n x_n)$. Furthermore, since all three notions of weak solution to the Monge–Ampère equation (Alexandrov, Brenier, and viscosity) are equivalent in this case, the following theorem classifies not only Brenier solutions to (1.1), but also Alexandrov and viscosity solutions to (1.1).

Theorem 1.4. *Let u be convex and such that $(\nabla u)_{\#} d^{\alpha}_{\partial\{x_n>0\}} = d^{\beta}_{\partial\{y_n>0\}}$, for two given constants $\alpha \geq 0$ and $\beta \geq 0$. Then*

$$u(x) = p_0 + p' \cdot x' + P'x' \cdot x' + p_n x_n^{1+\gamma}$$

for some $p_0 \in \mathbb{R}$, $p' \in \mathbb{R}^{n-1}$, positive definite matrix P', and constant $p_n > 0$.

This paper is organized as follows. The next section collects some facts from measure theory and convex analysis. In Sect. 3, we revisit Caffarelli's boundary regularity theory, and prove Theorem 1.1. In Sect. 4, we study the flat setting, and we prove our (Liouville) Theorem 1.4. In Sect. 5, we establish a pointwise "flat implies smooth" result. Finally, in Sect. 6, we prove Theorem 1.2.

2. Preliminaries

Throughout this work, c and C will denote positive constants that may change from line to line. It will be clear from the context, if any change occurs. Sometimes some of the quantities on which c and C depend will be explicit and denoted in parentheses or as subscripts; other times, especially when these quantities are contextually clear, these quantities will be implicit.

Let us start with a pair of definitions and an important lemma by John.

Definition 2.1. We say that a map T *pushes-forward* a measure f to another measure g, $T_{\#}f = g$, if

$$\int \varphi \circ T \, \mathrm{d}f = \int \varphi \, \mathrm{d}g \text{ for all } \varphi \text{ Borel and bounded.}$$

Definition 2.2. A nonnegative measure f is *doubling* (on bounded convex domains) if there is a constant $C \geq 1$ such that the following holds: given an open, bounded convex set S whose barycenter is contained in spt f,

$$f(S) \leq Cf(\tfrac{1}{2}S),$$

where $\frac{1}{2}S$ is the dilation of S with respect to its center of mass by $1/2$.

Definition 2.3. An *ellipsoid* is the image under a symmetric positive definite affine transformation of $B_1(0)$. In particular, let E be any symmetric positive definite matrix and $x \in \mathbb{R}^n$, the ellipsoid generated by E and centered at $x \in \mathbb{R}^n$ is

$$\mathcal{E}_{E,x} := x + E(B_1(0)).$$

Given $r > 0$, we let

$$r\mathcal{E} = r\mathcal{E}_{E,x} := \mathcal{E}_{rE,x}$$

be the dilation of $\mathcal{E}$ with respect to its center by r. (Given an ellipsoid $\mathcal{E}$, we can assume its generating matrix E can be diagonalized with a determinant 1 orthogonal matrix.)

Lemma 2.4. (John's Lemma) *Let $S \subset \mathbb{R}^n$ be a bounded convex set with nonempty interior and center of mass z. A unique ellipsoid $\mathcal{E}$ also with center of mass z exists such that*

$$\mathcal{E} \subset S \subset n^{\frac{3}{2}}\mathcal{E}.$$

With these definitions and John's lemma in hand (see, e.g., [19] for a proof), we prove that a measure that is doubling on ellipsoids is doubling.

Corollary 2.5. *Let f be a nonnegative measure. If f is doubling on ellipsoids, then f is doubling.*

Proof. Let S be an open, bounded convex set. Then, by John's lemma, S is comparable to an ellipsoid $\mathcal{E}$: $\mathcal{E} \subset S \subset n^{3/2}\mathcal{E}$, and the center of mass of $\mathcal{E}$ is the same as the center of mass of S. If $C \geq 1$ is the doubling constant for f on ellipsoids and $k \geq 1$ is chosen such that $n^{3/2}/2^k \leq 1/2$, then

$$f(S) \leq f(n^{\frac{3}{2}}\mathcal{E}) \leq Cf(\tfrac{1}{2}n^{\frac{3}{2}}\mathcal{E}) \leq \cdots \leq C^k f(\tfrac{1}{2^k}n^{\frac{3}{2}}\mathcal{E}) \leq C^k f(\tfrac{1}{2}\mathcal{E}) \leq C^k f(\tfrac{1}{2}S).$$

□

As a consequence of Lemma 2.5, we can show that measures that are comparable to the distance function to the boundary of a convex domain are doubling.

Lemma 2.6. *Let $X \subset \mathbb{R}^n$ be an open, bounded convex set. The density $f = ad_{\partial X}^{\alpha}$ defines a doubling measure on ellipsoids if $0 < \inf_X a, \sup_X a < \infty$. In particular, a constant $C \geq 1$ exists for which*

$$\int_{\mathcal{E}} f \leq C \int_{\frac{1}{2}\mathcal{E}} f$$

given any ellipsoid $\mathcal{E}$ centered in $\overline{X}$.

Proof. There are two cases to consider.

Case 1: $\mathcal{E} \subset X$. Let $d := d_{\partial X}(z)$, with z taken to be the center of $\mathcal{E}$. Up to a translation and rotation, $\mathcal{E}$, $X \subset \{x_n > 0\}$, and the origin is the closet point on ∂X to z.

First, note $d_{\partial X}(x) \leq 2d$ for all $x \in \partial\mathcal{E}$, and so, for all $x \in \mathcal{E}$. Indeed, if $x_n \leq z_n$, then there is nothing to show. If $x_n > z_n$, then

$$d_{\partial X}(x) \leq d_{\partial\{x_n>0\}}(x) = x_n = (x_n - z_n) + z_n = (z_n - x_n^*) + z_n \leq 2z_n,$$

since $x_n^*, x_n > 0$. Here $x^* \in \partial\mathcal{E}$ is the dual point to x. So

$$\int_{\mathcal{E}} f \leq 2^{\alpha}(\sup a)d^{\alpha}|\mathcal{E}|.$$

Second, for $i = 1, \ldots, n$, let r_i and $\mathbf{e}_i$ be the principle radii and directions of $\mathcal{E}$. Hence, $\triangle := \operatorname{conv}\{z \pm r_i\mathbf{e}_i : i = 1, \ldots, n\} \subset \mathcal{E}$. Also, $|\triangle|/|\mathcal{E}| \geq c(n)$. Now consider $\triangle_d := \operatorname{conv}\{z \pm \max\{r_i, d\}\mathbf{e}_i : i = 1, \ldots, n\}$, which contains $\triangle$ and is contained in X. For all $x \in \frac{1}{2}\triangle_d$ then, $\operatorname{dist}_{\partial\triangle_d}(x) \geq d/2^{\frac{3}{2}}$. In turn,

$$d_{\partial X}(x) \geq d_{\partial\triangle_d}(x) \geq \frac{d}{2^{\frac{3}{2}}} \text{ for all } x \in \tfrac{1}{2}\triangle.$$

It follows that

$$d^{\alpha}|\mathcal{E}| \leq \frac{2^{n+\frac{3\alpha}{2}}}{c(n)\inf a}(\inf a)\frac{d^{\alpha}}{2^{\frac{3\alpha}{2}}}|\tfrac{1}{2}\triangle| \leq \frac{2^{n+\frac{3\alpha}{2}}}{c(n)\inf a}\int_{\frac{1}{2}\triangle} f \leq C(n, \alpha, \inf a)\int_{\frac{1}{2}\mathcal{E}} f.$$

Finally, the above two inequalities together yield

$$\int_{\mathcal{E}} f \leq C(n, \alpha, \inf a, \sup a)\int_{\frac{1}{2}\mathcal{E}} f.$$

Case 2: $\mathcal{E} \setminus X$ is nonempty. Up to a translation, we can assume that the center of $\mathcal{E}$ is the origin. Let z be the center of mass of $(\frac{1}{2}\mathcal{E}) \cap X$, $d := d_{\partial X}(z)$, and suppose that the nearest point to z on ∂X lives on the plane $\{x \cdot \mathbf{e} = 0\}$ for some $|\mathbf{e}| = 1$.

Using that open, bounded convex sets are balanced with respect to their center of mass, $d_{\partial\{x\cdot\mathbf{e}>0\}}$ is 1-homogeneous, and arguing like we did to produce the first inequality above, we see that

$$d_{\partial X}(x) \leq C(n)d \text{ for all } x \in \mathcal{E} \cap X.$$

Now let $\mathcal{J}$ be the John ellipsoid of $(\frac{1}{2}\mathcal{E}) \cap X$, which also has center z. Notice that $\mathcal{J} \subset X$. So

$$\int_{\mathcal{E}} f \leq Cd^{\alpha}|\mathcal{E} \cap X| \leq Cd^{\alpha}2^n|(\tfrac{1}{2}\mathcal{E}) \cap X| \leq Cd^{\alpha}|\mathcal{J}| \leq C\int_{\frac{1}{2}\mathcal{J}} f \leq C\int_{\frac{1}{2}\mathcal{E}} f.$$

Here we have used the arguments of Case 1 on $\mathcal{J}$ and that $\mathcal{E} \cap X \subset 2[(\frac{1}{2}\mathcal{E}) \cap X]$, which uses that $0 \in \overline{X}$. □

Remark 2.7. By Corollary 2.5, if $0 < \inf_X a$, $\sup_X a < \infty$, then $f = ad_{\partial X}^{\alpha}$ defines a doubling measure.

We conclude this section with three lemmas. These lemmas use nothing about the optimal transport problem; they are facts about convex functions with centered sections. The first lemma's proof may be found in [4].

Lemma 2.8. (Centered Sections) *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function whose graph contains no complete lines. Then, for every $h > 0$ and $z \in \mathbb{R}^n$, there exists an affine function ℓ such that $\ell(z) = u(z) + h$ and the set*

$$S_h^c(z) := \{x \in \mathbb{R}^n : u < \ell\}$$

is centered at z.

Lemma 2.9. *Let $S_h^c(z)$ be a centered section for u at $z \in \mathbb{R}^n$. Let z_1 and z_2 be two opposite points on $\partial S_h^c(z)$, i.e., $z_2 = z + l(z - z_1)$ for some $l > 0$. Then*

$$n^{-\frac{3}{2}} \leq l \leq n^{\frac{3}{2}}.$$

Proof. Without loss of generality, we may assume that $z = 0$ and that z_1 and z_2 lie on the $\mathbf{e}_1$-axis. Thus, by John's lemma, $\mathcal{E} \subset S_h^c(0) \subset n^{3/2}\mathcal{E}$, for some ellipsoid $\mathcal{E}$, whose center of mass is also the origin. Let $e > 0$ denote the $\mathbf{e}_1$-component of $\mathcal{E} \cap \{\text{positive } \mathbf{e}_1\text{-axis}\}$. Abusing notation, we let $z_1 > 0$ and $-lz_1$ denote the $\mathbf{e}_1$-component of z_1 and z_2. In turn, $e \leq z_1$, $lz_1 \leq n^{3/2}e$, from which it follows that $n^{-3/2} \leq l \leq n^{3/2}$, as desired. □

Lemma 2.10. *Let $S_h^c(z)$ be a centered section for u at $z \in \mathbb{R}^n$. Then*

$$h \leq \max_{S_h^c(z)}(\ell - u) \leq (1 + n^{\frac{3}{2}})h.$$

Proof. Without loss of generality, $z = 0$.

Since $(\ell - u)(0) = h$, by definition, the first inequality is trivial.

Now let $z_h \in S_h^c(0)$ be a point at which $\ell - u$ achieves its maximum value; let z_1 and z_2 be the opposite points on $\partial S_h^c(0)$ for which the segment $[z_1, z_2]$ contains 0 and x_h; and let $\psi \geq 0$ be the one-dimensional concave function defined by $\ell - u$ on $[z_1, z_2]$. Notice that the lines $\ell_i \subset \mathbb{R}^2$ determined by $(z_i, 0)$ and $(0, h)$, for $i = 1, 2$, are secant lines for graph of ψ. Hence, the graph of ψ must live under the union of the subgraphs of these two lines. Assume that $z_h \in [0, z_2]$. Otherwise, swap the roles of z_1 and z_2 in what follows. Consider the triangle (in $\mathbb{R}^2$) over $[z_1, 0]$ with height h determined by the points $(z_1, 0)$, $(0, 0)$, and $(0, h)$. Its maximal self-similar enlargement over $[z_1, z_2]$, whose base has right end point $(z_2, 0)$ instead of $(0, 0)$, has height Ch with $1 + n^{-3/2} \leq C \leq 1 + n^{3/2}$ (see Lemma 2.9), from which the second inequality follows. □

3. Boundary Regularity of Maps with Convex Potentials Revisited

In this section, we prove Theorem 1.1, and list some geometric properties of convex potentials defining optimal transports between absolutely continuous doubling measures on convex domains. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a convex potential defining the optimal transport of Theorem 1.1. It will be convenient to replace u_0 with its minimal convex extension outside of X. More precisely, we consider the function

$$u(x) := \sup_{z \in X,\, p \in \partial u_0(z)} \{u_0(z) + p \cdot (x - z)\}.$$

Similarly, we let v_0 be the Legendre transform of u_0, and

$$v := \text{the minimal convex extension outside } Y \text{ of } v_0.$$

Thus, ∇v is the optimal transport taking g to f.

Given a centered section $S = S_h^c(z)$ for u, which exists at every $z \in \mathbb{R}^n$ ([4]), we define the normalized pair $(\tilde{u}, \tilde{S})$ by

$$\tilde{u}(x) := \frac{[u - \ell](A^{-1}x)}{h} \text{ and } \tilde{S} := A(S)$$

where $A(\mathcal{E}) = B_1(0)$ and $\mathcal{E}$ is the John ellipsoid of S. Moreover, we let $\tilde{f}$ and $\tilde{g}$ be the appropriate rescalings of f and g which ensure that $(\nabla \tilde{u})_{\#} \tilde{f} = \tilde{g}$. Similarly, we define $\tilde{X} := A(X)$ and $\tilde{Y} := A^{-t}(Y)$. Here and in the remainder of this work, we let

$$L^{-t} = (L^{-1})^t,$$

i.e., the transpose of the inverse of L, for any invertible transformation.

We first recall that optimal transports balance mass ([2,29]).

Lemma 3.1. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be convex and such that $(\nabla u)_\# f = g$, where f and g are two absolutely continuous measures. Then, for all Borel sets $B \subset \mathbb{R}^n$,*

$$\int_B f = \int_{\partial u(B)} g.$$

Next we prove an Alexandrov maximum principle for normalized pairs.

Lemma 3.2. *There is an increasing function $\vartheta : [0, \infty) \to [0, \infty)$, depending only on dimension and the doubling constants of f and g, such that $\lim_{d\to 0} \vartheta(d) = 0$ and*

$$|\tilde{u}(x)| \le \vartheta(d_{\partial \tilde{S}}(x)) \text{ for } x \in \tilde{S}.$$

Here $\tilde{S}$ is any normalized centered section based at any point in $\overline{X}$.

Proof. For notational simplicity, we suppress the dependence on x and set

$$d = d_{\partial \tilde{S}}(x) \text{ and } |\tilde{u}| = |\tilde{u}(x)|.$$

First, observe that

$$\partial \tilde{u}(\tfrac{1}{2}\tilde{S}) \subset B_R(0).$$

Also, considering the cone generated by $(x, \tilde{u}(x))$ and $\tilde{S}$,

$$\partial \tilde{u}(\tilde{S}) \supset K := \text{conv}(B_{r|\tilde{u}|}(0) \cup \tfrac{|\tilde{u}|}{d}\mathbf{e})$$

for some unit vector $\mathbf{e} = \mathbf{e}(x)$ and two positive constants R and r depending only on dimension. Since the slope of the plane which determines S_h^c is in Y, $0 \in \tilde{Y}$. By assumption, the center of $\tilde{S}$ is in the closure of $\tilde{X}$. In turn,

$$\int_K \tilde{g} \le \int_{\partial \tilde{u}(\tilde{S})} \tilde{g} = \int_{\tilde{S}} \tilde{f} \le C \int_{\frac{1}{2}\tilde{S}} \tilde{f} = C \int_{\partial \tilde{u}(\frac{1}{2}\tilde{S})} \tilde{g} \le C \int_{B_R(0)} \tilde{g}.$$

(Normalization affects neither the doubling property nor the doubling constants.)

Now let $B_{r_m}(t_m \mathbf{e}) \subset \text{conv}(B_r(0) \cup \frac{1}{d}\mathbf{e})$ for $m = 1, \dots, M$ be a sequence of balls chosen so that

$$\tfrac{1}{2}K_m \subset K_m \setminus K_{m-1} \text{ with } K_m := \text{conv}(B_{r_m}(t_m \mathbf{e}) \cup B_r(0)) \text{ and } K_0 := B_r(0).$$

By construction, $\{\frac{1}{2}K_m\}_{m=1}^M$ is a disjoint family, and

$$M = M(d) \to \infty \text{ as } d \to 0.$$

Hence, if we consider the collection

$$\{B_{r_m|\tilde{u}|}(t_m|\tilde{u}|\mathbf{e})\}_{m=1}^M \subset K \subset \tilde{Y},$$

and redefine $K_m := \mathrm{conv}(B_{r_m|\tilde{u}|}(t_m|\tilde{u}|\mathbf{e}) \cup B_{r|\tilde{u}|}(0))$, we see that the redefined family $\{\frac{1}{2}K_m\}_{m=1}^M$ is also disjoint. Thus,

$$M \int_{B_{r|\tilde{u}|}(0)} \tilde{g} \leq \sum_{m=1}^{M} \int_{K_m} \tilde{g} \leq C \sum_{m=1}^{M} \int_{\frac{1}{2}K_m} \tilde{g} \leq C \int_K \tilde{g}.$$

Combining the two chains of inequalities above, we find that

$$M \int_{B_{r|\tilde{u}|}(0)} \tilde{g} \leq C \int_{B_R(0)} \tilde{g} \leq C^k \int_{B_{r|\tilde{u}|}(0)} \tilde{g},$$

where

$$k = \left\lceil \frac{\log \frac{r}{R}|\tilde{u}|}{\log \frac{1}{2}} \right\rceil.$$

In turn, $M \leq C^k$. Solving for $|\tilde{u}|$ concludes the proof. □

With Lemma 3.2 in hand, Theorem 1.1 follows from Caffarelli's arguments ([4, 5]). Indeed, we first find that centered sections based at points in $\overline{X}$ have an engulfing property. To prove this property, we replace Caffarelli's modulus for normalized solutions $Cd^{1/n}$, i.e, the classical Alexandrov maximum principle modulus, with the modulus $\vartheta(d)$ from Lemma 3.2 in his proof.

Lemma 3.3. *For any pair of constants $0 \leq \underline{t} < \bar{t} \leq 1$, there exists a constant $0 < t_0 \leq 1$ such that*

$$S^c_{t_0 h}(z) \subset \bar{t} S^c_h(x)$$

for all $x \in \overline{X}$ and all $z \in \underline{t} S^c_h(x) \cap \overline{X}$. The constant t_0 depends on $\underline{t}$, $\bar{t}$, dimension, and the doubling constants of f and g.

Second, we obtain that u is strictly convex in $\overline{X}$ (cf., [5, Corollary 2.3]), and, by duality and iteration, Theorem 1.1.

Corollary 3.4. *A constant $c > 0$, depending only on dimension and the doubling constants of f and g, exists for which*

$$u(z) \geq u(x) + p \cdot (z - x) + ch$$

for all $x \in \overline{X}$, $p \in \partial u(x)$, and all $z \in \partial S^c_h(x) \cap \overline{X}$.

Proof of Theorem 1.1. Let $y, z \in \overline{Y}$, and let $h > 0$ be such that $z \in \partial S^c_h(v, y)$. By Corollary 3.4, for any $q \in \partial v(y)$,

$$v(z) \geq v(y) + q \cdot (z - y) + ch.$$

By compactness, $S^c_1(v, y) \subset B_{1/\tau}(y)$ for some $\tau \in (0, 1)$ depending only on the inner and outer diameters of X and Y. (See [4].)

Applying Lemma 3.3 iteratively, we see that

$$S^c_{t_0^j}(v, y) \subset \tfrac{1}{2^j} S^c_1(v, y) \text{ for all } j \in \mathbb{N} \text{ for some } t_0 \in (0, 1).$$

Let $k \in \mathbb{N}$ such that $t_0^{k+1} \leq h \leq t_0^k$. Then

$$z \in S^c_{t_0^k}(v, y) \text{ and } |y - z| \leq \frac{1}{\tau 2^k}.$$

In turn, for $M \geq \log t_0 / \log(1/2)$, we deduce that

$$v(z) \geq v(y) + q \cdot (z - y) + c t_0 \tau |y - z|^M.$$

Therefore, as $v|_{\overline{Y}}$ agrees with the Legendre transform of u in Y, $u \in C^{1+\sigma}(\overline{X})$ for some $\sigma \in (0, 1)$, as desired. □

Following the proof of [5, Corollary 2.2], we find a first volume product estimate.

Corollary 3.5. *Let $x \in \overline{X}$ and $S^c_h(x)$ be a centered section for u based at x. There is a constant $r > 0$, depending on dimension and the doubling constants of f and g, such that*

$$B_r(0) \subset \nabla \tilde{u}(\tilde{S}) \subset B_{1/r}(0).$$

Consequently,

$$r^n h^n \leq |S^c_h(x)||\nabla u(S^c_h(x))| \leq \frac{h^n}{r^n}.$$

Remark 3.6. An implication of Corollary 3.5 is that

$$\vartheta(d) = C d^{\frac{1}{n}},$$

for some $C > 0$ depending only on dimension and the doubling constants of f and g, is a candidate modulus for Lemma 3.2. Indeed, since $r^n \leq |\nabla \tilde{u}(\tilde{S})| \leq r^{-n}$, this follows from Alexandrov's maximum principle.

From Corollary 3.4, we also deduce that centered sections and classical sections

$$S_h(u, x, p) := \{z \in \overline{X} : u(z) < u(x) + p \cdot (z - x) + h\},$$

where $p \in \partial u(x)$, are comparable (see [7] for a proof). When u is differentiable at x, the set $\partial u(x)$ is a singleton, and we write $S_h(u, x)$ rather than $S_h(u, x, \nabla u(x))$. We often also suppress the dependence on u.

Corollary 3.7. *There are constants $c > 0$, depending only on dimension, and $C \geq 1$, depending only on dimension and the doubling constants of f and g, such that*

$$S^c_{ch}(x) \cap \overline{X} \subset S_h(x) \subset S^c_{Ch}(x) \cap \overline{X}$$

for all $x \in \overline{X}$.

In addition, we find that the image of (centered) sections of u of height h are comparable to (centered) sections of v height h.

Corollary 3.8. *There are constants $c > 0$ and $C \geq 1$, depending only on dimension and the doubling constants of f and g, such that*

$$S_{ch}(v, \nabla u(x)) \subset \nabla u(S_h(u, x)) \subset S_{Ch}(v, \nabla u(x))$$

and

$$S^c_{ch}(v, \nabla u(x)) \cap \overline{Y} \subset \nabla u(S^c_h(u, x)) \subset S^c_{Ch}(v, \nabla u(x)) \cap \overline{Y}$$

for any $x \in \overline{X}$.

Proof. Up to a translation, we assume that $x = 0$. Furthermore, up to subtracting off the tangent plane to u at 0, we assume that $u(0) = 0$ and $u \geq 0$.

We start with the second inclusion. Since v in $\overline{Y}$ agrees with the Legendre transform of u and $\nabla u(\overline{X}) = \overline{Y}$, $\nabla v(\nabla u(x)) = x$ for all $x \in \overline{X}$. In particular, $\nabla v(\nabla u(0)) = 0$. Moreover, $v(0) = 0$ and $v \geq 0$. Considering Corollary 3.7 then, it suffices to show that $v(\nabla u(x)) < Ch$ for all $x \in S^c_h(u, 0) \cap \overline{X}$, which follows from Corollary 3.5. Indeed, letting A be the John transformation that normalizes $S^c_h(u, 0)$, observe that

$$v(\nabla u(x)) = \nabla u(x) \cdot x - u(x) = h \nabla \tilde{u}(Ax) \cdot Ax - u(x) + \ell(x) < Ch,$$

as desired. (Recall Lemma 2.10.)

The first inclusion now follows from symmetry and duality. Specifically, reversing the roles of u and v in the second inclusion and applying ∇u, we see that

$$S_h(v, 0) = \nabla u(\nabla v(S_h(v, 0))) \subset \nabla u(S_{Ch}(u, 0)).$$

Replacing h by $C^{-1}h$ concludes the proof. □

Finally, again following the arguments of [5, Section 3], we obtain a uniform density estimate on centered sections as well as a second volume product estimate, this time on (centered) sections and their images when X is polynomially convex. For completeness, let us recall the definition of polynomially convex and an important remark, both taken directly from [5], which will be used in the proof of Theorem 1.2.

Definition 3.9. A domain X is polynomially convex at the origin provided $0 \in \partial X = \{x_n = \Gamma_X(x')\}$ (up to a rotation) and two constants $0 < \kappa_1, \kappa_2 < 1$ exist such that

$$\Gamma_X(x') \leq \frac{x' \cdot z'}{|z'|^2} \Gamma_X(z')$$

whenever $|x'|, |z'| \leq \delta$ and x' lies in the truncated cone

$$\begin{cases} |x'| < \kappa_1 |z'| \\ |\sin \angle(x', z')| \leq \kappa_2. \end{cases}$$

Here $\angle(x', z')$ denotes the angle between x' and z'. A domain is polynomially convex if it is polynomially convex at every point on its boundary.

Remark 3.10. In two dimensions, every convex domain is polynomially convex. In every dimension, given a polynomially convex domain, the constants κ_1 and κ_2 can be chosen uniformly for all points on its boundary depending only on the inner and outer diameters of the domain. (See [5, Section 3, Remark 2 and Lemma 3.1].)

Proposition 3.11. *Let X be polynomially convex. Then there are constants $C \geq 1$ and $c > 0$, depending on the inner and outer diameters of X and Y, dimension, and the doubling constants of f and g, such that*

$$C \frac{|S_h^c(x) \cap X|^{\frac{1}{n}}}{|S_h^c(x)|^{\frac{1}{n}}} \geq \frac{\operatorname{diam}(S_h^c(x) \cap X)}{\operatorname{diam}(S_h^c(x))} \geq c$$

for any $x \in \partial X$.

For the convenience of the reader, we sketch the proof of this proposition. First, however, we make a remark.

Remark 3.12. The polynomial convexity of X only plays a role in proving the first inequality, between normalized volume and normalized diameter, but for ellipsoids centered at points in ∂X rather than centered sections. This inequality is one about convex sets, and nothing more. In terms of the geometry of X and Y, the remainder of the proof uses only that X and Y are convex, and have boundaries that can be locally written as graphs of a Lipschitz functions.

We write $A \sim B$ if $cB \leq A \leq CB$ for some $c > 0$ and $C > 0$.

Sketch of Proof. Let $0 \in \partial X$ and consider $S = S_h^c(0)$. The first inequality is a consequence of [5, Lemma 3.2] and the comparability of S to an ellipsoid $\mathcal{E}$ centered at 0; and so, it suffices to show that the normalized diameter of S,

$$\delta := \frac{\operatorname{diam}(S \cap X)}{\operatorname{diam}(S)},$$

cannot be too small.

From Corollary 3.5, if r_i is the principle radius of $\mathcal{E}$ in the $\mathbf{e}_i$ direction, then $\nabla u(S)$ is comparable to an ellipsoid $\mathcal{E}^*$ that has principle radius h/r_i in the direction $\mathbf{e}_i$. Let $y = \nabla \ell \in Y$, where ℓ defines S. Up to a rotation, we assume that $r_1 \geq r_i$ for all $i \neq 1$. Let $x_\pm \in \partial S$ be such that $\nu_{\partial S}(x_+) = \mathbf{e}_1$ and $\nu_{\partial S}(x_-) = -\mathbf{e}_1$. In particular, $\nabla(\ell - u)$ at $x_\pm$ is parallel to $\mathbf{e}_1$. Since $|x_\pm| \sim r_1$, $x_\pm \notin X$ if $\delta > 0$ is sufficiently small. Hence, $y_\pm = \nabla u(x_\pm) = y \pm t_\pm \mathbf{e}_1 \in \partial Y$ with $t_\pm \sim h/r_1$.

Now ∂Y is locally the graph of a Lipschitz function in the direction ν. Let $t_1 > 0$ be the largest constant for which $y_1 := y - t_1 \nu \in \nabla u(S)$. Thus, $y_1 \in \partial Y$, as $\nabla u(S) \subset \overline{Y}$. Since $y_\pm \in \partial Y$ and ∂Y is Lipschitz function in the direction ν, we find the inequality $t_1 \leq Ch/r_1$. Let $t_2 > 0$ be such that $y_2 := y + t_2 \nu \in Y \cap \partial(\nabla u(S))$. By Corollary 3.5, it follows that $t_2 \leq Ch/r_1$. Thus, if $x_2 = (\nabla u)^{-1}(y_2)$, then $x_2 \in \partial S \cap X$. Moreover, by convexity and Lemma 2.10,

$$|x_2||y - y_2| = |x_2||\nabla(\ell - u)(x_2)| \geq (\ell - u)(0) = h.$$

In turn, $|x_2| \geq r_1/C$. But this contradicts $\delta > 0$ being arbitrarily small. □

Corollary 3.13. *Let X be polynomially convex. Then constants $C \geq 1$ and $c > 0$ exist, depending on the inner and outer diameters of X and Y, dimension, and the doubling constants of f and g, for which*

$$ch^n \leq |S_h^c(x) \cap X||\nabla u(S_h^c(x))| \sim |S_h(x)||\nabla u(S_h(x))| \leq Ch^n$$

for any $x \in \partial X$.

4. The Flat Case

Let u be convex and such that

$$(\nabla u)_{\#} d^{\alpha}_{\partial\{x_n>0\}} = d^{\beta}_{\partial\{y_n>0\}}, \tag{4.1}$$

for two nonnegative constants α and β. Then, by the arguments of Sect. 3 and classical regularity theory for the Monge–Ampère equation, we find that u is strictly convex in $\{x_n \geq 0\}$, $u \in C^{1,\sigma}_{\text{loc}}(\{x_n \geq 0\}) \cap C^{\infty}_{\text{loc}}(\{x_n > 0\})$, $u_n \geq 0$, and solves

$$\det D^2 u = \frac{x_n^{\alpha}}{u_n^{\beta}} \text{ in } \{x_n > 0\} \text{ and } u_n = 0 \text{ on } \{x_n = 0\}. \tag{4.2}$$

From this point forward, in this section, we assume that $\max\{\alpha, \beta\} > 0$. If $\alpha = \beta = 0$, then u must be a quadratic polynomial by the classical Liouville theorem for the Monge–Ampère equation; indeed, its even reflection over the set $\{x_n = 0\}$ solves $\det D^2 u = 1$ in $\mathbb{R}^n$.

First, we prove a Pogorelov estimate in x', which holds up to $\{x_n = 0\}$.

Proposition 4.1. *Let u be convex and satisfy $u_n \geq 0$ in $\{x_n > 0\}$ and (4.2). Let $x_0 \in \{x_n = 0\}$ and ℓ_{x_0} be the tangent plane to u at x_0. For any tangential direction $\mathbf{e}$, i.e., such that $\mathbf{e} \cdot \mathbf{e}_n = 0$,*

$$u_{\mathbf{ee}}|u - \ell_{x_0} - h| \leq C(n + \beta, \|\partial_{\mathbf{e}} u - \partial_{\mathbf{e}} \ell_{x_0}\|_{L^{\infty}(S_h(x_0))}). \tag{4.3}$$

Proof. Up to a translation and subtracting off the tangent plane to u at x_0, we assume that $x_0 = 0$ and $u(0) = \nabla u(0) = 0$. Now let $\epsilon > 0$, $\Upsilon := B \cap \{y_n > 0\}$, for some large ball B centered at the origin, and Ω be a dilation of $(\nabla u)^{-1}(\Upsilon)$ such that

$$\int_{\Omega} (x_n + \epsilon)^{\alpha} = \int_{\Upsilon} (y_n + \epsilon)^{\beta}.$$

Furthermore, let $\nabla\psi$ be the optimal transport taking $f = (x_n + \epsilon)^{\alpha} \llcorner \Omega$ to $g = (y_n + \epsilon)^{\beta} \llcorner \Upsilon$. Note that the even reflection in x_n of ψ, call it $\bar{\psi}$, is a potential whose gradient is the optimal transport taking the even reflection of f to the even reflection of g, in x_n and y_n respectively. So points along $\{x_n = 0\}$ can be turned into interior points. By [3] and symmetry, $D^2\bar{\psi}$ is locally Hölder continuous in $\bar{\Omega}$, the reflection of Ω over the x_n-axis, $\nabla\bar{\psi}(\Omega) \subset \{y_n > 0\}$, and $\bar{\psi}_n = 0$ on $\bar{\Omega} \cap \{x_n = 0\}$. In particular, we find that

$$\det D^2\psi = \frac{(x_n + \epsilon)^{\alpha}}{(\psi_n + \epsilon)^{\beta}} \text{ in } \Omega, \tag{4.4}$$

$$\psi_n = 0 \text{ on } \{x_n = 0\}, \tag{4.5}$$

and ψ is smooth in Ω and $D^2\psi$ is Hölder continuous and strictly positive definite up to $\{x_n = 0\}$. By Theorem 1.1, $\nabla\psi$ converges to ∇u locally uniformly in $\bar{\Omega} \cap \{x_n \geq 0\}$. (We can choose doubling constants for the denisites $(x_n + \epsilon)^\alpha$ and $(y_n + \epsilon)^\beta$ uniformly in ϵ.) So it suffices to prove (4.3) for ψ.

If we differentiate the log of (4.4) and (4.5) in any tangential direction $\mathbf{e}$ (with $\mathbf{e} \cdot \mathbf{e}_n = 0$), we have that

$$\psi^{ij}\partial_{ij}\psi_{\mathbf{e}} = -\beta\frac{\psi_{n\mathbf{e}}}{\psi_n + \epsilon} \text{ in } \{x_n > 0\} \tag{4.6}$$

and

$$\partial_n\psi_{\mathbf{e}} = 0 \text{ on } \{x_n = 0\}.$$

The right-hand side of this equation is Hölder continuous. Hence, $D^2\psi_{\mathbf{e}}$ is Hölder continuous across $\{x_n = 0\}$. Differentiating again in the $\mathbf{e}$ direction, we find that

$$\psi^{ij}\partial_{ij}\psi_{\mathbf{ee}} = \beta\frac{\psi_{n\mathbf{e}}^2}{(\psi_n + \epsilon)^2} - \beta\frac{\psi_{n\mathbf{ee}}}{\psi_n + \epsilon} + \psi^{ik}\psi^{jl}\psi_{ij\mathbf{e}}\psi_{kl\mathbf{e}} \tag{4.7}$$

and

$$\partial_n\psi_{\mathbf{ee}} = 0 \text{ on } \{x_n = 0\}.$$

The right-hand side of this equation is Hölder continuous, given the Hölder continuity of $D^2\psi_{\mathbf{e}}$ just observed. In conclusion, the fourth order derivatives $\bar{\psi}_{ijkl}$ are continuous across $\{x_n = 0\}$ provided that no more than two of the four indices are n, and so

$$M := \log|\bar{\psi}| + \log\bar{\psi}_{\mathbf{ee}} + \tfrac{1}{2}\bar{\psi}_{\mathbf{e}}^2$$

is $C^2(\bar{S})$ with $S = S_h(\bar{\psi}, 0)$. The ball B is chosen large enough so that $S_h(\bar{\psi}, 0) \Subset \bar{\Omega}$. Furthermore, up to subtracting off the tangent plane to $\bar{\psi}$ and h, we can assume that that $S = \{\bar{\psi} < 0\}$.

Let $z \in \{\bar{\psi} < 0\}$ be a point at which M achieves its maximum. (The point $z \notin \{\bar{\psi} = 0\}$ since e^M vanishes on $\{\bar{\psi} = 0\}$.) As $\bar{\psi}$ is even in x_n, we can assume that $z \in \{x_n \geq 0\}$. For notational simplicity, we identify $\bar{\psi}$ with ψ.

Case 1: M is achieved at $z \in \{x_n = 0\}$. By (4.5), at z,

$$\psi_{ni} = 0 \text{ for all } i < n.$$

Thus, after an orthogonal transformation in the tangential coordinates, which leaves the equation invariant, we can assume that $D^2\psi(z)$ is diagonal and $\mathbf{e} = \mathbf{e}_1$. (While the equality $\psi_{ni}|_{\{x_n=0\}} = 0$ simplifies some of the expressions below, we refrain from using it, so that Case 2 becomes evident.)

First, differentiating M twice in the $\mathbf{e}_i$ and evaluating at z, we find

$$\frac{\psi_i}{\psi} + \frac{\psi_{11i}}{\psi_{11}} + \psi_1\psi_{1i} = 0 \tag{4.8}$$

and

$$\frac{\psi_{ii}}{\psi} - \frac{\psi_i^2}{\psi^2} + \frac{\psi_{11ii}}{\psi_{11}} - \frac{\psi_{11i}^2}{\psi_{11}^2} + \psi_{1i}^2 + \psi_1\psi_{1ii} \le 0.$$

So multiplying by $\psi^{ii} = \psi_{ii}^{-1}$ and summing over i, we deduce that

$$\frac{n}{\psi} - \frac{\psi^{ii}\psi_i^2}{\psi^2} + \frac{\psi^{ii}\psi_{11ii}}{\psi_{11}} - \frac{\psi^{ii}\psi_{11i}^2}{\psi_{11}^2} + \psi_{11} + \psi_1\psi^{ii}\psi_{1ii} \le 0. \tag{4.9}$$

Second, considering (4.6), (4.7), and (4.9), we have that

$$\frac{n}{\psi} - \frac{\psi^{ii}\psi_i^2}{\psi^2} - \frac{\beta\psi_{n11}}{(\psi_n+\epsilon)\psi_{11}} + \frac{\psi^{ii}\psi^{jj}\psi_{1ij}^2}{\psi_{11}} - \frac{\psi^{ii}\psi_{11i}^2}{\psi_{11}^2} + \psi_{11} - \frac{\beta\psi_1\psi_{n1}}{\psi_n+\epsilon} \le 0.$$

Third,

$$\frac{\psi^{ii}\psi^{jj}\psi_{1ij}^2}{\psi_{11}} - \frac{\psi^{ii}\psi_{11i}^2}{\psi_{11}^2} = \frac{1}{\psi_{11}} \sum_{i=2,j=1}^{n} \frac{\psi_{1ij}^2}{\psi_{ii}\psi_{jj}},$$

and, by (4.8), we find that

$$\frac{\psi^{ii}\psi^{jj}\psi_{1ij}^2}{\psi_{11}} - \frac{\psi^{ii}\psi_{11i}^2}{\psi_{11}^2} \ge \sum_{i=2}^{n} \frac{\psi_i^2}{\psi_{ii}\psi^2}.$$

Our first three steps together yield that

$$\frac{n}{\psi} - \frac{\psi_1^2}{\psi_{11}\psi^2} - \frac{\beta}{\psi_n+\epsilon}\left(\frac{\psi_{n11}}{\psi_{11}} + \psi_1\psi_{n1}\right) + \psi_{11} \le 0.$$

Now, from (4.8) again, i.e.,

$$\frac{\psi_{11n}}{\psi_{11}} + \psi_1\psi_{n1} = -\frac{\psi_n}{\psi},$$

it follows (recall $\psi < 0$ and $\psi_n = 0$ at z) that

$$\frac{n}{\psi} - \frac{\psi_1^2}{\psi_{11}\psi^2} + \psi_{11} = \frac{1}{\psi}\left(n + \beta\frac{\psi_n}{\psi_n+\epsilon}\right) - \frac{\psi_1^2}{\psi_{11}\psi^2} + \psi_{11} \le 0.$$

Consequently,

$$|\psi|\psi_{11} \le C(n, \|\psi_1\|_{L^\infty(S)}).$$

Case 2: M is achieved at $z \in \{x_n > 0\}$. In this case, after a rotation $\mathbf{e} \mapsto \mathbf{e}_1$, a shearing transformation $x \mapsto (x_1 - s_i x_i, x_2, \dots, x_n)$ for $i = 2, \dots, n$, and then a rotation in the x_i variables for $i > 1$, we can assume $D^2\psi(z)$ is diagonal provided we replace (4.4) and (4.5) with

$$\det D^2\psi = \frac{(x \cdot {}^\backprime\xi + \epsilon)^\alpha}{(\psi_\xi + \epsilon)^\beta} \text{ in } \{x \cdot {}^\backprime\xi > 0\} \text{ with } {}^\backprime\xi \cdot \mathbf{e}_1 = 0,\ |\xi| = 1, \tag{4.10}$$

and

$$\psi_\xi := s_n\psi_1 + \psi_{\setminus\xi} = s_n\psi_1 + \nabla\psi \cdot {}^{\setminus}\xi = 0 \text{ on } \{x \cdot {}^{\setminus}\xi = 0\}. \tag{4.11}$$

Identical computations to those in Case 1 yield the same final inequality:

$$\frac{n+\beta}{\psi} - \frac{\psi_1^2}{\psi_{11}\psi^2} + \psi_{11} \le \frac{1}{\psi}\left(n + \beta\frac{\psi_\xi}{\psi_\xi + \epsilon}\right) - \frac{\psi_1^2}{\psi_{11}\psi^2} + \psi_{11} \le 0,$$

from which (4.3) follows for ψ, as desired. □

A an important consequence of Proposition 4.1 (applied to u and v the Legendre transform of u) is that u_n and x_n^γ are comparable. Recall,

$$\gamma := \frac{1+\alpha}{1+\beta}.$$

Since the rescalings

$$u_t(x) := \frac{u(D_t x)}{t} \text{ with } D_t := \operatorname{diag}(t^{\frac{1}{2}}\,\mathrm{Id}', t^{\frac{1}{1+\gamma}}) \tag{4.12}$$

leave the equation invariant, the correct geometry in which to work is defined by the cylinders

$$\mathcal{C}_r(z) := B'_{r^{1/2}}(z') \times (z_n - r^{\frac{1}{1+\gamma}}, z_n + r^{\frac{1}{1+\gamma}}) \text{ and } \mathcal{C}_r := \mathcal{C}_r(0).$$

We state our comparability estimate in this geometry.

Lemma 4.2. *Let u be convex and satisfy (4.1). Then two constants $c_0 > 0$ and $C_0 > 0$, depending on $\|\nabla_{x'}u\|_{L^\infty(\mathcal{C}_1\cap\{x_n\ge 0\})}$, $\|\nabla_{y'}v\|_{L^\infty(\nabla u(\mathcal{C}_1\cap\{x_n\ge 0\}))}$, α, β, and n, exist such that*

$$c_0 \le \frac{u_n}{x_n^\gamma} \le C_0 \text{ on } \mathcal{C}_1 \cap \{x_n \ge 0\}. \tag{4.13}$$

Proof. We claim that if

$$D^2_{x'}u(0) \ge \frac{1}{M}\,\mathrm{Id}' (\text{or} \le M\,\mathrm{Id}'),$$

then

$$u_n(0, x_n) \le C(M)x_n^\gamma \,(\text{or} \ge c(M)x_n^\gamma).$$

Before proving this claim, we use it to conclude the proof of our lemma. Up to subtracting the tangent plane to u at 0, assume that $u \ge 0$. Then, as 0 was arbitrary, our lemma follows, since Proposition 4.1 (applied to u and v the Legendre transform of u) tells us that

$$\frac{1}{M}\,\mathrm{Id}' \le D^2_{x'}u \le M\,\mathrm{Id}' \text{ in } B'_1 \times \{0\}.$$

Now we prove our claim. First, we show that $S_h(0)$ is comparable to an ellipsoid whose axes are parallel to the coordinate axes. Indeed, let

$$l_{i;h} = l_{i;h}(u, 0) := -\inf\{x_i : (x_i, 0) \in S_h(u, 0)\}$$

and

$$r_{i;h} = r_{i;h}(u, 0) := \sup\{x_i : (x_i, 0) \in S_h(u, 0)\}.$$

By Corollary 3.7,

$$cl_{i;h} \le r_{i;h} \le \frac{l_{i;h}}{c}$$

for some $c \le 1$, depending only on α, β, and n. Now define

$$w_{i;h} := r_{i;h} + l_{i;h}$$

and

$$d_h = d_h(u, 0) := \sup\{x_n : (0, x_n) \in S_h(u, 0)\},$$

and consider

$$T_{i;h} := \text{ the triangle determined by } (0, d_h) \text{ and } (\pm \min\{r_{i;h}, l_{i;h}\}, 0).$$

(The center of mass of $T_{i;h}$ is $(0, \frac{1}{3}d_h)$.) Note that $S_h(0) \cap \text{span}\{\mathbf{e}_i, \mathbf{e}_n\}$ is contained in the union of the subgraphs of the lines determined by $(0, d_h)$ and $(\pm \min\{r_{i;h}, l_{i;h}\}, 0)$ and inside the strip $[-l_{i;h}, r_{i;h}] \times [0, \infty)$. The heights of the intersections of these lines and the boundary of the strip is less than or equal to $d_h(1 + c^{-1})$, from which find a $C \ge 1$, depending only on α, β, and n, such that $CT_{i;h} \supset S_h(0) \cap \text{span}\{\mathbf{e}_i, \mathbf{e}_n\}$. In turn, if $\mathcal{E}_h$ is the John ellipsoid of $\text{conv}\{\cup_{i<n} T_{i;h}\}$, then

$$\delta \mathcal{E}_h \subset S_h(0) \subset \frac{1}{\delta}\mathcal{E}_h, \tag{4.14}$$

for some $0 < \delta < 1$ depending only on c. The ellipsoid $\mathcal{E}_h$ has axes parallel to the coordinate axes, and is our desired ellipsoid.

Since $\mathcal{E}_h$ has axes parallel to the coordinate axes the distance from its center to $\partial\{x_n > 0\}$ is the vertical height of the center, which is comparable to d_h. Then arguing as in Lemma 2.6, but using the ellipsoid just constructed above, we see that

$$\frac{1}{C} d_h^\alpha d_h w_{1;h} \cdots w_{n-1;h} \le \int_{S_h(u)} (x_n)_+^\alpha \le C d_h^\alpha d_h w_{1;h} \cdots w_{n-1;h}.$$

And, as $S_h(v)$ is dual to $S_h(u)$ (Corollary 3.8), we similarly find that

$$\frac{1}{C} \frac{h^\beta}{d_h^\beta} \frac{h}{d_h} \frac{h}{w_{1;h}} \cdots \frac{h}{w_{n-1;h}} \le \int_{S_h(v)} (y_n)_+^\beta \le C \frac{h^\beta}{d_h^\beta} \frac{h}{d_h} \frac{h}{w_{1;h}} \cdots \frac{h}{w_{n-1;h}}.$$

Moreover, by Lemma 3.1 and Corollary 3.8, again,

$$\int_{S_h(u)} (x_n)_+^\alpha \sim \int_{S_h(v)} (y_n)_+^\beta.$$

Therefore,

$$\frac{1}{C} \le \frac{d_h^{2+\alpha+\beta} w_{1;h}^2 \cdots w_{n-1;h}^2}{h^{n+\beta}} \le C. \tag{4.15}$$

By assumption, for all $i \le n-1$,

$$w_{i;h}^2 \le 2Mh.$$

In turn, $h^{1+\beta} \le C M^{n-1} d_h^{2+\alpha+\beta}$, or, equivalently,

$$h^{\frac{1}{1+\gamma}} \le C M^{\frac{n-1}{2+\alpha+\beta}} d_h.$$

Thus, $C(M) x_n^{1+\gamma} \ge u(0, x_n) \ge 0$. So our claim follows by the convexity of u; indeed,

$$C(M) 2^{1+\gamma} x_n^{1+\gamma} \ge u(0, 2x_n) \ge u(0, 2x_n) - u(0, x_n) \ge u_n(0, x_n) x_n.$$

□

Lemma 4.2 effectively gives us control over the second derivatives of u in the normal direction. And since u_n / x_n^γ is a solution to an elliptic equation, with Lemma 4.2 in hand, we can prove an oscillation decay estimate for u_n / x_n^γ. In particular,

$$\phi := \frac{u_n}{x_n^\gamma}$$

solves

$$u^{ij} \phi_{ij} + \frac{\beta x_n \delta^{in} + (1+\gamma) u_n u^{in}}{x_n u_n} \phi_i = 0.$$

Here $\delta^{ij} = 0$ if $i \ne j$ and $\delta^{ij} = 1$ if $i = j$.

Proposition 4.3. *Let u be convex and satisfy (4.1). A constant $\zeta \in (0,1)$ exists, depending only on α, β, and n, such that*

$$\operatorname{osc}_{\mathcal{C}_{1/2} \cap \{x_n \ge 0\}} \frac{u_n}{x_n^\gamma} \le (1-\zeta) \operatorname{osc}_{\mathcal{C}_1 \cap \{x_n \ge 0\}} \frac{u_n}{x_n^\gamma}. \tag{4.16}$$

(A similar and simpler version of the proof of this proposition can be found in the proof of Lemma 6.3.)

Proof. From (4.13),

$$\text{either } \frac{u_n}{x_n^\gamma}(2^{-\frac{1}{1+\gamma}}\mathbf{e}_n) < \frac{C_0+c_0}{2} \text{ or } \frac{u_n}{x_n^\gamma}(2^{-\frac{1}{1+\gamma}}\mathbf{e}_n) \geq \frac{C_0+c_0}{2}.$$

If the first inequality holds, we build a barrier that pulls u_n/x_n^γ down in $\mathcal{C}_{1/2}$. Whereas if the second inequality holds, we build a barrier that pulls u_n/x_n^γ up in $\mathcal{C}_{1/2}$.

Case 1: $\alpha < \beta$. If the first inequality holds, up to dividing by C_0, assume that $C_0 = 1$. Then let

$$\psi := (1-\epsilon)x_n^\gamma + c_1\epsilon|x'|^2 - \epsilon x_n^\kappa,$$

with $c_1 > 0$ and $1 > \kappa > \gamma$ to be chosen. (This is our upper barrier.) First, note that

$$\psi \geq x_n^\gamma \text{ on } \{x_n^\gamma \leq \tfrac{c_1}{2}|x'|^2\}.$$

Second, given any $\delta \in (0, 2^{-\frac{1}{1+\gamma}})$, applying the Harnack inequality to u_n/x_n^γ along a chain of overlapping balls, we find that

$$\sup_{\mathcal{C}_{1/2}\cap\{x_n\geq\delta\}} u_n \leq (1-c(\delta))x_n^\gamma,$$

for some small $c(\delta) > 0$. Third, let δ be small enough to ensure that

$$\Omega := \{x_n^\gamma > \tfrac{c_1}{2}|x'|^2\} \cap \{x_n < \delta\} \subset \mathcal{C}_{1/2} \cap \{x_n \geq 0\}.$$

Set

$$\epsilon := c(\delta).$$

Then

$$\psi \geq u_n \text{ on } \partial\Omega.$$

With our boundary values understood, we now turn to the interior of Ω and the equation.

Suppose that $u_n - \psi$ achieves its maximum at some point in Ω. Then, at this point, $u_n = \psi + s$ for some $s \geq 0$, and

$$0 \geq u^{ij}\partial_{ij}(u_n - \psi) \geq \frac{\alpha}{x_n} - \frac{\beta\psi_n}{\psi} - u^{ii}\psi_{ii}. \tag{4.17}$$

Using that $\nabla u_n = \nabla\psi$ at our distinguished point, we show that (4.17) is impossible provided $\delta > 0$ is sufficiently small, which forces our distinguished point to be very close to $\{x_n = 0\}$.

First, we compute an upper bound for the quotient ψ_n/ψ. Observe that

$$\frac{\psi_n}{\psi} \leq \frac{(1-\epsilon)\gamma x_n^{\gamma-1} - \epsilon\kappa x_n^{\kappa-1}}{(1-\epsilon)x_n^\gamma - \epsilon x_n^\kappa} = \frac{\gamma}{x_n}\frac{(1-\epsilon)x_n^\gamma - \epsilon\kappa\gamma^{-1}x_n^\kappa}{(1-\epsilon)x_n^\gamma - \epsilon x_n^\kappa}.$$

Now we look at the sum $u^{ii}\psi_{ii}$, which we break into two pieces: $u^{nn}\psi_{nn}$ and the remainder. First, observe that

$$\sum_{i<n} u^{ii}\psi_{ii} = 2c_1\epsilon \operatorname{tr}(D^2_{x'}u)^{-1}.$$

From Proposition 4.1 (applied to u and its Legendre transform), we have that

$$\frac{1}{C'} \leq (D^2_{x'}u)^{-1} \leq C'.$$

Therefore, choose

$$c_1 := \frac{1}{2C'(n-1)}.$$

Hence,

$$-\sum_{i<n} u^{ii}\psi_{ii} \geq -\epsilon.$$

Now considering D^2u as a block matrix, we see that

$$u^{nn} = \frac{1}{u_{nn} - \nabla_{x'}u_n(D^2_{x'}u)^{-1}\nabla_{x'}u_n} \geq \frac{1}{u_{nn}};$$

and so, provided $\delta > 0$ is small enough so that $\psi_{nn} \leq 0$,

$$-u^{nn}\psi_{nn} \geq -\frac{\gamma-1}{x_n}\frac{(1-\epsilon)x_n^\gamma - \epsilon\kappa\gamma^{-1}(\kappa-1)(\gamma-1)^{-1}x_n^\kappa}{(1-\epsilon)x_n^\gamma - \epsilon\kappa\gamma^{-1}x_n^\kappa}.$$

In turn, we have the following inequality, for the right-hand side of our equation,

$$\frac{\alpha}{x_n} - \frac{\beta\psi_n}{\psi} - u^{ii}\psi_{ii} \geq \frac{\alpha}{x_n} - \frac{\beta\gamma}{x_n}\mathrm{I} - \frac{\gamma-1}{x_n}\mathrm{II} - \epsilon$$

with

$$\mathrm{I} := \frac{(1-\epsilon)x_n^\gamma - \epsilon\kappa\gamma^{-1}x_n^\kappa}{(1-\epsilon)x_n^\gamma - \epsilon x_n^\kappa}$$

and

$$\mathrm{II} := \frac{(1-\epsilon)x_n^\gamma - \epsilon\kappa\gamma^{-1}(\kappa-1)(\gamma-1)^{-1}x_n^\kappa}{(1-\epsilon)x_n^\gamma - \epsilon\kappa\gamma^{-1}x_n^\kappa}.$$

As

$$\alpha - \beta\gamma = \gamma - 1,$$

we rearrange our lower bound as follows, splitting II into two pieces:

$$\frac{\alpha}{x_n} - \frac{\beta\gamma}{x_n}\mathrm{I} - \frac{\gamma-1}{x_n}\mathrm{II} = \frac{\alpha}{x_n}(1-\mathrm{II}) + \frac{\beta\gamma}{x_n}(\mathrm{II}-\mathrm{I}).$$

Now we estimate the two factors, above, in parentheses. Observe that

$$1 - \mathrm{II} \geq -\frac{\epsilon}{1-\epsilon} C_{0,\kappa,\gamma} x_n^{\kappa-\gamma} (1 + \bar{C}\epsilon x_n^{\kappa-\gamma})$$

with

$$C_{0,\kappa,\gamma} := \frac{\kappa(\kappa-\gamma)}{\gamma(1-\gamma)} > 0.$$

(Here and below $\bar{C} > 0$ is a large constant that may change from line to line; it depends only on κ and γ.) Similarly,

$$\mathrm{II} - \mathrm{I} \geq \frac{\epsilon}{1-\epsilon} C_{1,\kappa,\gamma} x_n^{\kappa-\gamma} (1 - \bar{C}\epsilon x_n^{\kappa-\gamma})$$

with

$$C_{1,\kappa,\gamma} := \frac{(\kappa-\gamma)(1+\kappa-\gamma)}{\gamma(1-\gamma)} > 0.$$

Finally, we conclude. From above,

$$\frac{\alpha}{x_n} - \frac{\beta\gamma}{x_n}\mathrm{I} - \frac{\gamma-1}{x_n}\mathrm{II} \geq \frac{\epsilon}{1-\epsilon}\frac{C_{0,\kappa,\gamma}}{\kappa} x_n^{\kappa-\gamma-1}((\beta\gamma+\kappa)(1-\gamma) - \bar{C}\epsilon).$$

Then, considering (4.17) and choosing $\epsilon > 0$ sufficiently small depending only on κ and γ, we find the inequality

$$0 \geq \delta^{\kappa-\gamma-1} - \bar{C},$$

which is impossible once $\delta > 0$ is sufficiently small, as desired.

Consequently,

$$\psi \geq u_n \text{ in } \mathcal{C}_{1/2} \cap \{x_n \geq 0\}.$$

In particular,

$$(1-\epsilon)x_n^\gamma \geq u_n \text{ along } \{x' = 0\}.$$

Translating the barrier ψ to any $z \in \mathcal{C}_{1/2} \cap \{x_n = 0\}$ and repeating the above argument, we find that

$$(1-\epsilon)x_n^\gamma \geq u_n \text{ in } \mathcal{C}_{1/2} \cap \{x_n \geq 0\}.$$

If, on the other hand, the second inequality holds, up to dividing by c_0 (so that $c_0 = 1$), consider

$$\psi := (1+\epsilon)x_n^\gamma - c_1\epsilon|x'|^2 + \epsilon x_n^\kappa.$$

An analogous argument proves that

$$(1+\epsilon)x_n^\gamma \leq u_n \text{ in } \mathcal{C}_{1/2} \cap \{x_n \geq 0\}.$$

In summary, (4.16) holds in Case 1.

Case 2: $\alpha = \beta$. Setting $\kappa = 2$ from the start, and following the same line of reasoning proves this case.

Case 3: $\alpha > \beta$. By duality, consider v (the Legendre transform of u). Reversing the roles of α and β, and applying the arguments of Case 1 and 2 proves this case. □

Iterating Proposition 4.3 (rescaling $\mathcal{C}_{1/2}$ to $\mathcal{C}_1$ leaves things unchanged), we find that u_n/x_n^γ is Hölder continuous at the origin. Translating this argument to other boundary points yields that u_n/x_n^γ is locally Hölder continuous up to $\{x_n = 0\}$. In particular, we have the following corollary.

Corollary 4.4. *Let u be convex and satisfy (4.1). A constant $\chi \in (0,1)$ exists, depending only on α, β, and n, such that*

$$\left[\frac{u_n}{x_n^\gamma}\right]_{C^{0,\chi}(\mathcal{C}_1\cap\{x_n\geq 0\})} \leq C(\alpha,\beta,n,\|u_n/x_n^\gamma\|_{L^\infty(\mathcal{C}_1\cap\{x_n\geq 0\})}).$$

From Corollary 4.4, we deduce Theorem 1.4.

Proof of Theorem 1.4. By Proposition 4.1, duality, and Lemma 4.2, there exists an $M > 0$ such that

$$\frac{1}{M}\,\mathrm{Id}' \leq D^2_{x'}u \leq M\,\mathrm{Id}' \text{ and } \frac{1}{M} \leq \frac{u_n}{x_n^\gamma} \leq M \text{ in } B_1' \times \{0\}.$$

After subtracting off the tangent plane to u at any point in $\mathcal{C}_1 \cap \{x_n = 0\}$ and a translation, if we can show that

$$\mathcal{C}_{1/R} \cap \{x_n \geq 0\} \subset S = \{\psi < 1\} \subset \mathcal{C}_R \cap \{x_n \geq 0\} \text{ with } \psi := u_t, \quad (4.18)$$

for some $R = R(M) > 0$, then (since $u_n = 0$ on $\{x_n = 0\}$)

$$\left\|\frac{\partial_n u_t}{x_n^\gamma}\right\|_{L^\infty(\mathcal{C}_1\cap\{x_n\geq 0\})} = \left\|\frac{\psi_n}{x_n^\gamma}\right\|_{L^\infty(\mathcal{C}_1\cap\{x_n\geq 0\})} \leq C(M).$$

(Recall that u_t is the rescaling of u defined in (4.12).) Therefore, by Corollary 4.4 and scaling,

$$\left[\frac{u_n}{x_n^\gamma}\right]_{C^{0,\chi}(\mathcal{C}_t\cap\{x_n\geq 0\})} \leq \frac{C(M)}{\min\{t^{\frac{\chi}{2}}, t^{\frac{\chi}{1+\gamma}}\}} \to 0 \text{ as } t \to \infty.$$

Hence, u_n/x_n^γ is constant in $\{x_n \geq 0\}$. In turn, $\det D^2_{x'}u$ is constant in $\{x_n > 0\}$. Thus, by Jörgens, Calabi, and Pogorelov's Liouville theorem ([6,21,26]),

$$u(x) = P(x') + px_n^{1+\gamma}$$

for some uniformly convex quadratic polynomial P, proving the theorem. Now we prove (4.18).

By the arguments of Lemma 4.2, we find that the linear map A that normalizes the pair (ψ, S) is such that

$$A = \mathrm{diag}(A', a) \text{ and } \frac{1}{C} \leq (\det A)^2 a^{\alpha+\beta} \leq C,$$

for some $C > 0$, depending only on α, β, and n. Let

$$\tilde{\psi}(x) := \psi(A^{-1}x),$$

so that

$$B_1(0) \subset \tilde{S} := A(S) \subset B_{n^{3/2}}(0) \text{ and } \nabla\tilde{\psi}(\tilde{S}) \subset B_{1/r}(0), \tag{4.19}$$

for some $r > 0$, depending only on α, β, and n. The inclusion concerning the gradient of $\tilde{\psi}$ follows from Corollary 3.5 and Corollary 3.7. So, by Proposition 4.1 and duality,

$$\frac{1}{\tilde{M}}\,\mathrm{Id}' \le D^2_{x'}\tilde{\psi}(0) \le \tilde{M}\,\mathrm{Id}'.$$

Furthermore, by Lemma 4.2,

$$\frac{1}{C(\tilde{M})} \le \frac{\tilde{\psi}_n}{x_n^\gamma}(0) \le C(\tilde{M}).$$

Here and above $\tilde{M} > 0$ and $C(\tilde{M}) > 0$ depend only on α, β, and n. It follows that

$$\frac{1}{M\tilde{M}}\,\mathrm{Id}' \le (A')^t A' \le M\tilde{M}\,\mathrm{Id}' \text{ and } \frac{1}{MC(\tilde{M})} \le a \le MC(\tilde{M}).$$

These inequalities, by [13, Lemma A.4], imply that

$$|A|, |A^{-1}| \le C(M).$$

In turn, from (4.19) we deduce (4.18), as desired. □

5. Flat Implies Smooth

In this section, we prove a pointwise "flat implies smooth" result. Before doing so, we reintroduce and introduce some notation essential to the statements and proofs of this section.

Recall,

$$\mathcal{C}_r(z) = B'_{r^{1/2}}(z') \times (z_n - r^{\frac{1}{1+\gamma}}, z_n + r^{\frac{1}{1+\gamma}}) \text{ and } \mathcal{C}_r = \mathcal{C}_r(0).$$

Let

$$\overline{\lambda} := \lambda + \frac{2}{1+\gamma} = 1 + \lambda + \frac{1-\gamma}{1+\gamma} \text{ and } \underline{\lambda} := \lambda + \frac{2\gamma}{1+\gamma} = 1 + \lambda - \frac{1-\gamma}{1+\gamma}.$$

Also, let X be an open set whose boundary in $\mathcal{C}_1$ is defined by be a nonnegative function $\Gamma_X = \Gamma_X(x')$ on $\mathbb{R}^{n-1}$:

$$X \cap \mathcal{C}_1 = \{x_n > \Gamma_X(x')\} \cap \mathcal{C}_1 \text{ and } \partial X \cap \mathcal{C}_1 = \{x_n = \Gamma_X(x')\} \cap \mathcal{C}_1.$$

Set

$$U(x) := \frac{|x'|^2}{2} + \gamma^{\frac{\beta}{1+\beta}} \frac{x_n^{1+\gamma}}{(1+\gamma)\gamma}.$$

Finally, define

$$\mathcal{C}^*_r(z) := B'_{r^{1/2}}(z') \times (z_n - r^{\frac{\gamma}{1+\gamma}}, z_n + r^{\frac{\gamma}{1+\gamma}}) \text{ and } \mathcal{C}^*_r := \mathcal{C}^*_r(0).$$

Proposition 5.1. *Assume that X is convex with $0 \in \partial X$. Let $Y \subset \{y_n > 0\}$ be an open set with $0 \in \partial Y$, and assume further that*

$$0 \leq x_n \leq \delta\epsilon|x'|^{\overline{\lambda}} \text{ on } \partial X \cap \mathcal{C}_2 \text{ and } 0 \leq y_n \leq \delta\epsilon|y'|^{\underline{\lambda}} \text{ on } \partial Y \cap \mathcal{C}^*_{1/\rho}. \tag{5.1}$$

Suppose that $u \in C^1(\overline{X} \cap \mathcal{C}_2)$ is a convex function such that

$$u(0) = 0 = |\nabla u(0)|, \tag{5.2}$$

$u_n \geq 0$, and

$$\nabla u(\partial X \cap \mathcal{C}_1) \subset \partial Y \cap \mathcal{C}^*_{1/\rho}.$$

In $X \cap \mathcal{C}_1$, assume that

$$\frac{1-\delta\epsilon|x|^\mu}{1+\delta\epsilon|\nabla u|^\omega}\frac{(x_n - 2\delta\epsilon|x'|^{\overline{\lambda}})_+^\alpha}{u_n^\beta} \leq \det D^2 u \leq \frac{1+\delta\epsilon|x|^\mu}{1-\delta\epsilon|\nabla u|^\omega}\frac{x_n^\alpha}{(u_n - 2\delta\epsilon|\nabla_{x'}u|^{\underline{\lambda}})_+^\beta}.$$

Furthermore, suppose that

$$|u - U| \leq \epsilon \text{ in } X \cap \mathcal{C}_2.$$

If $\delta, \epsilon, \rho > 0$ are sufficiently small, then

$$|u(Rx) - U| \leq C\epsilon(|x'|^2 + x_n^{1+\gamma})^{1+\frac{\lambda}{2}}$$

for some $R = \operatorname{diag}(R', r_n)$.

Our proposition will follow from an iteration of an improvement of flatness lemma. In order to state it, however, we need to define one more object:

$$U_{\tau'}(x) := U(x) + \tau' \cdot x' \text{ so that } U_0 = U.$$

Also, recall that

$$D_h := \operatorname{diag}(h^{\frac{1}{2}}\,\mathrm{Id}', h^{\frac{1}{1+\gamma}}).$$

Lemma 5.2. *Let $0 \in \partial X$ and assume that*

$$0 \leq x_n \leq \delta\epsilon \text{ on } \partial X \cap \mathcal{C}_1. \tag{5.3}$$

Furthermore, let $u \in C^1(\overline{X} \cap \mathcal{C}_1)$ be convex and such that

$$u(0) = 0, \tag{5.4}$$

$u_n \geq 0$, and

$$0 \leq u_n \leq \delta\epsilon \text{ on } \partial X \cap \mathcal{C}_1. \tag{5.5}$$

Suppose that

$$\frac{1-\delta\epsilon}{1+\delta\epsilon}\frac{(x_n - \delta\epsilon)_+^\alpha}{u_n^\beta} \leq \det D^2 u \leq \frac{1+\delta\epsilon}{1-\delta\epsilon}\frac{x_n^\alpha}{(u_n - \delta\epsilon)_+^\beta} \text{ in } X \cap \mathcal{C}_1.$$

For any $0 < \lambda < 1$*, there exist constants* $\delta_0 > 0$*,* $\epsilon_0 > 0$*, and* $h_0 > 0$ *such that the following holds: if*

$$|u - U_{\tau'}| \le \epsilon \text{ in } X \cap \mathcal{C}_1,$$

then

$$|\tilde{u} - U_{\tilde{\tau}'}| \le h_0^{\frac{\lambda}{2}} \epsilon \text{ in } \tilde{X} \cap \mathcal{C}_2$$

provided $0 < \delta \le \delta_0$ *and* $0 < \epsilon \le \epsilon_0$*. Here*

$$\tilde{u}(x) := \frac{u(QD_{h_0}x)}{h_0},\ \tilde{\tau}' := \frac{(Q'D'_{h_0})^t(\tau' + q')}{h_0},\ \text{and } \tilde{X} := (QD_{h_0})^{-1}X$$

for some Q *and* q' *such that*

$$Q = \operatorname{diag}(Q', q_n) \text{ and } |Q - \mathrm{Id}|, |q'| \le C_0\epsilon$$

and

$$q_n^{\alpha+\beta}(\det Q)^2 = 1. \tag{5.6}$$

To prove our improvement of flatness lemma, we approximate $u - U_{\tau'}$ by a solution to the Grushin type equation with singular drift $Lw = 0$ where L is defined by

$$Lw := \gamma^{\frac{\beta}{1+\beta}} x_n^{\gamma-1} \Delta_{x'} w + w_{nn} + \beta\gamma \frac{w_n}{x_n},$$

which, with the Neumann condition $w_n = 0$ on $\{x_n = 0\}$, has a rather nice regularity theory (see [11]).

Proof. Let

$$u_\epsilon := \frac{u - U_{\tau'}}{\epsilon}.$$

Step 1: We show that u_ϵ is well-approximated near the origin by a solution w to the linearized equation

$$\begin{cases} Lw = 0 & \text{in } \mathcal{C}_{1/4} \cap \{x_n > 0\} \\ w_n = 0 & \text{on } \mathcal{C}_{1/4} \cap \{x_n = 0\} \end{cases} \tag{5.7}$$

in the viscosity sense. In other words, for any $\eta > 0$, a solution w to (5.7) in $\mathcal{C}_{1/4} \cap \{x_n > 0\}$ exists such that

$$|u_\epsilon - w| \le \eta \text{ in } X \cap \mathcal{C}_{1/4},$$

provided that $\delta_0, \epsilon_0 > 0$ are sufficiently small, depending on η, ρ, n, α, and β.

Step 1.1: First, we derive the linearized equation.

To this end, by convexity, observe that if $x + t\mathbf{e} \in X \cap \mathcal{C}_1$, then

$$(\nabla u(x) - \nabla U_{\tau'}(x)) \cdot t\mathbf{e} \le 2\epsilon + (\nabla U_{\tau'}(x + t\mathbf{e}) - \nabla U_{\tau'}(x)) \cdot t\mathbf{e};$$

in particular,

$$|u_n(x) - U_n(x)| \le 2\epsilon^{\frac{1}{2}} + C_\gamma \max\{\epsilon^{\frac{\gamma}{2}}, \epsilon^{\frac{1}{2}}\}.$$

Consequently, $u_n \to \partial_n U_{\tau'} = U_n$ uniformly in compact subsets of $X \cap \mathcal{C}_1$ as $\epsilon \to 0$. Moreover,

$$\det D^2 u - \det D^2 U_{\tau'} = \operatorname{tr}(A_\epsilon D^2(u - U_{\tau'})),$$

with

$$A_\epsilon := \int_0^1 \operatorname{cof}(D^2 U + t(D^2 u - D^2 U))\,\mathrm{d}t.$$

Since $\det^{1/n}$ is concave on symmetric, positive semi-definite $n \times n$ matrices,

$$(\det A_\epsilon)^{1/n} \ge \int_0^1 (t(\det D^2 u)^{1/n} + (1-t)(\det D^2 U)^{1/n})\,\mathrm{d}t.$$

Therefore, $(\det A_\epsilon)^{1/n}$ is strictly positive and bounded on compact subsets of $X \cap \mathcal{C}_1$. Furthermore,

$$\begin{aligned}\frac{x_n^\alpha}{(u_n - \delta\epsilon)_+^\beta} - \frac{x_n^\alpha}{U_n^\beta} &= \frac{x_n^\alpha}{u_n^\beta} - \frac{x_n^\alpha}{U_n^\beta} + \frac{x_n^\alpha}{(u_n - \delta\epsilon)_+^\beta} - \frac{x_n^\alpha}{u_n^\beta} \\ &= -x_n^\alpha \int_0^1 \beta \frac{u_n - U_n}{(U_n + t(u_n - U_n))^{1+\beta}}\,\mathrm{d}t + \frac{x_n^\alpha}{(u_n - \delta\epsilon)_+^\beta} - \frac{x_n^\alpha}{u_n^\beta}.\end{aligned}$$

Thus,

$$\frac{1+\delta\epsilon}{1-\delta\epsilon}\frac{x_n^\alpha}{(u_n - \delta\epsilon)_+^\beta} - \frac{x_n^\alpha}{U_n^\beta} \le b_\epsilon^+ \partial_n(u - U_{\tau'}) + c_\delta^+ \epsilon$$

with

$$b_\epsilon^+ \to -\beta \frac{x_n^\alpha}{U_n^{1+\beta}} \text{ as } \epsilon \to 0 \text{ and } c_\delta^+ \to 0 \text{ as } \delta \to 0$$

locally uniformly in $X \cap \mathcal{C}_1$. Similarly,

$$\frac{1-\delta\epsilon}{1+\delta\epsilon}\frac{(x_n - \delta\epsilon)_+^\alpha}{u_n^\beta} - \frac{x_n^\alpha}{U_n^\beta} \ge b_\epsilon^- \partial_n(u - U_{\tau'}) + c_\delta^- \epsilon$$

with

$$b_\epsilon^- \to -\beta \frac{x_n^\alpha}{U_n^{1+\beta}} \text{ and } c_\delta^- \to 0$$

uniformly in compact subsets of $X \cap \mathcal{C}_1$ as $\epsilon, \delta \to 0$, respectively. In turn, by the ABP estimate and Schauder theory, $D^2 u \to D^2 U_{\tau'} = D^2 U$ locally uniformly in $X \cap \mathcal{C}_1$ as $\epsilon \to 0$. Hence,

$$A_\epsilon \to \operatorname{cof} D^2 U = \operatorname{diag}(\gamma^{\frac{\beta}{1+\beta}} x_n^{\gamma-1}\,\mathrm{Id}', 1)$$

as $\epsilon \to 0$ uniformly in compact subsets of $X \cap \mathcal{C}_1$.

Step 1.2: Second, we show that u_ϵ is uniformly Hölder continuous in an appropriate sense up to $\mathcal{C}_{1/4} \cap \{x_n = 0\}$ as $\delta, \epsilon \to 0$.

To start, we claim that two small constants $c_0 > 0$ and $\delta_0 > 0$ exist such that the following holds: for all $\delta \le \delta_0$, if

$$\operatorname{osc}_{X \cap \mathcal{C}_1} u_\epsilon \le 2,$$

then

$$\operatorname{osc}_{X \cap \mathcal{C}_{1/2}} u_\epsilon \le 2(1 - c_0).$$

We prove this claim with a barrier argument. For every $z \in \partial X \cap \mathcal{C}_{1/2}$, define

$$\phi_z(x) := -1 + c_2\left(1 + 2C_1\gamma^{\frac{\beta}{1+\beta}} \frac{x_n^{1+\gamma}}{(1+\gamma)\gamma} + x_n - C_1\frac{|x' - z'|^2}{2}\right)$$

and

$$\phi^z(x) := 1 - c_2\left(1 + 2C_1\gamma^{\frac{\beta}{1+\beta}} \frac{(x_n - \delta\epsilon)_+^{1+\gamma}}{(1+\gamma)\gamma} - C_1\frac{|x' - z'|^2}{2}\right),$$

for $C_1 \gg 1$ and $c_2 \ll 1$ with $c_2 C_1 \ll 1$ to be chosen (uniformly in z). Also, let

$$F^+_{\delta,\epsilon}(D^2\psi, \nabla\psi, x) := (1 - \delta\epsilon)(\psi_n - \delta\epsilon)_+^\beta \det D^2\psi - (1 + \delta\epsilon)x_n^\alpha$$

and

$$F^-_{\delta,\epsilon}(D^2\psi, \nabla\psi, x) := (1 + \delta\epsilon)\psi_n^\beta \det D^2\psi - (1 - \delta\epsilon)(x_n - \delta\epsilon)_+^\alpha.$$

Finally, define

$$w_z := U_{\tau'} + \epsilon\phi_z$$

and

$$w^z := \frac{|x'|^2}{2} + \gamma^{\frac{\beta}{1+\beta}} \frac{(x_n - \delta\epsilon)_+^{1+\gamma}}{(1+\gamma)\gamma} + \epsilon\phi^z + \tau' \cdot x'.$$

First, we show that if $\delta > 0$ is small enough (and $\epsilon < 1$), then

$$F^+_{\delta,\epsilon}(D^2 w_z, \nabla w_z, x) \ge 0 \ge F^+_{\delta,\epsilon}(D^2 u, \nabla u, x) \text{ in } X \cap \mathcal{C}_1$$

and

$$F^-_{\delta,\epsilon}(D^2 w^z, \nabla w^z, x) \le 0 \le F^-_{\delta,\epsilon}(D^2 u, \nabla u, x) \text{ in } X \cap \mathcal{C}_1.$$

Indeed, if $\delta \le c_2$ (recalling that $c_2 C_1 \ll 1$ and $C_1 \gg 1$), then

$$F^+_{\delta,\epsilon}(D^2 w_z, \nabla w_z, x) \ge [(1 - \delta\epsilon)(1 - c_2 C_1\epsilon)(1 + c_2 2C_1\epsilon) - (1 + \delta\epsilon)]x_n^\alpha \ge 0$$

and

$$F^-_{\delta,\epsilon}(D^2 w^z, \nabla w^z, x) \ \le [(1 + \delta\epsilon)(1 + c_2 C_1\epsilon)(1 - c_2 2C_1\epsilon)$$

$$-(1-\delta\epsilon)](x_n-\delta\epsilon)_+^{\alpha}\le 0.$$

Second, we address the boundary data. Note that

$$\text{either } u_\epsilon(2^{-\frac{1}{1+\gamma}}\mathbf{e}_n)>0 \text{ or } u_\epsilon(2^{-\frac{1}{1+\gamma}}\mathbf{e}_n)\le 0.$$

In the first case, we prove that

$$u\ge w_z \text{ on } X\cap\partial\mathcal{C}_{1/8}(z) \text{ and } \partial_n w_z\ge\partial_n u \text{ on } \partial X\cap\mathcal{C}_{1/8}(z).$$

While, in the second case, we prove that

$$u\le w^z \text{ on } X\cap\partial\mathcal{C}_{1/8}(z) \text{ and } \partial_n u\ge\partial_n w^z \text{ on } \partial X\cap\mathcal{C}_{1/8}(z).$$

In the first case, fix some small distance $\bar{d}>\delta\epsilon$, depending only on γ. If C_1 is sufficiently large, depending only on γ, then $\phi_z\le -1$ on $\partial\mathcal{C}_{1/8}(z)\cap\{x_n\le\bar{d}\}$. Hence, $w_z\le u$ here. By the Harnack inequality, choosing $c_2>0$ small, we can ensure that $w_z\le u$ on the remainder of $X\cap\partial\mathcal{C}_{1/8}(z)$. Also, by (5.5), once again if $\delta\le c_2$,

$$\partial_n w_z=U_n+\epsilon c_2 2C_1U_n+\epsilon c_2\ge\epsilon c_2\ge\partial_n u \text{ on } \partial X\cap\mathcal{C}_{1/8}(z).$$

In the second case, a similar Harnack inequality argument, yields the required inequality along $X\cap\partial\mathcal{C}_{1/8}(z)$. Also, by (5.3),

$$\partial_n w^z=\gamma^{-\frac{1}{1+\beta}}(1-\epsilon c_2 2C_1)(x_n-\delta\epsilon)_+^{\gamma}=0\le\partial_n u \text{ on } \partial X\cap\mathcal{C}_{1/8}(z).$$

By the maximum principle then, one of the two inequalities

$$w_z\le u\le w^z \text{ in } X\cap\mathcal{C}_{1/8}(z)$$

holds for all $z\in\partial X\cap\mathcal{C}_{1/2}$, from which the claim follows with $c_0=c_2/2$.

Iterating the claim, we find that if $\epsilon<1/2^{k-1}$ and $\delta_0>0$ is small, then

$$\operatorname{osc}_{X\cap\mathcal{C}_{2^{-k}}}u_\epsilon\le 2(1-c_0)^k.$$

Thus, after translating the above argument to any point $x_0\in\partial X\cap\mathcal{C}_{1/4}$, we see that u_ϵ converges uniformly in $\mathcal{C}_{1/4}\cap\{x_n\ge 0\}$, as ϵ and δ tend to zero, to some function w that solves

$$Lw=\gamma^{\frac{\beta}{1+\beta}}x_n^{\gamma-1}\Delta_{x'}w+w_{nn}+\beta\gamma\frac{w_n}{x_n}=0 \text{ in } \{x_n>0\}\cap\mathcal{C}_{1/4}$$

in the viscosity sense (and so, classically by elliptic theory).

Step 1.3: Third, we show that the Neumann condition $w_n=0$ is satisfied in the viscosity sense as defined in [11, Definition 7.1] on $\mathcal{C}_{1/4}\cap\{x_n=0\}$.

If $\beta\gamma\ge 1$, then $w_n=0$ on $\{x_n=0\}$ in the viscosity sense since $|w|\le 1$. This bound is a consequence of Step 1.2.

When $\beta\gamma<1$, we show that w can neither be touched from above at any point on $\mathcal{C}_{1/4}\cap\{x_n=0\}$ by any test function of the form

$$\frac{A}{2}|x'-z'|^2+B+2px_n^{1-\beta\gamma}$$

where $z' \in \mathbb{R}^{n-1}$, A, $B \in \mathbb{R}$, and

$$p < 0$$

(making w a viscosity subsolution) nor be touched from below at any point on $\mathcal{C}_{1/4} \cap \{x_n = 0\}$ by any test function of the form

$$\frac{A}{2}|x' - z'|^2 + B + 2px_n^{1-\beta\gamma}$$

where $z' \in \mathbb{R}^{n-1}$, A, $B \in \mathbb{R}$, and

$$p > 0$$

(making w a viscosity supersolution).

Suppose, to the contrary, that w can be touched from below at some $x_0 = (x_0', 0) \in \mathcal{C}_{1/4} \cap \{x_n = 0\}$ by

$$\frac{A}{2}|x' - z'|^2 + B + 2px_n^{1-\beta\gamma}$$

for some $z' \in \mathbb{R}^{n-1}$, A, $B \in \mathbb{R}$, and $p > 0$. Since $\beta\gamma < 1$, we can touch w at x_0 from below strictly by

$$\phi(x) := \frac{A}{2}|x' - z'|^2 + B + \frac{\gamma^{\frac{\beta}{1+\beta}}}{(1+\gamma)\gamma} C x_n^{1+\gamma} + px_n$$

with any $C \in \mathbb{R}$. Since $u_\epsilon \to w$ uniformly as $\epsilon, \delta \to 0$,

$$\Phi := U_{\tau'} + \epsilon(\phi + c_{\epsilon,\delta})$$

touches u from below strictly at some $x_\epsilon \in X$. Arguing as in Step 1.2, we find that

$$F^+_{\delta,\epsilon}(D^2\Phi(x_\epsilon), \nabla\Phi(x_\epsilon), x_\epsilon) > 0,$$

provided that $0 < \delta, \epsilon \ll 1$ and $C \gg 1$ (since $p > 0$), but then

$$0 \geq F^+_{\delta,\epsilon}(D^2u(x_\epsilon), \nabla u(x_\epsilon), x_\epsilon) \geq F^+_{\delta,\epsilon}(D^2\Phi(x_\epsilon), \nabla\Phi(x_\epsilon), x_\epsilon) > 0,$$

which is impossible. (The first inequality is an assumption on u, and the middle inequality holds since Φ touches u from below.) On the other hand, suppose, to the contrary, that w can be touched from above at some $x_0 = (x_0', 0) \in \mathcal{C}_{1/4} \cap \{x_n = 0\}$ by

$$\frac{A}{2}|x' - z'|^2 + B + 2px_n^{1-\beta\gamma}$$

for some $z' \in \mathbb{R}^{n-1}$, A, $B \in \mathbb{R}$, and $p < 0$. Since $\beta\gamma < 1$, we can touch w at x_0 from above strictly by

$$\phi(x) := \frac{A}{2}|x' - z'|^2 + B + \frac{\gamma^{\frac{\beta}{1+\beta}}}{(1+\gamma)\gamma} C (x_n - \delta\epsilon)_+^{1+\gamma} + px_n$$

with any $C \in \mathbb{R}$. Since $u_\epsilon \to w$ uniformly as $\delta, \epsilon \to 0$,

$$\Phi := \frac{|x'|^2}{2} + \gamma^{\frac{\beta}{1+\beta}} \frac{(x_n - \delta\epsilon)_+^{1+\gamma}}{(1+\gamma)\gamma} + \tau' \cdot x' + \epsilon(\phi + c_{\epsilon,\delta})$$

touches u from below strictly at some $x_\epsilon \in X$. Arguing as in Step 1.2, we find that

$$F^-_{\delta,\epsilon}(D^2\Phi(x_\epsilon), \nabla\Phi(x_\epsilon), x_\epsilon) < 0$$

provided $0 < \delta, \epsilon \ll 1$ and $C \ll -1$ (since $p < 0$). But again this inequality is impossible.

Step 2: Now we find the transformation Q and prove (5.6).

An application of the arguments of [11, Section 7.1] yields that $w \in C^{1+\gamma}_{\text{loc}}(\mathcal{C}_{1/4} \cap \{x_n \geq 0\})$. In particular, $w_n = 0$ is satisfied in the classical sense. Moreover, $D^k_{x'} w \in C^{1+\gamma}_{\text{loc}}(\mathcal{C}_{1/4} \cap \{x_n \geq 0\})$. Also, since w is the limit of a sequence of functions that vanish at the origin (by (5.4)),

$$w(0) = 0.$$

In turn, by Taylor's theorem and (5.7),

$$w(x) = p' \cdot x' + \frac{1}{2} P' x' \cdot x' + C_\gamma p_n x_n^{1+\gamma} + O(|x'|^3 + x_n^{2+2\gamma} + |x'| x_n^{1+\gamma})$$

where

$$|p'|, |P'|, |p_n| \leq C;$$

in particular,

$$p_n = -\frac{\operatorname{tr} P'}{1+\beta} \text{ and } C_\gamma = \frac{\gamma^{\frac{\beta}{1+\beta}}}{(1+\gamma)\gamma}.$$

It follows that, if $0 < \delta, \epsilon \ll 1$,

$$\left| u - U_\tau - \epsilon\left(p' \cdot x' + \frac{1}{2} P' x' \cdot x' + C_\gamma p_n x_n^{1+\gamma} \right) \right| \leq \epsilon\eta + \epsilon C(|x'|^3 + x_n^{2+2\gamma} + |x'| x_n^{1+\gamma}).$$

So

$$|u(\tilde{Q}'x', \tilde{q}_n x_n) - U(x) - (\tau' + q') \cdot \tilde{Q}'x'| \leq \eta\epsilon + Ch^{\frac{3}{2}}\epsilon + C\epsilon^2 \text{ in } \tilde{Q}^{-1}(X \cap \mathcal{C}_{4h})$$

with

$$\tilde{Q} = \operatorname{diag}(\tilde{Q}', \tilde{q}_n) := ((\mathrm{Id}' + \epsilon P')^{-\frac{1}{2}}, (1 + \epsilon p_n)^{-\frac{1}{1+\gamma}}) \text{ and } q' := \epsilon p'.$$

In turn, if $\epsilon, \eta \leq h^{3/2}$,

$$|u(\tilde{Q}'x', \tilde{q}_n x_n) - (\tau' + q') \cdot \tilde{Q}'x' - U(x)| \leq Ch^{\frac{3}{2}}\epsilon \text{ in } (\tilde{Q}^{-1}X) \cap \mathcal{C}_{2h}.$$

Now notice that

$$\frac{2+\alpha+\beta}{1+\gamma}p_n = -\frac{2+\alpha+\beta}{1+\gamma}\frac{\operatorname{tr} P'}{1+\beta} = -\operatorname{tr} P',$$

and so

$$\begin{aligned}\tilde{q}_n^{\alpha+\beta}(\det \tilde{Q})^2 &= (1+\epsilon p_n)^{-\frac{2+\alpha+\beta}{1+\gamma}}(\det(\mathrm{Id}' + \epsilon P'))^{-1} \\ &= \left(1-\epsilon\tfrac{2+\alpha+\beta}{1+\gamma}p_n + O(\epsilon^2)\right)\left(1-\epsilon\operatorname{tr} P' + O(\epsilon^2)\right) = 1 + O(\epsilon^2).\end{aligned}$$

Thus, we can find a Q that satisfies (5.6) and is ϵ^2 close to $\tilde{Q}$, i.e.,

$$|\tilde{Q} - Q| \le \tilde{C}\epsilon^2.$$

It follows that

$$|u(Q'x', q_n x_n) - (\tau' + q')\cdot Q'x' - U(x)| \le Ch^{\frac{3}{2}}\epsilon \text{ in } (Q^{-1}X)\cap \mathcal{C}_{2h},$$

since $\epsilon \le h^{3/2}$. Therefore, taking $h > 0$ sufficiently small (depending on $\lambda < 1$), we find that

$$\left|\frac{1}{h}u(QD_hx) - \frac{1}{h}(\tau' + q')\cdot Q'D_h'x' - U(x)\right| \le \epsilon h^{\frac{\lambda}{2}} \text{ in } ((QD_h)^{-1}X)\cap \mathcal{C}_2,$$

as desired. □

With Lemma 5.2 in hand, we prove our proposition.

Proof of Proposition 5.1. First, we claim, by induction, that a sequence of matrices $R_k := \operatorname{diag}(R_k', r_{k,n})$ and vectors $\tau_k' \in \mathbb{R}^{n-1}$ exit such that the rescalings of u at height $h_k = h_0^k$,

$$u_k(x) := \frac{u(R_kD_kx)}{h_k} \text{ for } x \in X_k := (R_kD_k)^{-1}X$$

with

$$D_k := \operatorname{diag}(h_k^{\frac{1}{2}}\,\mathrm{Id}', h_k^{\frac{1}{1+\gamma}})$$

satisfy

$$|u_k - U_{\tau_k'}| \le \epsilon_k := \epsilon h_k^{\frac{\lambda}{2}} \text{ in } X_k \cap \mathcal{C}_2 \tag{5.8}$$

provided

$$\delta = c\delta_0 \text{ and } \epsilon = \epsilon_0$$

for some small $c > 0$. Moreover,

$$r_{k,n}^{\alpha+\beta}(\det R_k)^2 = 1 \text{ and } |R_k - R_{k-1}| \le C\epsilon_{k-1}. \tag{5.9}$$

The base case $k = 0$ holds by assumption, with $R_0 = \mathrm{Id}$ and $\tau_0' = 0$.

Now suppose the claim holds for some $k \in \mathbb{N}$. Note that the second inequality in (5.9) implies that

$$|R_k - \mathrm{Id}\,| \le C\epsilon.$$

From (5.1), provided that $c > 0$ is sufficiently small, we see that

$$0 \le x_n \le |r_{k,n}^{-1}||R_k'|^{\lambda+\frac{2}{1+\gamma}}\delta\epsilon_k|x'|^{\lambda+\frac{2}{1+\gamma}} \le \delta_0\epsilon_k \text{ on } \partial X_k \cap \mathcal{C}_2.$$

Furthermore, since the segment between 0 and $\epsilon_k^{1/2}(\tau_k'|\tau_k'|^{-1} + \delta_0\epsilon_k \mathbf{e}_n)$ lives inside $X_k \cap \mathcal{C}_2$ (by the above inequality on the height of $\partial X_k \cap \mathcal{C}_2$ and the convexity of X), the function $w_k = u_k + \tau_k' \cdot x'$ is convex, and the second equality in (5.2), we deduce that

$$|\tau_k'| \le 2\epsilon_k^{\frac{1}{2}} + C_\gamma \max\{\epsilon_k^{\frac{\gamma}{2}}, \epsilon_k^{\frac{1}{2}}\}$$

(cf., the beginning of Step 1.1 in the proof of Lemma 5.2). In particular, this shows that $\tau_k' \to 0$ as $k \to \infty$. A similar argument, but also using that the family of slopes τ_k' is uniformly bounded, yields the inclusion

$$\nabla u_k(X_k \cap \mathcal{C}_1) \subset Y_k \cap \mathcal{C}_{1/\rho}^* \text{ with } Y_k := (R_k D_k)^t h_k^{-1} Y.$$

By construction, the boundary of X_k maps to the boundary of Y_k. From (5.1), we also see that

$$0 \le y_n \le |r_{k,n}||(R_k')^{-1}|^{\lambda+\frac{2\gamma}{1+\gamma}}\delta\epsilon_k|y'|^{\lambda+\frac{2\gamma}{1+\gamma}} \le \delta_0\epsilon_k \text{ on } \partial Y_k \cap \mathcal{C}_{1/\rho}^*.$$

In turn, in $X_k \cap \mathcal{C}_1$, $u_n \ge 0$ and, using (5.8),

$$\frac{1-\delta_0\epsilon_k}{1+\delta_0\epsilon_k}\frac{(x_n - \delta_0\epsilon_k)_+^\alpha}{(\partial_n u_k)^\beta} \le \det D^2 u_k \le \frac{1+\delta_0\epsilon_k}{1-\delta_0\epsilon_k}\frac{x_n^\alpha}{(\partial_n u_k - \delta_0\epsilon_k)_+^\beta},$$

taking $c > 0$ smaller if needed depending on $\rho > 0$. Therefore, by Lemma 5.2,

$$|\tilde{u}_k - U_{\tilde{\tau}_k'}| \le \epsilon_k h_0^{\frac{\lambda}{2}} = \epsilon_{k+1} \text{ in } \tilde{X}_k \cap \mathcal{C}_2$$

where

$$\tilde{u}_k(x) := \frac{u_k(Q D_{h_0} x)}{h_0} \text{ and } \tilde{X}_k := (Q D_{h_0})^{-1} X_k$$

for

$$Q = \mathrm{diag}(Q', q_n) \text{ with } |Q - \mathrm{Id}\,|, |q'| \le C_0\epsilon_k \text{ and } q_n^{\alpha+\beta}(\det Q)^2 = 1.$$

In other words, the inductive step holds taking

$$u_{k+1} = \tilde{u}_k,\ R_{k+1} = R_k Q, \text{ and } \tau_{k+1}' = \tilde{\tau}_k'.$$

Indeed,

$$r_{k+1,n}^{\alpha+\beta}(\det R_{k+1})^2 = q_n^{\alpha+\beta}(\det Q)^2 r_{k,n}^{\alpha+\beta}(\det R_k)^2 = 1,$$

and

$$|R_{k+1} - R_k| \leq |Q - \mathrm{Id}\,||R_k| \leq 2C_0\epsilon_k.$$

Second, we find R and conclude. By the inequality in (5.9), we see that R_k converges to some R, as k tends to infinity. In particular,

$$|R - R_k| \leq C\epsilon_k.$$

Thus, after replacing ϵ_k with $C\epsilon_k$, we can replace R_k by R in (5.8). In particular, considering the inductive manner in which each τ_k' is produced, we have that

$$|u(Rx) - r_k' \cdot R'x' - U(x)| \leq C\epsilon h_k^{1+\frac{\lambda}{2}} \text{ in } (R^{-1}X) \cap \mathcal{C}_{h_k}$$

with

$$r_k' := \sum_{i=1}^{k} h^{\frac{i-1}{2}} (R_{i-1}')^{-t} q_i'.$$

Here q_i' is the linear part of the polynomial found at each application of Lemma 5.2. Hence, r_k' converges to some r', and

$$|r' - r_k'| \leq 2\epsilon h_k^{\frac{1}{2}+\frac{\lambda}{2}}.$$

It follows that we can replace r_k' with r', as we replaced R_k with R: for all $k \in \mathbb{N}$,

$$|u(Rx) - r' \cdot R'x' - U(x)| \leq C\epsilon h_k^{1+\frac{\lambda}{2}} \text{ for all } x \in (R^{-1}X) \cap \mathcal{C}_{h_k}.$$

Finally, we claim that $(R')^t r' = 0$, which concludes the proof. Indeed, the inequality above tells us that the function $u(Rx) - r' \cdot R'x' - U(x)$ vanishes up to and including first order at the origin. From (5.2), we know that $\nabla u(0) = 0 = \nabla U(0)$, forcing $(R')^t r' = 0$. □

6. Proof of Theorem 1.2

The proof of Theorem 1.2 has three steps. First, we prove a strict obliqueness estimate. This key estimate allows us to find an affine transformation that aligns $\nu_{\partial X}(0)$ and $\nu_{\partial Y}(0)$, assuming $\nabla u(0) = 0$. Second, after a rotation which prescribes the now aligned normals at the origin, we blow-up to the global flat setting of Sect. 4. Finally, we apply Proposition 5.1 to find a pointwise expansion of u at 0. Since, in this procedure, the origin was fixed arbitrarily, Theorem 1.2 follows.

6.1. Strict Obliqueness

In this section, we prove our strict obliqueness estimate.

Lemma 6.1. *Let X and Y be open, bounded, and C^1 convex sets in $\mathbb{R}^2$. Suppose that $T_{min} = \nabla u$ is the optimal transport taking $f = ad^{\alpha}_{\partial X}$ to $g = bd^{\beta}_{\partial Y}$, where a and b are functions bounded away from zero and infinity in X and Y respectively, and $\max\{\alpha, \beta\} > 0$. Then*

$$\nu_{\partial X}(x) \cdot \nu_{\partial Y}(\nabla u(x)) \geq \theta > 0 \text{ for all } x \in \partial X,$$

where θ depends only on the inner and outer diameters of X and Y, α, β, and the upper and lower bounds of a and b.

The proof of this estimate follows the proof of the same estimate in the work of Savin and Yu ([27, Section 3]). We show that orthogonality (as opposed to strict obliqueness) is at odds with the volume estimate for sections.

Proof. By an approximation argument, we may assume that a and b are C^1 and that X and Y are C^2 and uniformly convex (cf. [27]).

Let us assume, without loss of generality, that $0 \in \partial X$, $\{x_2 = 0\}$ is tangent to X (at 0), $X \subset \{x_2 > 0\}$, $u(0) = 0$, and $\nabla u(0) = 0$. Now suppose, to the contrary, that we have orthogonality instead of strict obliqueness; then $\{y_1 = 0\}$ is tangent to Y (at $0 = \nabla u(0)$), and, without loss of generality, $Y \subset \{y_1 > 0\}$. Set

$$\Omega := \{x \in X : x_2 < d_0\} \cap \nabla v(\{y \in Y : y_2 > 0\}).$$

(Recall v is the minimal convex extension outside of Y of the Legendre transform of u.) Also, define

$$\psi := \frac{\operatorname{diam}(Y)}{d_0} x_2.$$

Notice that

$$u_2 \leq \psi \text{ on } \partial\Omega \cap X. \tag{6.1}$$

Moreover, if $u_{12} = u_{21}$ along ∂X, then

$$u_{21} \geq 0 \text{ along } \partial X \cap \overline{\Omega}. \tag{6.2}$$

Indeed, first, since $Y \subset \{y_1 > 0\}$ and Y is tangent of the positive y_2-axis, the image under ∇u moves to the left as we move along ∂X from the left toward the origin; if Γ_X determines ∂X near the origin,

$$u_1(x_1, \Gamma_X(x_1)) \geq u_1(z_1, \Gamma_X(z_1)) \text{ if } x_1 < z_1 \leq 0.$$

As $u_1(z_1, \Gamma_X(x_1)) \geq u_1(x_1, \Gamma_X(x_1))$, by the convexity of u, we deduce that

$$u_1(z_1, \Gamma_X(x_1)) \geq u_1(z_1, \Gamma_X(z_1)).$$

Finally, since $\Gamma_X(x_1) > \Gamma_X(z_1)$, it follows that

$$u_{21}(z_1, \Gamma_X(z_1)) = u_{12}(z_1, \Gamma_X(z_1)) \geq 0,$$

letting x_1 tend to z_1. Suppose the maximum of $u_2 - \psi$ is achieved at some $z \in \Omega$. Then $\nabla u_2(z) = \nabla \psi(z)$. And setting

$$Lw := u^{ij} \partial_{ij} w,$$

we find that, at z,

$$\begin{aligned} 0 \geq L(u_2 - \psi) &= \alpha \frac{\nu_{\partial X_d} \cdot \mathbf{e}_2}{d_{\partial X}} + \frac{a_2}{a} - \frac{\operatorname{diam}(Y)}{d_0} \left(\beta \frac{\nu_{\partial Y_d}(\nabla u) \cdot \mathbf{e}_2}{d_{\partial Y}(\nabla u)} + \frac{b_2(\nabla u)}{b(\nabla u)} \right) \\ &\geq \alpha \frac{\nu_{\partial X_d} \cdot \mathbf{e}_2}{d_{\partial X}} - C_a - \frac{\operatorname{diam}(Y)}{d_0} \left(\beta \frac{\nu_{\partial Y_d}(\nabla u) \cdot \mathbf{e}_2}{d_{\partial Y}(\nabla u)} + C_b \right) \\ &> 0. \end{aligned}$$

Indeed, $\nu_{\partial X_d} \cdot \mathbf{e}_2 > 0$, if $d_0 \ll 1$. Moreover, $\nu_{\partial Y_d}(\nabla u) \cdot \mathbf{e}_2 < 0$, as $\nabla u(\Omega) \subset \{y_2 > 0\}$ and, also, provided that $d_0 \ll 1$. Finally, if $d_0 \ll 1$, then the terms with $d_{\partial *}$ in the denominator will be large enough to absorb the constants C_a and C_b. (Recall $\max\{\alpha, \beta\} > 0$.) Here, for example,

$$\partial X_d := \{x \in X : d_{\partial X}(x) = d\}$$

and $\nu_{\partial X_d}$ is the unit normal to ∂X_d oriented to point inside $\{x \in X : d_{\partial X}(x) > d\}$. (See, e.g., [16].) This is a contradiction. In turn, by (6.1) and (6.2), the maximum of $u_2 - \psi$ is achieved on $\partial\Omega \cap X$, which implies that

$$u \leq \frac{\operatorname{diam}(Y)}{d_0} x_2^2 \text{ in } \Omega. \tag{6.3}$$

Unfortunately, we cannot guarantee that $u_{12} = u_{21}$ along ∂X. Therefore, we consider the following approximation scheme: let ∇u^k be the solution to the optimal transport problem taking

$$f^k := (1 - k^{-1}) f + k^{-1} \|f\|_{L^1(X)} \text{ to } g^k := (1 - k^{-1}) g + k^{-1} \|g\|_{L^1(Y)}.$$

By [27, Remark 2.1], ∇u^k converges to ∇u locally uniformly in $\mathbb{R}^2$, and, similarly, ∇v^k converges to ∇v locally uniformly in $\mathbb{R}^2$. (Here v^k is the minimal convex potential associated to the optimal transport problem taking g^k to f^k, and v^k, in Y, agrees with the Legendre transform of u^k.) Set

$$\Omega_k := \{x \in X : x_2 < d_0\} \cap \nabla v^k(\{y \in Y : y_2 > 0\}).$$

Then $\overline{\Omega}_k$ converges to $\overline{\Omega}$ in the Hausdorff sense. Since f_k and g_k are positive and Hölder continuous, $u^k \in C^2(\overline{X})$ by Caffarelli's boundary regularity theory ([5]). Thus $u^k_{21} = u^k_{12}$ along ∂X. In turn, by the formal maximum principle argument above,

$$u^k \leq \frac{\operatorname{diam}(Y)}{d_0} x_2^2 \text{ in } \Omega_k.$$

Taking the limit, $k \to \infty$, proves (6.3).

In summary, if we have orthogonality rather than strict obliqueness,

$$u \leq Cx_2^2 \text{ in } \Omega(u) \text{ and } v \leq Cy_1^2 \text{ in } \Omega(v), \tag{6.4}$$

where the estimate on v is by duality.

To conclude, let Γ_Y determine ∂Y near the origin. Corollaries 3.13, and 3.8 and (6.4), applied in succession, imply that

$$\begin{aligned} Ch^2 &\geq |S_h(u,0)||\nabla u(S_h(u,0))| \\ &\geq c|S_h(u,0)||S_h(v,0)| \\ &\geq c|\{x \in \overline{X} : x_1 \leq 0, x_2 \leq ch^{\frac{1}{2}}\}||\{y \in \overline{Y} : y_2 \leq 0, y_1 \leq ch^{\frac{1}{2}}\}| \\ &\geq ch^{\frac{1}{2}}\Gamma_X^{-1}(ch^{\frac{1}{2}})h^{\frac{1}{2}}\Gamma_Y^{-1}(ch^{\frac{1}{2}}). \end{aligned}$$

Dividing through by h^2 yields

$$C \geq \frac{\Gamma_X^{-1}(ch^{\frac{1}{2}})}{ch^{\frac{1}{2}}} \frac{\Gamma_Y^{-1}(ch^{\frac{1}{2}})}{ch^{\frac{1}{2}}}.$$

Since $\Gamma_X'(0) = \Gamma_Y'(0) = 0$, by assumption, $(\Gamma_X^{-1})'(0) = (\Gamma_Y^{-1})'(0) = +\infty$, but this implies that the right-hand side above tends to infinity when h tends to zero, which is impossible. □

6.2. Blow-Ups

In this section, we blow-up. That said, in order to blow-up to the flat setting studied in Sect. 4, we have to not only use Lemma 6.1 but choose the right transformation to normalize sections.

6.2.1. A First Normalization Up to a translation and subtracting an affine function, we assume that

$$0 \in \partial X \cap \partial Y, \; u(0) = 0 = v(0), \text{ and } \nabla u(0) = 0 = \nabla v(0).$$

From our strict obliqueness estimate, a shearing transformation exists that aligns the inner unit normals of ∂X and ∂Y at the origin, which after a rotation can be prescribed. In particular, a Θ exists such that

$$\tilde{X} := \Theta^{-1}X \subset \{x_n > 0\} \text{ and } \tilde{Y} := \Theta^t Y \subset \{y_n > 0\}$$

have $\{x_2 = 0\}$ and $\{y_2 = 0\}$ as tangent planes to their boundaries at 0 respectively. Moreover, $\det \Theta = 1$. Then defining

$$\tilde{u}(x) := u(\Theta x),$$

we find that

$$(\nabla \tilde{u})_{\#} \tilde{f} = \tilde{g}$$

where

$$\tilde{f} = \tilde{a} d_{\partial \tilde{X}}^{\alpha} \text{ with } \tilde{a}(x) := a(\Theta x) \left[\frac{d_{\partial X}(\Theta x)}{d_{\partial \tilde{X}}(x)} \right]^{\alpha}$$

and

$$\tilde{g} = \tilde{b} d_{\partial \tilde{Y}}^{\beta} \text{ with } \tilde{b}(y) := b(\Theta^{-t} y) \left[\frac{d_{\partial Y}(\Theta^{-t} y)}{d_{\partial \tilde{Y}}(y)} \right]^{\beta}$$

By [22, Lemma 6.1], $\tilde{a} \in C^{\mu}(\tilde{X})$ and $\tilde{a} > 0$. Similarly, $\tilde{b} \in C^{\omega}(\tilde{Y})$ and $\tilde{b} > 0$.

Remark 6.2. The restrictions on λ with respect to α and β explicitly, rather than via γ, come from this normalization. Indeed, $\tilde{a}$, for instance, as the product of two Hölder continuous functions, will be Hölder continuous. Yet between the two Hölder exponents it could inherit, it will inherit the smaller one.

It will be convenient to suppress the tildes in these definitions, and write u rather than $\tilde{u}$, for example.

6.2.2. A New Ellipsoid In the proof of Lemma 4.2, we found ellipsoids comparable to sections whose axes were parallel to the coordinate axes. The same construction applies here. The only difference is that the δ in (4.14) now depends on the doubling constants of f and g, dimension, and the Lipschitz semi-norm of ∂X. For simplicity, we let $w_h = w_{1;h}$, as defined a few lines above (4.14). It will be convenient to normalize these ellipsoids as we blow-up.

6.2.3. A Blow-Up Limit In this section, we show that the boundaries of X and Y flatten as we normalize. Since X and Y are $C^{1,1}$ and uniformly convex, near the origin,

$$\{x_2 > p^{-1} x_1^2\} \subset X \subset \{x_2 > p x_1^2\} \text{ and } \{y_2 > q^{-1} y_1^2\} \subset Y \subset \{y_2 > q y_1^2\}$$

for two constants $p, q > 0$.

Lemma 6.3. *For each $k \in \mathbb{N}$, there is an $h_k > 0$ such that $d_{h_k} > k w_{h_k}^2$.*

Proof. Suppose the lemma fails to hold; that is, a $\tilde{k} \in \mathbb{N}$ exists for which

$$d_h \leq \tilde{k} w_h^2 \text{ for all } h \leq \tilde{h}.$$

Let

$$A_h := \operatorname{diag}(w_h, d_h), \; X_h := A_h^{-1} X, \text{ and } Y_h := h^{-1} A_h^t Y,$$

and consider the normalized potentials

$$u_h(x) := \frac{u(A_h x)}{h} \text{ and } v_h(x) := \frac{v(A_h^{-t} h y)}{h}$$

along with their normalized densities

$$f_h(x) := (\det A_h) f(A_h x) \text{ and } g_h(y) := h^2 (\det A_h^{-t}) g(A_h^{-t} h y).$$

Now observe that

$$\{x_2 > d_h^{-1} w_h^2 p^{-1} x_1^2\} \subset X_h \subset \{x_2 > d_h^{-1} w_h^2 p x_1^2\}$$

and

$$\left\{y_2 > h d_h w_h^{-2} q^{-1} y_1^2\right\} \subset Y_h \subset \left\{y_2 > h d_h w_h^{-2} q y_1^2\right\}$$

By convexity,

$$d_h \geq cp w_h^2.$$

Thus, taking $\tilde{k}$ larger if needed,

$$\frac{1}{\tilde{k}} \leq cp \leq \frac{d_h}{w_h^2} \leq \tilde{k}.$$

Hence, up to a subsequence, as $h \to 0$, we find two limiting domains $\tilde{X}$ and $\tilde{Y}$ such that

$$\{x_2 > \tilde{k} p^{-1} x_1^2\} \subset \tilde{X} = \{x_2 > \tilde{p}(x_1)\} \subset \{x_2 > p\tilde{k}^{-1} x_1^2\},$$

for some convex $\tilde{p}$ that vanishes only at 0, and

$$\tilde{Y} = \{y_2 > 0\}.$$

Moreover, recalling our balancing condition (4.15),

$$\frac{1}{C} \leq m_h := \frac{d_h^{2+\alpha+\beta} w_h^2}{h^{2+\beta}} \leq C,$$

we find a convex function $\tilde{u}$, smooth in $\tilde{X}$, and such that

$$\det D^2 \tilde{u}(x) = \tilde{m} \frac{a(0)(x_2 - \tilde{p}(x_1))^\alpha}{b(0)\tilde{u}_2^\beta} \text{ in } \{x_2 > \tilde{p}(x_1)\}$$

and

$$\tilde{u}_2 = 0 \text{ along } \{x_2 = \tilde{p}(x_1)\}.$$

Up to multiplying $\tilde{u}$ by a constant, we may assume that $\tilde{m} a(0)/b(0) = 1$. (The constant $\tilde{m} = \lim_{h\to 0} m_h$.)

Set

$$Lw := \tilde{u}^{ij} \partial_{ij} w.$$

First, notice that there is a $C > 0$ such that

$$C x_2^\gamma \geq \tilde{u}_2 \text{ on } \partial S_1(\tilde{u}, 0).$$

If $s := \sup_{S_1(\tilde{u},0)}(\tilde{u}_2 - Cx_2^\gamma)$ is achieved at $\tilde{x} \in S_1(\tilde{u}, 0)$, then $\tilde{u}_2(\tilde{x}) = C\tilde{x}_2^\gamma + s$ for some $s \geq 0$, and

$$\begin{aligned} 0 &\geq L(\tilde{u}_2 - Cx_2^\gamma)(\tilde{x}) \\ &= \alpha \frac{1}{\tilde{x}_2 - \tilde{p}(\tilde{x}_1)} - \beta \frac{C\gamma \tilde{x}_2^{\gamma-1}}{C\tilde{x}_2^\gamma + s} - C^{-1}\gamma^{-1}\tilde{x}_2^{1-\gamma} C\gamma(\gamma-1)\tilde{x}_2^{\gamma-2} \\ &\geq \alpha \frac{1}{\tilde{x}_2 - \tilde{p}(\tilde{x}_1)} - (\beta\gamma + \gamma - 1)\frac{1}{\tilde{x}_2} \\ &> 0, \end{aligned}$$

provided $\alpha > 0$, an impossibility. (Because $\tilde{u}_2$ and $Cx_2^\gamma + s$ touch at $\tilde{x}$, their gradients agree at $\tilde{x}$. Since $\tilde{u}^{ij}$ is the inverse of $\tilde{u}_{ij}$, we have that $\tilde{u}_{12}\tilde{u}^{12} + \tilde{u}^{22}\tilde{u}_{22} = 1$. Moreover, $\tilde{u}_{12}(\tilde{x}) = (Cx_2^\gamma + s)_1(\tilde{x}) = 0$. In turn, $\tilde{u}^{22}(\tilde{x})\tilde{u}_{22}(\tilde{x}) = 1$, which explains the second equality line.) So we find that

$$\tilde{u} \leq Cx_2^{1+\gamma} \text{ in } S_1(\tilde{u}, 0).$$

If $\alpha = 0$, then consider the power $\gamma - \epsilon$ rather than γ, with $\epsilon > 0$ arbitrary but small. In particular, $Cx_2^{\gamma-\epsilon}$, for some $C > 0$ independent of ϵ, is an upper barrier; the right-hand side, in this case, becomes $\epsilon(1+\beta)\tilde{x}_2^{-1} > 0$. Applying the maximum principle and then sending ϵ to zero, yields the same inequality. Thus,

$$d_t(\tilde{u}, 0) \geq ct^{\frac{1}{1+\gamma}}.$$

Since u_h converges to $\tilde{u}$ locally uniformly,

$$d_t(u_h, 0) \geq \frac{c}{2} t^{\frac{1}{1+\gamma}} \text{ for all } h \ll 1.$$

Moreover,

$$d_t(u_h, 0) = \frac{d_{th}(u)}{d_h(u)} \text{ and } w_t(u_h, 0) = \frac{w_{th}(u)}{w_h(u)}.$$

So our balancing condition holds for u_h as well after replacing C by C^2. In turn,

$$w_t(u_h, 0)^2 \leq Ct.$$

Then

$$c\frac{t^{\frac{1}{1+\gamma}}}{t} \leq \frac{d_t(u_h, 0)}{w_t(u_h, 0)^2} = \frac{d_{th}}{d_h}\frac{w_h^2}{w_{th}^2} \leq \tilde{k}^2.$$

However, taking $t \ll 1$, we find this inequality impossible. □

Swapping the roles of u and v and α and β (also p and q), we find a dual lemma.

Lemma 6.4. *For each $k \in \mathbb{N}$, there is an $h_k > 0$ such that $kh_k d_{h_k} < w_{h_k}^2$.*

A consequence of Lemmas 6.3 and 6.4 is that the boundaries of X and Y flatten under A_h^{-1} and hA_h^t respectively; up to a subsequence, in the Hausdorff sense,

$$X_h \to \{x_2 > 0\} \text{ and } Y_h \to \{y_2 > 0\},$$

as $h \to 0$. In turn, up to multiplication by a constant and the same subsequence, locally uniformly in $\mathbb{R}^2$,

$$u_h \to \bar{u} = \bar{P}(x_1) + \bar{p}\frac{\gamma^{\frac{\beta}{1+\beta}}}{(1+\gamma)\gamma}x_2^{1+\gamma},$$

by Theorem 1.4. Furthermore,

$$|\nabla \bar{u}| \leq \frac{1}{r} \text{ in } S_1(\bar{u}, 0),$$

where $r > 0$ is the constant from Corollary 3.5.

6.3. Conclusion

Up to a multiplication by a constant and determinant 1 transformation in the x' variables (both depending on the constant $r > 0$ from Corollary 3.5, i.e., only on the doubling constants of f and g), we find that, after choosing $h > 0$ sufficiently small, the rescaling

$$\tilde{u} := u_h$$

satisfies the hypothesis of Proposition 5.1. First, notice that

$$\tilde{u}(0) = 0 = |\nabla \tilde{u}(0)|,$$

which is (5.2), since $u(0) = 0 = |\nabla u(0)|$. Second, for any $\epsilon > 0$,

$$|\tilde{u} - U| \leq \epsilon \text{ in } \mathcal{C}_2,$$

by the conclusion of the previous section. (The constant and determinant 1 transformation turn $\bar{u}$ into U.) By convexity,

$$\nabla \tilde{u}(\tilde{X} \cap \mathcal{C}_1) \subset \tilde{Y} \cap \mathcal{C}^*_{1/\rho}.$$

By construction, $\nabla \tilde{u}$ maps $\partial \tilde{X}$ to $\partial \tilde{Y}$. Also,

$$\{x_2 > d_h^{-1} w_h^2 t^{\frac{\gamma}{1+\gamma}} p^{-1} x_1^2\} \subset \tilde{X} \subset \{x_2 > 0\}$$

and

$$\{y_2 > h d_h w_h^{-2} t^{\frac{1}{1+\gamma}} q^{-1} y_1^2\} \subset \tilde{Y} \subset \{y_2 > 0\},$$

so (5.1) follows by Lemmas 6.3 and 6.4 as well as the definition of λ. These estimates on $\partial \tilde{X}$ and $\partial \tilde{Y}$ together with the estimates

$$|\tilde{a}(x) - 1| \leq \delta\epsilon|x|^\mu \text{ and } |\tilde{b}(y) - 1| \leq \delta\epsilon|y|^\omega$$

imply that the inequalities on $\det D^2\tilde{u}$ hold. Thus, applying Proposition 5.1 proves that u is $C^{2+\lambda}$ at 0 when $\gamma \geq 1$ and $C^{1+\gamma(1+\omega)}$ at 0 when $\gamma < 1$.

Since Θ and $r > 0$ are uniform over points in ∂X, Theorem 1.2 is proved.

References

1. BENAMOU, J.-D., BRENIER, Y.: Weak existence for the semigeostrophic equation formulated as a coupled Monge–Ampère/transport problem. *SIAM J. Appl. Math.* **58**, 1450–1461, 1998
2. BRENIER, Y.: Polar factorization and monotone rearrangement of vector-valued functions. *Commun. Pure Appl. Math.* **44**(4), 365–417, 1991
3. CAFFARELLI, L.A.: The regularity of mappings with a convex potential. *J. Am. Math. Soc.* **5**(1), 99–104, 1992
4. CAFFARELLI, L.A.: Boundary regularity of maps with convex potentials. *Commun. Pure Appl. Math.* **45**(9), 1141–1151, 1992
5. CAFFARELLI, L.A.: Boundary regularity of maps with convex potentials II. *Ann. Math. (2)* **144**(3), 453–496, 1996
6. CALABI, E.: Improper affine hypersurfaces of convex type and a generalization of a theorem by K. Jörgens. *Mich. Math. J.* **5**, 105–126, 1958
7. CHEN, S., LIU, J., WANG, X.-J.: Global regularity for the Monge–Ampère equation with natural boundary condition. *Ann. Math. (2)* **194**(3), 745–793, 2021
8. CHEN, S., LIU, J., WANG, X.-J.: *Boundary Regularity for the Second Boundary-Value Problem of Monge–Ampère Equations in Dimension Two*. arXiv:1806.09482
9. DELANOË, P.: Classical solvability in dimension two of the second boundary-value problem associated with the Monge–Ampère operator. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **8**(5), 443–457, 1991
10. DE PHILIPPIS, G., FIGALLI, A.: Partial regularity for optimal transport maps. *Publ. Math. Inst. Hautes Études Sci.* **121**, 81–112, 2015
11. DE SILVA, D., SAVIN, O.: On certain degenerate one-phase free boundary problems. *SIAM J. Math. Anal.* **53**(1), 649–680, 2021
12. FIGALLI, A.: Regularity properties of optimal maps between nonconvex domains in the plane. *Commun. Partial Differ. Equ.* **35**(3), 465–479, 2010
13. FIGALLI, A.: *"The Monge–Ampère equation and its applications", Zürich Lectures in Advanced Mathematics*. European Mathematical Society, Zürich, 2007. x+200 pp
14. FIGALLI, A., KIM, Y.-H.: Partial regularity of Brenier solutions of the Monge–Ampère equation. *Discrete Contin. Dyn. Syst.* **28**(2), 559–565, 2010
15. FIGALLI, A., MAGGI, F., PRATELLI, A.: A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.* **182**(1), 167–211, 2010
16. GIUSTI, E.: *"Minimal Surfaces and Functions of Bounded Variation", Monographs in Mathematics*, vol. 80. Birkhäuser Verlag, Basel, 1984. xii+240 pp
17. GOLDMAN, M.: An ϵ-regularity result for optimal transport maps between continuous densities. *Atti. Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **31**(4), 971–979, 2020
18. GOLDMAN, M., OTTO, F.: A variational proof of partial regularity for optimal transportation maps. *Ann. Sci. Éc. Norm. Supér. (4)* **53**(5), 1209–1233, 2020
19. GUTIÉRREZ, C.E.: *"The Monge–Ampè're Equation", Progress in Nonlinear Differential Equations and Their Applications*, vol. 44. Birkhäuser Boston, Boston, MA, 2001
20. JHAVERI, Y.: On the (in)stability of the identity map in optimal transportation. *Calc. Var. Partial Differ. Equ.* **58**(3), 96, 25, 2019
21. JÖRGENS, K.: Über die Lösungen der Differentialgleichung $rt - s^2 = 1$. *Math. Ann.* **127**, 130–134, 1954
22. LE, N.Q., SAVIN, O.: Schauder estimates for degenerate Monge–Ampère equations and smoothness of the eigenfunctions. *Invent. Math.* **207**(1), 389–423, 2017

23. Lott, J., Villani, C.: Ricci curvature for metric-measure spaces via optimal transport. *Ann. Math. (2)* **169**(3), 903–991, 2009
24. McCann, R.J.: Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.* **80**, 309–323, 1995
25. McCann, R.J.: Displacement convexity of Boltzmann's entropy characterizes the strong energy condition from general relativity. *Camb. J. Math.* **8**(3), 609–681, 2020
26. Pogorelov, A.V.: On the improper affine hypersurfaces. *Geom. Dedicata* **1**, 33–46, 1972
27. Savin, O., Yu, H.: Regularity of optimal transport between planar convex domains. *Duke Math. J.* **169**(7), 1305–1327, 2020
28. Urbas, J.: On the second boundary value problem for equations of Monge–Ampère type. *J. Reine Angew. Math.* **487**, 115–124, 1997
29. Villani, C.: *"Topics in optimal transportation", Graduate Studies in Mathematics*, vol. 58. American Mathematical Society, Providence, RI, 2003. xvi+370 pp
30. Villani, C.: *"Optimal Transport. Old and New." Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 338. Springer, Berlin, 2009. xxii+973 pp

Yash Jhaveri & Ovidiu Savin
Department of Mathematics,
Columbia University,
2990 Broadway,
New York
NY
10027 USA.
e-mail: yjhaveri@math.columbia.edu
e-mail: savin@math.columbia.edu

(*Received July 16, 2021 / Accepted May 5, 2022*)
Published online June 9, 2022