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ON CERTAIN DEGENERATE ONE-PHASE FREE BOUNDARY PROBLEMS*

DANIELA DE SILVA† AND OVIDIU SAVIN†

Abstract. We develop an existence and regularity theory for a class of degenerate one-phase free boundary problems. In this way we unify the basic theories in free boundary problems like the classical one-phase problem, the obstacle problem, or more generally for minimizers of the Alt–Phillips functional.

Key words. free boundary problems, regularity, viscosity solutions

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1. Introduction.

The most basic elliptic free boundary problems arise in the study of minimizers of energy functionals,

$$J(u,\Omega)=\int_\Omega \frac{1}{2}|\nabla u|^2+W(u)\,dx,$$

among functions u which are prescribed on the boundary

$$u=\varphi \quad \text{on } \partial\Omega.$$

The potential $W(t)\geq 0$ is assumed to be nonnegative and to vanish on $(-\infty,0]$.

If we restrict our attention to nonnegative boundary data $\varphi\geq 0$, then the conditions on W guarantee that minimizers must satisfy $u\geq 0$. The strict positivity of u in the interior of Ω can be deduced from the Euler–Lagrange equation

$$\Delta u=W'(u),$$

and the strong maximum principle, whenever W is of class $C^{1,1}$ at the origin. Otherwise $\{u=0\}$ can develop patches, and then interesting questions arise concerning the properties of the *free boundary* $\partial\{u>0\}$.

Historically the first such case that was analyzed systematically is the *obstacle problem*, that corresponds to

$$W(t)=t^+, \qquad \Delta u=\chi_{\{u>0\}}.$$

The optimal regularity of the solution was first obtained by Frehse in [F]. The general regularity theory of the free boundary was established by Caffarelli in [C] (see also [C4]). He made use of monotonicity and convexity estimates of the solution u to obtain the smoothness of the reduced part of the free boundary $\partial^*\{u>0\}$.

An important class of potentials which were studied later by Alt and Caffarelli are those which are discontinuous at 0, and in the simplest form correspond to

$$W(t)=\chi_{\{t>0\}}, \qquad \Delta u=0 \quad \text{in } \{u>0\}, \quad |\nabla u|=\sqrt{2} \quad \text{on } \partial\{u>0\}.$$

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†Department of Mathematics, Barnard College, Columbia University, New York, NY 10027 USA (desilva@math.columbia.edu, savin@math.columbia.edu).

This is known as the *one-phase* or *Bernoulli* free boundary problem and the smoothness of the reduced part of the free boundary was established by variational techniques in [AC]. Later Caffarelli developed an alternate viscosity theory approach for the regularity of the free boundary, based on the Harnack inequality and regularizations by sup-convolutions [C1, C2, C3]. A method based on Harnack inequality and compactness arguments was subsequently developed by the first author in [D].

Another general class of examples with free boundaries is given by the Alt–Phillips energy functional, which corresponds to the power-growth potentials

$$W(t) = (t^+)^\gamma \quad \text{with } \gamma \in (0,2), \qquad \Delta u = \gamma u^{\gamma-1}.$$

When $\gamma \in (0,1)$ these potentials interpolate between the one-phase problem $\gamma = 0$ and the obstacle problem $\gamma = 1$. Alt and Phillips showed in [AP] that a similar analysis as in the one-phase problem can be carried out in this case as well, and they established the smoothness of the reduced part of the free boundary.

As observed by Alt and Phillips, after a simple change of variables

$$w = u^{1/\beta}, \qquad \beta := \frac{2}{2-\gamma}, \qquad \beta \in (1,\infty),$$

the problem above can be viewed as a one-phase free boundary problem for w. It turns out that w is Lipschitz and it solves a degenerate equation of the type

$$\Delta w = \frac{h(\nabla w)}{w} \qquad \text{in } \{w > 0\} \tag{1.1}$$

with

$$\nabla w \subset \{h = 0\} \qquad \text{on } \partial\{w > 0\}, \tag{1.2}$$

where h is the quadratic polynomial

$$h(p) = \frac{\gamma}{\beta} - (\beta - 1)|p|^2.$$

A key feature of (1.1) is that it remains invariant under Lipschitz scaling $\tilde{w}(x) = w(rx)/r$. The right-hand side degenerates as w approaches 0 and the free boundary condition (1.2) can be understood as a natural balancing condition in order to seek out for Lipschitz solutions w.

In this paper we are interested in developing the viscosity theory for the degenerate class of one-phase free boundary problems (1.1)–(1.2) for general functions h. When h is not necessarily quadratic as in the examples above, then (1.1) cannot revert back to an Alt–Phillips equation by a change of variables. Our main assumptions are that $h \in C^1$ and $h \geq 0$ in a star-shaped domain D and $h \leq 0$ outside D. The free boundary condition (1.2) then reads as $\nabla w \in \partial D$. The interior regularity for solutions to (1.1) is not immediate as the right-hand side degenerates either as $w \to 0$ or $\nabla w \to \infty$. In our analysis we will make use of the results of Imbert and Silvestre [IS] in order to establish a uniform Hölder modulus of continuity for w. Similar estimates for a related nonvariational degenerate elliptic equation were obtained by Teixeira in [T].

Equations (1.1)–(1.2) do not necessarily have a variational structure, but can be thought of as interpolating free boundary conditions for different exponents γ depending on the behavior of h near ∂D, and the direction ν to the free boundary.

For example, a region around ∂D where h vanishes corresponds to the classical one-phase free boundary problem, while in a region where h is quadratic corresponds to solving the Alt–Phillips free boundary problem for some exponent γ.

We remark that the sign assumptions on the function h are crucial. When h changes sign across ∂D in the opposite directions, $h \le 0$ in D and $h \ge 0$ outside D, then the problem becomes completely different and it would correspond to the case of negative γ's in the Alt–Phillips functional. We will address this interesting case in a subsequent paper.

1.1. Setup and definitions. Let $D \subset \mathbb{R}^n$ be a bounded C^1 domain and let $h \in C^1(\mathbb{R}^n)$ vanish on $\Gamma := \partial D$. Assume that $0 \in D$ and

$$h \ge 0 \quad \text{in } D, \quad h \le 0 \quad \text{in } \bar{D}^c, \tag{1.3}$$

$$h(p) \ge -C|p|^2, \quad C > 0, \quad \text{as } |p| \to \infty. \tag{1.4}$$

Here and throughout the paper, the superscript c denotes the complement of the set in $\mathbb{R}^n$.

We ask for D to be star-shaped with respect to the origin. Precisely, given a unit direction $\nu \in \mathbb{S}^{n-1}$, we denote by $f(\nu) \in \mathbb{R}$ the positive number such that

$$f(\nu)\,\nu \ \in \ \Gamma = \partial D.$$

In view of the C^1 regularity of D, the function

$$f : \mathbb{S}^{n-1} \to \mathbb{R}$$

is also C^1. In particular there exists a $\delta > 0$ such that,

$$\delta \le f \le \delta^{-1}, \tag{1.5}$$

and if $x = f(\nu)\,\nu \in \Gamma$ and ω_x is the unit normal to Γ at x pointing towards $\bar{D}^c$, then

$$\omega_x \cdot \nu \ge \delta. \tag{1.6}$$

Without loss of generality we may relabel the constant C in (1.4), such that the inequality holds in the whole space

$$h(p) \ge -C_h|p|^2 \quad \forall p \in \mathbb{R}^n. \tag{1.7}$$

We are now ready to introduce our one-phase free boundary problem: find a continuous function $w \ge 0$ in $\bar{B}_1$ which is prescribed on ∂B_1 and solves

$$\begin{cases} \Delta w = \dfrac{h(\nabla w)}{w} & \text{on } B_1^+(w) := B_1 \cap \{w > 0\}, \\ \nabla w \in \Gamma & \text{on } F(w) := \partial B_1^+(w) \cap B_1. \end{cases} \tag{1.8}$$

The two conditions above are understood in the viscosity sense and we make them precise below. First we recall that given two continuous functions u, ψ in B_1, we say that ψ touches u from below (resp., above) at $x_0 \in B_1$ if

$$\psi \le u \quad (\text{resp., } \psi \ge u) \quad \text{near } x_0, \quad \psi(x_0) = u(x_0).$$

If the first inequality is strict (except at x_0), we say that ψ touches u strictly from below (resp., above).

The notion of viscosity solution for the interior equation is standard and in fact we will show that w is locally Lipschitz and it is a classical solution in the set $\{w > 0\}$. We therefore provide only the definition of a viscosity solution to the free boundary condition. From now on, for any continuous function ψ, we denote

$$F(\psi) := \partial\{\psi > 0\}.$$

DEFINITION 1.1. *We say that w satisfies the free boundary condition in* (1.8) *in the viscosity sense if, given $x_0 \in F(w)$, and $\psi \in C^2$ such that ψ^+ touches w from below (resp., from above) at x_0 with $|\nabla\psi(x_0)| \neq 0$, and ν denotes the unit normal to $F(\psi)$ at x_0 pointing towards $\{w > 0\}$, then*

$$|\nabla\psi(x_0)| \leq f(\nu), \quad \textit{i.e., } \nabla\psi(x_0) \in \bar{D}, \quad \textit{supersolution property}$$

$$\textit{(resp., } |\nabla\psi(x_0)| \geq f(\nu) \textit{ i.e., } \nabla\psi(x_0) \notin D, \quad \textit{subsolution property).}$$

As observed earlier on, this problem is invariant under Lipschitz rescaling:

$$\tilde{w}(x) := \frac{w(rx)}{r}, \quad x \in B_1,$$

a crucial ingredient in the body of the proofs.

1.2. Main results. We investigate here the question of existence and regularity of viscosity solutions to (1.8) together with qualitative properties of their free boundaries. The main difficulty comes from the fact that the equation is degenerate near the free boundary.

We summarize our main results below. The universal constants that appear in the theorems depend only on the dimension n, the C^1 norm of h in a neighborhood of Γ, the C^1 norm of f, the constant C_h in (1.7), and the constant δ in (1.5)–(1.6). In each section we will point out the precise dependence of the constants.

Existence of a nondegenerate viscosity solution (see section 3 for the precise definition of nondegeneracy) is obtained by Perron's method. Under appropriate regularity assumptions on D, the free boundary of the Perron solution has finite Hausdorff dimension. We obtain this precisely.

THEOREM 1.2 (existence and finite Hausdorff dimension). *Given $\phi \in C^{0,\alpha}(\partial B_1)$, there exists a viscosity solution to* (1.8) *in B_1 with $w = \phi$ on ∂B_1. Moreover, w is nondegenerate and if the set D is C^2 and convex, then*

$$\mathcal{H}^{n-1}(F(w) \cap B_{1/2}) \leq C \tag{1.9}$$

for a $C > 0$ universal (depending also on the C^2 norm of f).

In fact, estimate (1.9) holds for any viscosity solution which is nondegenerate, as long as D is convex and C^2 smooth.

Remark 1.3. In general uniqueness for this problem is not expected. The solution in Theorem 1.2 is obtained as the least supersolution in the appropriate class of competitors. This may differ from the energy minimizer, whenever a variational formulation exists.

Concerning the regularity of viscosity solutions we prove the following.

THEOREM 1.4 (Lipschitz regularity). *Let w be a viscosity solution to* (1.8) *in B_1, and assume that $F(w) \cap B_{1/8} \neq \emptyset$. Then w is locally Lipschitz in $B_{1/2}$, i.e.,*

$$\|\nabla w\|_{L^\infty(B_{1/2})} \leq C$$

with C universal. Moreover, $w \in C^{2,\alpha}_{loc}$ in $B_1^+(w)$, for some $0 < \alpha < 1$ universal.

Finally, we provide an improvement of flatness lemma which leads to the following "flatness implies $C^{1,\alpha}$" type result. The strategy follows the lines of [D].

THEOREM 1.5 (flatness implies regularity). *Let w be a viscosity solution to* (1.8) *in B_1 with $0 \in F(w)$. There exists ϵ_0 universal such that, if w is ϵ-flat, i.e.,*

$$(f(\nu)x \cdot \nu - \epsilon)^+ \le w(x) \le (f(\nu)x \cdot \nu + \epsilon)^+ \quad \text{in } B_1, \quad \epsilon \le \epsilon_0, \tag{1.10}$$

for some $\nu \in \mathbb{S}^{n-1}$, then $F(w) \cap B_{1/2}$ is a $C^{1,\alpha}$ graph in the ν direction with norm bounded by $C\epsilon$ for a universal $0 < \alpha < 1$.

This theorem gives the regularity of the reduced boundary $\partial^*\{w > 0\} \subset F(w)$ for solutions satisfying (1.9) since, after a sufficiently large dilation, the flatness hypothesis (1.10) is guaranteed (see Lemma 4.6 in section 4).

The paper is organized as follows. In section 2 we provide the proof of the existence statement in Theorem 1.2. The following section is dedicated to Theorem 1.4, while measure theoretic properties of the free boundary are determined in section 4, completing the proof of Theorem 1.2. The last three sections are devoted to Theorem 1.5. Precisely, in section 5 we obtain a Harnack type inequality for "flat" solutions of (1.8). This is the key ingredient which allows us to use a linearization method to obtain in section 6 an improvement of flatness lemma. Finally the last section is dedicated to the linear problem associated with (1.8).

2. Existence. In this section we use Perron's method to prove the existence of a nondegenerate viscosity solution to (1.8), with a given boundary data.

Let $\phi \ge 0$ be a $C^{0,\alpha}$ function on ∂B_1. We claim that, by choosing α possibly smaller, the functions

$$\psi_\phi(x) := \inf_{x_0 \in \partial B_1} (\phi(x_0) + C|(x - x_0) \cdot \nu_{x_0}|^{\alpha/2}), \quad x \in \bar{B}_1,$$

and

$$\varphi_\phi(x) := \sup_{x_0 \in \partial B_1} (\phi(x_0) - C|(x - x_0) \cdot \nu_{x_0}|^{\alpha/2})^+, \quad x \in \bar{B}_1,$$

are, respectively, a supersolution and a subsolution to (1.8). Moreover, it easily follows that

$$\psi_\phi = \varphi_\phi = \phi \quad \text{on } \partial B_1, \tag{2.1}$$

provided that we choose C large, depending on the $C^{0,\alpha}$ norm of ϕ. Here ν_{x_0} is the outer unit normal to ∂B_1 at x_0.

We prove the first claim. It is readily seen that the infimum of a family of supersolutions is again a supersolution, thus it is enough to show that

$$\Psi(x) := \phi(x_0) + C|(x - x_0) \cdot \nu_{x_0}|^{\alpha/2}, \quad x_0 \in \partial B_1,$$

is a supersolution. After a change of coordinates, let us assume that $x_0 = 0, \nu_{x_0} = e_n$. Then, using the quadratic bound (1.7) of h, and assumption (1.3), we get

$$\Delta\Psi(x) = C\frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right) x_n^{\frac{\alpha}{2}-2} \le \frac{1}{C x_n^{\frac{\alpha}{2}}} h\left(C\frac{\alpha}{2} x_n^{\frac{\alpha}{2}-1}\right) \le \frac{h(\nabla\Psi)}{\Psi},$$

as long as C is large enough so that $\nabla\Psi \notin D$, and α is small enough so that

$$\frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right) \le -\frac{1}{4} C_h \alpha^2.$$

The second claim follows similarly by noticing that

$$\Phi(x) := \Big(\phi(x_0) - C|(x-x_0)\cdot \nu_{x_0}|^{\alpha/2}\Big)^+$$

is a subsolution for our problem (1.8) since $\nabla\Phi \notin D$ and

$$\Delta\Phi > 0 \geq \frac{h(\nabla\Phi)}{\Phi}.$$

We can now prove our existence theorem. We refer to the solution achieved in the following theorem as the Perron solution associated with ϕ.

THEOREM 2.1. *Let*

$$A := \{\psi \in C(\bar{B}_1) : \psi \leq \psi_\phi \text{ is a supersolution to } (1.8),\ \psi = \phi \text{ on } \partial B_1\}$$

and set

$$w(x) := \inf_{\mathcal{A}} \psi(x).$$

Then $w \in C(\bar{B}_1)$ is a viscosity solution to (1.8) in B_1 with $w = \phi$ on ∂B_1.

Proof. First, since $\psi_\phi \in \mathcal{A}$, w is well defined. Furthermore, by the maximum principle it is easily seen that each $\psi \in \mathcal{A}$ satisfies

$$\psi \geq \varphi_\phi.$$

Indeed, this follows from the fact that $\Phi - t$, with Φ as above ant $t > 0$, cannot touch a supersolution ψ from below.

Now we show that we can restrict the minimization to elements in $\mathcal{A}$ that are uniformly Hölder continuous of exponent $\alpha/2$. Precisely, for each $\psi \in \mathcal{A}$ we can construct another element of $\mathcal{A}$, $\bar{\psi} \leq \psi$, which is uniformly Hölder continuous, and is given by the inf-convolution

$$\bar{\psi}(x) := \inf_{y\in\bar{B}_1} (\psi(y) + 2C|x-y|^{\alpha/2}) \quad x \in \bar{B}_1.$$

Clearly $\bar{\psi} \leq \psi$. We claim that $\bar{\psi} \in \mathcal{A}$.

To show this, first notice that since $\psi_\phi \geq \psi \geq \varphi_\phi$ and (2.1) holds, then for all $y \in \bar{B}_1$,

$$\psi(y) + 2C|x-y|^{\alpha/2} \geq \phi(x) \geq \psi(y) - 2C|x-y|^{\alpha/2} \quad \text{if } x \in \partial B_1 \tag{2.2}$$

with strict inequality if $y \in B_1$.

From this we deduce that

$$\bar{\psi} = \phi \quad \text{on } \partial B_1$$

and, moreover, if

$$\psi(y_0) = \bar{\psi}(x_0) - 2C|x_0 - y_0|^{\alpha/2}, \quad x_0 \in B_1, \tag{2.3}$$

then

$$y_0 \in B_1.$$

Now we prove that $\bar{\psi}$ satisfies the equation

$$\Delta\bar{\psi} \leq \frac{h(\nabla\bar{\psi})}{\bar{\psi}} \quad \text{in } B_1^+(\bar{\psi}).$$

A similar argument holds for the free boundary condition. To this aim, let P be a quadratic polynomial touching $\bar{\psi}$ (strictly) from below at $x_0 \in B_1^+(\bar{\psi})$, and let us show that

$$\Delta P(x_0) \leq \frac{h(\nabla P(x_0))}{P(x_0)}.$$

Let y_0 be as in (2.3). We distinguish two cases. If $y_0 = x_0$, the claim is obvious since P touches also ψ from below at x_0. Otherwise, notice that

$$\nabla P(x_0) \notin D, \tag{2.4}$$

let $\eta := x_0 - y_0$, and set

$$P_\eta(x) := P(x+\eta) - 2C|\eta|^{\alpha/2}.$$

It is easily verified that P_η touches ψ from below at y_0, hence in view of (2.4), we conclude $y_0 \in B_1 \cap \{\psi > 0\}$. Thus,

$$\Delta P(x_0) = \Delta P_\eta(y_0) \leq \frac{h(\nabla P_\eta(y_0))}{P_\eta(y_0)} = \frac{h(\nabla P(x_0))}{P(x_0) - 2C|\eta|^{\alpha/2}} \leq \frac{h(\nabla P(x_0))}{P(x_0)},$$

where in the last inequality we again used (2.4) and (1.3).

The claim is proved and we conclude that the function w defined above is also a Hölder continuous supersolution which coincides with ϕ on the boundary. It remains to show that w is a subsolution.

Let P be a quadratic polynomial touching w strictly from above at $x_0 \in B_1^+(w)$, and assume by contradiction that

$$\Delta P(x_0) < \frac{h(\nabla P(x_0))}{P(x_0)}.$$

Then, in a small neighborhood B_ρ of x_0, $P > 0$ and

$$\Delta P < \frac{h(\nabla P)}{P}.$$

Then for $\epsilon > 0$ small,

$$\psi := \begin{cases} w & \text{in } B_1 \setminus \bar{B}_\rho \\ \min\{w, P - \epsilon|x - x_0|^2\} & \text{in } B_\rho, \end{cases}$$

belongs to $\mathcal{A}$ and it is strictly below w. This contradicts the minimality of w.

Now we check the free boundary condition. If P^+ touches w from above at $x_0 \in F(w) \cap F(P)$ with $P \in C^2$, we need to show that $\nabla P(x_0) \notin D$. Suppose not, and let

$$g(x) := P(x) + C\epsilon d(x) - Cd^2(x) + |x - x_0|^2 - \epsilon^3, \quad x \in B_\epsilon(x_0),$$

with $d(x)$ the signed distance from x to $F(P)$, positive in $\{P > 0\}$. The constant $C > 0$ is chosen large enough so that

$$\Delta g \leq 0 \quad \text{in } B_\epsilon(x_0) \cap \{g > 0\},$$

while ϵ is small enough so that

$$\nabla g(x) \in D, \quad x \in B_\epsilon(x_0).$$

Notice that

$$g^+ \geq 0 = P^+ \quad \text{in } \{d \leq 0\} \cap B_\epsilon(x_0),$$

while

$$g > P \quad \text{on } \{d > 0\} \cap \partial B_\epsilon(x_0).$$

Thus g^+ is a supersolution in $B_\epsilon(x_0)$ and $g^+ \geq w$ on $\partial B_\epsilon(x_0)$. We conclude that

$$\psi := \begin{cases} w & \text{in } B_1 \setminus \bar{B}_\epsilon(x_0) \\ \min\{g^+, w\} & \text{in } B_\epsilon(x_0) \end{cases}$$

is still a supersolution which is less than w. Since (for ϵ small) $g^+ \equiv 0$ in a small neighborhood of $x_0 \in F(w)$, ψ does not coincide with w, and this contradicts the minimality of w. ∎

We conclude this section with a form of nondegeneracy satisfied by our solutions.

PROPOSITION 2.2. *Let w be the Perron solution in B_1 such that $0 \in F(w)$. Then,*

$$\max_{\partial B_r} w \geq cr, \quad r < 1,$$

with $c > 0$ universal.

Proof. By rescaling, it is enough to show that

$$\max w \geq c \quad \text{on } \partial B_1$$

for some $c > 0$ universal to be specified later. Assume not, and let v be defined as (say $n > 2$)

$$v = \bar{c}((1/2)^{2-n} - |x|^{2-n}) \quad \text{for } |x| \geq 1/2,$$

extended to zero in $B_{1/2}$, with $\bar{c}$ sufficiently small universal such that $|\nabla v|(x) \in D$ for $x \in \partial B_{1/2}$. We then conclude that v is a supersolution to (1.8) in B_1. By choosing c small enough we guarantee that $v \geq c > w$ on ∂B_1. Thus,

$$\psi := \min\{w, v\}$$

is a supersolution to (1.8) with the same boundary data as w, and by the minimality of w we find $\psi = w$. On the other hand, $w = \psi \equiv 0$ in $B_{1/2}$ and we contradict that $0 \in F(w)$. ∎

3. Lipschitz regularity. In this section we show that any viscosity solution to (1.8) is uniformly Lipschitz continuous near the free boundary. The strategy is to show first the Hölder continuity of the solutions in the set where w is positive and then to use the free boundary condition and the scaling of the equation to obtain the Lipschitz continuity.

The only hypotheses needed in this section are (with $D \subset B_M$ for some large $M > 0$),

$$-C_h|p|^2 \leq h(p) \leq C\chi_{B_M}. \tag{3.1}$$

The starting point is that w is superharmonic when $|\nabla w|$ is large, and that we can use the L^ϵ estimate due to Imbert and Silvestre [IS] for supersolutions of uniformly elliptic equations that hold only for large gradients. First we recall the following Theorem 5.1 from [IS], for the special case of the Laplace operator.

THEOREM 3.1 (Imbert–Silvestre). *There exist small constants $\eta_0, \xi > 0$, such that if $u \in C(B_1)$ is a nonnegative function satisfying (in the viscosity sense)*

$$\Delta u \leq 1 \quad \text{in } B_1 \cap \{|\nabla u| \geq \eta_0\},$$

and $\inf_{B_{1/2}} u \leq 1$, *then*

$$\|u\|_{L^\xi(B_{1/2})} \leq C \tag{3.2}$$

with C, η_0, ξ depending only on the dimension n.

With this result at hand, we can prove the following Harnack type inequality.

LEMMA 3.2. *Let $w > 0$ solve (in the viscosity sense)*

$$\Delta w = \frac{h(\nabla w)}{w} \quad \text{in } B_1.$$

Then given $\sigma \geq 0$, if $w \geq \sigma$,

$$\sup_{B_{1/2}} (w - \sigma) \leq C\Big(1 + \inf_{B_{1/2}} (w - \sigma)\Big) \tag{3.3}$$

with $C > 0$ universal.

Proof. From assumption (3.1), the function

$$u := \frac{w - \sigma}{M/\eta_0(1 + \inf_{B_{1/2}}(w - \sigma))}$$

satisfies the hypothesis of Theorem 3.1 (say we choose $M \geq \eta_0$), and

$$\|w - \sigma\|_{L^\xi(B_{1/2})} \leq C\Big(1 + \inf_{B_{1/2}} (w - \sigma)\Big). \tag{3.4}$$

On the other hand, using the equation together with assumption (3.1), we get (in the viscosity sense) in B_1,

$$\Delta(w - \sigma)^\gamma = \gamma \left(\frac{w - \sigma}{w} h(\nabla w) + (\gamma - 1)|\nabla w|^2 \right)$$

$$\geq \gamma \left(-C_h |\nabla w|^2 + (\gamma - 1)|\nabla w|^2 \right) \geq 0,$$

as long as $\gamma > 0$ is large enough. By the weak Harnack inequality for $(w - \sigma)^\gamma$ (Theorem 9.26 in [GT]),

$$\sup_{B_{1/4}} w - \sigma \leq C(\gamma, \xi) \|w - \sigma\|_{L^\xi},$$

which combined with (3.4) gives the desired bound. ∎

We are now ready to prove a Hölder continuity result for solutions to (1.8) with universal estimates. We start with an oscillation decay lemma. In what follows, given a continuous function w defined in a ball B_r we denote

$$\omega(r) := \sup_{B_r} w - \inf_{B_r} w$$

as the oscillation of w on B_r.

LEMMA 3.3. *Let $w > 0$ solve (in the viscosity sense)*

$$\Delta w = \frac{h(\nabla w)}{w} \quad \text{in } B_r.$$

If $\omega(r) \geq Kr$ for some K large universal, then

$$\omega(r/2) \leq \gamma\, \omega(r), \quad 0 < \gamma < 1.$$

Proof. Let

$$\tilde{w}(x) := \frac{w(rx) - \sigma_r}{r}, \quad x \in B_1, \quad \sigma_r := \inf_{B_r} w,$$

then $\tilde{w} \geq 0$ and according to Proposition 3.2,

$$\sup_{B_{1/2}} \tilde{w} \leq C\Big(1 + \inf_{B_{1/2}} \tilde{w}\Big). \tag{3.5}$$

Our desired claim follows if we show that for some $0 < \gamma < 1$,

$$osc_{B_{1/2}} \tilde{w} \leq \gamma\, osc_{B_1} \tilde{w}. \tag{3.6}$$

Notice that

$$osc_{B_1} \tilde{w} = \frac{\omega(r)}{r}, \quad \inf_{B_1} \tilde{w} = 0.$$

If for all $x \in B_{1/2}$, we have $\tilde{w}(x) \geq \frac{\omega(r)}{2Cr}$, the bound (3.6) trivially follows. Otherwise, $\tilde{w}(x_0) < \frac{\omega(r)}{2Cr}$ for some $x_0 \in B_{1/2}$, and (3.5) yields

$$\sup_{B_{1/2}} \tilde{w} \leq C + \frac{\omega(r)}{2r},$$

which again implies the desired bound if we choose $K > 2C$. ∎

We can now deduce our Hölder continuity estimate.

PROPOSITION 3.4. *Let $w > 0$ solve (in the viscosity sense)*

$$\Delta w = \frac{h(\nabla w)}{w} \quad \text{in } B_1.$$

Then, $w \in C^{0,\alpha}(B_{1/2})$ and

$$\|w\|_{C^{0,\alpha}(B_{1/2})} \leq C(1 + w(0))$$

with $C > 0$ universal and $0 < \alpha < 1$ universal.

Proof. We wish to show that

$$\omega(r) \leq \bar{C} r^{\alpha}, \quad r = r_k := 2^{-k} \quad \forall k \geq 0. \tag{3.7}$$

We choose $\bar{C} \geq \max\{\omega(1), 2K\}$ with K given by Lemma 3.3, and argue by induction. If

$$\omega(r) \geq Kr,$$

then

$$\omega\left(\frac{r}{2}\right) \leq \gamma\omega(r) \leq \gamma \bar{C} r^{\alpha} \leq \bar{C}\left(\frac{r}{2}\right)^{\alpha},$$

so by choosing $\alpha = \alpha(\gamma)$ appropriately, our claim is satisfied. If $\omega(r) < Kr$, the claim is obviously satisfied (by our choice of $\bar{C}$).

The final estimate follows as in view of Lemma 3.2,

$$\omega\left(\frac{1}{2}\right) \leq C(1 + w(0)).$$

∎

Next we deduce that solutions to our free boundary problem grow at most linearly away from the free boundary.

PROPOSITION 3.5. *Let w be a viscosity solution to* (1.8) *in B_1, then*

$$w(x) \leq Cd(x), \quad d(x) := dist(x, F(w)), \quad B_{d(x)}(x) \subset B_{3/4}$$

with $C > 0$ universal.

Proof. By a Lipschitz rescaling we can assume that $0 \in B^+_{4/3}(w)$, and that B_1 is the largest ball around 0 contained in $B^+_{4/3}(w)$, tangent to $F(w)$ say at x_0. Thus, we need to show that $w(0)$ is bounded above by a universal constant. We claim that if $w(0) \gg C$, with C the constant in Lemma 3.2 then it follows from that lemma (applied with $\sigma = 0$) that

$$w \geq c\, w(0) \quad \text{on } \partial B_{1/2}$$

for some c universal. Now set,

$$\psi(x) := M(|x|^{-n} - 1) \quad \text{in } |x| \geq 1/2,$$

and we have

$$\Delta\psi > 0, \quad |\nabla\psi| \geq M \quad \text{in } A := B_1 \setminus \bar{B}_{1/2},$$

hence (see (3.1)),

$$\Delta\psi > 0 \geq \frac{h(\nabla\psi)}{\psi} \quad \text{in } A.$$

If we assume by contradiction that $w(0)$ is sufficiently large, then $\psi - t$ with $t > 0$ cannot touch w from below in A or on $\partial B_{1/2}$, and it follows that

$$\psi \leq w \quad \text{on } A.$$

Thus ψ^+ touches w from below at $x_0 \in F(w) \cap F(\psi^+)$ and $|\nabla\psi| \in \mathbb{R}^n \setminus \bar{D}$. This contradicts the free boundary condition for w. ∎

Finally, we show that solutions are locally Lipschitz with universal bound, and $C^{1,\alpha}$ in their positive phase.

THEOREM 3.6. *Let w be a viscosity solution to* (1.8) *in B_1 with h satisfying* (3.1). *Assume that $F(w) \cap B_{1/4} \neq \emptyset$. Then w is locally Lipschitz in $B_{1/2}$ with universal Lipschitz norm. Moreover, $w \in C^{1,\alpha}_{loc}$ in $B^+_1(w)$ for some $0 < \alpha < 1$.*

Remark 3.7. Since $w \in C^{1,\alpha}_{loc}$ in the set $\{w > 0\}$, we can then apply Schauder estimates and conclude that if $h \in C^{\alpha}_{loc}$, then $w \in C^{2,\alpha}_{loc}$ is a classical solution in $B^+(w)$.

Proof. We prove the estimates near a point $x_0 \in B^+_{1/2}(w)$. In view of Proposition 3.5, the Lipschitz rescaling

$$\tilde{w}(x) := \lambda^{-1} w(x_0 + \lambda x), \qquad \lambda := w(x_0),$$

satisfies

$$\tilde{w}(0) = 1, \quad \tilde{w} > 0 \quad \text{in } B_r \quad \text{for some } r \text{ universal.}$$

Thus, by Proposition 3.4, we find $\tilde{w} \in C^{0,\alpha}$ with

$$\|\tilde{w}\|_{C^{0,\alpha}(B_{r/2})} \leq C$$

and in particular

$$\frac{1}{2} \leq \tilde{w} \leq 2 \quad \text{in } B_\rho, \text{ for all } \rho \text{ sufficiently small.}$$

Now set

$$\bar{w}(x) = \frac{\tilde{w}(\rho x) - 1}{C\rho^\alpha}, \quad x \in B_1,$$

and then

$$|\Delta \bar{w}| \leq 2C^{-1}\rho^{2-\alpha}|h(C\rho^{\alpha-1}\nabla \bar{w}(x))|, \qquad |\bar{w}| \leq 1 \quad \text{in } B_1.$$

Thus, in view of (1.7), we can choose ρ small universal, such that

$$|\Delta \bar{w}| \leq \eta \quad \text{when } |\nabla \bar{w}| \leq \frac{1}{\eta}$$

with $\eta(n) > 0$ the universal constant in Lemma 3.8 below. Hence,

$$\|\bar{w}\|_{C^{1,\alpha}(B_{1/2})} \leq C(n) \;\Rightarrow\; \|\tilde{w}\|_{C^{1,\alpha}(B_{\rho/2})} \leq C(n,\rho).$$

Since $\nabla w(x_0) = \nabla \tilde{w}(0)$, the desired conclusion easily follows. ∎

The following lemma says that if u is almost harmonic except possibly in the region where the gradients are large, then it is of class $C^{1,\alpha}$. Its proof follows from the perturbations arguments developed in [S]. Here we only sketch the main ideas.

LEMMA 3.8. *There exists $\eta > 0$ universal (i.e., depending only on n), such that if u solves in the viscosity sense in B_1,*

$$|\Delta u| \leq \eta \quad \textit{when } |\nabla u| \leq \frac{1}{\eta}, \tag{3.8}$$

and $\|u\|_{L^\infty} \leq 1$, then

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C(n)$$

for some $0 < \alpha < 1$ universal.

Proof. It suffices to show that there exists a linear function l with $|\nabla l| \leq C$ universal, such that for $r > 0$ small universal, and $0 < \alpha < 1$ universal,

$$|u - l| \leq r^{1+\alpha} \qquad \text{in } B_r. \tag{3.9}$$

Then it is enough to observe that

$$\tilde{u}(x) := \frac{(u-l)(rx)}{r^{1+\alpha}}, \quad x \in B_1,$$

satisfies the assumptions of the lemma, hence estimate (3.9) can be iterated indefinitely, leading to the $C^{1,\alpha}$ estimate.

Let $r > 0$ be fixed, to be made precise later. Assume by contradiction that there exists a sequence $\eta_j \to 0$ as $j \to \infty$ and a sequence u_j of solutions to (3.8) with $|u_j| \leq 1$, which do not satisfy the conclusion (3.9).

We divide the proof into two steps.

Step 1. *Improvement of oscillation.*

Claim 1: There exist universal constants $C_0, C_1 > 0$ such that if $u \geq 0$ and

$$u(x_0) \leq 1, \quad x_0 \in B_{1/2},$$

and

$$|\Delta u| \leq 1 \quad \text{when } |\nabla u| \leq C_1,$$

then

$$|\{u < C_0\} \cap B_{1/2}| \geq \frac{3}{4}|B_{1/2}|.$$

This follows from a version of the Alexandrov–Bakelman–Pucci estimate; see [S]. A precise reference is Theorem 2.4 in [DS], by noticing that u is a "supersolution" in the sense of Definition 2.1 of [DS] with $\Lambda = 4n$, r arbitrarily small, and $I = [1, +\infty)$: u cannot be touched from below in a B_r neighborhood by a polynomial of the form aP with $a \in I$, and

$$P := \frac{\Lambda}{2}(x \cdot \xi)^2 - \frac{1}{2}|x|^2 + L(x)$$

with ξ a unit direction and $L(x) := b \cdot x + d, |b|, |d| \leq 1$.

After dividing u by $4C_0$ we can restate the claim as follows.

Claim 2: There exist universal constants $c > 0$ (small) and $C_2 > 0$, such that if

$$u(x_0) \leq c, \quad x_0 \in B_{1/2},$$

and

$$|\Delta u| \leq c \quad \text{when } |\nabla u| \leq C_2, \tag{3.10}$$

then

$$\left|\left\{u < \frac{1}{4}\right\} \cap B_{1/2}\right| \geq \frac{3}{4}|B_{1/2}|.$$

Thus if $|u| \leq 1$, and if there exist $x_0, x_1 \in B_{1/2}$ such that

$$(1-u)(x_0) \leq c, \quad (1+u)(x_1) \leq c$$

we would reach a contradiction, as, in view of Claim 2,

$$\left|\left\{u > \frac{3}{4}\right\} \cap B_{1/2}\right|, \left|\left\{u < -\frac{3}{4}\right\} \cap B_{1/2}\right| \geq \frac{3}{4}|B_{1/2}|.$$

In conclusion either

$$u < 1 - c \quad \text{or} \quad u > -1 + c \quad \text{in } B_{1/2}.$$

Thus, the following diminishing of oscillation statement holds: if $osc_{B_1} u \leq 2$ and (3.10) holds, then

$$osc_{B_{1/2}} u \leq 2\left(1 - \frac{c}{2}\right).$$

Step 2. *Compactness.* Consider the rescalings

$$\tilde{u}_{j,k}(x) := \left(1 - \frac{c}{2}\right)^{-k} u_j(2^{-k}x), \quad k \geq 0, \quad x \in B_1,$$

and apply Claim 2 inductively on k. We have

$$|\Delta \tilde{u}_{j,k}|(x) = \frac{2^{-k}}{(2-c)^k} |\Delta u_j|(2^{-k}x) \leq \eta_j \leq c$$

when $|\nabla \tilde{u}_{j,k}|(x) \leq C_2$ as long as k satisfies

$$C_2 \leq (2-c)^{-k} \frac{1}{\eta_j}.$$

Thus, for all such k's, say $k \leq k_j^*$,

$$osc_{B_{2^{-k}}} u_j \leq 2\left(1 - \frac{c}{2}\right)^k \leq 2(2^{-k})^\beta$$

for some $\beta > 0$ small universal. Notice that $k^*(\eta_j) \to \infty$ as j goes to ∞. Hence the modulus of continuity $\omega_j(t) : [0,1] \to \mathbb{R}^+$ of the u_j's satisfies

$$\omega_j(t) \leq Ct^\beta,$$

as long as $t \geq 2^{-k_j^*}$. Thus, by the Ascoli–Arzela theorem, we conclude that u_j converges (up to a subsequence) uniformly on compacts to a Hölder continuous function u_∞. Passing to the limit in (3.8), we obtain that

$$\Delta u_\infty = 0 \quad \text{in } B_{1/2}.$$

By elliptic regularity,

$$|u_\infty - l| \leq Cr^2 \quad \text{in } B_r, \quad l \text{ linear and } |\nabla l| \leq C \text{ universal.}$$

From the uniform convergence,

$$|u_j - l| \leq 2Cr^2 \leq r^{1+\alpha} \quad \text{in } B_r$$

provided that r, α are chosen appropriately, and we have reached a contradiction. ∎

4. Measure theoretic properties of the free boundary. In this section we show that if D is convex and C^2 smooth and $h \in C^1$, then the free boundary of a nondegenerate viscosity solution to (1.8) has finite Hausdorff measure. We follow a strategy inspired by the work of Alt and Phillips in [AP]. The universal constants in this section depend on the C^2 norm of D and the C^1 norm of h in a neighborhood of ∂D.

We say that a solution w to (1.8) is nondegenerate if there exists a constant $\kappa > 0$ such that for any $x_0 \in F(w)$ and r such that $B_r(x_0) \subset B_1$ we have

$$x_0 \in F(w) \quad \Longrightarrow \quad \max_{\partial B_r(x_0)} w \geq \kappa r. \tag{4.1}$$

THEOREM 4.1. *Assume that D is a bounded convex set with C^2 boundary, and let w be a viscosity solution to* (1.8) *satisfying the nondegeneracy condition* (4.1). *Then*

$$\mathcal{H}^{n-1}(\partial\{w > 0\} \cap B_{1/2}) \leq C(\kappa)$$

for some $C(\kappa) > 0$ depending on the universal constants and κ.

For this we first prove the following lemma.

LEMMA 4.2. *Assume that w is a global Lipschitz solution to* (1.8). *Then* $\nabla w \in \overline{D}$.

Proof. Since D is convex, it suffices to show that ∇w belongs to the convex hull of $\overline{D}$. Let $L := \sup w_n$ and assume by contradiction that

$$L > \max_{y \in D} e_n \cdot y,$$

which means that

$$h(y) \leq 0 \text{ if } y_n \geq L. \tag{4.2}$$

Let x_k be a sequence of points for which w_n approaches the limit L. For each k we rescale w into

$$w^k(x) := \frac{1}{r_k} w(x_k + r_k x), \quad r_k = w(x_k), \tag{4.3}$$

so that

$$w^k(0) = 1, \quad \partial_n w^k(0) = w_n(x_k), \quad |\nabla w^k| \leq \|\nabla w\|_\infty, \quad w_n^k \leq L.$$

We can extract a subsequence of the w^k's which converges uniformly on compact sets to $\bar{w}$. Moreover, by Theorem 3.6, in a ball B_c with c universal, the convergence holds in the $C^{1,\alpha}$ norm due to the uniform $C^{1,\alpha}$ estimates. In conclusion, $\bar{w}$ solves the same equation, and

$$\bar{w}_n \leq L = \bar{w}_n(0).$$

Differentiating in the x_n direction we find

$$\Delta \bar{w}_n = \frac{\nabla h}{\bar{w}} \cdot \nabla \bar{w}_n - \frac{h}{\bar{w}^2} \bar{w}_n, \tag{4.4}$$

and $h, \nabla h$ are evaluated at $\nabla \bar{w}$.

At the origin $\bar{w}_n$ has a maximum and $h(\nabla \bar{w}) \leq 0$ by (4.2). The strong maximum principle implies that $\bar{w}_n$ is constant ($\bar{w}_n \equiv L$) in the connected component of $\{\bar{w} > 0\}$ which contains the origin. Thus, for all x' with $|x'|$ small,

$$\bar{w}(x', x_n) = \bar{w}(x', 0) + L x_n,$$

as long as the right-hand side is positive. In particular the point $x_0 := -e_n/L$ belongs to the free boundary of $\bar{w}$, given by $x_n = -L^{-1}\bar{w}(x', 0)$. Since $\bar{w} \in C^{2,\alpha}$ near the origin (in view of Theorem 3.6), we find that $\partial\{\bar{w} > 0\}$ is $C^{2,\alpha}$ in a neighborhood of x_0. This means that we can touch $\bar{w}$ at x_0 by a quadratic polynomial P with $P_n \geq L - \delta$, $\Delta P > 0$ in a neighborhood of x_0. Then we easily contradict the definition of viscosity solutions for the w^k's and reach a contradiction. ∎

We define a convex function η in a neighborhood of D which is comparable to the distance to D. Precisely η is such that

$$\eta = 0 \quad \text{on } D, \quad \eta(y) \sim dist(y, D), \quad \eta \in C^2(D^c),$$

and $\|D^2\eta\| \leq C$ universal, by the C^2 regularity of the domain D.

By compactness, from Lemma 4.2 above we obtain the following corollary.

COROLLARY 4.3. *Assume that w is a viscosity solution to* (1.8) *in* B_1 *with* $0 \in F(w)$. *For any* $\epsilon > 0$ *there exists* $\rho(\epsilon) > 0$ *small such that*

$$\eta(\nabla w) \leq \epsilon \quad \text{in} \quad B_\rho \cap \{w > 0\}.$$

Proof. Indeed, assume by contradiction that there exist sequences $\rho_j \to 0$ and w_j solutions to (1.8) with $0 \in F(w_j)$ such that

$$\eta(\nabla w_j(x_j)) > \epsilon \quad \text{for some } x_j \in B_{\rho_j} \cap \{w_j > 0\}.$$

We rescale w_j around x_j as in (4.3), and obtain a sequence that converges uniformly (up to a subsequence) to a global Lipschitz solution $\bar{w}$ with

$$\eta(\nabla \bar{w}(0)) \geq \epsilon \quad \text{with } \bar{w}(0) = 1.$$

This contradicts Lemma 4.2. ∎

Next, we show the following.

LEMMA 4.4. *Assume that w is a viscosity solution to* (1.8) *in* B_2 *and*

$$\eta(\nabla w) \leq \epsilon_0 \quad \text{in } B_2.$$

Then

$$\eta(\nabla w) \leq w^\xi \quad \text{in } B_1$$

with ϵ_0 and ξ sufficiently small.

Proof. Let φ be a nonnegative C^2 function which vanishes in B_1 and $\varphi = 1$ on ∂B_2. We show that

$$g(x) := \eta(\nabla w) - w^\xi - \varphi(x)$$

cannot have a positive maximum in the region $B_1 \cap \{w > 0\}$. Notice that $g \leq 0$ on ∂B_1 and, by Corollary 4.3, $\limsup g \leq 0$ as we approach $\partial\{w > 0\}$. Assume by contradiction that g achieves a positive maximum x_0. At x_0,

$$w^\xi \leq \eta(\nabla w) \leq \epsilon_0, \tag{4.5}$$

and $\nabla w(x_0)$ belongs to D^c and is sufficiently close to ∂D. Then $\nabla g = 0$ implies

$$\partial_s(\eta(\nabla w)) = \xi w^{\xi-1} w_s + \varphi_s. \tag{4.6}$$

At x_0 we compute (the functions η and h and their derivatives are evaluated at $\nabla w(x_0)$)

$$\Delta\eta(\nabla w) = \eta_k \, \Delta w_k + \eta_{kl} \, w_{ki} w_{li}.$$

The last term is nonnegative by the convexity of η, and after replacing Δw_k (see (4.4)) we obtain

$$\Delta\eta(\nabla w) \geq \frac{1}{w} \eta_k h_s w_{ks} - \frac{h}{w^2} \eta_k w_k.$$

Using $\eta_k(y) y_k \geq c$ and $h \leq 0$ in D^c we find that the second term is nonnegative hence

$$\Delta\eta(\nabla w) \geq \frac{1}{w} h_s \partial_s(\eta(\nabla w)) + c\frac{|h|}{w^2},$$

and by (4.6),

$$\Delta\eta(\nabla w) \geq \xi w^{\xi-2} h_s w_s - C w^{-1} + c\frac{|h|}{w^2}.$$

On the other hand

$$\Delta w^\xi = \xi w^{\xi-2}\left(h + (\xi-1)|\nabla w|^2\right) \le -c\xi w^{\xi-2},$$

hence

$$\Delta g \ge w^{\xi-2}\left(c\xi + cw^{-\xi}|h| + \xi h_s w_s - Cw^{1-\xi} - Cw^{2-\xi}\right) > 0,$$

and we reach a contradiction.

In the last inequality we used that w is sufficiently small, and then either $|h_s w_s| < c/2$ and the claim is clear or $C \ge |h_s w_s| \ge 1/2$ which together with $h = 0$ on ∂D and the C^1 smoothness of h gives (see (4.5)) $|h| \ge c\eta \ge cw^\xi$ and, again, the inequality follows provided that ξ is sufficiently small. ∎

The lemma above leads to the following integral estimate.

LEMMA 4.5. *Assume that w is a viscosity solution to* (1.8) *in B_2 and*

$$\eta(\nabla w) \le \epsilon_0 \quad \text{in } B_2.$$

Then

$$\int_{B_1 \cap \{w>0\}} \frac{(\eta(\nabla w))^+}{w} dx \le C.$$

Proof. Let $f(t)$ be a $C^{1,1}$ smoothing of $(t^+)^{1+\xi}$, i.e.,

$$f(0) = f'(0) = 0, \quad f'' = \min\{\epsilon^{-1}, t^{\xi-1}\}.$$

We have $h(\nabla w) \ge -C\epsilon_0$ so, if ϵ_0 is sufficiently small, then

$$\Delta f(w) = f''(w)\left(\frac{f'}{f''w}h + |\nabla w|^2\right) \ge cf''(w)|\nabla w|^2.$$

We integrate and use

$$\int_{B_1} \Delta f(w) dx = \int_{\partial B_1} \partial_\nu f(w) \le C,$$

hence

$$\int_{B_1 \cap \{|\nabla w| > c\}} f''(w) dx \le C.$$

Now the result follows by letting $\epsilon \to 0$ and noticing that by Lemma 4.4,

$$(\eta(\nabla w))^+ w^{-1} \le w^{\xi-1} = \lim_{\epsilon\to 0} f''(w),$$

and $\eta^+ = 0$ when $|\nabla w| < c$ with c small. ∎

We are now ready to show the proof of Theorem 4.1.

Proof. By Corollary 4.3 we may assume that after some initial dilation around a free boundary point we have $\eta(\nabla w) \le \epsilon_0$ in B_2. Let f be a smoothing of t^+, i.e., $f(0) = f'(0) = 0$ and $f'' \ge 0$ supported on $[\epsilon, 4\epsilon]$. From the computations above with this choice of f we find

$$\begin{aligned} C &\ge \int_{B_1} \Delta f(w) dx = \int_{B_1} f''(w)|\nabla w|^2 + f'(w)hw^{-1}\, dx \\ &\ge \int_{B_1} \frac{c}{\epsilon}\chi_{w\in[\epsilon,2\epsilon]}|\nabla w|^2 dx - C, \end{aligned} \tag{4.7}$$

where in the last inequality we have used $hw^{-1} \ge -C\eta^+ w^{-1}$ and Lemma 4.5.

On a ball of radius $C\epsilon$ around a free boundary point z we have due to nondegeneracy (4.1) and Lipschitz continuity

$$\frac{1}{\epsilon}\int_{B_{C\epsilon}(z)\cap\{\epsilon<w<2\epsilon\}} |\nabla w|^2 dx \geq c(\kappa)\epsilon^{n-1}.$$

We cover the free boundary in $B_{1/2}$ with N balls $B_{C\epsilon}(z_k)$ that have the finite overlapping property. By adding the inequalities above over k, we obtain

$$\frac{1}{\epsilon}\int_{B_{3/4}\cap\{\epsilon<w<2\epsilon\}} |\nabla w|^2 dx \geq c\ N(C\epsilon)^{n-1}.$$

This combined with (4.7) gives the claim. ∎

We conclude the section with the following lemma.

LEMMA 4.6. *Assume w satisfies the hypotheses of Theorem* 4.1 *and* $0 \in \partial^*\{w>0\}$. *If ν is the unit inner normal to $F(w)$ at* 0, *then*

$$(f(\nu)x\cdot\nu - r\sigma(r))^+ \leq w \leq (f(\nu)x\cdot\nu + r\sigma(r))^+ \quad \text{in } B_r$$

with $\sigma(r)\to 0$ as $r\to 0$.

Proof. We need to show that any blowup sequence of rescalings $w_r(x) = r^{-1}w(rx)$ with $r\to 0$ converges to $f(\nu)(x\cdot\nu)^+$. Let $\bar w$ be such a blowup limit. Assume for simplicity of notation that $\nu = e_n$ and $f(\nu) = 1$.

The nondegeneracy and Lipschitz continuity imply that the positive set $\{w>0\}$ has positive density in any ball centered at a free boundary point. This together with our assumption that $0\in\partial^*\{w>0\}$ gives that $0\in F(\bar w)$ and

$$\bar w = 0 \quad \text{in } x_n \leq 0. \tag{4.8}$$

On the other hand $\nabla\bar w \in \bar D$ by Lemma 4.2, and then we easily obtain

$$\bar w \leq x_n^+$$

from the convexity of D and (4.8). Assume by contradiction that $\bar w$ does not coincide with x_n^+. Then, by the strong maximum principle we have that $\bar w < x_n^+$ in $x_n>0$. In particular we can find $\epsilon>0$ such that

$$\bar w \leq x_n^+ - \epsilon \quad \text{on} \quad B_1\cap\{x_n = l\}$$

with l small, universal. Now we can argue as in the proof of Lemma 5.3 below and construct a barrier from above to conclude

$$\bar w \leq (x_n - c\epsilon)^+ \quad \text{near the origin.}$$

This shows that 0 is an interior point of $\{\bar w = 0\}$ which contradicts $0\in F(\bar w)$. ∎

5. Harnack inequality. In this section we prove a Harnack type inequality for viscosity solutions to (1.8), which satisfy a flatness assumption. This will be the key ingredient in the improvement of flatness argument leading to the $C^{1,\alpha}$ regularity of flat free boundaries, that is Theorem 1.5. We follow the strategy from [D].

The constants in this section depend on the dimension n, the C^1 norm of ∂D, the constant δ in (1.5),(1.6), and the Lipschitz norm of h in a neighborhood of ∂D.

Recall that h satisfies (1.3) which is important in our analysis since we can construct comparison subsolutions Ψ^+ with

$$\Delta\Psi > 0, \quad \nabla\Psi \notin \bar{D} \quad \Longrightarrow \quad \Delta\Psi^+ > 0 \geq \frac{h(\nabla\Psi^+)}{\Psi^+},$$

and supersolutions Φ^+ with

$$\Delta\Phi < 0, \quad \nabla\Phi \in D \quad \Longrightarrow \quad \Delta\Phi^+ < 0 \leq \frac{h(\nabla\Phi^+)}{\Phi^+}. \tag{5.1}$$

We also assume for simplicity that

$$e_n \in \partial D. \tag{5.2}$$

We wish to prove the following result.

THEOREM 5.1 (Harnack inequality). *Let w be a viscosity solution to* (1.8) *in B_2, and assume* (5.2) *holds. There exist universal constants $\bar{\epsilon}, \eta$, such that if w satisfies at some point $x_0 \in B_2$*

$$(x_n + a_0)^+ \leq w(x) \leq (x_n + b_0)^+ \quad \text{in } B_r(x_0) \subset B_2 \tag{5.3}$$

and

$$b_0 - a_0 \leq \epsilon r$$

for some $\epsilon \leq \bar{\epsilon}$, then

$$(x_n + a_1)^+ \leq w(x) \leq (x_n + b_1)^+ \quad \text{in } B_{r\eta}(x_0)$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1 - c)\epsilon r,$$

and $0 < c < 1$ universal.

Before giving the proof we deduce an important consequence.

If w satisfies (5.3) with, say $r = 1$, then we can apply the Harnack inequality repeatedly and obtain

$$(x_n + a_m)^+ \leq w(x) \leq (x_n + b_m)^+ \quad \text{in } B_{\eta^m}(x_0)$$

with

$$b_m - a_m \leq (1 - c)^m \epsilon$$

for all m's such that

$$(1 - c)^m \eta^{-m} \epsilon \leq \bar{\epsilon}.$$

This implies that for all such m's, the oscillation of the function

$$\tilde{w}_\epsilon(x) = \frac{w(x) - x_n}{\epsilon} \quad \text{in } B_2^+(w) \cup F(w)$$

in $B_\rho(x_0), \rho = \eta^m$, is less than $(1 - c)^m = \eta^{\gamma m} = \rho^\gamma$. Thus, the following corollary holds.

COROLLARY 5.2. *Let w be as in Theorem* 5.1 *satisfying* (5.3) *for $r = 1$. Then in $B_1(x_0)$, $\tilde{w}_\epsilon$ has a Hölder modulus of continuity at x_0, outside the ball of radius $\epsilon/\bar{\epsilon}$, i.e., for all $x \in B_1(x_0)$ with $|x - x_0| \geq \epsilon/\bar{\epsilon}$*

$$|\tilde{w}_\epsilon(x) - \tilde{w}_\epsilon(x_0)| \leq C|x - x_0|^\gamma.$$

The proof of the Harnack inequality relies on the following lemma.

LEMMA 5.3. *Let w be a viscosity solution to* (1.8) *in B_1 which satisfies*

$$(x_n + 2\epsilon)^+ \geq w(x) \geq x_n^+ \quad \text{in } B_1.$$

There exist universal constants $\bar{\epsilon}, \eta > 0$ such that if at $\bar{x} = \frac{1}{5}e_n$

$$w(\bar{x}) \geq (\bar{x}_n + \epsilon)^+, \quad \epsilon \leq \bar{\epsilon}, \tag{5.4}$$

then

$$w(x) \geq (x_n + c\epsilon)^+ \quad \text{in } \overline{B}_\eta \tag{5.5}$$

for some $0 < c < 1$ universal. Analogously, if

$$(x_n - 2\epsilon)^+ \leq w(x) \leq x_n^+ \quad \text{in } B_1$$

and

$$w(\bar{x}) \leq (\bar{x}_n - \epsilon)^+,$$

then

$$w(x) \leq (x_n - c\epsilon)^+ \quad \text{in } \overline{B}_\eta.$$

Proof. We prove the first statement. The second one follows from a similar argument.

First set

$$\tilde{w} := \frac{w - x_n}{\epsilon} \qquad \text{defined only in } B_2^+(w) \cup F(w),$$

and

$$\mathcal{C}_l := B'_{3/4} \times \left\{ \frac{l}{2} < x_n < \frac{1}{2} \right\} \subset B_1^+(w)$$

with l small, universal, to be made precise later. Using that $h(e_n) = 0$ and w is bounded below in $\mathcal{C}_l$, we have

$$|\Delta\tilde{w}| = \frac{1}{\epsilon}|\Delta w| = \frac{1}{\epsilon w}|h(e_n + \epsilon\nabla\tilde{w})| \leq C(l)|\nabla\tilde{w}| \quad \text{in } \mathcal{C}_l \cap \{|\nabla\tilde{w}| \leq c\epsilon^{-1}\}.$$

This means that a sufficiently large dilation of $\tilde{w}$ satisfies the hypotheses of Lemma 3.8 and we conclude that $|\nabla\tilde{w}| \leq C(l)$ in the interior of $\mathcal{C}_l$. Since

$$|\Delta\tilde{w}| \leq C(l)|\nabla\tilde{w}| \quad \text{and} \quad \tilde{w} \geq 0, \quad \tilde{w}(\bar{x}) \geq 1,$$

we can apply the Harnack inequality and obtain

$$\tilde{w} \geq c(l) \quad \text{in } T_l := B'_{1/2} \times \{x_n = l\},$$

that is,

$$w \geq x_n + \epsilon c(l) \quad \text{on } T_l. \tag{5.6}$$

Now, let ω be the unit normal to Γ at e_n pointing towards $\mathbb{R}^n \setminus \bar{D}$, which in view of (1.6) satisfies $\omega_n \geq \delta$. Set

$$Q(x) := -\left|x' - \frac{\omega'}{\omega_n} x_n\right|^2 + Ax_n^2 + x_n$$

with $A > (n-1) + \delta^{-2}$ universal and define ($c = c(l)$)

$$\Psi_t := x_n + \epsilon c(Q + t), \quad t \in \mathbb{R}.$$

Then for $t = \underline{t} < 0$ depending on δ,

$$\Psi_{\underline{t}} < x_n \leq w$$

on the region $\mathcal{C}_\epsilon := \bar{B}'_{1/2} \times \{-2\epsilon \leq x_n \leq l\}$. Let $\bar{t}$ be the largest t such that

$$\Psi_{\bar{t}} \leq w \quad \text{on } \mathcal{C}_\epsilon,$$

and let $\tilde{x} \in \mathcal{C}_\epsilon$ such that

$$\Psi_{\bar{t}}(\tilde{x}) = w(\tilde{x}).$$

We show that $\bar{t} \geq \frac{1}{8}$. Indeed if $\bar{t} < \frac{1}{8}$, then for ϵ, l small universal, we can guarantee that

$$\Psi_{\bar{t}} < 0 \leq w \quad \text{on } B'_{1/2} \times \{x_n = -2\epsilon\}, \quad \Psi_{\bar{t}} < x_n + \epsilon c \leq w \quad \text{on } T_l,$$

and

$$\Psi_{\bar{t}} < x_n \leq w \quad \text{on } \{|x'| = 1/2\} \times \{-2\epsilon \leq x_n \leq l\}.$$

We conclude that $\tilde{x} \in \mathcal{C}_\epsilon^+(\Psi_{\bar{t}}) \cup F(\Psi_{\bar{t}})$. On the other hand, we argue that $\Psi_{\bar{t}}$ is a strict subsolution to the interior equation, and w satisfies the free boundary condition, hence no touching can occur in $\mathcal{C}_\epsilon^+(\Psi_{\bar{t}}) \cup F(\Psi_{\bar{t}})$, as long as $\nabla\Psi_{\bar{t}} \notin D$. This leads to a contradiction.

To show our claim for $\Psi_{\bar{t}}^+$ we check that for ϵ small,

$$\Delta\Psi_{\bar{t}} = \epsilon c \Delta Q > 0,$$

which follows by our choice of A. We are left to prove that $\nabla\Psi_{\bar{t}} \notin D$. Since

$$\nabla\Psi_{\bar{t}} = e_n + \epsilon c \nabla Q,$$

and ω is perpendicular to Γ at e_n, it is enough to show that

$$\omega \cdot \nabla Q > 0.$$

A quick computation gives that for ϵ small,

$$\omega \cdot \nabla Q = 2A\omega_n x_n + \omega_n > 0 \quad \text{in } \mathcal{C}_\epsilon.$$

Thus,

$$w \geq x_n + \epsilon c \left(Q + \frac{1}{8}\right), \quad \text{on } \mathcal{C}_\epsilon,$$

and for η small universal,

$$Q \geq -\frac{1}{16} \quad \text{on } B_\eta.$$

This concludes our proof. ∎

We can now prove our Theorem 5.1.

Proof of Theorem 5.1. Without loss of generality, we can assume that $x_0 = 0, r = 1$. First notice that, for ϵ small, if $a_0 < -1/5$ then $B_{1/10}(0)$ belongs to the zero phase of w, and the conclusion is trivial. Thus we only need to distinguish two cases.

If $a_0 > 1/5$, then $B_{1/5} \subset \{w > 0\}$ and

$$0 \le v := \frac{w - (x_n + a_0)}{\epsilon} \le 1$$

satisfies (see proof of Lemma 5.3)

$$|\Delta v| \le C|\nabla v| \quad \text{in } B_{1/5}.$$

Therefore, the claim is deduced from the standard Harnack inequality for v.

If $|a_0| < 1/5$, we set

$$v(x) := w(x - a_0 e_n), \quad x \in B_{4/5}.$$

Then, v satisfies the assumptions of Lemma 5.3, and the desired conclusion follows. ∎

6. Improvement of flatness. In this section we prove our main improvement of flatness proposition, from which Theorem 1.5 follows by standard arguments. The universal constants in this section depend on the dimension n, the C^1 norm of ∂D, the constant δ in (1.5),(1.6), and the C^1 norm of h in a neighborhood of ∂D.

PROPOSITION 6.1. *Let w be a viscosity solution to* (1.8) *in B_1. There exist $\epsilon_0, r > 0$ universal, such that if w is ϵ-flat, i.e.,*

$$(f(e_n)x_n - \epsilon)^+ \le w(x) \le (f(e_n)x_n + \epsilon)^+ \quad \text{in } B_1, \quad \epsilon \le \epsilon_0, \tag{6.1}$$

with $0 \in F(w)$, then

$$\left(f(\nu)x \cdot \nu - \frac{\epsilon}{2}r\right)^+ \le w(x) \le \left(f(\nu)x \cdot \nu + \frac{\epsilon}{2}r\right)^+ \quad \text{in } B_r \tag{6.2}$$

with $|\nu| = 1$, and $|f(\nu)\nu - f(e_n)e_n| \le C\epsilon$ for $C > 0$ universal.

Proof of Theorem 1.5. By iterating Proposition 6.1, we obtain

$$(f(\nu_k)x \cdot \nu - \epsilon_k r_k)^+ \le w(x) \le (f(\nu_k)x \cdot \nu_k + \epsilon_k r_k)^+ \quad \text{in } B_{r_k} \tag{6.3}$$

with $|\nu_k| = 1$, and $|f(\nu_k)\nu_k - f(\nu_{k-1})\nu_{k-1}| \le C\epsilon_k$ for $C > 0$ universal,

$$\epsilon_k := 2^{-k}\epsilon, \quad r_k := r^k, \quad k \ge 1.$$

This implies that the free boundary is pointwise $C^{1,\alpha}$ at the origin. The same argument can be repeated at all free boundary points in $B_{1/2}$, and the conclusion follows (see for example [CC]). ∎

Proof of Proposition 6.1. Without loss of generality, we assume that $f(e_n) = 1$.

Let r be fixed small (to be made precise later.) Assume by contradiction that there exist a sequence $\epsilon_k \to 0$ and a sequence of domains D_k (and corresponding f_k), functions h_k (satisfying the same assumptions as f, h with the same bounds), and solutions w_k satisfying (6.1) but not the conclusion (6.2). Since h_k, D_k, f_k have a

uniformly bounded C^1 norm, and $\nabla h_k, \nabla f_k$ have a uniformly bounded modulus of continuity, up to extracting a subsequence,

$$h_k \to h^*, \quad D_k \to D^*, \quad f_k \to f^*$$

uniformly on compacts, with h^* defined only in a neighborhood of ∂D^*. The limits are also C^1 with

$$\nabla h_k \to \nabla h^*, \quad \nabla f_k \to \nabla f^*$$

uniformly on compacts.

Step 1. Let

$$\tilde{w}_k := \frac{w_k - x_n}{\epsilon_k} \quad \text{in } \Omega_k := B_1^+(w_k) \cup F(u_k).$$

Then, by (6.1)

$$-1 \le \tilde{w}_k \le 1 \quad \text{in } \Omega_k \tag{6.4}$$

and, moreover, $F(w_k)$ converges to $B_1 \cap \{x_n = 0\}$ in the Hausdorff distance.

By Corollary 5.2, and the Ascoli–Arzela theorem, it follows that as $\epsilon_k \to 0$, the graphs of the $\tilde{w}_k$'s over $B_{1/2} \cap \Omega_k$ converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{w}$ on $B_{1/2}^+$.

Step 2. We wish to show that $\tilde{w}$ is a viscosity solution to the linearized problem

$$\begin{cases} \Delta \tilde{w} + v \cdot \dfrac{\nabla \tilde{w}}{x_n} = 0 & \text{in } B_{1/2}^+, \\ \tilde{w}_\omega = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}, \end{cases} \tag{6.5}$$

where

$$v := -\nabla h^*(e_n) = |\nabla h^*(e_n)|\omega,$$

and ω is the outer unit normal to D^* at e_n. For the precise definition of a viscosity solution to (6.5) we refer to section 7, where the problem above is analyzed and the necessary properties which will be used later on in this proof, are established.

Since $\tilde{w}_k$ satisfies

$$\Delta \tilde{w}_k = \frac{1}{\epsilon_k} \frac{h_k(e_n + \epsilon_k \nabla \tilde{w}_k) - h_k(e_n)}{x_n + \epsilon_k \tilde{w}_k} \qquad \text{in } \Omega_k$$

and $\nabla h_k \to \nabla h^*$, Proposition 2.9 in [CC] implies that $\tilde{w}$ satisfies the equation in the interior.

We only need to verify the free boundary condition. Following the notation in subsection 7.2 we set

$$s = v_n$$

and notice that in our case

$$C \ge s \ge 0, \qquad \omega_n \ge \delta. \tag{6.6}$$

In view of (6.4), the case $s \ge 1$ is trivial. Consider the case $s < 1$ and assume by contradiction that there exists a test function

$$A \left| x' - \omega' \frac{x_n}{\omega_n} - \bar{x}' \right|^2 + B + p x_n^{1-s}, \quad A, B \in \mathbb{R}, \bar{x}' \in \mathbb{R}^{n-1},$$

with

$$p < 0,$$

which touches $\tilde{w}$ by above at $\bar{x} \in \{x_n = 0\}$. Notice that since $s \geq 0$ we can replace the test function above with

$$\phi := A \left| x' - \omega' \frac{x_n}{\omega_n} - \bar{x}' \right|^2 + B - C(A) x_n^2 + \frac{p}{2} x_n$$

which still touches $\tilde{w}$ strictly from above at $\bar{x}$ (in a small neighborhood) and has the property that (for $C(A)$ appropriately chosen),

$$\Delta \phi < 0.$$

Then, the convergence of the $\tilde{w}_k$'s to $\tilde{w}$ implies that there exist points in $B_{1/2} \cap \Omega_k$ with $x_k \to \bar{x}$ and constants $c_k \to 0$ such that

$$\phi(x_k) + c_k = \tilde{w}_k(x_k)$$

and ($\mathcal{N}$ a small neighborhood of x_k)

$$\tilde{w}_k < \phi + c_k \quad \text{in } \mathcal{N} \setminus \{x_k\}.$$

Equivalently,

$$w_k(x_k) = (x_k)_n + \epsilon_k \phi_k(x_k)$$

and

$$w_k < x_n + \epsilon_k(\phi + c_k) \quad \text{in } \mathcal{N} \setminus \{x_k\}.$$

Call

$$\Phi_k := x_n + \epsilon_k(\phi + c_k).$$

In order to reach a contradiction it suffices to show that Φ^+ is a strict supersolution to our problem. Indeed (see (5.1)),

$$\Delta \Phi_k = \epsilon_k \Delta \phi < 0,$$

and it remains to prove that

$$\nabla \Phi_k(x) \in D_k \quad \text{for } x \text{ near } x_k. \tag{6.7}$$

Notice that

$$\nabla \Phi_k = e_n + \epsilon_k \nabla \phi,$$

and using the convergence of D_k to D^* it suffices to check that

$$\omega \cdot \nabla \phi < 0$$

in a neighborhood of $\bar{x}$. It is easily verified that

$$\omega \cdot \nabla \psi = -2C(A) x_n \omega_n + \frac{p}{2} \omega_n,$$

and the conclusion follows since $p < 0$, $\omega_n > 0$.

Step 3. The limit function $\tilde{w}$ solves (6.5) and $\tilde{w}(0) = 0$ since $0 \in F(w_k)$. According to Theorem 7.11 and recalling (6.6) we find that $\tilde{w}$ satisfies the pointwise $C^{1,\mu}$ estimate

(7.20) with universal constants. Thus, by the convergence of the $\tilde{w}_k$, we conclude that

$$|\tilde{w}_k(x) - a \cdot x| \le C_1 r^{1+\mu} \quad \text{in } B_r \cap \Omega_k$$

with

$$|a| \le C_0, \quad a \cdot \omega = 0,$$

with C_0, C_1, μ universal. Hence for r small enough universal,

$$x_n + \epsilon_k a \cdot x - \epsilon_k \frac{r}{4} \le w_k(x) \le x_n + \epsilon_k a \cdot x + \epsilon_k \frac{r}{4} \quad \text{in } B_r \cap \Omega_k. \tag{6.8}$$

Since

$$a \cdot \omega = 0, \quad |a| \le C_0, \quad \Gamma^* \in C^1, \quad \Gamma_k \to \Gamma^*$$

we can write

$$e_n + \epsilon_k a = \sigma_k \nu_k, \quad |\nu_k| = 1,$$

with

$$|\sigma_k - f_k(\nu_k)| \le \epsilon_k \frac{r}{4}, \qquad f_k(\nu_k)\nu_k \in \Gamma_k,$$

as long as ϵ is small enough. Thus, (6.8) gives

$$(f_k(\nu_k)x \cdot \nu_k - \epsilon_k r/2)^+ \le w_k(x) \le (f_k(\nu_k)x \cdot \nu_k + r\epsilon_k/2)^+ \quad \text{in } B_r,$$

and we reach a contradiction. ∎

7. The linearized problem. In this section we study the linearized problem associated with the free boundary problem (1.8). This is a Neumann type problem in the upper half-ball, governed by the the degenerate equation

$$\Delta\varphi + v \cdot \frac{\nabla\varphi}{x_n} = 0$$

for some constant vector $v \in \mathbb{R}^n$. We develop the viscosity theory for such a problem.

We use the following notation:

$$B_r^+ := B_r \cap \{x_n \ge 0\}$$

and

$$B_r' = B_r \cap \{x_n = 0\}$$

denotes a ball in $\mathbb{R}^{n-1}$. Points in $\mathbb{R}^n$ are sometimes denoted by $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$.

7.1. The normalized linear problem. After an affine deformation, we reduce to the case when v is parallel to e_n, and the operator is given by a general constant coefficients linear operator.

Let $A = (a_{ij})_{i,j}$ be uniformly elliptic with ellipticity constants $0 < \lambda \le \Lambda$, $a_{nn} = 1$, and let $s > -1$.

Definition 7.1. *We say that φ is a viscosity subsolution in B_1^+ to*

$$\begin{cases} \mathcal{L}_s\varphi := \sum_{ij} a_{ij}\varphi_{ij} + s\dfrac{\varphi_n}{x_n} = 0 \quad \text{in } B_1 \cap \{x_n > 0\}, \\ \varphi_s := \lim_{t\to 0} \dfrac{\varphi(x_0', t) - \varphi(x_0', 0)}{t^{1-s}} = 0 \quad \text{on } B_1' \end{cases} \tag{7.1}$$

if it is continuous in B_1^+, $\mathcal{L}_s\varphi \ge 0$ in $B_1 \cap \{x_n > 0\}$ in the viscosity sense, and φ satisfies the boundary condition in the following sense:

i. *if $s \geq 1$, then φ is uniformly bounded in B_1^+;*
ii. *if $s < 1$, then φ is continuous in B_1^+ and it cannot be touched from above at a point $x_0 \in B_1'$ by a test function*

$$\phi := A|x' - y_0'|^2 + B + px_n^{1-s}, \quad A, B \in \mathbb{R}, y_0' \in \mathbb{R}^{n-1},$$

with

$$p < 0.$$

Similarly we can define the notion of viscosity supersolution and viscosity solution to (7.1).

The main result in this section is the following theorem. From now on,

$$\delta^{-1} \geq s \geq -1 + \delta,$$

and universal constants depend on $n, \delta, \lambda, \Lambda$.

THEOREM 7.2. *Let φ be a viscosity solution to* (7.1) *with $|\varphi| \leq 1$ in B_1^+. Then $\varphi \in C^{1,\mu}(B_{1/2}^+)$ with a universal bound on the $C^{1,\mu}$ norm. In particular, φ satisfies for any $x_0 \in B_{1/2}'$,*

$$|\varphi(x) - \varphi(x_0) - a' \cdot (x' - x_0')| \leq C|x - x_0|^{1+\mu}, \qquad |a'| \leq C, \tag{7.2}$$

for $C > 0$, $0 < \mu < 1$ universal, and a vector $a' \in \mathbb{R}^{n-1}$ depending on x_0.

First we need to prove a Hölder regularity result.

THEOREM 7.3. *Let φ be a viscosity solution to* (7.1) *with $|\varphi| \leq 1$ in B_1^+. Then $\varphi \in C^{\alpha}(B_{1/2}^+)$ with a universal bound on the C^{α} norm.*

The theorem above immediately follows from the next lemma.

LEMMA 7.4. *Let φ be a viscosity solution to* (7.1) *with $|\varphi| \leq 1$ in B_1^+. Assume that*

$$\varphi\left(\frac{1}{2}e_n\right) > 0. \tag{7.3}$$

Then, there exists a universal constant $c > 0$ such that

$$\varphi \geq -1 + c \quad \text{on } B_{1/2}^+.$$

Proof. From the Harnack inequality, and assumption (7.3), we get that for $l > 0$ small,

$$\varphi + 1 \geq c(l) \quad \text{on } \{|x'| \leq 3/4\} \times \{x_n = l\}. \tag{7.4}$$

We consider first the case when $s < 1$. Let ($c := c(l)$)

$$w := c\left(-|x'|^2 + Ax_n^2 + \frac{1}{32}x_n^{1-s}\right), \quad A > \Lambda\frac{(n-1)}{\delta}. \tag{7.5}$$

It is easy to verify that w is a strict subsolution to the interior equation in (7.1) in $B_1 \cap \{x_n > 0\}$. Moreover, if l is chosen sufficiently small (depending on A),

$$w \leq -\frac{1}{2}c \quad \text{on } \left\{|x|' = \frac{3}{4}\right\} \times \{0 \leq x_n \leq l\}, \tag{7.6}$$

$$w \le \frac{1}{2}c \quad \text{on } \{|x'| \le 3/4\} \times \{x_n = l\}. \tag{7.7}$$

Now, let

$$w_t := w + t, \quad t \ge -T,$$

with T large enough so that $w_T < \varphi + 1$ in $\mathcal{C} := \{|x'| \le 3/4\} \times \{0 \le x_n \le l\}$. Let $\bar{t}$ be the largest t such that $w_t \le \varphi + 1$ on $\mathcal{C}$ and let $\bar{x}$ be the first contact point. We wish to show that $\bar{t} \ge \frac{c}{2}$. Indeed, if that is the case then

$$w + \frac{c}{2} \le \varphi + 1 \quad \text{on } \mathcal{C}.$$

The desired claim then would follow since

$$w + \frac{c}{2} \ge \frac{c}{4} \quad \text{on } \{|x'| \le 1/2\} \times \{0 \le x_n \le l\}.$$

We are left with the proof that $\bar{t} \ge \frac{c}{2}$. Indeed if $\bar{t} < \frac{c}{2}$, then in view of (7.4), (7.6), (7.7), the first contact point for $w + \bar{t}$ cannot occur on $\{|x|' = \frac{3}{4}\} \times \{0 \le x_n \le l\}$ or on $\{|x'| \le 3/4\} \times \{x_n = l\}$. On the other hand, the first contact point cannot occur either on $\{x_n = 0\}$ (because of the free boundary condition), or in the interior of $\mathcal{C}$ (because $w + \bar{t}$ is a strict subsolution to the interior equation). We have reached a contradiction, hence the desired claim holds.

If $s \ge 1$, we set

$$w_\epsilon = c(-|x'|^2 + Ax_n^2 - \epsilon x_n^{1-s}), \quad s \ne 1,$$

$$w_\epsilon = c(-|x'|^2 + Ax_n^2 + \epsilon \ln x_n), \quad s = 1,$$

with

$$A > \Lambda \frac{n-1}{2}$$

and $\epsilon > 0$. We choose $d(\epsilon) > 0$ so that

$$w_\epsilon \le -\frac{c}{2} \quad \text{if } x_n \le d(\epsilon), \quad d(\epsilon) \to 0 \quad \text{as } \epsilon \to 0.$$

Then it is easy to check that for l small,

$$w_\epsilon \le \frac{c}{2} \quad \text{on } \{|x'| \le 3/4\} \times \{x_n = l\},$$

$$w_\epsilon \le -\frac{c}{2} \quad \text{on } \left\{|x|' = \frac{3}{4}\right\} \times \{0 \le x_n \le l\}.$$

Since $\mathcal{L}_s w_\epsilon > 0$, we conclude that

$$w_\epsilon + \frac{c}{2} \le \varphi + 1 \quad \text{in } \{|x'| \le 3/4\} \times \{d(\epsilon) \le x_n \le l\}.$$

By letting $\epsilon \to 0$,we obtain the desired estimate. ∎

One key ingredient in the proof of Theorem 7.2 is the next proposition, from which the subsequent corollary immediately follows . We postpone its proof till the end of the section.

PROPOSITION 7.5. *Let φ, ψ be subsolutions (resp., supersolutions) to* (7.1). *Then $\varphi + \psi$ is a subsolution (resp., supersolution) to* (7.1).

COROLLARY 7.6. *Let φ be a viscosity solution to* (7.1) *then for any unit vector e' in the x' direction,*

$$\frac{\varphi(x+\epsilon e') - \varphi(x)}{\epsilon}$$

is a viscosity solution to (7.1).

Combining Corollary 7.6 with the Hölder regularity of viscosity solutions, we obtain by standard techniques [CC] the following result.

THEOREM 7.7. *Let φ be a viscosity solution to* (7.1) *with $|\varphi| \le 1$ in B_1^+. Then for some $\mu \in (0,1)$ universal, $\varphi \in C^{k,\mu}$ in the x' direction in $B_{3/4}^+$, for all $k \ge 1$, with $C^{k,\mu}$ norm bounded by a universal constant (depending on k).*

We are now ready to provide the proof of our main theorem.

Proof of Theorem 7.2. We rewrite the interior equation in (7.1) as

$$\varphi_{nn} + s\frac{\varphi_n}{x_n} = g(x) + h(x)$$

with

$$g(x) := -\sum_{i,j\neq n} a_{ij}\varphi_{ij}, \quad h(x) := -\sum_{i\neq n} a_{in}\varphi_{in}.$$

By Theorem 7.7, the function $g(x', x_n)$ is smooth in the x'-direction and, in particular, it is uniformly bounded on $0 \le x_n \le 1/2$. Similarly, by interior estimates and Theorem 7.3, we conclude that for some $0 < \alpha < 1$,

$$|h(x', x_n)| \le C x_n^{\alpha-1} \quad \text{in } B_{1/2}^+.$$

Thus, for each fixed $x' \in B'_{1/2}$, we are led to consider the ODE

$$u'' + s\frac{u'}{t} = f(t), \quad t \in [0, 1/2],$$

with

$$|f(t)| \le C(1 + t^{\alpha-1}). \tag{7.8}$$

The general solution is given by

$$u(t) = c_1 t^{1-s} + c_2 + \bar{u}(t) \quad \text{for } s \neq 1$$

and

$$u(t) = c_1 \ln t + c_2 + \bar{u}(t) \quad \text{for } s = 1,$$

with $\bar{u}(t)$ a particular solution. It is easy to check that since f satisfies (7.8), we can choose a particular solution $\bar{u}$ that satisfies,

$$|\bar{u}| \le C t^{1+\alpha}.$$

In conclusion

$$|\varphi(x', x_n) - c_1(x') x_n^{1-s} - c_2(x')| \le C x_n^{1+\alpha}.$$

Using the smoothness of φ in the x' direction together with the free boundary condition, we conclude that $c_1 \equiv 0$, $c_2(x') = \varphi(x', 0)$, and (7.2) holds. ∎

In order to prove Proposition 7.5 we also need the following expansion lemma.

LEMMA 7.8 (expansion at regular points). *Let $s < 1$ and let $\varphi \in C(B_1^+)$ be a viscosity supersolution to (7.1) in B_1^+. Assume that $\varphi(x',0)$ is $C^{1,1}$ at 0 in the x'-direction. If $\tilde{\varphi}$ is a solution to $\mathcal{L}_s\tilde{\varphi} = 0$ in $B_1 \cap \{x_n > 0\}$ with $\tilde{\varphi} = \varphi$ on ∂B_1^+, then, $\tilde{\varphi}_s(0)$ is well defined and*

$$\tilde{\varphi}_s(0) \le 0.$$

Proof. Without loss of generality, we can assume that $\tilde{\varphi}(0,0) = 0, \nabla_{x'}\tilde{\varphi}(0,0) = 0$. Since $\tilde{\varphi}(x',0)$ is $C^{1,1}$ at 0, in a neighborhood of 0 we have that for some large constant $C > 0$,

$$-C|x'|^2 \le \tilde{\varphi}(x',0) \le C|x'|^2.$$

We define ($k \ge 0$)

$$p_k := \sup\{p : \tilde{\varphi} \ge -2C|x'|^2 + Ax_n^2 + px_n^{1-s} \quad \text{in } B^+_{2^{-k}}\},$$

$$m_k := \inf\{m : \tilde{\varphi} \le 2C|x'|^2 - Ax_n^2 + mx_n^{1-s} \quad \text{in } B^+_{2^{-k}}\}$$

with $A > 0$ chosen so that

$$\mathcal{L}_s(-2C|x'|^2 + Ax_n^2) = 0.$$

Notice that $\{p_k\}_k$ is an increasing sequence, while $\{m_k\}_k$ is decreasing. Thus,

$$\bar{p} = \sup p_k, \quad \bar{m} := \inf m_k,$$

are well defined.

We wish to show that

$$\bar{p} = \bar{m} \in (-\infty, +\infty), \tag{7.9}$$

from which our claims will follow immediately.

First, set

$$w = -2C|x'|^2 + Ax_n^2 - Mx_n^{1-s}$$

with A as above, and $M > 0$ large so that

$$w \le \tilde{\varphi} \quad \text{on } \partial B_1^+.$$

Thus, $w \le \tilde{\varphi}$ in B_1^+ and $\{p_k\}_k$ is bounded below. Similarly, $\{m_k\}_k$ is bounded above. In order to obtain (7.9),we prove by induction that there exist sequences $\{\bar{p}_k\}, \{\bar{m}_k\}$ with $\bar{p}_k \le p_k$ and $\bar{m}_k \ge m_k$ such that

$$m_k - p_k \le \bar{m}_k - \bar{p}_k = C_0(1 - c_0)^k \tag{7.10}$$

with $c_0 > 0$ universal to be specified later, and C_0 chosen universal so that the statement holds for $k = 0$. Towards this aim let $\mu := \frac{\bar{m}_k - \bar{p}_k}{2}$ and assume (7.10) holds for $k \ge 1$. If

$$\tilde{\varphi}\left(\frac{r}{2}e_n\right) \ge (\bar{p}_k + \mu)\left(\frac{r}{2}\right)^{1-s}, \quad r = 2^{-k}, \tag{7.11}$$

then we claim that

$$p_{k+1} \ge \bar{p}_k + c_1\mu. \tag{7.12}$$

Similarly, if

$$\tilde{\varphi}\left(\frac{r}{2}e_n\right) \le (\bar{m}_k - \mu)\left(\frac{r}{2}\right)^{1-s}, \quad r = 2^{-k}, \tag{7.13}$$

then

$$m_{k+1} \le \bar{m}_k - c_1\mu. \tag{7.14}$$

Thus assuming (7.10) holds for $k \ge 1$ with $c_0 = c_1/2$, if (7.11) is satisfied, we can choose $\bar{p}_{k+1} = \bar{p}_k + c_1\mu$ and $\bar{m}_{k+1} = \bar{m}_k$, otherwise we choose $\bar{p}_{k+1} = \bar{p}_k$ and $\bar{m}_{k+1} = \bar{m}_k - c_1\mu$.

To conclude our proof, let us assume that (7.11) hold and let us show that (7.12) follows.

Call

$$v_k := -2C|x'|^2 + Ax_n^2 + \bar{p}_k x_n^{1-s}$$

and

$$u_k(x) := r^{-1+s}(\tilde{\varphi} - v_k)(rx), \quad x \in B_1.$$

Then,

$$\mathcal{L}_s u_k = 0, \quad u_k \ge 0 \quad \text{in } B_1 \cap \{x_n > 0\},$$

and

$$u_k\left(\frac{1}{2}e_n\right) \ge \mu - A\left(\frac{r}{2}\right)^{s+1} \ge \frac{\mu}{2},$$

where in the last inequality we used that by the induction hypothesis

$$\mu = C_0(1-c_0)^k = C_0 r^\alpha$$

for some small α, and C_0, c_0 can be chosen possibly larger and smaller respectively (recall that $s + 1 > \delta$). By a standard barrier argument (see proof of Lemma 7.4) we conclude that

$$u_k \ge c_1\mu x_n^{1-s} \quad \text{in } B_{1/2}^+,$$

and the desired claim follows. ∎

Remark 7.9. The existence of the replacement

$$\tilde{\varphi} \in C^2(B_1 \cap \{x_n > 0\}) \cup C(\overline{B_1^+})$$

can be achieved via Perron's method. Using the barrier functions $\pm w$ in the proof above, one can guarantee the continuity up to the boundary.

We conclude this section with the proof of Proposition 7.5. First, let us introduce the following regularizations. Given a continuous function φ in B_1^+, we define for $\epsilon > 0$ the upper ϵ-envelope in the x' direction,

$$\varphi^\epsilon(y', y_n) = \sup_{x \in B_\rho^+ \cap \{x_n = y_n\}} \left\{\varphi(x', y_n) - \frac{1}{\epsilon}|x' - y'|^2\right\}, \quad y = (y', y_n) \in B_\rho^+.$$

The proof of the following facts is standard (see [CC]):

(1) $\varphi^\epsilon \in C(B_\rho^+)$ and $\varphi_\epsilon \to \varphi$ uniformly in B_ρ^+ as $\epsilon \to 0$.

(2) φ^ϵ is $C^{1,1}$ in the x'-direction from below in B_ρ^+. Thus, φ^ϵ is pointwise second order differentiable in the x'-direction at almost every point in B_ρ^+.

(3) If φ is a viscosity subsolution to (7.1) in B_1^+ and $r < \rho$, then for $\epsilon \le \epsilon_0$ (ϵ_0 depending on φ, ρ, r) φ^ϵ is a viscosity subsolution to (7.1) in B_r^+. This fact follows from the obvious remark that the maximum of solutions of (7.1) is a viscosity subsolution.

Analogously we can define φ_ϵ, the lower ϵ-envelope of u in the x'-direction which enjoys the corresponding properties.

We are now ready to prove our main proposition.

Proof of Proposition 7.5. In view of property (1) above, it is enough to show that

$$v := \varphi^\epsilon + \psi^\epsilon$$

is a subsolution to (7.1) on B_1^+. The case $s \ge 1$ is trivial, and the interior property is standard. We only need to check the boundary condition when $s < 1$.

Assume by contradiction that there exists $A > 0$ so that

$$\phi := A|x'|^2 + px_n^{1-s}$$

touches v from above, say at 0, and $p < 0$. Then $\varphi^\epsilon, \psi^\epsilon$ are $C^{1,1}$ at zero in the x'-direction. This follows from the fact that $\varphi^\epsilon, \psi^\epsilon$ are $C^{1,1}$ from below (see property (2)) and their sum is $C^{1,1}$ from above at the origin. According to Lemma 7.8, we can consider their replacements $\tilde\varphi^\epsilon, \tilde\psi^\epsilon$. Thus ϕ will touch $\tilde\varphi^\epsilon + \tilde\psi^\epsilon$ from above at zero and

$$\tilde\varphi_s^\epsilon(0) + \tilde\psi_s^\epsilon(0) \ge 0,$$

a contradiction. ∎

7.2. The linear problem. We now discuss the general case. Let $\omega \in \mathbb{S}^n$ and $v := \lambda\omega$ with $\lambda \in \mathbb{R}$. Denote

$$s := v \cdot e_n$$

and assume that for $\delta > 0$,

$$\delta^{-1} \ge s \ge -1 + \delta, \quad \omega_n \ge \delta. \tag{7.15}$$

Definition 7.10. *We say that φ is a viscosity subsolution to*

$$\begin{cases} \Delta\varphi + v \cdot \dfrac{\nabla\varphi}{x_n} = 0 \quad \text{in } B_1 \cap \{x_n > 0\}, \\ \varphi_\omega := \lim_{t\to 0} \dfrac{\varphi(x_0 + t\omega) - \varphi(x_0)}{t^{1-s}} = 0 \quad \text{on } B_1' \end{cases} \tag{7.16}$$

if it is continuous in B_2^+, it is a subsolution to the equation in $B_1 \cap \{x_n > 0\}$ in the viscosity sense, and

i. *if $s \ge 1$, then φ is uniformly bounded in B_1^+;*
ii. *if $s < 1$, then φ is continuous in B_1^+ and it cannot be touched from above at a point $x_0 \in B_1'$ by a test function*

$$\phi := A\left|x' - \frac{\omega'}{\omega_n}x_n - y_0'\right|^2 + B + px_n^{1-s}, \quad A, B \in \mathbb{R}, y_0' \in \mathbb{R}^{n-1},$$

with

$$p < 0.$$

We remark that, after performing the following domain variation,

$$\tilde{\varphi}(x', x_n) = \varphi\left(x' + \frac{\omega' x_n}{\omega_n}, x_n\right) \tag{7.17}$$

the function $\tilde{\varphi}$ satisfies the equation

$$\sum_{i,j\neq n} d_{ij}\tilde{\varphi}_{ij} + \sum_{i\neq n} b_i\tilde{\varphi}_{in} + \tilde{\varphi}_{nn} + s\frac{\tilde{\varphi}_n}{x_n} = 0 \quad \text{in } B_c^+, \tag{7.18}$$

where

$$d_{ij} = \frac{\omega_i\omega_j}{\omega_n^2}, \quad b_i = 2\frac{\omega_i}{\omega_n}. \tag{7.19}$$

In particular, in view of (7.15), (7.18) is uniformly elliptic with ellipticity constants depending only on δ. It is also easy to see that $\tilde{\varphi}$ satisfies the free boundary condition $\tilde{\varphi}_s = 0$ on B_c'. Thus, the next result follows from Theorem 7.2. Here constants depending on n, δ, are called universal.

THEOREM 7.11. *Let φ be a viscosity solution to (7.1) with $|\varphi| \leq 1$ in B_1^+. Then $\varphi \in C^{1,\mu}(B_{1/2}^+)$ with a universal bound on the $C^{1,\mu}$ norm. In particular, φ satisfies, for any $x_0 \in B_{1/2}'$,*

$$|\varphi(x) - \varphi(x_0) - a\cdot(x - x_0)| \leq C|x - x_0|^{1+\mu}, \qquad |a| \leq C, \tag{7.20}$$

with $C > 0$, $0 < \mu < 1$ universal, and a vector $a \in \mathbb{R}^{n-1}$ depending on x_0, with

$$a \cdot \omega = 0.$$

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