Letter

Optimality of Lindblad unfolding in measurement phase transitions

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Entanglement phase transitions in hybrid quantum circuits describe individual quantum trajectories rather than the measurement-averaged ensemble, despite the fact that results of measurements are not conventionally used for feedback. Here, we numerically demonstrate that a class of generalized measurements with identical measurement-averaged dynamics give different phases and phase transitions. We show that measurement-averaged destruction of Bell state entanglement is a useful proxy for determining which hybrid circuit yields the lowest-entanglement dynamics. We use this to argue that no unfolding of our model can avoid a volume law phase, which has implications for simulation of open quantum systems.

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Hybrid quantum circuits in which measurements are interspersed with unitary dynamics have been shown to yield novel nonequilibrium phases and phase transitions [1–19]. A core concept is that weak or infrequent measurements cut Bell pairs in a quantum circuit and can decrease entanglement from volume law to area law. After this was first shown numerically in [1,3], a variety of theoretical perspectives have emerged, including maps of the circuit dynamics to various statistical mechanics models [4,11,12,16] and replica tricks in which the steady state maps to the ground state of an effective Hamiltonian [20]. Meanwhile, classification of these phases can be extended to include not just entanglement properties, but also symmetry breaking, even within the volume law phase [15].

A consistent picture that emerges is that the equilibrium properties cannot be described by the measurement-averaged density matrix, which is a featureless infinite temperature state. This is true despite the fact that the quantities of interest are indeed averaged over measurements, with no measurement-dependent feedback. It has been argued that this arises because measurement phases and phase transitions are only found in quantities that are nonlinear in the density matrix, including Renyi entropies of the (pure state) trajectories. A complementary perspective is that measurement phases emerge as the $n \to 1$ limit of n replicas, whose measurement-averaged states encode higher moments of the probability distribution over pure state density matrices [20].

Despite this perspective, there are nevertheless potential connections between measurement phase transitions and quantum error correction thresholds which remain to be understood [21]. One potential connection comes from the Lindblad equation, which is often thought of as a quantum system that is continuously measured by its environment. Indeed, Lindblad dynamics not only describe the equilibrium properties of the steady state, but also its nonequilibrium dynamics through the quantum regression formula [22–24]. Such dissipative dynamics contain information about scrambling [25–27], and one of the perspectives on measurement phase transitions is in terms of a scrambling and nonscrambling phase [6]. There remain many important open cases, such as in what circumstances does measurement-averaged scrambling dynamics contain information about the underlying measurement phase transition?

In this paper, we study a family of generalized measurements ("unfoldings") such that the measurement-averaged dynamics are identical. Three conventional unfoldings that we consider give similar entanglement phase transitions in the steady state, but the exact value of entanglement and critical measurement strength differs. The fourth unfolding shows no phase transition, exhibiting a volume law phase independent of generalized measurement strength. We discuss general properties for such unfoldings to give different measurement phases and what general entanglement structure emerges. This result clarifies the applicability of measurement-averaged dynamics to understand scrambling and has implications for simulations of open quantum systems, for which our results imply that different unfoldings of the quantum master equation lead to different entanglement in the resulting trajectories. While similar results have been seen in the context of free fermion quantum circuits [28,29], our results generalize these ideas to the generic nonintegrable case, where a volume-law entangled phase is possible.

Model. We consider a quintessential model of measurement phase transition, in which random 2-qubit Haar unitaries are interspersed with on-site Z measurements, as illustrated in Fig. 1. For the simplest case of projective measurements with probability p, a number of papers have shown that a phase transition exists in this model between a volume law entangled phase at low p and an area law entangled phase at high p [2,30,31]. We can write such measurement dynamics in the language of Kraus operators:

$$M_0^P = \sqrt{p}|\uparrow\rangle\langle\uparrow|,$$
$$M_1^P = \sqrt{p}|\downarrow\rangle\langle\downarrow|,$$
$$M_2^P = \sqrt{1-p}\mathbb{1}.$$

For a generic pure state $|\psi\rangle$, each of these outcomes is obtained with probability $p_j = \langle\psi_j|\psi_j\rangle$, where $|\psi_j\rangle = M_j|\psi\rangle$. Averaging over measurement outcomes, the

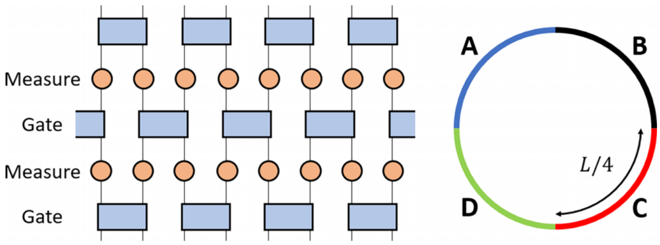

FIG. 1. (left) Illustration of one step of our hybrid circuit model. Boxes correspond to 2-site random unitaries drawn from the Haar measure. Circles correspond to generalized Z measurements, defined in the text. (right) The qubits are arranged on a ring with each quadrant labeled A-D.

post-measurement state is given by

$$\rho_f = \sum_j M_j \rho_i M_j^\dagger.$$

For such randomly placed projective measurements, we see that the result is a pure dephasing channel:

$$\rho_f^P = \begin{pmatrix} \rho_{i,\uparrow\uparrow} & (1-p)\rho_{i,\uparrow\downarrow} \\ (1-p)\rho_{i,\downarrow\uparrow} & \rho_{i,\downarrow\downarrow} \end{pmatrix}.$$

The advantage of this Kraus operator formalism is that it can be applied to generalized measurements. For instance, a simple description of weak measurements is given by [2]

$$M_0^{NP} = \frac{1+\lambda Z}{\sqrt{2(1+\lambda^2)}},$$
$$M_1^{NP} = \frac{1-\lambda Z}{\sqrt{2(1+\lambda^2)}}, \tag{1}$$

where the superscript "NP" indicates that the measurement is nonprojective. Considering the action on a single qubit, we can again see this corresponds to a dephasing channel. As shown in the Supplemental Material [32] (which includes Ref. [33]),

$$\rho_f^{NP} = \begin{pmatrix} \rho_{i,\uparrow\uparrow} & \left(\frac{1-\lambda^2}{1+\lambda^2}\right)\rho_{i,\uparrow\downarrow} \\ \left(\frac{1-\lambda^2}{1+\lambda^2}\right)\rho_{i,\downarrow\uparrow} & \rho_{i,\downarrow\downarrow} \end{pmatrix}.$$

Clearly the measurement-averaged dynamics match if $1 - p = \frac{1-\lambda^2}{1+\lambda^2}$, suggesting that generalized measurement strength λ corresponds to an effective measurement rate

$$p_{\text{eff}}^{NP} = \frac{2\lambda^2}{1+\lambda^2}. \tag{2}$$

As we will see in the next section, both projective and nonprojective measurements behave in a similar way, producing volume law phases at low p_{eff} and area law at high p_{eff}. It might then be tempting to suggest that the phase transition is indeed identical for different models of the same measurement-averaged dynamics. However, we now show that this is not the case by considering a third generalized measurement protocol, which we refer to as unitary unfolding. In this case, with probability q, the qubit undergoes a unitary kick with operator Z. This is represented by Kraus operators

$$M_0^U = \sqrt{q}Z,$$
$$M_1^U = \sqrt{1-q}\mathbb{1}.$$

Again, this corresponds to a pure dephasing channel, with identical measurement-averaged dynamics when

$$p_{\text{eff}}^U = 2q. \tag{3}$$

While such unitary kicks do not collect information about the qubit, they are valid Kraus operators and therefore we refer to this situation as "unitary measurements" and use the superscript "U" [34].

Note that the limit of weak continuous measurement corresponds to Lindblad dynamics, meaning that the strong generalized measurements above can be generated by finite time evolution under appropriate unfoldings of the Lindblad equation. Therefore, we refer to these measurement-averaged dynamics as "Lindblad equivalent" and use the term Lindblad to refer to any such dynamics, even if the measurement amplitudes are not small.

Results. To confirm these predictions, we numerically examine the steady state entanglement under these measurement protocols using exact diagonalization. The conventional measure defining the phase transition is half-system von Neumann entanglement entropy

$$S_{AB} = -\text{Tr}[\rho_{AB} \log_2 \rho_{AB}],$$

where ρ_{AB} is the reduced density matrix of subsystem AB, which has length $L/2$ (see Fig. 1). In principle, S_{AB} is proportional to L in the volume law phase and $O(1)$ in the area law phase. However, entanglement entropy has not been found to be a sensitive metric for the phase transition. Instead, we adopt the tripartite mutual information as used in [31]:

$$I_3 = S_A + S_B + S_C + S_D - S_{AB} - S_{BC} - S_{AC}. \tag{4}$$

While I_3 is extensive (and negative) in the volume law phase, it vanishes in the thermodynamic limit within the area law phase, since boundary contributions cancel. Therefore, it provides much more useful finite size scaling for detecting the phase transition on small system size.

The entanglement entropy and tripartite mutual information are seen in Fig. 2. The unitary unfolding is not shown explicitly, but for all p it matches the $p = 0$ limit of P and NP measurements. The first thing to note is that neither S_{AB} nor I_3 matches for the three different unfoldings. This implies that the steady state ensembles are not microscopically equivalent, but not necessarily that the phases of matter differ. However, analyzing crossings of I_3 clarifies that the three unfoldings indeed give different phases and phase transitions. Most notably, the unitary unfolding has *no* phase transition, exhibiting a volume law phase for arbitrary $q = p/2$. By contrast, both the strong and weak measurements do exhibit phase transitions. Therefore, we see, as anticipated elsewhere, that different unfoldings of the measurement-averaged dynamics generally give different measurement-induced phase of matter. We note that the phase transitions are not guaranteed to be in the same universality class because, for instance, the unitary unfolding has no phase transition. Whether universality class of the

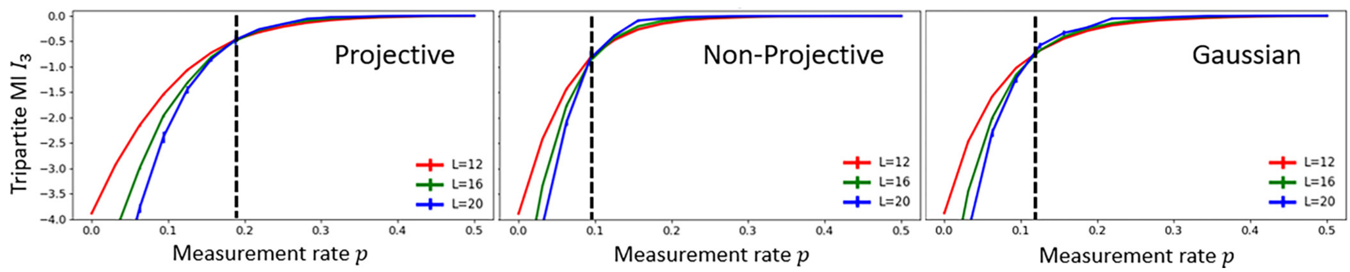

FIG. 2. Comparison of tripartite mutual information [Eq. (4)] between projective (P), nonprojective (NP), and Gaussian [G, see Eq. (5)] measurements over the same effective range of measurement rate p. Dashed lines show approximate p_c from finite size crossings, which clearly differs between measurement types.

phase transitions can differ away from the unitary limit is an open question for future work.

Having established that different unfoldings yield different measurement phase transitions, it is worth asking the question of which unfolding works best to minimize entanglement, allowing the area law phase to survive to the lowest p_c. To address this, we note that there is a general trend in the data: nonprojective measurement consistently yields the smallest entanglement entropy, followed by random projective, and of course unitary measurement has the largest entanglement. This suggests that, among the measurements considered, nonprojective would be the optimal unfolding for simulation by, e.g., matrix product states.

Before proceeding to argue that the nonprojective measurement specified in Eq. (1) is optimal, we need a simpler way to estimate the ability of a given measurement in terms of removing entanglement, under the assumption that a single measurement that removes entanglement will result in an overall lower entanglement within the many-body steady state. We propose a simple test, namely to determine how much entanglement is lost upon measuring one qubit in a maximally entangled state, such as the Bell state

$$|\psi_{\rm Bell}\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}.$$

The loss of entropy of the first qubit $\Delta S_{\rm Bell} = S_f - S_i$ is shown for various measurements in Fig. 3. Clearly it aligns with the results for steady state entropy; a smaller steady state entropy density corresponds to larger $|\Delta S|$. To further test this theory, we consider a slightly more accurate model of weak measurement in which the histograms of measurement results are Gaussian, distributed with a finite separation between $\uparrow$ and $\downarrow$ corresponding to the measurement strength α [30]

$$M^G(x) = 2^{1/2}\pi^{1/4}[G(x-\alpha)|\uparrow\rangle\langle\uparrow| + G(x+\alpha)|\downarrow\rangle\langle\downarrow|], \quad (5)$$

where $G(x)$ is a normalized Gaussian of mean 0 and variance 1, and $x \in (-\infty, \infty)$ are the possible measurement outcomes. These Gaussian measurements further support our idea, as both the Bell state entropy loss $\Delta S_{\rm Bell}$ and the steady state entropy S_{AB} are intermediate between nonprojective and projective measurements.

Clearly nonprojective measurements outperform random projective measurements in producing low-entanglement trajectories for the same Lindblad equation, i.e., are closer to the optimal unfolding for stochastic Schrödinger equation simulations. To argue that the measurements labeled "NP" are optimal, we consider the following generic family of generalized measurements:

$$M^{\rm gen}(p, x) = \beta_p(x)[\mathbb{1} + xZ],$$

a set of nonprojective measurements weighted by the real function $\beta_p(x) = \beta_p(-x)$. As shown in the Supplemental Material [32], the function β_p is constrained by a normalization condition, $\int_{-\infty}^{\infty}(1+x^2)\beta_p(x)^2 = 1$, and our goal of matching the measurement-averaged dynamics, which sets $\int_{-\infty}^{\infty} x^2\beta_p(x)^2 dx = p/2$. Note that all four of the measurement

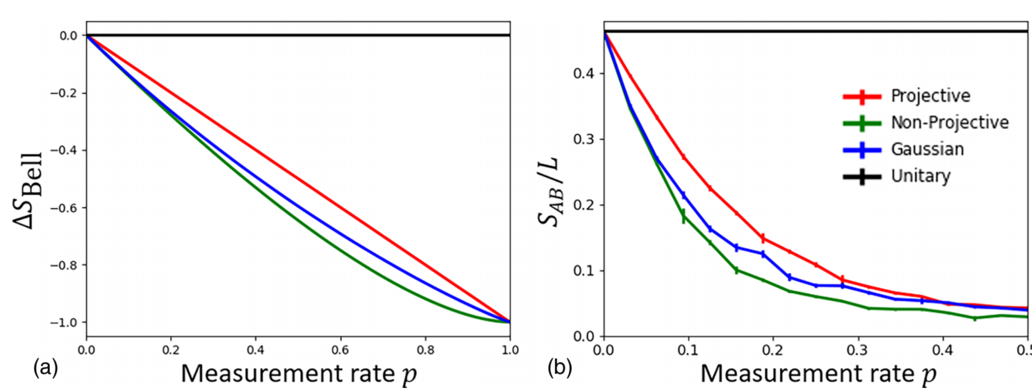

FIG. 3. (a) Entanglement loss $\Delta S_{\rm Bell}$ after measurement of qubit 1 in a Bell pair and (b) steady state half-system entanglement entropy density S_{AB}/L. Many-body entropy S_{AB} lines up precisely with $\Delta S_{\rm Bell}$ for all measurements considered, suggesting $\Delta S_{\rm Bell}$ as a useful proxy for optimal unfolding of the measurement-averaged dynamics.

types considered so far fall within this family with appropriate choices of β_p. In order to better understand which β_p will maximize $|\Delta S_{\rm Bell}|$, we start by noticing that, for each x, the measurement matches $M^{NP}(\lambda)$. As seen in Fig. 3 and shown analytically in the Appendix, Bell state entropy loss is a convex function in the range $x \in [0, 1]$, going from $\Delta S_{\rm Bell} = 0$ at $x = 0$ to $\Delta S_{\rm Bell} = 1$ at $x = 1$, which corresponds to a projective measurement. The precise opposite happens for $x > 1$, as $\Delta S(x) = \Delta S(1/x)$. Therefore, the optimal entropy loss will be given by a δ function peaked at whatever value is necessary to match p, i.e., the nonprojective (NP) measurement. While this argument is specific to our class of measurements and this particular system, we expect a similar line of logic to hold in attempting to determine optimal unfolding of more general Lindblad dynamics.

Discussion. We have shown explicitly that different unfoldings of the same measurement-averaged (Lindblad-type) dynamics give rise to different values of entanglement and the entanglement phase transition in the equivalent hybrid quantum circuit. We find that destruction of entanglement in a Bell pair is a useful proxy for many-body steady state entanglement for our class of hybrid circuits. We use this to show that a nonprojective measurement of the form $\mathbb{1} \pm \lambda Z$ is optimal for minimizing entanglement.

The most immediate consequences of this work are for numerical simulations of open quantum systems via the stochastic Schrödinger equation, particularly for entanglement-sensitive methods such as matrix product states. Our work suggests to use an unfolding of the form $\mathbb{1} + \lambda Z$ for dephasing channels, which are commonly found experimentally. We expect that a similar analysis can be applied for other Lindblad operators as well. Interestingly, our results imply that entanglement complexity of the stochastic Schrödinger equation is not equivalent to that of the Lindblad evolution, for example by simulating the density matrix directly as a matrix product operator. In particular, [35] showed that for unital quantum channels—like the ones we examine here—density matrices always flow to the area law phase and are thus efficiently representable. This implies that, for sufficiently slow Lindblad operators (small p), direct Lindblad evolution of the density matrix is more efficient than stochastic evolution of even a single pure state trajectory. The potential efficiency of density matrix evolution over trajectories was noted in [3]; this work adds to the picture by arguing that no trajectory unfolding can be as efficient as density matrix evolution.

In the longer term, this work may provide an interesting perspective on open quantum systems directly. In particular, the circuit models studied here are similar to models of noisy quantum devices, with environment playing the role of measurement, for which quantum error correction displays phase transitions at finite error rate [21]. It is clear from our results that no direct connection exists between measurement phase transitions and error correction in general, as error correction schemes must handle open quantum systems, e.g., Lindblad dynamics, whose measurement-induced phases behave differently for different unfoldings. However, there are clear similarities between these schemes which remain to be explored (cf. [36]). Further discussion of the general case in which syndrome measurement combined with environmental dissipation and error-correcting feedback can be interpreted through the lens of measurement phase transitions will be the subject of future work.

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