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ORIGINAL PAPER

A New Proof of the Erdős–Simonovits Conjecture on Walks

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Abstract
Let G^n be a graph on n vertices and let $w_k(G^n)$ denote the number of walks of length k in G^n divided by n. Erdős and Simonovits conjectured that $w_k(G^n)^t \geq w_t(G^n)^k$ when $k \geq t$ and both t and k are odd. In 2018, Sağlam proved this conjecture. We give a new shorter proof of this result.

Keywords Graph homomorphism inequalities · Paths · Walks

1 Introduction

Let G^n be a graph on n vertices, let $e(G^n)$ be the number of edges in G^n, and let $w_k(G^n)$ denote the number of walks of length k (i.e., with k edges) in G^n divided by n. In [2], Conjecture 6 reads as follows:

Conjecture 1 (Erdős–Simonovits, 1982) If d is the average degree in G^n, i.e., $d = \frac{2e(G^n)}{n}$ then

$$w_k(G^n) \geq d^k,$$

further if $k \geq t$, and both t and k are odd, then

$$w_k(G^n)^t \geq w_t(G^n)^k.$$

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Erdős and Simonovits mention that the first inequality in the conjecture had already been proven in [1, 4, 5]. Today, it is known as the Blakley-Roy inequality. They then go on to remark that the second inequality in the conjecture is a generalization of the first, and it is known to hold when k is even and they give a proof due to C.D. Godsil. The authors finally point out that the second inequality does not hold when k is odd and t is even.

In this paper, we prove the remaining case of the conjecture: we prove the second inequality when both t and k are odd. To do so, we reformulate the conjecture in terms of numbers of graph homomorphisms, and then apply a theorem from Kopparty and Rossman [3]. We present the background needed in Sect. 2, and the proof in Sect. 3. In Sect. 4, we rewrite the proof in the language of entropy. After putting a first version of this paper on the arXiv, we learned that the conjecture was no more: Sağlam had recently turned it into a theorem in [6]. The proof we present here is substantially different and quite simple; we believe it deserves consideration.

2 Reformulation and a Theorem by Kopparty and Rossman

Let $V(G)$ and $E(G)$ denote respectively the vertex set and the edge set of a graph G. A *graph homomorphism* from a graph F to a graph G is a map from the vertex set of F to the vertex set of G that sends edges to edges, i.e., that preserves adjacency. More precisely, a graph homomorphism is a function $\varphi : V(F) \to V(G)$ such that for any edge $\{v_1, v_2\} \in E(F)$, $\{\varphi(v_1), \varphi(v_2)\} \in E(G)$. Let $\mathrm{Hom}(F; G)$ be the set of homomorphisms from F to G. Let $t(F; G)$ be the homomorphism density of F in G, i.e., the probability that a random map from the $V(F)$ to $V(G)$ is a graph homomorphism. Note that $t(F; G) = \frac{|\mathrm{Hom}(F;G)|}{|V(G)|^{|V(F)|}}$. One well-known property is that $|\mathrm{Hom}(F_1; G)| \cdot |\mathrm{Hom}(F_2; G)| = |\mathrm{Hom}(F_1F_2; G)|$ and $t(F_1; G) \cdot t(F_2; G) = t(F_1F_2; G)$ where F_1F_2 denotes the disjoint union of F_1 and F_2.

In this paper, G will normally vary over all graphs on n vertices. To lighten the notation, in inequalities, we will write F to mean the function that can be evaluated on graphs G by taking the number of homomorphisms from F to G. The property $|\mathrm{Hom}(F_1; G)| \cdot |\mathrm{Hom}(F_2; G)| = |\mathrm{Hom}(F_1F_2; G)|$ thus becomes $F_1 \cdot F_2 = F_1F_2$.

Let P_k be the function that evaluates the number of homomorphisms from a path with k edges to some graph G on n vertices. Note that $\frac{P_k}{n} = w_k(G)$. When $k = 0$, P_0 is a single vertex (i.e., a 0-path), and thus $P_0 = n$. The second part of Conjecture 6 from [2] can thus be reformulated as

$$\left(\frac{P_k}{n}\right)^t \geq \left(\frac{P_t}{n}\right)^k$$

when $k \geq t$ and both t and k are odd. Another way to formulate the conjecture is to say that $n^{k-t}{P_k}^t \geq {P_t}^k$ or that $P_0^{k-t}P_k^t \geq P_t^k$ for all $k \geq t$ where t and k are both odd. Finally, observe that by dividing $n^{k-t}P_k^t \geq P_t^k$ by $n^{k(t+1)}$ on both sides, we obtain $t(P_k^t; G^n) \geq t(P_t^k; G^n)$ or equivalently $t(P_k; G^n)^t \geq t(P_t; G^n)^k$, which thus yields another way of formulating the conjecture.

Lemma 1 *To prove the conjecture, it suffices to show that* $P_0^2 P_{t+2}^t \geq P_t^{t+2}$ *for any odd* $t+2$.

Proof Suppose that we know that $P_0^2 P_{t+2}^t \geq P_t^{t+2}$ for any odd $t+2$. This is equivalent to knowing that $t(P_{t+2}; G^n) \geq t(P_t; G^n)^{\frac{t+2}{t}}$. If $k = t+2i$ where $i > 1$, then we have

$$\begin{aligned} t(P_k; G^n) &= t(P_{t+2i}; G^n) \\ &\geq t(P_{t+2i-2}; G^n)^{\frac{t+2i}{t+2i-2}} \\ &\geq \left(t(P_{t+2i-4}; G^n)^{\frac{t-2i-2}{t-2i-4}}\right)^{\frac{t+2i}{t+2i-2}} \\ &\geq \dots \\ &\geq \left(\left(\left(t(P_t; G^n)^{\frac{t+2}{t}}\right)^{\frac{t+4}{t+2}} \dots\right)^{\frac{t-2i-2}{t-2i-4}}\right)^{\frac{t+2i}{t+2i-2}} \\ &= t(P_t; G^n)^{\frac{t+2i}{t}} = t(P_t; G^n)^{\frac{k}{t}} \end{aligned}$$

as desired. □

The concept of homomorphism domination exponent was introduced in [3], though the idea behind it had been central to many problems in extremal graph theory for a long time. Let the *homomorphism domination exponent* of a pair of graphs F_1 and F_2, denoted by $\mathrm{HDE}(F_1; F_2)$, be the maximum value of c such that $|\operatorname{Hom}(F_1; G)| \geq |\operatorname{Hom}(F_2; G)|^c$ for every graph G. Thus, by Lemma 1, to prove the conjecture, it suffices to show that $\mathrm{HDE}(P_0^2 P_{t+2}^t; P_t) = t+2$ for any odd t (where we now think simply of P_i as a graph, namely the path with i edges, and not as a function).

In [3], Kopparty and Rossman showed that $\mathrm{HDE}(F_1; F_2)$ can be found by solving a linear program when F_1 is chordal and F_2 is series–parallel. Since this is the case when $F_1 = P_0^2 P_{t+2}^t$ and $F_2 = P_t$, we will use this linear program to prove the conjecture. We now briefly describe Kopparty and Rossman's result which is based on comparing the entropies of different distributions on $\operatorname{Hom}(F_2; G)$. We later pull back such distributions, and in particular the uniform distribution on all homomorphisms.

Let $\mathcal{P}(F_2)$ be the polytope consisting of normalized F_2-polymatroidal functions, which is defined to be

$$\begin{aligned} \mathcal{P}(F_2) = \big\{p \in \mathbb{R}^{2^{|V(F_2)|}} \,|\, & p(\emptyset) = 0 \\ & p(V(F_2)) = 1 \\ & p(A) \leq p(B) && \forall\, A \subseteq B \subseteq V(F_2) \\ & p(A \cap B) + p(A \cup B) \leq p(A) + p(B) && \forall\, A, B \subseteq V(F_2) \\ & p(A \cap B) + p(A \cup B) = p(A) + p(B) && \forall\, A, B \subseteq V(F_2) \text{ such that } A \cap B \\ & && \text{separates } A \backslash B \text{ and } B \backslash A \big\}. \end{aligned}$$

Note that $A \cap B$ is said to separate $A \backslash B$ and $B \backslash A$ if there are no edges in F_2 between $A \backslash B$ and $B \backslash A$.

Theorem 1 (Kopparty–Rossman, 2011) *Let F_1 be a chordal graph and let F_2 be a series–parallel graph. Then*

$$\mathrm{HDE}(F_1, F_2) = \min_{p\in\mathcal{P}(F_2)} \max_{\varphi\in\mathrm{Hom}(F_1;F_2)} \sum_{S\subseteq\mathrm{MaxCliques}(F_1)} -(-1)^{|S|} p(\varphi(\cap S))$$

where $\mathrm{MaxCliques}(F_1)$ *is the set of maximal cliques of* F_1 *and* $\cap S$ *is the intersection of the maximal cliques in* S.

3 Proof of the Conjecture

Let $[m] := \{1, 2, \ldots, m\}$, $V(P_t) = [t+1]$, and let $E(P_t) = \{\{i, i+1\} | i \in [t]\}$. Lemma 2.5 of [3] implies that for any $p \in \mathcal{P}(P_t)$,

$$p(V(P_t)) = \sum_{S\subseteq\mathrm{MaxCliques}(P_t)} -(-1)^{|S|} p(\cap S).$$

For completeness, we give a short argument.

Lemma 2 *For any $p \in \mathcal{P}(P_t)$ for some $t \geq 1$ (not necessarily odd),*

$$p(V(P_t)) = \sum_{\{i,i+1\}\in E(P_t)} p(\{i, i+1\}) - \sum_{i\in\{2,\ldots,t\}} p(\{i\}).$$

Proof We prove it by induction on t. If $t = 1$, it is trivially true since there are no negative terms to consider. Suppose it is true for t. Consider P_{t+1}. Let $A = [t+1]$ and let $B = \{t+1, t+2\}$. Then $A \cup B = [t+2] = V(P_{t+1})$, and $A \cap B = \{t+1\}$. Note that $A \cap B$ separates $A \backslash B$ and $B \backslash A$, so $p(A \cup B) = p(A) + p(B) - p(A \cap B)$. Thus

$$\begin{aligned} p(V(P_{t+1})) &= p(V(P_t)) + p(\{t+1, t+2\}) - p(\{t+1\}) \\ &= p(\{1, 2\}) + \ldots + p(\{t, t+1\}) - p(\{2\}) - \ldots \\ &\quad - p(\{t\}) + p(\{t+1, t+2\}) - p(\{t+1\}) \\ &= \sum_{\{i,i+1\}\in E(P_{t+1})} p(\{i, i+1\}) - \sum_{i\in\{2,\ldots,t+1\}} p(\{i\}), \end{aligned}$$

where the second line follows from the induction hypothesis. □

Theorem 2 *We have that* $\mathrm{HDE}(P_0^2 P_{t+2}^t; P_t) = t+2$*, and thus that Conjecture 1 holds.*

Proof We first show that $\mathrm{HDE}(P_0^2 P_{t+2}^t; P_t) \leq t+2$. For $i \in [t+1]$ and $S \subseteq [t+1]$, let $p_i \in \mathbb{R}^{2^{t+1}}$ be such that $p_i(S) = 1$ if S contains i, and $p_i(S) = 0$ otherwise. It's easy to check that $p_i \in \mathcal{P}(P_t)$. Let p^* be the average of the p_i's, i.e., $p^* = \frac{1}{t+1} \sum_{i\in[t+1]} p_i$. In particular, this means that $p^*(\{i\}) = \frac{1}{t+1}$ for any $i \in [t+1]$, and $p^*(\{i, i+1\}) = \frac{2}{t+1}$ for any $i \in [t]$. Since p^* is a convex combination of the p_i's, $p^* \in \mathcal{P}(P_t)$. For any homomorphism φ from $P_0^2 P_{t+2}^t$ to P_t,

$$\begin{aligned}\sum_{S \subseteq \mathrm{MaxCliques}(P_0^2 P_{t+2}^t)} & -(-1)^{|S|} p^*(\varphi(\cap S)) \\ &= t \cdot (t+2)\frac{2}{t+1} - t \cdot (t+1)\frac{1}{t+1} + 2\frac{1}{t+1} = t+2,\end{aligned}$$

which implies that the optimal value of the linear program is at most $t+2$.

We now show that $\mathrm{HDE}(P_0^2 P_{t+2}^t; P_t) \geq t+2$. For $1 \leq i \leq t$ let φ_i be the homomorphism from P_{t+2} to P_t such that $\varphi_i(j) = j$ for all $j \leq i+1$, and $\varphi_i(j) = j-2$ for all $j \geq i+2$. In other words, every edge of P_t is visited by P_{t+2} once, except for $\{i, i+1\}$ which is visited three times. Let ψ be the homomorphism from $P_0^2 P_{t+2}^t$ to P_t such that one copy of P_0 gets sent to vertex 1 in P_t, the other copy of P_0 is sent to vertex $t+1$ of P_t (i.e., the two copies of P_0 are sent to the end vertices of P_t), and the i-th copy of P_{t+2} is mapped to P_t via φ_i for $1 \leq i \leq t$.

Now for any $p \in \mathcal{P}(P_t)$, we compute

$$\sum_{S \subseteq \mathrm{MaxCliques}(P_0^2 P_{t+2}^t)} -(-1)^{|S|} p(\psi(\cap S)).$$

Observe that only sets S of size one or two contribute in the above sum since no three maximal cliques of $P_0^2 P_{t+2}^t$ intersect. Every edge of P_t is covered by an image of an edge of P_{t+2} via ψ exactly $t+2$ times. Every inner (non-end) vertex of P_t is covered by an image of an inner (non-end) vertex of P_{t+2} via ψ exactly $t+2$ times. Note that each inner vertex of some copy of P_{t+2} is the intersection of two maximal cliques (i.e., edges) of P_{t+2}, and thus the coefficient will be negative. Finally, the end vertices of P_t are covered by an image of an inner (non-end) vertex of P_{t+2} via ψ exactly once each (which brings again a negative coefficient as it is the intersection of two maximal cliques), as well as once each by one copy of P_0 (which brings a positive coefficient as each P_0 is a maximal clique in itself). Thus the coefficients for the end vertices of P_t are zero. Accordingly we have

$$\begin{aligned}\sum_{S \subseteq \mathrm{MaxCliques}(P_0^2 P_{t+2}^t)} -(-1)^{|S|} p(\psi(\cap S)) &= (t+2)\left(\sum_{\{i,i+1\} \in E(P_t)} p(\{i, i+1\}) - \sum_{i \in \{2,\ldots,t\}} p(\{i\})\right) \\ &= (t+2) p(V(P_t)) \\ &= t+2.\end{aligned}$$

The second line follows from Lemma 2, and the third line follows from $p(V(P_t)) = 1$ since $p \in \mathcal{P}(P_t)$. Therefore, for every $p \in \mathcal{P}(P_t)$, there is an homomorphism that yields $t+2$, so we see that $\mathrm{HDE}(P_0^2 P_{t+2}^t; P_t) \geq t+2$. This proves that $\mathrm{HDE}(P_0^2 P_{t+2}^t; P_t) = t+2$, and therefore the conjecture holds. □

Corollary 1 *We also have that* $t(P_k; G^n)^t \geq t(P_t; G^n)^k$ *holds.*

4 Proof in the Language of Entropy

We now rewrite the proof of the preceding section in the language of entropy.

Given a discrete random variable X taking values in S, its *entropy* is $H(X) := \sum_{s \in S} -\mathbb{P}(X = s) \log_2 \mathbb{P}(X = s)$. One can think of entropy as recording the amount of surprise in the possible outcomes of X. Given jointly distributed random variables X and Y, the *conditional entropy*

$$H(X|Y) := \sum_{y \in \text{support}(Y)} \mathbb{P}(Y = y) \sum_{x \in \text{support}(X)} -\mathbb{P}(X = x|Y = y) \log_2 \mathbb{P}(X = x|Y = y).$$

One can think of conditional entropy as recording the amount of additional surprise in X not contained in Y. Here are a few well-known properties of entropy, some of which we will use in the proof.

1. We have that $H(X) \leq \log_2 |\text{support}(X)|$, and equality holds if and only if X is uniformly distributed.
2. If X and Y are independent random variables, then $H(X, Y) = H(X) + H(Y)$.
3. The chain rule states that $H(X, Y) = H(X) + H(Y|X)$.
4. For jointly distributed random variables $X_1, X_2, \ldots, X_l$, we have that $H(X_1, \ldots, X_l) \leq H(X_1) + \ldots + H(X_l)$.
5. We have that $H(X|Y) \leq H(X)$.

Fix an arbitrary graph G with at least one edge, and let $X = (X_0, \ldots, X_t)$ be a uniform random walk of length t in G, that is, a uniform random element of $\text{Hom}(P_t; G)$. By (1), we have that $H(X) = \log_2 |\text{Hom}(P_t; G)|$.

Let ψ be the homomorphism from $P_0^2 P_{t+2}^t$ to P_t defined in the previous section. Consider the pullback of X via ψ. This is the unique distribution Y on $\text{Hom}(P_0^2 P_{t+2}^t; G)$ whose marginals satisfy $Y_u = X_{\psi(u)}$ and $(Y_v, Y_w) = (X_{\psi(v)}, X_{\psi(w)})$ for each vertex u and edge $\{v, w\}$ in $P_0^2 P_{t+2}^t$ and where Y is a Markov random field over $P_0^2 P_{t+2}^t$. This means that the marginals Y_I and Y_J are conditionally independent over Y_K for all $I, J, K \subseteq V(P_0^2 P_{t+2}^t)$ such that K separates I and J.

It follows from the choice of ψ and the definition of Y that the entropy of Y is given by

$$H(Y) = H(X_0) + H(X_t) + \sum_{i=1}^{t} \left(H(X_{i-1}, X_i) + H(X_0, \ldots, X_{i-1}|X_i) + H(X_i, \ldots, X_t|X_{i-1})\right).$$

Rearranging the terms, we get

$$H(Y) = (H(X_0) + H(X_1, \ldots, X_t|X_0)) + (H(X_0, X_1) + H(X_2, \ldots, X_t|X_1))$$

$$+\sum_{i=2}^{t-1}\big(H(X_{i-1},X_i)+H(X_0,\ldots,X_{i-1}|X_i)+H(X_i,\ldots,X_t|X_{i-1})\big)$$
$$+\big(H(X_{t-1},X_t)+H(X_0,\ldots,X_{t-2}|X_{t-1})\big)+\big(H(X_t)+H(X_0,\ldots,X_{t-1}|X_t)\big)$$

By the conditional independence of $(X_0,\ldots,X_{i-1})$ and $(X_{i+1},\ldots,X_t)$ over X_i, we have that each of the $t+2$ sums between parentheses are equal to $H(X_0,\ldots,X_t)$. Thus

$$\log|\operatorname{Hom}(P_0^2P_{t+2}^t;G)|\geq H(Y)=(t+2)H(X)=(t+2)\log|\operatorname{Hom}(P_t;G)|$$

where the first inequality holds by (1). Therefore, $\operatorname{HDE}(P_0P_{t+2}^t;P_t)\geq t+2$. To show that $\operatorname{HDE}(P_0P_{t+2}^t;P_t)\leq t+2$, one can use a construction corresponding to p^* in the previous section: let G^* be a $(t+1)$-partite graph where parts 1 through $t+1$ all have the same number of vertices, say n, and where every vertex in part i forms an edge with every vertex in part $i+1$ for $1\leq i\leq t$. Then $\hom(P_0P_{t+2}^t;G^*)=O(n^{2+t(t+3)})=O(n^{t^2+3t+2})$ and $\hom(P_t;G^*)=O(n^{t+1})$. Thus, as $n\to\infty$, we see that $\operatorname{HDE}(P_0P_{t+2}^t;P_t)\leq\frac{t^2+3t+2}{t+1}=t+2$. More simply, one could simply take G^* to be K_n or $K_{n,n}$ as $n\to\infty$.

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Declarations

Conflict of interest The authors have not disclosed any competing interests.

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