A Terminal Set Feasibility Governor for Linear Model Predictive Control

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Abstract—**The Feasibility Governor (FG) is an add-on unit for Model Predictive Controllers (MPC) that increases the closed-loop region of attraction by manipulating the applied reference to ensure the underlying optimal control problem is always feasible. The FG requires an estimate of the feasible set of the optimal control problem that underlies the MPC; obtaining this estimate can be computationally intractable for high-dimensional systems. This paper proposes a modified FG that bypasses the need for an explicit estimate, instead relying entirely on the MPC terminal set. The proposed FG formulation is proven to be asymptotically stable, exhibits zero-offset tracking, satisfies constraints, and achieves finite-time convergence of the reference. Numerical comparisons featuring an MPC with a long prediction horizon show that the FG+MPC system can achieve comparable closed-loop performance to long-horizon MPC at a significantly reduced computational cost by suitably detuning the terminal controller to enlarge the terminal set.**

I. Introduction

Model Predictive Control [1], [2] (MPC) is a high-performance control strategy that defines a feedback policy by solving a receding horizon optimal control problem (OCP) at every sampling instant. MPC is used widely in applications since it can efficiently handle input and state constraints and is supported by a robust theoretical literature. Due to the numerous methods for designing MPC controllers, stability guarantees are useful to ensure that the closed-loop system will behave as desired. A typical method for ensuring stability and constraint satisfaction is to incorporate terminal elements into the OCP, e.g. an invariant terminal set and a terminal cost [3].

Often, MPC controllers require the ability to track piecewise-constant references [4] and to transition between them. Although this is a simple modification to the OCP, if the reference command being tracked has a large change in value, the OCP may become infeasible since the terminal set is no longer reachable within the desired prediction horizon. While simply increasing the prediction horizon is an intuitive method for avoiding this problem, it is not always applicable since it increases the computational complexity of the underlying OCP. This can result in significantly longer computation times that may even exceed the real-time requirements.

Another technique for avoiding infeasibility is to treat aspects of the terminal set as optimization variables inside the OCP; these additional degrees of freedom are used to enlarge the terminal set. This method has been applied to various tracking problems [4], [5] as well as economic operation of nonlinear systems with terminal state constraints [6]. A different approach is employed in [7], where a contractive sequence of terminal sets is computed offline. This sequence is then incorporated into the OCP in order to increase the region of attraction (ROA) of the MPC controller. A drawback of all these methods is that they require modifications to the OCP, and are therefore hard to implement on "off-the-shelf" MPC toolkits.

An alternative approach is to modify the reference in such a way to ensure that the OCP always remains feasible. The dual-mode controller in [8] incorporates a recovery mode that simultaneously computes a modified reference and control input. Although this method converges in finite-time, it may reduce performance. Ad hoc methods such as rate limiting the reference are commonly used but are suboptimal, and do not come with theoretical guarantees. A governor-like algorithm in [9] uses ellipsoidal terminal sets and a specific reference parameterization in order to avoid infeasibility. A different governor-like algorithm is proposed in [10] and is used as an intermediate design stage for a explicit MPC. Unfortunately, all these methods also require modifications to the OCP.

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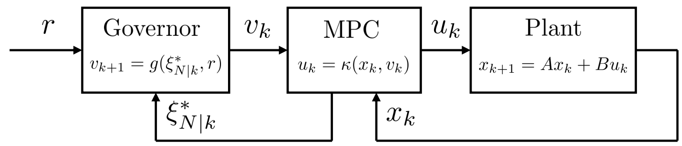

Fig. 1. A block diagram of the closed loop FG+MPC architecture. Given a desired reference r, the FG uses a predicted optimal state ξ^*_N from the MPC to modify the auxiliary reference v ensuring the MPC can produce a valid control input u.

The feasibility governor [11], is an add-on unit inspired by reference/command governors [12], [13] that guarantees safe transitions between set-points while avoiding infeasibility of the OCP. As shown in Figure 1, this is achieved by filtering the applied reference as opposed to re-designing the MPC. This makes the FG an attractive solution for practitioners interested in a modular framework. Notably, the FG add-on unit expands the region of attraction of the underlying MPC to the set of initial conditions that can reach the terminal set of *any* steady state admissible reference. This is accomplished by solving an optimization problem that uses information about the feasible set (also known as the N-step backwards reachable set) of the MPC. While this method is modular and avoids infeasibility of the OCP, computing the feasible set can be challenging.

Using explicit feasible sets is an intuitive approach for constructing a FG but comes with a number of downsides from an implementation perspective. Typically, feasible sets are either ellipsoidal or polyhedral. While, in general, ellipsoidal sets can be easier to compute, they are typically conservative and lead to quadratic constraints when implemented. Polyhedral feasible sets are relatively easy to compute by orthogonal projections [14], and several toolboxes are readily available to compute them offline [15], [16]. The downside is that projection methods suffer from the curse of dimensionality, and inner-approximations [17], [18] are usually required even for moderately sized systems.

In this paper, we propose an updated version of the feasibility governor that does not require an explicit estimate of the feasible set while still maintaining the same safety, asymptotic stability, and finite-time convergence properties of the original FG. This is done by using an implicit representation of the feasible set. Moreover, we discuss a simple heuristic for increasing the size of the terminal set to recover some of the performance of long prediction horizon MPC.

Notation: For vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $(x, y) = [x^T \;\; y^T]^T \in \mathbb{R}^{n+m}$. Consider the set $\Gamma \subseteq \mathbb{R}^{n+m}$, the slice (or cross-section) operation is $S_y(\Gamma, x) = \{y \mid (x,y) \in \Gamma\}$. For $x \in \mathbb{R}^n$, and $\delta \geq 0$, $\mathcal{B}_\alpha(x) = \{y \mid \|y - x\| \leq \alpha\}$. The identity and zero matrices are denoted $I_N \in \mathbb{R}^{N\times N}$ and $0_{N\times M} \in \mathbb{R}^{N\times M}$, respectively with the subscripts omitted whenever the dimensions are clear from context. Given $M \in \mathbb{R}^{m\times n}$, $\text{Ker } M = \{x \in \mathbb{R}^n \mid Mx = 0\}$. Given $\mathcal{U} \subseteq \mathbb{R}^n$, $\text{Int } \mathcal{U}$ denotes the interior of $\mathcal{U}$. Positive (semi) definiteness of a matrix $P \in \mathbb{R}^{n\times n}$ is denoted by $(P \succeq 0)$ $P \succ 0$ and the induced norm is $\|x\|_P = \sqrt{x^TPx}$ for $x \in \mathbb{R}^n$. The natural numbers (without zero) are denoted $\mathbb{N}$ ($\mathbb{N}_+$). Our use of comparison functions, i.e., class $\mathcal{K}, \mathcal{K}_\infty$ and $\mathcal{KL}$ functions, follows [19].

II. Problem Setting

Similarly to [11], in this paper we address a Linear Time Invariant (LTI) system in the form

$$x_{k+1} = Ax_k + Bu_k \tag{1a}$$
$$y_k = Cx_k + Du_k \tag{1b}$$
$$z_k = Ex_k + Fu_k, \tag{1c}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$, and $z_k \in \mathbb{R}^{n_z}$ are the states, control inputs, constrained outputs, and tracking outputs, respectively, and $k \in \mathbb{N}$ is the discrete-time index.

The following assumptions ensure that the system (1) admits a well-posed tracking problem.

Assumption 1. *The pair (A, B) is stabilizable.*

System (1) is subject to the pointwise-in-time output constraints

$$\forall k \in \mathbb{N} \quad y_k \in \mathcal{Y}, \tag{2}$$

with $\mathcal{Y} \subseteq \mathbb{R}^{n_y}$.

Assumption 2. *The set $\mathcal{Y}$ is a compact polyhedron with representation $\mathcal{Y} = \{y \mid Yy \leq h\}$, and satisfies $0 \in \text{Int } \mathcal{Y}$.*

In order to track a target reference $r \in \mathbb{R}^{n_z}$, the steady-state of (1) can be characterized as follows

$$\begin{bmatrix} A - I & B & 0 \\ E & F & -I \end{bmatrix} \begin{bmatrix} \bar{x}_v \\ \bar{u}_v \\ \bar{z}_v \end{bmatrix} = Z \begin{bmatrix} \bar{x}_v \\ \bar{u}_v \\ \bar{z}_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

Assumption 1 is necessary and sufficient (see e.g. [4]) to ensure that $\text{Ker } Z \neq \emptyset$. This property allows us to introduce an auxiliary reference $v \in \mathbb{R}^{n_z}$ that parameterizes the equilibrium manifold. Every solution to (3) can be written as

$$\bar{x}_v = G_x v, \quad \bar{u}_v = G_u v, \quad \bar{z}_v = G_z v, \tag{4}$$

where $G^T \equiv \begin{bmatrix} G_x^T & G_u^T & G_z^T \end{bmatrix}$ is a basis for $\text{Ker } Z$. Using the auxiliary reference v instead of r allows us to handle over/under parameterization of the equilibrium manifold. The following assumption excludes pathological cases.

Assumption 3. *The matrix G_z is full rank.*

When G_z is invertible, it is simple to recover $v = r$ using a change of basis that maps $G_z \to I$. In the case that G_z is not invertible, $G_z v$ is either an under or over parameterization of the reference r. In either case, the inverse can be replaced with the pseudo inverse to either minimize the error $\|G_z v - r\|^2$, or minimize the norm $\|v\|^2$ while satisfying $r = G_z v$.

Since MPC cannot stabilize points on the boundary of the feasible set, we introduce a design parameter $\epsilon \in (0, 1)$ and define the set of strictly steady-state admissible auxiliary references as

$$\mathcal{V}_\epsilon \equiv \{v \mid G_y v \in (1 - \epsilon)\mathcal{Y}\}, \tag{5}$$

where $G_y = CG_x + DG_u$. The set of strictly admissible target references is then

$$\mathcal{R}_\epsilon \equiv G_z \mathcal{V}_\epsilon = \{G_z v \mid v \in \mathcal{V}_\epsilon\}. \tag{6}$$

Finally, we define the strictly steady-state admissible equilibria as

$$\Sigma \equiv \{(x, v) \mid x = G_x v, \ v \in \mathcal{V}_\epsilon\}. \tag{7}$$

Given Assumptions 1–2 as the only limitations to our problem setting, we now state the control objectives for this paper.

Control Objectives: Given the LTI system (1), let $\mathcal{Y} \subseteq \mathbb{R}^{n_y}$ be a constraint set, and let $r \in \mathbb{R}^{n_z}$ be a target reference. The goal of this paper is to design a full state feedback law that achieves the following objectives:

- *Safety:* Ensure that $y_k \in \mathcal{Y} \quad \forall k \geq 0$;
- *Asymptotic Stability:* $\lim_{k\to\infty}(x_k, v_k) = (x_r^*, v_r^*)$ where $(x_r^*, v_r^*) = (G_x v_r^*, v_r^*)$ is a stable equilibrium point satisfying $\lim_{k\to\infty} z_k = r^* = G_z v_r^*$ with

$$r^* = \underset{s \in \mathcal{R}_\epsilon}{\arg\min} \ \|s - r\|.$$

Remark 1. *The proposed formulation handles steady-state inadmissible targets r by steering the system to the closest strictly steady-state admissible target reference r^*. When the tracking problem is well posed, i.e., when $r \in \mathcal{R}_\epsilon$, we recover the more intuitive objective* $\lim_{k\to\infty} z_k = r$.

III. Control Strategy

We approach the control objective using a classic MPC formulation. Notably, the feedback is obtained by solving the following parameterized optimal control problem (POCP)

$$\min_{\xi,\mu} \ ||\xi_N - \bar{x}_v||_P^2 + \sum_{i=0}^{N-1} ||\xi_i - \bar{x}_v||_Q^2 + ||\mu_i - \bar{u}_v||_R^2 \tag{8a}$$
$$\text{s.t.} \ \ \xi_0 = x, \tag{8b}$$
$$\xi_{i+1} = A\xi_i + B\mu_i, \quad i = 1, \ldots, N-1, \tag{8c}$$
$$C\xi_i + D\mu_i \in \mathcal{Y}, \quad i = 1, \ldots, N-1, \tag{8d}$$
$$(\xi_N, v) \in \mathcal{T}, \tag{8e}$$

where $N \in \mathbb{N}_+$ is the prediction horizon, $\mu = (\mu_0, \ldots \mu_{N-1})$ are the decision variables, $P, Q \in \mathbb{R}^{n_x\times n_x}$, and $R \in \mathbb{R}^{n_u\times n_u}$ are the cost matrices, and $\mathcal{T} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_v}$ is a polyhedral terminal set

$$\mathcal{T} = \{(x, v) \mid T_x x + T_v v \leq c\}. \tag{9}$$

The POCP (8) contains reference tracking objectives by letting $Q = E^TE$ (see e.g. [2, Section 2.4.4]). To ensure that (8) is a well-posed MPC problem, the following assumptions are required.

Assumption 4. *The stage cost matrices satisfy $Q = Q^T \succeq 0$, with (A, Q) detectable, and $R = R^T \succ 0$.*

Given the stage cost matrices, the terminal cost P and terminal set $\mathcal{T}$ can be designed starting from the terminal control law

$$\kappa_N(x, v) \equiv \bar{u}_v - K(x - \bar{x}_v). \tag{10}$$

Assumption 5. *The terminal set $\mathcal{T}$ is constraint admissible, positively invariant under* (10)*, and satisfies $S_x(\Sigma, v) \subset \text{Int } S_x(\mathcal{T}, v)$, $\forall v \in \mathcal{V}_\epsilon$. Moreover, given $(x, v) \in \mathcal{T}$ and the gain K, the terminal cost satisfies $P = P^T \succeq 0$ and the LMI*

$$(A - BK)^T P(A - BK) - P + K^TRK \preceq -Q. \tag{11}$$

Clearly, the most straightforward (and most used) choice for the terminal cost P and terminal feedback gain K is the LQR solution

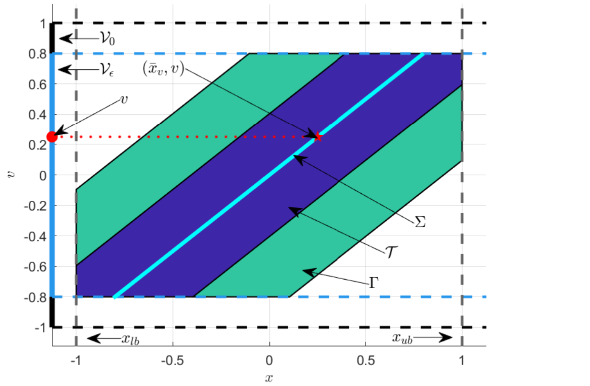

Fig. 2. The sets defined in this paper for the integrator $x_{k+1} = x_k + u_k$ subject to $|x_k| \leq 1$, $|u_k| \leq 0.25$ and with $\epsilon = 0.2$, $\mathcal{T} = \tilde{O}_\infty^{0.2}$ and $N = 2$.

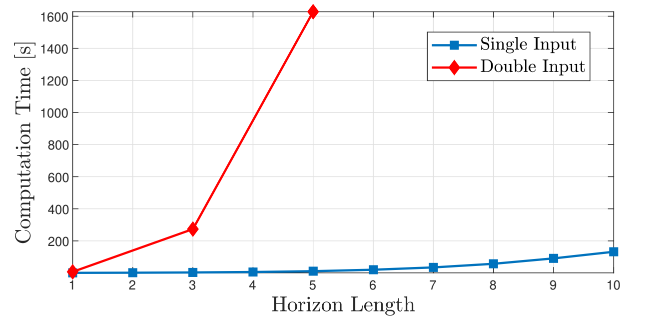

Fig. 3. Feasible set computation time for the single and double input models used in this paper. We were unable to compute the feasible set for the double input (a low dimensional example) for $N > 5$ in a reasonable amount of time.

$P = A^T P A - (A^T P B)(R + B^T P B)^{-1}(B^T P A) + Q$ and $K = (R + B^T P B)^{-1} B^T P A$. Another choice is detailed in Remark 3, which provides a heuristic for increasing performance.

As for the terminal set, a suitable choice is $\mathcal{T} = O_\infty$, where O_∞ is the maximal output admissible set [20] of the terminal closed-loop system $x_{k+1} = (A - BK)x_k + B(G_u + KG_x)v$. Given the previously defined parameter $\epsilon \in (0, 1)$, another choice is $\mathcal{T} = \tilde{O}_\infty^\epsilon$, where $\tilde{O}_\infty^\epsilon \subset O_\infty$ is a positively invariant inner approximation of O_∞, which can be computed efficiently using the algorithm detailed in [20, Sec. III] and implemented in [15]. Figure 2 illustrates the sets defined in this paper.

The MPC control action can be computed if and only if (8) admits a solution. The set of all parameters for which the POCP admits a solution, i.e., the feasible set, is

$$\Gamma_N = \{(x, v) \mid \exists\, \mu : \text{ (8b)} - \text{(8e)}\} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_v}, \qquad (12)$$

which is the N-step backwards reachable set of $\mathcal{T}$. Given $(x, v) \in \Gamma_N$, it is possible to compute the MPC feedback policy

$$\kappa(x, v) \equiv \mu_0^*(x, v) \qquad (13)$$

where $\mu_0^*(x, v)$ selects the value of μ_0^* from the minimizer $\zeta^*(x, v) = (\mu_0^*, \ldots, \mu_{N-1}^*, x, \xi_1^*, \ldots, \xi_N^*)$ of (8).

Given $(x, v) \notin \Gamma_N$, the POCP becomes infeasible because x_0 cannot be steered to $S_x(\Sigma, v_0)$ within N steps. As discussed in the introduction, increasing N is a potential solution, but the computation time needed to solve (8) scales unfavorably with N and may simply exceed the desired sampling time.

This work follows from [11] which introduced the feasibility governor (FG), an *add-on* unit that expands the region of attraction of the closed-loop without increasing the prediction horizon or modifying the OCP. The philosophy behind the FG is to modify the reference so that the MPC problem remains feasible. This concept follows from command governor (CG) literature [13] where it is assumed the inner-loop controller is well designed and we do not wish to modify it. The action of the FG in can be written as the following optimization problem.

$$g(x, r) \equiv \underset{v \in \mathcal{V}_\epsilon}{\arg\min} \ \ \{\psi(v, r) \mid (x, v) \in \Gamma_N\} \qquad (14)$$

where

$$\psi(v, r) \equiv \begin{cases} \|G_z v - r\|_L^2 & \text{if } G_z \text{ is injective} \\ \|v - v_r^*\|_L^2 & \text{otherwise,} \end{cases} \qquad (15)$$

and the designer can select any v_r^* satisfying

$$v_r^* \in \mathcal{V}_r^* \equiv \underset{v \in \mathcal{V}_\epsilon}{\arg\min} \ \ \|G_z v - r\|_L^2, \qquad (16)$$

where $L \succ 0$ and the additional step in (16) is necessary to ensure that ψ is a strongly convex function.

Although this formulation achieves the desired control objectives and is backed by rigorous proofs, it comes with the drawback of requiring an *explicit* representation of the feasible set. Indeed, as shown in Figure 3, computing Γ_N can quickly become intractable when dealing with systems that have a high dimensional auxiliary reference space. One possible way to alleviate this issue is to under-approximate the feasible set using a safe and strongly returnable set [11]. This approach helps manage the costly polyhedral projections required for computing Γ_N, but the approximations may be more conservative while still being difficult to find. Moreover, the under approximated sets make this method a more conservative approach since the sets are only guaranteed to be strongly returnable and safe. In the following section, we introduce a new feasibility governor that requires no additional set computations beyond those used in traditional MPC. This formulation of the FG achieves the desired control objectives of asymptotic stability, safety/recursive feasibility, and finite-time convergence of the auxiliary reference.

IV. Terminal Set Feasibility Governor

In this section, we propose a new feasibility governor that relies entirely on the terminal set $\mathcal{T}$, which is present in many MPC formulations. The new feasibility governor consists of a slight modification to the original FG formulation (14), i.e.

$$g(\xi_N^*, r) \equiv \underset{v \in \mathcal{V}_\epsilon}{\arg\min} \ \ \{\psi(v, r) \mid (\xi_N^*, v) \in \mathcal{T}\}. \qquad (17)$$

However, proving that (17) achieves the control objectives is challenging and requires significant changes to the theoretical analysis of the original FG [11], as detailed in the following section.

In practice, given the state predicted at the previous timestep $\xi_{N|k-1}^* \equiv \xi_N^*(x_{k-1}, v_{k-1})$, the governor computes the virtual reference $v_k = g(\xi_{N|k-1}^*, r)$, which is then passed to the MPC controller to obtain a control action $u_k = \kappa(x_k, v_k)$. We present Figure 4 to give the reader some intuition behind how the FG operates. Assuming that the pair (x_k, v_k) is feasible, the blue line represents the optimal predicted trajectory at time k. The feasibility governor then determines v_{k+1} such that $\xi_{N|k}^*$ is inside the slice of the terminal set at v_{k+1}. We then use a one-step shift to construct a feasible trajectory (red) at time $k+1$, thus ensuring that the POCP admits at least one solution.

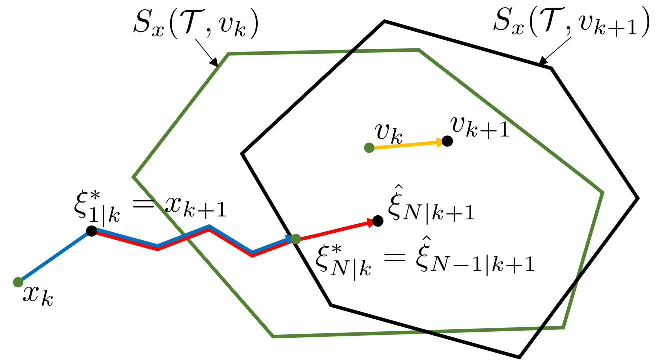

Fig. 4. Illustration of how the feasibility governor assigns the new auxiliary reference v_{k+1} based on the current optimal trajectory (blue) and why, at the next timestep, x_{k+1} is guaranteed to admit a feasible trajectory (red).

While we claim that the FG requires no modification to the closed loop MPC, it does require one additional output signal from the MPC controller. Specifically, it requires $\xi^*_{N|k}$, which is the current prediction of the state of the system at the end of the prediction horizon. This additional signal is readily available as it is generated by the MPC controller at every time step, either explicitly or implicitly, depending on the specific solver being used.

Remark 2. *In the presence of disturbances, the proposed approach can be generalized by a) replacing* (8) *with an MPC that is robustly recursively feasible (e.g., [21], [22]), and b) designing the terminal set $\mathcal{T}$ to be robustly positively invariant [22, Definition 3.2]. Doing so enables the FG to be used without any additional modifications.*

V. Theoretical Analysis

In this section, we analyze the closed-loop properties of the combined FG and MPC. The target reference is assumed constant throughout this section, which allows us to suppress any dependence on r to simply the notation. Nevertheless, the results are easily extended to piecewise-constant references.

First, the feasible set of the FG is defined as

$$\Lambda = \Gamma_N \cap (\mathbb{R}^{n_x} \times \mathcal{V}_\epsilon). \tag{18}$$

The closed loop dynamics of (1) under the combined FG and MPC policy are

$$v_{k+1} = g(\xi^*_N(x_k, v_k)), \tag{19a}$$
$$x_{k+1} = f(x_k, v_k), \tag{19b}$$
$$y_k = Cx_k + D\kappa(x_k, v_k), \tag{19c}$$

where $f(x, v) = Ax + B\kappa(x, v)$. We begin by showing the set Λ is forward invariant and that it satisfies the *Safety* control objective.

Theorem 1. *(Safety and Recursive Feasibility). Let Assumptions 1–5 hold, given $(x_k, v_k) \in \Lambda$ the solution to* (19a) *exists and is such that* (19b) *satisfies $(x_{k+1}, v_{k+1}) \in \Lambda$. Moreover, $y_k \in \mathcal{Y}, \forall k \in \mathbb{N}$.*

Proof. To show that the FG+MPC is recursively feasible, we need to show that given $(x_k, v_k) \in \Lambda$ it is possible to compute a solution at time $k+1$. By the definition of Λ, we know the optimal solution to (8) at time k exists, is unique by Assumption 4, and can be written as

$$\mathcal{U}^*_k = \left\{\mu^*_{0|k}, \dots, \mu^*_{N-1|k}\right\}, \tag{20a}$$
$$\mathcal{X}^*_k = \left\{x_k, \xi^*_{1|k}, \dots, \xi^*_{N|k}\right\}. \tag{20b}$$

Moreover, since $(x_k, v_k) \in \Lambda$ implies $(\xi^*_{N|k}, v_k) \in \mathcal{T}$, it is obvious that $v_{k+1} = v_k$ is a feasible solution to (17), thus proving that (19a) admits a solution. Given $(\xi^*_{N|k}, v_{k+1}) \in \mathcal{T}$, we proceed by constructing the following sequences for time $k+1$

$$\mathcal{U}_{k+1} = \left\{\mu^*_{1|k}, \dots, \mu^*_{N-1|k}, \kappa_N(\xi^*_{N|k}, v_{k+1})\right\},$$
$$\mathcal{X}_{k+1} = \left\{\xi^*_{1|k}, \dots, \xi^*_{N|k}, A\xi^*_{N|k} + B\kappa_N(\xi^*_{N|k}, v_{k+1})\right\},$$

where $\kappa_N(\xi^*_N, v_{k+1})$ is the terminal control law in (10). Since the terminal set is invariant under the terminal control law, the sequences $\mathcal{U}_{k+1}, \mathcal{X}_{k+1}$ are feasible solutions to (8) at time $k+1$. Since (19b) entails $x_{k+1} = \xi^*_{1|k}$, we show that $(x_{k+1}, v_{k+1}) \in \Lambda$.

Finally, since $(x_k, v_k) \in \Lambda,\ \forall k \in \mathbb{N}$, the MPC feedback policy $\kappa(x, v)$ is such that $y_k \in \mathcal{Y},\ \forall k \in \mathbb{N}$. □

The following well-known result proves Input-to-State Stability (ISS) for the nominal MPC with a varying v. This paper leverages a direct consequence of the ISS property.

Theorem 2. *([11, Lemma 5]) Given Assumptions 1–5, and the system $x_{k+1} = f(x_k, v_k)$, the error signal $e_k = x_k - G_x v_k$ is ISS [23] with respect to the input $\Delta v_k = v_{k+1} - v_k$, i.e., there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that*

$$\|x_k - G_x v_k\|_Q \leq \beta(k, \|x_0 - G_x v_0\|) + \gamma\left(\sup_{j \geq 0} \|\Delta v_j\|\right).$$

Moreover, γ is an asymptotic gain, i.e.,

$$\limsup_{k\to\infty} \|x_k - G_x v_k\|_Q \leq \gamma\left(\limsup_{k\to\infty} \|\Delta v_k\|\right). \tag{21}$$

Corollary 1. *Let Assumptions 1–5 hold, then it is also true that*

$$\limsup_{k\to\infty} \|\xi^*_{N|k} - G_x v_k\|_P \leq \bar{\gamma}\left(\limsup_{k\to\infty} \|\Delta v_k\|\right) \tag{22}$$

where $\bar{\gamma} = \alpha_u \circ \gamma \in \mathcal{K}$.

Proof. Theorem 2 implies the existance of an ISS Lyapunov function $J : \Gamma_N \to \mathbb{R}$ for the system $x_{k+1} = f(x_k, v_k)$, that satisfies

$$J(f(x, v), v) - J(x, v) \leq -\alpha(\|x - G_x v\|_Q), \tag{23}$$
$$\alpha_l(\|x - G_x v\|_Q) \leq J(x, v) \leq \alpha_u(\|x - G_x v\|_Q) \tag{24}$$

for all $(x, v) \in \Gamma_N$ where J is the cost function (8a) evaluated at the optimal solution $\zeta^*(x, v)$, and $\alpha, \alpha_l, \alpha_u \in \mathcal{K}$.

Let $W : \mathcal{T} \to \mathbb{R}$ be defined as $W(\xi^*_N, v) = \|\xi^*_N - G_x v\|^2_P$. Since W is the terminal cost in J, and the remaining terms are positive (semi-)definite, we can bound W as

$$W(\xi^*_N(x, v), v) \leq J(x, v) \leq \alpha_u(\|x - G_x v\|_Q) \tag{25}$$

for all $(x, v) \in \Lambda$. Moreover, since J is a ISS-Lyapunov function we know that

$$\limsup_{k\to\infty}\ J(x_k, v_k) \leq \alpha_u \circ \gamma\left(\limsup_{k\to\infty} \|\Delta v_k\|\right). \tag{26}$$

Combining (25) and (26) leads to the result. □

The following sequence of four Lemmas build up to the result that the auxiliary reference produced by the FG necessarily converges in finite time. We begin by showing in the next two Lemmas that if the FG has not yet converged, the rate of change of the auxiliary reference can be infinitesimal for only a finite amount of time.

Lemma 1. *Under Assumptions 1–5 define,*

$$mcS_\delta(\Sigma) \equiv \{(x, v) \mid v \in \mathcal{V}_\epsilon,\ \|x - G_x v\|_P \leq \delta\}, \tag{27}$$

where $\Sigma = \mathcal{S}_0(\Sigma) = \{(x, v) \mid x = G_x v,\ v \in \mathcal{V}_\epsilon\}$. *Then, given* $\delta > 0$ *satisfying* $\mathcal{S}_\delta(\Sigma) \subset \text{Int}\ \mathcal{T}$, $\exists \alpha > 0$ *such that*

$$\begin{cases} \|v_{k+1} - v_k\| \geq \alpha & \text{if } \|v_k - v^*\| > \alpha, \\ v_{k+1} = v^* & \text{if } \|v_k - v^*\| \leq \alpha, \end{cases} \tag{28}$$

for any $(\xi^*_{N|k}, v_k) \in \mathcal{S}_\delta(\Sigma)$.

Proof. We approach this proof by constructing a point $v'(t)$ such that $\psi(v'(t)) < \psi(v)$ for each case. Since $\mathcal{S}_\delta(\Sigma) \subset \text{Int}\ \mathcal{T}$ for any $(\xi^*_N, v) \in \mathcal{S}_\delta(\Sigma)$, $\exists \alpha = \alpha(\delta) > 0$ such that $\mathcal{B}_\alpha(v) \subseteq S_v(\mathcal{T}, \xi^*_N)$ for all $v \in \mathcal{V}_\epsilon$.

Fix α and define the set $\mathcal{C}_\alpha = \mathcal{V}_\epsilon \cap \mathcal{B}_\alpha(v)$ and the ray $v'(t) = v + t(v^* - v)$ for $t \in [0, 1]$ and assume $v \neq v^*$. To show that ψ decreases along $v'(t)$ recall that ψ is a convex function, therefore

$$\psi(v'(t)) = \psi((1-t)v + tv^*) \leq \psi(v) - t[\psi(v) - \psi(v^*)]$$

for all $v \in \mathcal{V}_\epsilon \setminus v^*$ and $t \in [0, 1]$.

Given $\|v_k - v^*\| > \alpha$, choose $\rho = \frac{\alpha}{\|v_k - v^*\|} < 1$, and it is clear that $v'(\rho) \in \mathcal{C}_\alpha$. Additionally, since $\psi(v^*) < \psi(v)$ for all $v \in \mathcal{V}_\epsilon \setminus v^*$ and $\rho \in (0, 1)$ we have from the convexity of $\mathcal{V}_\epsilon$ that $\psi(v'(\rho)) < \psi(v)$. This implies

$$\psi(g(\xi^*_N)) = \min_{s \in S_v(\mathcal{T}, \xi^*_N)} \psi(s) \leq \psi(v'(\rho)) < \psi(v). \tag{29}$$

The strong convexity of ψ along with (29) entails $\|v_{k+1} - v_k\| \geq \alpha$.

Given $\|v_k - v^*\| \leq \alpha$ it is clear that $v^* \in \mathcal{C}_\alpha$ This implies $\psi(g(\xi^*_N)) = \psi(v^*) = 0$, which further implies $v_{k+1} = v^*$ by the strong convexity of ψ. □

Lemma 2. *Under Assumptions 1–5, let* $(\xi^*_{N|k_0}, v_{k_0}) \in \mathcal{T}$, *then there always exists a finite time* $k_i \geq k_0$ *such that*

$$\|g(\xi^*_{N|k_i}, v_{k_i}) - v_{k_i}\| \geq \eta \quad \text{or} \quad g(\xi^*_{N|k_i}, v_{k_i}) = v^* \tag{30}$$

with $\eta \in (0, \min(\alpha(\delta), \bar{\gamma}^{-1}(\delta)))$.

Proof. Obviously, if k_0 satisfies either condition then we can choose $k_i = k_0$. To prove the existence of k_i for the nontrivial case we proceed by contradiction.

Assume that $\nexists k_i$ satisfying (30). Then our system must satisfy

$$\|v_{k+1} - v_k\| < \eta < \bar{\gamma}^{-1}(\delta), \quad \forall k > k_0.$$

Following from Corollary 1 we have

$$\limsup_{k \to \infty} \|\xi^*_{N|k} - G_x v_k\|_P < \bar{\gamma}(\eta) < \bar{\gamma} \circ \bar{\gamma}^{-1}(\delta) = \delta.$$

Thus, there exists a finite k_i such that $(\xi^*_{N|k_i}, v_{k_i}) \in \mathcal{S}_\delta(\Sigma)$. It then follows from Lemma 1 and the choice $\eta < \alpha$ that $\|v_{k_i+1} - v_{k_i}\| \geq \eta$, which satisfies (30). □

The remaining two Lemmas show that if there are enough "large" jumps of the auxiliary reference, then it must converge to r.

Lemma 3. *Given Assumptions 1–5, for all* $(x_k, v_k) \in \Lambda$ *there exists* $h > 0$ *such that the FG satisfies*

$$\|v_{k+1} - v^*\|^2 \leq \|v_k - v^*\|^2 - h\|v_k - v_{k+1}\|^2. \tag{31}$$

where $v_{k+1} = g(\xi^*_N(x_k, v_k))$.

Proof. Recall that ψ is strongly convex so there exists $h > 0$ such that

$$\psi(v) \geq \psi(v') + \nabla\psi(v')^T(v - v') + h\|v - v'\|^2 \tag{32}$$

for all $v, v' \in \mathbb{R}^{n_v}$. The optimally conditions associated with $g(\xi^*_N) = \arg\min_{s \in S_v(\mathcal{T}, \xi^*_N)} \psi(s)$ are

$$\nabla\psi(g(\xi^*_N))^T(v - g(\xi^*_N)) \geq 0 \quad \forall v \in S_v(\mathcal{T}, \xi^*_N). \tag{33}$$

Substituting $v' = g(\xi^*_N)$, plugging (33) into (32) and rearranging completes the proof. □

Lemma 4. *Let Assumptions 1–5 hold and define* $M = \text{ceil}\left(\frac{\|v_0 - v^*\|^2}{h\eta^2}\right)$. *If there exists a sequence of* M *time instants* $k_1, k_2, \ldots, k_M$ *such that* (30) *is satisfied at each* k_i, *then* $v_k = v^*,\ \forall k \geq k_M + 1$.

Proof. Clearly, if $g(\xi^*_{N|k_i}, v_{k_i}) = v^*$ for any $i^* < M$, we obtain $g(\xi^*_{N|k_i}, v_{k_i}) = v^*\ \forall i > i^*$. Thus, the worst-case scenario is if $\|g(\xi^*_{N|k_i}, v_{k_i}) - v_{k_i}\| \geq \eta\ \forall i \in \{1, \ldots, M\}$. In this case, it follows from Lemma 3 that

$$\|g(\xi^*_{N|k_M}, v_{k_M}) - v^*\|^2 \leq \|v_0 - v^*\|^2 - Mh\eta^2 \leq 0.$$

Which is sufficient to show $\|v_{k_M+1} - v^*\|^2 = 0$. □

Having assembled all the required components, we prove the following Theorems.

Theorem 3. *(Finite Time Convergence). Given Assumptions 1–5 and* $(\xi^*_{N|0}, v_0) \in \mathcal{T}$, $\exists k^* \geq 0$ *such that* $v_k = v^*\ \forall k \geq k^*$.

Proof. By virtue of Lemma 2, (30) can only be violated for a finite amount of time. Thus, given a sufficiently long wait, there always exists a finite sequence of times instants $k_1, k_2, \ldots, k_M$ that satisfy Lemma 4, which completes the proof. □

Using the finite time convergence of FG, we can finally prove asymptotic stability of the FG+MPC.

Theorem 4. *(Asymptotic Stability) Let Assumptions 1–5 hold. For the coupled FG+MPC, the point* (x^*, v^*) *is asymptotically stable.*

Proof. Theorem 2 shows the closed-loop MPC is ISS with respect to the changing reference v_k. Therefore, to prove asymptotic stability (AS), it is sufficient to show that $\limsup_{k \to \infty} \|\Delta v_k\| = 0$. This follows directly from Theorem 3. □

Remark 3. *While the FG is an effective tool for enforcing feasibility at low computational cost, it is typically less performant than long horizon MPC. A potential method for recovering performance from the FG+MPC is to leverage the degree of freedom enabled by Assumption 5, which does not require* (K, P) *to satisfy the infinite-horizon LQR solution* $K = (R + B^T PB)^{-1} B^T PA$. *Indeed, the terminal gain* K *can be obtained using a "less aggressive" control strategy that increases the size of the terminal set* $\mathcal{T}$. *This can be done without modifying terminal cost* P, *as long as the LMI* (11) *is satisfied. A possible option for obtaining the terminal control gain* K *is to solve a different LQR problem where the input cost matrix is re-scaled using* $R_\rho = \rho R$, *with* $\rho > 1$. *After checking if the pair* (K, P) *satisfies* (11) *the terminal set* $\mathcal{T}$ *can then be computed normally.*

VI. Numerical Examples

The goal of this section is to demonstrate the benefits of the proposed feasibility governor using two examples. The first shows that there is no decrease in performance when compared to the FG in [11] (denoted FG2), which relies on the explicit computation of Γ_N. The second showcases a system where computing Γ_N is intractable for even small prediction horizons. Comparisons with a long-horizon MPC show the significant reduction in computational cost at the expense of closed-loop performance. We then conclude the section by illustrating how to potentially recover most of the performance loss by de-tuning the terminal controller to increase the size of the terminal set.

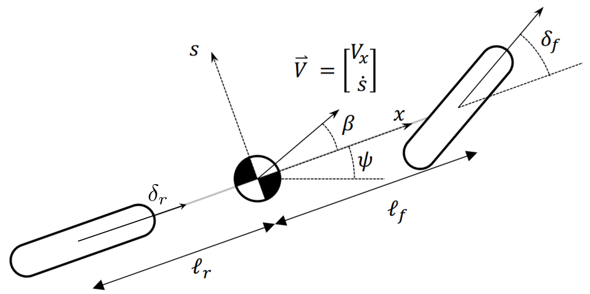

Fig. 5. The bicycle model of the lateral vehicle dynamics.

A. Single Input Model

Consider the lateral dynamics of a car moving forward at a constant speed of $V_x = 30m/s$. The model is based on the one in [24], [25] without rear steering and roughly represents a 2017 BMW 740i sedan and is identical to the example in [11]. Figure 5 contains a diagram of the bicycle model used to represent the system.

The state of the system is $x^T = [s\ \phi\ \beta\ \omega]$ where s is the lateral position of the vehicle, ψ is the yaw angle, $\beta = \dot{s}/V_x$ is the sideslip angle, and $\omega = \dot{\psi}$ is the yaw rate. The control input is the front steering angle $u = \delta_f$. The system is subject to constraints on $y^T = [\alpha_f\ \alpha_r\ \delta_f]$ where α_f and α_r are the front and rear slip angle, and the tracking output is $z = s$. The system matrices are

$$A = \begin{bmatrix} 0 & V_x & V_x & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{2C_\alpha}{mV_x} & \frac{C_\alpha(\ell_r - \ell_f)}{mV_x^2} - 1 \\ 0 & 0 & \frac{C_\alpha(\ell_r - \ell_f)}{I_{zz}} & -\frac{C_\alpha(\ell_r^2 + \ell_f^2)}{I_{zz}V_x} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \frac{C_\alpha}{mV_x} \\ \frac{C_\alpha \ell_f}{I_{zz}} \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & -1 & -\frac{\ell_f}{V_x} \\ 0 & 0 & -1 & \frac{\ell_r}{V_x} \\ 0 & 0 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$E = [1\ 0\ 0\ 0]$, and $F = 0$, where $\ell_f = 1.56\ m$ and $\ell_r = 1.64\ m$ are the moment arms of the front and rear wheels relative to the center of mass, and $C_\alpha = 246994\ N/rad$ is the tire cornering stiffness, $m = 2041\ kg$ is the mass of the vehicle, and $I_{zz} = 4964\ kg \cdot m^2$ is the moment of inertia about the yaw axis. The continuous time system matrices are converted to discrete-time using a zero-order hold with a sampling time of $t_s = 0.01$ seconds. The constraint set is

$$\mathcal{Y} = [-8^\circ,\ 8^\circ] \times [-8^\circ,\ 8^\circ] \times [-30^\circ,\ 30^\circ], \tag{34}$$

which are limits on the front and rear slip angles (to prevent drifting) and mechanical limits on the steering angle. The control horizon for this example is $N = 15$, the initial condition is $x_0 = 0$ and the target position $r = 5\ m$ is chosen such that $x_0 \notin S_x(\Gamma_{15}, r)$. The weighting matrices are $Q = E^T E$, $R = 0.1$, $L = 1$, and the terminal cost and gain are computed using the linear quadratic regulator. Finally, the choice of terminal set is $\mathcal{T} = \tilde{O}_\infty^{0.01}$.

Figure 6 compares the response between FG2 and the new FG presented in this paper. In both cases, the FG+MPC systems produce desirable closed loop responses, and the lack of knowledge of the feasible set in the new FG formulation does not decrease performance.

B. Double Input Model

In this section we address the case where the rear wheel steering angle is used as an additional control input, as detailed in [24], [25]. The state of the system remains the same, but the input (and output) are modified to include $u^T = [\delta_f\ \delta_r]$. Thus, the (B, C, D) system matrices are adjusted to

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{C_\alpha}{mV_x} & \frac{C_\alpha}{mV_x} \\ \frac{C_\alpha \ell_f}{I_{zz}} & \frac{-C_\alpha \ell_r}{I_{zz}} \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & -1 & -\frac{\ell_f}{V_x} \\ 0 & 0 & -1 & \frac{\ell_r}{V_x} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $D = [I_{n_u}\ I_{n_u}]^T$ to include the additional control input. We also use the same sampling time of $t_s = 0.01$ seconds to convert the continuous time system matrices to discrete-time. The updated constraint set is

$$\mathcal{Y} = [-10^\circ,\ 10^\circ]^2 \times [-35^\circ,\ 35^\circ] \times [-6^\circ,\ 6^\circ], \tag{35}$$

and the weighting matrices are $Q = E^T E$, $R = \text{diag}([0.01\ 1])$, and $L = \text{diag}([1\ 10])$.

As discussed in Section III, we were unable to compute Γ_N for this system due to the excessive computational requirements, making the FG2 intractable. Figure 7 shows a comparison between the FG+MPC feedback law with $N = 10$ and an ungoverned MPC with $N = N^* = 67$ where N^* is the shortest prediction horizon length such that $(x_0, r) \in \Gamma_N$. Although the ungoverned MPC clearly outperforms

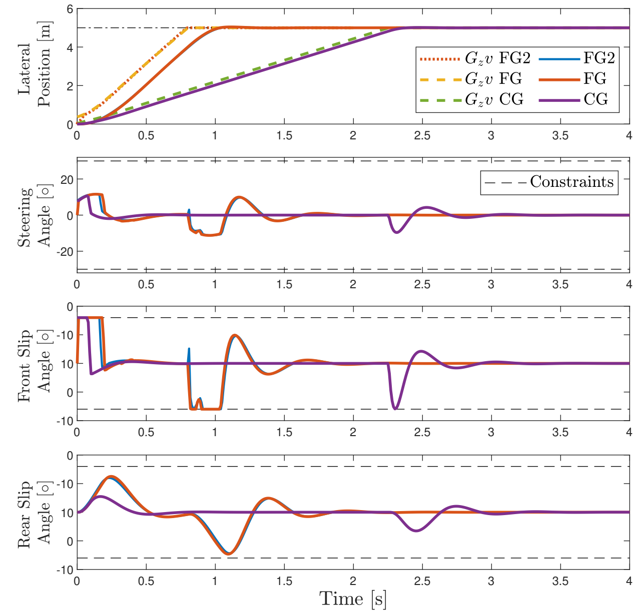

Fig. 6. Comparison between the two different feasibility governors and an LQR plus command governor [13] where the control objective is to track the lateral position command.
FG (orange) is from this paper and FG2 (blue) is the older version. Both perform in a nearly identical manner.

TABLE I
EXECUTION TIME FOR THE LATERAL VEHICLE MODEL WITH THE STANDARD AND MODIFIED TERMINAL SET

	Nominal $\mathcal{T}$				
	MPC (N=67)	FG	MPC (N=10)	FG+MPC (N=10)	$\mathcal{T}$ (Computed Offline)
TAVE [ms]	236	0.098	2.89	2.99	1510
TMAX [ms]	914	1.35	9.33	9.43	2840
	Modified $\mathcal{T}$				
	MPC (N=53)	FG	MPC (N=10)	FG+MPC (N=10)	$\mathcal{T}$ (Computed Offline)
TAVE [ms]	139	0.057	2.40	2.45	2250
TMAX [ms]	475	1.12	9.96	9.98	3710

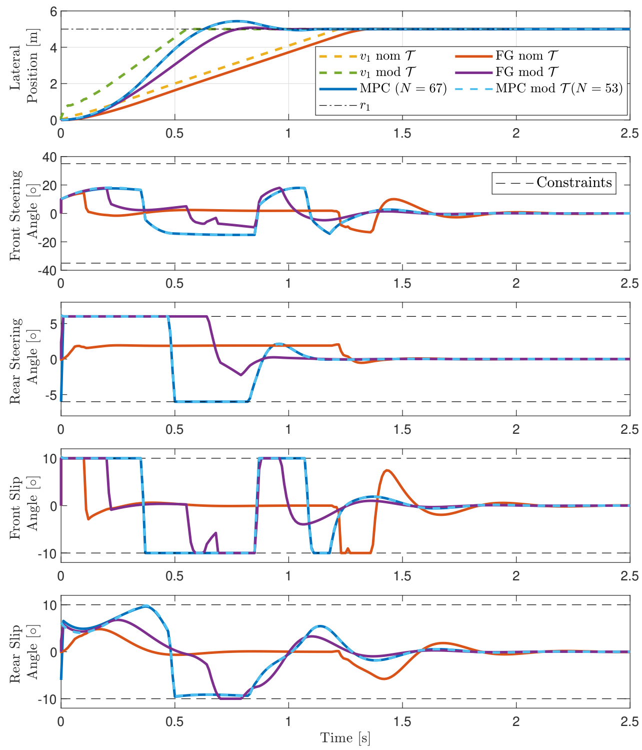

Fig. 7. Closed-loop two input lateral vehicle dynamic responses for the FG+MPC with $N = 10$ and both nominal and modified terminal set vs. an ungoverned long horizon MPC controller with $N = N^* = 67$. The rise/settle-time of the FG+MPC with nominal $\mathcal{T}$ is only marginally worse than the ungoverned MPC while the FG with modified $\mathcal{T}$ outperforms the MPC.

the FG+MPC in rise and settle times, Table I shows that both its average and max computation times[1] are over 50 times slower than the combined FG+MPC.

Figure 7 also demonstrates the behavior obtained using $R_\rho = 50\,R$ to determine $(K, \mathcal{T})$ as described in Remark 3. Due to the increased size of the terminal set, the prediction horizon required by the ungoverned MPC is now $N^* = 53$, which entails a slight reduction in computational cost with absolutely no loss in performance compared to the previous ungoverned MPC. When combined with the FG+MPC, the new terminal set leads to a significant increase in performance while maintaining a reduced computational cost.

VII. Conclusions

This paper introduced an add-on unit that guarantees the feasibility of finite-horizon MPC by manipulating its applied reference so that the final prediction is always contained in the terminal constraint set. The proposed method requires no modifications to the primary MPC controller and is supported by rigorous proofs of feasibility, safety, and convergence. Numerical comparisons between a traditional MPC with a long prediction horizon and a governed MPC with a significantly shorter prediction horizon show that the feasibility governor drastically reduces computational costs. The potential loss in closed-loop can partially be recovered by suitably modifying the terminal controller to increase the size of the terminal set.

[1]Timings are done on an ASUS-UX550VE (2.8 GHz i7, 16GB RAM) running MATLAB 2021b using `tic` and `toc`. The computation time for the terminal set (which does not depend on the prediction horizon N) is also included as an average over fifty test runs in Table I.

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